HIGH SCHOOL STUDENTS' MEANINGS OF CARTESIAN COORDINATE SYSTEM

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Boğaziçi University, 2016

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfilment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics and Science Education
Boğaziçi University
2022

ACKNOWLEDGEMENTS

I would like to begin by expressing my gratitude to my advisor, Assoc. Prof. Gülseren Karagöz Akar, for her endless support, guidance, constructive feedback and expertise throughout my research project. Learning from and with her was a wonderful experience.

I would like to thank my thesis committee members: Assoc. Prof. Sevda Yerdelen Damar and Assoc. Prof. Burak Karabey for allocating their valuable time and improving my research study with their insightful comments and suggestions.

I would like to thank all instructors at Boğaziçi University Science and Mathematics Education program. Special thanks to Assoc. Prof. Fatma Aslan Tutak for her mentorship, encouragement, and support in several ways throughout my undergraduate and graduate years.

Most importantly, I owe my deepest gratitude to my family who has always been there for me with their endless love and support. Especially I would like to thank my mother, Şafer Güven, for always supporting me in my decisions and encouraging me in every possible way.

ABSTRACT

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The aim of this study was to investigate high school students' reasoning of spatial and quantitative Cartesian coordinate systems, their meanings of a point on a graph, outputs of a function and graphing within spatial and quantitative coordinate systems. The study was conducted at a private high school in Istanbul towards the end of Spring semester in 2021-22 academic year. Participants consisted of 229 high school students from different grade levels who learned different kinds of functions and graphing in Cartesian coordinate system. The instrument with nine open ended questions was developed by the researcher based on the literature. Data were categorized by coded analysis and descriptively analyzed mainly according to framework for reasoning about graphs in spatial and quantitative Cartesian coordinate system and framework for representing a multiplicative object in the context of graphing. Analysis of the results showed that students had critical difficulty in conceiving axes as frame of reference to represent quantities as horizontal and vertical directed distances from origin in describing location of a point and coordinating quantities. Additionally, results showed that significant number of student students viewed x and y-coordinates and output of a function as a point on a graph hence carried non-normative meanings for point in terms of multiplicative object. Besides, students' difficulties with function and function notation were found to be an issue in their reasoning about points. Relatedly, results regarding graphing in Cartesian coordinate system pointed to students' difficulties in envisioning a graph as an emergent trace of multiplicative object representing changes in two quantities simultaneously. Students either focused on gross variation of the quantities and sketched a memorized graph, sketched a discrete graph, could not envision how the graph behaves between landmark points or sketched two separate graphs considering time as a secondary variable.

ÖZET

LİSE ÖĞRENCİLERİNİN KARTEZYEN KOORDİNAT SİSTEMİNİ ANLAMLANDIRMA BİÇİMLERİ

Bu çalışmanın amacı lise öğrencilerinin uzamsal ve niceliksel Kartezyen koordinat sistemlerini oluşturmadaki muhakemelerini, grafik üzerindeki bir noktanın ve bir fonksiyonun çıktılarını anlamlandırma biçimlerini, uzamsal ve niceliksel koordinat sistemlerinde grafikler hakkındaki muhakemelerini incelemektir. Araştırma 2021-22 eğitim öğretim yılı bahar döneminin sonuna doğru İstanbul'da özel bir lisede gerçekleştirilmiştir. Katılımcılar, farklı fonksiyon türlerini ve bu fonksiyonların koordinat sistemindeki grafiklerini öğrenmiş, farklı sınıf seviyelerinden olan 229 lise öğrencisinden oluşmaktadır. Dokuz açık uçlu sorudan oluşan ölçme aracı literatüre dayalı olarak araştırmacı tarafından geliştirilmiştir. Veriler literatürde bu alandaki teorik çerçevelere göre kategorilere ayrılarak betimsel istatistiklerle analiz edilmiştir. Analizin sonuçları öğrencilerin bir noktanın konumunu belirlemede ve nicelikleri koordine etmede eksenleri, nicelikleri orijinden yatay ve dikey yönlü uzaklıklar olarak temsil etmek için kullanılan bir referans çerçevesi olarak anlamlandırmada ciddi zorluklar yaşadıklarını göstermiştir. Ek olarak öğrencilerin önemli bir kısmının x ve y-koordinatlarını ve bir fonksiyonun çıktısını grafik üzerinde bir nokta olarak gördüklerini ve bu nedenle noktayı anlamlandırmada çarpımsal nesne açısından normatif olmayan anlamlar taşıdıklarını göstermiştir. Ayrıca fonksiyon ve fonksiyon notasyonuna dair yaşanılan zorlukların öğrencilerin noktalara ilişkin muhakemesinde sorun oluşturduğu tespit edilmiştir. Bununla bağlantılı olarak, sonuçlar öğrencilerin grafiği her iki niceliğin birlikte/ilişkili değişimini ifade eden ve çarpımsal nesnenin bu değişim sonucu koordinat sisteminde bıraktığı bir iz olarak görmede zorlandıklarına işaret etmiştir. Öğrenciler ya niceliklerin brüt değişimine odaklanarak bunu temsil eden ezberlenmiş bir grafik çizdiler, ayrık noktalardan oluşan grafik çizdiler, grafiğin çizilen belirli noktalar arasında nasıl davrandığını tasavvur edemediler ya da zamanı ikincil bir değişken olarak alan iki ayrı grafik çizdiler.

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LIST OF SYMBOLS

m	Meter
$\mathbb{R}$	Real Numbers
$\times$	Cartesian Product

LIST OF ACRONYMS/ ABBREVIATIONS

3D	Three Dimensional
A	UGA Arch
AFT	The Ant Farm Task
B.C.	Before Christ
C	Double-Barreled Cannon
CBMS	Conference Board of Mathematical Sciences
CCSM	Common Core Standards of Mathematics
DfA	Distance from Arch
DfC	Distance from Cannon
FAB	First American Bank
GT	Georgia Theatre
IEEE	Institute for Electrical and Electronic Engineering
NCTM	National Council of Teachers of Mathematics
NMO	Non-multiplicative Object
PTs	Preservice Teachers
QMO	Quantitative Multiplicative Object
S	Starbucks
STEM	Science, Technology, Engineering, and Mathematics
SoA	Statue of Athena
SQMO	Spatial-quantitative Multiplicative Object
WFB	Wells Fargo Bank

1. INTRODUCTION

Representations lie at the heart of learning and doing mathematics because representation and symbolization are at the core of the mathematical content and hence mental actions involved in mathematical activities (Kaput, 1987). Throughout K-12 education, students are expected to develop representational skills to model various phenomena, communicate mathematical language, select and apply mathematical representations fluently in problem solving (National Council of Teacher of Mathematics [NCTM], 2000). Coordinate systems are one of the most common used representational tools not only in the learning and doing mathematics but also in science, technology, mathematics, and engineering (STEM) areas (Paoletti, Rahman, Vishnubhotla, Seventko, & Basu, 2016; Roth, Bowen & McGinn, 1999). Research studies put forth significance of using Cartesian coordinate system in conceiving and representing relationships between two quantities' values and magnitudes, hence in students' development of *covariational reasoning*. Covariational reasoning is defined as "the cognitive activities [one] involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen & Hsu, 2002, p. 354) and is of great importance in reasoning about graphs in coordinate systems (Moore, Paoletti, Gammaro, & Musgrave, 2013) and precalculus and calculus topics such as exponential functions (Castillo-Garsow, 2010; Confrey & Smith, 1995), trigonometric functions (Moore, 2012), rate of change (Carlson et al., 2002; Thompson, 1994), function (Oertman, Carlson, & Thompson, 2008) and the fundamental theorem of calculus (Thompson, 1994). Primarily, Cartesian coordinate system allows representing attributes of two or more quantities on axes and uniting them by their orthogonal projection, hereby forming a *multiplicative object* in one's mind. Then, students can operate on these quantities and represent relationship between them (Thompson, Hatfield, Yoon, Joshua, & Byereley, 2017) which is also preliminary for covariational reasoning (Thompson, 2011). In support of this, several researchers (Frank, 2016, 2017; Stalvey & Vidakovic, 2015; Stevens & Moore, 2017; Thompson et al., 2017) reported that students face difficulties due to not conceiving points on a graph as multiplicative objects which therefore is not a trivial task. According to Thompson et al. (2017), forming a multiplicative object is a central ability for mathematical ideas of function, rate of change, accumulation, vector space as well as several physical quantities such as force, momentum, energy and so on. Thus, Cartesian coordinate system lays foundations for function and rate of change ideas (Thompson et al., 2017) as well as to reason about ratios and proportional relationships, number systems, geometry, algebra, functions, vector, and matrix quantities (Common Core Standards for Mathematics [CCSM], 2010; MEB, 2018). Additionally, Cartesian coordinate system establishes one-to-one mapping between points and real numbers that allows algebraic operations for spatial transformations such as rotations to be applied to geometric objects. Therefore, Cartesian coordinate system is used not only for representation but also transformation of geometric objects. So, with the help of coordinate plane, spatial processes can be described mathematically (Just & Carpenter, 1985). Since high school students use Cartesian coordinate system and related concepts in learning various fundamental mathematical concepts as well as science concepts (Paoletti et al., 2016; Potgieter, Harding & Engelbrecht, 2008; Roth et al., 1999), forming a solid understanding of Cartesian coordinate system and its relation to different mathematical concepts is of great significance particularly at high school level.

Despite their widespread use in mathematics, mathematics and science education, little instructional time is devoted to introducing the Cartesian coordinate system and plotting points (Lee, 2020; Schoenfeld, Smith, & Arcavi, 1993). Yet, to move flexibly between representations such as graphical and algebraic, and to understand abstract invariance between different representational systems such as different coordinate systems, representations must be symbolized at mental operations level (Lee, Moore, & Tasova, 2019). Researchers point that neither quantities and relationships among them nor students' representations of them in coordinate systems should not be taken as granted to problem situations (Moore, 2013; Moore & Carlson, 2012). Researchers further emphasize that no matter how easy it seems, even plotting a point in the coordinate plane at a quantitative level is a nontrivial task. For example, students can consider the point (2, 3) as counting jumps, 1-2 units over and 1-2-3 units up, whereas they are supposed to consider coordinates as projection of two quantities' magnitudes represented on the axes (Frank, 2016).

Unfortunately, review of the literature pointed out to several difficulties in constructing and interpreting graphs in Cartesian coordinate plane that students face from middle school to undergraduate level (Clement, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Moore & Thompson, 2015). In addition, undergraduate students (Montiel, Vidakovic, &

Kabael, 2008) and preservice teachers (Moore, Silverman, Paoletti, Liss, & Musgrave, 2019) carried conventions of Cartesian coordinate system to Polar coordinate system. Also, Frank (2017) reported that undergraduate students struggled in constructing graphs as emergent trace of points that unite two varying quantities' magnitudes. Similarly, Thompson (2016) reported that 29 of 111 high school teachers had static shape thinking, which can negatively impact students' thinking, because teachers considered graphs as wire-like shapes instead of emergent trace of covariation of quantities. On the other hand, even though preservice teachers were able to conceive quantitative relationships, unconventional aspects of graphs or coordinate system such as thinking y as the horizontal, x as the vertical axis perplexed them, and affected their ability to understand and evaluate student work (Moore et al., 2013). Several research studies (Frank, 2016, 2017; Stalvey & Vidakovic, 2015; Stevens & Moore, 2017; Thompson et al., 2017; Thompson & Carlson, 2017) reported that part of the difficulties described above stemmed from inability to conceive points as multiplicative objects. In their study with middle school students, Tasova and Moore (2020) presented a framework for different ways that a multiplicative object can be conceptualized and presented alternative meanings of a coordinate system and coordinate points students can have in graphing activity (Tasova & Moore, 2021). However, there is limited number of studies on what kind of meanings students hold for coordinate systems, especially at secondary level. Goldin and Shteingold (2001) stated that for effective mathematics teaching, we should have access to how students understand external representations internally, what meanings and relationships students form and how this reflects on their mathematical activities. This is important especially because a certain graph in Cartesian coordinate system can only make sense when meanings and conventions are established for the Cartesian coordinates on the part of students.

Therefore, the goal of this study is to investigate how high school students reason about Cartesian coordinate system, points and graphs within Cartesian coordinate system. Particularly, characteristics of students' reasoning in Cartesian coordinate system with different uses, their meanings of a point, outputs of a function on a graph and a graph as a whole will be investigated. Maher and Davis (1990) also stated that it is essential for teachers to be aware of the nature of their students' thinking and representations. In this respect, this study can help teachers turn their classroom into more effective learning environments and help prevent possible misunderstandings regarding Cartesian coordinates. Besides, most of the previous research studies were conducted with preservice teachers or middle school students. There seems to be a need for more research studies at high school level with a larger sample size, as many of the studies are small scale teaching experiments. Findings of this study can provide insight for practitioners, curriculum developers and mathematics educators and fill this gap in mathematics education literature.

2. LITERATURE REVIEW

The purpose of this section is to understand conceptions and uses of coordinate systems and to present a brief review of the literature on studies related to coordinate systems in mathematics education. Firstly, history of coordinate systems will be presented with an emphasis on their significance in mathematics and mathematics education. Then a summary and a critique of the literature will be presented.

2.1. History and Definition of Coordinate Systems

Coordinate systems have played a critical role in the development of science and mathematics. They still have a widespread use in several branches of mathematics and constitute an integral part of mathematics education at various grade levels. Coordinate systems are highly abstracted mathematical tools that are developed throughout history. Around 3200 B.C. the Egyptians used coordinates to survey their lands and to show location of points in real space. Also, circa 1500 Leonardo Da Vinci used rectangular coordinates to analyze velocity of falling objects (Beniger & Robyn, 1978). Historically, coordinates were mainly used in surveying and map projections as a tool to record positions in two-dimensional or three-dimensional space. For instance, geographical coordinates had been used to define a position on a three-dimensional spherical or ellipsoid body. Every map projection creates a different reference system and representations for points. Amongst all possibilities, coordinate systems that are easy to understand and simple to express are preferred (Maling, 1992). In addition to surveying and map projections, by representing positions and space, coordinate systems have been used to study determine patterns, relationship between quantities and study functions. For instance, in the 14^{th} century, Thomas Bradwardine used sort of a histogram to depict the relationship between velocity, acceleration and distance. In the same century, with the intent of studying rate of change, Nicole Oresme used columns to show value of velocity at a given point in time. This graphical representation was a useful tool in analyzing how variable changes over time and relationships between variables (Anderson, 2008). These examples put forth the significance of coordinate systems and why they are integral part of mathematics and mathematics education.

There are different types of coordinate systems that are used in mathematics, engineering, physics, applied sciences and technology such as Cartesian, Polar, Homogeneous, Spherical, Curvilinear, and Log-polar coordinate system. Commonly, a coordinate system can be defined as set of rules for mapping pairs of [real] numbers to points of the plane or the space (Borji & Voskoglou, 2016). Alternatively, a coordinate system for a plane can be determined by associating a coordinate set/label (u, v), where u and $v \in \mathbb{R}$, to each point of the plane. A *coordinate set* consists of an ordered pair of numbers that are measured according to the rule of the coordinate system. In general, a coordinate system should satisfy the following conditions (Singer, 1941):

> Each point of a coordinate plane should have at least one label.
> If P and Q are distinct points, no label attached to P should be the same as any label attached to Q.
> If a point P moves continuously in the plane, the coordinates of any one of its labels should change continuously (p. 50).

Cartesian and polar coordinate systems are usually referred as conventional coordinate systems since conventional tools are used to designate coordinates of a point of the plane (Maling, 1992). Between the two, Cartesian coordinate system has a substantial place in both STEM areas and science and mathematics education. In the seventeenth-century, Descartes and Fermat connected algebra and geometry by means of coordinate systems as follows. Curves can be represented algebraically in two variable equations and graph of such equations can be represented as curves in the coordinate system. Thus, curves and equations are different notions of the same idea (Bos, 1993). The connection between coordinate system and algebra laid the foundations for analytic geometry (Anderson, 2008). Although Pierre de Fermat was the first mathematician to use orthogonal axes system in graphing equations, the conventional Cartesian coordinate system was named in 1673 after Rene Descartes whose Latin name is pronounced as "Cartesius". The Cartesian plane defined for two-dimensional space is constructed by the intersection of two orthogonal real number lines. The Cartesian plane is defined as the set of ordered pairs (x, y) with real numbers x and y, namely the product set $\mathbb{R} \times \mathbb{R} = \{(x, y) | \ x \ \in \mathbb{R}, y \ \in \mathbb{R}\}$. Thus, it can also be referred as real coordinate system, denoted by $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$ (Batschelet, 2012) where the horizontal axis is usually denoted by x and the vertical axis is denoted by y and the intersection point of x and y- axis is called the Origin, $(0,0)$. Thus, a certain point of the coordinate system is represented by (x, y). This coordinate system applies to two and three-dimensional space (Levenberg, 2015).

Coordinate system serves as a tool to construct quantities, obtain measures by partitioning quantities based on a unit magnitude and representing these values simultaneously to represent relationships between quantities (Moore, Stevens, Paoletti, Hobson, & Liang, 2019). Mathematical concepts related to two-dimensional orthogonal space, such as coordinates, grids, rectangles etc. are considered as part of foundations of applications of mathematics in real life, advanced mathematics, analytic geometry and the concept of functions in general (Sarama, Clements, Swaminathan, McMillen & González Gómez, 2003). Therefore, Cartesian coordinate system is vital laying the foundation of several mathematical concepts. Also, Cartesian coordinate system is an integral part of mathematics curriculum (CCSM, 2010; MEB, 2018). In the Common Core Standards of Mathematics (2010), students are introduced with the coordinate system at the 5th grade with the expectation to graph points on the coordinate plane and solve real-life problems using the Cartesian coordinate system where it is defined as follows:

> Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p.38)

In the Common Core Standards for Mathematics, students are expected to use Cartesian coordinate system effectively in geometry, algebra, and statistics for different purposes starting from early grades of schooling (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Similarly, in the Turkish National High School curriculum, Cartesian coordinate system is introduced at 8th grade for the first time and several objectives require the use of Cartesian coordinate system at different grade levels in high school. For instance, at the 9^{th} grade, students are expected to explain that geometric representation of $\mathbb{R}$ is number line, and representation of $\mathbb{R} \times \mathbb{R}$ is Cartesian coordinate plane. They are also expected to define unit circle and form a relationship between trigonometric ratios and the coordinates of a point on the unit circle. In addition, at the 11^{th} grade, they are expected to reason about the relationship between a point and a line and line segment as well as doing computations with lines in the Cartesian plane called also as analytical plane. Finally, at the 12^{th} grade, using the points in the plane, students are further expected to form and understand isometries such as translation, rotation, and reflection as functions in the Cartesian plane.

In addition, as stated in The Preparation of High School Mathematics Teachers (Conference Board of Mathematical Sciences [CBMS], 2012), one of the most powerful tools in mathematics is the connection between algebra and geometry. Algebraic computations can be used in reasoning about geometry. On the other side, geometric visualizations clarify algebraic methods, some calculus concepts such as limits, probability and visualization of data in statistics. Since coordinate systems play an essential role in this reciprocal connection, teachers need to have quantitative understanding and facilitate variety of methods associated with coordinate systems. This way, students can build meaningful understanding of coordinate systems.

2.2. The Uses and Importance of Conventional Coordinate Systems

In this section, I describe coordinate system as a representational tool along with the importance of representations in mathematics education, two different uses of coordinate system as spatial and quantitative, graphing in spatial and quantitative Cartesian coordinate systems, and conceptions of multiplicative object as graphical representation of a point in Cartesian coordinate system.

Representations are pointed out as essential tools in supporting students' understanding of mathematical concepts and relationships, communicating mathematics, applying mathematics to realistic problem-solving situations. Conventional coordinate systems are common representational tools in learning and doing mathematics (NCTM, 2000). In this regard, especially Cartesian coordinate system is a significant tool in reasoning about number systems, functions, equations, inequalities, geometric figures, ratios, and proportional relationships in mathematics (CCSM, 2010). Lee (2017) defined *coordinate system* as “a system through which one quantitatively organizes (Piaget, Inhelder & Szeminska, 1960) or coordinatizes points in the space being re-presented” (p. 3). Similarly, Lee, Hardison, Kandasamy and Guajardo (2020) referred to coordinate system as a representational space in which one systematically coordinates quantities to organize some phenomenon.

Representation and visualization are central to understanding and doing mathematics (Duval, 1999; Arcavi, 2003). Largely history of mathematics is about constructing and revising representational systems and much of the teaching of mathematics is about learning these representational systems and to use them in problem solving (Lesh, Landau & Hamilton, 1983). Duval (1999) asserts that unlike some other disciplines (such as biology, physics, geography etc.), objects in the study of mathematics are not directly available to senses. Thus, perceiving mathematical concepts relies only on the use of semiotic/external representations such as sentences, equations or geometric configurations since they are means of mathematical thinking processes. In mathematics, there are several *registers* of representations such as tables, graphs, diagrams, symbolic expressions and formal language. Understanding mathematics implies clear distinction between the representation and the mathematical object being re-presented as well as successful transition between different registers used to represent the same mathematical object (Duval, 1999). As an example, the internal organization of a graphical representation is different than internal organization of an algebraic equation. They provide diverse meanings regarding a mathematical object such as functions. In *Principles and Standards for School Mathematics*, NCTM (2000) underlines the significance of representation beginning from early stages of schooling in the mathematical Processes Standards. According to Representation Standard:

> Instructional programs from prekindergarten through grade 12 should enable all students to: Create and use representations to organize, record and communicate mathematic ideas; Select, apply, and translate among mathematical representations to solve problems; Use representations to model and interpret physical, social, and mathematical phenomena (p. 67).

Nevertheless, development of representations and constituting a meaningful understanding for them is not an easy process. It takes time and effort although it may not seem like it, as they are inherent part of doing and learning mathematics (NCTM, 2000). Several research studies (Artigue, 1992; Brenner et al., 1997; Hiebert & Carpenter, 1992; Hollar & Norwood, 1999; Leinhardt et al., 1990; Moon, Brenner, Jacob, & Okamoto, 2013; Mousoulides & Gagatsis, 2004; Thompson, 1994) also revealed the significant role representations play in mathematical understanding and problem solving. However, as Von Glasersfeld (1987) put forth: “A representation does not represent by itself — it needs interpreting and, to be interpreted, it needs an interpreter” (p. 216). Learning mathematics with understanding requires conceptual understanding of representations utilized. For instance, although high school students had experience with Cartesian coordinate system, Lee (2020) found that using or constructing a Cartesian coordinate system was not their first choice in a task providing experientially real context to reason about objects in space. This implies that constructing a coordinate system is not a trivial task. In addition, Knuth (2000) pointed out the difficulties high school students had in forming the connection between algebraic and geometric representations which he referred to as “Cartesian connection” in his study with 178 first year algebra high school students. Accordingly, 75% of the students relied on algebraic solutions over graphical solutions even though questions were designed to encourage use of graphical solutions. Many students did not even recognize graphical solution as an option; only one third of the students utilized graphical register as their first or alternative solution method. Some possible reasons of students’ failure in perceiving representations from the process perspective were listed as instructional/curricular overreliance on algebraic methods and students’ inability to conceive points on the lines as solutions to algebraic equations. Similarly, many students between age of 15-17 could not distinguish $y = 2x$ and $y = x + 2$ by looking at their Cartesian graphs although they were successful in standard tasks like sketching graphs from equations and vice versa (Duval, 1988). Hence, we can conclude that regardless of grade level, all students need to develop a solid understanding of mathematical ideas captured in conventional representations (NCTM, 2000) such as Cartesian coordinate system and related notions.

Coordinate system is constructed and interpreted by a cognizing subject in a goal directed activity (Lee et al., 2020). Lee, Moore and Tasova (2019) defined coordinate system as “a mental system of coordinated measurements [of quantities] obtained through coordinating multiple frames of reference" and a *frame of reference* as "a mental structure through which an individual situates a quantity where the structure is constructed through the process of committing to a reference point, a unit measure, directionality of measure comparison" (p. 82). In this cognitive activity, preceding quantitative organization of space, qualitative organization of space with the use of frame of reference is needed. A frame of reference can be considered as mental structure that an individual uses to determine location of the objects of the perceptual space (Rock, 1992). Similarly, Soetching and Flanders (1992) indicated that location of a point can be re-presented within a frame of reference through coordinate systems or using vectors. Joshua, Musgrave, Hatfield and Thompson (2015) described frame of reference as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 32). According to Tversky (2003), mental representations of space are constructed based on elements in the space and spatial relations between them relative to a frame of reference. Therefore, frame of reference serves as a scaffolding to define qualitative spatial relations among elements in the space and hence qualitative organization of space (Piaget et al., 1960). Moreover, Joshua et al. (2015) distinguished frame of reference from coordinate systems such that conceptualizing a frame of reference corresponds to mental activity whereas coordinate system is the product of mental activities involved in conceptualizing and coordinating multiple frames of reference. In this study, as in Joshua et al. (2015), Lee (2017) and Sarama et al. (2003), I assumed frame of reference to be a mental construction established by an individual which is abstracted from elements of the space being re-presented.

2.3. Different Uses of Coordinate Systems

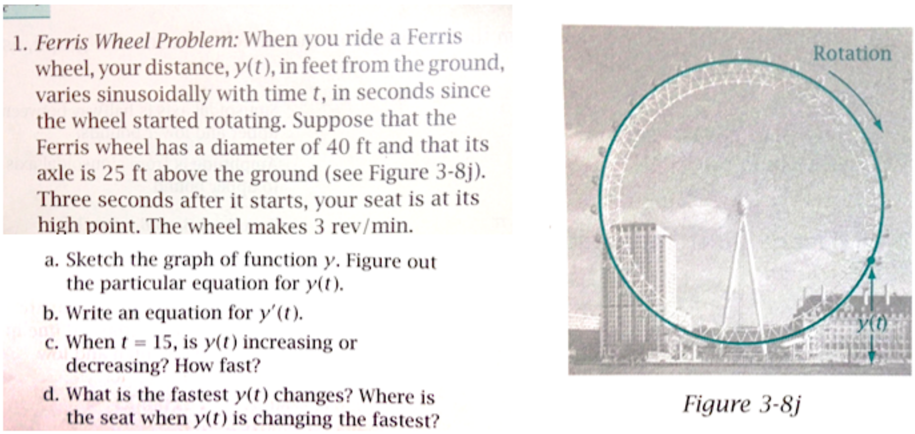
Lee and Hardison (Lee, 2016; Lee & Hardison, 2016; Lee, 2017) described two different uses of coordinate systems in students' thinking: situational/spatial and quantitative coordination.

In *spatial coordination,* coordinate system is used to "re-present or mathematize a space or physical phenomena." Constructing situational coordinate system entails establishing frames of reference to measure various attributes of the objects in the space and coordinating these quantities to locate points within the space. In a situational/spatial coordinate system, frames of reference are established and used to construct and label quantities onto the situational space or physical phenomena (Lee, Hardison, & Paoletti, 2018). For instance, in Task-I (see Figure 2.1) a spatial coordinate system is laid onto the Ferris Wheel such that the axle is located at the origin. Then the location of the car at a specific moment is described by Cartesian coordinates found by orthogonal distance from the origin. Some other examples of spatial coordination can be listed as using a map to describe location of specific objects, working on geometric transformations in the Cartesian coordinate system or using coordinate system to mathematize a physics problem such as describing position of an object with position vector in relative to the origin.

31. **Entertainment** The Ferris Wheel first appeared at the 1893 Chicago Exposition. Its axle was 45 feet long. Spokes radiated from it that supported 36 wooden cars, which could hold 60 people each. The diameter of the wheel itself was 250 feet. Suppose the axle was located at the origin. Find the coordinates of the car located at the loading platform. Then find the location of the car at the 90° counterclockwise, 180°, and 270° counterclockwise rotation positions.

Figure 2.1. Task-I (Holliday et al., 2006, p. 95).

In *quantitative coordination,* coordinate system is used to obtain a geometrical representation of the product of the measure spaces by coordinating sets of quantities. In this system, quantities are established within and extracted from the space/situation and projected onto a new space. For instance, in Task-II (see Figure 2.2), the height of the car from the ground and time are extracted from the problem situation and their varying magnitudes are represented on number lines. By orthogonal intersection of two lines, Cartesian coordinate plane is constructed where time is represented by the horizontal axis and distance from the ground by the vertical axis. By Cartesian product of two measures, namely distance and time, $\{time \times distance\}$ a two-dimensional coordinate system is constructed which is different than the space containing the problem situation.



1. *Ferris Wheel Problem:* When you ride a Ferris wheel, your distance, $y(t)$, in feet from the ground, varies sinusoidally with time t, in seconds since the wheel started rotating. Suppose that the Ferris wheel has a diameter of 40 ft and that its axle is 25 ft above the ground (see Figure 3-8j). Three seconds after it starts, your seat is at its high point. The wheel makes 3 rev/min.
   a. Sketch the graph of function y. Figure out the particular equation for $y(t)$.
   b. Write an equation for $y'(t)$.
   c. When $t = 15$, is $y(t)$ increasing or decreasing? How fast?
   d. What is the fastest $y(t)$ changes? Where is the seat when $y(t)$ is changing the fastest?

Figure 3-8j

Figure 2.2. Task-II (Foerster, 2005, p. 112).

Therewith, Lee et al., (2018, 2020) distinguished between the two types of coordinate systems based on their intended uses as *spatial* and *quantitative* coordinate systems. According to this, both coordinate systems necessitate coordination of quantities. The distinction between spatial and quantitative coordinate systems foregrounds the mental actions involved in coordination of quantities. Constructing *quantitative coordinate system* requires first establishing spatial coordinate system in terms of relevant quantities in the situation/space before extracting them. A *spatial Cartesian coordinate system* can be constructed by a reference point and two orthogonal lines passing through reference point. An individual can quantitatively describe locations of points or objects within the space in terms of horizontal and vertical distance from the reference point (Lee et al., 2020) as shown in Figure 2.3. In this case, the location of the point is described as logical multiplication (Piaget et al., 1960) of horizontal and vertical displacement from the reference point such as the intersection of black orthogonal lines in Figure 2.3. Also, question 6a in the inventory (see Appendix A) is another example necessitates spatial Cartesian coordinate system as it asks to represent the location of a person in a rectangular garden mathematically. In quantitative Cartesian coordinate system, two quantities are disembedded from a situation/phenomenon and overlaid onto two orthogonal number lines. A point is formed by the intersection of orthogonal projections from each quantity on each number line (see Figure 2.4 and 2.5) and corresponds to the representation of "multiplicative object" in the context of graphing (Tasova & Moore, 2020).

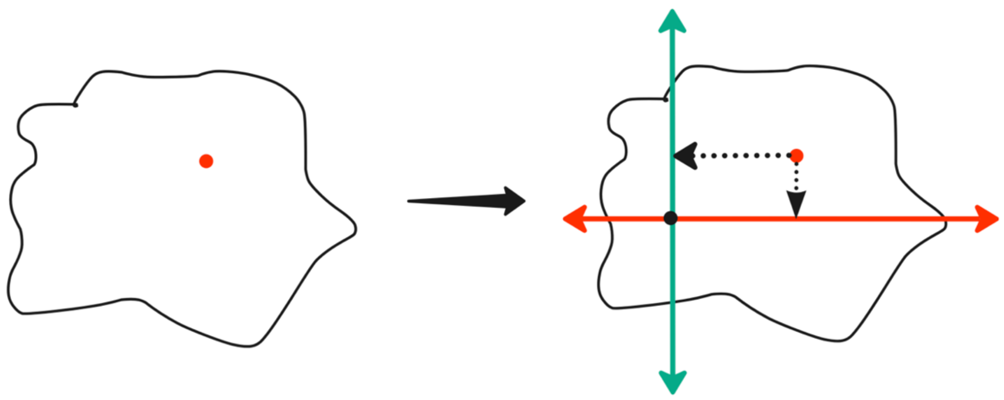


Figure 2.3. Spatial Cartesian coordination (constructed by Lee et al., 2020, p. 932 and drawn by the researcher).

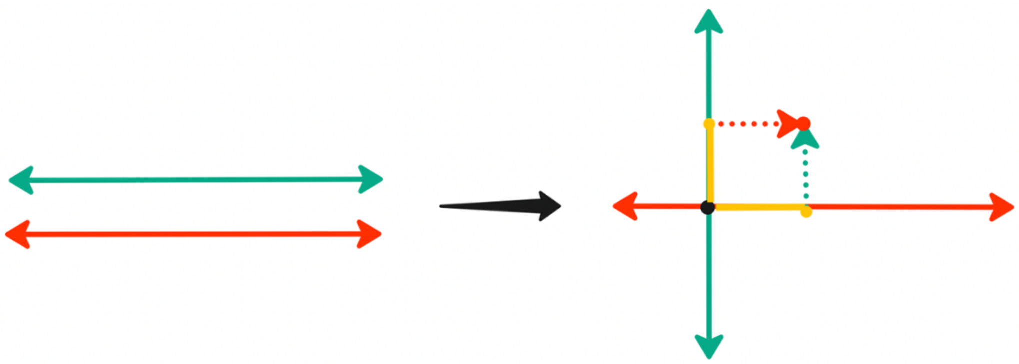


Figure 2.4. Quantitative Cartesian coordination (constructed by Lee et al., 2020, p. 932 and drawn by the researcher).

Multiplicative object can be seen as a conceptual object formed by "holding both quantities' values (magnitudes) simultaneously" (Saldanha & Thompson, 1998, p. 1). More broadly, it is formed as a result of uniting two or more quantities' values or magnitudes in mind simultaneously (Thompson, 2011; Thompson & Carlson, 2017; Thompson et al., 2017). In this respect, Frank (2016) provided a model for understanding a point in the Cartesian coordinate plane. According to the model, a point represents two quantities simultaneously by projecting orthogonally onto horizontal and vertical axes (see Figure 2.5). Frank further elaborated on the quantities using the model as the two quantities represented as directed distance from the [reference point] origin to the end point of the segment along the respective axes and can be discussed in terms of frames of reference. In Cartesian coordinate system, understanding point as a representation of a multiplicative object provides productive meanings for points and graphing (Thompson & Carlson, 2017; Frank, 2017; Stalvey & Vidakovic, 2015; Stevens & Moore, 2017; & Thompson, Hatfield et al., 2017).

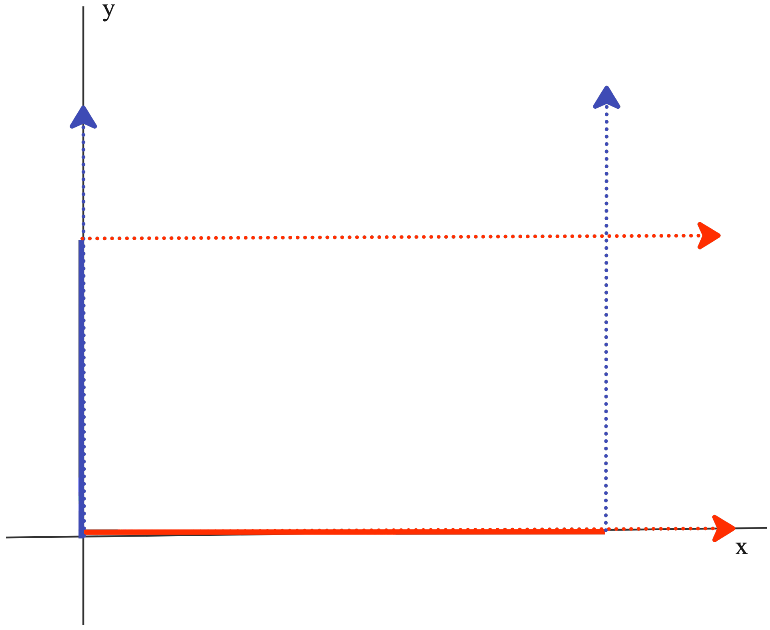


Figure 2.5. Representation of a point as projection of two quantities' magnitudes extended from the axes (constructed by Frank, 2016, p. 574 and drawn by the researcher).

Similarly, Karagöz Akar, Zembat, Arslan and Belin (2022) suggested conceptualizing a point, for instance (a, b), as combination of directed lengths between origin and $(a, 0)$, and origin and $(0, b)$ rather than combination of just two labels. In other words, by coupling magnitudes of two quantities, namely vectors $\binom{a}{0}$ and $\binom{0}{b}$, one can form a multiplicative object by uniting their orthogonal projection in mind. Furthermore, they proposed that conceptualizing a point this way may help students develop understanding of $\mathbb{R}^2$ as made up of such points and every point is a subset of $\mathbb{R}^2$.

In this regard, a framework for conceptualizing multiplicative object was shared (Tasova & Moore, 2020; Tasova, 2021): quantitative multiplicative object, spatial-quantitative multiplicative object and non-multiplicative object.

Quantitative Multiplicative Object (QMO): There are two types of quantitative multiplicative object one can envision.

Type-1: This corresponds to a student who "envisions points as a circular dot that represents two quantities' magnitudes or values simultaneously and envision that points on a graph do not exist until they are physically and visually plotted." (p. 241). In this case, line can be perceived as a direction or route for points instead of an object constructed by emergent trace of quantities.

Type-2: This corresponds to a student who "envisions a point as an abstract object that represents two quantities' magnitudes or values simultaneously, and envision a graph as composed of infinitely points, each of which represent two quantities' values or magnitudes, which is an indication of emergent shape thinking." (p. 241). As visualized in Figure 2.4 and Figure 2.5, students would identify the quantities, extract them from the situation and represent them as magnitudes on orthogonal axes. Then a point would be formed by uniting orthogonal projection of these two quantities' magnitudes.

Spatial-Quantitative Multiplicative Object (SQMO): In this category, students represent object's location by coordinating and representing two (measurable) attributes of the object *in* the plane, instead of representing those attributes *on* the axes of the plane. For instance, in question 7 a map with several locations marked is given (see Appendix A) and students were asked to describe what each point in the coordinate plane corresponds to on the map. A student who envisions a point as SQMO might describe points not in terms of magnitudes on the axis, but in terms of the distances on the map. The only difference between QMO and SQMO is that, in SQMO non-normative reference point or frame of reference is used to measure or represent quantities. Figure 2.6. illustrates an example response from a preliminary study by Tasova (2021) where a student extracted distance from Arch and Canon from the physical space/map but did not represent them *on* the axes and form the multiplicative object by taking their orthogonal projection. Instead, the student considered Arch and Cannon as actual points on the vertical and horizontal axis respectively, then coordinated distance from Arch and distance from Cannon *in* the plane in reference to these non-normative reference points. Lastly, envisioning a point as a spatial quantitative multiplicative object does not necessarily imply representing relationship on spatial coordinate system (Tasova & Moore, 2020).

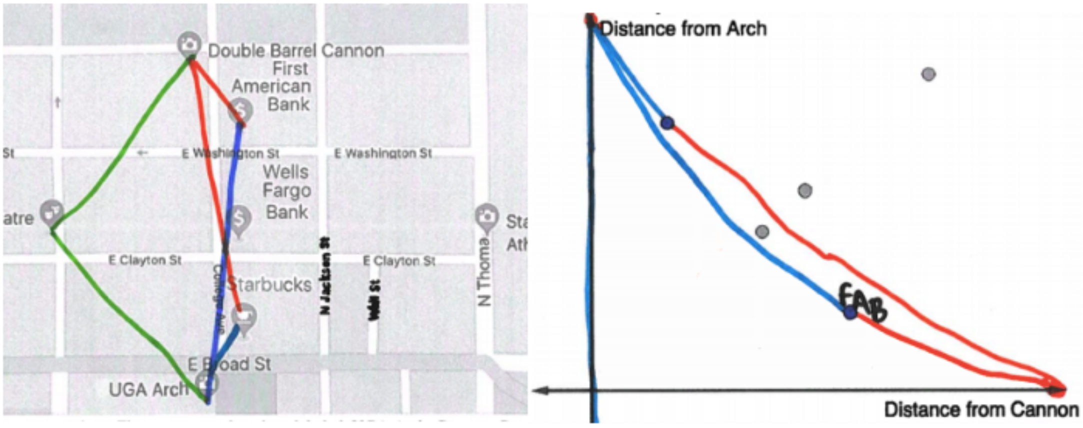


Figure 2.6. Example for a spatial-quantitative multiplicative object (Tasova, 2021, p. 600).

Non-Multiplicative Object (NMO): Representing a non-multiplicative object entails envisioning points on a coordinate plane as a location/object by assimilating figurative and perceptual aspects of the situation. In this case, point is not constructed by uniting magnitudes of two quantities and students do not consider point as a representation of two quantities laid on orthogonal axes. Also in this category, a point only represents an ordered pair of two numbers and location of a point in the coordinate plane rather than coupling of magnitudes of two quantities. Some examples are perceiving $f(a)$ as a location of the point, $(a, f(a))$ as the value of the function whereas in fact "a" is magnitude on x-axis and "$f(a)$" is a magnitude on the y axis. Tasova (2021) distinguished two different meanings for NMO in the coordinate system: *iconic translation* and *transformed iconic translation*. In iconic translation, points reflect literal pictures of the situation. In other words, students assimilate points onto the coordinate plane according to perceptual features of the situation. Here students basically copy-paste an image of the phenomenon onto the coordinate plane. Students do not construct points by quantitative reasoning. For instance, in question 7a (see Figure 5.17) locations on the map might be paired with points in the coordinate plane according to their literal position on the map rather than using quantities extracted from map. On the other hand, when points are represented as NMO and transformed iconic translation, students translate a transformed version of a situation on the map whereas iconic translation involves translation of the situation as it is. Students might rotate, reflect a coordinate plane or the phenomenon itself to represent the points. For instance, in question 7a (see Figure 5.18) students might rotate the map or coordinate plane so that points in the coordinate plane match with the locations labeled on the Downtown Athens map.

In order to construct a two-dimensional Cartesian coordinate system quantitatively, one needs to measure two quantities by combining two frames of reference simultaneously (Drimalla et al., 2020). As Lee, Moore and Tasova (2019) indicated, this is a nontrivial task. Not constructing meaningful mental structures for coordinate systems may retain students from constructing and interpreting graphs. Furthermore, it may give rise to lack of understanding of some other mathematical ideas such as expressions from calculus statements (Parr, 2021).

2.4. Graphing in Spatial and Quantitative Cartesian Coordinate Systems

Moore and Thompson (2015) differentiated between students' static and emergent shape thinking while producing and interpreting graphs within coordinate systems. Students who engage in *static shape thinking* conceive graphs as static shapes that can be rotated, shifted, and they conceive equations and functions as property of the shape. In this case, students' thoughts are dominated by figurative thinking where students rely on perceptual information in interpreting and producing graphs. For instance, a line going down from left to right would indicate negative slope even if it does not in a non-canonical coordinate system where y-axis refers to the horizontal and x-axis refers to the vertical axis. On the contrary, in *emergent shape thinking,* students reason about graphs in terms of quantities and their simultaneous covariation which is dominated by operative thinking where students focus on quantities and operations on quantities rather than perceptual conventions.

Paoletti, Lee and Hardison (2018) combined students' reasoning about producing and interpreting graphs within coordinate systems: static and emergent shape thinking (Moore & Thompson, 2015) with the two uses of coordinate systems: spatial and quantitative coordinate systems (Lee, 2016; Lee & Hardison, 2016; Lee, 2017) as shown in Table 2.1.

Table 2.1. Framework for reasoning about graphs in spatial and quantitative Cartesian coordinate systems (Paoletti et al., 2018, p. 1318).

		Uses of Coordinate Systems (Lee & Hardison, 2016)	
		Spatial Coordination	**Quantitative Coordination**
Ways of Reasoning About a Graph (Moore & Thompson, 2015)	**Emergent Reasoning**	Emergent thinking within a spatial coordinate system	Emergent thinking within a quantitative coordinate system
	Static Reasoning	Static thinking within a spatial coordinate system	Static thinking within a quantitative coordinate system

A student who reasons *emergently in spatial coordinate system*, perceives a phenomenon happening and leaving a trace in the space. Student overlays a coordinate system onto the space and constructs a multiplicative object as simultaneously uniting two quantities represented in the spatial coordinate system. Then the graph represents emergent trace of two quantities represented on horizontal and vertical axis. For instance, in question 6b (see Appendix A), where students are asked to describe the location of the person (represented by red dot/point in the problem situation) throughout its journey toward the exit gate, a student who thinks emergently in spatial coordinate system is expected to describe the location of the red point in terms of the point's horizontal and vertical distances from the origin. Similarly in question 6a, the person's location is described by coupling magnitudes of two quantities which are measured according to frame of reference laid onto the space (i.e. rectangular garden) forming a multiplicative object.

In static *shape thinking in spatial coordinate system*, a student establishes frames of reference (horizontal, vertical axis, reference point), yet describes the graph as a memorized mathematical formula rather than covariation of two simultaneously represented quantities. For instance, a student who thinks statically in spatial coordinate system might describe the logo on the pool table given in question 8a in the inventory (see Appendix A) with a semi-circle equation.

Emergent shape thinking in quantitative coordinate system entails conceiving two covarying quantities in the situation, disembedding them from the situation and then representing these magnitudes on two orthogonal number lines to represent the relationship between them. For instance, in question 9 (see Appendix A), a student would first measure distance of the car from Bolu and Istanbul on the route, lay those quantities on the horizontal and vertical axis respectively and form a multiplicative object by uniting their orthogonal projection. Then by tracing changes in two quantities simultaneously, one would envision the graph as emergent trace of multiplicative object representing two quantities.

A student who engages in *static shape thinking in quantitative coordinate system* constructs a graph resembling visual features of phenomenon or assimilating thematic features of the situation. For instance, in question 8b and question 9 in the inventory, a student could graph relationship between two quantities by resembling it to physical traces of the ball and the car respectively.

2.5. Previous Research on Coordinate Systems

Despite their extensive use in teaching and learning in mathematics, coordinate systems are taken for granted by researchers, teachers, and curriculum developers (Lee 2017; Lee et al., 2018). Usually, little instructional time is devoted for construction of Cartesian coordinate systems (Schoenfeld et al., 1993). Instead, the rule of Cartesian Coordinate system is introduced, and then students are expected to use it as a tool to reason about number systems, functions and geometric figures (CCSM, 2010; Lee, 2020; MEB, 2018).

Despite the substantial use of coordinate systems, students face several challenges in constructing and interpreting graphs (Clement, 1989; Leinhardt, Zaslavsky & Stein, 1990; Moore & Thompson, 2015) both in mathematics and science courses (Potgieter, Harding & Engelbrecht, 2008). Many middle grade students have difficulty in reading and plotting points correctly in Cartesian coordinate system such as reversing the x and y-coordinate (Battista, 2007; Tillema & Gatza, 2007). Frank (2016) found that undergraduate students could not plot points when the values were given on the axis, even though they successfully labeled the given points. In construction of graphs, students found to struggle establishing and scaling axes, making the transition from discrete to continuous graphs (Herscovics, 1989) or connecting points without understanding the continuous relationship between quantities (Yavuz, 2010). In sketching and interpreting, students may conceive graphs as representing physical features of a situation (Clement, 1989; Oertman et al., 2008) or focus only on one variable whereas they are expected to interpret the relationship between two quantities represented by Cartesian graphs (Leinhardt et al. 1990; Oertman et al. 2008). In their study examining the role of Cartesian connection for conceptualization of trigonometric functions, Demir (2012) and Marchi (2012) stated that most students failed to visualize graphical representations and to connect them correctly to Cartesian plane. Similarly, in another research study (Moon, 2019) majority of preservice secondary teachers used rules like greater is upper and less is lower rather than reasoning about the concept of variable or Cartesian connection. Aforementioned research studies highlight once again that introducing the rules of Cartesian coordinate system is not sufficient for using it effectively as a tool in learning and doing mathematics. Thus, investigating meanings that students hold for coordinate systems are important. This way we can help students use coordinate systems productively in doing mathematics and learning new concepts. Investigating how high school students reason about Cartesian coordinate system, point as multiplicative object and graphing in Cartesian coordinate system may also help them express their mathematical thinking and understanding in more proper and formal ways unlike what Moon (2019) revealed for preservice teachers.

Aforementioned studies also point out that from middle school to undergraduate level, students experience similar obstacles working with Cartesian coordinate system. However, until not long-ago researchers did not view coordinate systems as mental structures to be developed by students (Lee et al., 2019). Therefore, the number of research studies investigating students' understanding and construction of Cartesian coordinate system is limited, especially at high school level. In this respect, Lee et al. (2020) conducted a teaching experiment The Ant Farm Task (AFT) with four preservice teachers (PTs) who were in middle or elementary grade teacher preparation program. The major goal of this research study was to investigate how preservice teachers construct Cartesian coordination and reason about coordinate systems and to find-out how their thinking changed throughout the teaching sessions. Either individually or in pairs, preservice teachers attended in total of eight 60-minute-long teaching sessions. PTs were given two plastic tubes accompanied by a dynamic geometry environment to model two ant farms in which giant two ants were moving haphazardly. The task required PTs to represent two ants' location with a single point. Results on one of the PTs, Ginny, showed that manipulating tubes as non-fixed objects was a critical cognitive step in her thinking. First, Ginny established a spatial coordinate system by placing two number lines (zero in the middle) on parallel tubes to describe ants' location in the tube-1 and tube-2 separately. Afterwards, with the help of attempting to capture both ants' locations, variation in ants' positions and prior graphing experiences, she established a quantitative coordinate system. In this process she arranged the tubes orthogonally and represented location of two ants as one static point. Then with the researcher's prompts, she started to consider the point dynamically. In sum, by constructing number lines on each tube, then extracting them from the space and arranging orthogonally she constructed a quantitative coordinate system to represent the location of two ants as one single point by using horizontal and vertical projections from vertical and horizontal lines respectively. Establishing this Cartesian coordination was novel to the students and Ginny conceived the coordinate system she established different than the ant space. This study highlights once more that constructing spatial and quantitative Cartesian coordinate system is not trivial and conceiving points as dynamic representation of measures of two quantities is important for meaningful graphing activities within spatial and quantitative coordinate systems.

Regarding students' experiences with spatial and quantitative coordinate systems, in a study, 1129 tasks from three major grade 6-8 textbook series were analyzed according to the coordinate types and graphing tasks presented (Lee & Guajardo, 2021). Tasks included either directly a two-dimensional coordinate plane blank or with a graph on it or referred to a coordinate plane in a previous problem. The coordinate types presented were categorized into four as: *spatial, quantitative, both* or *neither*. Tasks were categorized as: *create*, *interpret, both* or *neither*. Create was used if the problem required to plot points or to create a graph by plotting points. A task was coded as interpret if it required interpreting or forming an equation for a given graph as well as describing relationship between variables. Results showed that mostly quantitative coordinate systems and tasks that require interpreting graphs were used. When the type of coordinate system was compared across grade levels, in grade 6, spatial and quantitative coordinate systems were close in number. In grade 7, quantitative coordinate system presented 128 times whereas, spatial coordinate system was only presented 8 times. In grade 8, percentages were close with quantitative taking 52% and spatial 48%. No task used both coordinate types together and 22% of the total tasks were coded as neither. The use of interpreting decontextualized graphs increased as the grade level increased (15, 19, 124 for grade 6-8 respectively). In a similar research study, Paoletti et al. (2016, 2022) asserted that most calculus textbooks present decontextualized graphing tasks which lead to disparity between graphing activity in mathematics class and in STEM areas. Particularly, Paoletti et al. (2022) found that in four mathematics textbooks, one chemistry textbook and IEEE/Physics journals 85% or more used coordinate systems for quantitative coordination. On the other hand, spatial coordinate system was used almost exclusively in the Statistics textbook and 15% in the Physics textbook. When quantitative coordinate systems were further analyzed in mathematics, science, and engineering textbooks and practitioner journals, majority (95% or more) of the science and engineering textbooks and journals used coordinate systems to represent relationships between contextualized quantities. On the contrary, when in total four Calculus and Precalculus textbooks were analyzed, three books had 90% or more and one precalculus textbook had 70% of the time decontextualized quantities. Considering the results of these studies, more balanced use of coordinate systems and a smoother transition from spatial to quantitative coordinate system is required. Also, contextualized *create* tasks should appear more in textbooks. In this respect, this research study might shed light on students' activities in spatial and quantitative coordinate systems and may help us understand better the significance of students' prior experience with spatial and quantitative Cartesian coordinate systems.

Although the number of studies examining students' construction and understanding of coordinate systems is scarce, some research studies inform us about students' understanding through their graphing activities in coordinate plane. For instance, Tasova and Moore (2020) explored four 7th grade students' meanings of a point on a plane in terms of multiplicative object in a semester-long teaching experiment at a public school in the southeast US. According to Tasova and Moore, representation of a multiplicative object is not limited to plotting a point on a coordinate plane; it refers to conceiving a point as "simultaneous representation of two attributes of the same object" (p. 237). Yet, they indicated that although students could plot points correctly, meaning of a point was limited to an ordered pair of numbers rather than uniting two quantities' magnitudes. Tasova (2021) found that students can conceive of a point as non-multiplicative object, spatial-quantitative multiplicative object and quantitative multiplicative object with two different types, Type-1 and Type-2. Similarly, in a teaching experiment (Tasova, Liang & Moore, 2020) with four secondary students (lasted 7 weeks 16x1 hour sessions), how students' meanings for points and lines affect their emergent shape thinking was investigated. Particularly, in the teaching experiment teacher researcher focused on students' thinking in conceiving situations quantitatively and representing relationships on number lines and coordinate systems. In the study one student, namely Zane, was asked to graph the relationship between amount of water and depth of water as the swimming pool filled with water. Despite constructing a point as a multiplicative object, Zane was unable to conceive line as a continuous emergent shape due to meanings he held for points and lines. Instead, he defined the line as the direction of movement of the point. Therefore, investigating students' meanings for points as multiplicative object as well as meanings for graphs may help understand their graphing activities in coordinate systems.

As part of a larger study which examines four 7th grade students' graphing activities, one student's, namely Ella's, meanings for graphs and change in her meanings throughout the teaching experiment is presented (Tasova & Moore, 2021). The student attended approximately one-hour long 6 teaching sessions. In the study, Ella was given a map with seven locations labeled and a coordinate plane with seven points plotted. Horizontal axis of the coordinate plane was labeled as Distance from Cannon (DfC) and vertical axis was labeled as Distance from Arch (DfA). When Ella was asked to explain what each point on the coordinate plane might represent, she used quantitative properties and locations of the points in the plane. In the subsequent part, Ella was provided with a dynamic tool representing quantities varying magnitudes as directed bars on empty lines in order to trigger magnitude reasoning in contrary to value or numerical reasoning. The question was to sketch the relationship between DfA and DfC as the bike traveled between two locations on the map. Results of this study indicated that focusing on quantities and representing their magnitudes on empty number lines were helpful in re-organizing the space in align with Cartesian coordinate plane. That being said, this study may contribute mathematics education by providing data from a larger sample of high school students about their abilities to use quantities and their magnitudes while working in Cartesian coordinate system.

Similarly, Knuth (2000) examined 178 high school students' understanding of algebraic and graphical representations of functions since use of multiple representations plays critical role in students' mathematical development. More than three quarters of the participants chose algebraic solution method and many of them could not even suggest a graphical solution. Unfortunately, moving flexibly between graphical and algebraic representations is assumed to be straightforward for students. Although Cartesian connection and fluent use of multiple representations are fundamental in mathematics, little instructional time is devoted to Cartesian connection (Schoenfeld et al., 1993). Therefore, once the Cartesian Coordinate is introduced, students are assumed to understand and use it easily. Knuth argues that students' difficulties may stem from their inability to recognize that points used in calculation of slope are also solutions for the equation, hence they are points forming the graph of the line. This highlights significance of meanings students attribute for coordinate plane, point, line etc. One of the major reasons for students' overreliance on algebraic solution methods although they were pushed to use visual methods, was stated as instructional emphasis on the former method. Therefore, students are found to struggle in forming a connection from graphs to equations.

With the aim of investigating students' reasoning about graphing in non-canonical coordinate systems, Lee et al. (2019) investigated how a preservice teacher Lydia constructed and reasoned within frames of reference, particularly *committing to a reference point* and *directionality of measure comparison,* when graphing in non-canonical coordinate system. During 12 teaching sessions, four tasks (A, B, C, D) had been conducted. Tasks A and B required committing to directionality of change in measure and task C and D required committing directionality of a measure comparison in accumulating quantities. Throughout teaching sessions, Lydia's frames of reference shifted from figurative to operative frames of reference and hence the Cartesian coordinate system became an operative structure. For instance, when a coordinate system was given with non-canonical orientation of axes, Lydia was able to compute the slope or when a graph was rotated, she concluded the invariant relationship between quantities. Whereas at the beginning her mental actions were associated with sensorimotor activity. Initially, Lydia experienced perturbations and constraints regarding her understanding of slope and rate of change ideas in non-canonical Cartesian coordinate system. Therefore, examining the meanings hold for coordinate systems, graphs and points are critical. Lydia's activities in non-canonical coordinate system asserts that students may conceive working in coordinate systems as following a certain set of rules regardless of the context and orientation. Not focusing on quantities, not conceiving a point as a multiplicative object uniting two magnitudes, disregarding orientation of the axes, the context of the problem and alike might hinder students from building meaningful understanding of coordinate system as a tool to learn and do mathematics.

3. STATEMENT OF THE PROBLEM

3.1. Research Question

The aim of this research study was to investigate high school students' meanings of Cartesian coordinate system. Investigating students' meanings of Cartesian coordinate system entails investigating their construction and reasoning of spatial and quantitative Coordinate systems using multiple frames of reference. Besides, conceptualizing Cartesian coordinate system cannot be considered apart from reasoning about points and graphs within coordinate system because conceptual understanding of Cartesian coordinate system requires understanding how frames of reference such as orthogonal axes, directionality, reference point, is used to describe location or situate quantities to represent relationship between them by means of point and graphs. In addition, at high school level Cartesian coordinate system is a common representational tool for reasoning about functions and their graphs. In this respect, investigating students' meanings of a point on a graph and output of a function are also very significant. Therefore, I sought to answer the following research questions:

How do high school students reason about Cartesian coordinate system and graphs within spatial and quantitative Cartesian coordinate systems?

What are high school students' meanings of a point in terms of multiplicative object, point on a graph, and outputs of a function?

3.2. Significance of the Research Study

Cartesian coordinate system plays critical role in learning domains such as geometry, algebra, statistics, precalculus, calculus (CCSM, 2010; MEB, 2018; NCTM, 2000) as well as it is widely used in science, technology and engineering to communicate information (Paoletti et al., 2016; Roth et al., 1999; Rybarczyk, 2011). In mathematics education, research studies point to significance of using Cartesian coordinate system in conceiving and representing relationships between two quantities' values and magnitudes, hence students' development of covariational reasoning. Cartesian coordinate system allows representing attributes of two or more quantities on axes and by means of forming a multiplicative object students can operate on these quantities and representing relationship between them (Thompson et al., 2017) which is also preliminary for covariational reasoning (Thompson, 2011).

Although conceptual understanding of Cartesian coordinate system and its related notions such as graphing and multiplicative object play crucial role in students' understanding in several domains of mathematics and science, there are limited research studies which are focused on students' meanings of Cartesian coordinate system. Often, studies are centered around graphing and covariational reasoning, but some of the challenges students face might stem from their conceptions of coordinate system, and unfortunately research studies in this regard are scarce. Therefore, this study extends the existing literature by investigating high school students' reasonings of Cartesian coordinate system in a more comprehensive way by investigating their construction of spatial and quantitative coordinate systems, reasoning of a point in terms of multiplicative object and graphing within spatial and quantitative coordinate systems, which are all part of robust and coherent understanding of Cartesian coordinate system. This study opens a new window to the literature by testing the hypothetical model (Paoletti et al., 2018) aimed to address students' reasoning for graphs in spatial and quantitative coordinate system, informs us about students' reasoning and suggests resources to improve in that regard. Also, Lee and Hardison (2016) asserted that establishing situational coordinate system precedes quantitative coordinate system. Considering lack of studies regarding spatial coordinate system, this study contributes to the literature as students' construction and graphing within both spatial and quantitative coordinate system was explored.

Moreover, in the coordinate system literature often teaching experiments were conducted either with small sample of preservice teachers or middle school students. Meanwhile, high school students use coordinate systems in learning of several fundamental ideas in mathematics which is grounding for more complex mathematical ideas in undergraduate level and STEM areas. This signifies a need for research studies regarding high school students' understanding of coordinate systems. Thus, this study might shed light on students' weaknesses and strengths, hence impart implications for learning and teaching mathematics at various academic levels. In addition, this study also fills the gap for a large-scale study regarding students' meanings of Cartesian coordinate system which potentially provides insights into difficulties students experience when engaged in graphing or forming a multiplicative object in spatial and quantitative Cartesian coordinate systems.

Furthermore, by bringing together frameworks related to Cartesian coordinate system, an instrument was developed by the researcher in the light of literature. This study has an alternative to suggest new items or instruments to investigate students' meanings of spatial and quantitative Cartesian coordinate system. Based on the results, new factors can be identified, or the instrument might be adapted. The results of this study can contribute to educators and researchers in development of curricular materials to support students in explicitly understanding the difference between spatial and quantitative coordinate system as well as graphs within these systems and using coordinate systems for each purpose as they are relevant for STEM areas.

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4. METHODOLOGY

This chapter presents detailed information about the methodology of the research study. In this regard, research design, participants of the study, sampling method, data collection and data analysis will be discussed.

4.1. Research Design

The current study aimed to investigate high school students' meanings for Cartesian coordinate system, their meanings of a point on a graph, outputs of a function and graph as a whole. For this purpose, data were collected by an inventory consisting of nine open-ended questions some of which with sub-questions. Since the goal of the study was to identify and categorize students' meanings of Cartesian coordinate system, phenomenography was used as research design (Orgill, 2012). Orgill indicated that "The central aim of a phenomenographic study is to identify the different ways in which people experience, interpret, understand, perceive, or conceptualize a certain phenomenon." (p. 2608). Students' approaches and understandings may vary from one learning context to another, and teachers should discern learning from different perspectives and address differences in meanings and conceptions adopted by students (Cheng, 2016). In this regard, content-rich phenomenographic research study can illuminate variations in students' conceptions and potential learning outcomes so that teaching and learning strategies can be developed to shift students from fragmented meanings to more coherent ones (Han & Ellis, 2019).

Phenomenographic data can be collected in multiple ways including semi-structured interviews, open-ended questionnaires, think-aloud methods and observation, each of which has different potentials and limitations for the research study. Among these data collection methods, using an open-ended questionnaire is advantageous when there is relatively high number of participants because it enables collection of a wide range of experiences of phenomenon and is easier to conduct (Han & Ellis, 2019). In this study data regarding 229 high school students' conceptions and understandings of Cartesian coordinate system were analyzed through their written responses to the open-ended inventory. As outlined (Marton, 1994; Bowden, 2000), in data analysis students' responses were categorized and results were presented as descriptions of each category accompanied by illustrative sample student responses.

4.2. Participants

The sample of this study was selected by convenient sampling technique from a private high school in Istanbul. The school has high standards and follow student-centered approach. Students enrolled at this school have very high performance at the high school entrance exam, among the top first percentile, and mostly come from high socioeconomic status. The sample of this study consists of 229 voluntary students from grade level 9 through 12 who have been introduced to Cartesian coordinate system and have been using it in mathematics classes. More specifically, all students were introduced the function concept in detail and had experience with graphing, modeling, and forming various types of functions such as linear, quadratic, and exponential. In addition, students in the sample had experiences with graphing functions using technology and dynamic learning environments. In general students have high potential and capability in learning and doing mathematics. These topics are covered in mathematics curriculum in Turkey (MEB, 2018) in grade level 10 through 12 to a more limited scope. In the beginning, 126 9^{th} grade, 51 10^{th} grade, 53 11^{th} grade and six 12^{th} grade students, adding up to 236 students, participated in the study. Among initial participants, seven of them left most of the questions blank and therefore taken out of the sample. The final sample consists of 229 students. Although 16 of them did not answer around four questions excluding sub-questions, they answered other questions properly. Thus, their responses can be informative in terms of which questions were solved and which ones were skipped as well as understanding their reasoning for the questions responded.

4.3. Instrument

The purpose of this research study was to investigate high school students' reasoning of Cartesian coordinate system. Firstly, the relevant literature was extensively examined to determine how to achieve this purpose. Then in the light of the literature, significant notions related to Cartesian coordinate system was determined and the research question was elaborated. As a result of substantial review of the existing research studies, an item pool was developed by the researcher within the scope of the study. In further detail, the instrument aims to investigate students' meanings of Cartesian coordinate system by investigating their construction of spatial and quantitative Cartesian coordinate system, their meanings for related notions such as point as a multiplicative object, point on a graph, output of a function and graphing in spatial and quantitative systems.

Firstly, I put the questions together by examining the existing research and conference proceedings (David et al., 2019; Lee, 2017; Johnson et al., 2020; Lee et al., 2019, 2020; Moore, Silverman, et al., 2019; Paoletti et al., 2018; Moore, Stevens et al., 2019; Parr, 2021; Patterson & McGraw, 2018; Sencindiver, 2020; Tasova & Moore, 2020, 2021) investigating students' understanding of Cartesian coordinate system, graphing activities withing Cartesian coordinate system and meanings for point as a multiplicative object. During development of the instrument, first I narrowed down the research question and collected related items, problems, tasks and cut downsize to 30 questions at the beginning. Then, my advisor and I came together on a regular basis to examine each of the questions based on the research question. After our analysis, there were a total of 11 questions. As the next step, experts' opinion was taken on the initial instrument before running a pilot data collection with high school students. The experts are four mathematics educators who have conducted research on students' understanding of Cartesian coordinate systems. The researchers were sent a document that presented the purpose of the research study, theoretical framework, the research question, and the sample. Each question was presented together with the evaluation criteria in tabular form (see Table 4.1) and the experts were requested to evaluate the instrument and give feedback on (i) suitability of the questions with aim, (ii) the appropriateness of the question with the aim and the knowledge area for reasoning in coordinate systems and (iii) the clarity/relevance of the questions in terms of language. The experts evaluated each question as either not sufficient, partially sufficient or sufficient and stated their opinions and suggestions in the table. This way, content and face validity was ensured regarding the appropriateness of the content and the language of the instrument.

Table 4.1. Expert evaluation chart.

Question:			
Knowledge Domain:			
Aim of the question:			
Evaluation Chart			
Sufficiency to reveal…(objective of the question)	*Not Sufficient*	*Partially sufficient*	*Sufficient*
Comments:			
The suitability of the question with the aim and the knowledge domain for assessing …… (objective of the question)	*Not Relevant*	*Partially relevant*	*Relevant*
Comments:			
The suitability of the question regarding language	*Not Sufficient*	*Partially sufficient*	*Sufficient*
Comments:			

Upon gathering the expert opinion, questions were revised according to the feedback gathered from experts. Some of the questions were taken out, some of them were edited and some new parts were included. Lastly, opinions of three high school mathematics teachers were taken about the content and the language of the instrument. Afterwards, questions were piloted with two high school students and revised again by the researcher and the advisor accordingly to bring the inventory to the final version.

The finalized version of the inventory consists of nine open ended mathematics questions (see Appendix A) with question 6 through 8 including sub-questions (6a, 6b; 7a, 7b, 7c; 8a 8b). While some of the questions were taken from previous research studies (Lee, et al., 2020; Paoletti et al., 2018; Sencindiver, 2020), some of them were adapted in accordance with the current research study based on previous studies (Moore, Stevens, et al., 2019; Paoletti et al., 2018; Sencindiver, 2020; Tasova & Moore, 2020, 2021; Tasova,

2021) and some of them are written by the researcher. The table 4.2. indicates how each question matches with the goal of research question.

Table 4.2. Goals of the questions in the inventory.

Goal		**Question**
Students' construction of spatial and quantitative Cartesian coordinate systems		1
Students' meanings of a point	as a multiplicative object	3, 4, 5, 7a, 7b, 7c
	on a graph & output of a function	2
Students' graphing	within spatial coordinate system	6a, 6b, 8a
	within quantitative coordinate system	8b, 9

The following paragraph explains how questions are aligned with the aim of the study and the framework for the uses of coordinate systems (Lee et al., 2018) and framework for representing a multiplicative object in the context of graphing (Tasova & Moore, 2020).

Question 1 (Q1), adapted from Lee et al. (2020, p. 934), aims to explore how students construct and reason about spatial and quantitative coordinate systems. More specifically, how students use their spatial coordination skills to orient the perceptual space and then construct quantitative coordinate system. In the problem two ants (represented by points) move haphazardly in tubes that can be rotated and moved in a dynamic geometry environment. The task is to indicate the location of two ants with one single point that moves along with the ants. To achieve this, first students are expected to spatially coordinate the tubes orthogonally. Then treating the tubes as *x*- and *y*- axis, they need to mark the directed distance of each ant from the intersection of tubes which can be considered as the origin (reference point). Afterwards by uniting the magnitudes of vertical and horizontal directed distances as one single point, forming a multiplicative object, the location can be described mathematically.

Questions 2 through 5 and question 7 mainly focus on students' meanings of a point. The goal of each question and how they measure students' meanings of a point in terms of multiplicative object are further described in the following paragraphs.

Question 2 (Q2) is adopted from Sencindiver (2020, p. 99) and David et al. (2019) and aims to explore not only students' meanings of a point in terms of a multiplicative object, but also output of a function on a graph. In other words, it seeks to explore how students conceive outputs of a function (such as $f(h)\ and\ f(a+h)$) and difference of outputs of a function (such as $f(a+h)-f(h)$) to be represented on a graph. Do students understand the output of a function as a *location* on the graph versus as a *value* whose magnitude is represented on the y-axis or as a directed distance from the x-axis? Investigating this is important as previous research indicated that students and prospective teachers are not necessarily aware of the fact that output of a function is a value whose magnitude is represented on the y-axis (David et al., 2019; Parr, 2021). In In addition to conceptions of output of a function, questioning students' conceptions of difference of output can reinforce the implications on their location and value thinking (Sencindiver, 2020). Additionally, high school students conceptualize different types of functions such as liner, quadratic, exponential, logarithmic, trigonometric and related graphs within Cartesian coordinate system. Cartesian coordinate system is also used for modelling with various functions using function notation. Thus, it is of great significance to perceive a point as a multiplicative object $(a, f(a))$ where a is the quantity represented (conventionally) on the x-axis and $f(a)$ is the quantity represented (conventionally) on the y-axis.

Question 3 (Q3) is a conventional question written by the researcher to assess if students can correctly plot points in four quadrants of the Cartesian coordinate plane and what meanings they hold for x and y-coordinates.

Question 4 (Q4) and question 5 (Q5) are adapted from a Calculus content quiz (Sencindiver, 2020, p. 7-8) and new parts are added based on the feedback received from the experts consulted. The goal is to investigate students' meanings of point as a multiplicative object by assessing their meanings for y-coordinate in Q4 and x-coordinate in Q5. More precisely, whether students conceive the point in Cartesian coordinate system as coupling of two magnitudes: x-coordinate as magnitude on the x-axis and y-coordinate as magnitude on the y-axis is analyzed. Students are given a set of statements and asked to select all correct statements. In order to conceive x-coordinate as a magnitude on the x-axis, they should also select the directed distance from the origin on the x-axis, the distance between graph and the y-axis. Similarly, in order to conceive y-coordinate as a magnitude on the y-axis, they should select the distance from the origin on the y-axis, the height of the graph, the distance between graph and the x-axis.

Question 7 (Q7) is adapted from research studies (Tasova & Moore, 2020, 2021) investigating middle school students' representation of multiplicative object and graphing in Cartesian coordinate system. Question 7a presents a specific map with seven locations pinned and a Cartesian coordinate plane with the orthogonal axes representing distance from two specific locations on the map with some points plotted (p. 88). The main aim is to investigate what quantities/magnitudes students attend while determining what each point in the coordinate plane corresponds to on the map as well as to observe what meanings they hold for points in Cartesian coordinate system. Question 7b explores how students reason when producing a point in Cartesian coordinate system (p. 148). How do they construct a point and what quantities they attend on the map and on the coordinate plane? Question 7c is prepared by the researcher (H. Tasova, personal communications, May 2022). Like Q7a, Q7c presents the map with seven locations pinned and a Cartesian coordinate plane with the same orthogonal axis. Different than question 7a, in question 7c coordinate plane is drawn on a grid space with a numerical, scaled axes and the distance between two cities is stated in the question. Students are asked to determine what each point in the coordinate plane (different points than Q7a) corresponds to on the map. This way, whether students are perplexed or reason about points better when numerical values are given can be investigated.

Question 6, 8 and 9 are mainly about students' meanings of graphing in Cartesian coordinate system. Question 6 (Q6) and 8a (Q8a) are aimed to explore students' reasoning in spatial Cartesian coordinate system in producing and interpreting a graph. Whereas question 8b (Q8b) and question 9 (Q9) explore students' graphing in quantitative Cartesian coordinate system in producing and interpreting a graph.

Q6 is written by the researcher in the light of previous research studies (Lee, 2017, 2020; Paoletti et al., 2018; Piaget et al., 1960). In Q6a, the goal is to explore if students use frames of reference to establish a spatial Cartesian coordinate system onto the figure (garden), draw orthogonal axes, then produce quantities to locate a point (represents a person in a garden) as a multiplicative object. On top of the requirements in Q6a, Q6b further explores if students reason about the path of the person toward the exit gate as an emergent or static shape thinking. As in Lee (2017), in this study spatial frame of reference corresponds to mental structures (such as axis or reference point) that one constructs and place onto perceptual space with the aim of representing relative positions of the elements (Piaget & Inhelder, 1967). In this question students might describe the graph as representing how the distance from the axes change or as a static shape without focusing on the trace of change in quantities forming the line.

Question 8a requires mathematically describing the logo on a pool table. If students insert a spatial coordinate plane onto the pool table and give an equation of a semicircle, then they can be considered as interpreting the graph as static shape thinking. If students describe the shape as emergent trace of two quantities' magnitudes, then their reasoning about graph can be considered as emergent shape thinking. Otherwise, they might be unable to construct a spatial coordinate system. This question was adapted from Paoletti et al. (2018, p. 1319).

In question 8b students' reasoning about graphs in quantitative coordinate system is investigated. By exploring the interactive animation, students are asked to create a graph for the red ball's distance from the yellow ball and its distance from the blue ball as it moves along the path as shown in the animation and on the inventory. No coordinate plane is given in this question. This means that students are supposed to construct the coordinate plane, label axes correctly and extract the quantities from the problem situation and project on the quantitative coordinate system. This question is adapted from Paoletti et al. (2018, p. 1319).

Question 9, which is adapted from Moore, Stevens, et al. (2019, p. 4), explores high school students' graphing in quantitative Cartesian coordinate system in producing and interpreting a graph. An interactive animation is provided to show that a car is traveling back and forth along a certain path. The task is to graph the relationship between the car's distance from Istanbul and its' distance from Bolu during the trip. What magnitudes students attend and whether they draw the graph as a result of emergent or static shape thinking are aimed to be explored. Different than Q8b, a coordinate plane is provided in this question. The starting point of the graph will be on the x-axis and first it goes from right to left upward which are different than conventional graphing. Therefore, students need to attend quantities, form a multiplicative object by uniting the distance from Bolu and the distance from Istanbul, and finally envision their covariation to be able to sketch the graph correctly. No numerical value is given so that students could focus on magnitudes and quantities rather than numbers.

4.4. Data Collection

The data were collected at a private high school in İstanbul towards the end of the Spring semester in 2021-2022 academic year. The data collection was completed before the last exam period to increase the participation and students' attention on the questions. Before the data collection process, first the ethics committee at Boğaziçi University was informed about the research study and consent was taken. Then, the school was informed about the details of the study and necessary permissions were also taken via informed consent forms from students. Then, the data were collected through hardcopies of the inventory consisting of nine open ended mathematics questions. The instrument was administered to each class at a single point in time under the supervision of volunteer teachers at different time slots. Teachers were also informed in detail about the research study and data collection process beforehand. Instructions were also attached on as a list on the first page of the instrument to inform the students. Students completed the inventory in about 70-80 min. within their two back-to-back classes. Students were required to use their computers, tablets or mobile phones to explore the animations in question 1, 8b and 9. The study sought to explore students' reasoning skills with quantities, thus most of the questions did not provide numerical values or information about numerical relationships. Instead, scissors, wires and papers to use as straight edge were provided as optional tools to use in questions. Since the research strives to understand students' reasonings of Cartesian coordinate system, in the instructions students were specifically encouraged to explain their reasoning and label their work clearly, not to erase any of prior solution and to use an extra sheet of paper in that case.

4.5. Data Analysis

In this study, the data source consisted of high school students' written responses to the inventory which was developed to investigate their reasoning of Cartesian coordinate system. First of all, the analysis was done separately for each question and not for each participant. In other words, question 1 was analyzed for all participants, then question 2 was analyzed for all participants and so on. In order to obtain more consistent and reliable results, the researcher and the advisor came together several times to discuss and review the analysis of each item. For the same purpose, the analysis of each response was also reviewed multiple times by the researcher and responses to sub-questions making up the same question were also compared with each other. While doing the analysis per question, responses were mainly categorized according to two frameworks as demonstrated in Table 4.3. In this regard, focusing on the descriptions students provided for each question, coded analysis was used (Clement, 2000) for categorizing the data. Also, in case students left a question blank or expressed "I don't know", "I couldn't solve", "no time left" etc., their responses were categorized as *no answer*. The results were then presented descriptively providing the percentage of each category together with qualitative descriptions through sample responses in an explanatory way.

However, when students provided insufficient or no explanation at all to justify their answer, I was restricted to categorize their reasoning. Therefore, such responses were presented as *insufficient justification*, but this necessitated a further analysis of such responses as they may have significant implications for students' meanings of Cartesian coordinate system. In this case, such responses were further described to reflect students' thinking. Also, if there was any striking observation, it was also exemplified and elaborated in the results section.

Table 4.3. Table of specification for analysis.

Goal	Questions	Analysis
Students' meanings of a point in Cartesian coordinate system as a multiplicative object	Q2, Q3, Q4, Q5, Q7	Framework for representing a multiplicative object in the context of graphing (Tasova & Moore, 2020; Tasova, 2021)
Students' graphing in Cartesian coordinate system	Q6a, Q6b, Q8a (in spatial coordinate system)	Framework for reasoning about graph in spatial Cartesian coordinate system (Paoletti, Lee and Hardison, 2018)
	Q8b, Q9 (in quantitative coordinate system)	

In further detail, according to framework for representing a multiplicative object in the context of graphing, question 7 was categorized as non-multiplicative object, spatial-quantitative multiplicative object or quantitative multiplicative object. According to framework for reasoning about graph in spatial Cartesian coordinate system, responses to question 6, 8 and 9 were categorized as static shape thinking and emergent shape thinking. Aside from these two frameworks, in question 1, students' spatial and quantitative coordination was investigated according to their uses of frames of reference (Joshua et al. 2015). Students established either *spatial or quantitative coordinate system,* else gave *no answer* or *incorrect answer*. In question 2, in addition to the frameworks in Table 4.3, framework for ways of thinking about points was utilized to categorize students' meanings of a point on a graph as location-thinking, value-thinking (David et al., 2019) or arc-thinking (Sencindiver, 2020). This way, those who view output of a function as a location on a graph (location-thinking) were associated with non-multiplicative object and those who view output of a function as a magnitude on the y-axis or vertical distance from the x-axis and point as coupling of two quantities (value-thinking) were associated with quantitative multiplicative object in the context of the problem. Question 4 and 5 were more conventional questions in comparison to other questions. Therefore, students' performance was described in general terms based on percentage of *blank*, *incorrect*, *partial: correct & incorrect*, *some correct* and *all correct* responses. If all the correct statements were chosen with no mistake, then it was categorized as all correct. If some of the correct statements were chosen with no mistake, then it was categorized as some correct. If both correct and incorrect statements were chosen, it was categorized as partial: correct & incorrect. If only incorrect statements were chosen it was categorized as incorrect and lastly if question was left blank, then it was categorized as no answer. As in question 2, descriptive analysis of question 4 and 5 reveals significant implications for students' meanings of a point in terms of multiplicative object. Those who selected all the correct items in the question were associated with quantitative multiplicative object and value-thinking. On the other hand, by looking into incorrect answers selected, students' meanings of a point were interpreted in terms of non-multiplicative object and location-thinking.

5. RESULTS

In the following sections, results of the coded analysis are presented both descriptively with percentages and qualitatively through sample student responses in three sections: students' construction of Cartesian coordinate system, students' meanings of a point and students' graphing within Cartesian coordinate system. In this study, data were collected from 229 high school students through an inventory consisting of open-ended mathematical questions.

5.1. Students' Construction of Cartesian Coordinate System

In this section, results for Q1, which required spatial and quantitative coordination to establish a Cartesian coordinate system, will be presented. Particularly, how students leverage their ways of spatial coordination to coordinate quantitatively was explored. In Q1, students first explored a simulation of two ant farms (represented by tubes) in each of which one ant (represented by point) was moving haphazardly. Students were able to rotate and shift the tubes and stop the animation as they wish. The task was to describe mathematically locations of the two ants with a single point that moves along with the ants. According to Table 5.1, question 1 had the lowest success rate among all questions.

Table 5.1. Students' responses to question 1.

Responses to Question 1	**Frequency**	**Percentage**
Quantitative coordinate system	12	5%
Spatial coordinate system	20	9%
No answer	129	56%
Incorrect	68	30%
Total	229	100%

Particularly, more than half of the students (56%) could not answer the question which was categorized as no answer. They either wrote "I don't know", "no time left", "I couldn't solve" or left the question blank. Also, 30% of the students came up with an incorrect answer and could not establish a quantitative Cartesian coordinate system. In total, only 12 students, which corresponds to 5%, answered the question successfully by establishing a quantitative coordinate system as illustrated in Figure 5.1 below.

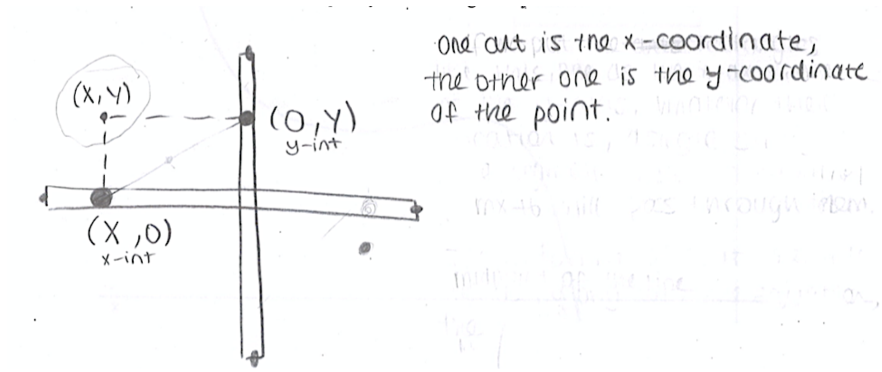

Figure 5.1. Sample student response for quantitative Cartesian coordinate system.

As shown in Figure 5.1, these students arranged the tubes orthogonally as number lines and represented locations of the two ants as one single point by combining horizontal and vertical distance from a reference point. Although not specifically mentioned by the student, the reference point seemed to be $(0, 0)$ from the statements of $(x, 0)$, $(y, 0)$ and (x, y). Also, most notably two students constructed a three-dimensional coordinate system where axes represented x-coordinate, y-coordinate and time as exemplified in Figure 5.2 below.

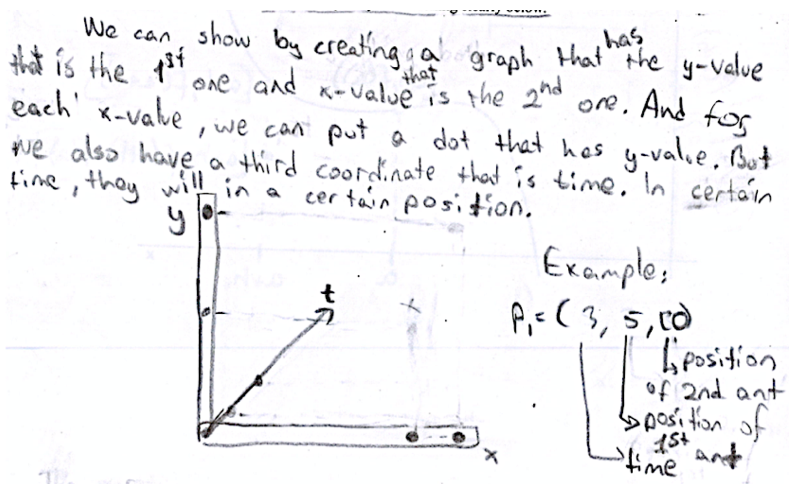

Figure 5.2. Sample student response for 3D quantitative Cartesian coordinate system.

As Figure 5.2 shows, this student took into consideration not only the position of two ants but also time of the position. This might suggest the following: The student seemed to be thinking that the position of one ant was on the x-axis with a distance from $(0, 0)$ and the position of the other ant was on the y-axis with a distance from $(0, 0)$. This suggests that the position of the ants was considered as one single point by combining horizontal and vertical distance from the reference point, namely $(0, 0)$. Consideration of time as another quantity involved in the situation further suggests that student might be thinking of the position of the two ants corresponding to one embedded composite unit which in his mind was united with another quantity, namely time. That is how he considered and made sense of using a three dimension to position of the ants.

In addition, as shown in Table 5.1, approximately 9% of the students formed a spatial Cartesian coordinate system but failed to describe the location of two ants with one single point that moves along with the ants. As an example, Figure 5.3 below illustrates a response where a student established spatial Cartesian coordinate system and took the mid-point to represent location of both ants with one single point.

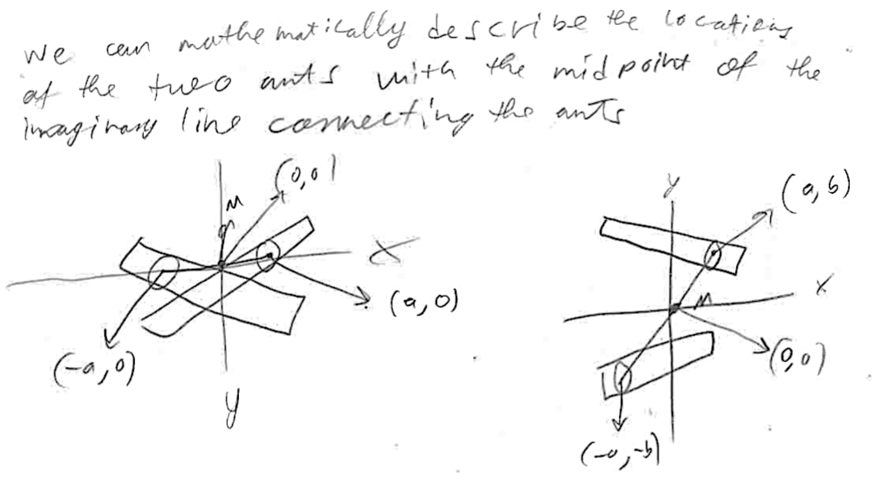

Figure 5.3. Sample spatial Cartesian coordinate system with mid-point as response.

Since students placed a Cartesian coordinate system onto the figure provided in the question, and assigned coordinates of the points accordingly, their responses can be considered as an example of establishing a spatial Cartesian coordinate system. As depicted in Figure 5.3 above, in this category specifically students drew a Cartesian coordinate system to assign coordinates to each ant but could not unite the coordinates of the points. They either attempted to take the mid-point, connected two points by line or did not conclude with a single point. Students seemed to use coordinate plane for conventional tasks that they were familiar from prior mathematics classes. Therefore, they might not have conceptualized using each axis to represent directed distance from origin and coupling these quantities to represent location of a point. Finally, when incorrect answers were examined (see Figure 5.4), differences in students' reasoning were further depicted.

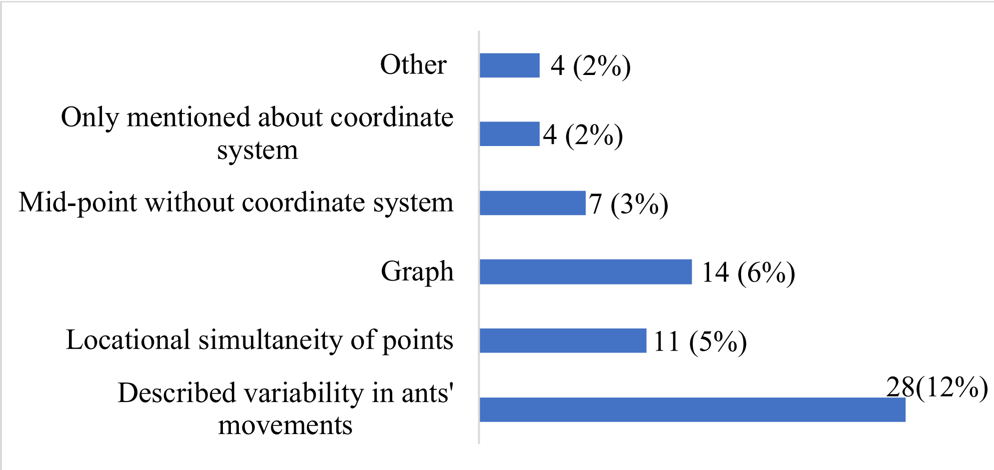

Figure 5.4. Incorrect responses to question 1.

As Figure 5.4 above shows, 12% of the participants focused on variability in ants' movements and described their movement in tubes. Lee and Hardison (2020) identified attention to variability of ants' locations coupled with imagining with the single point moving along with them as a significant cognitive resource in establishing a Cartesian connection. Yet, these students seemed to be perturbed since ants displayed different variations in their movements. In addition, 5% of the participants thought that single point can be achieved only if ants (points) were overlaid. So, single point was described as locational simultaneity of the ants. Also, as shown in Figure 5.5a below, 6% of the students sketched a graph (or two separate graphs) of distance versus time or mentioned a function or a graph to depict the single point uniting both ants' locations. Intriguingly, one student sketched a hypothetical graph on two intersecting coordinate systems as shown in Figure 5.5b below. The claim was that the graph would have two different meanings according to each coordinate system indicating the location of each ant separately. But she seemed unaware of frames of reference in her coordinate systems and the hypothetical graph possibly does not cover all locations of ants. Also, these results show that instead of identifying quantities and using axes to represent them, students attempted to find rule of a function or graph where mostly time was an inherent quantity.

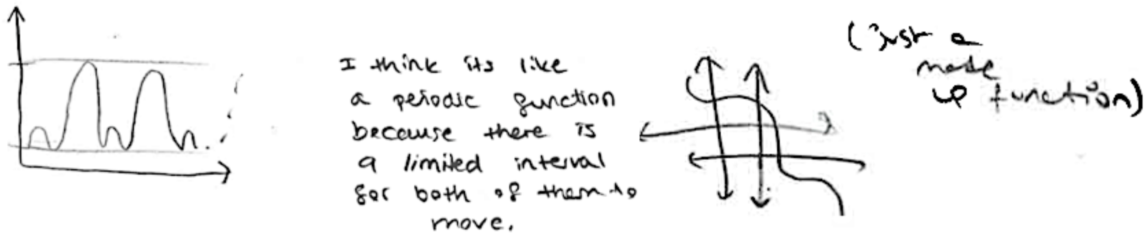

Figure 5.5. (a) Graph (b) Two intersecting coordinate systems containing a graph.

Moreover, approximately 3% of the students mentioned taking mid-point of the two points without referring to a coordinate system and 2% only mentioned coordinate system as an idea but did not really make use of it in their solution. Lastly, 4 responses were categorized as other because they did not fall into common categories and were not related to construction of coordinate system idea directly. These responses indicate that even if students thought of the coordinate system in their approach, they were not aware of how to construct a quantitative coordinate system to describe location mathematically.

5.2. Students' Meanings of a Point

As shown in Table 4.2, there were five questions in the inventory with the goal of investigating students' meanings of a point. In the following subsections students' responses to question 2, followed by question 4, 5 and 7 will be presented respectively. Question 3 will not be presented detailly in a subsection, because it was a conventional question with which students encounter very often in their mathematics classes both in middle and high school level. Thus, majority of the students were successful in plotting points in different quadrants of a Cartesian coordinate system.

5.2.1. Students' Responses to Question 2

The aim of question 2 was to investigate students' meanings of a point on a graph and output of a function. In this regard, students' answers were categorized according to the framework for ways of students' interpretation of point as *location-thinking*, *value-thinking*

(David et al., 2019) and *arc-thinking* (Sencindiver, 2020). In addition to these categories, new categories have emerged such as *hybrid value and location thinking* and *graph-thinking.* Since arc-thinking and graph thinking are similar in some aspects, which will be elaborated later, I reported them together in one category as shown in Table 5.2.

In general, students appeared to have difficulty conceiving points on a graph as union of two quantities' magnitudes because only a small portion of students had value thinking for representing points on a graph, and significant number of students had hybrid value and location thinking which is an indicator of fragmented meanings of a point. Also, about half of the students either stayed at unproductive ways of thinking such as location, arc or graph-thinking or could not answer the question.

Table 5.2. Students' responses to question 2.

Responses to Question 2	**Frequency**	**Percentage**
Value-thinking	47	20.5%
Location-thinking	67	29.7%
Hybrid value and location thinking	32	13.5%
Arc-thinking or Graph-thinking	15	6.6%
No answer	57	24.9%
Incorrect answer	11	4.8%
Total	229	100%

In detail, even though all students have worked with functions and function notation before, about one quarter (24.9%) of the students did not answer the question and 4.8% of the students gave incorrect, irrelevant responses. Besides, approximately 30% of the students demonstrated location-thinking and marked the outputs of the function and difference of outputs as a point on the graph as shown in Figure 5.6b. This shows that most of the students, adding up to approximately 60%, did not conceptualize point on graph of a function as quantitative multiplicative object.

Considering value-thinking and hybrid value and location thinking, while in sum 34% of the students incorporated quantities in their thinking about a point on a graph, only one fifth of the participants (20.5%) viewed point as quantitative multiplicative object combining two quantities as $(a, f(a))$, referring to value-thinking. Whereas 13.5% of the students had fragmented meanings for point on a graph and output of a function showing hybrid value and location thinking.

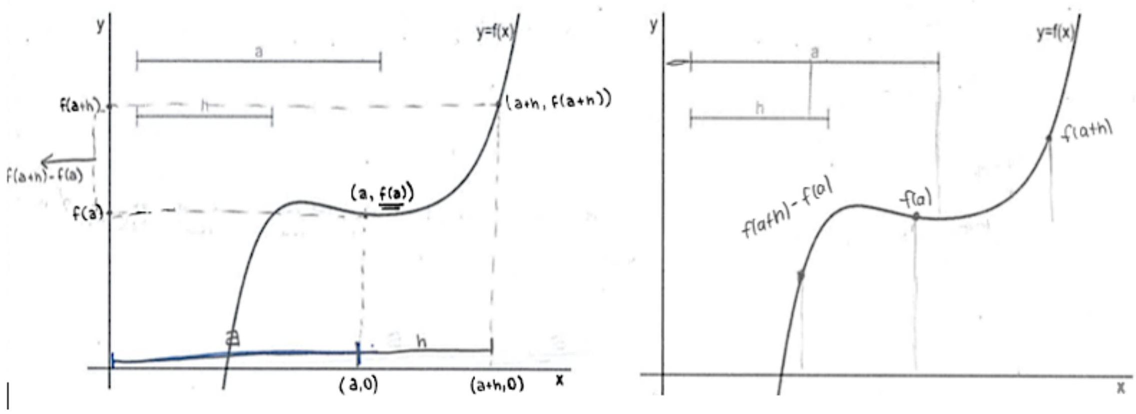

Figure 5.6. (a) Value thinking (b) Location thinking.

As Table 5.2 showed, only approximately 20.5% of the students demonstrated value thinking. In this group students either marked the outputs $f(a)$ and $f(a+h)$ on the y-axis or labeled the points on the graph as an ordered pair such as $(a, f(a))$. Students with value-thinking represented difference of outputs either as a quantity on the y-axis or as vertical distance between points on the graph (see Figure 5.6a above).

Different from existing research (David et al., 2019; Sencindiver, 2020), some students (13.5%) demonstrated new ways of reasoning about aspects of a graph. In line with location-thinking, they labeled outputs of functions $f(a)$ and $f(a+h)$ as points on the graph. Whereas they labeled the difference of outputs as a quantity measured by the vertical distance between y-coordinates of the outputs which was suggestive of value-thinking. This group of responses were categorized as *hybrid value and location thinking*. One other indicator of hybrid value and location thinking was when students labeled the outputs of the function as directed distance from the x-axis but then labeled the difference of outputs as a point on the graph. Figure 5.7 exemplifies these two cases for hybrid value and location thinking.

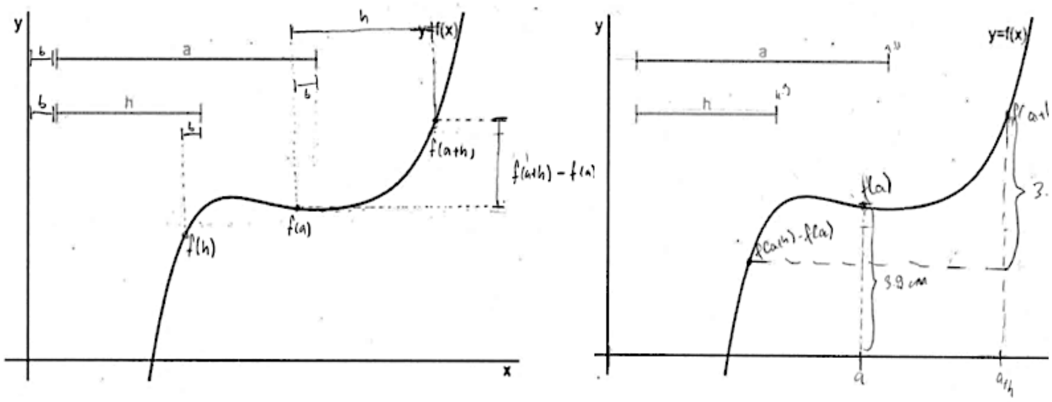

Figure 5.7. Hybrid value and location thinking.

About 7% percent of the students (15 students) either marked arc/arc length of the graph (arc-thinking) or sketched new graphs to represent outputs of function (graph-thinking) as demonstrated in Figure 5.8 below. In arc-thinking, since no detailed written explanation was given by these students, it was not possible to distinguish whether they meant arc (i.e. part of a graph) or arc length of the graph. Considering graph-thinking, those who sketched separate graphs might have considered output of the function as a transformation of the original function, f, because they either stretched or shrank the graph of function f (see Figure 5.8a below) or shifted the graph horizontally along the x-axis. Since these students either marked part of a graph or draw a whole new graph, their reasoning was similar as they associated graph itself with outputs of a function. Therefore, descriptive results of arc-thinking and graph-thinking were presented together in Table 5.2.

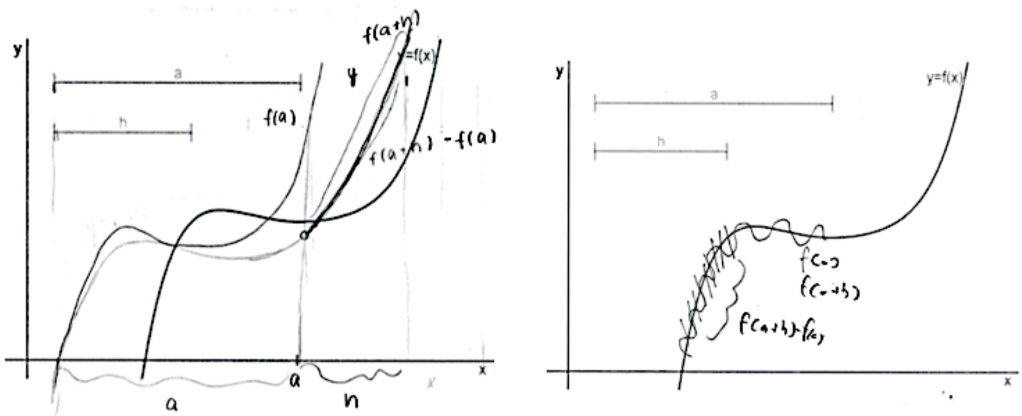

Figure 5.8. (a) Graph-thinking (b) Arc-thinking.

In addition, although not shown among the categories in Table 5.2, some students' responses pointed to difficulties regarding understanding and representing difference of outputs of a function. Firstly, even though 28 students answered part (i) and (ii) of question 2, they could not label $f(a+h)-f(a)$ as requied in part (iii). More precisely, 13 students from value thinking, 10 from location thinking, 2 from graph and 3 from hybrid value and location thinking were perturbed in labeling difference of outputs in the coordinate plane. In this respect, conceiving output of a function as a location on the graph might hinder students from representing difference of outputs of a function in a coordinate system. Also, working with quantities instead of numbers was possibly a novel problem situation and students might have had difficulty reasoning through quantities. Although no numerical values were given, labeling output of a function $f(a)$ could be more analogous to their previous mathematical activities compared to labeling difference of outputs.

Secondly, 18 students incorrectly stated that $f(a+h)-f(a)=f(h)$. Some of the difficulties in comprehending a graph and its' related aspects might stem from students' difficulties with function notation, operations on functions and how it is related to point on a graph and graph itself. Relatedly, two students tried to measure lengths of a and h and could not label the difference of outputs because the input (magnitude of x-coordinate) was less than the x-intercept where the graph crossed the x-axis. In consequence, they indicated that there is no graph and hence no corresponding y value for the input: "$f(a+h)-f(a)$ does not exist".

5.2.2. Students' Responses to Question 4 and 5

In Q4 and Q5, students were given the function $y=-3x+2$ and expected to select all correct statements that represent the y-value when $x=2$ and the solution(s) to $y=2$ respectively. This way, students' meanings of point as a multiplicative object were examined by assessing their meanings for y-coordinate in Q4 and x-coordinate in Q5. Students' responses were categorized as *blank*, *incorrect*, *partial: correct & incorrect*, *some correct* and *all correct* as mentioned in the data analysis section. Afterwards, based on the number of correct, partial and incorrect responses, students' meanings of point in terms of multiplicative object were analyzed. Those who selected all the correct items were associated with quantitative multiplicative object and value-thinking. Depending on the incorrect answers selected, students' meanings of a point were associated with non-multiplicative object and location-thinking.

In general, results of question 4 showed that for majority of the students, meanings of a point as a multiplicative object and meanings for y-coordinate were either incomplete or compartmentalized containing correct and incorrect meanings. Unfortunately, only few students demonstrated a thorough understanding for various representations of y-coordinate. Most of them were not aware of representing y-coordinate as vertical distance from origin. Instead, they considered it as a point on the graph.

Table 5.3. Students' responses to question 4.

Responses to Question 4	**Frequency**	**Percentage**
All correct	14	6%
Some correct	77	34%
Partial: correct & incorrect	122	53%
No answer	16	7%
Total	229	100%

According to Table 5.3 above, data shows that more than half of the students (53%) gave partially correct answers. Although in total 40% of the students chose correct answers with no mistake (some correct + all correct), only 6% of them were able to identify all the correct indicators of y-coordinate. This implies that most of the students did not have a thorough understanding of various representations of y-coordinate. In addition, 7% of the students left the question blank. Having no frequency/response in the incorrect category shows that students selected at least one the of correct statements in their answer.

Table 5.4 displays students' responses to question 5. Like question 4, majority of the students carried incorrect meanings for point as a multiplicative object and representation of x-coordinate in coordinate system. About half of them did not consider horizontal distance from origin as representation of x-coordinate.

Table 5.4. Students' responses to question 5.

Responses to Question 5	**Frequency**	**Percentage**
All correct	25	11%
Some correct	18	8%
Partial: correct & incorrect	147	64%
Incorrect	12	5%
No answer	27	12%
Total	229	100%

In detail, regarding students' responses to question 5 (see Table 5.4), majority of the students (64%) gave partial: correct & incorrect answer. Only about one fifth of the students were able to find correct (11%) and some of the correct answers (8%). While 12% of them did not answer the question, 5% made only incorrect selections. These results suggest that students' meanings for x-coordinate may not be coherent and solid. This could lead to problems in understanding point as a multiplicative object and reasoning about points and graphs within coordinate system. Besides, students might have more difficulty in representing the x-coordinate than the y-coordinate as the percentage of correct answers decreased in Q5 compared to Q4. Moreover, in Q5 percentage of some correct answer is much lower and percentage of no answer is higher than in Q4.

Apart from these general results, it might be very propitious to further examine some of the items students selected in Q4 and Q5 (see Figure 5.9 and 5.10). These figures display correct statements in green and incorrect statements in red together with the number of responses.

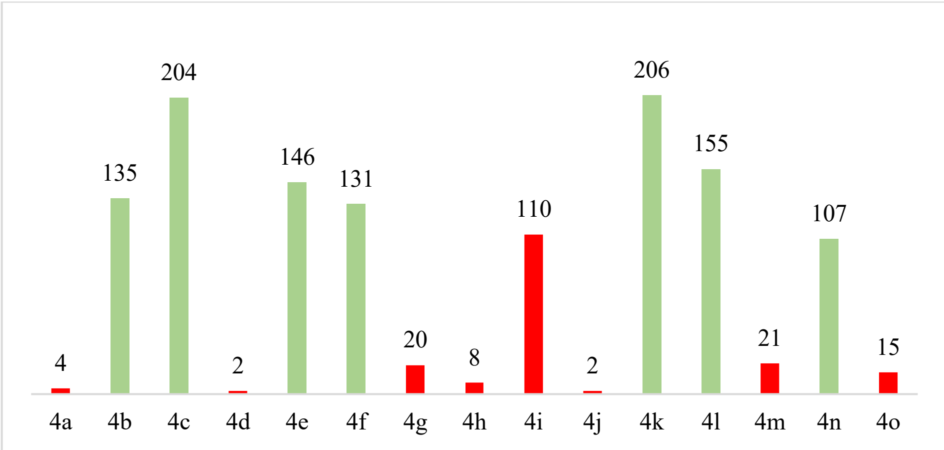

Figure 5.9. Students' responses to each item in question 4.

According to the detailed statistics of question 4 (see Figure 5.9 above), nearly half of the students (48%) selected the statement 4i. This suggests that half of the students perceived y-coordinate as a point on the graph. This also aligns with the results of question 2 where about 30% of the students used location-thinking and 14% used hybrid value and location thinking in representing output of a function in a coordinate plane. That is, the sum of 30% and 14% also shows that students considered y-coordinate of the function (i.e. the value of the function) not as a magnitude on the vertical axis but as a point on the graph. In addition, about 9% of the students selected the statement 4g which corresponds to distance between the graph and the y-axis. So, they confused the y-coordinate with the x-coordinate.

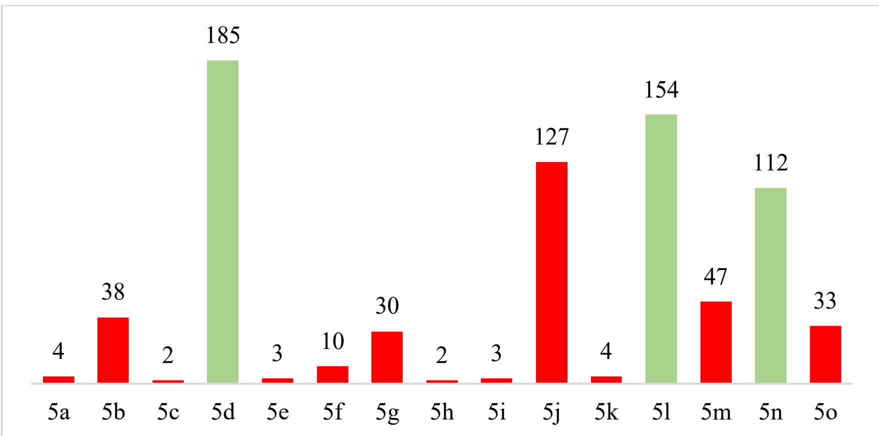

Figure 5.10. Students' responses to each item in question 5.

In a similar vein, according to the detailed statistics of question 5 (see Figure 5.10), in general the number of students who selected incorrect statements regarding the meaning of x-coordinate was higher than the number of students who selected incorrect statements regarding the meaning of y-coordinate. Like in Q4, more than half of the students (55.4%) selected the statement 5j indicating that most of the students considered x-coordinate as a point on the graph, which can potentially be associated with non-multiplicative object. Also, 16.5% of the students could not make sense of the statement 5b: "function evaluated at 2". Additionally, around 13% chose the distance between the graph of the function and the y-axis as representation of the x-coordinate, but the x value was incorrect. Similarly, 20.5% of the students selected the statement 5m: "the value of 2 on the y-axis" and 14.4% selected the statement 5o: "the distance between the origin and the value of 2 on the y-axis" as indicators of the x-coordinate of the function. That is, in these cases, students selected representations of y-coordinate whereas the question was to identify statements representing the x-coordinate.

5.2.3. Students' Responses to Question 7

In question 7 (Q7) students' meanings of a point on a coordinate plane in terms of multiplicative object was explored. Although they slightly differed, in each sub-question the map of Downtown Athens was given with seven locations pinned: UGA Arch (A), Double-Barreled Cannon (C), First American Bank (FAB), Georgia Theater (GT), Wells Fargo Bank (WFB), Statue of Athena (SoA), and Starbucks (S) (see Appendix A).

Specifically, in Q7a and Q7c, students were required to determine what each point on the given coordinate plane corresponds to on the Downtown Athens map. Q7a was in a quantitative context such that students were to dis-embed the quantities in the problem situation and match them with the points given in a non-numerical coordinate system. Whereas, in Q7c the same task was presented with a coordinate system depicted on a grid with numbers and the distance between Arch and Cannon was also mentioned in the question. While Q7a and Q7c required students to interpret meanings of the points to match them with locations on the map, Q7b required students to produce a point that represented a crow's distance from Arch and Cannon at a certain moment, again in a non-numerical coordinate system. Depending on what magnitudes students attended, whether they conceived a point as a quantitative multiplicative object (QMO), spatial-quantitative multiplicative object (SQMO) or non-multiplicative object (NMO) was decided (see Table 5.5).

Data from Table 5.5 shows that in general about half of the students performed successfully interpreting points as QMO, yet non-negligible number of students viewed points as either NMO or SQMO. Besides, some students were not able to answer the question by reflecting their justifications and reasoning.

Table 5.5. Students' responses to question 7.

	Frequency - Percentage		
Responses to Question 7	**Q7a**	**Q7b**	**Q7c**
Quantitative multiplicative object (QMO)	98 (42.8%)	159 (69.4%)	124 (54.1%)
Spatial-quantitative multiplicative object (SQMO)	39 (17%)	24 (10.5%)	13 (5.7%)
Non-multiplicative object (NMO)	30 (13.1%)	15 (6.6%)	14 (6.1%)
Insufficient justification for QMO, SQMO and NMO	47 (20.5%)	16 (7%)	48 (21%)
No answer	15 (6.6%)	15 (6.6%)	30 (13.1%)
Total	229 (100%)	229 (100%)	229 (100%)

Particularly regarding QMO, while only about half of the students successfully interpreted the points as QMO in Q7a (42.8%) and in Q7c (54.1%), most of the students envisioned point as QMO (69%) in producing/plotting a point task in Q7b as demonstrated in Table 5.5. Yet, these percentages should have been significantly higher, especially at high school level.

Moreover, about 17% of the students used SQMO, in other words they relied on non-normative frames of reference in organizing quantities in Q7a. For instance, instead of representing distance from Arch and Canon on axes to form a multiplicative object, students might have coordinated these quantities *in* the coordinate plane in reference to two points considering that the points are actual Arch and Cannon (e.g. see Figure 2.6). The percentage of SQMO decreased to 10.5% in point-plotting task (Q7b) and to much lower (5.7%) when scale and numerical values were given (Q7c). This suggests that when coordinate system was non-numerical, students focused less on using axes as frame of reference to represent quantities and more on spatial features such as being on the right or left side of the map resulting in using non-normative frames of reference to coordinate quantities. Also, despite non-numerical context of Q7b, producing a point might have necessitated using quantities and representing them along axes more compared to Q7a where students only required to determine what each point in the coordinate plane corresponds to on the map.

On the other hand, approximately one fifth of the students could not think through quantities in reasoning about points in Q7a, as 13.1% of them viewed points as NMO, and 6.6% of the students gave no answer to this question. These values seem to be the opposite for Q7c as there is approximately 13.1% no answer and 6.1% NMO. When students were given numerical context in the problem situation in Q7c, even though percentage of QMO increased and NMO decreased, at the same time percentage of students who could not give an answer also increased in comparison to Q7a. This may point that those who interpreted points as SQMO in Q7a were perturbed and could not answer Q7c.

Aside from these results, due to insufficient written explanations some of the responses (20.5% in Q7a, 7% in Q7b and 21% in Q7c) were not categorized as QMO, SQMO or NMO and hence represented as *insufficient justification for SQMO, QMO and NMO* in Table 5.5. Nonetheless, analyzing these responses for Q7a and Q7c, might provide insights about students' possible reasonings for interpreting points in Cartesian coordinate system.

In Q7a, almost half of them (49% of insufficient and 10% of all participants) labeled GT or WFB interchangeably as their distances to Arch and Cannon were approximately equal, and also labeled other locations on the map such as Starbucks, Arch etc. correctly. On the other hand, 17% (3% of all participants) could not distinguish magnitudes of distance from WFB, GT and SoA from each other, although SoA was way further from Arch and Cannon in comparison to WFB and GT. This might further emphasize that students may not pay attention to differences in magnitudes as well as not lay quantities on the axes attentively that might hinder their graphing skills in coordinate system.

In Q7c, 77% percent of the responses with insufficient justification (16% of all participants) successfully matched points with locations on the map. Only 10% (2% of all participants) could not distinguish magnitudes of distance from WFB, GT and SoA from each other. On the other hand, 13% (3% of all participants) failed to label points with corresponding locations on the coordinate system. All these results suggest that identifying magnitudes of the quantities on the map and embedding them to the coordinate system by attending to their magnitudes seemed to play a significant role in interpreting the points more accurately.

In what follows, students' responses for quantitative multiplicative object, spatial-quantitative multiplicative object and non-multiplicative object will be elaborated through some sample responses on each sub-question from the data.

<u>*Quantiative Multiplicative Object (QMO)*</u>

In representing quantitative multiplicative object, students mostly established the quantities of distance from Arch (DfA) and distance from Cannon (DfC) on the map (see Figure 5.11 on the left) and then interpreted the points in relation to these two quantities represented on x and y-axis (see Figure 5.11 on the right).

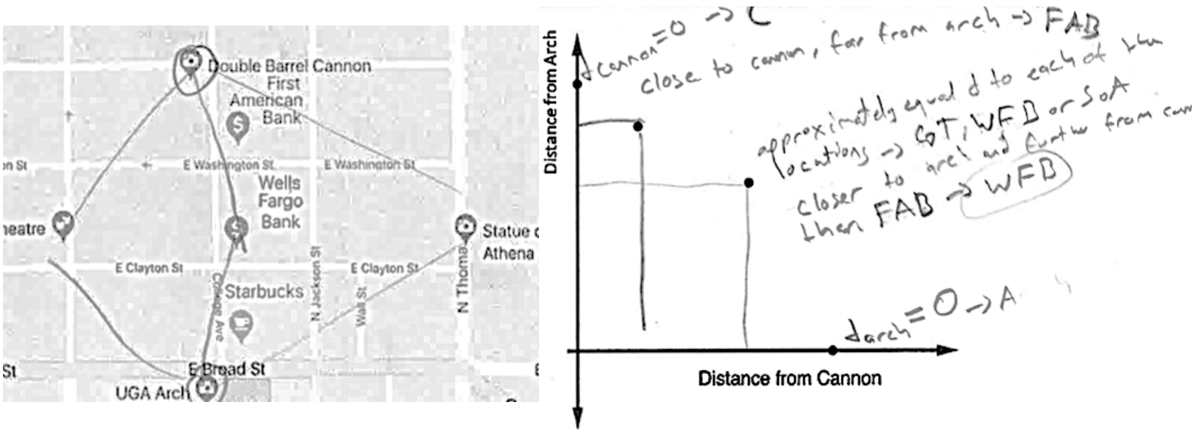

Figure 5.11. Sample quantitative multiplicative object response to question 7a.

Yet, not all responses under QMO category clearly labeled quantities on the axes or as horizontal and vertical distances from y and x-axis respectively. Usually, students labeled

Cannon on the y-axis indicating that distance from Arch was zero and similarly Arch on the x-axis as the other coordinate, i.e. distance from Arch, was zero. Then, they generally described FAB and GT/WFB by comparing their distance to Arch and Cannon. One sample student answer for this case was as follows "Double Barrel Cannon → The distance between the point and the Cannon is 0 [zero]; FAB → distance between point and Cannon is much smaller than the distance between point and Arch; WFB → distances from the point to Arch and Cannon are equal; Arch → distance between the point and the Arch is 0 [zero]".

As indicated in Table 5.5, QMO had the highest percentage in plotting the crows' distance from Arch (DfA) and distance from Cannon (DfC) in the given coordinate plane. Students might have performed better as they were more familiar with plotting point task from their mathematics classes. Besides, this task might have driven students more towards forming quantities and representing them on the axes as exemplified in Figure 5.12a below. Majority of the students in this group measured the distances on the map and then plotted the point accordingly even though magnitudes of x and y-coordinates were not clearly labeled in the diagram. Some students also used vertical distances from crow to A and C as measures of DfA and DfC ignoring the fact that the crow and the locations were not colinear.

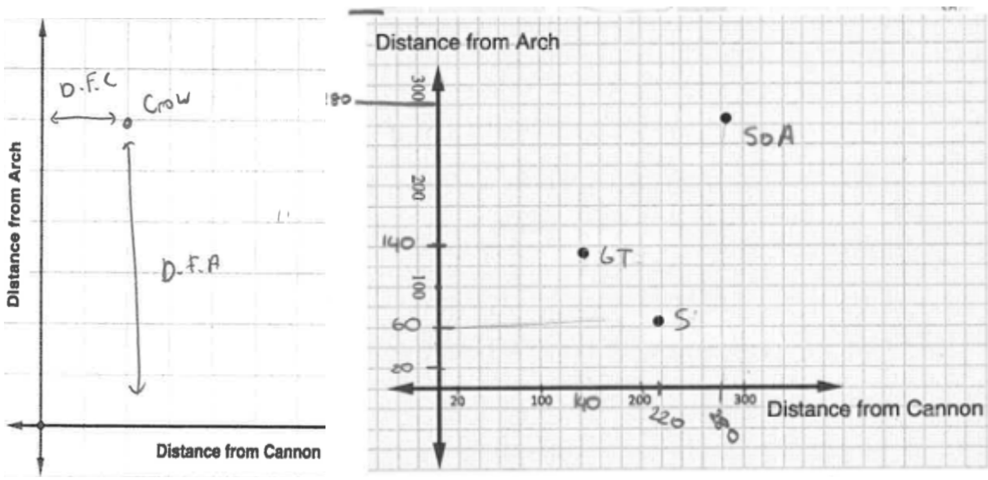

Figure 5.12. (a) Sample QMO response in Q7b (b) Sample QMO response in Q7c.

In Q7c, students either labeled values on the axes (see Figure 5.12b) or wrote an ordered pair such as (220, 60) for describing Starbucks. Moreover, the number of students who envisioned points as QMO increased in Q7c (54.1%) in comparison to Q7a (43%).

While only 7% of the students made insufficient justification in Q7a, 21% of the students thought that there was no need for an explanation for their reasoning in Q7c. These might suggest that given the grid-numerical coordinate system students were more prone to show a point as a quantitative multiplicative object. From a different perspective, students might be perturbed in non-numerical context due to their inadequate experience with working in quantitative problem contexts.

<u>*Spatial-quantitative Multiplicative Object (SQMO)*</u>

In general, when students interpreted points as SQMO, they determined location of a point by coordinating quantitative features *in* the plane rather than representing the quantities *on* the axes and uniting them as in quantitative multiplicative object. For instance, in Figure 5.13, a student considered Arch to be location on the y-axis or the y-axis itself. Then other points were determined by their relative distance to the designated object/location for Arch.

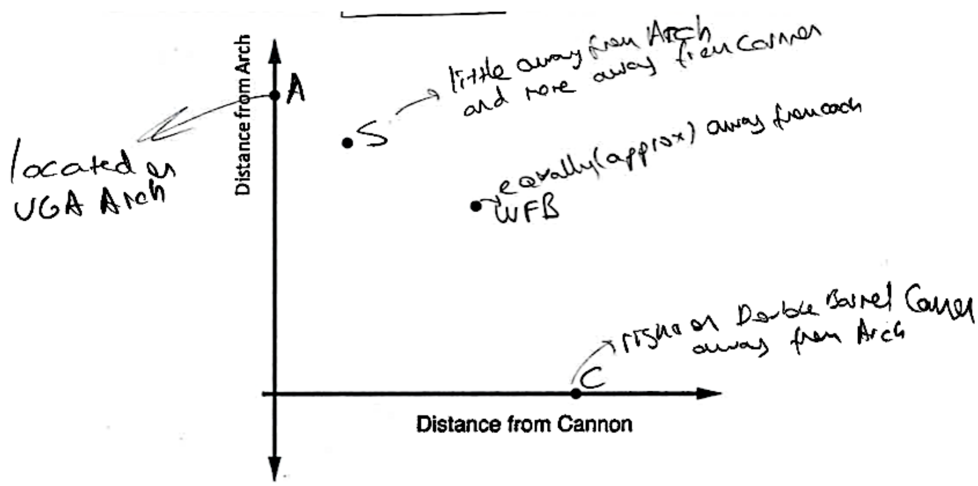

Figure 5.13. First sample student response for SQMO in question 7a.

As seen in Figure 5.14, another student assimilated Arch to y-axis and Cannon to x-axis and then labeled the points on the axes as Arch and Cannon. This supports that student conceived first the axes and then points A and C as reference to assign locations for the remaining points in the coordinate plane. Afterwards, S and WFB were labeled according to their distances from point A. I infer that this student focused on the direct distances between points when s/he coordinated distances. Although the student focused on distances between different locations, s/he described them in a non-normative way.

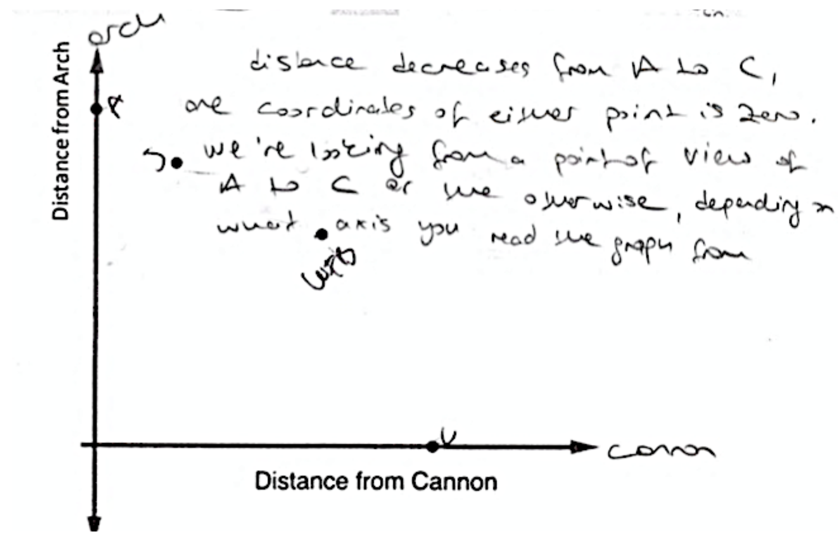

Figure 5.14. Second sample student response for SQMO in question 7a.

Apart from these cases above, I suspect that several students coordinated distances from Arc and Cannon by conceiving Arc and Cannon as locations/objects outside of the coordinate plane. In the sample response below (see Figure 5.15), FAB and S were located in reference to their distance from Arch and Cannon which showed that student coordinated quantities to identify what each point represented. However, if FAB and S were on the axes, then their distance to Cannon and Arch should have been zero respectively assuming the axes were used to represent magnitudes and to identify what each point represents. Also in this particular response, Arch and Cannon were circled, so student might have conceived them as actual A and C.

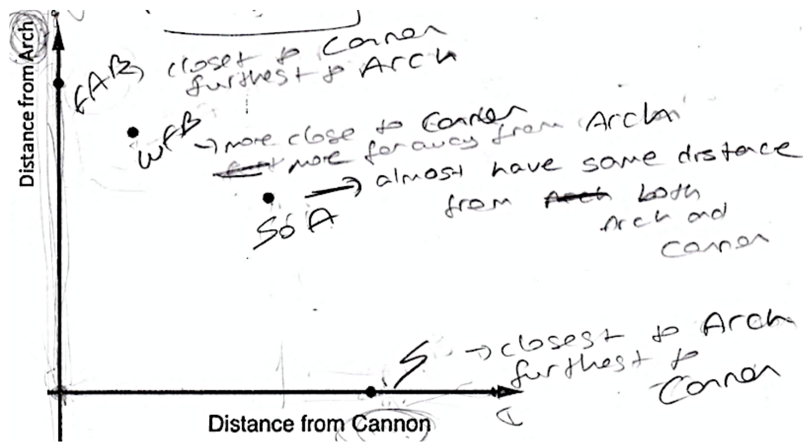

Figure 5.15. Third sample response for SQMO in question 7a.

Furthermore, some students stated that the x value represented distance from Cannon, but instead of using Cartesian coordinates, they focused on spatial features or alternative reference points to measure and coordinate distances. This implies that knowing conventional definition of x and y-coordinates may not necessarily be sufficient for representing points as quantitative multiplicative object. In line with this, the following (see Figure 5.16) shows a sample student response where distance from Cannon (DfC) was measured horizontally instead of distance as crow flies, most probably because DfC is the horizontal axis. By the same token, distance from Arch was measured vertically rather than the direct distance. Although quantities were laid on the axes and the point was formed by coupling these wo magnitudes, their measurement was affected by the perceptual orientation of the axes and hence SQMO was produced.

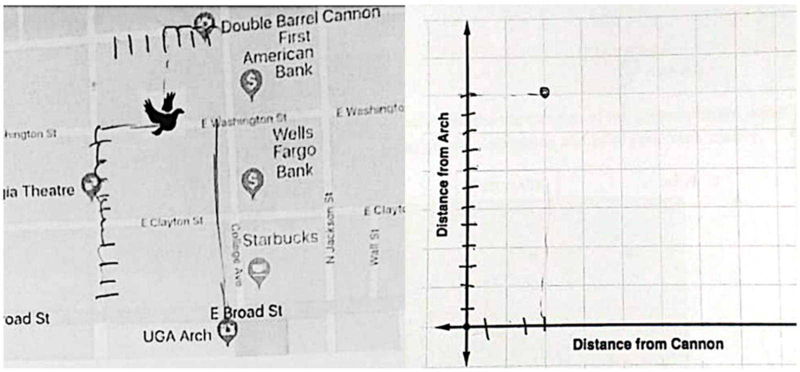

Figure 5.16. Spatial-quantitative multiplicative object in question 7b.

Non-multiplicative Object (NMO)

When students envisioned points as objects or locations by figuratively associating their location on the map instead of uniting two quantities' magnitudes, they viewed points as non-multiplicative object. Students can assimilate the points on the plane as a location/object by engaging in iconic translation (see Figure 5.17) or transformed iconic translation (see Figure 5.18). When students engage in iconic translation, they conceive graphs as representation of the literal pictures of the situation (Clement, 1989; Monk 1992). That is, in case of the Downton Athens task, students would view a point as a location/object by focusing on its position (right, left, south, north) and positional relation between objects/locations on the map (street, point, line etc.). As exemplified in Figure 5.17 (on the left), a student connected FAB, WFB, S and A and preserved this perceptual property of the situation on the plane by labeling the points in the same order. In this case, student assimilated the point on the vertical axis as physical Arch as it appears to be at the top of the map and then labeled WFB, S and A in the same order as appears on the map.

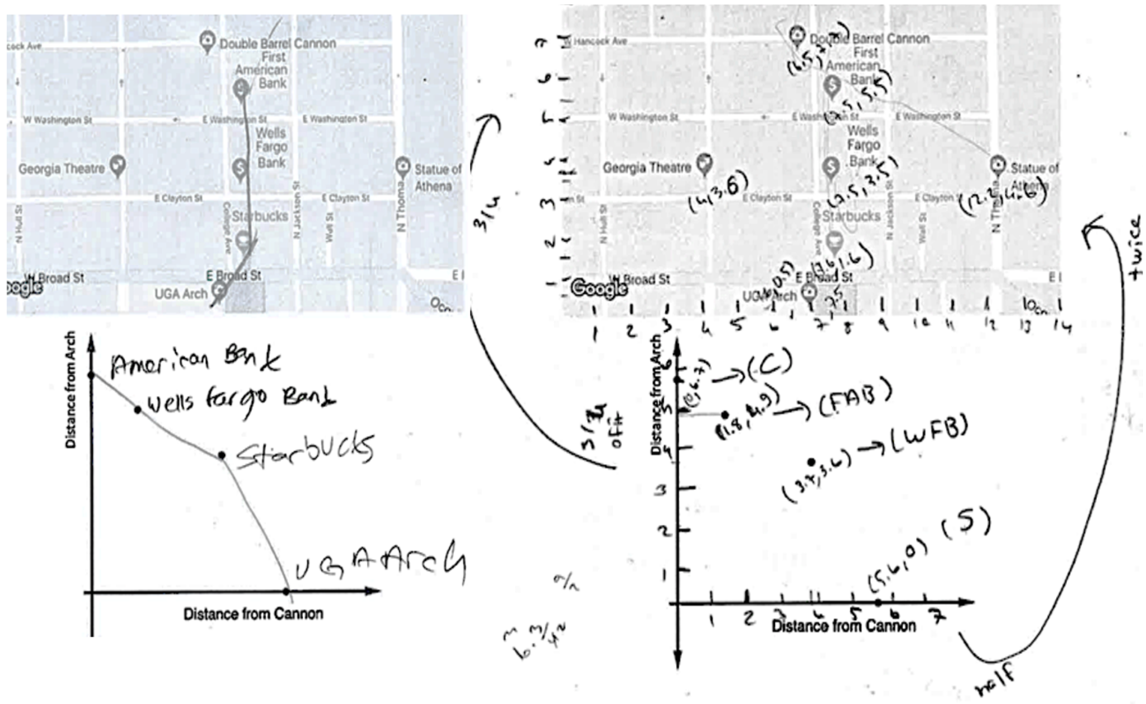

Figure 5.17. Sample student response for NMO-iconic translation in Q7a.

As another example of iconic translation of NMO, some students superimposed a Spatial coordinate system (see Figure 5.17 on the right) on the map and marked locations of the points with ordered pairs. In this spatial coordinate system, axes represented two streets and origin was their intersection. Then, by adjusting the scale of the coordinate system, points on the coordinate plane were assimilated to the points on the map with the same perceptual properties. Thus, points in the coordinate plane represented objects/locations, not coupling of two quantities represented on the axis. In the iconic translation, perceptual properties are translated on the coordinate plane exactly as they appear on the map.

Furthermore, Figure 5.17 (on the right) brought up the distinction between spatial coordinate system and SQMO/NMO. Tasova and Moore (2020) suggested that conceiving a point as SQMO does not necessarily entail representing the relationship on a spatial coordinate system. Supportively, in this example spatial coordinate system was used to assign ordered pairs to locations but student perceived the points in the given coordinate plane as NMO. So, it may not be possible to conclude a particular relationship between spatial coordinate system and representation of a point as a multiplicative object considering these results.

Regarding translated iconic translation for NMO, students might have rotated, reflected the graph or the map to translate perceptual properties as illustrated in Figure 5.18 below. In this sample response, student assimilated the x and y-axis as two streets and rotated them 90° in counterclockwise direction to obtain the same perceptual properties and positional relationship between the points as on the map.

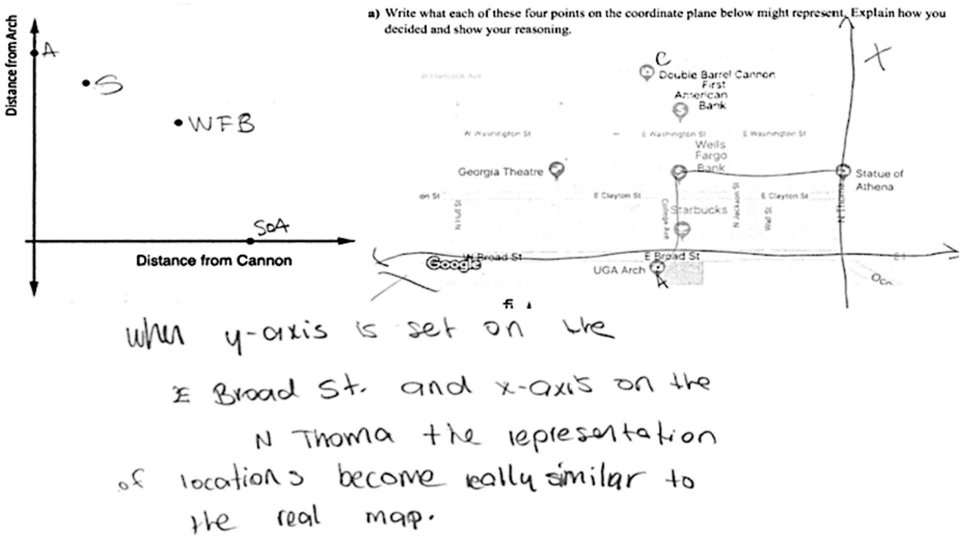

Figure 5.18. Sample student response for NMO-transformed iconic translation in Q7a.

Although it was only 7%, students also formed NMO in plotting point task in Q7b. For instance, one student first established points C and A in Q7a by considering their distance to A and C. Then, he directly plotted points C and A as in Q7b and connected them by a line segment which was assimilated to a street on the map (see Figure 5.19 on the left). He also stated that "The line I drew is vertical street between C and A. Crow is closer to C than A, and leftly located to vertical street". Therefore, we can deduce that the student was engaged in transformed iconic translation to produce the point as NMO.

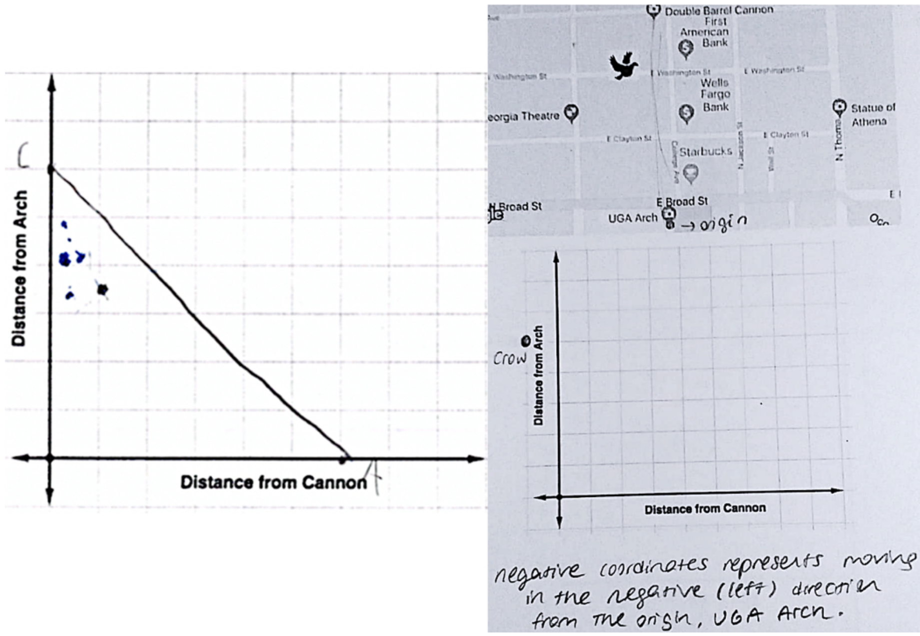

Figure 5.19. (a) NMO-transformed iconic translation in question 7b (b) NMO-iconic translation in question 7b.

Another student connected Arch and Cannon on the map with a vertical line and associated Arch with the origin and y-axis with the line connecting Arch and Cannon on the map (see Figure 5.19b). Then, student plotted the point on the left side of the y-axis since the crow was on the left side of the line connecting Arc and Cannon on the map. The student interpreted the negative sign as direction of the movement. In this example, student translated literal image of the map on the coordinate plane and produced the point as a physical location rather than union of quantities. Thus, student was engaged in iconic translation and envisioned the point as NMO. Also, some other students used locations found in Q7a as reference to decide location of the crow in comparison to their position on the map. So, they viewed the points as physical locations in the coordinate plane and did not use quantities on the axes to plot the point.

5.3. Students' Graphing within Cartesian Coordinate System

In what follows, results of question 6, 8 and 9 will be shared respectively. While Q6a, Q6b and Q8a focused on students' graphing within spatial Cartesian coordinate system, Q8b and Q9 were about students' producing and interpreting graph within quantitative Cartesian coordinate system.

5.3.1. Students' Responses to Question 6a

In general, Q6 explored students' construction of coordinate system in a spatial context and their reasoning about graphs within spatial coordinate system. In the first part, in Q6a, the task was to describe location of a person (represented by point) in a rectangle-like shaped garden for someone else so that they exactly know where the person is. The purpose was to explore if students use frames of reference to establish a spatial Cartesian coordinate system and produce quantities to locate a point as a multiplicative object as coupling of two quantities' measures based on spatial components. Students' responses were analyzed based on whether they superimposed a rectangular frame of reference onto the figure, their description of the point, coordination perspective and committing to reference point, units, and directionality.

Rectangular Frame of Reference

First, students' responses were categorized based on establishing a rectangular frame of reference consisting of vertical and horizontal lines intersecting orthogonally. Results of the analysis (see Table 5.6) show that most of the students established rectangular frame of reference to describe location of a point mathematically. Nevertheless, about one quarter of the students described the point without constructing a coordinate system and small portion of the participants did not answer the question.

Table 5.6. Students' responses to question 6a.

Responses to Question 6a	**Frequency**	**Percentage**
Spatial coordinate system (SCS)	142	62%
No coordinate system (No CS)	63	28%
No answer	23	10%
Quantitative coordinate system (QCS)	1	–
Total	229	229

As summarized in Table 5.6, 62% (142 students) spatially oriented rectangular frame of reference to form a spatial Cartesian coordinate system which was categorized as *Spatial CS,* standing for spatial coordinate system. In Spatial CS, students either laid orthogonal axes onto the figure (126 students), described x and y-axis verbally without drawing them, or labeled an ordered pair without drawing the axes explicitly (16 students). Although I anticipated outline of the garden to be suggestive of axes and make it easier for students as in Piaget's task (1960, p. 153), 10% of the participants (23 students) gave no answer to the question and 28% of the participants (63 students) did not construct a rectangular frame of reference with orthogonal axes to describe location of the point, which was categorized as *No CS,* standing for no coordinate system. Among No CS category, only two students drew axes onto the diagram but did not utilize them as part of their solution to construct a spatial Cartesian coordinate system. Although so few in numbers, this might point that those students did not think of an axis as a tool to represent horizontal and vertical directed distances from the reference point, origin. In other words, the coordinate system was dysfunctional in their solution.

Description of the Point and Coordination Perspective

Next, students' responses were categorized based on how they described the location of the point in No CS and Spatial CS. Table 5.7 summarizes the responses given to describe the exact location of the point where in spatial coordinate system mostly ordered pair and in no coordinate system verbal description was preferred.

Table 5.7. Descriptions of point in question 6a.

Description of point in Q6a	**No CS**	**Spatial CS**	**Total**
Ordered pair	2	102	104
Verbal description	57	33	90
No description	4	7	11
Total	63	142	205

Regarding Spatial CS, majority of the students (72%) described the point mathematically with an ordered pair. However, 23% of them (14% of all participants) gave verbal description instead of forming a multiplicative object by coupling directed distances from the origin represented on vertical and horizontal axis. For example, "it is in the 2nd quarter (meant quadrant) and we in the origin the person is a bit higher than the x-axis but exactly on the side" or "once you go to the origin (left bottom corner of the garden) turn to the direction y and move 10 units" were some of the verbal descriptions. This also supports that students may not have an integral understanding of spatial Cartesian coordinate system and forming multiplicative object within spatial coordinate system. Similarly, 5% of the students in Spatial CS (3% of all participants) gave no specific answer to describe the location such as "Cartesian system as someone can tell y and x values respect to themselves as they are the origin" and "I use coordinate system to show where the lost person is. Because coordinate system is clear and understandable for everyone" or made no explanation at all. So, their understanding of Cartesian coordinate system seems to be incomplete as they could not set directionality or units to describe the point mathematically. Also, they did not conceptualize spatial coordinate system and axes to represent vertical and horizontal directed distance from origin to describe location of an object in perceptual space.

In addition, majority of the students in No CS (83% of No CS and 24% of all participants) gave verbal descriptions as I expected. Some examples were as such: "If you want to reach them, go to the upper left side and walk down 2.75 meters vertically", "2x units from the northern edge of the garden, on the eastern edge of the garden" and etc. For these students, mathematically describing location of a point was not associated with a coordinate system. This may be related to their lack of prior experience in spatial coordinate system. Actually, utilizing spatial coordinate system in such tasks in math classes might leverage students' conceptualization of coordinate system and their uses.

Regarding students' coordination perspective, in their verbal description for location of a point, both in Spatial CS and No CS, majority of the students coordinated an environment-centered frame of reference from *above the ground perspective* (Lee, 2017, p. 36; Taylor & Tversky, 1996). In terms of directionality, they usually pointed to left, right, up, down and west, east, south, north as if the figure was seen from above the ground. A few students attained *embedded within the space perspective* (Lee, 2017, p. 36) making explanations in reference to the listener.

Incorporation of Units and Directionality

After categorizing whether students superimposed a rectangular frame of reference onto the figure and how they described the point mathematically within that frame, students' committing to a unit, a reference point and directionality was investigated as these are entailed for an individual to conceive measures as existing within a frame of reference (Joshua et al., 2015). If students partitioned an axis into small units to measure quantities, their response was categorized as *partitioned*. If they directly assigned numerical values without any partitioning, then the response was categorized as *value*. Lastly, if no numerical value or partitioning was incorporated in students' description, then the response was categorized as *no value* (see Figure. 5.20). Results of Q6a showed that those who established a spatial coordinate system seemed to incorporate partitioning and units more in describing location of a point compared to responses in No CS. Nonetheless, many of the students did not use any numerical value at all.

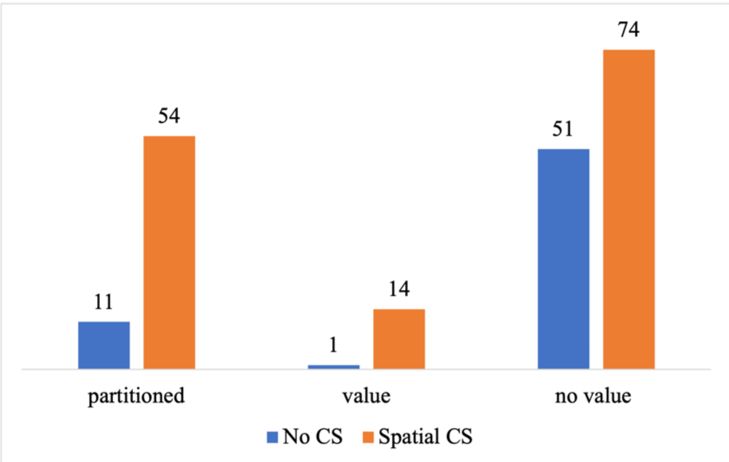

Figure 5.20. Results for committing to units in Cartesian coordination.

Particularly, in terms of committing to units, about half of all the students (55%) did not use any numerical value or partitioning in their description (*no value*). Instead, they used parameters such as (a, b), $(-a, -h)$, x meters up etc. In total, only 28% of the students used partitioning in their description. In No CS, students partitioned side lengths so that width and length of the garden are proportional, and description of the point is more precise. In Spatial CS, students partitioned x and y-axis (44 students) or only y-axis with zero x-

coordinate (10 students) into smaller parts to identify the precise location of the point. In this case, students tried to have the same scale for both axes. In Spatial CS, they usually showed this by putting tick marks on the axes and in No CS by assigning values to side lengths of the garden proportionally. Around 7% of the students used values such as 2 meter, 10 units without partitioning the axes or sides of the figure into smaller units (*value*).

In terms of committing to directionality, only around 10% of the students indicated positive and negative directions clearly as part of their answer or used negative values in ordered pairs. Most of them used lower left corner of the garden (according to above the ground perspective) as reference point (origin) and hence ended up with positive Cartesian coordinates. Also, some of the students did not draw the negative side of the axes. In addition to these results, Figure 5.21 displays a sample student response where the axes were oriented in the opposite direction of conventional Cartesian coordinate system. This might indicate that some students might have associated only positive values for describing location and be reluctant to use negative sign for that purpose. This also aligns with the results of Q4 and Q5 where some students commented that quantities for x and y-coordinates cannot be considered as distance as distance cannot be negative. Therefore, students might potentially struggle about directionality and meaning of negative in a coordinate system. This also highlights conceptualizing Cartesian coordinates as directed distance from origin and discussing meaning of negative sign in different problem contexts.

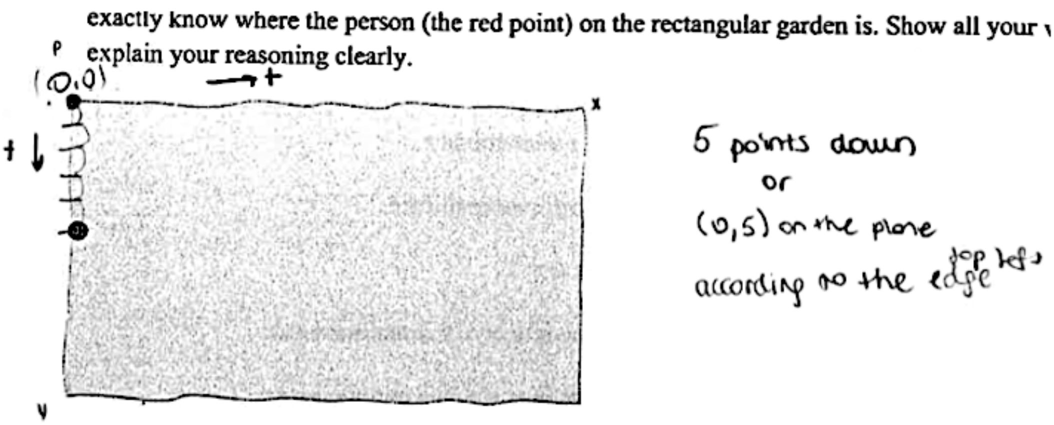

Figure 5.21. Unconventional orientation of axes in Q6a.

5.3.2. Students' Responses to Question 6b

Q6b was about students' reasoning in spatial coordinate system in producing and interpreting a graph. The question was to describe the location of the person (represented by point) throughout its journey toward the exit gate in the garden. In addition to constructing a spatial coordinate system and conceptualizing the point by uniting x-coordinate representing the horizontal distance and y-coordinate representing the vertical distance from the origin as in Q6a, they were further required to describe the graph (path) mathematically. In this case, students might reason about the graph as representing how vertical and horizontal distance from reference point change which is indicator of emergent shape thinking or as a static shape without focusing on the trace of changes in quantities forming the line. Results of Q6b (see Table 5.8) reveal that unfortunately only small of percentage of students were able to reason emergently about the graph within spatial coordinate system. Majority of the students either considered the graph as a static, stable shape or gave no answer.

Table 5.8. Students' reasoning about graph in spatial coordinate system.

Reasoning about graph in Spatial CS	**Frequency**	**Percentage**
Emergent shape thinking	19	8%
Static shape thinking	141	62%
No answer	40	17%
Insufficient justification	29	13%

Particularly, only 8% percent of the participants demonstrated emergent shape thinking in reasoning about the graph. Most of the participants (62%) were engaged in static shape thinking where the graph given in the problem situation was interpreted as a static linear path. Apart from static shape thinking, some students stated that they did not have enough time or know how to procced, therefore 17% of the students gave no answer. Lastly, 13% of the students came up with linear equations, mainly in the form of $y = mx + n$. However, due to insufficient explanation or labeling, their responses were inconclusive and represented in Table 5.8 as *insufficient justification*.

In the following, students emergent and static shape thinking about graphs within spatial coordinate system will be elaborated through sample student responses.

Emergent Shape Thinking in Spatial Coordinate System

Reasoning about the graph emergently in spatial coordinate system entails conceiving of the location of the person (represented by point) as multiplicative object which is simultaneously composed of horizontal and vertical distances from the reference point, and the graph as varying horizontal and vertical distances (Paoletti et al., 2018). For this reason, even though students usually did not write detailed explanations, if a student indicated that the graph consists of emergent trace of infinitely many points representing varying horizontal and vertical components/quantities, their answers were categorized as emergent shape thinking (e.g. see Figure 5.22 below).

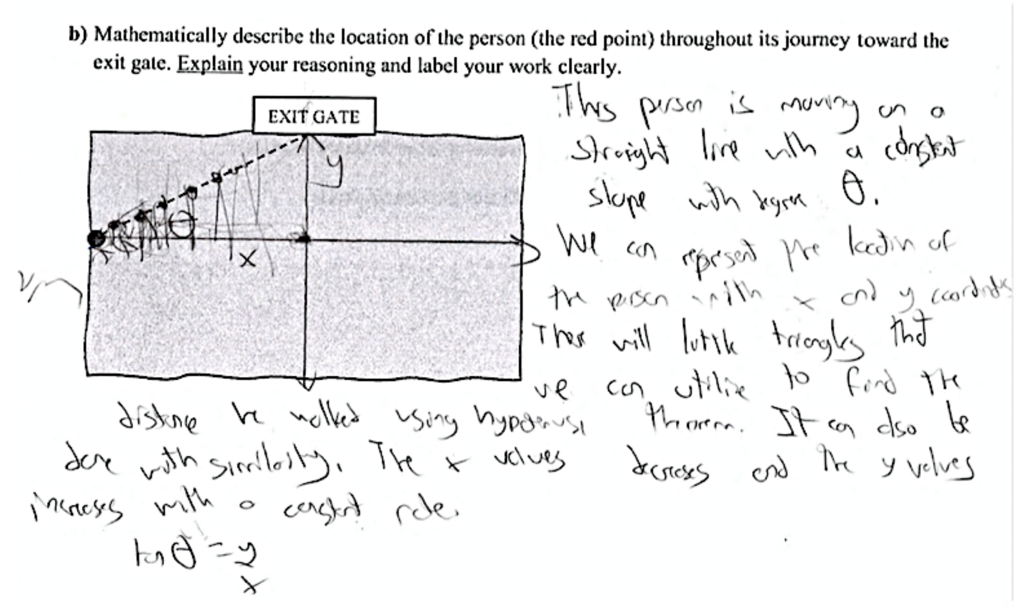

Figure 5.22. First sample student response for emergent shape thinking in Q6b.

Figure 5.22 exemplifies that despite not coming up with an equation, some particular students were aware of the horizontal and vertical displacement from the origin and the fact that two quantities simultaneously changed resulting in new points, new locations and creating a function. In the figure, student expressed that “The x values decreases and y

values increases at a constant rate" and additionally marked some of the y-coordinates on the diagram to demonstrate increasing quantities of the y-coordinate. So, the graph seems to be envisioned as an emergent trace of multiplicative object combining two quantities. Similarly Figure 5.23 exemplifies another emergent shape thinking response.

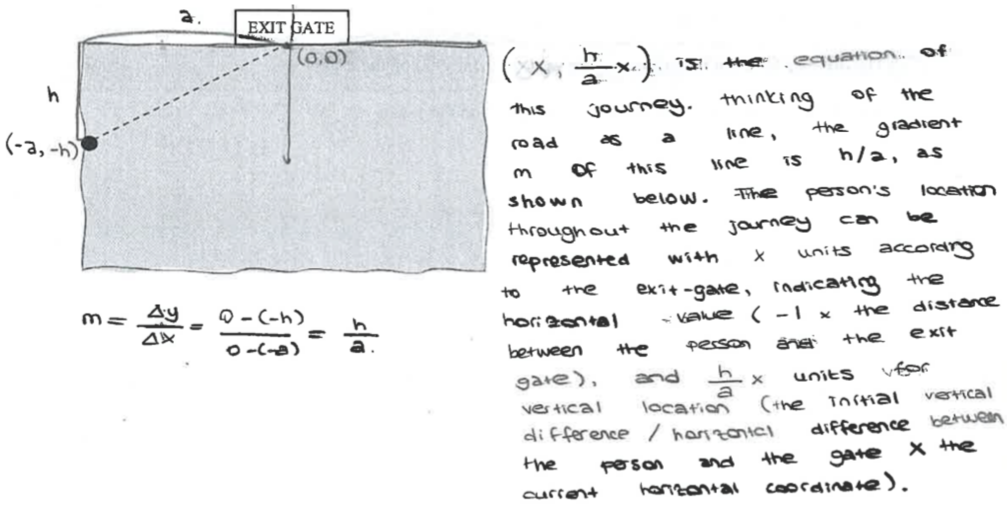

Figure 5.23. Second sample student response for emergent shape thinking in Q6b.

In Figure 5.23 shown above, student described the x-coordinate as the horizontal value ($-1\times$the distance between the person and the exit gate) and the y-coordinate as vertical location. Although initial location was indicated as $(-a, -h)$, the path was described in terms of variables which is suggestive of graph consisting of varying quantities of horizontal and vertical components.

Besides, the following statements and similar responses were considered as indicator of emergent shape thinking as students mentioned about continuous change of x and y-coordinates throughout journey which ultimately formed the graph such as: "The person would be traveling on the hypotenuse of the triangle which has (x, y) coordinates throughout the journey of person", "We should make a function as person moves and so we can be able to find person wherever he goes. $f(x) = x + 5$" or "He displaces on x-axis and y-axis towards positive direction simultaneously". Similarly in the following Figure 5.24, line was described by a parametric equation where initial x and y-coordinates change throughout time continuously and the line is formed as a result of emergent trace of these points.

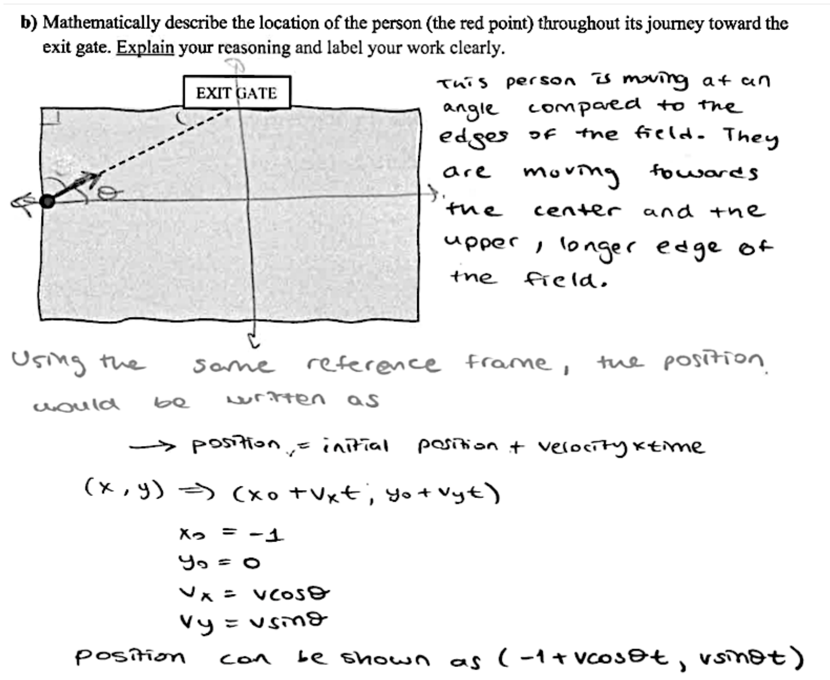

Figure 5.24. Third sample student response for emergent shape thinking in Q6b.

<u>*Static-Shape Thinking in Spatial Coordinate System*</u>

Among static shape thinking category, 62 students (44% of this group and 27% of all participants) described the graph as a linear path/route to the exit gate such as “Using the coordinate plane drawn above [*referring to Q6a*], this red dot will have to go with a slope of 4/7 until reaching the edge of the garden, where the exit gate is. This would mean that there should be an angle of about 60.3° calculated from triangle” or “In the plane, person goes to {c, o} from the point {b, a} using a linear path (slope)”. Strikingly, in the second response student used the terms slope and linear interchangeably. This suggests that student could not distinguish slope and linear function and used these terms in reference to a static linear shape. Using curly brackets instead of parenthesis also highlights the fact that this student might not have strong understanding of Cartesian coordinate system and its components.

In parallel, 26 students (18% of this group and 11% of all students) described the path as distance. 38 students (27% of this group and 17% of all participants) conceived of the graph as a static shape/route to the exit gate and this time preferred to describe it as a rectangular path (see Figure 5.25) rather than describing the line as emergent trace of points representing multiplicative object. Here although Cartesian coordinate system was drawn onto the figure, it was only used as a measurement tool to quantify horizontal and vertical distance along the side of the garden. So, the coordinate system was not used to lay quantities on the axes and to form multiplicative object by uniting them. Furthermore, since perceptual shape and the shape itself are the focus of static thinking (Paoletti et al., 2018), the following response supports the finding of static shape thinking since the student described person to stand on the line: "According to the origin that is set at the middle of the exit gate, the person stands at the 3rd quarter on the line of $y = 5/7x$ when x is equal to -7." Another response in this category was that "Because the person is walking in a straight line their movement can be defined by a linear function.".

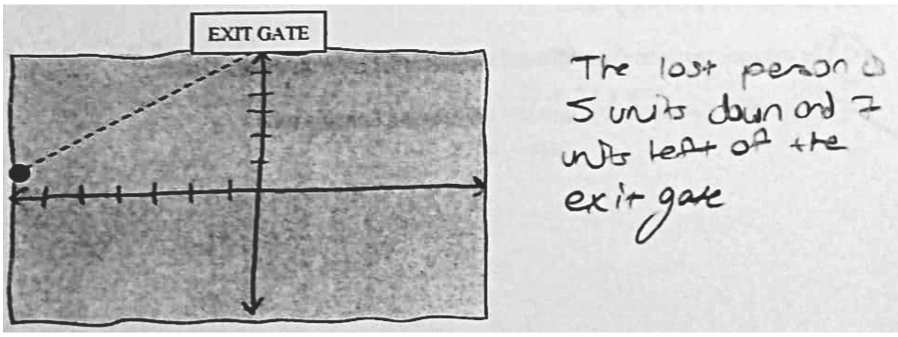

Figure 5.25. Sample student response for static shape thinking in spatial quantitative coordinate system.

Consider the following response "If I had a coordinate plane, I could write an equation $y = mx + b$. Find the midpoint of the left side and the upper side. Connect those points with an imaginary line. The person is moving along that line". In this response, points worked as landmarks rather than forming a line consisting of infinitely many points. Besides, the line equation served to connect the points as a static shape. Considering "$f(x) = ax +$, the person will have a constant slope and will go on a straight line", the student possibly thought of line equation but associated it with perceptual feature by "straight line". In these examples, x and y variables were not conceptualized as quantities' measure based on spatial features. This puts forth that students' reasoning should be always questioned by teacher in the learning environment as unlike Figure 5.24, in these examples coming up with an equation was not an indicator of emergent shape thinking.

Subsequently, students' reasoning about producing and interpreting graphs were analyzed according to their construction of coordinate system. In Q6b, excluding 39 blank answers, students either established spatial Cartesian coordinate system, quantitative Cartesian coordinate system, polar coordinate system or no coordinate system in their responses (see Table 5.9 and Figure 5.26). By establishing coordinate systems, I only mean laying rectangular or polar frame of reference onto perceptual space and identifying a reference point since not all students integrated units and directionality into their coordinate plane.

Table 5.9. Students' graphing in spatial coordinate system.

	No answer	**No CS**	**Spatial CS**	**Quantitative CS**	**Polar CS**
Blank	39		1		
Static shape thinking		79	61		1
Emergent shape thinking			17	2	
In sufficient justification (might be emergent or spatial)		2	27		
Total	39	81	106	2	1

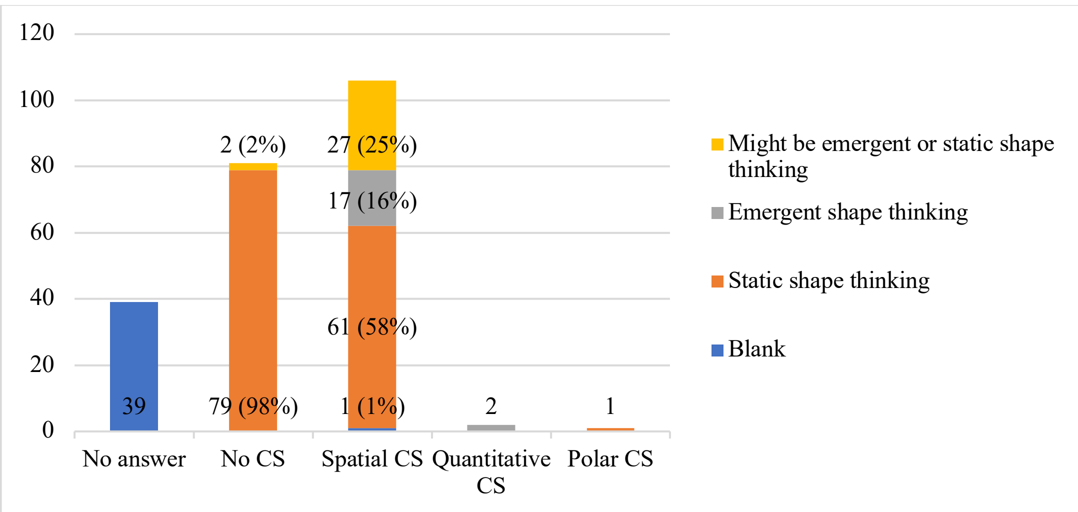

Figure 5.26. Percentages of students' graphing in spatial coordinate system.

Table 5.9 and the Figure 5.26 demonstrate that all the students in emergent shape thinking built either a spatial or quantitative coordinate system. Almost all No CS responses (98% among No CS) attained static shape thinking in interpreting and producing graph. Among those who had spatial coordinate system, more than half of them (58%) had static shape thinking, one fourth of them might be categorized under insufficient justification and 16% demonstrated emergent shape thinking. Only one student stated that "I don't know" and could not utilize the spatial coordinate system.

5.3.3. Students Responses to Question 8a

Q8a in the inventory was about students' graphing within spatial Cartesian coordinate system where students were asked to mathematically describe the logo on a pool table. In this question, I expected majority of the students to construct a spatial coordinate system and reason about the logo with static shape thinking. On the contrary, majority of the students did not establish any coordinate system to describe the logo mathematically in reference to (see Table 5.10).

Table 5.10. Students' responses to question 8a.

Responses to Question 8a	**Frequency**	**Percentage**
Spatial coordinate system (Spatial CS)	47	21%
No coordinate system (No CS)	149	65%
No answer	33	14%
Total	229	100%

More specifically, Table 5.10 indicates that large percent of the students (65%) did not establish a spatial coordinate system (No CS) to describe the logo mathematically. 21% of the students inserted a spatial Cartesian coordinate plane onto the pool table (Spatial CS) and 14% of the students either had not enough time to answer or left the question blank. When responses in Spatial CS and No CS are further investigated, in general students described the logo of the pool table either as *shape* such as semicircle, arc, parabola or half a circle, *length (*πr*) or* with different kinds of *equation/expressions.* In general students who used semicircle or circle equation to describe the logo demonstrated static shape thinking in interpreting the graph in spatial coordinate system as they used a general formula that they correctly or incorrectly remembered from their mathematics class.

Particularly in No CS, 110 students (74% of No CS and 48% of all participants) described the logo as a shape mainly as semicircle (93 students) and arc (8 students). Similarly, 11 students described the shape as length πr and 28 students with some kind of an equation in No CS. Within Spatial CS, only 11 students (23% of spatial CS 5% of all participants) described the logo as semicircle without utilizing the spatial coordinate system and 2 students described it as length, πr. The remaining 34 students described the logo with some kind of an equation in Spatial CS. Especially the mistakes students made describing the graph with an equation imply that students used a memorized mathematical formula rather than covariation of two simultaneously represented quantities in Spatial CS in this problem.

In sum, firstly when No CS and Spatial CS compared, it might point out that when no coordinate system is constructed, students more tend to give mathematical description based on the general shape of the logo or conceive it as a length, not consisting of emergent trace of points. Secondly, those who gave general semicircle and circle equations did not use coordinate system to set directionality and units.

5.3.4. Students Responses to Question 8b and 9

Q8b and Q9 were about students' graphing in quantitative coordinate system. Specifically, in Q8b students were required to construct a quantitative coordinate system to graph the red ball's distance from the blue ball and its distance from the yellow ball as it moves toward pocket of a pool table (see Appendix A). In Q9 the task was to graph the relationship between the car's distance from Istanbul and its distance from Bolu during the trip from Istanbul to Ankara (see Appendix A). Compared to Q9, Q8b was more challenging because students were not provided with any constructed coordinate system. Tracing covariation of two quantities was also more challenging since in some intervals, quantities did not increase or decrease simultaneously. According to results in Table 5.11 below, students had major difficulty in graphing relationships as emergent trace of quantities (emergent shape thinking) in quantitative Cartesian coordinate system.

Table 5.11. Students' graphing in quantitative Cartesian coordinate system.

	Frequency - Percentage	
Responses	**Q8b**	**Q9**
Emergent shape thinking	25 (11%)	91 (40%)
Static shape thinking	44 (19%)	40 (17%)
Inchoate emergent shape thinking	76 (33%)	25 (11%)
No answer	84 (37%)	73 (32%)
Total	229 (100%)	229 (100%)

Specifically, in Q8b, 19% of the students engaged in static shape thinking and 37% of the students could not give an answer. Therefore, more than half of the students (adding up to 56%) did not engage in productive reasoning for graphing. Similarly, in Q9 17% of the students demonstrated static shape thinking and 32% of the students gave no answer, showing that about half of the students (49%) had poor meanings for graphing in quantitative Cartesian coordinate system.

On the other hand, as anticipated, students performed more successfully in Q9 where 40% of the students engaged in emergent shape thinking whereas strikingly in Q8b only 11% of the students were able to produce the graph as an emergent trace of infinitely many points that represent two quantities' covariation simultaneously. So, students had critical difficulty in establishing a Cartesian coordinate system and engaging emergent shape thinking in Q8b.

Lastly, some students (33% in Q8b and 11% in Q9) showed inchoate emergent shape thinking for graphing in Cartesian coordinate system as shown in Table 5.11. Unfolding these cases where students engaged in other forms of reasoning for graphing might shed light on how to improve students' reasoning for graphing in coordinate system. In what follows, students' emergent shape thinking, static shape thinking, and inchoate forms of emergent shape thinking will be elaborated through sample student responses.

Emergent Shape Thinking in Quantitative Coordinate System

The following Figure 5.27 illustrates two different examples where students successfully formed a multiplicative object to unite two quantities, namely distance from yellow ball and distance from blue ball, and then depicted the relationship between two quantities by envisioning how they covaried. The statement "Then I tried to combine them [two quantities] in a graph" (see Figure 5.27a), implies that student constructed a multiplicative object by "combining" the two quantities and then envisioned how they varied together. Mostly students plotted points and tried to envision how the graph varied between these marks as shown in Figure 5.27b. It seems like by plotting several points over small interval, the student was able to depict the relationship between two quantities successfully.

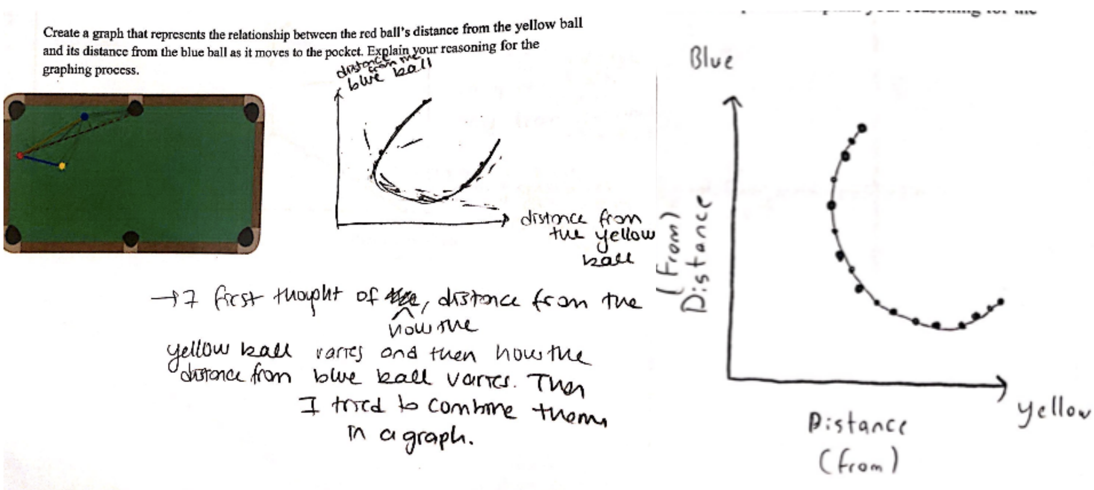

Figure 5.27. (a) Emergent shape thinking-I (b) Emergent shape thinking-II in Q8b.

Static Shape Thinking in Quantitative Coordinate System

Students who engaged in static shape thinking in graphing engaged in iconic or thematic translation (Lee, Hardison & Paoletti, 2018).

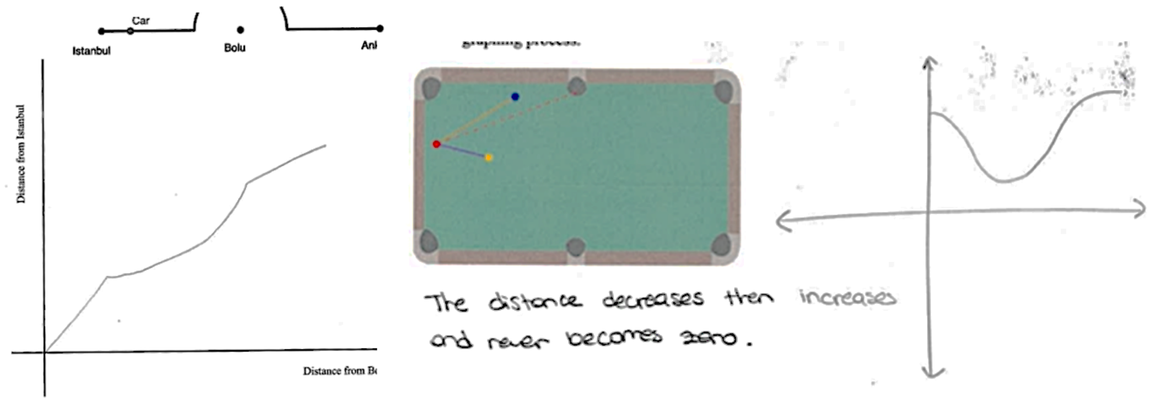

Figure 5.28. (a) Iconic translation in static shape thinking (b) Thematic association in static shape thinking in quantitative Cartesian coordinate system.

Figure 5.28a particularly depicts an example where a student engaged in iconic translation. In this case, student focused on figurative aspects of the route and conceived the graph as a static shape in and of itself. More specifically, s/he constructed the graph by assimilating the circular path with the graph itself.

As another way of static shape thinking, Figure 5.28b exemplifies thematic association since student seemed to use sinusoidal graph that s/he retrieved from prior mathematics classes to model increase and decrease of a variable in experiential time. In a similar manner, many students drew parabola or a linear graph showing distance from yellow ball and distance from blue ball first decreased then increased together. These students used graphs that they were familiar from prior experiences not modeling the problem situation and did not construct a multiplicative object to represent quantities simultaneously. Whereas the task was to decide how two quantities covary by first visualizing change in each quantity and then tracing multiplicative object representing their union.

Inchoate Forms of Emergent Shape Thinking

In this category, students showed various unproductive ways of thinking about graphs in Cartesian coordinate system. According to Table 5.11, 76 students in Q8b (33%) and 25 students in Q9 (11%) engaged in inchoate emergent shape thinking.

Regarding question 8b, in total 33 students (44% of inchoate emergent shape thinking and 15% of all participants) sketched two different graphs, either as curve or in linear form, to represent distance from yellow ball and distance from blue ball separately using time as secondary variable as shown in Figure 5.29a below. These students failed to construct the multiplicative object to track varying quantities.

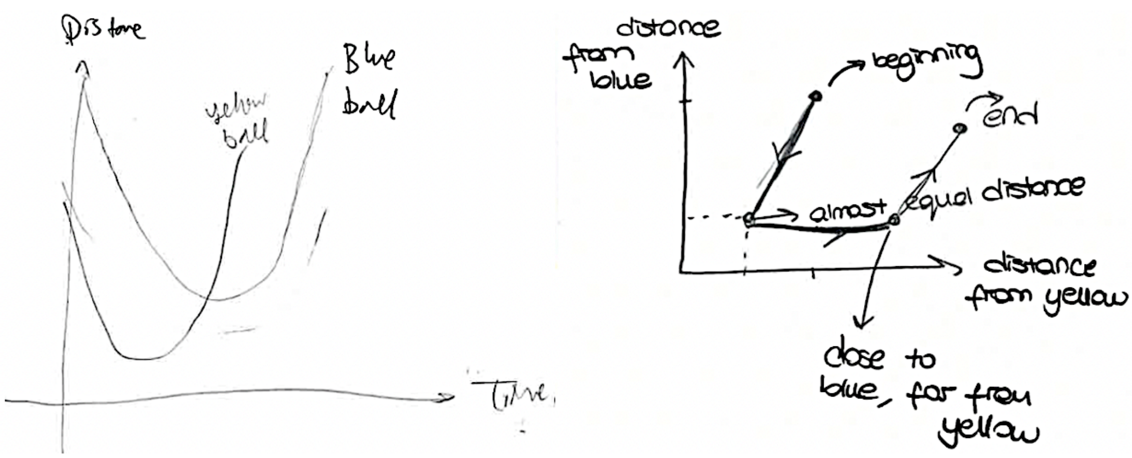

Figure 5.29. (a) Graphs of one quantity (b) Emergent shape thinking with constant rate of change.

On the other hand, 15 students (20% inchoate emergent shape thinking and 7% of all participants) managed to construct multiplicative objects, identified critical intervals where variation in quantities changed direction but failed to envision how graph behaved between these landmark points. As shown in Figure 5.29b above, a student connected the points directly by line segments. Thus, they do not seem to attend to continuous covariation of two quantities.

Slightly different than the last case (Figure 5.29b), 16 students (21% of inchoate emergent shape thinking and 7% of all participants) marked two or three landmark points but were inattentive to all intervals where quantities increased or decreased asynchronously. Here they assumed that distance from the blue ball and distance from the yellow ball increased or decreased together at the same rate. 7 of the students connected the points by line and the remaining 9 student connected by curve as shown in Figure 5.30.

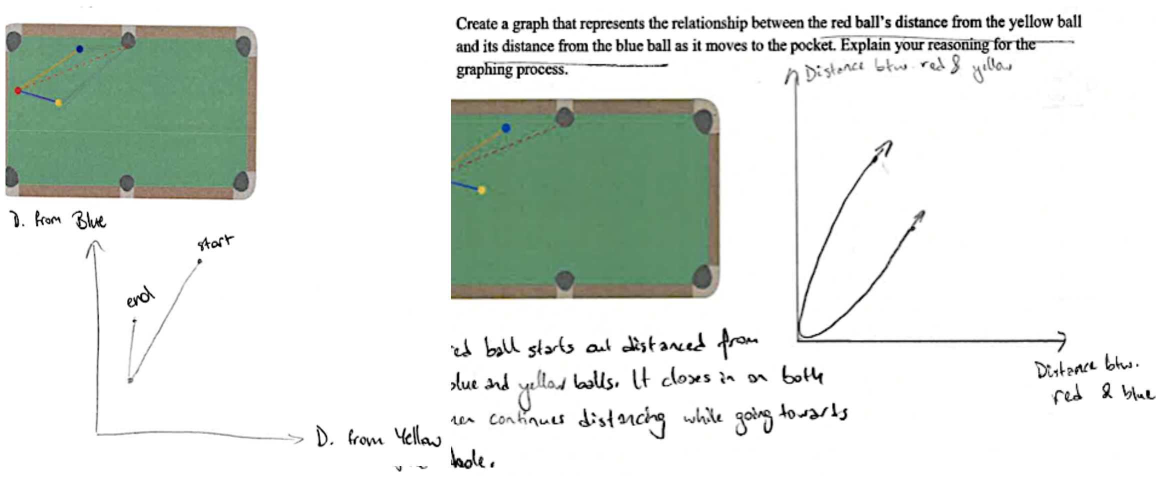

Figure 5.30. Graphs that contain a few points and represent synchronous change of two quantities.

Considering the remaining responses in Q8b, students either graphed a discrete graph with several landmark points disconnected, plotted one point but perturbed to coordinate the changes in both quantities or they connected a few landmarks resulting in a very irrelevant graph for the problem situation.

Regarding inchoate ways of reasoning in Q9, as shown in Figure 5.31 below, students mainly failed to graph when the distance from Bolu did not change for some interval but the distance from Istanbul increased at the same time.

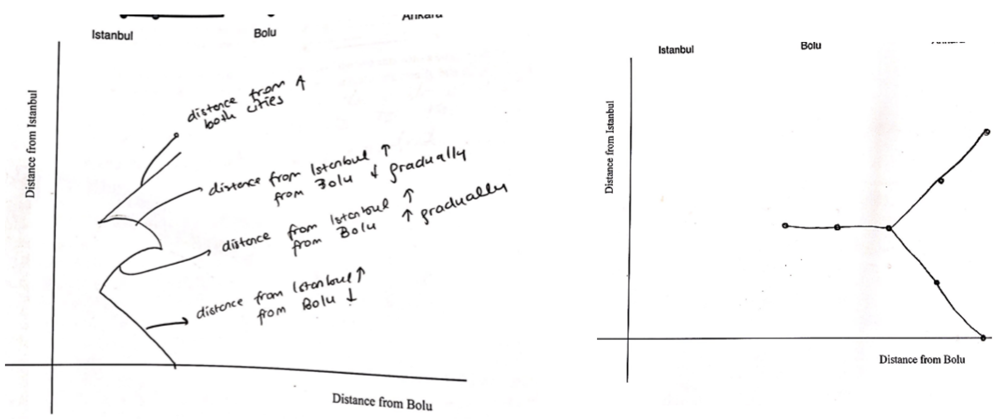

Figure 5.31. Inchoate ways of emergent shape thinking in Q9.

All these inchoate ways of emergent shape thinking provide insights about why students might be hindered in graphing in Cartesian coordinate system. Particularly, students seemed to struggle envisioning covariation of two quantities when time is not one of the quantities. Some students attended to gross variation in quantities such as both quantities decreased or increased synchronously, thus they tried to associate the relationship with a memorized graph of a function. Some of them did not conceive quantities represented on axes and imagine how they changed in mind. Even if some students succeeded to form multiplicative objects, they focused on discrete moments to plot points and did not envision how the graph behaved in between these points. Non-canonical aspects of a graph such as graph starting on the x-axis going upward to left in Q9, might have also perturbed students in their graphing activities. Overall, conceiving points as quantitative multiplicative object combining quantities represented as directed distances from the origin, coordinating magnitudes of two quantities simultaneously and tracing them emergently in graphing within Cartesian coordinate system seem to be crucial based on the results obtained and students' written explanations.

6. DISCUSSION

In this section, first the findings of the study will be discussed with regard to the research questions and taking the relevant literature into consideration. Then, limitations of the study and implications for further research will be explained.

This study set out to answer how high school students reason about Cartesian coordinate system and graphs within Cartesian coordinate system. For this purpose, an inventory was developed based on the existing literature to investigate how high school students reason about spatial and quantitative Cartesian coordinate systems, their meanings of a point on a graph, outputs of a function and graphs within these coordinate systems. The data which composed of graphs, diagrams, written explanations and mathematical expressions were analyzed mainly according to two frameworks: the framework for reasoning about graph in spatial and quantitative Cartesian coordinate system and the framework for representing a multiplicative object in the context of graphing. In what follows, at the outset, findings related to students' spatial and quantitative coordination will be discussed. Then findings regarding students' meanings of a point as a multiplicative object and graphing within spatial and quantitative coordinate systems will be discussed.

Overall, analysis of the results revealed that students had crucial difficulty in spatial and quantitative coordination, particularly in conceiving axes as frame of reference to represent horizontal and vertical directed distance from the origin, standing for x and y-coordinates, in describing the location of the point. In this respect, when students' meanings for points in Cartesian coordinate system was investigated, results showed that almost half of the students viewed x and y-coordinates of a function such as $y = mx + n$ and output of a function, $f(h)$, as a point on the graph rather than conceiving point on a graph as a multiplicative object formed by combining orthogonal projection of two quantities' magnitudes represented on axes. That is, when asked to show what x and y-coordinates of a function refer to, they showed a point on the graph rather than pointing to the orthogonal projections on the axes. Students' preference to use Cartesian coordinate system for conventional tasks such as computing distance between two points, finding mid-point, or using axis for counting purposes also suggest that those students lacked conceptual understanding of frame of references used for Cartesian coordination as they could not establish quantitative Cartesian coordinate system. Relatedly, when students' reasoning for graphing in Cartesian coordinate system was investigated, results pointed to student difficulties for envisioning a graph through emergent shape thinking. First, students were not prone to think quantities as directed distance from origin represented on axes. Then they were challenged in forming multiplicative object, hence tracing changes in both quantities simultaneously to view graph as an emergent trace of two quantities' representation. In what follows these conclusions and related arguments will be discussed in regard to prior research studies more in detail.

6.1. Discussion Regarding Students' Meanings of Cartesian Coordinate System

Overall, findings of the study regarding students' construction of spatial and quantitative Cartesian coordinate systems showed that in Q1 only 5% established quantitative Cartesian coordinate system and 9% established spatial coordinate system which did not serve beyond carrying out conventional tasks. Also, in Q6a 62% and in Q8a 21% of the students established a spatial coordinate system. However, regarding those who constructed spatial rectangular frame of reference, they either had some issues with committing to directionality and units or did not form an ordered pair to describe location of an object in Q6a and majorly they relied on static shape thinking when graphing in spatial coordinate system. Therefore, especially spatial coordinate system should also be integrated into middle school curriculum when students are first introduced with Cartesian coordinate system. This is important as previous research also pointed that both middle and high school students should be provided with balanced understanding of both spatial and quantitative coordinate systems and their meanings for Cartesian coordinate system should not be taken for granted (Lee et al., 2018).

In particular, results from Q1 pointed to important findings regarding students' meanings of Cartesian coordinate system. Coordinate systems are established by coordinating multiple frames of reference (Lee et al., 2019) by which one can organize processes and products of quantitative reasoning (Joshua et al., 2015) and determine location of objects within perceptual space (Rock, 1992). In that sense, firstly, in this study students had crucial difficulty particularly with spatial and quantitative coordination required to describe location of a point since they did not consider axis as a tool, thereby did not represent directed distances from origin and did not view point as union of orthogonal projections of these quantities represented on axes. Instead, students relied on procedural tasks that they were familiar from their mathematics classes such as finding mid-point or drawing graphs to depict distance traveled by time. Secondly, in line with the previous research (Drimalla et al., 2020; Lee et al., 2019; Lee et al., 2020), this study further showed that constructing a spatial and quantitative coordinate system by coordinating multiple frames of reference is also a non-trivial task for even high school students who have high potential and capability in learning and doing mathematics.

Moreover, results from Q1 shed light on some ways of thinking that leveraged students in establishing quantitative Cartesian coordinate system. When students intersected the tubes/lines orthogonally at their end point, one of the (ants) points' x and the other's y-coordinate were zero. This enabled students to focus only on nonzero coordinates of the ants and to perceive those magnitudes on the horizontal and vertical axis as representation of x and y-coordinates of the single point. I mark this finding as a significant cognitive resource for students' reasoning about Cartesian coordinate system because it showed that arranging the tubes in a way that bringing students' attention on directed distances from the origin was productive to conceive x and y-coordinates. Thus, findings of this study can inspire mathematics educators and teachers for learning design and developing curriculum materials to support students' productive struggle (Boston, Dillon, Smith, & Miller, 2017) by showing ways to focus students' attention on Cartesian coordinates as quantities represented as directed distances along axes in constructing coordinate system (Karagöz Akar et al., 2022).

In addition, this study contributed to the literature by elaborating and deepening results of prior studies. In a preliminary study, Lee et al. (2020) highlighted *attention to variability in the ants' locations* coupled with *imagining the single point as moving along with the two ants as* cognitive resource critical for constructing a Cartesian coordination given two lines that would enable holding a sustained image of two locations simultaneously for random positions of points on each line. As a follow up to this outcome, in the current study Q1 was phrased as "Can you describe mathematically the locations of the two ants with a single point that moves along with the ants?" to emphasize that the single point is not a static point. Despite students' attention to variability in the ants' locations and being informed of the dynamic nature of the single point, only 5% of the students were able to answer the question correctly. Another highlighted cognitive resource was recognizing the tubes as objects that can be manipulated and rearranged (Lee et al., 2020). However not all students who manipulated the axes resembling Cartesian coordinate plane performed successfully. These results showed that very few number of students were able to use cognitive resources. Therefore, results suggest that students' thinking needs to be supported by probing questions at these critical moments during instruction so that they can attend to conceptual meanings of the frames of references to engage in productive reasoning processes of quantitative coordination and describing locations of points mathematically. Otherwise, these significant cognitive resources may not be sufficient for spatial and quantitative coordination. By taking findings of this study into account, future studies can investigate how to support students' reasoning in transitioning toward productive meanings for Cartesian coordinate system.

According to results regarding students' meanings for spatial coordinate system, although most of the students (62%) were successful in laying orthogonal axes onto the figure in the problem situation for example in Q6a, committing to a unit to measure quantities and committing to directionality appeared to be an issue among high school students. Some students specifically indicated that they arranged the axes and the reference point such that resulting Cartesian coordinates do not take negative values to describe location of a point (standing for location of a person). I think this is a critical finding not only for mathematics but also for science education since for instance signed quantities are ubiquitous in physics and attain various significant meanings (Bing & Redish, 2007). Similarly, Brahmia et al. (2020) discussed that orientation along an axis and sense of positive-negative are not always explicit in coordinate system which impacts students' understandings in physics. As an example, aligning positive axis with direction of motion eliminates the necessity for signed quantities in discussion of velocity. Thus, students' committing to directionality by defining what positive and negative sign mean in a coordinate system is crucial. In this regard, results of this study pointing to students' lack of committing to a unit to measure quantities and committing to directionality suggest that future research studies can investigate meanings of Cartesian coordinate system and its related notions with students from STEM fields. This is important as enhancing students' uses of frames of reference in spatial and quantitative coordination may also enhance their understandings in science classes.

Researchers (Lee & Hardison, 2016; Paoletti et al., 2018) advocate constructing situational/spatial coordinate system prior to constructing quantitative coordinate system. Results from this study showing that in Q6a almost %40 and in Q8a 79% of students could not lay orthogonal axes onto the figure in the problem situation suggest that constructing spatial coordinate system prior to constructing quantitative coordinate system can be useful. Specifically, students can be provided with tasks to conceptualize axes as real number lines to represent magnitudes of quantities, form multiplicative object and construct graphs as emergent trace of points where and coordinates represent directed distances from the origin. Otherwise, "students might see coordinates not as [directed] lengths but as markers on a map." (Battista, 2007, p. 902). This way, directionality and meaning of positive and negative directions can also be elaborated. Coordinate systems in pre-calculus and calculus books can be extended from decontextualized to involving quantity referents as well including some situational coordination (Lee & Guajardo, 2021; Paoletti et al., 2016).

As also highlighted in the literature (Lee et al., 2020), in general curricular materials focus on rules to generate Cartesian plane. In parallel to this, results of the study showed that almost exclusively all students performed successfully in generating a Cartesian coordinate system and conventional point plotting task in four quadrants. On the contrary, in a problem situation like Q1 only 5% of the students constructed a quantitative coordinate system and 9% constructed spatial coordinate system but failed to construct a single point as multiplicative object. Also, in Q6a only 62% of the students constructed a spatial coordinate system and 40% of those who formed a spatial coordinate system was unable to form an ordered pair to describe location of a point. Therefore, students should be provided with realistic problem situations to construct both spatial and quantitative Cartesian coordinate systems and operationalize why we need a coordinate pair and what it represents in the problem situation. I believe that the most significant part is to equip students with necessary skills and conceptual meanings so that they can flexibly use different coordinate systems, because studies (e.g. Sayre & Wittman, 2007) reported that undergraduate students persisted using inappropriate Cartesian coordinate system in mechanics course and some students carried conventions of Cartesian coordinate system to polar coordinate system even though they are completely different (Montiel et al., 2008). Eventually, focusing on conceptual meanings of the frames of reference may help reducing figurative aspects in students' thinking and prevent applying properties of Cartesian coordinate system into other coordinate systems such as polar coordinate system.

6.2. Discussion Regarding Students' Meanings of a Point

Researchers argue that constructing a coordinate pair is nontrivial although it is taken for granted (Thompson et al., 2016) and plays a critical role in learning of various fundamental topics in mathematics (Whitmire, 2014). In support of this, results of this study also showed that conceiving a point as quantitative multiplicative object was nontrivial for even high school students who have high potential and capability in learning and doing mathematics. Although in general students performed well in conventional point plotting task, about half of the students demonstrated fragmented meanings for points in terms of a multiplicative object. For instance, in Q4 and Q5 students were aware of some of the correct representations of Cartesian coordinates in the coordinate system but at the same time they viewed x (55%) and y coordinate (48%) as points on the graph. Moreover, they had more difficulty in identifying various representations of x-coordinate compared to y coordinate. Data from Q7 further showed that not only forming a multiplicative object but also the form of multiplicative object matters in students' reasonings for points and graphs because only 42.8% in Q7a, 69.4% in Q7b and 54.1% of the students in Q7c were able to conceive points as quantitative multiplicative objects in the coordinate plane. Unfortunately, significant number of students (17% in Q7a, 10.5% in Q7b and 5.7% in Q7c) viewed points as spatial-quantitative multiplicative object or non-multiplicative object (13.1% in Q7a, 6.6% in Q7b and 6.1% in Q7c). Besides, in Q2, 29.7% of the students conceived output of a function unproductively as a location/point on the graph while a point on a graph entails values of both input and output of a function. Also, some students (13.5%) demonstrated hybrid value and location thinking and arc or graph-thinking (6.6%) in labeling outputs and difference of outputs of a function in the coordinate plane. All these results indicated that students may not have a coherent and consistent understanding of a point in terms of a multiplicative object. In what follows findings related to students' meanings of a point in terms of multiplicative object and outputs of a function will be further elaborated

Particularly, findings from Q7 provided some insights regarding students' interpretation of points in terms of multiplicative object in numerical and non-numerical

Cartesian coordinate systems. Overall students seemed to perform better when scale and numerical values were provided in coordinate system for interpreting points in terms of multiplicative object. Although small in amount, 12% of the students constructed spatial-quantitative or non-multiplicative object in quantitative problem context (Q7a) and then switched to quantitative multiplicative object in numerical problem context (Q7c). Yet, this does not conclude that students perform better in constructing quantitative multiplicative object when numerical values are given. These results might be interpreted in several ways: First, these students' reasoning might have been positively affected from Q7b where students mostly measured quantities to produce a point. Second, students might have wanted to give more accurate and precise answers due to given numerical values and scaled coordinate system, hence they measured quantities approximately and used axes to represent them. Third, the increase in the number of students who envisioned points as quantitative multiplicative object in Q7c compared to Q7a may stem from students' lack of in experience in working with non-numerical quantities because in general numerical values are marked on the axis when working with Cartesian coordinate system. Therefore, students might not be aware that x and y-coordinates can also be represented as distance from the origin to the end point of the segment along the axes (Frank, 2016). Often, marking coordinates as a point or an integer value on the axes is more standard practice than perceiving them as directed line segments represented on the axes. Besides, when no numerical values are provided, students' attention could be more oriented toward spatial features of the problem situation. This highlights once more the need for integrating quantitative problem situations into learning and applications of Cartesian coordinates. Further research studies can investigate how different problem contexts influence students' meanings for a point and what helps or prevents students from forming a quantitative multiplicative object.

Also, students who constructed non-multiplicative or spatial quantitative multiplicative objects sometimes struggled to interpret some of the given points in the coordinate plane. I reckon that their interpretation of graphs might have been negatively affected by perceptual features and potentially be figurative in nature as some studies (Frank, 2017; Lee et al., 2019; Moore et al., 2019) highlight difficulties stemming from non-normative graphing schemes using coordinate system.

Moreover, findings from Q2 aligns with the previous research (David et al., 2019; Moore & Thompson, 2015; Parr, 2021; Sencindiver, 2020) that students might hold different understandings of output of a function. Particularly, in this study students demonstrated value thinking (20.5%), location thinking (29.7%), hybrid value and location thinking (13.5%), and arc or graph thinking (6.6%). While these studies were conducted at undergraduate level, this study extends the literature and points that some of the non-normative understandings regarding output of a function and hence representation of a point on a graph might emerge at high school and carried through undergraduate years. Findings of this study additionally provided evidence for new conceptions of outputs of function which I called *hybrid value and location* and *graph-thinking.* Regarding the latter, students conveyed outputs of functions and difference of outputs by sketching transformations of the original function, f, in the coordinate plane (see Figure 5.18). These results underline the need for working in quantitative context and reasoning through quantities for high school students to build a more comprehensive understanding of coordinate system, graphs, and related notions.

By the same token, students' difficulties pertaining to functions seemed to affect their meanings of a point on a graph negatively. The non-numerical context of the question (i.e. giving quantities such as a and h as inputs of function) might have also led students to interpret inputs of a function as variables rather than quantities. Likewise, students' reasoning of a point in terms of multiplicative object might also possibly left a negative impact on students' understanding of functions and function notation. These results suggest that future research studies can investigate the relationship between meanings of a point and output of a function, operations on functions, graphs of functions and so on as these concepts are fundamental in high school mathematics.

Besides, Sencindiver (2020) found that students conveyed stable meanings for output and difference of output of a function whether normative or nonnormative and hypothesized that stable meanings come from stable nature of students' reasoning. However, findings of this study challenged this hypothesis because some students conveyed *hybrid value and location thinking*: location-thinking for outputs and value-thinking for difference of outputs, and vice versa. I anticipate that some of these inconsistencies in students' reasoning may be related to their habits and prior experiences from mathematics classes as well as their lack of understanding about function notation or graphs of functions. One concrete step to prevent compartmentalized meanings for representing output of a function on a graph could be labeling a point not only as (x, y) but also $(x, f(x))$ as suggested by Skordoulis et al. (2009) since a lot of students associated points on a graph only with $f(x)$ disregarding that a point entails information about two quantities: x and $f(x)$. As also suggested by other researchers (e.g. Chapin, O'Connor & Anderson, 2013) a problem-based approach can be used to introduce the Cartesian coordinate system that unfolds meanings of frames of references and provides students opportunities to reason about magnitudes on axes in constructing and interpreting points, graphs, etc. Moreover, results of this study suggest that educators should find ways to integrate dynamic learning tools in conceptualizing magnitudes on axes and their dynamic change in forming multiplicative object, producing and interpreting graphs of functions for both numerical and non-numerical contexts.

6.3. Discussion Regarding Students' Meanings for Graphs within Spatial and Quantitative Coordinate System

Analysis of the results regarding students' graphing within spatial and quantitative coordinate systems set forth that graphing might seem more straightforward than it is (Gravemeijer, 2020) for high school students as in the case for conceptualizing points. In general, results illustrated the need to improve students' understanding of frame of reference in establishing spatial coordinate system and to enhance their emergent shape thinking for graphs within both spatial and quantitative coordinate systems. Particularly, regarding spatial coordination in Q6b only 8%, regarding quantitative coordination in Q8b 11% and in Q9 40% of the students reasoned emergently about graphs. Considering the capability and the level of the students, I expected these percentages to be higher. Therefore, it is valuable to discuss the findings of the study as they can shed light on students' challenges and strengths in reasoning about graphs within both spatial and quantitative coordinate systems.

Especially in spatial coordinate system, most of the students interpreted the line as static shape thinking (62% in Q6b) by utilizing coordinate system either for measuring distance or identifying coordinates of the end points of the route. This supports the argument (Diezmann & Lowrie, 2006) that using an axis or real number line solely as a tool for counting model in teaching could be limiting for students. As reported in previous studies (Kerslake, 1981;

Leinhardt et al., 1990), some of the participants in this study perceived the line as a path for a point rather than consisting of infinitely many points in spatial coordinate system. In this respect, when learning and teaching graphs of functions in class, evaluating functions only for a couple of values most of which are integers and labeling corresponding points on the graph might prevent students from viewing axis as a continuous real number line to represent measures of quantities and the graph as consisting of infinitely many points. Findings of this study point that when working with different kinds of functions and graphs, magnitudes and their continuous change should be presented on the real number line so that students do not associate axes only with counting or procedural tasks. In order to facilitate students to conceptualize graphs as representation of continuous covariation between variables which consist of infinitely many points, dynamic geometry environments or technological tools can be integrated to simulate continuous change of quantities on axes together with emergent trace of multiplicative object in the coordinate plane. In addition to using spatial coordinate system to describe locations of points mathematically, graphing relationships or interpreting graphs within spatial coordinate system might provoke students to think quantitatively and focus their thinking on meanings of points, graphs and coordinate system.

Analysis of the results also pointed to some challenges high school students face when graphing in quantitative Cartesian coordinate system. Regarding Q8b, firstly some students failed to establish quantitative coordinate system (Herscovics, 1989) and some could not even produce a point representing the initial distance of the red ball from the yellow and blue balls. Second, in line with previous research while some students produced graphs only consisting of discrete points (Herscovics, 1989), 14% of the students connected landmark points linearly without understanding the relationship between them (Yavuz, 2010). Some of these difficulties may stem from students' lack of covariational reasoning skills as those students could not envision the continuous covariation of quantities but only reasoned through certain landmark points. Also, 15% of all students graphed the distance of red ball from the blue ball and its distance from the yellow ball by sketching two separate graphs using time as an inherent variable. These students challenged to sustain changes of both variables simultaneously (Stalvey & Vidakovic, 2015) in mind. Thus, in mathematics classes students should also graph relationships between variables different than time and without relying on a rule of a function. Similar to Moore et al. (2019), I conjecture that some of the students might also have carried figurative-dominated graphing meanings such as starting the graph at the origin and from right the left in Q8b and Q9. Therefore, students' meanings for graphing might potentially influence how they coordinate quantities within the coordinate system and produce and interpret point in terms of multiplicative object and vice versa.

On the other hand, 19% of the students reasoned statically about graphs in Q8b. Within this group, students mostly focused on the overall changes in two quantities such as "both decreased and then increased" by forming a non-multiplicative link between changes in the quantities and without attending to values of each quantity. As a result, they retrieved graphs of functions from prior math classes such as parabola, sinusoidal or absolute value to graph the relationship between quantities. I think this is a significant finding about students' meanings for graphs because students might associate graphs as properties of certain types of functions and hence their reasoning about graphs might be dominated by rules of functions. Students who attempted to come up with an equation of a function in Q1 and Q8b also reveals that students can think about graphs and functions as inseparable, therefore struggle to reason productively about graphs. From this perspective, this finding also aligns with Knuth (2000) as he stated that due to overreliance on algebraic methods in curricula and teaching methods, students prefer algebraic solutions over graphical ones even in problems that encourage graphical solution as it is more convenient. To avoid such unproductive way of thinking about graphs, results of this study suggest that in high school mathematics curriculum, graphical thinking can precede functional thinking. In other words, students should learn about graphing relationships between variables through tasks that require to identify and observe quantities and focus students' thinking on relationship between quantities and their covariation. After students form a solid understanding about Cartesian coordinate system, how it is used to form a multiplicative object and graph relationships between quantities, students can more productively reason about types of functions and their graphs.

6.4. Limitations and Implications for Further Research and Teaching

This section provides limitations of the study and their implications for further research on coordinate systems and related notions. This study explored high school students' meanings for Cartesian coordinate system, producing and interpreting a point, outputs of a function on a graph and graphing in spatial and quantitative coordinate systems. To this end, an inventory consisting of open-ended questions was developed corresponding to the existing literature. Further research studies can involve additional notions of reasoning about Cartesian coordinate system such as including non-canonical coordinate systems and improve the inventory accordingly. It can also be adapted to reasoning about other coordinate systems such as polar coordinate system as building a grounded understanding about different coordinate systems as well as similarities and differences between them would improve the overall understanding on coordinate systems.

This study is subject to some limitations. First, participants of the study were selected by convenient sampling from a high school located in Istanbul. Therefore, results of the study may not be representative of all high school students in Turkey. On the other hand, since large number of participants from different grade levels participated, various levels of students' reasoning were included. Participants consist of high achieving students with high potential which also created an opportunity to unfold possible productive student reasonings. Plus, identifying the struggles high achieving students face could better inform teaching practices and elucidate students' mathematical learning processes

Second, duration of the data collection and novelty of the problem situations in the inventory might be limiting to this study since some students stated that they needed more time to work on the questions or could not completely answer all questions. Although not being able to complete the inventory within data collection duration also provides insight regarding students' challenges in reasoning about coordinate systems, further studies could consider this limitation in their research design to collect more data.

Another limitation of the study originates from not having full access to student reasoning despite the open-ended questions in the inventory. Lack of detailed explanations on some of the answers was one of the major challenges in data analysis. To enhance the data analysis process, my advisor and I came together on a regular basis to review data analysis process. Besides, common student responses were elaborated detailly in the results section and opened for discussion to make the best sense out of the data.

Lastly, due to size of the sample and format of the study, results were analyzed question by question and unfortunately individual participants' overall performance is not elaborated. Further studies can investigate students' performance in different goals and look for a relationship between them by conducting interviews.

Findings of this study can inform teachers, teacher educators, and curriculum developers regarding teaching and learning Cartesian coordinate system in several aspects as also explained in the previous discussion section. First of all, more balanced use of spatial and coordinate systems should be attained in the curriculum according to findings of the study. Cartesian coordinate systems should be constructed through problem solving rather than introducing the rules of generating a coordinate system and plotting points. At this point, focusing students' attention on meanings of frames of reference, representing Cartesian coordinates as distance from origin on the axes and forming a point by coupling orthogonal distances from the origin are fundamental steps in guiding students to build a meaningful understanding of Cartesian coordinate system. This way students understanding and use of frames of reference, meanings of positive and negative and committing to units in coordinate systems can potentially be enhanced. In addition, problem situations that require use of coordinate systems should be extended from decontextualized to contextualized problem situations where students can also work with non-numerical quantities and experience real life problems as students seemed to challenge in non-numerical contexts. Introducing graphical thinking before introducing types of functions and their graphs might also improve students' covariational reasoning, functional and graphical thinking. I believe integrating technological tools and dynamic learning environments are also integral part of such fruitful learning environments. In this respect, excluding time from the variables in graphing can also improve students' thinking about point as a multiplicative object and about graphs emergently in coordinate system. Because when time is one of the variables, it is easier to focus on changes in the other quantity. On the contrary, when time is excluded, envisioning magnitudes of both quantities and tracing their simultaneous covariation requires more effort. Moreover, teachers should be more aware of their habits in the classroom. For instance, labeling points as $(2, f(2))$ instead of $f(2)$ on the graph or labeling $f(2)$ in the vertical axis can emphasize the two components of a point on a graph and contribute to students' understandings of function notation.

Furthermore, the study shows directions for further research studies. First of all, in order to obtain more representative results, further studies can be conducted in a larger scale including different high schools. In addition, they can be extended to middle school, undergraduate level, preservice and in-service teachers as well as STEM fields to investigate their meanings for Cartesian coordinate system and to question any potential relationship between these levels. Since context of the problems differ among various sciences such as physics, chemistry, as well as meanings of positive and negative sign in coordinate system, investigating meanings of Cartesian coordinate system with participants form STEM fields is also significant.

Second, further studies can approach the same research question with interviews to obtain in-depth data for students' meanings for Cartesian coordinate system. Alternatively, students' construction of Cartesian coordinate system, meanings of a point in terms of multiplicative object, their reasoning about point on a graph and output of a function and graphing in spatial and quantitative coordinate systems can be investigated through separate research studies to allocate more time for students' thinking.

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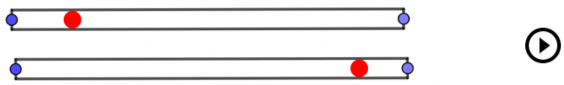
APPENDIX A: INVENTORY

1. Two long, thin rectangles represent two ant farms, each containing a point (ant) moving haphazardly (randomly). The points' movement could be activated and paused by play button, and the rectangles could be moved or rotated using the blue dots. Explore the ant farms via the following link (hit play button to start the animation) and answer the following question. https://tinyurl.com/antfarm1

Note: in the animation

Left blue dot moves the rectangle

Right blue dot rotates the rectangle



Play button

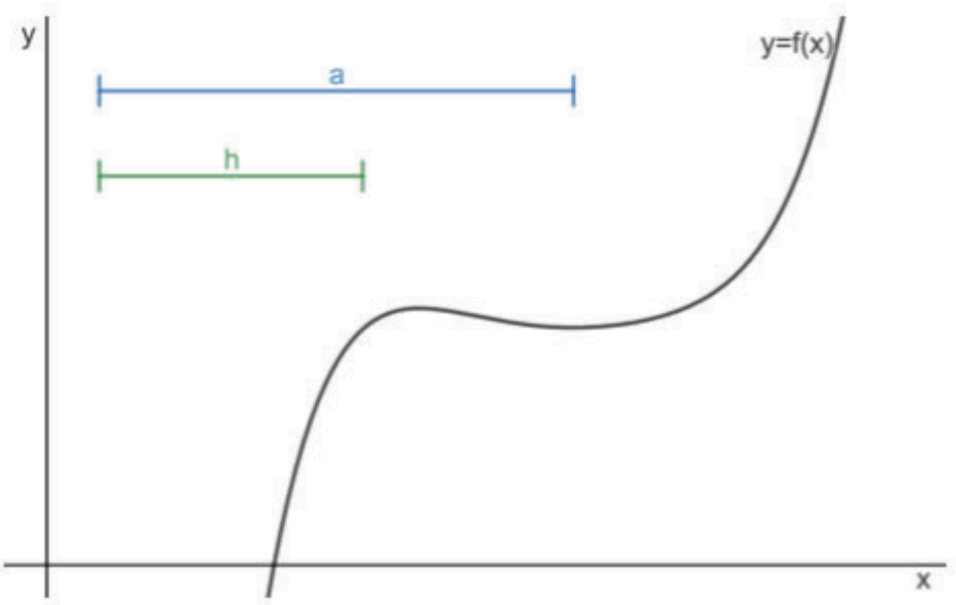
Can you describe mathematically the locations of the two ants with a single point that moves along with the ants? Show all your work and explain your reasoning clearly below.

2. Consider the graph of the function $y = f(x)$. The lengths of colored line segments represent quantities a and h. Both the x- and y-axes have the same scale length. Represent each of the following quantities on the diagram given below. Label clearly.

i) $f(a)$

ii) $f(a + h)$

iii) $f(a + h) - f(a)$



3. a) Draw a Cartesian coordinate plane and plot points A$(-2, 1)$, B$\left(0, -\frac{7}{3}\right)$ and C$(-4, -\sqrt{2})$.

b) What does a point represent in the Cartesian coordinate plane?

c) What do coordinates of a point (x −coordinate, y −coordinate) represent in the Cartesian coordinate plane?

4. Consider the function $y = -3x + 2$. What does the y value when $x = 2$ represent? Select all that apply. For each selection you made, please explain your reasoning.

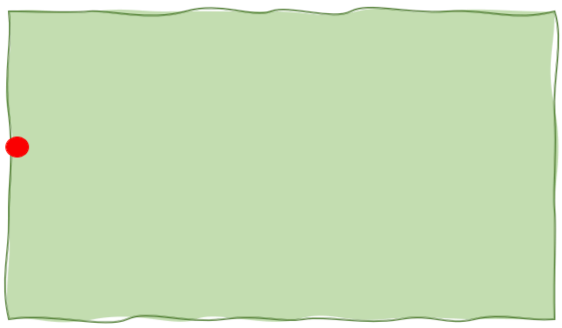
a) The function gets multiplied by 2.
b) The function evaluated at 2.
c) The y-value on the graph of the function with x-coordinate 2.
d) The x-value on the graph of the function with y-coordinate 2.
e) The height of the graph of the function at $x = 2$.
f) The distance between the graph of the function at $x = 2$ and the x-axis.
g) The distance between the graph of the function at $x = 2$ and the y-axis.
h) The slope of the graph of the function at $x = 2$.
i) The point on the graph at $x = 2$.
j) The point on the graph when $y = 2$.
k) $-3(2) + 2$.
l) The value of y (at $x = 2$) on the y-axis.
m) The value of 2 on the x-axis.
n) The distance between the origin and the value of y (at $x = 2$) on the y-axis.
o) The distance between the origin and the value of 2 on the x-axis.

5. Consider the function $y = -3x + 2$. What does the solution(s) to $y = 2$ represent? Select all that apply. For each selection you made, please explain your reasoning.

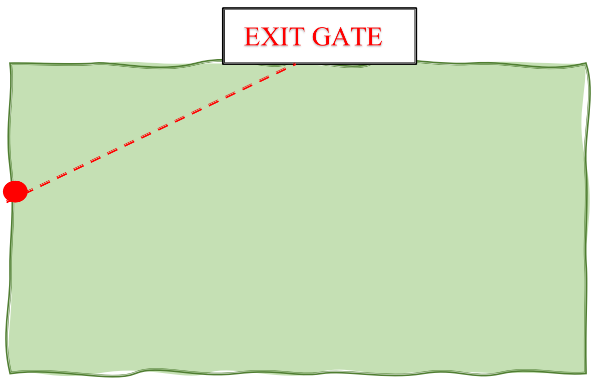
a) The function gets multiplied by 2.
b) The function evaluated at 2.
c) The y-value on the graph of function with x-coordinate 2.
d) The x-value on the graph of the function with y-coordinate 2.
e) The height of the graph of the function at $x = 2$.
f) The distance between the graph of the function at $x = 2$ and the x-axis.
g) The distance between the graph of the function at $x = 2$ and the y-axis.
h) The slope of the graph of the function at $x = 2$.
i) The point on the graph at $x = 2$.
j) The point on the graph when $y = 2$
k) $-3(2) + 2$
l) The value of x (at y=2) on the x-axis
m) The value of 2 on the y-axis
n) The distance between the origin and the value of x (at y=2) on the x-axis
o) The distance between the origin and the value of 2 on the y-axis

6.

a) The red point represents a person who got lost in a green rectangular garden as shown below. Mathematically describe the location of the person (the red point) for someone else so that they exactly know where the person (the red point) on the rectangular garden is. Show all your work and explain your reasoning clearly.

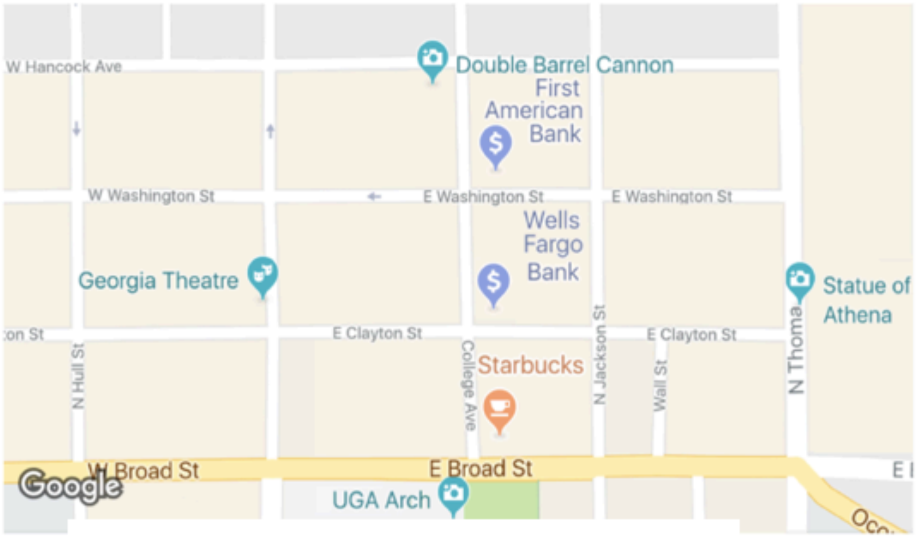


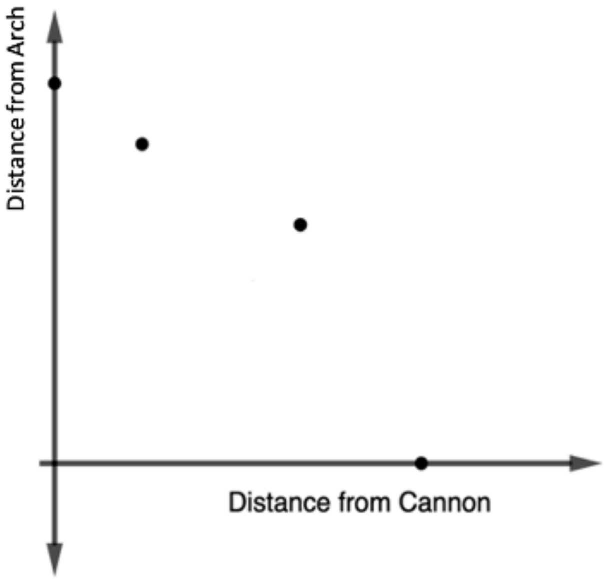
b) Mathematically describe the location of the person (the red point) throughout its journey toward the exit gate. <u>Explain</u> your reasoning and label your work clearly.



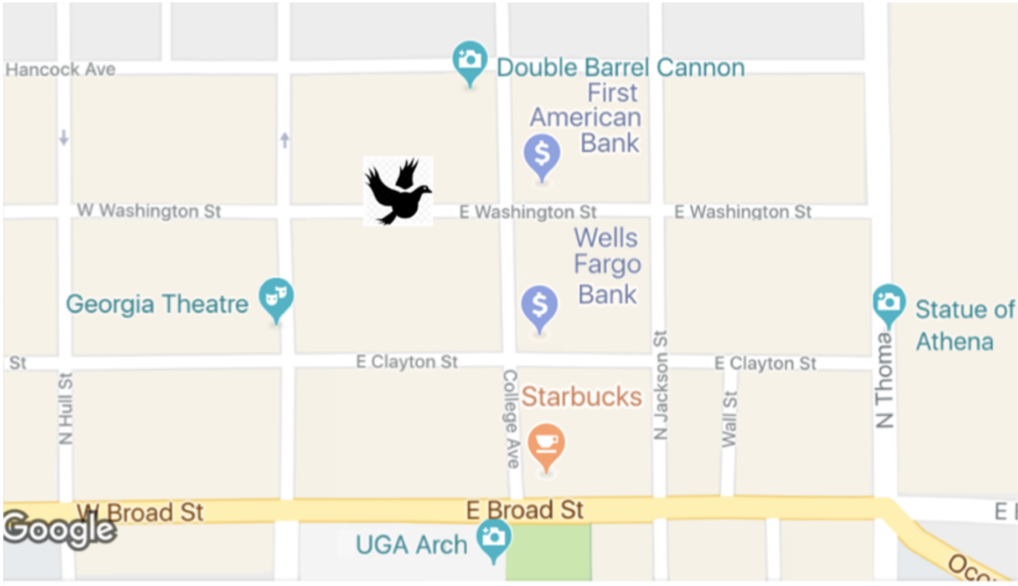
7. The map of Downtown Athens is given below with seven locations pinned: UGA Arch (A), Double-Barreled Cannon (C), First American Bank (FAB), Georgia Theater (GT), Wells Fargo Bank (WFB), Statue of Athena (SoA), and Starbucks (S).

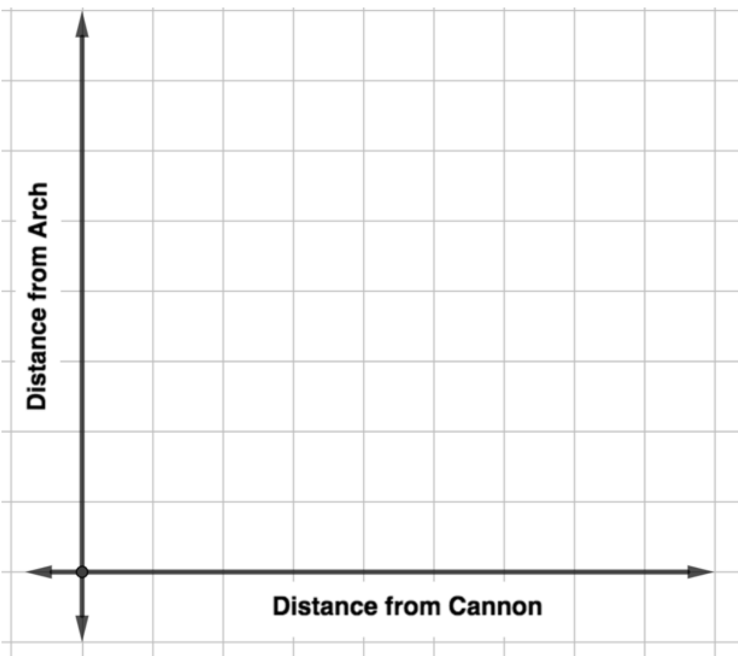
   a) Write what each of these four points on the coordinate plane below might represent. Explain how you decided and show your reasoning.



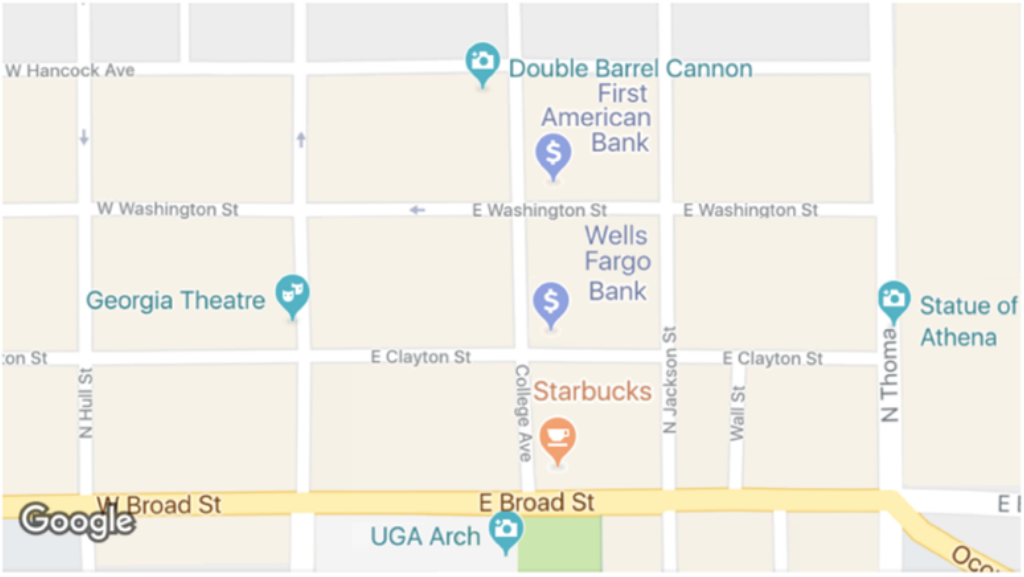


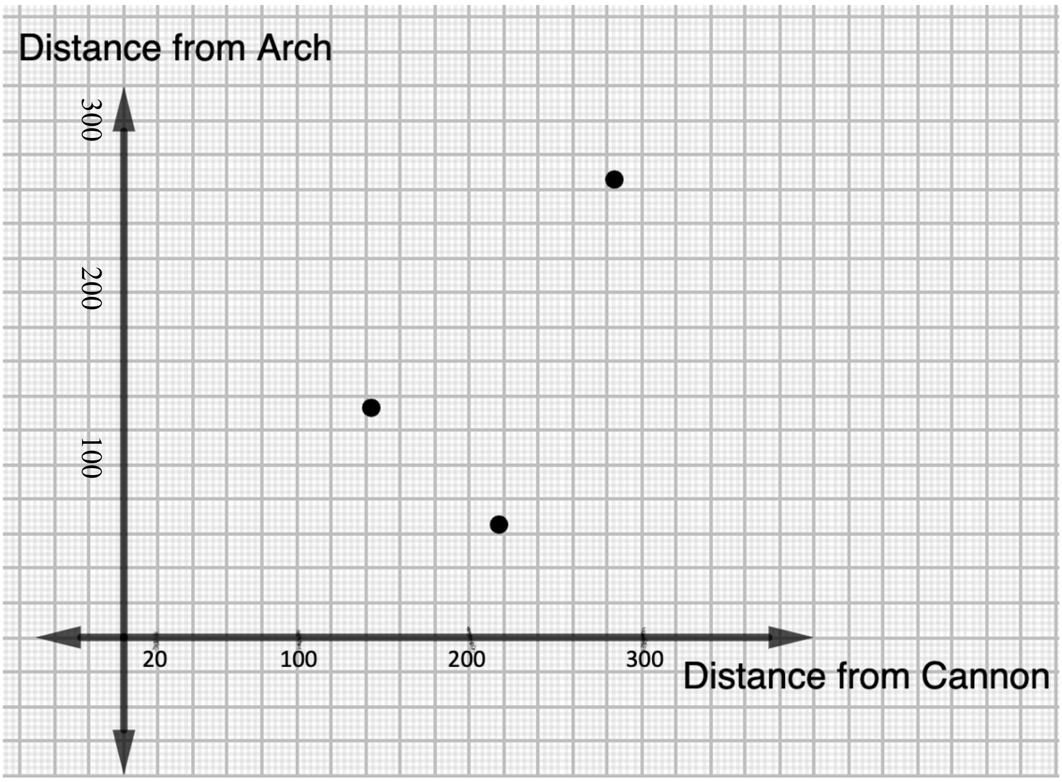
b) Plot a point that represents the crow's distance from Arch and distance from Cannon when the crow is in a place on the map as shown below. Explain how you decided using the map/diagram/in words.





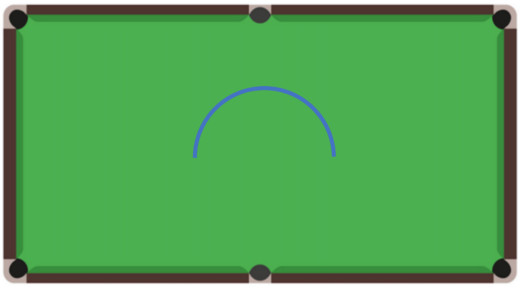
c) Given that the distance between Arch and Cannon is 260 meters, write what each of the three points on the coordinate plane might represent. Explain how you decided and show your reasoning.





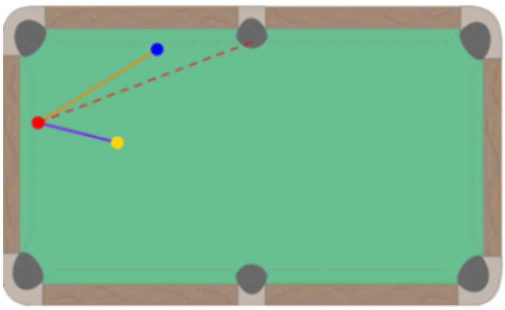
8.

a) Consider the blue logo of a Pool Hall shown on the pool table below. Mathematically describe the shape of this logo. Show your work and reasoning clearly.

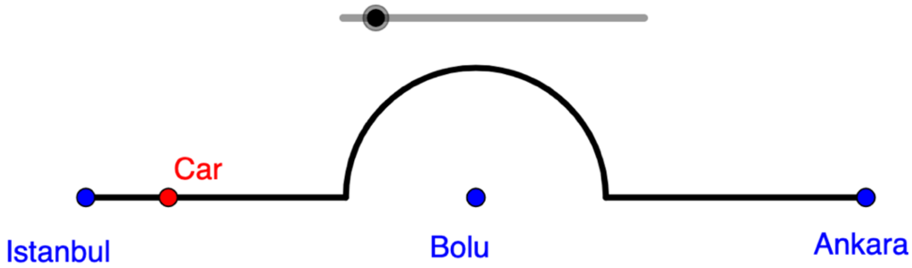


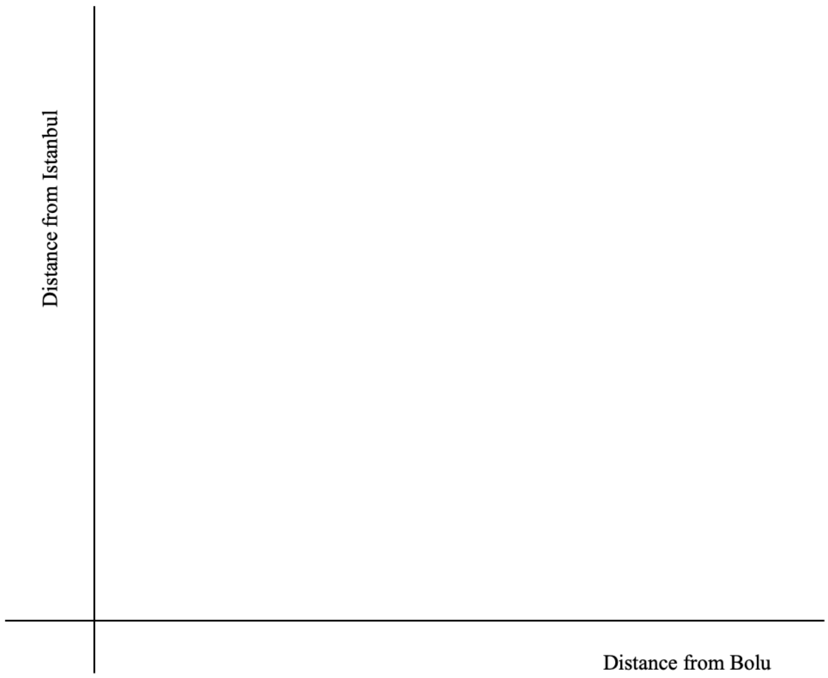
b) As the red ball is moving toward the middle pocket, the blue and yellow balls stay steady. Use the following link to animate the movement of the ball with the play button or use the slider to move the ball. https://tinyurl.com/pool-table

Create a graph that represents the relationship between the red ball's distance from the yellow ball and its distance from the blue ball as it moves to the pocket. Explain your reasoning for the graphing process.



9. Using the following link, explore an animation that shows that a car is traveling back and forth between Istanbul and Ankara. Graph the relationship between the car's distance from Istanbul and its distance from Bolu during the trip. Explain your reasoning for the graphing process. https://tinyurl.com/travelbycar





APPENDIX B: ETHICAL APPROVAL FORM

Evrak Tarih ve Sayısı: 14.12.2021-42469

T.C.
BOĞAZİÇİ ÜNİVERSİTESİ REKTÖRLÜĞÜ
Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik Kurulu
(FMİNAREK)

Sayı : E-84391427-050.01.04-42469 14.12.2021
Konu : 2021/23 Kayıt no'lu başvurunuz hakkında

Sayın Doç. Dr. Gülseren KARAGÖZ AKAR
Matematik ve Fen Bilimleri Eğitimi Bölüm Başkanlığı - Öğretim Üyesi

"Lise Öğrencilerinin Koordinat Sistemlerini Anlamlandırma Yöntemleri" başlıklı projeniz ile Boğaziçi Üniversitesi Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik Kurulu (FMİNAREK)'e yaptığınız 2021/23 kayıt numaralı başvuru 06.12.2021 tarihli ve 2021/10 No.lu kurul toplantısında incelenerek etik onay verilmesi uygun bulunmuştur. Bu karar tüm üyelerin toplantıya on-line olarak katılımıyla ve oybirliği ile alınmıştır.

COVID-19 önlemleri nedeniyle üyelerden ıslak imza alınamadığından bu onam mektubu tüm üyeler adına Komisyon Başkanı tarafından e-imzalanmıştır.

Saygılarımızla bilginize sunarız.

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