PROJECTIVE CHARACTERS OF FINITE GROUPS

by

Said Yılmaz

B.S., Mathematics, Boğaziçi University, 2018

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University

2022

ACKNOWLEDGEMENTS

First and foremost, I would like to express my gratitude to my thesis advisor, Prof. Olcay Coşkun of the Department of Mathematics at Boğaziçi University, who guided me throughout this thesis with an endless patience.

I would like to thank Prof. Kazım İlhan İkeda and Assoc. Prof. Ergün Yaraneri for participating in my thesis committee.

This thesis is partially supported by TÜBİTAK through 1001-119F422.

I am grateful for my family for their emotional and educational support for my whole life.

I am deeply thankful to my friends Şeyma, Ceren and Ebru for their devoted help with the office work and the joyfull environment they create in the office.

Finally, my special thanks to Yusuf for providing me with never ending support and continuous encouragement through the process of studying, researching and writing this thesis.

ABSTRACT

PROJECTIVE CHARACTERS OF FINITE GROUPS

The purpose of this thesis to analyze the basic facts about projective characters of finite groups and compare them to the facts about ordinary characters of finite groups. We start with review of basic facts about the twisted group algebras and projective representations of finite groups over a field. Then we study the properties of the projective characters. Finally, we will study these properties on the complex projective characters.

ÖZET

SONLU GRUPLARIN PROJEKTİF KARAKTERLERİ

Bu tezin amacı sonlu gruplar üzerine tanımlanan projektif karakterlerin temel özelliklerini analiz etmek ve bu özellikleri klasik karakter teorisindeki özelliklerle karşılaştırmaktır. İlk olarak bükümlü grup cebirlerinin ve projektif temsillerin tanımları ve özellikleri hatırlatılacak. Daha sonra projektif karakterlerin tanımları ve özellikleri verilecektir.

TABLE OF CONTENTS

LIST OF SYMBOLS

$B^2(G, A)$	the set of 2-coboundaries of G with coefficients in A
G	finite group
G'	the derived group of G
$H^2(G, A)$	the second cohomology group of G with coefficients in A
k	field
k^*	the set of elements of invertible elements of k
kG	the group algebra of G with coefficients in k
$k_\alpha G$	the twisted group algebra of G
$Z^2(G, A)$	the set of 2-cocycles of G with coefficients in A

1. TWISTED GROUP ALGEBRAS

Throughout this chapter we'll assume that G is an arbitrary group and A is an abelian multiplicative group where G acts on A from left, written as ${}^{a}g$, where $a \in A$ and $g \in G$. All the results are from Markus Linckelmann's book [1].

Definition 1.1. *A map $\alpha : G \times G \to A$ is called a 2-cocycle of G with coefficients in A if*

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)^{x}\alpha(y, z) \tag{1.1}$$

for every x, y and z in G.

Notice that if G acts trivially on A, the equation (1.1) becomes

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z) \tag{1.2}$$

for every x, y and z in G.

The set of all 2-cocycles of G with coefficients in A forms an abelian group, denoted by $Z^2(G, A)$, and the identity element is the constant map that is sending every $(x, y) \in G$ to the identity element 1_A of A.

Proposition 1.2. *Let G be a group and A be an abelian group on which G acts. Let $\alpha \in Z^2(G, A)$. Then we have $\alpha(1, x) = \alpha(1, 1)$ and $\alpha(x, 1) = {}^{x}\alpha(1, 1)$.*

Proof. Put $x = 1, y = 1, z = x$ in the equation (1.1) to get $\alpha(1, x)\alpha(1, 1) = \alpha(1, x)\alpha(1, x)$, so we get $\alpha(1, x) = \alpha(1, 1)$. Similarly, put $x = x, y = 1, z = 1$ in the equation (1.1) to get $\alpha(x, 1)\alpha(x, 1) = \alpha(x, 1)^{x}\alpha(1, 1)$, and hence $\alpha(x, 1) =^{x} \alpha(1, 1)$.

Definition 1.3. *Let G be a group and k be a commutative ring with unity on which G acts. Let $\alpha \in Z^2(G, k^*)$. The twisted group algebra of G by α is denoted by $k_\alpha G$ and the* *product is defined by $x.y = \alpha(x, y)xy$ for all $x, y \in G$, where $x.y$ is the multiplication in $k_\alpha G$ and xy is the usual group multiplication.*

Note that in the twisted group algebra $k_\alpha G$, the unit element need not be equal to the unit element 1 of the group G. For any $g \in G$, we have $1.g = \alpha(1, g)g$ and even though $\alpha(1, g)$ yields an identity element in $k_\alpha G$, it need not be equal to the unit element of k^*.

Proposition 1.4. *Let G be a group and k be a commutative ring with unity on which G acts trivially. Let $\alpha \in Z^2(G, k^*)$. For any $g \in G$ we have*

(i) $\alpha(1, g) = \alpha(g, 1) = \alpha(1, 1)$.
(ii) $\alpha(g, g^{-1}) = \alpha(g^{-1}, g)$.
(iii) *the unit element of $k_\alpha G$ is equal to $\alpha(1, 1)^{-1}1_G$.*
(iv) *the inverse of g in $k_\alpha G$ is equal to $\alpha(1, 1)^{-1}\alpha(g, g^{-1})^{-1}g^{-1}$.*

Proof. The statement (i) follow from Proposition 1.2 and the fact that G acts on k^* trivially. If we put $x = g$, $y = g^{-1}$ and $z = g$ in Equation (1.2) we get $\alpha(1, g)\alpha(g, g^{-1}) = \alpha(g, 1)\alpha(g^{-1}, g)$. Using (i) we get the desired equality in (ii). Now consider $\alpha(1, 1)^{-1}1.g = \alpha(1, 1)^{-1}\alpha(1, g)g = g$ for all $g \in G$. Hence $\alpha(1, 1)^{-1}1$ is the identity element in $k_\alpha G$. For the last part, consider $\alpha(1, 1)^{-1}\alpha(g, g^{-1})^{-1}g.g^{-1} = \alpha(1, 1)^{-1}\alpha(g, g^{-1})^{-1}\alpha(g, g^{-1})gg^{-1} = \alpha(1, 1)^{-1}1$ which is the identity element by (iii).

Proposition 1.5. *Let G be a group and $\alpha, \beta \in Z^2(G, k^*)$. Then $k_\alpha G$ is isomorphic to $k_\beta G$ as k-algebras where g is mapped to $\gamma(g)g$ for some $\gamma(g) \in k^*$ if and only if there exists a map $\gamma : G \to k^*$ such that $\alpha(g, h) = \beta(g, h)\gamma(g)\gamma(h)\gamma(gh)^{-1}$ for all $g, h \in G$.*

Proof. The image of $g._\alpha h = \alpha(g, h)gh$ under the given isomorphism is $\alpha(g, h)\gamma(gh)gh$. The product of the images of g and h under the isomorphism is $\gamma(g)\gamma(h)g._\beta h = \gamma(g)\gamma(h)\beta(gh)gh$. These two elements are equal if and only if the equality $\alpha(g, h) =$

$\beta(g,h)\gamma(g)\gamma(h)\gamma(gh)^{-1}$ holds for all $g, h \in G$.

Definition 1.6. *Consider the set of all maps $\alpha \in Z^2(G, A)$ such that there exits a map $\gamma : G \to A$ such that $\alpha(g,h) = \gamma(g)^g\gamma(h)\gamma(gh)^{-1}$ for all $g, h \in G$. This set is called the set of 2-coboundaries of G with coefficients in A and denoted by $B^2(G, A)$. The set $B^2(G, A)$ is a subgroup of $Z^2(G, A)$. The quotient group*

$$H^2(G, A) = Z^2(G, A)/B^2(G, A)$$

is called the second cohomology group of G with coefficients in A.

Definition 1.7. *Let $\alpha, \beta \in Z^2(G, k^*)$. If there exists a map $t : G \to k^*$ such that $t(1) = 1$ and $\alpha(g,h) = \beta(g,h)t(g)t(h)t(gh)^{-1}$ for all $g, h \in G$, then α and β are called cohomologous.*

Corollary 1.8. *Let G be a finite group and $\alpha \in Z^2(G; k^*)$. Then the followings are equivalent:*

(i) There is an algebra isomorphism between $k_\alpha G$ and $k_\beta G$.
(ii) The classes of α and β is are equal in $H^2(G, k^*)$
(iii) α and β are called cohomologous.

Proof. It follows from Proposition 1.5.

Note that if α is in the trivial class of $H^2(G, k^*)$, then there is an algebra isomorphism between $k_\alpha G$ and kG that sends $g \in G$ to a nonzero scalar multiple of g. This simple fact will be used later.

Proposition 1.9. *Let G be a finite group and $\alpha \in Z^2(G; k^*)$. The class of α in $H^2(G; k^*)$ is trivial if and only if $k_\alpha G$ has a module that is isomorphic to k as a k-module.*

Proof. Assume that $k_\alpha G$ has a module that is isomorphic to k as a k-module. Then we can define an algebra homomorhism $\gamma : k_\alpha G \to k$. Thus we have $\gamma(g.h) = \gamma(g)\gamma(h) = \gamma(\alpha(g,h)gh) = \alpha(g,h)\gamma(gh)$ for all $g, h \in G$. Therefore $\alpha(g,h) = \gamma(g)\gamma(h)\gamma(gh)^{-1}$ is a 2-coboundary, implying that α is in the trivial class of $H^2(G, k^*)$. Now assume that α is in the trivial class of $H^2(G, k^*)$, then $k_\alpha G$ is isomorphic to kG. Considering the trivial kG-module, it is isomorphic to k as a k-module.

Proposition 1.10. *Let G be a finite group.*

(i) Suppose that k is an algebraically closed field; consider $k^\times$ with the trivial action of G. Let Z be the group of $|G|$-th roots of unity in $k^\times$. The inclusion $Z \to k^\times$ induces a surjective group homomorphism $H^2(G; Z) \to H^2(G; k^\times)$; in particular, the abelian group $H^2(G; k^\times)$ is finite.

(ii) Suppose that k is a perfect field of prime characteristic p. Let P be a finite p-group. Then $H^2(P; k^\times)$ is trivial.

Proof. The proof is omitted. It is Proposition 1.2.9 in [1].

Proposition 1.11. *Let G be a finite cyclic group and k an algebraically closed field. Consider $k^\times$ with the trivial action of G. The group $H^2(G; k^\times)$ is trivial.*

Proof. The proof is omitted. It is Proposition 1.2.10 in [1].

Theorem 1.12. *Let $\alpha \in Z^2(G, F^*)$ and let E be a field extension of F. Then the map*

$$\begin{cases} E \otimes_F F_\alpha G \to E_\alpha G \\ \quad\lambda \otimes \bar{g} \mapsto \lambda \bar{g} \end{cases}$$

is an isomorphism of E-algebras.

Proof. Let $\{\bar{g} \mid g \in G\}$ be an F-basis of $F_\alpha G$ with $\bar{x}\bar{y} = \alpha(x,y)\overline{xy}$ for all $x, y \in G$. For each $g \in G$, put $\tilde{g} = 1 \otimes \bar{g}$. Then $\{\tilde{g} \mid g \in G\}$ is an E-basis of $E \otimes_F F_\alpha G$ and for all $x, y \in G$,

$$\begin{aligned} \tilde{x}\tilde{y} &= (1 \otimes \bar{x})(1 \otimes \bar{y}) = 1 \otimes \alpha(x,y)\overline{xy} \\ &= \alpha(x,y)(1 \otimes \overline{xy}) = \alpha(x,y)\widetilde{xy} \end{aligned}$$

as required.

2. PROJECTIVE REPRESENTATIONS

Definition 2.1. *Let G be a finite group and let V be a finite dimensional vector field over a field k. A map $\rho : G \to GL(V)$ is called a projective representation (or an α-representation) of G over k if there exists a map $\alpha : G \times G \to k^*$ such that*

(i) $\rho(g)\rho(h) = \alpha(g,h)\rho(gh)$
(ii) $\rho(1) = 1_V$*, where* 1_V *is the identity transformation*

for all $g, h \in G$.

Observe that $\rho((xy)\,z) = \alpha((xy)\,,z)\rho(xy)\rho(z) = \alpha((xy)\,,z)\alpha(x,y)\rho(x)\rho(y)\rho(z)$. Also $\rho(x\,(yz)) = \alpha(x,(yz))\rho(x)\rho(yz) = \alpha(x,(yz))\alpha(y,z)\rho(x)\rho(y)\rho(z)$. Therefore, in order to have associativity of the action of G to hold, one can observe that α must be an element of $Z^2(G,k^*)$ with the trivial action of G on k^*.

Definition 2.2. *Two projective representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are called projectively equivalent if there exists a map $\mu : G \to k^*$ with $\mu(1) = 1$ and there exists a vector space homomorphism $f : V_1 \to V_2$ so that $\rho_2(g) = \mu(g)f\rho_1(g)f^{-1}$ for all $g \in G$. If $\mu(g) = 1$ for all $g \in G$, they are called linearly equivalent.*

Here is an important relation between α-representations and $k_\alpha G$-modules. Let ρ be an α-representation. Then we can define a homomorphism $f : k_\alpha G \to End_k(V)$ such that $\bar{g} \mapsto \rho(g)$ for all $g \in G$. By extending it linearly, V becomes $k_\alpha G$-module by $\left(\sum_{g\in G} x_g \bar{g}\right) v = \sum_{g\in G} x_g \rho(g) v$. Conversely, let V be a $k_\alpha G$-module. Then there exists a homomorphism $f : k_\alpha G \to End_k(V)$. By defining $\rho(g) = f(\bar{g})$, ρ becomes an α-representation of G. This gives a bijection between α-representations and $k_\alpha G$-modules.

Definition 2.3. *Let $g \in G$ such that $\alpha(g, h) = \alpha(h, g)$ for all $h \in C_G(g)$, where $C_G(g)$ is the centralizer of h in G. Then the element g is called an α-regular element of G.*

We have $g \in G$ is α-regular if and only if $\bar{g}\bar{h} = \bar{h}\bar{g}$ for all $h \in C_G(g)$. Also if g is an α-regular element, then so is any conjugate of it. Therefore, if an element $g \in C$ is α-regular, where C is the conjugacy class of g, then every element in C is α-regular. In that case we say that C is α-regular.

Here are the two propositions that clarify some facts on the case where G is cyclic. We omit the proofs.

Proposition 2.4. *Assume that k is an algebraically closed field and $\alpha \in Z^2(G, k^*)$. If G is cyclic, then $k_\alpha G$ is isomorphic to kG.*

Proposition 2.5. *Let G be a cyclic group of order m and generated by $g \in G$ and let $\alpha \in Z^2(G, k^*)$. Also let $\lambda = \prod_{i=1}^{m} \alpha(g, g^i)$. Then $k_\alpha G$ is isomorphic to $k[x]/(x^m - \lambda)$.*

Lemma 2.6. *Let $\rho_1 : G \to GL(V), \rho_2 : G \to GL(W)$ be α and β representations of G, respectively. Then the map $\rho_1 \otimes \rho_2 : G \to GL(V \otimes_k W)$ defined by $(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$ is an $\alpha\beta$-representation of G.*

Proof. For all $x, y \in G$, we have

$$\begin{aligned}
(\rho_1 \otimes \rho_2)(x)(\rho_1 \otimes \rho_2)(y) &= (\rho_1(x) \otimes \rho_2(x))(\rho_1(y) \otimes \rho_2(y)) \\
&= (\rho_1(x)\rho_1(y) \otimes \rho_2(x)\rho_2(y)) \\
&= \alpha(x, y)\rho_1(xy) \otimes \beta(x, y)\rho_2(xy) \\
&= \alpha(x, y)\beta(x, y)(\rho_1(xy) \otimes \rho_2(xy)) \\
&= \alpha(x, y)\beta(x, y)((\rho_1 \otimes \rho_2)(xy))
\end{aligned}$$

as desired.

Here are some theorems that will be referred to later on the next chapter.

Theorem 2.7. *(Karpilovsky 1985) Let N be a normal subgroup of a finite group G, let k be an algebraically closed field of an arbitrary characteristic and let $\alpha \in Z^2(G, k^*)$.Then, for any simple $k_\alpha G$-module W, $dim_k W$ divides $(G : N)d$, where d is the dimension of a simple submodule of W_N.*

Proof. Proof is omitted. (Theorem 5.3.1 in [2]).

Theorem 2.8. *(Mangold 1966, Tappe 1977). Let G be a finite group, let k be an algebraically closed field of characteristic $p \geq 0$ and let $\alpha \in Z^2(G, k^*)$. Then the number of projectively nonequivalent irreducible α-representations of G over k is equal to the number of α-regular conjugacy classes of p'-elements of G contained in G'.*

Proof. Proof is omitted. (Theorem 6.4.1 in [2]).

Theorem 2.9. *Let G be a finite group, let k be an algebraically closed field of characteristic $p \geq 0$ and let $\alpha \in Z^2(G, k^*)$. Denote by r the number of nonisomorphic simple $k_\alpha G$-moudles. Then*

(i) r equals the number of α-regular conjugacy classes of G if $p = 0$.

(ii) r equals the number of α-regular conjugacy classes of p'-elements of G if $p \neq 0$.

Proof. Proof is omitted. (Theorem 6.1.1 in [2])

3. PROJECTIVE CHARACTERS

Definition 3.1. *Let $\alpha \in Z^2(G, k^*)$ and $\rho : G \to GL(V)$ be an α-representation. The character of ρ is the map $\chi : G \to k$ defined by $\chi(g) = tr(\rho(g))$ for all $g \in G$, where $tr(\rho(g))$ is the trace of the linear transformation $\rho(g)$ of V. We call χ the α-character of G over k (or projective character) of the α-representation of G over k.*

Let V be a $k_\alpha G$-module. Then the map $f : k_\alpha G \to End_k(V)$ is a homomorphism. Now the character of V is $\chi_V : k_\alpha G \to k$ as $\chi_V(g) = tr(f(g))$ for all $g \in G$. Then the map $\chi : G \to k$, where $\chi(g) = \chi_V(\bar{g})$ for all $g \in G$ is the α-character of G afforded by V. Conversely, if χ is a character of an α-representation $\rho : G \to GL(V)$, then χ is afforded by the $k_\alpha G$-module V corresponding to ρ. It follows from the equality

$$\chi_V\left(\sum_{g\in G} x_g \bar{g}\right) = \sum_{g\in G} x_g \chi_V(\bar{g}) = \sum_{g\in G} x_g \chi(g)$$

where $x_g \in k$. In conclusion, the character of a $k_\alpha G$-module V is determined by the α-character of G afforded by V.

Definition 3.2. *If the α-character χ is the character of an irreducible α-representation of G over the field k, then it is called an irreducible α-character of G over k.*

Let V and W be two $k_\alpha G$-modules, and let χ_V and χ_W be their α-characters respectively. Define $\chi_V + \chi_W$ the sum of two characters as

$$(\ \chi_V + \chi_W)\ (g) = \chi_V(g) + \chi_W(g)$$

for all $g \in G$. Therefore, we have $\chi_{V\oplus W} = \chi_V + \chi_W$ which is in fact an α-character.

However, we will see that the product of two α-characters may not be an α-character.

Lemma 3.3. *Suppose that V is a $k_\alpha G$-module and that*

$$V = V_m \supset V_{k-1} \supset \cdots \supset V_1 \supset V_0 = 0$$

is a chain of submodules of V. Denote by χ_i the α-character of G afforded by $V_i/V_{i-1}, 1 \leq i \leq m$, and let χ be the α-character of G afforded by V. Then

$$\chi = \chi_1 + \chi_2 + \cdots + \chi_m.$$

In particular, every α-character of the group G is a sum of irreducible α characters of G.

Proof. Define f and f_i as the representations of $k_\alpha G$ afforded by V and V_i/V_{i-1} respectively. For $g \in k_\alpha G$, we can choose an k-basis of V such that

$$f(g) = \begin{bmatrix} f_1(g) & & & \\ & f_2(g) & & \\ & & \cdots & \\ & & & f_m(g) \end{bmatrix}$$

where the blank entries of the above matrix are zero. Therefore, for $g \in G$, $\chi(g) = tr(f(\bar{g})) = tr(f_1(\bar{g})) + tr(f_2(\bar{g})) + ... + tr(f_m(\bar{g})) = \chi_1(g) + \chi_2(g) + ... + \chi_m(g)$. If we choose the above chain as a composition series of V, then the last part of the lemma follows.

Lemma 3.4. *Let $\rho_\alpha : G \to GL(V)$ and $\rho_\beta : G \to GL(W)$ be α and β-representations of G, respectively. If χ_α and χ_β are the characters of ρ_α and ρ_β, respectively, then the* *map*

$$\chi_\alpha \chi_\beta : G \to F$$

defined by

$$(\chi_\alpha \chi_\beta)(g) = \chi_\alpha(g)\chi_\beta(g) \quad \textit{for all } g \in G$$

is the character of the tensor product $\rho_\alpha \otimes \rho_\beta$.

Proof. Let $\{v_1, v_2, ..., v_n\}$ and $\{w_1, w_2, ..., w_m\}$ be bases for V and W. Then we can write $\rho_\alpha(g)v_i = \sum_{j=1}^{n} a_{ji}v_j$ and $\rho_\beta(g)w_t = \sum_{r=1}^{m} b_{rt}w_r$, where $a_{ji}, b_{rt} \in k$ and $g \in G$. Thus, $\chi_\alpha(g) = \sum_{i=1}^{n} a_{ii}$ and $\chi_\beta(g) = \sum_{r=1}^{m} b_{rr}$. Let's denote the character of $\rho_\alpha \otimes \rho_\beta$ as χ. Since $(\rho_\alpha \otimes \rho_\beta)(g)(v_i \otimes w_t) = \rho_\alpha(g)v_i \otimes \rho_\beta(g)w_t = \sum_{j,r} a_{ji}b_{rt}(v_i \otimes w_r)$, we have $\chi(g) = \sum_{i,t} a_{ii}b_{tt} = \sum_i a_{ii} \sum_t b_{tt} = \chi_\alpha(g)\chi_\beta(g)$.

Corollary 3.5. *Let* $\alpha, \beta \in Z^2(G, k^*)$ *and let* χ_α *and* χ_β *be* α *and* β *characters of* G*, respectively. Then* $\chi_\alpha\chi_\beta$ *is an* $\alpha\beta$*-character of* G.

Proof. It follows from Lemma 2.6 and Lemma 3.4.

Lemma 3.6. *The followings hold.*

(i) The α*-representations* ρ_1 *and* ρ_2 *have the same characters if they are linearly equivalent.*

(ii) The characters of V *and* W*, namely* χ_V *and* χ_W *are equal if* V *is isomorphic to* W *as* $k_\alpha G$*-modules.*

Proof. (i) is true since the conjugate matrices have the same trace. (ii) is equivalent to (i).

The theory of α-characters of G over k can differ from β-characters of G over k as the ordinary character theory can be different for two different groups. However, if α and β are cohomologous, then there is a bijection between α and β-characters of G over k. The following lemma illustrates this fact.

Lemma 3.7. *Let $\alpha, \beta \in Z^2(G, k^*)$ be cohomologous, meaning there is a map $t : G \to k^*$ such that $t(1) = 1$ and $\alpha(g, h) = \beta(g, h)t(g)t(h)t(gh)^{-1}$ for all $g, h \in G$. Let χ be an α-character and define $\chi' : G \to k$ such that $\chi'(g) = t(g)\chi(g)$ for all $g \in G$. Then the map that sends χ to χ' gives a bijective correspondence between α and β-characters that maps irreducible ones to irreducible ones.*

Proof. Let χ be the character of an α-representation $\rho : G \to GL(V)$. Define $\rho' : G \to GL(V)$ such that $\rho'(g) = t(g)\rho(g)$ for all $g \in G$. This map is clearly a β-representation with character χ'. We know that ρ is irreducible if and only if ρ' is irreducible, so that is true for χ and χ' as well. For the injectivity of the map, if $\chi_1' = \chi_2'$, then clearly $\chi_1 = \chi_2$ because the map t is nonzero for every $g \in G$. For the surjectivity, let δ be a β-character. Then define $\chi : G \to k^*$ so that $\chi(g) = t(g)^{-1}\delta(g)$ for all $g \in G$. Obviously, χ is an α-character and $\delta = \chi'$.

Now the following results illuminate the facts about the values of projective characters.

For a given ordinary character χ of G over k, for any $g \in G$, $\chi(g)$ is equal to a sum of roots of unity over k. This is no longer true for projective characters.

Lemma 3.8. *Let V be a $k_\alpha G$-module and $g \in G$. Let $\lambda = \alpha(g,g)\alpha(g^2,g)...\alpha(g^{n-1},g)$, where n is the order of g. Let χ be the α-character of G over k afforded by V. Then the followings hold:*

(i) *$\chi(g)$ is a sum of n^{th} roots of λ over k.*

(ii) *If k is an algebraically closed field with characteristic not dividing n, then there is a basis of V, $\{v_1, v_2, ...v_n\}$, such that $\bar{g}v_i = \lambda_i v_i$, where $i = 1, 2, .., m$ and each λ_i is an n^{th} root of λ.*

Proof. (i) We can see that $\bar{g}^n = \alpha(g,g)\alpha(g^2,g)...\alpha(g^{n-1},g)\bar{g^n}$ applying an induction on n. Hence we get $\bar{g}^n = \lambda\bar{1}$. Let $f : k_\alpha G \to End_k(V)$ be the homomorphism afforded by V. Then we have $f(\bar{g}^n) = f(\bar{g})^n = \lambda 1_V$. Since $\chi(g)$ is the sum of characteristic roots of $f(\bar{g})$, we have χ(g) is a sum of n^{th} roots of λ over k.

(ii) Let $H =< g >$, where $g \in G$. Since k is algebraically closed, by Proposition 2.4, we have $k_\alpha H$ is isomorphic to kH. And since characteristic of k does not divide the order of H, we have that $k_\alpha H$ is semisimple, meaning all simple $k_\alpha H$-modules are one-dimensional. Thus as a $k_\alpha H$-module, V is a direct sum of one-dimensional submodules, say $V = \oplus_{i=1}^m V_i$. Then for any nonzero $v_i \in V_i$, the set $\{v_1, v_2, ...v_m\}$ is a basis for V that satisfies $\bar{g}v_i = \lambda_i v_i$ for some $\lambda \in k$. Since $\bar{g}^n = \lambda\bar{1}$, each λ_i is an n^{th} roos of λ.

Definition 3.9. *The twisted group algebra $k_\alpha G$ is itself a $k_\alpha G$-module which is called the regular module. The corresponding α-representations are called the regular α-representations of G and the α-characters are called the regular α-characters of G over k.*

Lemma 3.10. *Let k be a field and $\alpha \in Z^2(G, k^*)$ and χ be the regular α-character of G over k. Then*

$$\chi(g) = \begin{cases} 0, & \text{if } g \neq 1 \\ |G|, & \text{if } g = 1 \end{cases} \tag{3.1}$$

Proof. Choose the elements $\{\bar{x} \mid x \in G\}$ as a k-basis for the regular module $k_\alpha G$. Then, for each $g \in G$, left multiplication by $\bar{g}$ permutes the basis elements up to nonzero scalar factors. Thus, if ρ is the regular α-representation of G, then each $\rho(x)$ has precisely one nonzero entry in each row and column. Moreover, if $g \neq 1$, then $\bar{g}\bar{x}$ is not a scalar multiple of $\bar{x}$ for all $x \in G$, so that $\rho(g)$ has only zero entries on its main diagonal. Hence $\chi(g) = \operatorname{tr} \rho(g) = 0$ for all $g \neq 1$. On the other hand, $\rho(1)$ is the identity matrix so that $\chi(1) = \operatorname{tr} \rho(1) = \dim_k k_\alpha G = |G|$ as required.

For any group $G \neq 1$, the ordinary regular character of G cannot be irreducible. However, it is quite possible for the projective characters.

Example 3.11. *Let G be a cyclic group of order 2. Then there is $\alpha \in Z^2(G, \mathbb{Q}^*)$ so that the regular α-character of G over $\mathbb{Q}$ is irreducible.*

Proof. By taking $m = 2$ and $k = \mathbb{Q}$ in the Proposition 2.5, we see that there exist $\alpha \in Z^2(G, \mathbb{Q}^*)$ such that $\mathbb{Q}_\alpha G$ is isomorphic to $\mathbb{Q}(\sqrt{2})$. Since $\mathbb{Q}_\alpha G$ is a field, it is simple as the regular $\mathbb{Q}_\alpha G$-module. Hence, the corresponding regular α-character of G over $\mathbb{Q}$ is irreducible.

Let's state an important theorem on linear independence of irreducible projective characters. However we omit the proof. It is Theorem 1.3.1 in [3].

Theorem 3.12. *Let k be a field with characteristic zero. Then the α-characters of G afforded by all non-isomorphic simple $k_\alpha G$-modules, say $\chi_1, \chi_2, ..., \chi_r$, are linearly independent.*

Theorem 3.13. *Let k be a field with characteristic zero. Let V and W be $k_\alpha G$-modules, and let ρ_V and ρ_W be their α-representations of G over k. Let χ_V and χ_W be their α-characters. Then the following are equivalent:*

(i) ρ_V *and* ρ_W *are linearly equivalent.*

(ii) $\chi_V = \chi_W$.

(iii) V *is isomorphic to* W.

Proof. We have (i) is equivalent to (iii) and (i) implies (ii) by Lemma 3.6. We only need to show that (ii) implies (iii). So assume that $\chi_V = \chi_W$. Let V_1, V_2, ..., V_r be the full set of non-isomorphic simple $k_\alpha G$-modules and let $\chi_1, \chi_2, ..., \chi_r$ be their corresponding α-characters of G over k. Let $n_i \geq 0$ (resp. $m_i \geq 0$) be the multiplicity of V_i as a composition factor of V (resp. W). Then by Lemma 3.3, $\chi_V = \sum_{i=1}^{r} n_i \chi_i$ and $\chi_W = \sum_{i=1}^{r} m_i \chi_i$. Consider $\chi_V - \chi_W = \sum_{i=1}^{r} (n_i - m_i)\chi_i = 0$. Since the given α-characters are linearly independent, we have $n_i = m_i$ for all $i = 1, ..., r$. Since char k $= 0$, it does not divide $|G|$ and $k_\alpha G$ is semi simple. Hence we have that V is isomorphic to W.

Theorem 3.14. *Let ρ_1 and ρ_2 be two α-representations of G and χ_1 and χ_2 be their α-characters. Assume that characteristic of k is zero. Then ρ_1 and ρ_2 are projectively equivalent if and only if there exists a homomorphism $\mu : G \to k^*$ such that $\chi_2(g) = \mu(g)\chi_1(g)$ for all $g \in G$.*

Proof. Assume that ρ_1 and ρ_2 are projectively equivalent. Then there exists a homomorphism $\mu : G \to k^*$ and an invertible matrix A such that $\rho_2(g) = \mu(g)A\rho_1(g)A^{-1}$ for all $g \in G$. By taking traces of both sides, we see that $\chi_2(g) = \mu(g)\chi_1(g)$ for all $g \in G$. Now assume that $\chi_2(g) = \mu(g)\chi_1(g)$ for all $g \in G$. Define $\rho(g) = \mu^{-1}(g)\rho_2(g)$, which is also an α-representation of G which has character χ_1. If ρ_2 is irreducible, then so is ρ. Hence by Theorem 3.13, ρ and ρ_2 are linearly equivalent. So we have $\mu^{-1}(g)\rho_2(g) = \rho(g) = A\rho_1(g)A^{-1}$. Therefore, ρ_1 and ρ_2 are projectively equivalent.

Definition 3.15. *Let p be a prime number or 0. An element $g \in G$ is called a p'-element if $p = 0$ or if $p > 0$ and does not divide the order of g.*

Now we can look at the following results about the number of irreducible projective characters.

Theorem 3.16. *Let F be a field of characteristic $p \geq 0$ so that it is a splitting field of $F_\alpha G$ for some $\alpha \in Z^2(G, F^*)$. Then the number of irreducible α-characters of G over F is equal to the number of α-regular conjugacy classes of p'-elements of G.*

Proof. Let E be an algebraic closure of F. Viewing α as an element of $Z^2(G, E^*)$, by Theorem 1.12, we have that $E \otimes_F F_\alpha G = E_\alpha G$. Since F is a splitting field of $F_\alpha G$, the number of non-isomorphic simple $E_\alpha G$-modules is equal to the number of non-isomorphic simple $F_\alpha G$-modules, which is also equal to the number of irreducible α-characters of G. By Theorem 2.9, the number of non-isomorphic simple $E_\alpha G$-modules is equal to the number of α-regular conjugacy classes of p'-elements of G. However, the α-regularity of an element of G does not depend on whether α is viewed as an element of $Z^2(G, F^*)$ or $Z^2(G, E^*)$. Therefore, theorem follows.

The following theorems are about the degree of an projective character.

Definition 3.17. *Let χ be an irreducible α-character of G over k. Assume that k is a splitting field of $k_\alpha G$ or that $char k = 0$. Then the degree of χ is defined to be the k-dimension of a simple $k_\alpha G$-module which affords χ, denoted by $deg\chi$.*

Theorem 3.18. *Let k be an algebraically closed field and let χ be an irreducible α-character of G over k. Let A be an abelian normal subgroup of G such that restriction of α to $A \times A$ is a coboundary. Then $deg\chi$ divides $(G : A)$.*

Proof. Let W be simple $k_\alpha G$-module and let d be the dimension of a simple submodule of W_A. Since restriction of α is a coboundary, we have that $k_\alpha A$ is isomorphic to kA by Proposition 2.4. Since A is abelian and k is algebraically closed, we have that $d = 1$. Then by Theorem 2.7, $deg\chi$ divides $(G : A)$.

Theorem 3.19. *Let k be an algebraically closed field and let G be abelian. Then all the irreducible α-characters of G over k have the same degree. In fact, if χ is an irreducible α-character, then any other such character is of the form $\chi\lambda$ for some $\lambda \in Hom(G, F^*)$, where $(\chi\lambda)(g) = \chi(g)\lambda(g)$.*

Proof. By Theorem 2.8, all irreducible α-representations of G are projectively equivalent. The result follows from Theorem 3.13.

Lemma 3.20. *Let k be a splitting field of $k_\alpha G$ and let $n_1, ..., n_r$ be the degrees of irreducible α-characters of G over k. Then $\sum_{i=1}^{r}(n_i)^2 = |G| - dim_k J(k_\alpha G)$. If $char F$ does not divide $|G|$, then $\sum_{i=1}^{r}(n_i)^2 = |G|$.*

Proof. The first equation is given by Artin-Wedderburn. If $char k$ does not divide the order of G, then $J(k_\alpha G) = 0$ and the result follows.

Theorem 3.21. *Let G be an abelian group and let k be an algebraically closed field with char$k$ does not dividing the order of G. Let G_0 be the subgroup of G consisting of all the α-regular elements of G. Then $(G : G_0)$ is a square and for any irreducible α-character χ of G over k, $deg\chi = \sqrt{(G : G_0)}$*

Proof. Let $\chi_1, \ldots, \chi_r$ be all the irreducible α-characters of G over k. Then, by Theorem 3.16, $r = |G_0|$ and by Theorem 3.19, we have $deg\chi_1 = \ldots = deg\chi_r = n$. Therefore, by Lemma 3.20, we have $|G_0|n^2 = |G|$ and the result follows.

REFERENCES

1. Linckelmann, M., "The block theory of finite groups I/II", *London Math. Soc. Student Texts*, Vol. 91, p. 92, 2018.

2. Karpilovsky, G., *Group Representations: Volume 2]*, North-Holland mathematics studies, North-Holland, 1993.

3. Karpilovsky, G., *Group representations*, Vol. 3, Elsevier, 1994.