# INVESTIGATING MATHEMATICS TEACHER EDUCATORS' SPECIALISED KNOWLEDGE FOR TEACHING GEOMETRIC TRANSFORMATIONS

by

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# ABSTRACT

## INVESTIGATING MATHEMATICS TEACHER EDUCATORS' SPECIALISED KNOWLEDGE FOR TEACHING GEOMETRIC TRANSFORMATIONS

In this study, three mathematics teacher educators'(MTEs) specialised knowledge for geometric transformations was explored. In this regard, MTEs' mathematical knowledge about geometric transformations including the definitions and properties and their' pedagogical content knowledge regarding the thinking ways of students about geometric transformations, the teaching strategies to develop students' understandings and to overcome students' difficulties were examined. Data were collected qualitatively in one-hour long structured interviews. Results showed that all participants defined geometric transformations in two ways: namely, as a motion and as a function. MTEs pointed that motion conception of geometric transformations is seen in school curricula; while, the function understanding is mostly delayed until the university level. MTEs also defined geometric transformations by using APOS theory in which they considered motion understanding at the action level, and function understanding both at the process level and the object level. Results further pointed to what MTEs consider as important in terms of the difficulties learners might possibly have and what strategies might be useful to overcome them. Particularly, results indicated that MTEs consider learners' need to get used to studying with different functions in different spaces as well as their understanding of plane, $\mathrm{R}^2$, as important to conceptualize the domain and the range of geometric transformations as the whole plane. In addition, Results further showed that MTEs consider that reflection is the easiest transformation for the learners and rotation is the hardest one. Thus, they recommend that it might be a good strategy to start teaching transformations with reflection.

# ÖZET

## MATEMATİK ÖĞRETMEN EĞİTİMCİLERİNİN GEOMETRİK DÖNÜŞÜMLERİ ÖĞRETME KONUSUNDAKİ UZMAN BİLGİSİNİN İNCELENMESİ

Bu çalışma üç matematik öğretmen eğitimcisinin geometrik dönüşümler konusundaki uzman bilgisini incelenmiştir. Bu amaçla, matematik öğretmen eğitimcilerinin geometrik dönüşümlerin tanımları ve özellikleri hakkındaki matematiksel bilgileri ve konu ile ilgili düşünme biçimlerine ve öğrencilerin anlamalarını geliştirmeye ve öğrencilerin zorluklarını aşmaya yönelik öğretim stratejilerine ilişkin pedagojik alan bilgileri incelenmiştir. Bu çalışmada nitel araştırma deseni kullanılmıştır ve matematik öğretmenlerinin uzmanlık bilgisi (MTSK) modeli çerçeve olarak kullanılmıştır. Matematik öğretmen eğitimcilerin bu konuyu nasıl kavramlaştırdıklarını anlamak amacıyla, katılımcılarla yaklaşık bir saat süren görüşmeler yapılmıştır. Sonuçlar, tüm katılımcıların geometrik dönüşümleri hareket ve fonksiyon olarak iki şekilde tanımladığını gösterdi. Matematik öğretmen eğitimcileri geometrik dönüşümlerin hareket anlayışının okul müfredatlarında görüldüğüne dikkat çekti. Diğer yandan, geometrik dönüşümleri APOS teorisini kullanarak açıkladılar. Hareket anlayışının eylem düzeyinde olduğu ve fonksiyon anlayışının hem süreç düzeyinde hem de nesne düzeyinde olabileceği bulgulandı. Sonuçlar öğrencilerin düzlemin geometrik dönüşümlerin tanım kümesi olduğunu kavramsallaştırmaları için farklı uzaylardaki fonksiyonlarla çalışmaya alışmaları gerektiğini gösterdi. Sonuçlar, matematik eğitimcilerinin bakış açısından, yansımanın öğrenciler için en kolay ve dönmenin en zor dönüşüm olduğunu göstermiştir. Bu nedenle, matematik eğitimcileri, dönüşümleri yansıma ile öğretmeye başlamanın iyi bir başlangıç olabileceğini savunmaktadırlar.

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| | |
|---|---|
| APOS | Action Process Object Schema |
| CCK | Common Content Knowledge |
| KMTEd | Knowledge of Mathematic Teacher Educations |
| MKT | Mathematics Knowledge for Teaching |
| MKTT | Mathematical Knowledge for Teaching Teachers |
| MTE | Mathematics Teacher Educator |
| MTE-PCK | Mathematics Teacher Educators' Pedagogical Content Knowledge |
| MTE-SMK | Mathematics Teacher Educators' Subject Matter Knowledge |
| MTESK | Mathematics Teacher Educators' Specialized Knowledge |
| MTSK | Mathematics Teachers' Specialized Knowledge |
| NCTM | National Council of Teachers of Mathematics |
| PCK | Pedagogical Content Knowledge |

# 1. INTRODUCTION

Geometric transformations are one-to-one and onto functions from plane to plane (Fife *et al.*, 2019; Martin, 1982). Thus, geometric transformations are an overarching content that connect algebra and geometry (Flanagan, 2002) such that geometric transformations combine algebraic ideas including functions, domain, and range with geometrical objects including points and lines. That is, geometric transformations are functions, but their inputs and outputs are not real numbers instead are points that their coordinates consist of two real numbers (Hollebrands, 2003; Martin, 1982). Understanding geometric transformations as functions are significant because first, by learning transformations as functions, learners understand the role of parameters on transformations better (Hollebrands, 2003). Second, they understand the fixed points of transformations without any confusion (Hollebrands, 2003). Moreover, learning geometric transformations as a function has a dual relationship, which means learning geometric transformations helps to understand functions better and vice versa (Fife *et al.*, 2019). Furthermore, when learners gain an understanding of geometric transformations, they are expected to improve their mathematical abilities namely finding patterns, making generalizations, mathematical reasoning, justifications, spatial competencies, and critical thinking (Hollebrands, McCulloch and Okumus, 2021; Yanik, 2014). Therefore, it is a crucial content in different curriculum standards (CCSSM, 2010; MEB, 2018) as transformations help to learn different mathematical ideas and competences in a connected way (Hollebrands *et al.*, 2021).

In middle school, students start to learn geometrics transformations at the $8^{th}$ grade (CCSSM, 2010; MEB, 2018). According to the CCSSM (2010), the concept of geometrical transformation at middle school level depends on more visual approximations. At this level, students are only expected to identify the properties of translation, reflection, rotation, and dilation experimentally. They also learn congruency and similarity of two-dimensional figures by using geometric transformations. In addition, using coordinates, they learn how to describe the effects of geometric transformations on two-dimensional figures. Similarly, in Turkish National curricula, students at the middle school level are expected to construct the preimage and image of geometrical figures under translation and reflection. They learn transformation of the point first followed by line segments and polygons. The one-to-one correspondence of the points on the preimage and image figures is also highlighted. Whereas, at the high school level, students begin to learn geometric transformations as functions (CCSSM, 2010; MEB, 2018). At science high schools in Turkey, students also learn to solve composition of transformation in modelling scenarios. They are expected to learn input and output points and apply a transformation to given input points. They are also expected to learn congruency of triangles by using geometric transformations. In the curricula of both countries, students are not exposed to parameters while learning transformations, so they have limited experience in learning transformation as functions. Therefore, although students are expected to learn transformations as functions, they seem to only focus on the motion conception.

Despite the emphasis on the gradual shift in the learning of geometric transformations in different grade levels from a motion perspective towards a function perspective, students have difficulties understanding the concept of geometric transformations, and they have alternative conceptions (Aktaş and Ünlü, 2017; Emre-Akdoğan *et al.*, 2018; Guven, 2012; Hollebrands, 2003, 2004, 2007; Hollebrands *et al.*, 2021; Sünker and Zembat, 2012; Xistouri *et al.*, 2014; Yanik, 2014). Specifically, students have difficulty in understanding parameters, vectors, and geometric transformations as functions. Prior understandings of preservice teachers about geometric transformations (Avcu and Çetinkaya, 2019; Uygun, 2020; Yanık, 2011; Yanık and Flores, 2009) also point to similar difficulties and alternative conceptions. These results indicate the persistency in the difficulty of understanding geometric transformations as functions on the part of learners at different ages. That is, the limited and alternative understandings seem to transit from the middle school to the high school and, from the high school to the college years. Both students' and prospective teachers' similar alternative and limited understandings of geometric transformations seem to suggest that the reason behind these understandings might be related to how learners at different ages are taught. In retrospect, as the future school teachers, if preservice teachers' limited understandings do not resolve during their teacher education programs, they might transfer these understandings back to their prospective students (Yanik, 2011). Thus, it is important to ensure the conceptualization of preservice teachers in the context of geometric transformations before they begin their teachings. This in turn suggest that doing research on the expert approach about geometric transformations might be informative about how a person who already knows geometric transformations conceptually might understand transformations as well as how prospective and in-service teachers might come to know. Mathematics teacher educators are people who are responsible of the education of preservice and in-service mathematics teachers (Cetinkaya *et al.*, 2017). In this respect, mathematics teacher educators (MTE) who do research on geometric transformations or teach geometric transformations might be considered as experts. The knowledge of mathematics teacher educators who are expected to have both the subject matter knowledge and the pedagogical knowledge required to educate both school students and preservice teachers might enlighten teaching and learning pathways of geometric transformations. Therefore, in this study, I aimed to investigate the schema of mathematics teacher educators to explicate both how an expert conceptualize geometric transformations and also in what ways they consider teachers to know geometric transformations and how they might assist their students.

## 1.1. Research Questions

In this respect, in this study, the following research questions will be scrutinized:

(i) What specialized knowledge for teaching do teacher educators have?
(ii) How can the mathematics teacher educators' specialized knowledge for teaching in the context of geometric transformations be characterized?

## 1.2. Significance of the Study

The aim of the study was to investigate how a person who knows teaching and learning geometric transformations might reason about geometric transformations. That is, I aimed to construct the schema of people as experts who know well geometric transformations. Therefore, this study provides an alternative perspective for both research on the knowledge of MTEs and research on the conceptualization of geometric transformations.

In addition, existing research (Aktaş and Ünlü, 2017; Emre-Akdoğan *et al.*, 2018; Guven, 2012; Hollebrands, 2003, 2004, 2007; Hollebrands *et al.*, 2021; Sünker and Zembat, 2012; Xistouri *et al.*, 2014; Yanik, 2014) solely focus on how learners who have difficulty in understanding geometric transformations might think and possible strategies to overcome these difficulties. Building on and extending previous research, this study provides a comprehensive view for how a person as an expert who knows teaching and learning geometric transformations might think about geometric transformations. Therefore, the findings of this study might contribute to the field in two related ways: by profoundly informing what and how experts think about geometric transformations and by pointing to in what ways teachers and mathematics teacher educators might plan their lessons to provide a high level of understanding on the part of their students in the context of geometric transformations.

# 2. LITERATURE REVIEW

## 2.1. Understanding of Geometric Transformations

Geometric transformations are special functions (Hollebrands, 2003) and can be defined as one-to-one and onto mappings from plane to plane (Fife *et al.*, 2019; Martin, 1982). From the definition, one of the distinctive features of geometric transformations from other functions is that the inputs and outputs of geometric transformations are points in the plane rather that real numbers (Steketee and Scher, 2011). Also, under any transformations, every preimage point in the plane has a corresponding and unique image point in the plane and vice versa (Steketee and Scher, 2012). The relationships among input, output and parameters help identifying characteristics of geometric transformations. Thus, parameters play a decisive role in determining the type of geometric transformations. There are different kinds of geometric transformations. For instance, the geometric transformations that preserve the distance between the points; thus, preserve the size and shape of the figures, are called isometries (Martin, 1982). As isometries, translation, reflection, and rotation are the fundamental ones.

Particularly, translations are one type of geometric transformations, the parameter of which is a vector (Flanagan, 2002; Martin, 1982). The translation can be vertical, horizontal, or diagonal according to the parameter vector. Meaning that if the parameter of the translation is a vector , say $(0,a)$, where a is any real number, then the translation is vertical; if the parameter of the translation is a vector, say $(b,0)$, where b is any real number, the translation is horizontal; and if the parameter of the translation is vector, say $(a,b)$, where a and b are any real numbers, the translation is diagonal. The main point is that there is a unique vector that maps any point P to its corresponding point Q (Martin, 1982). Next, reflection is a kind of transformation in which the orientation of preimage and image points are different (Hollebrands, 2004). The parameter of reflection is the line of reflection. The line of reflection is the perpendicular bisectors of the preimages and their corresponding image points. Therefore, perpendicular bisectors, equidistance, and perpendicularity are the properties related to reflection. Furthermore, rotations have two parameters, the angle of rotation and the center of rotation. The angle of the rotation is the angle that is constructed by a preimage point, the center of the rotation, and the corresponding image point. In addition, the center of rotation is the point about which the preimage points rotate. The distance between the preimage points and the center is the same as the distance between the corresponding image points and the center.

The concept of geometric transformations is also related to the concept of similarity and congruency (Jones, 2002). The similarity and congruency of the geometric figures can be proven by using the composition of geometric transformations. In particular, two figures are congruent if one figure maps on the other figure by a composition of transformation. Also, similar functions can be found via transformations by checking if they map onto each other. For example, all parabolas are similar because under a transformation or a composition of transformations, one parabola can be mapped to the other (Jones, 2002).

## 2.2. Research on Geometric Transformations

In this part I explain previous research on K-12 students' conceptions of geometric transformations and preservice teachers' conceptions of geometric transformations separately.

### 2.2.1. Research on K-12 Students' Conceptions of Geometric Transformations

There is some research on the middle school students' (Aktaş and Ünlü, 2017; Güven, 2012; Sünker and Zembat, 2012; Yanık, 2014, Xistouri *et al.*, 2014) and high school students' (Emre-Akdoğan *et al.*, 2018; Gülkılık *et al.*, 2015; Hollebrands, 2003, 2007; Hollebrands *et al.*, 2021; Kainose Mhlolo and Schafer, 2013) understanding of geometric transformations. Results from these studies point that students' understanding of geometric transformations are similar with respect to different grade levels.

Particularly, Sünker and Zembat (2012) examined conceptual understandings of four $6^{th}$ grade students on the concept of translation. Researchers developed a curriculum in which they used Wingeom-tr to teach translations to $6^{th}$ graders. Results showed that students were not able to understand translations as relationships between the sets of preimage and image values. Indeed, results showed that students did not know the meaning of the parameter, vector. They interpreted vectors differently on plotting paper versus on plain paper. Particularly, on the plotting paper, they interpreted vectors as line segments whereas on plain paper they considered vectors as rays. Researchers pointed that if students do not understand the role of parameter, vector, they can only conceptualize translations as a movement of a figure.

Yanık (2014) also conducted a study with 110 sixth-grade students who already were taught a unit on the geometric translations using the same textbook. Utilizing a written instrument, first he examined the sixth-grade students' concept images. Following, he conducted semi-structured interviews with twenty-two of them and their teachers. He also examined the textbook that those students have used. Results showed that, 46% of sixth graders conceptualized translation as translational motion; and the rest of them conceptualized translation as both translational and rotational motion. In fact, in the former category, students thought of translation as physical action such that they interpreted translation as sliding. Most of the students who considered translation as translational motion with only one parameter focused on the direction of the action rather than the magnitude of the translation. The students with this conception further believed that translation is a continuous action. The rest of the students (54% of sixth graders) who considered translation of a single object as both translational motion and rotational motion, used real-life examples to explain their understandings. However, students holding this conception did not have an agreement about whether the rotational motion of circular shapes and noncircular shapes could be thought as translation. Particularly, some of the students thought that only figures with circular shapes can be translated via rotation because they thought that the only way to translate a circular object is rotation. This suggested that they imagined the translation as a physical movement such that they needed rotational motion to translate any circular object. In addition, some of the students considered that if the figure has a circular shape, after any physical movement its direction will always be the same. Also, results showed that students held different meanings for vector such that they thought that vector can be a reference line, a symmetry line, or a direction indicator. Some of the participants could not figure out the role of vectors for translations either.

Some other studies (Aktaş and Ünlü, 2017; Guven, 2012; Xistouri *et al.*, 2014) also supported the findings of Sünker and Zembat (2012) and Yanık (2014). According to Guven (2012) and Xistouri *et al.* (2014), the participants, who were eight graders and primary school students, respectively, had motion conception regarding geometric transformations. The foci of the middle school students were on the geometrical shapes (Aktaş and Ünlü, 2017; Guven, 2012; Xistouri *et al.*, 2014). Particularly, students could not think of the geometrical shapes as a part of the coordinate plane, and they did not think of the geometrical shapes as a union of the points. Therefore, students considered that the image shape is the same as preimage shape, so they could not comment on what it meant when the preimage points and the corresponding image points were equidistant to the parameter. Moreover, the participants had difficulty differentiating different geometric transformations, making connections between them, and applying combinations of transformations (Aktaş and Ünlü, 2017; Guven, 2012).

Research focusing on high school students' understanding of translations (Emre-Akdoğan *et al.*, 2018; Gülkılık *et al.*, 2015; Hollebrands, 2003, 2007; Hollebrands *et al.*, 2021; Kainose Mhlolo and Schafer, 2013) also pointed to similar results. Hollebrands (2003) focused on six tenth grade students' conceptual understanding of transformation as function. The researcher classified the thoughts of students about the domain of a transformation in three ways; "(1) labeled points belonging to the preimage, (2) all points on the preimage, and (3) all points in the plane". (p. 60). Results showed that before the instructions, all participants thought that the transformations were applied only to a single object. These thoughts restricted their understandings of fixed points in the context of reflection, especially if the fixed point was not on the object. In particular, the participants, who thought of the domain as all points on the preimage, reasoned that the fixed points of a reflection needed to be the points of the preimage on the reflection line. Besides, students who had motion conception about translations seemed to have difficulty in understanding identity mappings (Hollebrands, 2003). According to these students "...no movement implied no translation". (p. 67). Students who had an opinion that a domain was a single object also considered that inputs, parameters, and variables cannot vary. Moreover, students had difficulty distinguishing a vector from a ray. Thus, Hollebrands (2003) concluded that understanding domain is essential to understand fixed points and see transformations as a function. She further concluded that the conception of translation as a function for tenth-grade students was challenging since students had difficulties identifying the preserved and unpreserved relations and properties under transformations. Hollebrands (2003) stated that "to consider all points in the plane as the domain, students may have been operating from the theoretical definition of the point, the figure, rather than only labeled points on the screen, the drawing" (p.70). That is, students who view all points in the plane as a domain may have begun to think of points as theoretical objects (figure) rather than physical drawings (drawing).

Hollebrands (2007) also conducted task-based interviews with four tenth-grade high school students who were at least at the third Van Hiele level of geometrical thinking. She focused on the preserved and unpreserved properties of transformations. Results of the study showed that labeled points in technological environments caused students to think that changing the location of a point does not change the point. Therefore, students could not interpret points as variable.

The main goal of the study of Emre-Akdoğan, Güçler, and Argün (2018) was to analyze how the teacher's discourse about geometric translations might have affected two tenth-grade students' understandings. They found that even though the teacher did not mention geometric transformations as motion students thought of the transformations as motion. One of the students did not use vectors to visually describe the translation. The other used it but she confused vectors with lines. She particularly considered vectors as any line that passes through the coordinates of the vector.

In another study Hollebrands, McCulloch, and Okumus (2021) intended to analyze how students developed an understanding of functions in the context of geometric transformations. Participants were eleven 15-16 years-old students. Results of the study showed that the participants recognized the independent and dependent variables by randomly dragging the points on the screen. Even if they noticed the non-function and the functions, they had difficulty in identifying their distinctions. Thus, the results suggested that understanding geometric transformations as functions seems continuing to be a problematic context for high school students.

In sum, studies focusing on primary and middle school students' understanding of translations (Sünker and Zembat, 2012; Yanık, 2014; Aktaş and Ünlü, 2017; Guven, 2012; Xistouri *et al.*, 2014) pointed that middle school students held mathematically incorrect meanings for vectors such as rays, line segments, a reference line, and a direction indicator. They did have a motion conception of translations involving both sliding and rotating. Also, they envisioned the figures and the plane separately. Similarly, studies with high school students also showed that high school students tended to interpret geometric translations as a movement (Emre-Akdoğan *et al.*, 2018; Gülkılık *et al.*, 2015; Hollebrands, 2003, 2007; Hollebrands *et al.*, 2021; Kainose Mhlolo and Schafer, 2013; Yanik, 2011). Students could not conceptualize translation as functions, and they had alternative conceptions about domain, variables, vectors, and parameters. Researchers concluded that when students had insufficient understandings of prerequisite basic concepts, they had problems in learning transformations (Gülkılık *et al.*, 2015). So, they suggested that mapping understanding may help students to think of transformations different from motion conception (Gülkılık *et al.*, 2015; Kainose Mhlolo and Schafer, 2013).

### 2.2.2. Research on Pre-service Teachers' Conceptions of Geometric Transformations

Results from research on preservice teachers' understandings of geometric transformations (Avcu and Çetinkaya, 2019; Uygun, 2020; Yanik, 2011; Yanik and Flores, 2009) were almost identical with the K-12 students. Particularly, Yanık (2011) conducted a study to describe the existing understandings and conceptions of 44 preservice middle school mathematics teachers about geometric transformations. Results showed that, twenty participants thought of translation as a rotational motion by referring to their physics courses. These participants gave rolling shapes as examples. Also, twenty-two participants considered translation based on linear motion and displacement. Only one participant saw translation as mapping. Though, he interpreted the domain of a translation as a single object. Results also showed that participants considered the parameter, vector, as a force (six participants), a line of symmetry, or a direction indicator (four participants). None of the pre-service teachers mentioned the importance of vectors when identifying the distance between pre-image and image points either. However, although some participants knew that a vector had a magnitude and a direction, they did not mathematize translation by using vectors either.

In addition, Yanik and Flores (2009) analyzed one master student in mathematics education, Jeff's, conceptualizing of translation. Results showed that although he knew that the shapes, sizes, distance, and angle measures will be preserved during and after the translation, he thought of translation as motion. Also, he had an unclear understanding about vectors. He confused the meanings of vectors and lines and did not know how to use the vector as a parameter for a translation. Jeff also thought of the domain of a translation as a single figure. In addition, Jeff thought that the coordinate system was also translated as a whole with the x and y axes and the origin. Therefore, he considered that the location of the object did not change. He could not perform finding the inverse of a translation either. Jeff stated that in his past experiences his teachers had shown geometrical concrete shapes on flat surfaces. Thus, Yanik and Flores (2009) argued that his experiences might have caused him to think that geometrical shapes are just located physically on the plane. Moreover, the researchers have speculated that because of the physics courses students have taken, they may have a tendency to think that the figures are independent from the plane (Yanık and Flores, 2009). Thus, researchers (Yanık and Flores 2009; Yanık, 2011) concluded that, past and daily life experiences might have affected preservice teachers' construction of transformation conceptions. They further stated that, direction signs and the daily life definitions of translation might also be a possible cause for preservice teachers to think of translation as an action (Yanık, 2011).

In other studies, Avcu and Çetinkaya (2019) and Uygun (2020) also pointed out that preservice middle school mathematics teachers had difficulties with parameters. The goal of Avcu and Çetinkaya (2019)'s study was to examine to what extent the understandings of preservice middle school mathematics teachers improve when they design and implement an instructional unit about geometric transformations. Results showed that preservice middle school mathematics teachers did not know the importance of parameters, nor were they able to define the parameters of specific transformations. Researchers emphasized that understanding geometric transformations as a mapping was the hardest on the part of preservice teachers (Avcu and Çetinkaya, 2019). In addition, Uygun (2020) also found that at the beginning of the study preservice teachers did not define geometric transformation sufficiently, nor did they did realize the importance of using vectors in translations.

Consequently, the results of all the aforementioned studies in different age groups point to quite similar findings. Students from all different age groups including preservice middle school mathematics teachers had difficulty understanding the role of parameters, domain of the geometric transformations, and the function conception of transformations. Particularly, researchers have found that preservice teachers had difficulty in conceptualizing the role of the parameter of the translation or they had misconceptions or unclear understandings of parameters. Specifically, although they could consider that the size, angles, and shapes have been preserved after translation, they could not explain where and to what extent the translation would be executed using the parameter (Yanık and Flores, 2009). Also, they confused lines, rays, and vectors (Yanık and Flores, 2009; Yanık, 2011). This confusion also led preservice teachers to consider vectors as symmetry lines. Therefore, researchers have concluded that all these limited understandings seem to be affected by their prior experiences, which also may be the reason behind the same misunderstandings at each grade level. They further argued these limited understandings might have an effect on their teachings in the future (Yanık, 2011). Thus, results from all these studies suggest the need for preservice teachers to learn geometric transformations comprehensively to help students to learn them conceptually.

One way for what and how teachers need to be taught can be determined by examining the knowledge of teachers of preservice teachers, namely, the knowledge of mathematics teacher educators (Escudero-Ávila *et al.*, 2021; Goos and Beswick, 2021; Masingila *et al.*, 2018; Muir *et al.*, 2021; Superfine *et al.*, 2020). For example, to be able to classify the knowledge MTEs should have, Escudero-Ávila *et al.* (2021) focused on the knowledge to be acquired by future mathematics teachers, because they considered that mathematics teacher educators' knowledge need to encompass the knowledge of preservice teachers albeit not limited with this knowledge. It is in this respect that, in this study, I will be investigating the mathematics teacher educators' knowledge in the context of geometric transformations by focusing on a model developed on the knowledge of mathematics teachers. In the next section, I first operationally define mathematics teacher educator and the knowledge of mathematics teacher educators by also pointing to the need for it. Then, I report on previous research on examining mathematics teacher educators' knowledge.

### 2.3. Knowledge of Mathematics Teacher Educators

Mathematics teacher educators (MTEs) are thought of people who focus on providing better learning opportunities for students who learn mathematics by working with both school teachers and preservice teachers (Jaworski, 2008). Thus, a wide variety of sub-identities can be defined as MTEs (Beswick and Goos, 2018; Erbilgin, 2019;

Goos and Beswick, 2021). Researchers point to these sub-identities as

> "...university academics, from the disciplines of mathematics and mathematics education, who teach in prospective or practising teacher education programmes or who engage in research with teachers; practising teachers who supervise and mentor prospective teachers during their school placement; officers of local or national education authorities who are involved in professional development programmes; and private providers of educational consultancy services" (Goos and Beswick, 2021, p. 2).

In this study I will consider university academics from the disciplines of mathematics and mathematics education as MTEs.

When defining the knowledge of MTE, the mathematics teachers' knowledge is considered as a base for MTEs' knowledge (Chapman, 2021; Escudero-Ávila *et al.*, 2021) since it is the knowledge required of MTEs to teach mathematic teachers to develop knowledge that is needed to teach mathematics. Therefore, to identify the knowledge of MTEs, I will first explain the knowledge of mathematics teachers.

There are well-documented research about the knowledge types that mathematics teachers need to have (Ball, Thames, and Phelps, 2008; Rowland *et al.*, 2009; Shulman, 1986). Shulman (1987) created a theoretical framework that forms the foundation for describing knowledge of mathematics teachers. This framework consists of seven categories which are content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational context, and knowledge of educational ends, purposes, and values, and their philosophical and historical grounds. According to Ball *et al.* (2008), pedagogical content knowledge that is described as a bridge between knowledge of content and teaching practice does not explain crystal clear, so they created a practice-based approach based on Shulman's (1987) framework of pedagogical content knowledge. I discuss the theoretical framework of the content knowledge of teaching of Ball *et al.* (2008) in theoretical framework subsection. At this point it is important to mention that Ball *et al.* (2008) created a model namely mathematics knowledge for teaching (MKT) and they defined the term as "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (p. 399). This model has two sub-domains: subject matter knowledge and pedagogical content knowledge. Subject matter knowledge is the mathematics-based knowledge required by teachers to teach mathematics, including the knowledge of curriculum or the knowledge of how mathematical contents are linked throughout mathematics included in the curriculum, the content knowledge, and the specialized content knowledge. Pedagogical content knowledge is the knowledge that connects the knowledge of content with the knowledge of teaching and learning, for example, being aware of possible student errors for specific mathematical subjects or mathematical knowledge to be able to design instructions.

Using the knowledge types asserted by Shulman (1987) and Ball, Thames, and Phelps (2008), Goos and Beswick ( 2021) define mathematics teacher educators' knowledge as a meta-knowledge. According to Goos and Beswick (2021) the knowledge of MTEs is "...a kind of meta-knowledge which could be described as knowledge for teaching knowledge for teaching mathematics". (p. 3). This meta-knowledge consists of subject matter knowledge and pedagogical content knowledge. This suggests that the knowledge of MTEs is very similar to MKT yet, the knowledge of MTEs differs from the knowledge of mathematics teachers at certain levels (Chapman, 2021; Jaworski, 2008; Masingila *et al.*, 2018; Superfine *et al.*, 2020).

The differences are generally about knowledge of mathematics teacher education (KMTEd) considered as the knowledge of MTEs required to engage preservice teachers in their learning (Chapman, 2021). Firstly, pedagogical content knowledge (PCK) of teaching adults does not have the same requirements as PCK of teaching children. PCK was defined by Shulman (1987) as finding ways to make a subject comprehensible to the people who want to learn it. To find these ways, teachers need to know possible student misconceptions, how to organize the topics so that they become comprehensible when they are taught, the variety of representations, examples to illustrate a subject, and how to be flexible to adjust the lesson plan in order to address all types of learners. Mathematics teacher educators' PCK is built on teachers' PCK, but it also involves the pedagogical knowledge of promoting preservice teachers' learning and reconstructing mathematics (Chapman, 2021; Superfine *et al.*, 2020). Chick and Beswick (2018) adapted the existing search about PCK to the mathematics teacher educator PCK (MTEPCK). For instance, the knowledge of the student thinking or misconceptions in mathematics in PCK was changed with preservice teacher thinking or misconception in PCK concepts (Chapman, 2021). Also, knowledge of cognitive demand of task was changed from tasks for learning mathematics for students to tasks for learning PCK for preservice teachers. In addition, MTEs need to know the preservice teachers' struggles within their learning processes. They should consider the psychological features of their students who are future mathematics teachers. In general, the aim of the MTEs is "to create a suitable environment for preservice teachers to construct their own rich and integrated knowledge" (p.24) (Escudero-Ávila *et al.*, 2021) so that preservice teacher can transform their knowledge into knowledge required to teach and learn.

To create such a learning environment, MTEs' knowledge must also include more mathematical knowledge than school mathematics. This knowledge encompasses

> "...having a clear understanding of the structuring ideas underlying mathematics, the connections which serve to simplify or increase the complexity of an item (Montes, Ribeiro, Carrillo, and Kilpatrick, 2016) and cross-curricular connections" (Escudero-Ávila, Montes, and Contreras, 2021, p.27).

Besides, the other categories of knowledge of MTEs are the components of efficient teaching mathematics practices, professional noticing, classroom preparation, standards and teaching contents of mathematics teacher education programs ( Escudero-Ávila, Montes, and Contreras, 2021). Teachers of school mathematics need to have some knowledge that is different from the knowledge of MTEs, as well (Chick and Beswick, 2018; Jaworski, 2008). For example, MTEs may not need to know the curriculum of school mathematics as detailed as teachers of school mathematics, but they need to know the curriculum as much as how detailed they want their students (preservice teachers) to know (Chick and Beswick, 2018).

The knowledge of MTEs is significant to study because MTEs are the people who have the responsibility to promote the professional development of the future or in-service mathematics teachers (Escudero-Ávila *et al.*, 2021; Goos and Beswick, 2021; Masingila *et al.*, 2018; Muir *et al.*, 2021; Superfine *et al.*, 2020; Tzur, 2001). However, the number of research on the knowledge and the development of MTEs is limited, although it is recently increasing (Beswick and Goos, 2018; Chapman, 2021; Masingila *et al.*, 2018; Superfine *et al.*, 2020). Particularly, Superfine, Prasad, Welder, Olanoff, and Eubanks-Turner (2020) conducted a study in which they elaborated on the existing models of mathematical knowledge for teaching teachers (MKTT) to construct a way to support the learning process of elementary preservice teachers. They considered that elementary preservice teachers often have difficulty in re-designing their mathematics knowledge to teach. According to their results, supporting elementary preservice teachers' learning and re-learning, and constructing mathematical tasks and pedagogical applications are the work of MTEs. Moreover, the study of Masingila, Olanoff, and Kimani (2018) used problem-solving to help preservice teachers to relearn their own mathematical knowledge and understand the connections between the interrelated knowledge. By doing so, they aimed preservice teachers to reconceptualize their knowledge as a teaching object.

Aforementioned discussion points that MTEs are the responsible people for the education of preservice mathematics teachers. Besides, MTEs knowledge includes knowing mathematics, teaching mathematics, student misconceptions, and knowing knowledge to teach mathematics. It is important to note that knowing the entire undergraduate level mathematical content is not expected from the MTEs; however, experiences on teaching or doing research in the specific mathematical topics might be an indicator for the subject matter knowledge of MTEs (Cetinkaya *et al.*, 2017). Therefore, in this study, I will scrutinize MTEs' ways of thinking about the required knowledge of preservice teachers to comprehend geometric transformations. As previous research has pointed, geometric transformations is a fundamental concept about which both students from different grade levels and preservice teachers have difficulties and misconceptions. To help students and preservice teachers to overcome their difficulties and misconceptions about geometric transformations, adjusting preservice teachers' education about geometric transformations may be beneficial, because preservice mathematics teachers are the future schoolteachers who will teach geometric transformations to students. In particular, I will search for the answers to questions like which concepts are crucial for the understanding of geometric transformations by middle school or high school preservice teachers, to what extent these concepts need to be known by them, how they need to think about these concepts to comprehend transformations, and how to teach geometric transformations so that learners' difficulties as previous research points to might have been overcome. Examining the thinking ways of MTEs might provide valuable information about how mathematics teacher education with respect can teaching geometric transformations be designed.

### 2.4. Conceptual Framework

There are different theoretical lenses examining the MTEs knowledge (Chick and Beswick, 2018; Escudero-Ávila *et al.*, 2021; Leikin, 2021; Muir *et al.*, 2021; Superfine *et al.*, 2020). Albeit different, all these lenses have been developed by considering already existing frameworks on knowledge for teaching (e.g., Ball *et al.*, 2008; Schulman, 1986).

In particular, Chick and Beswick (2018) focused on how the MTEs knowledge might be depicted in the act of MTE's teaching to preservice teachers. Therefore, they created a framework on pedagogical content knowledge of MTEs based on Chick's (2007) idea of PCK for school mathematics teacher (SMTPCK). In addition, Muir *et al.* (2021) used Knowledge Quartet framework to analyze MTEs knowledge in the context of joint teaching of an MTE and a primary school teacher to preservice teachers focusing on algebraic thinking and measurement. Moreover, considering that MTEs are responsible for the professional development of mathematics teachers, Escudero-Ávila *et al.* (2021) referred to Shulman's (1986) PCK while examining MTEs' knowledge. They argued that MTEs should know the theories of teaching, teaching mathematics, and learning mathematics. They should also be familiar with the application of the theories, the teaching strategies, educational resources, learning standards and hypothetical learning trajectories. MTEs should also have some specific knowledge about the mathematics teaching practice and skills. Furthermore, Leikin (2021) framed MTEs knowledge by converting the mathematical potential and mathematical challenges of students to the professional potential and challenges of a mathematics teacher. She analyzed this framework in an act of teaching. She contended that MTEs' goal is to improve the proficiency of mathematics teachers and the realization of their professional potentials. Therefore, she emphasized that the features of the knowledge of MTEs include both students' and teachers' mathematical potential and students' and teachers' challenges for the content. Hence, by considering the challenges of mathematics teachers, MTEs need to support developments of mathematics teachers about comprehending the content of the mathematical potential and challenges of students in detail. Besides, Superfine *et al.* (2020) referred to MTEs knowledge as mathematical knowledge for teaching teachers (MKTT). They based their research on the mathematical knowledge for teaching (Ball *et al.*, 2008), and the main two components of which consisted of MTE subject matter knowledge (MTE-SMK) and MTE pedagogical content knowledge (MTE-PCK). They included Ball and her colleagues' notion of mathematical knowledge for teaching into MTE's knowledge of mathematical knowledge for teaching in MTE-SMK. They expanded the idea of learning and relearning (Zazkis, 2011) and they used this idea as a base for MTE-PCK. Finally, Ferretti, Martignone, and Rodríguez-Muñiz (2021) created a model[1] by using Mathematics Teacher Specialized Knowledge model of Carrillo-Yañez *et al.* (2018). They focused on pedagogical content knowledge on the specialized knowledge of MTEs model.

In the existing research (Muir *et al.*, 2021; Superfine *et al.*, 2020), if researchers included a specific mathematics context in their study, they used the context as a tool to provide in-depth explanations for the knowledge of MTEs. In particular, Muir *et al.* (2021) investigated the knowledge of MTEs in the act of teaching algebraic thinking and measurement. They observed the lesson of an MTE and did interviews with the MTE. According to their data, they analyzed the knowledge of MTEs regardless of

[1] In this study, the terminologies model and framework was used attending to the researchers use. I have left the terminologies "model" and "framework" as used in the original cited studies.

the content of MTE's course. Superfine *et al.* (2020) also explored the knowledge of MTEs in the teaching of subtraction, fraction comparisons, and growing visual patterns. They analyzed the students' works and MTEs' reflections to their students' works. They also analyzed the data to investigate the knowledge of MTEs to support preservice teachers to relearn the mathematics regardless of the mathematical content. Therefore, researchers in these studies used mathematical content to set light on MTEs' knowledge that is not specific to the used mathematical content. In this study, my aim is to investigate the knowledge of MTEs that is specific to teaching geometric transformations. Hence, my study extends the existing research differentiating the topic it is focusing on.

In sum, aforementioned research on MTEs' knowledge (Chick and Beswick, 2018; Escudero-Ávila *et al.*, 2021; Leikin, 2021; Muir *et al.*, 2021; Superfine *et al.*, 2020) have utilized pre-existing frameworks regarding the mathematical knowledge for teaching (Ball *et al.*, 2008; Rowland *et al.*, 2009; Shulman, 1986) and provided valuable results on the different aspects of the knowledge of MTEs. Similarly, my study will be based on Carrillo-Yañez *et al.*'s (2018) framework of mathematics teachers specialized knowledge. Since mathematics teachers specialized knowledge model was built on Ball *et al.* (2008)'s mathematical knowledge for teaching model, in the following subsections, I first explain that. Then, I clarify mathematics teachers specialized knowledge model (Carrillo-Yañez *et al.*, 2018).

### 2.4.1. Mathematical Knowledge for Teaching

Ball, Thames and Phelps (2008) conducted an empirical study by building on the notion of pedagogical content knowledge (Shulman, 1986) to develop the idea of mathematical knowledge for teaching. According to Ball *et al.* (2008), the term pedagogical content knowledge in the Shulman's (1986) study was "remained underdeveloped" (p.389), so they created a practice-based model to specify the meaning of pedagogical content knowledge and develop content knowledge of teaching. They concluded that Shulman's content knowledge has subdomains as common content knowledge, horizon content knowledge, and specialized content knowledge (Figure 2.1). Also, pedagogical content knowledge in the study of Shulman (1986) has subdomains as knowledge of content and students and knowledge of content and teaching. They categorized the curricular knowledge in the study of Shulman (1986) under pedagogical content knowledge as knowledge of content and curriculum.

| Subject Matter Knowledge | Pedagogical Content |
|---|---|
| Common content knowledge | Knowledge of content and students |
| Specialized content knowledge | Knowledge of content and teaching |
| Horizon content knowledge | Knowledge of content and curriculum |

Figure 2.1. Domains of Mathematical Knowledge for Teaching (constructed by Ball et al., 2008, p.403 and drawn by the researcher).

Ball and her colleagues (2008) defined common content knowledge (CCK) as mathematical knowledge and skills to do procedural calculations and solve mathematical problems. So, they considered common content knowledge as the general mathematical knowledge that is not unique to teaching whereas, they defined specialized content knowledge as the mathematical knowledge and skills that are only used in teaching. For example, realizing the patterns in student misconceptions, knowing and leading students' way of the usage of mathematical language, and knowing to choose, create or use mathematical representations are the components of specialized content knowledge of teachers. Additionally, they referred horizon knowledge as the knowledge about the relations of the concepts and topics of mathematics. They have specified that such knowledge provides teachers with the ability to order the mathematical concepts in instructional designs or to decide how and when they can teach the concepts. Regarding pedagogical content knowledge, they have considered knowledge of content and students as also a combination of knowledge of students and knowledge about mathematics. They have stated that this knowledge expresses knowing the mathematical levels and possible misconceptions of students. For instance, when choosing a task, a teacher needs to know about whether the students might find the task challenging or easy. Knowledge of content and teaching, on the other hand, includes the knowledge of instructional decisions. For example, they have stated that knowledge of content and teaching involves the order of tasks and questions that require some specific knowledge by students. Finally, they have categorized the knowledge of content and curriculum as the knowledge of all possible programs for a subject or topic for each grade and student level. Also, they have stated that it includes knowing the variety of materials that can be used to apply for such programs.

### 2.4.2. Mathematics Teacher's Specialized Knowledge Model

Mathematics teacher's specialized knowledge model was developed by Carrillo-Yañez *et al.* (2018). When developing the model, researchers referred to the notions of pedagogical content knowledge by Shulman (1986) and mathematical knowledge for teaching by Ball *et al.*.(2008). Though, for providing a rationale on the need of a new model, they stated that Ball *et al.* (2008) and Rowland *et al.* (2009) models for mathematical knowledge for teaching "focus their attention on practice as carried out in class, ignoring the knowledge that teachers might bring into play when carrying out any other kind of activity as a teacher" (p.3). Moreover, they pointed to the difficulty in differentiating the common content knowledge and the specialized content knowledge in the MKT framework (Ball *et al.*., 2008). Thus, defining and building the subdomains of the mathematical knowledge of MTSK framework based on mathematics itself, they aimed to point to the importance of knowledge that teachers might bring into play when carrying out any other kind of activity as teachers and also aimed to overcome the difficulty in differentiating common content knowledge and specialized content knowledge. They have stated that this model intends to provide a holistic approach to the specialized nature of teachers' knowledge. Therefore, based on the aforementioned reasons, I base my research on the MTSK model instead of the MKT model.

The MTSK model consist of two wide areas of knowledge: mathematical knowledge and pedagogical content knowledge. Researchers also included beliefs on mathematics and beliefs on mathematics teaching and learning as core in their model.

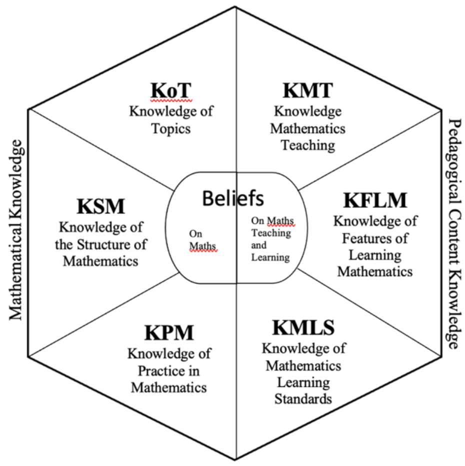


Figure 2.2. The Mathematics Teacher's Specialized Knowledge Model (constructed by Carrillo *et al.*, 2018, p.241 and drawn by the researcher).

Carrillo-Yañez *et al.* (2018) defined mathematical knowledge as a network of structured systematic knowledge. They have argued that knowing the rules and features of this systematic knowledge and the connections within the knowledge provide teachers with opportunity to teach mathematics in a connected way. They divided mathematical knowledge into three subdomains: *knowledge of topics, knowledge of the structure of mathematics* and *knowledge of practices in mathematics. Knowledge of topics* includes the knowledge of theorems, procedures, definitions, and facts of the mathematical contents, knowledge of connections within the contents, and knowledge about representing the contents. Teachers also need to know the types of different problems that the content can be applied. For example, for the concept of rectangles, teachers should know varying definitions of rectangles, its properties, and different examples and problems for rectangles. The knowledge of connections between the mathematical concepts are classified as *knowledge of the structure of mathematics.* These connections can be established by connecting the content with a previous easier content (simplification) or by connecting the content with later content (complexity). The last subdomain of mathematical knowledge is knowledge of practices in mathematics. The researchers have stated that the focus in the subdomain is on mathematics rather than teaching during the teaching practices and classroom practices. This subdomain has the teacher's knowledge of "...demonstrating, justifying, defining, making deductions and inductions, giving examples and understanding the role of counterexamples" (Carrillo-Yañez *et al.* 2018, p.244).

Carrillo-Yañez *et al.* (2018) defined the pedagogical content knowledge again based on mathematics. Pedagogical content knowledge has three subdomains as well: *knowledge of features of learning mathematics, knowledge of mathematics teaching*, and *knowledge of mathematics learning standards.* The first subdomain involves teachers' knowledge of student misconceptions, errors, and difficulties. Besides, teachers should know students' thinking ways on activities and tasks regarding the mathematics of interest. They also should know the different learning styles and the theories about cognitive developments' of students. Next, the knowledge of mathematics teaching is the subdomain that contains the personal experiences of teachers, their reflections, and literatures about mathematics education. For example, knowing the materials such as textbooks, technological resources, and so on is a component of this subdomain. Also, the information obtained from research literature like teaching strategies and techniques for different mathematical concepts, possible activities that can be used to teach these concepts is a part of knowledge that teachers need to know. The last subdomain is the knowledge of mathematical learning standards. This subdomain is defined as the knowledge of mathematics curriculum for different grade levels. It includes the knowledge of expected learning outcomes, knowledge of desired development of students after they are taught according to these learning standards, and sequencing of contents in this instrument.

In this study, I will be using Carrillo-Yañez *et al.*'s (2018) MTSK model for the following reasons. First, there are different theoretical lenses used in examining the MTEs knowledge (Chick and Beswick, 2018; Escudero-Ávila *et al.*, 2021; Leikin, 2021; Muir *et al.*, 2021; Superfine *et al.*, 2020). Though, these studies have utilized pre-existing frameworks regarding the mathematical knowledge for teaching (Ball *et al.*, 2008; Rowland *et al.*, 2009; Shulman, 1986) as there is a tight connection between the MTEs knowledge and teachers knowledge. In addition, researchers pointed out that MTEs are the people responsible for assisting preservice and practicing teachers to develop their mathematical knowledge for teaching (Chapman, 2021; Chick and Beswick, 2018; Flores *et al.*, 2013; Jaworski, 2008; Leikin, 2021; Masingila *et al.*, 2018; Muir *et al.*, 2021). Taking into consideration, such argument in this study in a similar vein I will be utilizing the MTSK model to characterize the MTEs' specialized knowledge of geometric transformations by focusing and elaborating on the key ideas and possible difficulties/misconceptions the learners (i.e. preservice teachers and students) at different grade levels might have. Secondly, previous research on MTE's knowledge mostly focused on MTEs' knowledge in the act of teaching to preservice teachers. Carrillo et al argued that there is a need to go beyond describing and interpreting teachers' knowledge by also focusing on their theoretical knowledge about their practices. They particularly stated

> "Note that the object of analysis in this model is not the mathematical knowledge used by teachers to carry out their work, but rather the assessment of the mathematical knowledge needed to do so (e.g., Ball, Hill, and Bass, 2005). Hence, the MKT model, and the work of Rowland *et al.* (2009), focus their attention on practice as carried out in class, ignoring the knowledge that teachers might bring into play when carrying out any other kind of activity as a teacher". (p. 238).

Third, Carrillo-Yañez *et al.* (2018) further highlighted, "Given that our approach to teacher observation promotes reflection on practice with the teachers, our goal in employing the MTSK model to analyze teachers' specialized knowledge is one of comprehension and interpretation rather than evaluation" (p. 237). Therefore, in this study, I specifically aim to analyze MTE's specialized knowledge for mathematics teaching in the context of geometric transformations by focusing on their practices, including their reflections on doing research on teaching and learning of geometric transformations, their reflections on reading related research articles, and their reflections on their experience of teaching and learning geometric transformations.

# 3. METHODOLOGY

In this section, I explain the method of this study in detail. I identify research design, participant information, data collection and data analysis for this study.

## 3.1. Research Design and Data Collection

Design of this study is qualitative research design (Merriam and Tisdell, 2016). In qualitative research design, researchers focus on the ways of identifying participants' experiences, the ways of creating their worlds, and the meanings that they attribute to their experiences (Merriam and Tisdell, 2016). The focus of my study is on how the participants make sense of geometric transformations and how they describe their experiences in teaching geometrical transformations or their research results in teaching geometrical transformations. Therefore, I did the study using qualitative research design. I collected the data by conducting structured interviews. In the structured interviews the interview questions are predetermined (Merriam and Tisdell, 2016) (See Appendix A). However, to understand the participants' perspectives more deeply, I will also ask follow-up questions.

## 3.2. Participants

In this study, I researched how mathematics teacher educators who are faculty members in different universities from different countries, such as Turkey and the USA, conceptualize and interpret teaching and learning of geometric transformations. Therefore, the participants of this study are mathematics teacher educators. Since I was interested in understanding and gaining insight into the phenomenon of interest, I used purposive sampling as a method of choice (Merriam and Tisdell, 2016).

Teacher educators can be described as people who train both preservice teachers and preservice teacher educators and do research about teaching and learning

(Cetinkaya *et al.*, 2017). In particular, mathematics teacher educators learn mathematics when they are students, and then they learn how to teach mathematics as a teacher. When they become a mathematics teacher educator, they learn how to train mathematics teachers and how to train mathematics teacher educators (Tzur, 2001). Mathematics teacher educators' development occurs by being involved in all stages of learning and teaching mathematics so, they are expected to know the appropriate subject matter knowledge of teaching school students and teaching preservice mathematics teachers.

Even though the mathematics teacher educators do not need to have the knowledge of undergrad level mathematical contents entirely, teaching or researching experiences in the specific mathematical content can be seen as an indicator for their subject matter knowledge (Cetinkaya *et al.*, 2017). Therefore, purposeful sampling was utilized by selecting specifically be the mathematics teacher educators who have taught courses on geometric transformations in their universities or have done research about learning and teaching of geometric transformations.

In this paragraph, I have provided detailed participant information so as not to reveal the identity of the participants. One of the participants from the USA gave two different courses related to geometric transformations to middle school preservice teachers more than one time and did more than one research related to geometric transformations. The participants of the research were middle school preservice teachers. The other participant from the USA gave several courses related to geometric transformations to the preservice high school teachers and high school students. Also, this participant has several publications that have participants as preservice high school teachers and high school students. The last participant was from Turkey and offered more than one lesson in geometric transformations to preservice high school teachers and did research on geometric transformations.

### 3.2.1. Interview and Interview Protocol

The interview protocol in this study consists of ten open-ended questions. These questions are written by examining the existing research that analyzed the understanding and conceptualizing geometric transformations of middle school and high school students and preservice teachers (Aktaş and Ünlü, 2017; Avcu and Çetinkaya, 2019; Emre-Akdoğan *et al.*, 2018; Guven, 2012; Hollebrands, 2003, 2004, 2007; Hollebrands *et al.*, 2021; Sünker and Zembat, 2012; Uygun, 2020; Xistouri *et al.*, 2014; Yanik, 2011, 2014; Yanik and Flores, 2009). First, I examined the existing research and noted the key concepts that the students were usually struggling with. Next, I wrote the questions, and then a colleague and I read and examined the questions. Re-examining the questions once a week for four weeks period, we finally came to a consensus. Then the questions were sent to an expert view. After that we finalized the interview questions according to feedback of the experts. In the following paragraphs, I explain the questions and their relations to the MTSK model in detail (See Table 3.1).

The interviews lasted approximately one hour for each participant. The interviews were conducted synchronously as one-to-one online interviews via Zoom, a video communication tool. Interviews were videotaped. The data source was the transcripts of the video recordings of the participants.

As the Table 3.1 shows, two questions are related to the subdomain knowledge of topics. Since the knowledge of topics subdomain identify what and how mathematics teachers know the contents they teach, with these questions MTEs are expected to explain the procedures, definitions, properties, and differences of geometric transformations. Four questions are related with the *knowledge of the structure of mathematics* subdomain that focusses on inter-conceptual connections. These include connections between mathematical items or connections to topics from other disciplines to increase or decrease the complexity of the mathematical ideas. For the *knowledge of practices in mathematics* subdomain, there is not any specific question but it is related to "knowing about demonstrating, justifying, defining, making deductions and inductions, giving examples and understanding the role of counterexamples". (Carrillo-Yañez *et al.*, 2018, p.9). Therefore, when explaining the questions, the codes about this subdomain may emerge. Five questions are related with the *knowledge of features of learning mathematics*. The knowledge of difficulties and strengths in learning geometric transformations are expected to be examined via these questions. One question corresponds to the *knowledge of mathematics teaching*. This subdomain includes the knowledge of strategies, activities, and teachings with knowing their pros and cons. Moreover, one question is written to reach data about the knowledge of *mathematics learning standards*. With this question, the knowledge of sequencing the related contents with geometric transformations can be obtained.

Table 3.1. Summary of Questions.

| **Subdomain of the MTSK model** | **Question** |
|---|---|
| Knowledge of Topics | "How would you define geometric transformations?" |
| | "What are the differences and similarities between geometric transformations?" |

Table 3.1. Summary of Questions. (cont.)

| **Subdomain of the MTSK model** | **Question** |
|---|---|
| Knowledge of the Structure of Mathematics | In the mathematics education literature, it is stated that learners think of the domain in three different conceptions (corners of the given shape, the shape itself, the plane). What is the information that should be known for the transitions between these three different thinking? |
| | What is the role of parameters in the definition? Can you explain? |
| | "What prior knowledge does one need to understand geometric transformations as functions?" |

Table 3.1. Summary of Questions. (cont.)

| **Subdomain of the MTSK model** | **Question** |
|---|---|
| Knowledge of Features of Learning Mathematics | "Mathematics education literature states that "knowing that the domain and range of geometric transformations is R2" is an important mathematical knowledge for learners. How does the person who knows that the domain is R2 thinks, can you explain?" |
| | "What kind of difficulties do you think students may have about the domain and the range?" |
| | "What kind of difficulties do you think students may have about the parameters?" |
| | "How does people that conceptualize geometric transformations as functions think so that they know it?" |
| | "Why learning geometric transformations is important?" |

Table 3.1. Summary of Questions. (cont.)

| **Subdomain of the MTSK model** | **Question** |
|---|---|
| Knowledge of Mathematics Teaching | "What should prospective teachers or practicing teachers be aware of when teaching this subject, can you explain with the reasons?" |
| Knowledge of Mathematics Learning Standards | "What should be the minimum and maximum information required for learning geometric transformation?" |

## 3.3. Data Analysis

The focus of this study is on characterizing mathematics teacher educators' specialized knowledge in teaching geometric transformations by elaborating on the key ideas and possible difficulties and misconceptions learners might have. So, I concentrated on mathematics teacher educators' current conceptions of teaching and learning geometric transformations.

First of all, all interviews were videotaped and transcribed. I read the transcripts of each participant's interview line-by-line and several times to look for MTEs' explanations regarding the teaching and learning geometric transformations. I used some predetermined categories in the mathematics teachers specialized knowledge model (Carrillo-Yañez *et al.*, 2018) to create initial codes. For this, I used coded analysis

(Clement, 2000). In this analysis, the researchers first code the transcript after formulating criteria for identifying the phenomenon. Then they note the places where they encounter the phenomenon in the transcript. The focus of coded analysis is on the fixed observation categories (Clement, 2000). In this respect, I analyzed the data to reach a conclusion from observation categories with reference to the framework of the mathematics teachers specialized knowledge (Carrillo-Yañez *et al.*, 2018).

The categories are given in Carrillo-Yañez *et al.* (2018) study separately. I constructed two tables to demonstrate the mathematical knowledge and the pedagogical content knowledge. Although the categories for *knowledge of practices in mathematics* sub-domain were not given in a table, I created the categories from the written text in Carrillo-Yañez *et al.* (2018) study (See Table 3.2 and Table 3.3).

Table 3.2. Mathematical Knowledge Domain of MTSK model (Carrillo *et al.*, 2018).

| **MATHEMATICAL KNOWLEDGE** | |
|---|---|
| **Knowledge of Topics (KoT)** | |
| **Categories** | Procedures |
| | Definitions, properties, and foundations (intra-conceptual connections) |
| | Registers of representation |
| | Phenomenology and applications |
| **Knowledge of Structure of Mathematics (KSM)** | |
| **Categories** | Connections based on simplification |
| | Connections based on increased complexity |
| | Auxiliary connections |
| | Transverse connections |

Table 3.2. Mathematical Knowledge Domain of MTSK model (Carrillo *et al.*, 2018). (cont.)

| **Knowledge of Practices in Mathematics (KPM)** | |
|---|---|
| **Categories** | Knowing about demonstrating |
| | Knowing about justifying |
| | Knowing about defining |
| | Making deductions and inductions |
| | Giving examples |
| | Understanding the role of counterexamples |
| | Understanding of the logic underpinning of the mathematical practices |
| | Knowledge about how mathematics is developed beyond any particular concept |
| | Knowledge of the type of proof for testing the truth-value of a proposition |
| | Knowing how to explore and generate new knowledge in mathematics |

Table 3.3. Pedagogical Content Knowledge Domain of MTSK model (Carrillo *et al.*, 2018).

| **PEDAGOGICAL CONTENT KNOWLEDGE** | |
|---|---|
| **Knowledge of Features of Learning Mathematics (KFLM)** | |
| **Categories** | Theories of mathematical learning |
| | Strengths and weakness in learning mathematics |
| | Ways pupils interact with mathematical content |
| | Emotional aspects of learning mathematics |
| **Knowledge of Mathematics Teaching (KMT)** | |
| **Categories** | Theories of mathematics teaching |
| | Teaching resources (Physical and digital) |
| | Strategies, techniques, tasks and examples |
| **Knowledge of Mathematics Learning Standards (KMLS)** | |
| **Categories** | Expected learning outcomes |
| | Expected level of conceptual or procedural development |
| | Sequencing of topics |

For the analysis, I also engaged in open coding to determine further sub-codes regarding mathematics teachers specialized knowledge in the domain of geometric transformations. That is, I used the constant comparative analysis (Corbin and Strauss, 2008) to look for the similarities and differences among the responses of the participants. I conducted the analysis as follows.

First, I read and examined each participant's whole transcript separately. Then I categorized and analyze the responses of each participant using the Carrillo-Yañez *et al.*'s (2018) framework. I turned back and created additional codes inside the subcategories of the framework to be able to analyze MTE's knowledge based on MTSK model in terms of geometric transformations. Besides, I also turned back and read the same transcripts several times in case the existing codes do not fit the data obtained from the other participants. This way, I also created new additional codes. After completing the analysis of the transcript of each participant, to determine the dominant conceptions of participants, I identified the frequency of the use of relevant explanations during the interviews. By using these determined dominant explanations that might emerge from the data, I clustered the data from different MTEs that shows a particular explanation. Then, I wrote the narrative depicting different codes with evidence from data.

### 3.4. Trustworthiness: Validity and Reliability

In this study, I provided trustworthiness by examining appropriate strategies for dealing with internal validity (i.e. credibility), reliability (i.e. consistency/ dependability) and external validity (i.e. transferability). In this subsection, I declare how these issues are implemented in this study.

Internal validity (i.e., credibility) deals with how much findings match with reality (Merriam and Tisdell, 2015). To verify internal validity, I used the suggestion of Merriam and Tisdell (2015) that is the use of expert evaluation. After I created the interview protocol with my advisor, three experts from mathematics education department who study teaching geometric transformations and/or teacher education examined the questions to evaluate whether the questions were appropriate to investigate research questions. Also, they examined the questions by considering if matching of the questions and the subcategories of the framework were appropriate. Then, we concluded the interview protocol by considering their feedback and suggestions. I also applied a pilot study to examine if the questions are understandable by a participant.

Regarding internal validity, I first wrote transcript of the data for all participants. Then, I and my advisor worked together and analyzed the half of the data that came from the first participant. After that I finished analyzing the data of the first participant. My advisor read my analysis and we came to a consensus by discussing where we thought differently in coding. Then, I followed the similar processes for analyzing the data of the other participants. One of the data was collected in Turkish. I wrote the transcript of the data and analyzed the data in the original language. I translated the results that provided evidence into English. My advisor and a bilingual person checked my translation. Then, I finalized the translation of the findings based on their feedback.

To provide the reliability, the consistency of the collected data with the reality needs to be ensured (Merriam and Tisdell, 2015). Therefore, in this study, although the data source is only an interview, I conducted the interview to more than one participant, so I provided a triangulation. Finally, external validity (transferability) (Merriam and Tisdell, 2015, p.257) is whether the study can be generalized for elsewhere. To support transferability, Merriam and Tisdell (2015) stated that there need to use a rich and thick description. They defined thick description as “a description of the setting and participants of the study, as well as a detailed description of the findings with adequate evidence presented in the form of quotes from participant interviews, field notes, and documents” (p.257). Therefore, to increase the likelihood of the findings’ transferability to other applications, in this study, I ensured the detailed descriptions of the participants as well as the findings with excerpts from the participants. I also provided a detailed description of the analysis process.

# 4. RESULTS

In this section, I share findings relevant to the mathematics teacher educators' knowledge about teaching and learning geometric transformations. I demonstrate the key ideas that MTEs held for the most frequently seen subcategories. I share findings for each participant, MTE1, MTE2, and MTE3, separately and respectively to depict the knowledge of each participant clearly. In the following sub-sections, first I share the frequency of the subcategories of MTEs' mathematical knowledge and pedagogical content knowledge (See Figures 3 and 4 for MTE1, Figures 5 and 6 for MTE2, and Figures 7 and 8 for MTE3). Then, to be able to depict the data from the mathematical knowledge and the pedagogical content knowledge for each MTE coherently and relatedly, I share the data together regarding different subcategories under both mathematical and pedagogical content knowledge.

## 4.1. Knowledge of MTE1

Figure 4.1 points to the frequency of the subcategories for MTE1's mathematical knowledge. The sizes of the rectangles in the table change according to the frequencies of the subcategories: the bigger the rectangle is, the greater the number of frequencies. In particular, regarding knowledge of topics there were 18 codes in the data. Twelve of them were on *definitions, properties, and foundations.* Also, 13 subcategories emerged from MTE1's data for the category of knowledge of structure of mathematics. MTE1 mentioned *connections based on simplification* more frequently. In addition, 13 subcategories about knowledge of practice in mathematics emerged in the data. Some of the subcategories such as phenomenology and applications under the category of knowledge of topics and *making induction and deduction and understanding the role of counterexamples* under the category of knowledge of practice in mathematics that are not shown on the table did not emerge from the data.

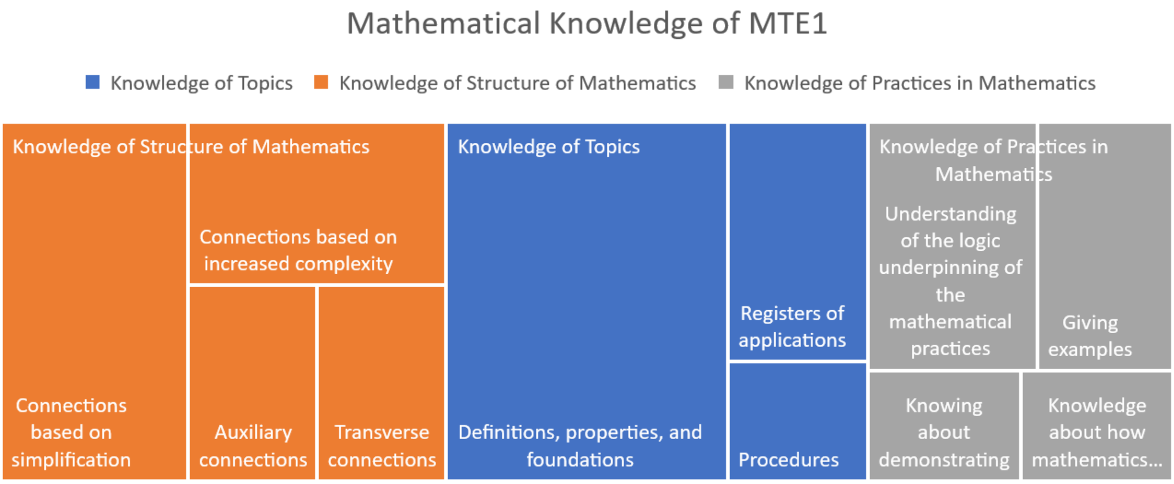


Figure 4.1. Visualize Code Occurrence for MTE1's Mathematical Knowledge.

Also, the frequencies of the subcategories of pedagogical content knowledge are given in Figure 4.2. The categories namely the knowledge of mathematical teaching, the knowledge of features of learning mathematics and the knowledge of mathematics learning standards emerged in the data 47, 43, and 12 times respectively. In particular, the subcategory of strategies, techniques, tasks, and examples was the most frequently seen subcategory of the knowledge of mathematics teaching with a frequency of 39. Regarding the knowledge of features of learning mathematics, *ways pupils interact with mathematical content* was shown 25 times. Besides, with a frequency of 10, *expected level of conceptual or procedural development* was the subcategory that emerged the most.

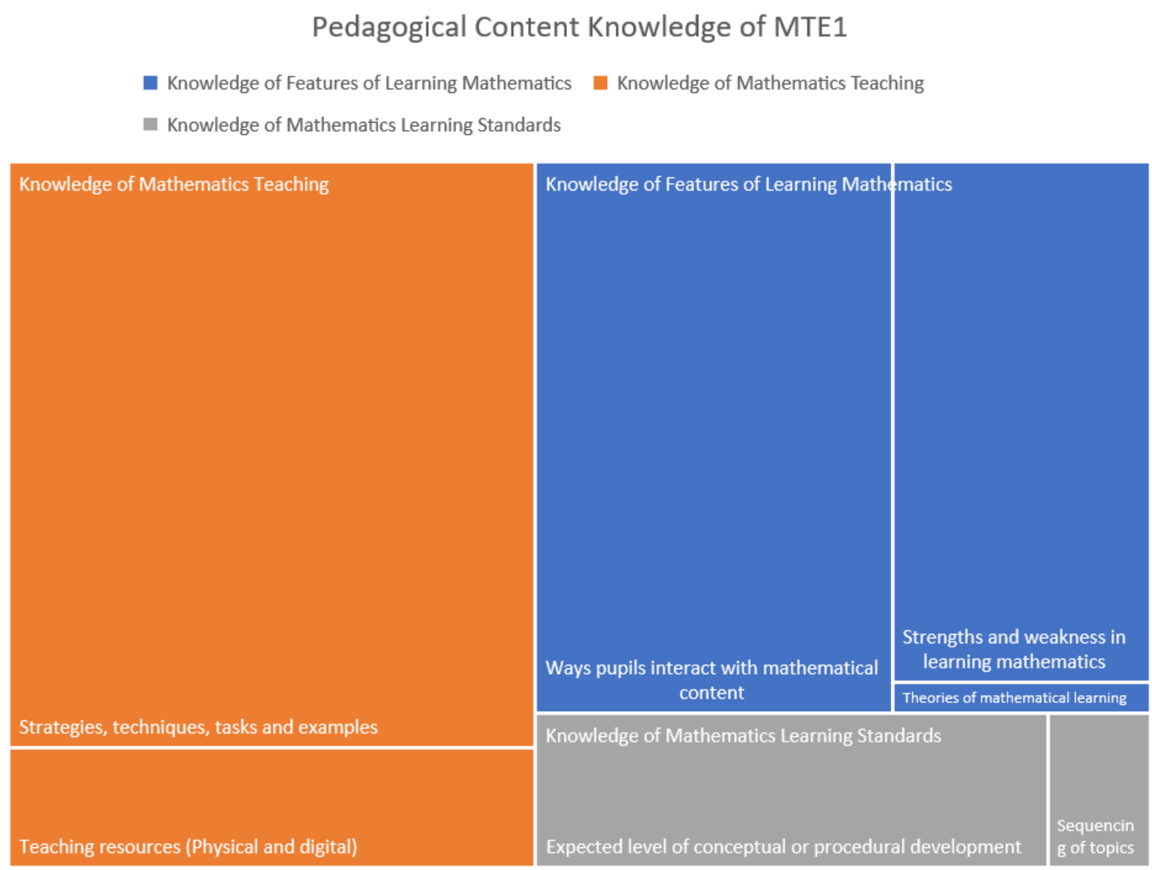


Figure 4.2. Visualize Code Occurrence for MTE1's Pedagogical Content Knowledge.

The following quotation by MTE1 about the definition of geometric transformations reveals data on MTE1's mathematical knowledge. MTE1 explained:

> "There are two different ways to think about geometric transformations. When, you know mathematically, you can think about the geometric transformations as a function mapping a point in $R^2$ to $R^2$. So, mapping the entire plane, so we can define that in that particular way. But in school curriculum you can see that that's not the typical approach. A lot of times we first introduce transformation as motion, so we can talk about, transformation is as a motion, translation as a slide, reflection as a flip and rotation as a turn".

As the data indicates MTE1 mentioned two different ways of defining geometric transformations: as a function from a mathematical point of view and as a motion from the school curricula point of view. While defining geometric transformations as a function she pointed to function as mapping the entire plane, $R^2$ to itself. Besides, she also mentioned how transformation is utilized in school curricula and embraced the idea of transformation as a motion. MTE1 also described the properties of geometric transformations under the subcategory of *definitions, properties, and foundations*:

> "The property. Okay, the distance preserving. so that's a general approach or my understanding of transformation. I map $R^2$ to $R^2$ such that the distance is preserved that's in general. ...Think about translation. There's no fixed point so all the points are moving right. And reflection, all the points on the reflection line fixed. And a rotation there's only one point in the entire plane that is fixed. ...We can also talk about orientations. Orientation of the vertex and orientation of the shapes, so for translation, it doesn't change the orientation of the vertex, and also the orientation of the shape, and for reflection it changes the orientation of the vertex, and for rotation it changes the orientation of the shape but not the orientation of the vertex. So, I mean, there are other properties, right, of translations, for example, or, you know, differences in isometries. So, for example, if you think about translation, all the segments connecting to the corresponding points parallel to each other, and but which is not true for reflection and rotation. right? So, there are many of the other like for reflection, right? And you can think about the line of reflection is actually a perpendicular bisector of the segment connecting to any corresponding pairs of corresponding points, right, the line segments formed by any pairs of corresponding points. And for rotation, you can think about it as the segments formed by connecting corresponding points to the center are congruent right?"

As the excerpt suggests, MTE1 pointed to properties both differentiating and making the geometric transformations such as translation, rotation, and reflection common. For instance, she distinguished those transformations by the number of fixed points: translation with no fixed points, refection with only the fixed points on the reflection line and rotation with only one fixed point. Moreover, according to MTE1 a second distinctive property for transformations is orientation. She mentioned that the orientations of the shapes are the same for a translation, whereas the orientation of the shapes might be changed in a rotation and a reflection. A third distinctive property she mentioned was about the type of segment pointing to the relationship between the points under transformation and their corresponding points in the image set. For example, the segments between the points and their corresponding points under translation are parallel lines, the segments between the points and their corresponding points under reflection are perpendicular bisectors and the segments between the points and the center of rotation and the segments between their corresponding points and the center of rotation are congruent segments. She also mentioned some common property among different transformations such that from her point of view distance preserving between the corresponding points in mapping from $R^2$ to $R^2$ is one of the common properties of geometric transformations.

These are important as the data regarding the pedagogical content knowledge shared below further indicate that MTE1 suggests that for one to understand geometric transformations as a function both as a process and as an object, one needs to understand these properties as the common regularities for each transformation. In particular, under the pedagogical content knowledge, about the category of knowledge of features of learning mathematics, MTE1 talked about the different levels of understandings in the context of geometric transformations. She mentioned:

> "There are different levels of understanding so it's hard to see how students who understand geometric transformations as a function understand transformations in a particular way. I am thinking about in different levels. If students have more like understanding transformation as a process, very much like a function. If you understand functions as an input and output, it's more like a process. Well, if you perform those actions, you will see the effects of those actions. They may understand transformation as a function, but they think about function as an input, output right. But if they think about function as an object, or as a covariation of dependent and independent variables, I think they probably, understand not just the properties of geometric transformations like those properties we mentioned, but also be able to actually reason about the composition of geometric transformations. So, they would be able to see that a reflection, followed by another reflection, could be a translation or a rotation. Right depends on the dependent on the number of fixed points. Right. So, if there's only one fixed point after those 2 transformations, then it's going to be a rotation. If there are no fixed points, then it's going to be a translation. That's how I think, so in general I think there are 2 different ways. Even you understand transformations as a function, but you might also think function as a process or action, input, and output. And then in this case I think it's really difficult for them to understand compositions of transformations. But if they understand function as an object, and if they do understand to geometric transformations as function, they should be able to talk about compositions of transformations and be able to actually reason properties of geometric transformations without actually performing those actions".

The excerpt together with considering how MTE1 defined geometric transformations earlier suggests that MTE1 considers that students might understand geometric transformations as motions or as functions. Particularly, from MTE1's point of view in the excerpt, understanding transformations as functions has three levels: as an action, as a process and as an object. Although not explicit from the data, understanding geometric transformations as functions as an action, might relate to MTE1's earlier defining transformations in school curricula as motion. Yet, MTE1 emphasized: When students have a process level of understanding transformations as functions, this might relate to students' acting with the properties such that the relationship between input and output values result in distance preserving, variance or invariancy in the number of fixed points and the orientation depending on the type of transformation and realizing them as commonalities. On the other hand, from MTE1's perspective, once students have an object level understanding of functions, she considers that students might perform actions without really doing the actions. That is, from MTE1s point of view, students with the understanding of transformations as object are aware of the properties those different transformations hold as common as well as they realize that they can reason with those properties to decide on the type of transformation while thinking about different compositions. All these suggests that although the data does not give the name, APOS theory directly, MTE1 seems to be mentioning APOS theory to explain the different levels of understanding geometric transformations. As well as detailing the data on how MTE1 defined geometric transformations, this also indicates that MTE1's explanations refer to *theories of mathematical learning* under the category of knowledge of features of learning mathematics. In addition, these data pointed that under the subcategory of *ways pupils interact with mathematical content*, MTE1 described ways pupils interact with geometric transformations according to these levels. Particularly, the excerpt above shows that if students have action or process level of understanding on geometric transformations as functions, they might have difficulty especially in understanding compositions of transformations. However, if students conceptualize functions as an object, they may overcome their difficulties with the composition of functions and they may consider performing the actions that require to practice properties of geometric transformations without actually doing them.

Moreover, some explanations by MTE1 exemplifying *connections based on increased complexity*, which is to connect a basic concept with more complex concepts, under the category of knowledge of structure of mathematics further detailed how she considered geometric transformations as functions as process and object. She stated:

> "...seeing transformation as a function is not something new. It's more like an abstract level of understanding of transformation. So, you are basically taking that actions or process. And then try to think about the process self as the object. So, you have, instead of studying this pre-image and image, you are studying the those, the properties of those actions".

As the data suggests, the two thinking ways of geometric transformations, as a process or as an object, that MTE1 mentioned are not two separate ideas from her point of view. Function conception of geometric transformations is more abstract level of understanding geometric transformations as a process. That is, data seems to suggest that MTE1 considers of those actions or processes as the object of thought such that once the learner realizes the properties of those actions as the commonalities of the mapping then the learner might abstract the mapping between pre-image and image as a function at an object level.

As well as pointing to different ways of defining and conceptualizing geometric transformations, under the subcategory of *ways pupils interact with mathematical content*, data from MTE1 further referred to how she considers what might hinder students' understanding of transformations as functions as well as how she considers what possible difficulties students might have and what propitious ways might contribute to their overcoming such difficulties. For the possible difficulties she mentioned three main difficulties: one-to-one correspondence between preimage and image points as a function, guessing the result without performing the action, and shifting the perspective from motion conception to function conception.

About the hindrances, she mentioned:

> "...And once they understand, really think about transformation as an object, you can study its properties. I mean, I think only by that time it makes sense to talk about the domain and range. Because if you are not really talk about

> transformation as a function. Really you cannot talk about a domain and range right just doesn't make any sense for them to talk about".

This excerpt with the previous two excerpts suggests that only after students conceptualize transformations at an object level of understanding functions such that they start thinking of the properties as object of thought, then students might be ready to study the domain and range of those functions.

She also added that students have difficulty in:

> "...understanding the idea of the dependence of the image and the preimage. Every point on the image is actually dependent on the point on the preimage. Even students who have this kind of like the point-by-point conception still don't see that kind of dependence. Some of the technology actually reinforce that kind of misconception".

As the excerpt indicates again together with the earlier data such that students have difficulty in considering one-to-one correspondence between preimage and image points as a function even though they might perform (act) with those values at a point-wise level. This suggests that MTE1 considers that the motion (or action) level, reinforced with the use of technology, might hinder students' understanding of those properties as commonalities to be deduced so that students understand that those properties remain invariant, yielding the relationship between pre-image and image points, creating a function. She further pointed to some other difficulty:

> "They may not be able to actually really perform or predict the results without actually performing those actions. So, they may not understand the idea of undo a transformation to get the pre-image. That's also difficult, I think, student may encounter. It's okay for them to go from one direction, but not be able to actually go back".

Difficulties MTE1 referred to in the previous two excerpts show that students' experience on how they perform geometric transformations might have an effect on their understanding of geometric transformations as functions at an object level. In particular, MTE1 states that trying to actually perform transformations might cause difficulty when undoing the transformations. Besides, data seems to suggest that if technological tools reinforce students' action conceptions, they might have difficulty in transitioning to an abstract level of understanding geometric transformations. She further stated, "in school curriculum that's not the typical approach" and described the most difficult part for students:

> "A lot of times we first introduce transformation is as motion, so we can talk about, transformation is as motion, translation as a slide, reflection as a flip and rotation as a turn so that's kind of like the approach lots of time in the school curriculum took. ...I think it's more sort of relates to intuition or experience. Sometimes, it's well to have those intuitions, but when it comes to develop a formal understanding of transformations, then those intuitions may give you know some difficulties for students. ...It's just like we see those motions in our life. But we only see the effect, right? We focus on the pre-image and image. Not necessary on those actions and it's property. So, we focus more on the impact if I perform this, not we're analyzing those actions ...I think that's the most difficult part of a student to shift to that perspective. Okay, to think about transformations as motion, and then moving to see transformation as a function. If you see transformation as a function, you are more thinking about the transformation as an object you can act on. But if you think about transformation as a motion, it's the action you actually perform, it's not an object. So, I think that's the hard one the student need to actually overcome".

First, data shows that daily life experiences about applications of transformations might be helpful for students at times; yet it is important as educators to realize that students' attention might be only on the effects of those actions rather than the relationship between actions and their effects such that students realize that the relationship between actions and their effects result in some commonality which are the properties. Secondly, as the excerpt shows MTE1 thinks that the transition from motion conception to function conception of geometric transformations is the most difficult for students. Besides according to MTE1, the difficulty in shifting the perspective "...has frequently been documented in the literature. Students see transformation is as motion, not as a function". This explanation MTE1 shares together with her explanations in the excerpt above and the earlier excerpts suggest the following: MTE1 seems to be thinking of students' understanding of geometric transformations in three different levels: one refers to the motion understanding of geometric transformations as emphasized in the high school curriculum where students only act at the action level of APOS such that they just engage in doing the actions of transformations (e.g., in her own words, "translation as a slide, reflection as a flip and rotation as a turn"). The second level is the understanding of geometric transformations as a function at two levels: as a process and as an object within the APOS theory.

MTE1 also compared the difficulties of specific transformations from the students' point of view, and she explained the reason behind this comparison as:

> "It's easy for student to perform the translation but it's really difficult for them to have a really conceptual understanding of translation. ... And I found that in my own research reflection seems to be the easiest one for student to understand, and because there is actually just one defining parameter which is not the case, both for actually for translation, rotation, and the dilation. So, they're more they need to consider but for reflection, it's just a reflection line. For rotation, it's the center and the angle of rotation and I think that's also it's certainly a very difficult one for student to understand. They may have an intuitive sense of the turn, but a lot of times, if the center is outside of the shape, they would have difficulty to perform the rotation, and that's which has already been documented in research and for dilation, it's the center of the dilation and the scale factor".

As the excerpt shows, from MTE1's perspective, reflection is the easiest transformation among reflection, translation, rotation, and dilation because reflection have one parameter, the others such as rotation and dilation have two parameters or two properties for the parameter. She further pointed that students might have an intuitive sense of rotation; yet, once the center of rotation is taken outside of the shape, this also makes the understanding difficult. Since MTE1 referred to the related literature to support her claim, this part of the excerpt further points to *teaching resources* under the category of knowledge of mathematics teaching.

With the following excerpts, MTE1 further explained both the reasons for the possible difficulties students might have in understanding geometric transformations as functions and also pointed to how to overcome such difficulties. These data referred to *strategies, techniques, tasks, and examples*. However, the subcategories of *expected level of conceptual or procedural development* in the category of knowledge of mathematics learning standards and *inter-conceptual connections* and transverse connections under the category of knowledge of structure of mathematics also emerged. She stated:

> "Think about K12 curriculum. We really don't talk functions with two variables. Lots of times it is function with one variable. So, I think this really makes it difficult for them to really think about the function. 'What do you mean by that? What do you mean by transformation as a function?' It doesn't make any sense right the only thing they see in the K12 education is always f(x) equals something it's not f(x, y) equals something. ...It does not just understand function in one variable, but understand a function in two variables, so they probably need to see some examples and so that they know what a function with two variables looks like right, and then move to think about transformations".

First, data indicated that MTE1 considered understanding a function with two variables as an important background knowledge for students to understand geometric transformations as functions. This is important as different geometric transformations such as translation, dilation, reflection, and rotation are functions with two variables. Secondly, data pointed that MTE1 considered that in K12 schools, students do not learn functions with two variables. So, this might be one reason causing difficulty for them in classifying transformations as functions. Therefore, MTE1 stated that encountering functions with two variables via examples might help students to understand function representations of transformations.

Moreover, MTE1 pointed that the hindrances regarding the understanding of geometric transformations as functions might be related to how students conceptualize the plane. She also gave advice for how to overcome such hinderances. She stated:

> "A lot of times we don't spend enough time to really help students to understand figures in the plane and seeing things about figures as part of the plane. So, I think that might create some difficulty for students to really understand. Of course, they would see if you thought about coordinate geometry a lot of times, you would think about shapes that is actually on top of a plane. But if you really think about, plane geometry (Euclidean geometry) you can geometry and you probably don't think about that in a particular way".

Data suggests that because of learning geometric figures in a coordinate plane, students may be mistaken that the figure in the coordinate plane is actually on the top of the plane. According to her, if students learn about the plane in Euclidean geometry, they might overcome this difficulty. Unfortunately, she did not further mention how she envisions what students might know more about the plane geometry so that they might overcome their thinking that figures are on top of the plane rather than being a part of it.

To sum up, MTE1 describe geometric transformations in two different ways, as a function and as a motion under the subcategory of *definitions, properties, and foundations.* Motion definition is in the action level understanding in APOS theory. Referring to *theories of mathematical learning*, she considered function definition as divided into two parts in the context of APOS theory, function as an object and function as a process. She also added that understanding geometric transformations as function is more abstract level for geometric transformations under the subcategory of *connections based on increased complexity. Referring to the strength and weakness of learning mathematical content*, MTE1 further mentioned that shifting the perspective from the motion conception to function conception is the most difficult part of learning geometric transformations. To overcome the difficulties, she also added some advice under the subcategory of *strategies, techniques, tasks, and examples.* Learning functions with two variables might help how to write geometric transformations by using function notation. Using the word 'motion' might cause hinderance when learning geometric transformations. The other strategy to overcome the difficulties in understanding that geometrical figures are a subset of the plane, plane geometry, which is described as Euclidean geometry by MTE1, can be taught more detailly. Lastly, among translation, dilation, reflection, and rotation, she pointed that while rotation being the most difficult transformation, reflection is the easiest one for students to understand.

## 4.2. Knowledge of MTE2

Figure 4.3 shows the frequencies of the mathematical knowledge of MTE2. According to the figure, the knowledge of topics has the most frequent occurrence compared with other types of knowledge subcategory. Also, there are seven subcategories in knowledge of mathematics teaching, six codes in knowledge of structure mathematics and five codes in knowledge of practice in mathematics. The subcategories that do not appear in the figure do not emerge from the data as well.

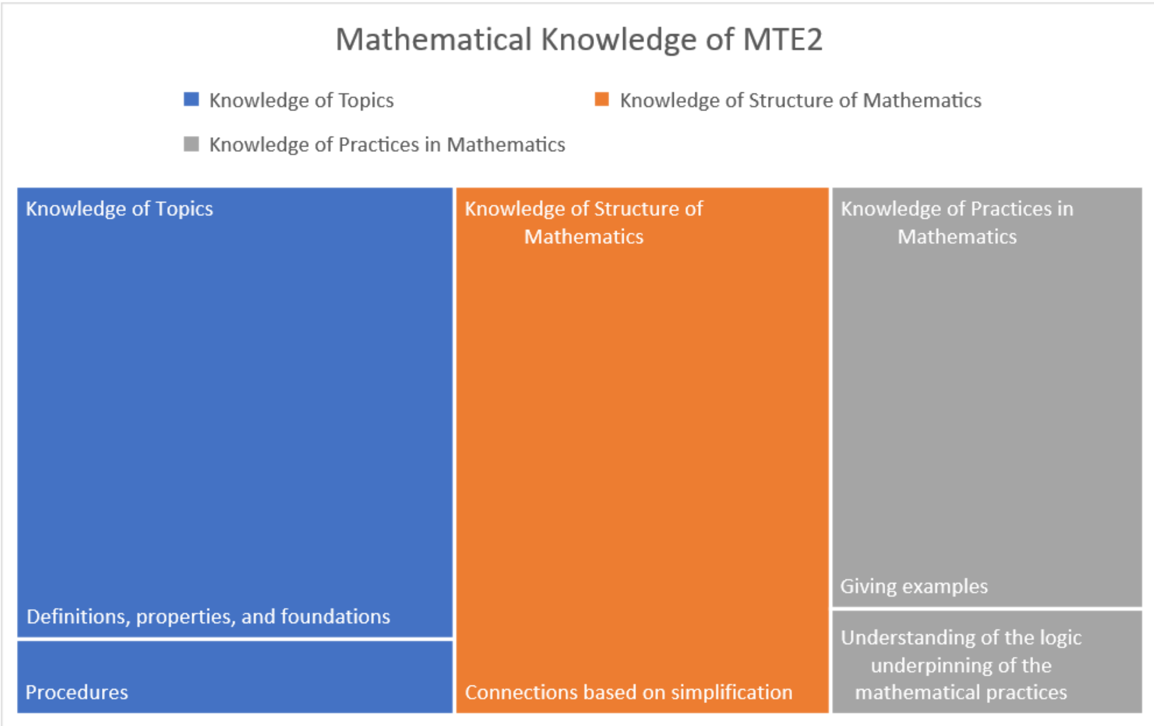


Figure 4.3. Visualize Code Occurrence for MTE2's Mathematical Knowledge.

Figure 4.4 shows the frequencies of the pedagogical content knowledge of MTE2. The most common category of pedagogical content knowledge in the data is the knowledge of mathematics teaching. Among the subcategories of knowledge of mathematics teaching, MTE2 mentioned *strategies, techniques, tasks, and examples* the most. Knowledge of features of learning mathematics is the second most frequently seen category in the data. Also, MTE2 talk about *strengths and weakness in learning mathematics* the most among the subcategories of knowledge of features of learning mathematics.

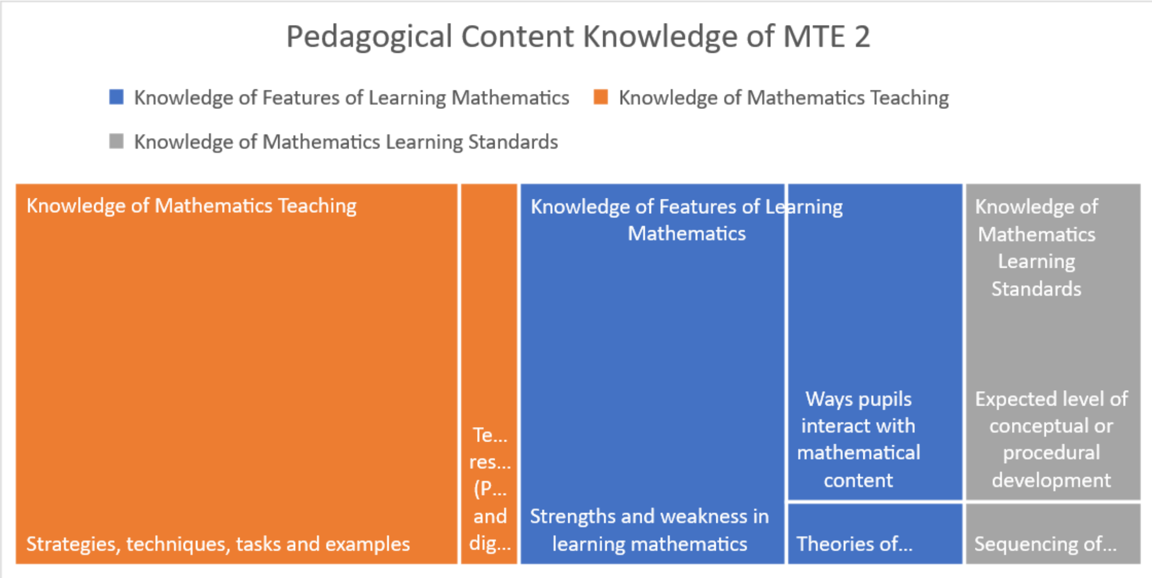


Figure 4.4. Visualize Code Occurrence for MTE2's Mathematical Knowledge.

In what follows, first I share data pointing to MTE2's knowledge of the definitions and properties of geometric transformations under *definitions, properties, and foundations*. Next, I share data indicating the possible difficulties of students with geometric transformations under *strengths and weaknesses in learning mathematics*. Then, I analyze MTE2's knowledge under the *strategies, techniques, tasks, and examples* to point to how she considers overcoming such possible difficulties students might have. If any excerpt related to other categories is reported, I also indicated their subcategories. First, she defined geometric transformations according to two approaches:

> "There are two different approaches when it comes to geometric transformations. One approach defines these transformations as functions or mappings which is an approach mostly accepted by mathematicians. The other approach defines geometric transformations with the motion of geometric figures this approach is widely accepted in curriculum especially in the education of younger children who are at kindergarten, primary school, middle school and even 9th grade. However, transformations are functions from a mathematician's point of view. There is actually a functional relationship. It moves all points from a plane into a different plane along with the geometric object that these points are part of. All points in the plane are actually transformed to the other plane. There is a mapping, a functional relationship in this transformation".

As the excerpt shows MTE2 defines geometric transformations as a function and as a motion under the subcategory of *definitions, properties, and foundations.* She added that mostly mathematicians define transformations as functions and transformations are defined as motion in school curriculum. She also mentioned that geometric transformations are one-to-one mapping of all points in a plane to another plane. In addition, she said that in the curricula from kindergarten to $9^{th}$ grade, geometric transformations are taught to students as motions. This also refers to the subcategory *expected level of conceptual or procedural development* under the knowledge of mathematics learning standards. For these two approaches of geometric transformations, she further pointed:

> "That is not true that considering transformations as functions is correct and considering transformations as motion is wrong. These two ideas do not create a contradiction, on the contrary, they complete each other. To learn transformations algebraically, students need to learn them as functions. However, students also need to conceptualize transformations as motions geometrically. That is, these two views are completing each other. They are not contradictory concepts. There is no such thing as transformations should not be known as motions or they should be known as functions".

Data shows that form MTE2's perspective, the two approaches of geometric transformations are not separate from each other; on the contrary, they are complementary ideas. She stated that to learn transformations algebraically, students need to learn them as functions, and to learn transformations geometrically students need to know them as motions. These all refer to *definitions, properties, and foundations.* She also explained these two approaches by referring *theories of mathematical learning*:

> "A student can consider transformations in two different ways. I will support this using a learning theory. A student may consider a transformation as a motion like as a process. She can think transformations as operations, and she can say that I substitute the points into the function and find the results. But if the student knows transformations as an object, I mean if she conceptualizes transformations like mathematicians conceptualize, how to say that, like we might see transformations as a process or as an object. Or we can think according to procedural and conceptual knowledge. If a student has a conceptual knowledge about geometric transformations, she actually can consider that all points on a geometrical object are moved into another place via a functional relationship".

Although not explicitly stated in the data, data suggested that MTE2 mentioned APOS theory to explain students' thinking ways because she talked about having a process or object level of understanding for geometric transformations. She explained process conception as finding the result of an input value for a function by substituting this value into the function without really knowing the reasons of the processes. Also, she stated that a student with an object conception may think abstractly and comprehensively like a mathematician. A student who conceptually knows geometric transformation may consider that there is a functional relationship between the points and their corresponding points in the image.

Further in the data, three main difficulties of students emerged referring to the subcategory *strength and weakness of learning mathematics* under the category of knowledge of features of learning mathematics. These difficulties MTE2 mentions are related to one-to-one correspondence, set theory, and parameters. About one-to-one correspondence and set theory she stated:

> "Students may not be able to reveal which point in the domain matches which point in the image set. As I said already a little bit ago, geometrically, students take an object (She takes a coaster in her hand.) and move it to another place. However, there is a function that do this motion. What is an element of the domain of this function when we work on R, $R^2$, or $R^3$? For instance, I have given this shape. I said, "what is this point?" (She holds a coaster in a rectangular shape and points to the right top corner) and for instance I did not give this shape on the coordinate system. So, like, that point is not shown in the grid, and I ask "what is this point" the student might have difficulty. They have difficulty in determining the elements of the domain. For instance, is this element (referring to the same corner in the coaster she holds) 1? But, If I give this on the coordinate plane like say (1,2), then she can understand it, I think. Or, if I give $R^2$, $R^3$ again some may still have difficulty in determining which point on the object goes to which point in the image of the function. These difficulties are a little bit related to the relations of sets and their elements and the relations of space and elements".

Data suggested that MTE2 considers that students might have difficulties in understanding the corresponding point pairs. Since students work on objects from real life to observe transformations, they might not conceptualize that these relationships might be defined by functions. Also, according to MTE2, students might have difficulty in determining one element of R, $R^2$, and $R^3$, so they might have difficulty to identify the one-to-one correspondence between the elements of the domain and the elements of the range. In addition, for difficulties in parameters, she pointed:

> "Students may have difficulty in knowing how to define functions via parameters. Yes, here the most critical point is that how to define a function via a parameter and what is the meaning of it geometrically... As the related literature also shows translation is difficult for students because of vectors. Similarly, when students have difficulty understanding the concept of the angle, they also struggle with the angle of the rotation. There are angle of rotation and center of rotation in rotation as critical and we can say students have difficulty because of angle concept... In reflection, there is line of reflection. It can be horizontal or vertical. When it is diagonal students have difficulty in taking the points of the figure perpendicular to the line of reflection. This is again another difficulty... Parameters are taught in high school and universities. Even in high school I'm not so sure if the function is defined by the parameter. It does not seem very critical. The critical thing here is that the mapping conception. There is actually a functional relationship. The meaning of transformations both algebraically and geometrically is also critical. This function can be explained by parameters, but it can also be explained by variables".

Data shows that MTE2 considers that students might have difficulty in determining how to use parameters to define functions and what is the meaning of it geometrically. She gave examples for the difficulties of the students in specific geometric transformations. For instance, since students have difficulty in understanding the concept of vectors, they might also have difficulty in understanding translations. A similar process is valid for rotation as well. Since students have difficulty in understanding angles, they might have difficulty in understanding rotations. However, she stated that since in the school curricula students are not expected to learn parameters, she does not consider critical to use parameters to define transformations. She considers critical to conceptualize the one-to-one correspondence of the preimage and image points (in her own words "the mapping conception"). That is, according to her, the function form of geometric transformations can also be defined by variables without using parameters.

To overcome the difficulties students might have, she suggested some strategies which refers to subcategory *strategies, techniques, tasks, and examples* under knowledge of mathematics teaching. For the conceptualization of $R^2$ she pointed:

> "R cartesian R. That is a two-dimensional space. And what is the element of this space? Students need to know the elements of this space. Showing this via different representations might be good to be conceptualized by students better. Algebraically, an element of $R^2$ is as (x,y), and an element of R is as x. Geometrically if we focus on $R^2$ which is a two-dimensional space, any object on this space does not hang on above the space. For example, this is my two-dimensional plane (she showed an A4 paper.). There is a square (she showed a coaster) on this plane. These does not stand like this (shows the gap between A4 and the coaster.). When I draw the square on the plane, they are not separate. When I keep the coaster on the paper, they become separate, and I can move the coaster. That is, if I move an object that is a part of the plane, the first place of the object will be empty. There need students to know the concept of infinity. Using an A4 paper to represent the space restricts the perceptions of the students about the plane. ...Teachers need to bring the ideas of 'What is $R^2$?,' 'What does it mean to have a shape on $R^2$?' into question".

Data suggest that MTE2 considers that to conceptualize $R^2$, students need to observe it via different representations both algebraically and geometrically. However, when representing $R^2$ geometrically, teachers need to consider the way they represent it. For example, representing plane via an A4 paper might cause students to think planes as quadrilaterals. Also, students need to learn the concept of infinity as the use of A4 paper might hinder the idea that any plane extends indefinitely. In addition, according to MTE2, teachers need to prepare the lessons by aiming to conceptualize the answers to the questions of 'What is $R^2$?' and 'What does it mean to have a shape on $R^2$?' and what its elements are both algebraically and geometrically. For another way to help students learning geometric transformations, she stated:

> "Students are learning geometric transformations as motions until 9th grade. After that they start learning as a function. For instance, this was the case in my thesis too. Teachers said that students already know functions and they learned geometric transformations at the middle school and the 9th grade, so they directly started teaching transformations as functions. However, students did not know that geometric transformations are functions, but they only knew that transformations are motions. There happens a disconnection in transition from motion conception to function conception. There is a need for some applications to resolve that disconnection... How can we understand that students conceptualize as a function? When we think about it, if we prepare action-oriented tasks we cannot understand it. If we prepare tasks with such routine problems involving functions, we still cannot understand it. Students might find the results directly as a process by substituting the given values into the given functions. We need to prepare tasks that help us to understand that students understand transformations conceptually. Only through these tasks we can understand better. Maybe

> if we prepare tasks that includes dialogs and scenarios or tasks that include students' misconceptions... For example, all points on the figures are actually translated with all points in the plane, but for now we are only interested in the transformation of all points on the geometric figures through a function. ... The questions that we prepare for the tasks are very critical to understand students' understandings and I think this is not easy... Very useful teaching situations are designed about resolving the disconnection. If I were to say roughly what comes to my mind right now, I would first mention prior knowledge about the geometric transformations in the curriculum. Then, I would make applications to show that transformations are functions. The most basically, for example, a square is translated by a vector. There is an algebraic equivalent of this translation. Like matching in natural numbers, I can do a study with examples where you map each point individually into the set of images. Because teachers do not teach algebraically if they teach geometrically, and vice versa. That is, I would describe geometric transformations both algebraically and geometrically, and I would separately show how some certain points are moved. The points between the certain points are also moved. For example, we usually show the integers, but reel numbers can also be showed. By using different representation, awareness of the one-to-one correspondence between the points can be provided. We can also benefit from scenarios in which student misconceptions exist in the literature".

As the excerpt pointed MTE2 considers that students learn geometric transformations as a motion until the end of $9^{th}$ grade, which refers to *expected level of conceptual or procedural development* under the knowledge of mathematics learning standards. After that they are expected to start learning transformations as functions. Data suggests that MTE2 considers that students might have difficulty in understanding transformations as functions if teachers do not provide a smooth transition from motion conception to function conception. Therefore, tasks preparation is a crucial step for students' conceptualization of geometric transformations, and it is significant for teachers to understand whether students get the desired level of understanding of the transformations. She further pointed that when designing the tasks, students' alternative conceptions about geometric transformations from the related literature may be used as scenarios. Then teachers might discuss the alternative conceptions of the students to lead students to find the correct thinking ways. Moreover, according to MTE2, different representations need to be used when teaching geometric transformations. For example, geometrically teachers might give a square translated via a vector. However, algebraically it is a function so they might for instance use tabular representation so that geometric transformations might be represented as one-to-one correspondence of the points. More importantly, MTE2 suggests that both algebraically and geometrically the transformations need to be examined by students as teachers focus their attention to the corresponding points as integer values and then the corresponding points in between those integer values. This is important as students also need to realize that not only some points on a shape but also all the points on a shape are transformed via a functional relationship. She also highlighted that after students are aware of the domain and range of the transformations, there is no problem using the given geometric figures for the procedural operations.

In sum, data shows the following: MTE2 described geometric transformations in two different ways, she pointed the students' difficulties and how to overcome these difficulties. First, MTE2 defined geometric transformations in two ways namely as a motion and as a function under the category of *definitions, properties, and foundations*. She said usually in K12 curricula, geometric transformations are taught as a motion, which refers to *expected level of conceptual and procedural development*. However, she also explained that geometric transformations are special functions that moves each point in the plane to another plane and there is a one-to-one correspondence between input and output point pairs. She further stated that to be able to understand transformations comprehensively, students need to learn both of these conceptions of transformations. By referring the mathematical learning theory, namely APOS theory, data suggested that MTE2 classified motion conception in action level and function conception in process level. However, she also pointed out that tasks that want students to only substitute the given values into the given functions of geometric transformations cause students to stay at the process level as these tasks might only yield to students' procedural knowledge. Secondly, according to MTE2, in the learning process, students may encounter some difficulties, which refer to *strength and weakness of learning mathematics*. Some of these difficulties are related to parameters. From her perspective, in general, students might have problems defining functions using parameters. Also, students might have difficulty in understanding the parameters of some certain transformations such as understanding vectors or understanding the concept of angle, so they might encounter difficulty with these transformations. For example, students might have difficulty with translation because they might have difficulty in understanding vectors. Therefore, these difficulties effect the geometric transformations indirectly. MTE2 stated that students might have difficulty in understanding the corresponding point pairs. Under the subcategory of *strategies, techniques, tasks, and examples*, she further explained that proper tasks might be helpful to teach transformations as an object and to understand whether students understand geometric transformations as functions. The disconnectedness between motion conception and function conception might be one of the reasons behind the difficulties of students encounter while learning geometric transformations. Therefore, MTE2 suggested that when preparing tasks providing such transition for students need to be considered by teachers. She gave the example of a task that includes tabular representation of the corresponding pairs of preimage and image points, algebraic representation as a function, and geometric representation.

## 4.3. Knowledge of MTE3

Figure 4.5 shows the most frequently seen categories for the mathematical knowledge of MTE3. Under the mathematical knowledge, knowledge of topics is seen in the data the most frequently. Among the subcategories of knowledge of topics, *definitions, properties, and foundations* have the highest frequency. The subcategories that did not appear in the data are not exhibited in figures.

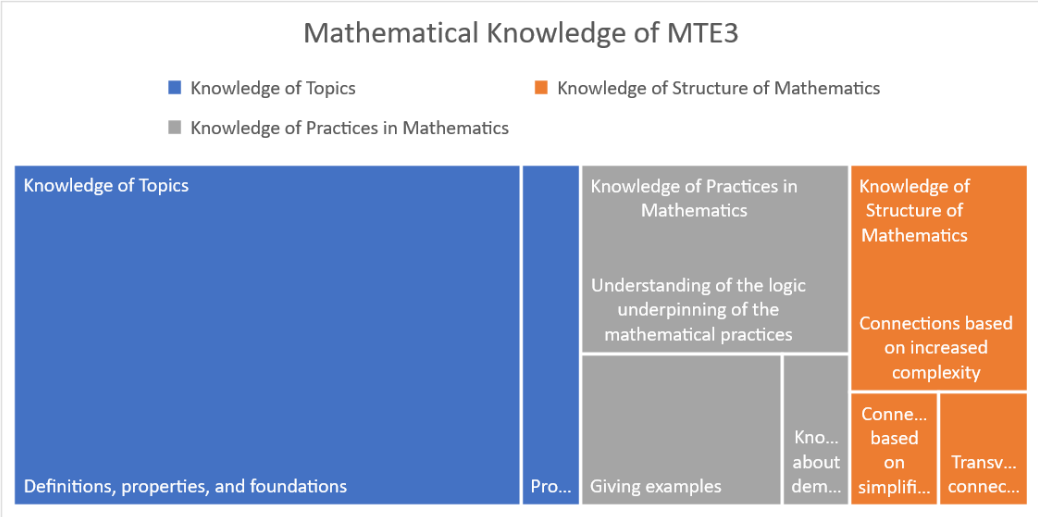


Figure 4.5. Visualize Code Occurrence for MTE3's Mathematical Knowledge.

In addition, Figure 4.6 show the most frequently seen categories of pedagogical content knowledge for MTE3. MTE3 stated about the subcategory of *strategies, techniques, tasks, and examples* the most frequently. Strategies, techniques, tasks, and examples emerge in the data 29 times.

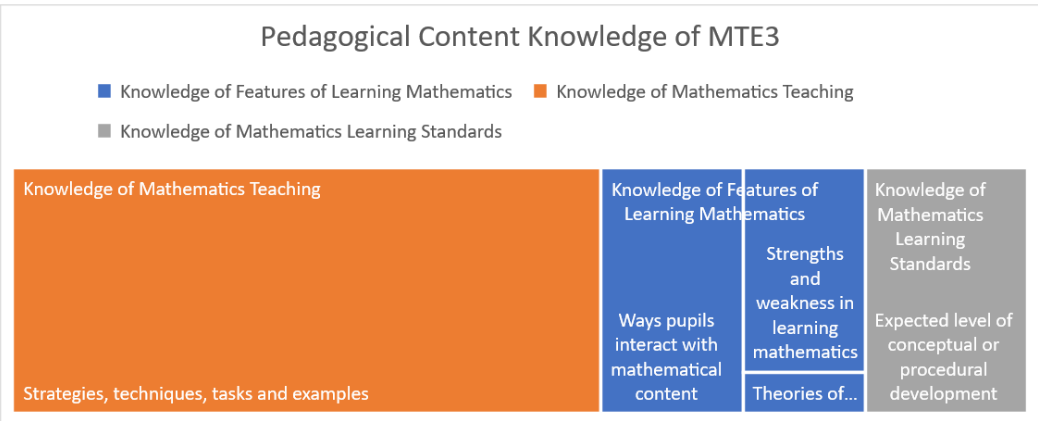


Figure 4.6. Visualize Code Occurrence for MTE3's Pedagogical Content Knowledge.

First, I share the data related to definition of geometric transformations and classification of specific geometric transformations under the subcategory of *definitions, properties, and foundations*. Then, I share the analysis of the possible difficulties

MTE3 points to students might have and the strategies that teachers might follow to teach geometric transformations that refers to *strengths and weaknesses in learning mathematics* and *strategies, techniques, tasks, and examples* respectively. First of all, MTE3 defined geometric transformations as "a function whose domain and range are geometric figures" under the subcategory of on *definitions, properties, and foundations.* She continued:

> "There are some who define geometric transformations in a static way, and some who define it in a what you might be called an active way, in a motion way. This is particularly true of the congruence transformations. So, there are some people who think of rotations, and they think of a rotation as it's actually the physical turn, so that's an active motion. And then others will say 'Oh, no no it's a function, it has an image a pre-image'. Let's say the domain is the pre-image the range is the image, and so it another aspect. Another aspect is when you think of a transformation, are you thinking of a transformation of a single figure, of such as a triangle or a quadrilateral, or a region, or some figure? Maybe a stick figure whatever or are you thinking always of a transformation of the whole plane. And you're only dealing looking at a subset what happens to a subset, so is a reflection. Is your pre-image the whole plane or is it a triangle or a quadrilateral? Is your image the whole plane, or is it just? So normally we like to specify, we just images, usually a figure, and so on. But it's embedded in the fact that the entire plane. If it's a three-dimensional transformation, the entire space is being transformed. But we're only looking at what the figures are".

Data showed that MTE3 considers that there are two ways to define geometric transformations namely: in a static way and in an active way. According to her, thinking in an active way, which is to think geometric transformations as a motion, is valid for congruence transformations such as rotation. Data also suggests that function definition of geometric transformation is the static way to define geometric transformations. Moreover, she stated that although it is known that the preimage and the image are the entire plane, the subset of the plane, which is the given figure, is the focus when applying geometric transformations.

Also, she further explained properties geometric transformations hold:

> "...the biggest differences and similarities before I am trying to classify them is what properties do they preserve? Does it map a figure onto a parallel line? Does a mapping a line onto a parallel line for instance, is a property of a transforma-

> tion? Does it preserve area? Obviously, dilation does not necessarily preserve area and they're usually not studied in school until you unless you study geometry at the college level. And fundamentally, the almost the first thing that people see is orientation. And so, reflections and glide reflections switch orientation... We had postulates and part of our postulates were the preservation properties of reflections. We assumed that reflections preserve angle measure, betweenness, collinearity and distance. We assumed all, and that they reverse orientation. One of the key things was that the preservation of betweenness not just collinearity, but betweenness. And what was important about that was that mean not just the endpoints of a triangle we're being transformed, but the segments the sides were and inside the sides were also preserved. So, the segments were preserved very consciously".

The excerpt suggested that MTE3 considers that to classify the geometric transformations the preserved properties such as size, betweenness, angle measures, measurements, collinearity, and orientation, need to be taken into consideration. Dilation does not always preserve area, and reflection and glide reflection might change the orientation. According to MTE3, the preservation of betweenness is a key property, and provides a conscious way to understand that figures and the segments are preserved. This data also refers to the subcategory of *definitions, properties, and foundations.*

Since the knowledge of MTE3 about the difficulties of the students and the strategies for teachers intertwine in data, I provide them together. The difficulties are under *strength and weakness of learning mathematics* and the strategies and techniques for teachers to teach transformations are under *strategies, techniques, tasks, and examples.* Besides, I additionally indicate the other subcategories from both mathematical knowledge and pedagogical knowledge if they appear within the data. First, she highlighted the strength of learning geometric transformations under the subcategory of *strength and weakness of learning mathematics* under the category of knowledge of features of learning mathematics:

> "It makes geometry accessible to slower student to students who are not generally as if it was polished, which you want to say they have more trouble learning, or they're slower. Or whatever you want to say. It is makes it accessible and some of the very first curriculum in the United States. We are called stretches and shrinkers the idea was. They didn't do transformations in theory, but they did not do them. Essentially it was that at the same time and this is what's really interesting".

Excerpt pointed that geometric transformations might be a strength for slower learners when learning geometry. Stretches and shrinkers makes the geometry accessible because it is so natural, so students do transformations without really knowing that they do transformations. She pointed also to the difficulties in understanding of parameters:

> "Because in understanding functions, there is a question of difficulties and parameters in understanding functions and real functions. We are talking functions of real numbers. Now we're saying what about this if we consider geometric transformations as functions, then, the difficulties that learners have with parameters in dealing with real functions, do they? Do these difficulties that students have with parameters which are in from studies of real functions are not studies of geometric functions? ... I know I do not think that these difficulties relate to geometric transformations. Not in my experience. ...When I said what geometric transformations do for functions that I talked about domain and range and composition. I did not talk about parameters and did not talk about variables. That was not what I do not think that that's the avenue the geometric transformations play. That is not the road that they play in the understanding of functions or of themselves as functions. How I think I don't see that students who deal with geometric transformations think of the triangles as variables. And I will give you an example I mean when you just identify triangle $ABC$, students don't think of $A$, $B$, and $C$ as variables, but they are. It could be any point $B$ could be any point $C$, could be any point, and they're just like $x$, $y$, and $z$. But we don't think of them as variables and therefore I think you know, and we don't want students necessary to think of them as variables. We can I mean that's imposing in language, and a structure that they don't need at that time. Let's take a specific example; when we say in triangle $ABC$, the sum of the measures of the angles is a 180 degrees. Do we think of a being triangle $ABC$ is a variable? We don't think of it, but it is".

She further stated:

> "Yes, there's a theoretical role but in in practice they're simply defining characteristics. They're what you need to have to define the transformation. And I don't see them as the same as let's say in describing a line in the plane that you have the slope and the intercept. These are parameters in defining the equation of a line let's say. But in transformations. The feel is not the same. You're just describing what is this rotation. 'Oh, it's a rotation of 90 degrees around that point'. Then you tell them parameters, but I mean you don't call them parameters. That is, you need to have them, but you don't want to impose theory. But it doesn't need to be imposed unless you're doing a study at the graduate level".

As the above excerpts showed according to MTE3, parameters and variables are not the difficulties specific to geometric transformations. They are in fact difficulties related to functions with real numbers. She further explained that students do not need to think that a triangle under a reflection is a variable to deal with the reflection. Also, she additionally explained that to define a linear equation, the slope and the y-intercept is needed, but when defining transformations, there is no need to specify the parameters. Therefore, she considers that for example rotations might be taught without naming the angle and the point that provide the figure to be rotated as parameters. She considered there is no need to impose extra vocabulary that they do not need when learning the transformations as an action. She further points that such vocabulary is appropriate at the graduate level.

MTE3 also pointed to the strength of geometric transformations in function conception. She stated "I believe this is easier than functions of real numbers. I actually think function notation is easier with geometric transformations because it's so visual". This data also referred to the subcategory of *connection based on simplifications* under knowledge of structure of mathematics. She pointed that since geometric transformations help visualize the function concept, the function notation in the context of geometric transformations is easier to understand. She also highlighted the importance of geometric transformations in understanding compositions of functions. She stated:

> "You can do composition of functions. To do composition of transformations is natural. I am going to follow this reflection by another reflection. Oh, look at the composite is a rotation, or the composite is a translation. Always one of the two if it's two reflections. But the whole notion of following a transformation by another transformations is natural. Whereas with functions in functions of real numbers that's not so natural. Students have trouble with composition of functions with real numbers. So, I think someone who understands geometric transformations as function really is someone, I'm talking about a student who understands geometric transformations as functions, has an advantage because with that student you can talk about composition of functions more easily if you have done that with transformations earlier".

The data together with the previous data suggested that from MTE3's perspective learning function notation and composition of functions with geometric transformations is easier than function notation and composition of functions with real numbers because function concept is so natural and visual in the context of geometric transformations. She emphasized that students who conceptualize geometric transformations might have advantageous to learn the composition of functions more easily. In addition, MTE3 explained using matrices to describe certain transformations under the subcategory of *connections based on simplifications* under knowledge of structure of mathematics:

> "Of course, you can describe these certain transformations with matrices. So, you could do matrices very quickly and instead of just matrices being to self-systems, which is where they have theoretic, have traditionally been introduced in the United States to self-systems. Now you've got matrices, standing for transformations and I might say it was only then that the students started saying 'Oh, my goodness, transformations are an object.' And if you're talking about getting to functions and thinking of functions as objects, this was really something, because it came late, it was only because the matrix is an object and by having an object that students for the transformation. Now they understood the transformation was an object, and we started seeing students calling the matrix, the transformation, I mean, it was really interesting. Oh, that you know matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ that's a reflection. That matrix is a reflection".

Data pointed that using matrices in geometric transformations helps students to think transformations as an object. That is, once students think about how to represent geometric transformations via matrices this might help them realize transformations as an object, which in turn helps them understand transformations as functions. Also, learners might realize the special matrices for certain transformations. For people who think that considering geometric transformations in matrices form is difficult, she said "it's not difficult it's intuitive, it's simple". Therefore, data suggested that using matrices in geometric transformations is an intuitive way yielding students' understanding of transformations as objects, which are functions. To sum up the previous excerpts point that geometric transformations help to understand function notations and function compositions in an easier way, and matrices helps to understand geometric transformations at an object level understanding of functions.

About how early geometric transformations be taught she pointed:

> "A study was done how early can you teach transformations? and certainly symmetry is done very early. You can do symmetry in second grade quite easily, particularly symmetry vertical lines, and people tend to be symmetric in the order. But rotations turn out to be the most difficult".

She also stated:

> "I am talking about a student who doesn't know what a transformation is, and I like to start with a reflection. Some people like to start with translation. I think so worst place to start. Because the figure looks too much like its image you haven't done anything you didn't transform anything You just moved it that's not what you want. What you want is something that actually has some sort of, actually think about it's actually transformed. And with reflection you get this reversal of orientation. So there really is a thing a transformation of some. The word transformation fits with reflection more than it does with rotation or translation. ...I mean this is not difficult".

The previous two excerpts suggested that rotation is the most difficult transformation and reflection is the easiest one because people might conceptualize symmetry very early. Therefore, reflection is a transformation that can be learned earliest such as at the second grade. Moreover, from MTE3's point of view, starting to learn geometric transformations with a certain transformation that provide a transformation in the figure helps to conceptualize geometric transformations. Reflection provides a transformation in the orientation of the figure, and it is easier than rotation, so starting with reflection is the most suitable strategy. When thinking whether it would be more appropriate to start learning transformations geometrically or algebraically, she explained:

> "In some schoolbooks you will find the first transformations, may be that I studied, is reflections. Reflections over the x-axis, or over the y-axis and so on. Well, that is to me is not a good way to start, because it's algebra, it takes what is a fundamentally geometric idea and starts it as an algebraic idea. I don't like that. So, to me you start with geometric ideas".

She pointed that since geometric transformations essentially a geometric idea, teachers need to start teaching transformations form a geometrical point of view rather than taking an algebraic approach. Since she elaborated her ideas, her personal knowledge, and her knowledge based on the literature about how and when students need to be started to learn geometric transformations, the previous three data shows examples *for the expected level of conceptual and procedural development* under the category of knowledge of mathematics learning standards.

To sum up, under the subcategory of *definitions, properties, and foundations*, MTE3 described geometric transformations in two ways: a static approach and an active approach. She described the properties of transformations according to preservation or exclusion of betweenness, collinearity, orientations, sizes, angle measures, and shapes. Under the subcategory of *strategies, techniques, tasks, and examples*, she pointed that although we use parameters to do transformations, we do not need to name them because naming parameters is imposing an extra vocabulary for students who recently start to learn geometric transformations. Similarly, she stated that "Yes, I reflected this triangle. 'Oh no, you reflected the entire plane'. No, you didn't. You were just reflecting the triangle I mean let's not impose structure on something that is in theory". The data shows that from MTE3's point of view the new learners of geometric transformations do not need to know that all plane is transformed. She considered this as imposing structure. She considered learning these ideas as necessary when students learn geometric transformations at the graduate level. Moreover, data shows that students might learn function notation with geometric transformations more easily because it is visual. Learning composition of functions is also easier after learning composition of geometric transformations because composition of geometric transformations can be observed, and it is natural. The idea of learning functions as easier with geometric transformations refers to *strength and weakness in learning mathematics* and using geometric transformations in function context refers to connections *based on simplifications*. Moreover, from MTE3's perspective, rotation is the most difficult geometric transformation, and the reflection is the easiest. Therefore, starting to teach geometric transformations with reflections is the most appropriate way for MTE3. She also added starting to teach transformations geometrically will help students to understand the content better under the subcategory of *strategies,* *techniques, tasks, and examples.*

Regarding the findings, all participants most frequently pointed to *definitions, properties, and foundations* in Mathematical Knowledge, and *strategies, techniques, tasks, and examples, ways pupils interact with mathematical content*, and *strength and weakness in learning mathematics* in Pedagogical Content Knowledge. I provide an extended table for the results matching with the subcategories (Table 4.1).

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge.

| **MATHEMATICAL KNOWLEDGE** | | | |
|---|---|---|---|
| **Knowledge of Topics (KOT)** | | **Description** | **Examples From MTEs' Excerpts** |
| **Categories** | Procedures | How do mathematical operations do? What to do at what stage of the procedure? Why the mathematical operations do in that way? What are the properties of the result of the operations? | "If there's only one fixed point after those two transformations, then it's going to be a rotation. if there is no fixed point, then it's going to be a translation" MTE1 |
| | Definitions, properties, and foundations (intra-conceptual connections) | Knowing the intra conceptual connections. Choosing the most suitable | "if you think about translation, they all the segments connecting to the corresponding points parallel to each other, and |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | properties to define a mathematical object. | but which is not true for reflection and rotation". MTE1<br><br>"Geometric transformations are similar in the sense that they are motion and in the sense that they specify functions". MTE2<br><br>"Transformations are not motions, but they have applications, obvious motions, gears, and all sorts of turning and motors and things. Biology, the beautiful symmetry of plants and animals, and in the sea, where you see three-dimensional symmetry of the gorgeous sea creatures". MTE3 |
| | Registers of representation | Knowing the graphic, arithmetic, algebraic, pictographic representations of mathematical contents. | "There are different representations of the vector. And also, there are two kind of variables the distance |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | | and or the magnitude, and also the direction" MTE1 |
| | Phenomenology and applications | Knowledge of the models that define a mathematical knowledge, knowledge of the phenomenon that helps to create a mathematical knowledge | |
| **Knowledge of Structure of Mathematics (KSM)** | | | |
| **Categories** | • Connections based on simplification | Knowing to connecting a mathematical content with a more basic mathematical content. The prerequisite knowledge that is needed to be known to learn a mathematical content. | "I think, first students need to understand function. Really understand function as a covariation. And so how the change of one variable right may change the other. and also, not just understand function in one variable, but understand a function in two variables". MTE1 "Students need to know sets, and elements of a set, and relations". MTE2 |
| | Connections based on increased complexity | Knowing to connect a previous basic mathematical content with a more advanced | "We see transformation as a function is not something new. it's just like you really, it's more like a like more abstract |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | mathematical content. Connected what is learned with a later content. | level of understanding of transformation". MTE1<br><br>"I believe this is easier than functions of real numbers. I actually think function notation is easier with geometric transformations because it's so visual". MTE3 |
| | Auxiliary connections | Knowing the type of connections between the concepts. - Incorporations mathematical concepts into larger processes | "You can talk about a congruency and similarity from transformations right and also It collects to also, they tessellation and also symmetries So all those topics are connected, all relates to transformations". MTE1 |
| | Transverse connections | Knowing the type of connections between the concepts – The case of different | "So not just a function, well, one variable, but actually function of two variables. Right. So it's |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | concepts having common properties | actually an ordered pair and they really understand that the idea of function, I think it's really essential for understanding transformations as a function, and also at like, I said the understanding they function in and two variables right is important". MTE1 |
| **• Knowledge of Practices in Mathematics (KPM)** | | | |
| **Categories** | Mathematical practice (syntactic knowledge of mathematics) | Knowing the proofs, justifications, how to define, how to do induction and deduction, and to give examples and non-examples. Understanding the reasons behind the mathematical practices. | "Identity function is an interesting function. It's an interesting function because it doesn't really play a role until you get to similarity. " MTE3 |
| | (Specific to mathematical content) Knowledge | Understanding how mathematics works | "A number line is actually one dimensional. And then, if we |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | about how mathematics is developed beyond any particular concept | behind any mathematical contents | use another number line that is perpendicular to the given number line, and we actually created two dimensional right. And even though we know that those two number lines doesn't have to be always has to be perpendicular to each other right. But we choose them to make them perpendicular for some reasons". MTE1 |
| | Mathematical reasoning | Knowing how to explore and extent a new mathematical knowledge | |
| **PEDAGOGICAL CONTENT KNOWLEDGE** | | | |
| **Knowledge of Features of Learning Mathematics (KFLM)** | | **Descriptions** | **Examples From MTEs Excerpts** |
| **Categories** | Theories of mathematical learning | Knowledge the personal or corporate theories related to students' cognitive development | "If students have a more like understanding transformation as a process, like very much like a function. If you understand |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | | function as an input and output right it's more like a process. If you perform those actions, you will see the effects of those actions. They may understand transformation as a function, but they think about function as an input, output. But if they think about function as an object, or as a covariation of dependent and independent variables, they probably, understand not just the properties of geometric transformations like those properties we mentioned, but also be able to actually reason about the composition of geometric transformations" MTE1 |
| | Strengths and weakness in learning mathematics | Awareness of the strengths and weaknesses | "The most difficult thing, like the learners have, while which has all |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | of the students in mathematical contents | has frequently been documented in the literature, is that students see transformation is as motion. Okay, not as a function. I think that's the most difficult part of a student to shift to that perspective". MTE1<br><br>"They may difficulty in revealing which point in the domain matches which point in the image set". MTE2 |
| | Ways pupils interact with mathematical content | Knowing the mathematical terminologies and applications that students familiar with | "if they understand function as an object, and if they do understand to geometric transformations as function, they should be able to talk about compositions of transformations and be able to actually reason properties of geometric transformations without action performing those actions". MTE1 |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | | "The student who evaluate this as a process sees transformations as a procedure, I substitute the points into the function, and it gives me the result". MTE2<br><br>"I think someone who understands geometric transformations as function has an advantage because with that student you can talk about composition of functions more easily if you have done that with transformations earlier". MTE3 |
| | Emotional aspects of learning mathematics | Knowledge of the contents and expressions that make students motivated or knowledge of the cases that makes students demotivated. | |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| **Knowledge ofs Mathematics Teaching (KMT)** | | | |
|---|---|---|---|
| **Categories** | • Theories of mathematics teaching | Theories special to teaching mathematics | |
| | Teaching resources (Physical and digital) | | In some school books you will find the first transformations may be reflections, reflections over the x-axis, or over the y-axis and so on". MTE2 |
| | Strategies, techniques, tasks and examples | Awareness of the strategies, techniques, tasks, and examples that needed to teach specific contents. And awareness of the limitations. | "How would students learn in secondary level, you know, geometrical transformations how to actually connect to college year level, math, right? For example, they would see geometric transformations as a group You know just a particular case, right? So then, when they started group, in abstract algebra, they would say, |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | | Oh, well, now I see I am now. They have some kind of concrete examples to draw from isometry. would be perfect example for them to know if these really have a good understanding of those isometries, then I think they help them too. So I think it's for teachers it's important for them to know how this piece of knowledge connects to different levels of mathematics. Then they are just teaching that single topic". MTE1<br><br>"The critical thing here is that the reasoning about the mapping is actually a functional relationship, what does the mapping mean algebraically, how can I carry it geometrically with the plane, as you just mentioned in your questions. |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | | In fact, he can establish that relationship. You know, this function can be explained with parameters or with normal variables". MTE2<br><br>"So many examples. Teachers need examples, and they need examples and non-examples so they need transformations. That preserve distance and transformations that do not preserve distance. Transformations give you similar figures, and here are transformations that don't give you similar figures". MTE3 |
| • **Knowledge of Mathematics Learning Standards (KMLS)** | | | |
| **Categories** | Expected learning outcomes | Knowledge of expected learning outcomes in the curricula | |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | Expected level of conceptual or procedural development | Knowledge of the subject to be covered and the depth of the subject. | "In school curriculum you can see that that's not the typical approach. A lot of times we first introduce transformation is as motion, right? So we can talk about, you know transformation is as motion, translation as a slide, reflection as a flip and rotation as a turn right so that's kind of like the approach lots of time in the school curriculum took". MTE1 "Another approach is the motion of geometric objects. We see this definition mostly in its reflections on the curriculum. Especially with younger children, we consider transformations as a motion until kindergarten, primary school, middle school and even 9th grade". MTE2 |

Table 4.1. Extended Table for Anaylsis of MTEs' Specialised Knowledge. (cont.)

| | | | |
|---|---|---|---|
| | | | "If you look at school geometry generally, the transformations that I studied are the transformations that lead to congruent or are similar figures" MTE3 |
| | Sequencing of topics | Positioning of the newly learned subject according to the previously acquired knowledge and the knowledge required to learn the subjects to be covered in the future | "They learn geometric transformations as a motion until the 9th grade. Then they start learning it as a function". MTE2 |

# 5. DISCUSSION

In this study, I investigated the mathematics teacher educators' specialised knowledge for three mathematics teacher educators regarding geometric transformations. I specifically examined MTEs' mathematical knowledge about geometric transformations regarding their definitions and properties. Besides, I examined MTEs' pedagogical content knowledge regarding the thinking ways of students about geometric transformations and the teaching strategies to develop students' understandings and to overcome students' difficulties. All three MTEs used their personal experiences and their knowledge that comes from reading the related literature to explain their reasonings. MTEs also included their reflections about their personal experiences about teaching geometric transformations. Therefore, studying with MTEs allowed to get first-hand comprehensive information from people who have done research about teaching geometric transformation, taught geometric transformations, and read related articles related to geometric transformations. Therefore, in the following paragraphs I discuss results pointing to all three MTEs' reflections on what learners need to know about geometric transformations, what difficulties learners might have as well as how to overcome such possible difficulties they might encounter while learning geometric transformations, and the teaching strategies for both students' and teachers' learning about geometric transformations.

Fife *et al.*, (2019) defined geometric transformation as "a function from the plane to the plane, that is, a function f: $R^2$ $R^2$" (p. 1). Also, they described "the transformations that are fundamental to geometry the rigid motions—translations, reflections, and rotations and compositions of these—which preserve distance and angles, together with dilations, which expand or contract". (p.1). In accordance with these explanations, in this study, MTEs also defined geometric transformations both mathematically as a function or mapping and geometrically as a motion of geometric figures. In particular, MTE1 defined "mathematically, the geometric transformations are functions mapping a point in $R^2$ to $R^2$, so mapping the entire plane. But in school curriculum you can see that that's not the typical approach. A lot of times we first introduce transformation as motion". MTE2 explained: "one approach defines these transformations as functions or mappings... The other approach defines geometric transformations with the motion of geometric figures". Lastly, MTE3 defined as "some defines it in a what you might be called an active way, in a motion way. This is particularly true of the congruence transformations. And then others will say 'Oh, no it's a function, it has an image, a pre-image".

Previous research emphasized that APOS theory has a positive effect on learning outcomes of geometric transformations and students' responses in lectures (Hanifah and Aliyyah, 2017). Although it is not explicit for all data, all participants also described geometric transformations regarding the learning theory, the APOS theory. Particularly, MTE1 considered that the motion conception of geometric transformations is at an action level, whereas the function conception of geometric transformations might be at a process level or an object level of understanding. According to MTE1 students with an action level of understanding consider geometric transformations as the inputs and outputs of a motion. Students with a process level of understanding might make a mental connection between input and output values via the properties of transformations and see the properties as commonalities. Students with an object level of understanding might be able to reason about the properties of geometric transformations without need to perform and see the effect of these properties. In addition, once they have the object level of understanding, students might reason about the properties of geometric transformations as well as composition of transformations. Similarly, MTE2 defined that motion conception for the understanding geometric transformation as function is a process and explained that students with motion conception might substitute the values into the function and find its result. Besides data of MTE2 suggested that students with object level of understanding might understand geometric transformations as one to one mapping from plane to plane. Previous research also used APOS theory to scrutinize understanding of geometric transformation (Cetinkaya *et al.*, 2017; Figueroa *et al.*, 2018; Hollebrands, 2003). Specifically, Hollebrands (2003) explain that geometric transformations as an action are to consider no more than the output value of substituting the input value into the function. Geometric transformations as a process are to consider complete activity as a geometric transformation that includes the preimage, preforming the action, and the image. Geometric transformations as an object are to consider geometric transformations as one to one and onto mappings from plane to plane. As MTE1 and MTE2 stated, Hollebrands (2003) pointed that students with object level of understanding might conceptualize composition of transformations and the properties that preserved after the composition of transformations. Moreover, MTE3 highlighted that matrices might provide students to learn geometric transformations as an object. Figueroa *et al.* (2018) also emphasized the advantage of learning matrix multiplication at an object level when learning linear geometric transformations. In addition, aligning with Karagöz Akar *et al.*, (in press) and Steketee and Scher (2011) studies, MTE1 and MTE2 pointed that understanding geometric transformations at action or process level might be either a precursor to the understanding of geometric transformations at an object level or that these two different ways of thinking about transformations are complementary rather than contrary. In fact, all MTEs highlighted that motion conception of geometric transformations is encountered in school curriculum. According to them students are first exposed to geometric transformations as rigid motions in school. Though, MTE2 pointed that "the meaning of transformations both algebraically and geometrically is also critical. as having both help students to have a comprehensive understanding of geometric transformations. Previous research also students need to have action level of understanding as well because action level of understanding transformations is crucial to construct the geometric transformations (Figueroa *et al.*, 2018). On the other hand, MTE3 also pointed that geometric definition of the transformation is the primarily definition, and transformations are a concept that is fundamental to geometry. In addition, she added that the vocabulary usage affect students' understanding. This means that using rigid motions for transformations lead students to only think geometric transformations as a motion or an action. Emre-Akdoğan *et al.* (2018), Yanik (2011), and Yanik (2014) also found that word preferences used during the teaching of transformations might canalize students to think geometric transformations only as an action. Results on how MTEs consider what and how learners need to know is important as previous research on students' (Aktaş and Ünlü, 2017; Emre-Akdoğan *et al.*, 2018; Guven, 2012; Gülkılık *et al.*, 2015; Hollebrands, 2003, 2004, 2007; Hollebrands *et al.*, 2021; Sünker and Zembat, 2012; Xistouri *et al.*, 2014; Yanik, 2014) and college and pre-service teachers' (Avcu and Çetinkaya, 2019; Uygun, 2020; Yanik, 2011; Yanik and Flores, 2009) point that they do have difficulties and lack of understanding of geometric transformations as functions. Particularly, all MTEs' detailed explanations further pointed to what students need to realize and how they need to think so that they might switch from the motion to function understanding of geometric transformations. students' (Aktaş and Ünlü, 2017; Güven, 2012; Sünker and Zembat, 2012; Yanık, 2014, Xistouri *et al.*, 2014) and high school students' (Emre-Akdoğan *et al.*, 2018; Gülkılık *et al.*, 2015; Hollebrands, 2003, 2007; Hollebrands *et al.*, 2021; Kainose Mhlolo and Schafer, 2013)

Specifically, according to MTE1 and MTE2, knowing point-by-point correspondence, the properties of the corresponding preimage and image point pairs, and knowing the variances and invariances helps to understand parameters and compositions of transformations because focusing on these help students to reflect on the properties of specific geometric transformation which might lead to their learning the interested geometric transformation as an object. In fact, although MTE3 considered that there is no need to impose extra vocabulary for the parameters to the students at the high school level, she pointed that transformations are taught by using the properties of transformations without naming them as parameters. Therefore, teaching the properties of parameters stands at a significant point for all participants. These results also align with (Karagöz Akar *et al.*, in press) study such that they propose that to understand that vectors are the parameters of translations, students need to understand the interrelationship between the preimage and their corresponding image points (Karagöz Akar *et al.*, n.d.). (Karagöz Akar *et al.*, in press) further highlighted that interrelationships might be observed via the slopes and the distances between these point pairs. In particular, they stated that learners need to find the slope and the distances and realize that these properties are common for each pair of points. In this study, MTE1 also described one of the properties for a transformation as the relationship between the points under a transformation and their corresponding points in the image set. Besides, she added that if students realize that the distance between each preimage point and its corresponding image point are equal and the segments that are constructed by the corresponding point pairs are parallel then they might understand transformations as functions like an object. That is, students need to observe that such properties are common. Moreover, Karagöz Akar *et al.* (in press) proposes to use continuos co-variation to helps students conceptualize isometries. In this study MTE1 also stated that understanding geometric transformations as functions at an object level depends on understanding the covariation of independent and dependent variables. That is, to help students to transit from a process level to an object level understanding, students need to think function relationship as a covariation of dependent and independent variables of the points and their corresponding points. MTE1 further pointed that in this way, learners can understand the properties of transformations and composition of transformations.

Regarding the teaching strategies, results pointed that all MTEs link and propose the strategies to students' difficulties and their background knowledge expected to study geometric transformations at the object level. First, MTE1 pointed that learning geometric figures only at coordinate geometry hinder understanding about geometric figures being a part of the plane. She further stated that to eliminate this hinderance students also need to learn geometric figures at plane geometry as well. By this way, students might understand that figures are a part of the plane. This is in align with a previous study (Karagöz Akar *et al.*, in press) as they also emphasized the importance of seeing geometric figures as a relative quantitiy to the plane. They stated "These conceptualizations are needed for learners to consider any set of points or geometric shapes as quantities relative to the whole, $R^2$, rather than isolated and independent entities". (p.) Secondly, in this study, MTE2 also summarized this by stating: "Teachers need to bring the ideas of 'What is $R^2$?,' 'What does it mean to have a shape on $R^2$?' and what the elements of $R^2$ are both algebraically and geometrically into question. Similarly, Karagöz Akar *et al.* (in press) pointed that learners' reflections on the answers to these kinds of questions are expected so that they understand geometric transformations quantitatively.

# 6. LIMITATIONS AND SUGGESTIONS

This study was conducted with three participants. Although revealing some important aspects in terms of learning and teaching of geometric transformations, further research can be conducted with more participants with similar background to understand and explore mathematics teacher educators' conceptions on geometric transformations more in-depth. It is also important to note that the beliefs of the mathematics teacher educators also affect their conception of geometric transformations. Belief systems of mathematics teacher educators were not examined in this study. Further research might be conducted to examine the belief systems of the mathematics teacher educators about geometric transformations.

In addition, in this study, the interview data were analyzed. To extend the schema of the mathematics teacher educators about teaching and learning geometric transformations, lessons of the mathematics teacher educators about geometric transformations might be observed. Also, as a suggestion for further research, research about exploring mathematics teacher educators' conceptions for different mathematical concepts might be conducted. Knowing how an expert explores concepts might provide a way for mathematics educators to shape the learners process of learning the concepts.

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# APPENDIX A: INTERVIEW PROTOCOL

VERİ TOPLAMA ARAÇLARI

Mülakat Protokolü

Tarih:

Yer ve zaman:

Görüşmeyi yapan:

Katılımcı:

> Geldiğiniz için teşekkür ederiz. Size daha önce de bildirdiğimiz üzere, burada bulunmamızın sebebi, sizlerin üzerinde çalıştığınız konular hakkındaki düşüncelerinizi incelemek. Bu sebeple, sorduğumuz sorularda, sizin nasıl düşündüğünüz ve kavramlara nasıl yaklaştığınız benim için önemli olacak. Amacımız sizin nasıl düşündüğünüz hakkında bilgi sahibi olmak. Dolayısıyla, size bazı ek (sonda) soruları sorabilirim. Bu sorularımın amacı sadece ne düşündüğünüzü daha net anlayabilmek. Ayrıca, cevap vermek istemediğiniz sorularda lütfen bildirebilirsiniz. Bir diğer soruya geçiş yapabiliriz böylelikle.
> Görüşmeyi istediğiniz zaman sonlandırma hakkına sahipsiniz. Bir sorudan rahatsız olursanız da lütfen bildirin. Geldiğiniz için tekrar teşekkür ederim. Bu bizim için çok önemli ve katkınızdan dolayı çok mutluyum.

Figure A.1. Interview Protocol.

Mülakat Soruları:

- Geometrik dönüşümler nasıl tanımlıyorsunuz?
  (i) Bu tanımı açar mısınız? Bu tanımın bileşenleri nelerdir? Biraz açıklar mısınız?
- "Tanım kümesinin R2 olduğunu bilir". tanım kümesinin R2 olduğunu bilen kişi nasıl düşünüyor da bunu biliyor, açıklar mısınız?
  (i) Bu fikre varabilmesi için bilen kişinin nasıl düşünmesi gerekiyor?
  (ii) Kişinin, tanım kümesinin ve görüntü kümesinin R2 olduğunu bilmesi için ne

tür bilgilere ihtiyacı vardır?

(iii) (Eğer katılımcı noktadan bahsederse) kişinin noktayı nasıl anlamlandırması gerekir?

- Sizce öğrenenler tanım kümesi ile ilgili ne tur zorluklar yaşıyor olabilirler?
  (i) Şeklin düzlemden ayrık olmaması demek ne demek?
  (ii) Alanyazın öğrencilerin tanım kümesini üç farklı şekilde düşündüklerini söylüyor (verilen şeklin köşeleri, şeklin kendisi, düzlem) bu üç başlık arasındaki geçişler için bilinmesi gereken bilgiler nelerdir? siz ne düşünüyorsunuz?
- Sizce öğrenenler görüntü kümesi ile ilgili ne tür zorluklar yaşıyor olabilirler?
- Parametrelerin ve vektörlerin tanımdaki rolu nedir? açıklar mısınız? (tanımın bileşenlerinde) parametreler ve vektörler için bilen kişi nasil düşünür ki bu rollerin farkındadır?
  (i) Öğrenenler parametrelerle ilgili ne tür zorluklar yaşıyor olabilirler? (Eğer katılımcı alan yazındaki zorluklara değinirse: neden sizce, neden zorluklar yaşanıyor olabilir, nasıl çözüm önerileri önerirsiniz? gibi açımlayıcı sorular sorulacak.)
  (ii) (Eğer katılımcı alan yazındaki zorluklara değinmezse parametrelerle ilgili zorlukları sor)
  - Öğrenenler parametreleri değişken olarak görmekte sorun yaşıyor, sizce neden?
  - Öğrenenler değişkenleri sabit olarak algılıyor, sizce neden?
  - Öğrenenler parametrelerin dönüşüme uğrayan geometrik objelerden farklı olduklarını düşünüyorlar, sizce neden?
  - Sıfır vektörü ve birim fonksiyon olma durumunu anlamlandıramıyorlar, sizce neden?
- Geometrik dönüşümleri fonksiyon olarak anlayan kişi nasıl düşünür?
- Kişinin geometrik dönüşümleri fonksiyon olarak anlaması için hangi ön bilgilere ihtiyacı var?
- Öğreticinin geometrik dönüşüm öğretisi için gereken optimum üst bilgisi ne olmalı?
  (i) Dönüşümlerin grup olarak düşünülmesi ne anlama geliyor?

- Geometrik dönüşümlerin birbirlerinden farklılıkları nelerdir?
- Öğretmen adayları ya da öğreticiler bu konuyu öğretirken nelerin farkında olmalı, nedenleri ile açıklar mısınız?
- Geometrik dönüşümleri öğrenmek neden önemlidir?
- Eklemek istediğiniz bir konu var mı?

# APPENDIX B: CONSENT FORM

KATILIMCI BİLGİ ve ONAM FORMU

**Araştırmayı destekleyen kurum:** Boğaziçi Üniversitesi

**Araştırmanın adı:** Matematik Öğretmen Eğitimcilerinin Geometrik Dönüşümler Konusunda Alan Eğitimi Bilgilerinin İncelenmesi

**Proje Yürütücüsü/Araştırmacının adı:** Doç. Dr. Gülseren Karagöz Akar/Hüsniye Aybüke Balcı

**Adresi:** Boğaziçi Üniversitesi-Eğitim Fakültesi-Matematik ve Fen Eğitimi Bölümü

**E-mail adresi:** gulseren.akar@boun.edu.tr

**Telefonu:** 0212 359 69 01

**Proje konusu:** Geometrik dönüşümler konusunun anlamlandırılmasında öğretmen adayları ve hem lise hem de orta okul öğrencileri zorluklar yaşamaktadırlar ve bu konuyu öğrenme sürecinde kavram yanılgıları oluşmaktadır. Matematik öğretmen eğitimcileri hem daha kapsamlı bir matematik bilgisine hem de öğretmen adaylarının ve öğrencilerin olmak üzere iki ayrı grubun alan eğitimi bilgisine sahip oldukları için, matematik öğretmen eğitimcilerinin bilgisinin incelenmesine bu zorlukların ve kavram yanılgılarının giderilmesinde ihtiyaç duyulmuştur.

Bu çalışma, Boğaziçi Üniversitesi Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik kurulu onayı ile yapılmaktadır.

**Onam:** Sayın öğretmen eğitimcisi,

Bu çalışma kapsamında, geometrik dönüşümler konusunun anlamlandırılması sürecini bilen kişilerin gözüyle incelemek üzere mülakatlar yapılacaktır. Böylece, araştırmaya katılmayı kabul ettiğiniz takdirde, araştırmacı ile 40-60 dakika sürecek olan yapılandırılmış mülakatlar yapmanız istenecektir. Mülakatlar boyunca konuşulanlar sizin de izniniz doğrultusunda kısa notlar, video ve ses kaydı ile kayıt altına alınacaktır ve ayrıca yazılı olarak sunduğunuz belgeler saklanacaktır.

Çalışmaya katılmanız tamamen isteğe bağlıdır. Bu bilgiler sadece araştırmacının erişiminde olacak ve sadece araştırma amaçlı kullanılacaktır. Bu çalışmada kullanıldığı takdirde, hiçbir isim açıklanmayacağı için size ait olup olmadıklarını başkalarının bilmesine imkân yoktur. Bu bilgiler araştırmadan 7 yıl sonra imha edilecektir. Ayrıca, istediğiniz zaman çalışmaya katılmaktan vazgeçebilirsiniz. Bu durumda da sizden almış olduğumuz veriler imha edilecektir.

Yapmak istediğimiz araştırmanın size risk getirmesi beklenmemektedir. Çünkü paylaştığınız fikirleriniz sizlerin nasıl düşündüğünüzü ifade etmektedir. Ayrıca paylaştığınız bilgiler isimleriniz değiştirilerek analiz edileceği için, sizlerin kariyerini etkileyecek bir durum da söz konusu değildir. Başlangıçtan itibaren veya daha sonrasında çalışmaya onay vermezseniz, hiçbir şekilde videoya kaydınız alınmayacak ve kullanılmayacaktır. Bu durumda da isimler hiçbir şekilde bilinmeyecektir.

Araştırma sonucunda aranan bilgi elde edildiği takdirde, tüm dünya öğretmenlerine katkıda bulunmuş olacaksınız. Belli teorik temellere dayalı olarak yapılacak olan bu çalışma teorinin bazı noktalarına ışık tutabileceği gibi, sizlerin de kanaatimizce yüksek düzeyde bir çalışma ortamında bulunmanıza ve bilginizi değerlendirmenize katkı sağlayacaktır. Araştırmanın ileride hem bizim ülkemizde ve hem de başka ülkelerdeki öğretmenler ve dolayısı ile öğrencilere yarar sağlaması muhtemeldir.

Bu formu imzalamadan önce, çalışmayla ilgili sorularınız varsa lütfen sorun. Daha sonra sorunuz olursa, Gülseren Karagöz Akar'a (0212 359 69 01) ve Hüsniye Aybüke Balcı'ya (Telefon: 0 544 391 51 23) sorabilirsiniz. Araştırmayla ilgili haklarınız konusunda Boğaziçi Üniversitesi Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik Kurulu'na (fminarek@boun.edu.tr) danışabilirsiniz.

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Bana anlatılanları ve yukarıda yazılanları anladım. Bu formun bir kopyasını aldım.

Çalışmaya katılmayı kabul ediyorum.

◯ Yapılacak mülakatta video kaydımın alınmasını kabul ediyorum.

◯ Yapılacak mülakatta ses kaydımın alınmasını kabul ediyorum.

Katılımcı Adı-Soyadı:..............

İmzası:.................

Tarih (gün/ay/yıl): ........./.........../..........

Evrak Tarih ve Sayısı: 14.12.2021-42473

T.C.
BOĞAZİÇİ ÜNİVERSİTESİ REKTÖRLÜĞÜ
Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik Kurulu
(FMİNAREK)

Sayı : E-84391427-050.01.04-42473 14.12.2021
Konu : 2021/24 Kayıt no'lu başvurunuz hakkında

Sayın Doç. Dr. Gülseren KARAGÖZ AKAR
Matematik ve Fen Bilimleri Eğitimi Bölüm Başkanlığı - Öğretim Üyesi

"Matematik Öğretmen Eğitimcilerinin Geometrik Dönüşümler Konusunda Alan Eğitimi Bilgilerinin İncelenmesi" başlıklı projeniz ile Boğaziçi Üniversitesi Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik Kurulu (FMİNAREK)'e yaptığınız 2021/24 kayıt numaralı başvuru 06.12.2021 tarihli ve 2021/10 No.lu kurul toplantısında incelenerek etik onay verilmesi uygun bulunmuştur. Bu karar tüm üyelerin toplantıya on-line olarak katılımıyla ve oybirliği ile alınmıştır.

COVID-19 önlemleri nedeniyle üyelerden ıslak imza alınamadığından bu onam mektubu tüm üyeler adına Komisyon Başkanı tarafından e-imzalanmıştır.

Saygılarımızla bilginize sunarız.

Prof. Dr. Tınaz EKİM AŞICI
Başkan

Bu belge, güvenli elektronik İmza ile imzalanmıştır.

Doğrulama Kodu :BSR3ZV7KKZ Pin Kodu :84403 Belge Takip Adresi : https://turkiye.gov.tr/ebd?eK=4787&eD=BSR3ZV7KKZ&eS=42473

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Bilgi için: Nurşen MUNAR
Unvan: Mühendis

**Bu belge, güvenli elektronik imza ile imzalanmıştır.**

Figure B.1. Ethical approval.