# SIMPLE SECTION BISET FUNCTORS

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# ABSTRACT


## SIMPLE SECTION BISET FUNCTORS

Let $G$ and $H$ be finite groups and $k$ be a commutative unitary ring. The Burnside group $B(G, H)$ is the Grothendieck group of the category of finite $(G, H)$ - bisets. The biset category $k\mathcal{C}$ of finite groups is the category defined over finite groups, whose morphism sets are given by the $kB(G, H)$ groups. A biset functor defined on $k\mathcal{C}$, with values in $k$-Mod is a $k$-linear functor from $k\mathcal{C}$ to the category of $k$-Mod. The remarkable results as the evaluation of the Dade group of endopermutation modules of a $p$-group and finding the unit group of the Burnside ring of a $p$-group are done using the theory of biset functors. Looking for ring objects in the category of biset functors one gets a more sophisticated structure, which is called a Green Biset Functor. Serge Bouc introduced the slice Burnside ring and the section Burnside ring for a finite group $G$. He also showed that these two rings have a natural structure of a Green Biset Functor. In our work we classify simple modules over the section Burnside ring of $G$ using the approach of the paper Fibered Biset Functors by Robert Boltje and Olcay Coşkun.

# ÖZET

## BASİT BÖLÜM İKİLİ KÜME İZLEÇLERİ

$G$ ve $H$ sonlu gruplar ve $k$ değişmeli bir halka olsun. Burnside grubu $B(G, H)$, sonlu $(G, H)$ ikili kümeler kategorisinin Grothendieck grubudur. Sonlu grupların ikili küme kategorisi $k\mathcal{C}$, nesneleri sonlu gruplar olan ve morfizm kümeleri $kB(G, H)$ grupları tarafından verilen kategoridir. İkili küme izleci, $k\mathcal{C}$ kategorisinden $k$-Mod kategorisine $k$-doğrusal bir izleçtir. Örneğin $p$-grubunun iç permutasyon modüllerinin Dade grubunun hesaplanması veya $p$-grubunun Burnside halkasının birim grubunun bulunması gibi dikkate değer sonuçlar, ikili küme izleci teorisi kullanılarak yapılmıştır. İkili küme izleçleri kategorisinde halka nesneleri dikkate alındığında, Green ikili küme izleci adlandırılan daha gelişmiş bir yapı elde edilir. Serge Bouc, sonlu bir grup $G$ için dilim Burnside halkasını ve bölüm Burnside halkasını tanıttı. Ayrıca Bouc bu iki halkanın bir Green ikili küme izlecinin doğal bir yapısına sahip olduğunu dahi gösterdi. Çalışmamızda, Robert Boltje ve Olcay Coşkun tarafından yazılan fiberli ikili küme izleçleri makalesinin yaklaşımını kullanarak bölüm Burnside halkası üzerinde basit modülleri sınıflandırdık.

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# LIST OF SYMBOLS

| | |
|---|---|
| $(A, G, \partial)$ | Crossed module |
| $\mathrm{Aut}_G(X)$ | Automorphism group of $G$-equivariant maps of a $G$-set $X$ |
| $B(H, G)$ | Double Burnside group |
| $C_G(H)$ | Centralizer of subgroup $H$ in $G$ |
| $\mathrm{Def}^G_{G/N}$ | Deflation section biset |
| $G - \mathrm{Mor}$ | Category of the class of morphisms of $G$-sets |
| $G - \mathrm{Mor}^{Gal}$ | Category of the class of Galois morphisms of $G$-sets |
| $G_x$ | The stabilizer of $x$ in $G$ |
| $H \leqslant G$ | $H$ is a subgroup of $G$ |
| $H \trianglelefteq G$ | $H$ is a normal subgroup of $G$ |
| $\mathrm{Ind}^G_H$ | Induction section biset |
| $\mathrm{Inf}^G_{G/N}$ | Inflation section biset |
| $\mathrm{Res}^G_H$ | Restriction section biset |
| | |
| $\Gamma(G)$ | Section Burnside ring of $G$ |
| $\Gamma(G \times H)$ | Section Burnside ring of $G \times H$ |
| $\Delta(A)$ | $\{(a, a) \mid a \in A\}$ |
| $\Xi(G)$ | The slice Burnside group of $G$ |
| $\Sigma_G$ | The set of sections of $G$ |
| | |
| $\mathcal{G}_G$ | $\{(K, P) \mid K \trianglelefteq G, P \trianglelefteq G \; and \; K \leqslant C_G(P)\}$ |

# 1. INTRODUCTION

Classification of simple modules over Green Biset Functors is an open problem more than three decades. Although there are partial solutions for some specific examples of Green Biset Functors like biset functors or fibered biset functors there is no general classification theorem or conjecture for simple modules over Green Biset Functors.

The classification of simple biset functors is done by Serge Bouc in [1]. He showed that there is a bijection between isomorphism classes of simple biset functors and equivalence classes of pairs $(H, V)$, where $H$ is a finite group and $V$ is a simple $R\mathrm{Out}(H)$-module. He also conjectured that it can be generalized to the case of Green Biset Functors. In her paper [2] Nadia Romero studied three examples of Green Biset Functors for which their simple modules can be parametrized in the way as Bouc conjectured. These examples are rhetorical biset functors ($kR_{\mathbb{Q}}$), the functor of complex representations with coefficients in the complex field ($\mathbb{C}R_{\mathbb{C}}$), and the Yoneda-Dress construction at a group $C$ of prime order of the Burnside functor ($RB_C$). However, recently Robert Boltje and Olcay Coşkun, come up with a counterexample to Bouc's conjecture classifying the simple modules of fibered biset functor, which also has a structure of a Green Biset Functor.

Our aim is to solve the classification problem for the functor of section Burnside rings. It is shown in [3] that basic morphisms of $G$-sets are given by the natural maps $G/S \to G/T$ associated to subgroups $S \leqslant T \leqslant G$ and basic Galois morphisms are those associated to subgroups $S \trianglelefteq T \leqslant G$ (a pair of this form is called a *section* of $G$). With this result, the section Burnside ring $\Gamma(G)$ is free as an abelian group on the $G$-conjugacy classes of sections of $G$. It is also shown in [3] that the functor $\Gamma$ associating $\Gamma(G)$ to the finite group $G$ induces a Green Biset Functor.

One of the important observations for classification is that section Burnside ring has a similar structure with the Burnside ring of fibered bisets and so it is most likely to have analogous results. First of all, in the article [4] the isomorphism class of transitive fibered bisets are parametrized with the conjugacy classes of some tuples. Similarly basis elements of section Burnside ring are in bijection with the representatives of conjugacy classes of sections of the base group.

We start this thesis with the preliminaries chapter where we summarize the theory of Green Biset Functors and mention facts about the section Burnside ring. In Chapter 3 we generalize the idea, given in [4], for evaluation of essential algebra of Green Biset Functors. We carry out some facts like Mackey formula, which describes how to compose two basis elements of the section Burnside ring in Chapter 4. Then we show that the category of modules over the Green Biset Functor $\Gamma$ is equivalent to the category of functors from the category $\mathcal{P}_{k\Gamma}$ of finite groups in which composition is given by the linear extension of the composition product. Here $k$ is a commutative ring of coefficients for $\Gamma$. We call a module over $k\Gamma$ a section biset functor (over $k$).

Then we parametrize some idempotents in the endomorphism ring $E_G := \Gamma(G, G)$ and establish linkage classes between the sections corresponding to the transitive elements of section Burnside ring. Following [4] we use these idempotent elements in $E_G$ to prove that the essential algebra $\bar{E}_G$ is isomorphic to a product of matrix algebras over certain group algebras associated to these idempotents. Essential algebra $\bar{E}_G$ is the quotient of $E_G$ by the ideal generated by all the morphisms that factor through a group of smaller order. In our case, the idempotents of interest in $E_G$ are parameterized by pairs $(K, P)$ of normal subgroups of $G$ that centralizes each other, which are also reduced. Given such a pair $(K, P)$, the section $(\Delta_K(G), \Delta(P))$ of $G \times G$ gives the idempotent element $e_{(K,P)}$ in $E_G$. Here $\Delta_K(G) = \{(g, h) \in G \times G | h^{-1}g \in K\}$ and $\Delta(P)$ is the diagonal inclusion of $P$ in $G \times G$. In particular, we associate simple $E_G$-modules to each reduced pair $(K, P)$.

As mentioned in Chapter 5, it is also possible to associate crossed modules to pairs $(K, P)$ in a natural way and it turns out that two reduced pairs induce isomorphic $E_G$-modules if and only if the corresponding crossed modules are isomorphic.

With this result, we classify simple modules over the algebra $E_G$, see Chapter 8 for details. Finally, in Chapter 9, we show that the relation defined on reduced pairs can be extended to an equivalence relation on the set of quadruples $(G, K, P, [V])$ where $G$ is a finite group, $(K, P)$ is a reduced pair for $G$ and $V$ is an irreducible $E_G$-module associated to the pair $(K, P)$. Moreover there is a bijective correspondence between the equivalence classes of these quadruples and the isomorphism classes of simple section biset functors.

In order to complete the parametrization one needs to know the complete set of reduced pairs for each finite group $G$. Although we do not know the complete answer, we have some necessary and also some sufficient conditions to ensure that a pair $(K, P)$ is reduced. For example $(K, P)$ is reduced if $K \leqslant P$. Hence the pairs $(1, P)$ for any $P \trianglelefteq G$ and the pairs $(K, G)$ for any $K \leqslant Z(G)$ are reduced. In particular, the algebra $\bar{E}_G$ is non-zero for any finite group $G$.

Our reason for attempting to classify simple modules over the Green Biset Functor $k\Gamma$ is that it is different from the previous known examples in the following sense. The previous examples are known to have strong connections with the theory of representations of finite groups and the classification in these cases uses these connections in crucial ways. On the other hand, section Burnside rings are relatively new and these kind of deep connections are not known yet. Therefore it is close to be an abstract example. Having a successful application of the techniques from [4] signals a path towards a more general theory of simple modules over Green Biset Functors.

A result that might be of general interest is the version of Goursat's Theorem for sections. The well-known Goursat's Theorem for subgroups states that there is a bijective correspondence between subgroups of a direct product $G \times H$ and the quintuples $(P, K, \eta, L, Q)$ where $K \trianglelefteq P \leqslant G$ and $L \trianglelefteq Q \leqslant H$ and $\eta : Q/L \to P/K$ a group isomorphism. Our version, given as Theorem 4.1.1, determines exact conditions on the pairs of quintuples coming from Goursat's Theorem so that the induced map between sections of $G \times H$ and the pairs satisfying these conditions is bijective.

# 2. NOTATION AND PRELIMINARIES

In this chapter we fix some notation that will be used throughout this thesis and recall some definitions from [3].

## 2.1. $G$-Sets

**Definition 2.1.1.** *Let $G$ be a finite group. A left $G$-set $X$ is a finite set with an action of group $G$ on it. In other words, there is a map $G \times X \to X$, which satisfies the following conditions*

$$g \cdot (h \cdot x) = (gh) \cdot x, \tag{2.1}$$

$$1_G \cdot x = x \tag{2.2}$$

*for any $g, h \in G$ and $x \in X$.*

Similarly one can define a right $G$-set. Unless otherwise stated we consider left $G$-sets in this thesis.

**Definition 2.1.2.** *Let $G$ be a finite group. Given $G$-sets $X$ and $Y$, we define a morphism ($G$-equivariant map) between $X$ and $Y$ as a map $f : X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$. Morphisms of $G$-sets can be composed and the identity map from $X$ to itself is an identity morphism. A $G$-equivariant map is an isomorphism if it is bijection.*

**Definition 2.1.3.** *Let $G$ be a finite group. We call a $G$-set $X$ transitive if for any two elements $x, y \in X$ there is a group element $g \in G$ such that $g \cdot x = y$.*

The following Lemma from [5] describes *transitive* $G$-sets.

**Lemma 2.1.4.** *Let $G$ be a finite group.*

*(a) For any transitive $G$-set $X$ there is a subgroup $H$ of $G$ such that $X$ is isomorphic to $G/H$ as a $G$-set.*
*(b) For subgroups $H$ and $K$ of $G$ there is an isomorphism between $G$-sets $G/H$ and $G/K$ if and only if $H$ and $K$ are conjugate in $G$.*

## 2.2. Morphisms of $G$-Sets

**Notation 2.2.1.** *Let $G$ be a finite group. Denote by $\Sigma_G$ the set of sections of $G$. That is, the set of all pairs $(T, S)$ where $S$ and $T$ are subgroups of $G$ and $S$ is normal subgroup of $T$.*

**Notation 2.2.2.** *We use the notation $\mathcal{G}_G$ for the set of all pairs $(K, P)$ where $K$ and $P$ are normal subgroups of finite group $G$ and $K \leqslant C_G(P)$. That is,*

$$\mathcal{G}_G = \{(K, P) \mid K \trianglelefteq G, P \trianglelefteq G \, and \, K \leqslant C_G(P)\}. \tag{2.3}$$

*The set $\mathcal{G}_G$ has a poset structure given by $(K, P) \preceq (L, Q)$ if $K \leqslant L$ and $P \geqslant Q$.*

**Definition 2.2.3.** *Let $G$ be a group. If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are morphisms of $G$-sets, a morphism from $f_1$ to $f_2$ is a pair of morphisms of $G$-sets $\alpha : X_1 \to X_2$ and $\alpha' : Y_1 \to Y_2$ such that $\alpha'(f_1(x)) = f_2(\alpha(x))$ for any $x$ in $X_1$.*

If $f_3 : X_3 \to Y_3$ is also a morphism of $G$-sets and $(\beta, \beta')$ is a morphism from $f_2$ to $f_3$ then the composition $(\alpha \circ \beta, \alpha' \circ \beta')$ obviously is a morphism from $f_1$ to $f_3$. With the composition described above the class of morphisms of $G$-sets become category, which we denote by $G - \mathrm{Mor}$.

**Definition 2.2.4.** *Let $G$ be a group. A $G$-set morphism $f : X \to Y$ is called a Galois morphism if $\mathrm{Aut}_G(X)$ acts transitively on the fibers $f^{-1}(y)$. In other words, $f$ is a Galois morphism if for any $x, x' \in X$ with $f(x) = f(x')$, there exists $\phi \in \mathrm{Aut}_G(X)$ such that $f \circ \phi = f$ and $\phi(x) = x'$.*

**Proposition 2.2.5.** *Let $G$ be a group. A morphism $f : X \to Y$ of $G$-sets is a Galois morphism if for any $y \in f(X)$ one can find a normal subgroup $M_y$ of $G_y$ such that $G_x = M_y$ for any $x \in f^{-1}(y)$.*

**Remark 2.2.6.** *If $S$ and $T$ are subgroups of a finite group $G$ such that $S \leqslant T$, then the projection morphism $G/S \to G/T$ is a Galois morphism of $G$-sets if and only if $S \trianglelefteq T$. Moreover by [3, Section 9], the class of all Galois morphisms of $G$-sets forms a subcategory of $G-\mathrm{Mor}$ and the product and coproduct in $G-\mathrm{Mor}$ restricts to a product and a coproduct in this subcategory. We denote the category of Galois morphisms by $G - \mathrm{Mor}^{Gal}$.*

## 2.3. Slice and Section Burnside Rings

In Proposition 2.2 of [3] it's shown that disjoint union and direct product of $G$-sets induces coproduct and product respectively in the category $G - \mathrm{Mor}$. Hence the slice Burnside group can be defined as follows.

**Definition 2.3.1.** *Let $G$ be a finite group. The slice Burnside group $\Xi(G)$ of $G$ is the Grothendieck group of the category $G - \mathrm{Mor}$. As usual it is defined as the quotient of the free abelian group on the set of isomorphism classes $[X \xrightarrow{f} Y]$ of morphisms of finite $G$-sets, by the subgroup generated by elements of the form*

$$[(X_1 \sqcup X_2) \xrightarrow{f_1 \sqcup f_2} Y] - [X_1 \xrightarrow{f_1} f(X_1)] - [X_2 \xrightarrow{f_2} f(X_2)], \tag{2.4}$$

*whenever $X \xrightarrow{f} Y$ is a morphism of finite $G$-sets with a decomposition $X = X_1 \sqcup X_2$ as a disjoint union of $G$-sets, where $f_1 = f_{|X_1}$ and $f_2 = f_{|X_2}$. Moreover, the product of morphisms induces a commutative unital ring structure on $\Xi(G)$.*

**Notation 2.3.2.** *We use $\pi(f)$ to denote the image of the isomorphism class of $[X \xrightarrow{f} Y]$ in $\Xi(G)$. Also for the section $(T,S) \in \Sigma_G$, set $\langle T,S\rangle_G = \pi(G/S \to G/T)$.*

**Definition 2.3.3** ( [3], Definition 10.2)**.** *Let $G$ be a finite group. The section Burnside ring $\Gamma(G)$ of $G$ is the subring of the slice Burnside ring $\Xi(G)$ generated by the classes of Galois morphisms of $G$-sets.*

By Corollary 10.4 of [3] we know that the elements $\langle T,S\rangle_G$, where $(T,S)$ runs through a set $[\Sigma_G]$ of representatives of conjugacy classes of sections of $G$, form a basis of $\Gamma(G)$.

**Notation 2.3.4. The group $\Gamma(G,H)$.** *Let $G$ and $H$ be finite groups. For $(T,S) \in \Sigma_{G\times H}$ we denote the corresponding Galois morphism $(G\times H)/S \to (G\times H)/T$ and its isomorphism class by*

$$\left(\frac{G\times H}{S \trianglelefteq T}\right) \quad \textit{and} \quad \left[\frac{G\times H}{S \trianglelefteq T}\right] \tag{2.5}$$

*respectively.*

*Moreover we write $\Gamma(G,H)$ for the section Burnside ring of $G\times H$. It has a basis parameterized by the representatives of $G\times H$-conjugacy classes of sections of the direct product $G\times H$. As usual we shall consider the elements of $\Gamma(G,H)$ as morphisms from $H$ to $G$.*

**Notation 2.3.5.** *Let $A$ be a subgroup of $G\times H$, then we set*

- $p_1(A) = \{g \in G \mid \exists h \in H,\ (g,h) \in A\}$,
- $p_2(A) = \{h \in H \mid \exists g \in G,\ (g,h) \in A\}$,
- $k_1(A) = \{g \in G \mid (g,1) \in A\}$,
- $k_2(A) = \{h \in H \mid (1,h) \in A\}$.
- $\eta_A : p_2(A)/k_2(A) \to p_1(A)/k_1(A), h\cdot k_2(A) \mapsto g\cdot k_1(A)$ *provided* $(g,h) \in A$.

*By Goursat Theorem, the map $\eta_A$ above is a group isomorphism and the correspondence mapping $A$ to the quintuple $(p_1(A), k_1(A), \eta_A, k_2(A), p_2(A))$ establishes a bijective correspondence between the subgroups $A$ of $G \times H$ and the quintuples $(P, K, \eta, L, Q)$ with $K \trianglelefteq P \leqslant G$ and $L \trianglelefteq Q \leqslant H$ and $\eta : Q/L \to P/K$ a group isomorphism. We call $(p_1(A), k_1(A), \eta_A, k_2(A), p_2(A))$ the Goursat correspondent of $A$. Also we call a quintuple $(P, K, \eta, L, Q)$ satisfying the above conditions a Goursat quintuple.*

**Notation 2.3.6.** *Let $G$, $H$ and $K$ be finite groups. For the subgroups $A$ and $B$ of $G \times H$ and $H \times K$ respectively, we set*

$$A * B = \{(g, k) \in G \times K \mid \textit{there exist}\ h \in H\ \textit{with}\ (g, h) \in A\ \textit{and}\ (h, k) \in B\}. \quad (2.6)$$

**Notation 2.3.7. Invariants of sections.** *Let $(T, S) \in \Sigma_{G \times H}$. We associate $(T, S)$ with its left and right invariants*

$$l(T, S) := (p_1(T), k_1(T), p_1(S), k_1(S)), \quad (2.7)$$

$$r(T, S) := (p_2(T), k_2(T), p_2(S), k_2(S)). \quad (2.8)$$

*Moreover, by $l_0(T, S)$ and $r_0(T, S)$ we denote the pairs $(k_1(T), p_1(S))$ and $(k_2(T), p_2(S))$ respectively.*

**Notation 2.3.8.** *Let $f : X \to Y$ be a Galois morphism in $\Gamma(G, H)$. We define the opposite $f^{op} \in \Gamma(H, G)$ of $f$ as the Galois morphism $f^{op} : X^{op} \to Y^{op}$ where $f^{op}(x) = f(x)$ for any $x \in X$. Here the opposite biset $X^{op}$ is the $(H, G)$-biset equal to $X$ as a set with the action defined in the following way. For all $h \in H$, $x \in X$, $g \in G$, $h \cdot x \cdot g$ (in $X^{op}$) $= g^{-1} \cdot x \cdot h^{-1}$(in $X$). In particular, if $f = \left(\frac{G \times H}{S \trianglelefteq T}\right)$ then $f^{op} \cong \left(\frac{H \times G}{S^{op} \trianglelefteq T^{op}}\right)$, where $S^{op} := \{(h, g) \in H \times G \mid (g, h) \in S\}$.*

**Notation 2.3.9.** *Let $G$ and $H$ be finite groups. For any subgroups $A \leqslant B \leqslant H$ and any group homomorphism $\alpha : B \to G$, we set*

$$ {}_{\alpha}\Delta(A) := \{(\alpha(a), a) \mid a \in A\} \leqslant G \times H. $$

*For any subgroups $C \leqslant D \leqslant G$ and any group homomorphism $\alpha : D \to H$, we set*

$$ \Delta_{\alpha}(C) := \{(c, \alpha(c)) \mid c \in C\} \leqslant G \times H. $$

*If $\alpha$ is the inclusion map of a subgroup, we write $\Delta(A)$ and $\Delta(C)$ respectively.*

**Notation 2.3.10.** *Let $G$ be a finite group and $N$ be a normal subgroup of $G$. Denote by*

$$ \Delta_N(G) := \{(g_1, g_2) \in G \times G \mid g_1 N = g_2 N\} = (N \times \{1\})\Delta(G) = (\{1\} \times N)\Delta(G). \quad (2.9) $$

*Note that $\Delta_N(G)$ is a subgroup of $G \times G$, since $N$ is normal in $G$.*

Furthermore, for a subgroup $H$ of $G$ if $N \leq C_G(H)$, then for any $(g_1, g_2) \in \Delta_N(G)$ we have $g_1^{-1}g_2 \in N$ and hence $g_1^{-1}g_2 \in C_G(H)$. So $\Delta(H)$ is normal in $\Delta_N(G)$. In particular, $(\Delta_N(G), \Delta(H)) \in \Sigma_{G\times G}$, i.e., it is a section of $G \times G$.

**Notation 2.3.11.** *Let $G$ be a finite group. $H$ and $N$ subgroups of $G$ such that $N \trianglelefteq G$ and $N \leqslant C_G(H)$. We write*

$$ J_N^G := \mathrm{Inf}_{G/N}^G \times_{G/N} \mathrm{Def}_{G/N}^G \in B(G, G) \quad (2.10) $$

$$ \textit{and} \quad I_H^G := \mathrm{Ind}_H^G \times_H \mathrm{Res}_H^G \in B(G, G). \quad (2.11) $$

*Here, $B(G,G)$ is the biset Burnside group $B(G \times G^{op})$, which is the Grothendieck group of the category of finite $(G \times G^{op})$-sets.*

**Definition 2.3.12.** *For a finite group $G$ and a pair $(K,P) \in \mathcal{G}_G$ we define Galois morphism $E_{(K,P)}$ in $\Gamma(G,G)$ as*

$$E_{(K,P)} = \left( \frac{G \times G}{\Delta(P) \trianglelefteq \Delta_K(G)} \right), \tag{2.12}$$

*then the map $I_P^G \to J_K^G$ is a Galois morphism and its image in $\Gamma(G,G)$ is $[E_{(K,P)}]$. The left and the right invariants of the section $(\Delta_K(G), \Delta(P))$ are both given by $(G,K,P,1)$.*

**Definition 2.3.13.** *For each pair $(K,P) \in \mathcal{G}_G$ we set*

$$e_{(K,P)} := \frac{1}{|G:P|} \left[ E_{(K,P)} \right] \in E_G. \tag{2.13}$$

## 2.4. Crossed Modules

Since a basis for $\Gamma(G,H)$ is given in terms of sections of $G \times H$, we first tried to parametrize sections of direct products. For this result, which we will demonstrate in the subsequent section, we need the crossed modules. Crossed modules has been defined by J. H. C. Whitehead in 1946, and then Katherine Norrie in her paper [6] generalized some aspects of the theory of automorphisms from groups to crossed modules. Here we give briefly the necessary information about crossed modules. See [6] for further details.

**Definition 2.4.1.** *Let $G$ and $A$ be finite groups. We call the triple $(A, G, \partial)$ crossed module if $G$ acts on $A$, which we denote by $(g, a) \mapsto {}^g a$ and $\partial : A \to G$ is a group homomorphism such that*

(i) $\partial({}^g a) = g\partial(a)g^{-1}$,
(ii) ${}^{\partial(b)}a = bab^{-1}$

*for all $g \in G$ and $a, b \in A$.*

**Definition 2.4.2.** *Given crossed modules $(A, G, \partial)$ and $(A', G', \partial')$, we define a morphism from $(A, G, \partial)$ to $(A', G', \partial')$ as a pair $(\alpha, \beta)$ of group homomorphisms $\alpha : A \to A'$ and $\beta : G \to G'$ satisfying*

(a) $\partial'(\alpha(a)) = \beta(\partial(a))$ *for any $a \in A$ and*
(b) $\alpha({}^g a) = {}^{\beta(g)}\alpha(a)$ *for any $g \in G$ and $a \in A$.*

**Notation 2.4.3.** *The pair $(\alpha, \beta)$ is called an isomorphism, a monomorphism, an epimorphism or an automorphism of crossed modules if the homomorphisms $(\alpha, \beta)$ are both isomorphisms, monomorphisms, epimorphisms or automorphisms, respectively. The group of automorphisms of $(A, G, \partial)$ is defined and denoted by $\mathrm{Aut}(A, G, \partial)$.*

**Definition 2.4.4.** *Let $\alpha : G \to \mathrm{Aut}(A), g \mapsto \alpha_g$ be the action of $G$ on $A$ and $c : G \to \mathrm{Aut}(G), g \mapsto c_g$ be the conjugation action of $G$ on itself. For any $g \in G$ the pair $(\alpha_g, c_g)$ is an automorphism of $(A, G, \partial)$ and it is called an inner automorphism. The induced function $\theta : G \to \mathrm{Aut}(A, G, \partial)$ is a homomorphism, its image is a normal subgroup, denoted by $\mathrm{Inn}(A, G, \partial)$. As usual the quotient $\mathrm{Out}(A, G, \partial) = \mathrm{Aut}(A, G, \partial)/\mathrm{Inn}(A, G, \partial)$ is called the group of outer automorphisms of $(A, G, \partial)$.*

## 2.5. Green Biset Functor

**Definition 2.5.1.** *Let $k$ be a commutative ring with unity. We define $\mathcal{C}_k$ to be the following category.*

- *The objects of $\mathcal{C}_k$ are finite groups.*
- *If $G$ and $H$ are finite groups, then $Hom_{\mathcal{C}_k}(G, H) := kB(H, G)$.*
- *If $G$, $H$ and $K$ are finite groups, then the composition $x \circ y$ of the morphisms $x \in kB(H, G)$ and $y \in kB(K, H)$ is equal to $y \times_H x$.*
- *For any finite group $G$, the identity morphism of $G$ in $\mathcal{C}_k$ is equal to $\left[\mathrm{Id}_G\right]$.*

We name this category $\mathcal{C}_k$ as the biset category of finite groups. The category $\mathcal{C}_k$ is a preadditive category, and it is generated as preadditive category by the five types of morphisms, Ind, Res, Inf, Def and Iso, which we call elementary morphisms. Moreover, in [5] it is shown that any preadditive category whose objects are finite groups, and morphisms are generated by elementary morphisms is equivalent to the biset category.

**Notation 2.5.2.** *A $k$-linear functor $F : \mathcal{C}_k \to k$-mod is called a* biset functor over *$k$. We denote the functor category of all biset functors over $k$ by $\mathcal{F}_k$.*

For an example, let $\mathcal{M}$ be the subcategory of $\mathcal{C}_k$ such that the objects of $\mathcal{M}$ are all finite groups and $\mathrm{Hom}_{\mathcal{M}}(G, H)$ is the subgroup of $kB(H, G)$ generated by the classes of finite left and right free $(H, G)$-bisets. Then the biset functors on $\mathcal{M}$ are the global Mackey functors.

**Definition 2.5.3.** *Let $A$ be a biset functor over $k$. We call $A$ a* Green Biset Functor *over $k$ if there are bilinear maps $A(G) \times A(H) \to A(G \times H), (a, b) \mapsto a \times b$ for each pair of finite groups $G$ and $H$ and an element $\epsilon_A \in A(1)$ with the following properties.*

*(i) For any finite groups $G, H$ and $K$ and any $(a, b, c) \in A(G) \times A(H) \times A(K)$ we have*

$$(a \times b) \times c = A(\mathrm{Iso}^{\alpha})(a \times (b \times c)), \tag{2.14}$$

*where $\alpha$ is the canonical isomorphism from $G \times (H \times K)$ to $(G \times H) \times K$.*

*(ii) For any finite group $G$ and any $a \in A(G)$ we have*

$$A(\mathrm{Iso}^{\lambda})(\epsilon_A \times a) = a = A(\mathrm{Iso}^{\lambda'})(a \times \epsilon_A), \tag{2.15}$$

*where $\lambda : 1 \times G \to G$ and $\lambda' : G \times 1 \to G$ denote the canonical isomorphisms.*

*(iii) Let $X \in kB(G', G)$ and $Y \in kB(H', H)$. For any $(a, b) \in A(G) \times A(H)$ we have*

$$A(X \times Y)(a \times b) = A(X)(a) \times A(Y)(b). \tag{2.16}$$

A Green Biset Functor is a biset functor with a ring structure. More precisely, a Green Biset Functor on $\mathcal{C}_k$, with values in $k$-Mod is a monoid in the monoidal category $\mathcal{F}_k$. This means that a Green Biset Functor is an object $A$ of $\mathcal{F}_k$ together with a bilinear product

$$A(G) \times A(H) \to A(G \times H) \qquad (a, b) \mapsto a \times b \tag{2.17}$$

satisfying associativity, functoriality, and identity element conditions for objects $G$, $H$, $K$ of $\mathcal{C}_k$, and for any elements $a$, $b$, $c$ of $A(G)$, $A(H)$ and $A(K)$ respectively and $\epsilon_A \in A(1)$.

**Associativity** $(a \times b) \times c = \mathrm{Iso}(\alpha)(a \times (b \times c))$, where $\alpha$ is the canonical group isomorphism from $G \times (H \times K)$ to the group $(G \times H) \times K$.

**Identity Element** $a = \mathrm{Iso}(\lambda_G)(\epsilon_A \times a) = \mathrm{Iso}(\rho_G)(a \times \epsilon_A)$, where $\lambda_G : 1 \times G \to G$ and $\rho_G : G \times 1 \to G$ denote the canonical group isomorphisms.

**Functoriality** $A(\phi \times \psi)(a \times b) = A(\phi)(a) \times A(\psi)(b)$, where $\phi : G \to G'$ and $\psi : H \to H'$ are morphisms in $\mathcal{C}_k$.

Given two Green Biset Functors $A$ and $A'$ on $\mathcal{C}_k$ over $k$, a morphism of Green Biset Functors from $A$ to $A'$ is a morphism of biset functors $f : A \to A'$ such that

$$f_G(a) \times f_H(b) = f_{G\times H}(a \times b) \tag{2.18}$$

for any finite groups $G$, $H \in Ob(\mathcal{C}_k)$ and any $a \in A(G)$, $b \in A(H)$.

Then Green Biset Functors on $\mathcal{C}_k$ with values on $k$-Mod form a category $\mathrm{Green}_k$.

**Example 2.5.4.** *For instance, the Burnside functor $B$ is a Green Biset Functor on $\mathcal{C}_{\mathbb{Z}}$ with values in $\mathbb{Z}$-Mod. For finite groups $G$ and $H$, bilinear product $B(G) \times B(H) \to B(G \times H)$ is defined by sending $(X, Y)$, where $X$ is a $G$-set and $Y$ is an $H$-set, to the $(G \times H)$-set $X \times Y$. The identity element is $1 \in B(1) = \mathbb{Z}$.*

# 3. DETERMINATION OF SIMPLE MODULES OVER GREEN BISET FUNCTOR

## 3.1. $A$-Modules

**3.1.1. The category $\mathcal{P}_A$.** *Let $A$ be a Green Biset Functor. We associate a category $\mathcal{P}_A$ to $A$ as follows ( [5, Section 8.6]). The objects of $\mathcal{P}_A$ are all finite groups.*

*(i) For finite groups $G$ and $H$, we put $\mathrm{Hom}_{\mathcal{P}_A}(H, G) = A(G \times H)$.*

*(ii) For finite groups $G, H, K$, and morphisms $\alpha \in A(H \times K)$ and $\beta \in A(G \times H)$, the composition is defined by*

$$\beta \circ \alpha = A(\mathrm{Def}^{G\times\Delta(H)\times K}_{G\times K}\mathrm{Res}^{G\times H\times H\times K}_{G\times\Delta(H)\times K})(\beta \times \alpha). \tag{3.1}$$

*Here we identify $G \times K$ with $(G \times \Delta(H) \times K)/(1 \times \Delta(H) \times 1)$ via the obvious canonical isomorphism.*

*(iii) For a finite group $G$, the identity morphism of $G$ in $\mathcal{P}_A$ is*

$$A(\mathrm{Ind}^{G\times G}_{\Delta(G)}\mathrm{Inf}^{\Delta(G)}_{1})(\epsilon_A). \tag{3.2}$$

*Note that the unique morphism $kB \to A$ of Green Biset Functors induces a functor $\mathcal{C}_k \to \mathcal{P}_A$. In particular we may consider the basic operations associated to bisets as operations in $\mathcal{P}_A$.*

**Definition 3.1.2. $A$-modules.** *An* $A$-module *is a $k$-linear functor $\mathcal{P}_A \to k$-mod. Then $A$-Mod stands for the category of all $A$-modules. In particular if we take $A = kB$ then $\mathcal{F}_k$ and $kB$-Mod are the same categories. Moreover, note that any $A$-module is a biset functor.*

**Notation 3.1.3.** *Let $G$ be a finite group. Denote by $E_G^A = \mathrm{End}_{\mathcal{P}_A}(G)$ the endomorphism ring of $G$ in $\mathcal{P}_A$. Let $I_G^A$ stands for the ideal of $E_G^A$ generated by the endomorphisms that factor through a group of smaller order. Then we call the quotient $E_G^A/I_G^A$ the* essential algebra of $A$ at $G$ *and denote it by $\hat{A}(G)$.*

**Remark 3.1.4.** *For an $A$-module $F$ of a Green Biset Functor $A$ and finite group $G$, the evaluation $F(G)$ is a module over $E_G^A$. Furthermore, if $G$ is a minimal group such that $F(G)$ is non-zero, then $F(G)$ becomes an $\hat{A}(G)$-module.*

## 3.2. Simple Modules

**Definition 3.2.1.** *For a simple $A$-module $S$ the pair $(G, S(G))$ is called a seed for $S$, where $G$ is the minimal for $S$. By the general results in [5], we know that simple $A$-module $S$ is determined by its seed. Note that $S(G)$ is a simple $\hat{A}(G)$-module. So we denote the simple $A$-module determined by the seed $(G, V)$ by $S_{G,V}^A$. We call two seeds equivalent if the corresponding simple $A$-modules are isomorphic.*

### 3.2.2. Construction of simple $A$-module from the given seed.

*Let $V$ be a module of the endomorphism ring $E_G$. Construct an $A$-module $L_{G,V}$ such that its evaluation at any finite group $H$ is given as follows*

$$L_{G,V}(H) = A(H \times G) \otimes_{E_G} V. \tag{3.3}$$

*Note that the endomorphism ring acts by composition on the right and the action of $A$ is given by composition on the left. So by [1, Lemma 1], if $V$ is a simple module then the $A$-module $L_{G,V}$ has a unique maximal submodule $J_{G,V}$ given by*

$$J_{G,V}(H) = \{\sum_i x_i \otimes v_i \in L_{G,V}(H) \mid \sum_i (y \circ x_i)v_i = 0 \, for \, any \, y \in A(G \times H)\}. \tag{3.4}$$

*Thus we construct the simple head $S_{G,V}$ as the quotient $L_{G,V}/J_{G,V}$, furthermore it satisfies $S_{G,V}(G) \cong V$.*

### 3.3. Covering Algebra

**Definition 3.3.1.** *Let $A$ be a Green Biset Functor and $G$ a finite group. If $E'_G$ is a subalgebra of the endomorphism ring $E^A_G$ such that*

$$E^A_G = E'_G + I^A_G, \tag{3.5}$$

*then we call it a covering algebra for $E^A_G$.*

Observe that by the definition of covering algebra we can approximate the essential algebra $\hat{A}(G)$

$$\hat{A}(G) = E'_G/(E'_G \cap I^A_G). \tag{3.6}$$

Then under some assumptions we generalize very basic results about the covering algebra using its central idempotents.

**Notation 3.3.2.** *Let $E^c_G$ be the covering algebra for $A$. Let $(X, \leqslant)$ be a finite lattice with minimum element $x_0$. Suppose there is a function*

$$e : X \to E^c_G, \quad x \mapsto e_x, \quad \textit{such that} \tag{3.7}$$

$$\text{(a)}\, e_{x_0} = 1_{E^c_G} \quad \text{and} \quad \text{(b)}\, e_x \cdot e_y = e_{x \vee y}. \tag{3.8}$$

*In particular $e_x$ is an idempotent in $E^c_G$ for each $x \in X$. We define*

$$f_x = \sum_{x \leqslant y \in X} \mu_{x,y} e_y, \tag{3.9}$$

*where $\mu_{x,y}$ is the Mobius function of the poset $X$.*

*Then by the Mobius inversion formula, we also get*

$$e_x = \sum_{x \leqslant y \in X} f_y \quad \text{and} \quad 1_{E_G^c} = e_{x_0} = \sum_{x \in X} f_x. \tag{3.10}$$

The following result is a generalization of Proposition 4.4 from [4] and Theorem 10.1 from [7].

**Proposition 3.3.3.** *For all $x, y \in X$, we have*

$$e_x \cdot f_y = f_y \cdot e_x = \begin{cases} f_y & \text{if} \quad x \leq y, \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

$$\textit{and} \quad f_x \cdot f_y = \begin{cases} f_x & \text{if} \quad x = y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

*Proof.* The proof is almost identical to the proof of [4, Proposition 4.4]. We repeat it for completeness. For the first case, if $x \leqslant y$, then

$$e_x \cdot f_y = \sum_{y \leqslant z} \mu_{y,z} e_x \cdot e_z = \sum_{y \leqslant z} \mu_{y,z} e_z = f_y.$$

Here the second equality holds since $x \leqslant z$ implies $e_x \cdot e_z = e_z$. Also it is clear that $e_x$ and $f_y$ commute. For the rest of the claims, we argue by induction on $d = d_x + d_y$ where for any $x \in X$, the natural number $d_x$ is the largest $n \in \mathbb{N}_0$ such that there exists a chain $x = a_0 < a_1 < \cdots < a_n$ in $X$. Now if $d = 0$, then both $x$ and $y$ are maximal in $X$, hence we have $y = x$. Therefore $e_x \cdot f_y = e_x = f_y$. If $d = 1$, either $x$ is maximal and $y < x$ or $y$ is maximal and so $x < y$. If $x$ is maximal then $e_x \cdot f_y = e_x \cdot (e_y - e_x) = e_{x \vee y} - e_x = 0$. If $y$ is maximal and $x < y$ then $e_x \cdot f_y = e_x \cdot e_y = e_{x \vee y} = e_y = f_y$. Next suppose $d > 1$ and that the proposition holds for all smaller $d$.

We claim that if $f_x \cdot f_y$ is non-zero, then $x = y$. Indeed we have

$$f_x \cdot f_y \quad = \sum_{x \leqslant z, y \leqslant t} \mu_{x,z}\mu_{y,t} e_z \cdot e_t.$$

Here the product $e_z \cdot e_t = e_{z \vee t}$ and $x \vee y \leqslant z \vee t$. Hence the above equality becomes

$$f_x \cdot f_y \quad = \sum_{x \vee y \leqslant z} \mu_{x,z}\mu_{y,z} e_z.$$

In particular $e_{x \vee y} \cdot f_x \cdot f_y = f_x \cdot f_y \neq 0$. Hence by the first part, either $x < x \vee y$ or $y < x \vee y$ and we get $e_{x \vee y} \cdot f_x \cdot f_y = 0$ by induction. This is a contradiction, so we must have $x = y$.

Finally if $x = y$, then

$$e_x = e_x^2 = f_x^2 + \sum_{x < y} f_y$$

and hence $f_x^2 = f_x$ as required. $\square$

Note that the elements $e_x, x \in X$ generate a commutative subalgebra $\bigoplus_{x \in X} \mathbb{Z} e_x$ of $E_G^c$ and the set $\{f_x \mid x \in X\}$ is a basis of this algebra consisting of orthogonal idempotents summing up to the identity element.

# 4. THEOREMS FOR SECTIONS AND SECTION BURNSIDE RINGS

## 4.1. Goursat Theorem

**Theorem 4.1.1** (Goursat Theorem for sections)**.** *Let $G$ and $H$ be finite groups. Then there is a bijective correspondence between*

*(a) the set of all sections $(T, S)$ of $G \times H$ and*

*(b) the set of all pairs $((P_1, K_1, \eta_1, L_1, Q_1), (P_2, K_2, \eta_2, L_2, Q_2))$ of Goursat quintuples satisfying the following conditions.*

*(i) $P_2 \subseteq P_1, K_2 \subseteq K_1, L_2 \subseteq L_1, Q_2 \subseteq Q_1$.*

*(ii) $(P_2/K_2, P_1/K_1, \partial)$ and $(Q_2/L_2, Q_1/L_1, \partial')$ are crossed modules where $\partial$ and $\partial'$ are given by $\partial(xK_2) = xK_1$ and $\partial(xL_2) = xL_1$ and the actions are given by*

$$aK_1 \cdot cK_2 = aca^{-1}K_2, \quad a'L_1 \cdot c'L_2 = a'c'a'^{-1}L_2. \tag{4.1}$$

*(iii) $(\eta_2, \eta_1) : (Q_2/L_2, Q_1/L_1, \partial') \to (P_2/K_2, P_1/K_1, \partial)$ is an isomorphism of crossed modules.*

*The correspondence is given by mapping a section $(T, S)$ of $G\times H$ to the pair of Goursat correspondents of $T$ and $S$.*

*Proof.* Let $(P_1, K_1, \eta_1, L_1, Q_1)$ and $(P_2, K_2, \eta_2, L_2, Q_2)$ be the Goursat quintuples of $T$ and $S$ respectively. Then by Goursat Theorem, we have

(*) $1 \leqslant K_X \trianglelefteq P_X \leqslant G$ and $1 \leqslant L_X \trianglelefteq Q_X \leqslant H$ for $X \in \{1, 2\}$,

(**) $\eta_1 : Q_1/L_1 \to P_1/K_1$ and $\eta_2 : Q_2/L_2 \to P_2/K_2$ induced respectively by $T$ and $S$ are isomorphisms $\big($ i.e., $\eta_1(yL_1) = xK_1$ iff $(x, y) \in T$ $\big)$.

In addition, the condition $S \trianglelefteq T$ implies that the following statements also hold:

$$K_2 \trianglelefteq K_1,\ L_2 \trianglelefteq L_1,\ P_2 \trianglelefteq P_1 \quad and \quad Q_2 \trianglelefteq Q_1. \tag{4.2}$$

It is straightforward to check that $K_2 \trianglelefteq K_1$ and $L_2 \trianglelefteq L_1$. To prove that $P_2 \trianglelefteq P_1$, let $x \in P_2$ and $(g,h) \in T$ with $(x,y) \in S$ then since $S$ is normal in $T$, $({}^{g}x, {}^{h}y) ={}^{(g,h)} (x,y) \in S$, that is, ${}^{g}x \in P_2$. The rest is proved in the same way. So we have the condition (i) of (b). To prove that $(P_2/K_2, P_1/K_1, \partial)$ is a crossed module first observe that $P_1/K_2$ contains $P_2/K_2$ as a subgroup and $P_1/K_1$ as a quotient. Hence we have the following diagram

$$\begin{tikzcd} & P_1/K_2 \arrow[dr, "\pi"] & \\ P_2/K_2 \arrow[ur, "\iota"] & & P_1/K_1. \end{tikzcd} \tag{4.3}$$

That is the composition is $\partial = \pi \circ \iota$, which is given by $\partial(xK_2) = xK_1$. Note that we can restrict a conjugation action of $P_1/K_2$ on $P_2/K_2$ to a conjugation action of $P_1/K_1$ on $P_2/K_2$ if $K_1/K_2$ is a subgroup of the centralizer of $P_2/K_2$ in $P_1/K_2$. It is easy to observe that the last condition is equivalent to $[K_1, P_2] \leqslant K_2$. Indeed let $x \in P_2$ with $(x,y) \in S$ and $g \in K_1$. Note that $(g,1) \in T$. Then since S is normal in T, we get ${}^{(g,1)}(x,y) = ({}^{g}x, y) \in S$. In particular $x^{-1} \cdot {}^{g}x \in K_2$ which implies $xK_2 = {}^{g}xK_2$, as required. Hence to prove that $(P_2/K_2, P_1/K_1, \partial)$ is a crossed module, one needs to justify the two conditions given in Definition 2.4.1. For the first condition, let $gK_1 \in P_1/K_1$ and $xK_2 \in P_2/K_2$. Then

$$\partial({}^{gK_1}xK_2) = \partial({}^{g}xK_2) = {}^{g}xK_1 = {}^{gK_1}\partial(xK_2), \tag{4.4}$$

hence the first condition of the definition holds.

For the second condition, let also $yK_2 \in P_2/K_2$. Then

$$^{\partial(xK_2)}yK_2 = {}^{xK_1}yK_2 = {}^{x}yK_2 = {}^{xK_2}yK_2, \tag{4.5}$$

which proves the second condition. Thus $(P_2/K_2, P_1/K_1, \partial)$ is a crossed module. Replacing $K$ with $L$ and $P$ with $Q$ in the above arguments, one can also prove that $(Q_2/L_2, Q_1/L_1, \partial')$ is a crossed module.

Lastly, since $\eta_1$ and $\eta_2$ are already isomorphisms, we only need to check that the pair $(\eta_2, \eta_1)$ is a morphism of crossed modules, see Definition 2.4.2. Let $qL_2 \in Q_2/L_2$ and let $\eta_2(qL_2) = pK_2$. Then

$$\eta_1(\partial'(qL_2)) = \eta_1(qL_1) = pK_1, \tag{4.6}$$

where the last equation holds since $(p, q) \in S$ implies it is in $T$. On the other hand

$$\partial(\eta_2(qL_2)) = \partial(pK_2) = pK_1 \tag{4.7}$$

and hence the diagram in Definition 2.4.2 commutes. For the other condition, let also $(g, h) \in T$ so that $\eta_1(hL_1) = gK_1$. Then

$$\eta_2({}^{hL_1}qL_2) = \eta_2({}^{h}qL_2) = {}^{g}pK_2. \tag{4.8}$$

Here the last equality holds since $S \trianglelefteq T$. We clearly have

$${}^{g}pK_2 = {}^{\eta_1(hL_1)}\eta_2(qL_2). \tag{4.9}$$

Hence $(\eta_2, \eta_1)$ is a morphism of crossed modules and the mapping is well-defined and injective. Hence it is sufficient to prove that any element of the later set corresponds to a section in $G \times H$.

Let $((P_1, K_1, \eta_1, L_1, Q_1), (P_2, K_2, \eta_2, L_2, Q_2))$ be a pair of Goursat quintuples satisfying the above conditions. Also let $(T, S)$ be the pair of subgroups of $G \times H$ corresponding to these Goursat quintuples. Explicitly we have

$$S = \{(x, y) \in P_2 \times Q_2 \mid \eta_2(yL_2) = xK_2\} \text{ and } T = \{(g, h) \in P_1 \times Q_1 \mid \eta_1(hL_1) = gK_1\}. \tag{4.10}$$

We have to show that $S \trianglelefteq T$. Let $(x, y)$ be an element of $S$. Since by condition $(i)$ $P_2 \subseteq P_1$ and $Q_2 \subseteq Q_1$, the pair $(x, y)$ is also an element of $P_1 \times Q_1$. Moreover, by condition $(ii)$ we have $\partial(\eta_2(yL_2)) = \eta_1(\partial'(yL_2))$ and so $xK_1 = \eta_1(yL_1)$, hence $(x, y) \in T$. Then take an arbitrary element $(g, h)$ of $T$. Now since the action given in $(ii)$ is well defined $gxg^{-1}K_2 \in P_2/K_2$. Hence $gxg^{-1}$ is in $P_2$ and $P_2 \trianglelefteq P_1$. Similarly $Q_2$ is normal in $Q_1$. So ${}^{(g,h)}(x, y) \in P_2 \times Q_2$ and the condition $(iii)$ implies that $\eta_2({}^{hL_1}yL_2) = {}^{gK_1}\, \eta_2(yL_2)$, that is $\eta_2({}^{h}yL_2) = {}^{g}xK_2$ and therefore $S$ is normal in $T$. $\square$

## 4.2. Decomposition of Sections

**Proposition 4.2.1.** *Let $G$ and $H$ be finite groups, $(T, S) \in \Sigma_{G\times H}$ with Goursat correspondents $(P_T, K_T, \eta_T, L_T, Q_T)$ and $(P_S, K_S, \eta_S, L_S, Q_S)$. Then there are group isomorphisms*

*(a)*

$$\frac{P_S}{P_S \cap K_T} \cong \frac{Q_S}{Q_S \cap L_T}, \tag{4.11}$$

*(b)*

$$\frac{P_T}{P_S K_T} \cong \frac{Q_T}{Q_S L_T} \tag{4.12}$$

*and*

*(c)*

$$\frac{P_S \cap K_T}{K_S} \cong \frac{Q_S \cap L_T}{L_S}. \tag{4.13}$$

*In particular if* $P_T = G$, $K_S = 1$ *and* $K_T \leqslant P_S$ *then* $|G| \leqslant |H|$.

*Proof.* (a) First note that since $K_T$ and $P_S$ are both normal in $P_T$, the intersection $P_S \cap K_T$ is normal in $P_S$. Hence the isomorphism $\eta_S : Q_S/L_S \to P_S/K_S$ pre-composed by $\pi : Q_S \to Q_S/L_S$ and post-composed with $\pi : P_S/K_S \to P_S/(P_S \cap K_T)$ becomes a surjective homomorphism

$$\tilde{\eta}_S : Q_S \to P_S/(P_S \cap K_T).$$

The kernel of $\tilde{\eta}_S$ is clearly $Q_S \cap L_T$. Note that by the second isomorphism theorem we also have the isomorphisms

$$P_S K_T/K_T \cong P_S/(P_S \cap K_T) \cong Q_S/(Q_S \cap L_T) \cong Q_S L_T/L_T.$$

(b) As in the above case, post-compose $\eta_T$ with the canonical homomorphism $\pi : P_T/K_T \to P_T/(P_S K_T)$ to obtain

$$\tilde{\eta}_T : Q_T/L_T \to P_T/(P_S K_T).$$

Let $(p, q) \in T$. Then we have $qL_T \in \ker \tilde{\eta}_T$ if and only if $pK_T \in P_S K_T/K_T$. We claim that the last condition is equivalent to $qL_T \in Q_S L_T/L_T$. There is nothing to prove if $p \in K_T$. Suppose $p \in P_S$. It is sufficient to prove that $qL_T \in Q_S L_T/L_T$ and this follows directly from the compatibility of $\eta_S$ and $\eta_T$. Hence the kernel of $\tilde{\eta}_T$ is $Q_S L_T/Q_S$ and the result follows from the first and the third isomorphism theorems.

(c) Consider the restriction $\bar{\eta}_S$ of $\eta_S$ to the subgroup $(Q_S \cap L_T)/L_S$. Since $\eta_S$ is an isomorphism, the restriction of $\bar{\eta}_S$ to its image is an isomorphism. To determine its image, let $q \in Q_S \cap L_T$ and let $p \in G$ be such that $(p, q) \in S$. Then since $(1, q) \in T$, we also get $(p, 1) \in T$, that is, $p \in K_T$. Therefore $pK_S$ is contained in $(P_S \cap K_T)/K_S$, as required.

For the final statement, we write the order of $G$ as

$$|G| = |G : P_T| \cdot |P_T : P_S K_T| \cdot |P_S K_T : K_T| \cdot |K_T : P_S \cap K_T| \cdot |P_S \cap K_T : K_S| \cdot |K_S|$$

and similarly the order of $H$ as

$$|H| = |H : Q_T| \cdot |Q_T : Q_S L_T| \cdot |Q_S L_T : L_T| \cdot |L_T : Q_S \cap L_T| \cdot |Q_S \cap L_T : L_S| \cdot |L_S|$$

Then with the above isomorphisms we have

$$\frac{|G|}{|H|} = \frac{|G : P_T| \cdot |K_T : P_S \cap K_T| \cdot |K_S|}{|H : Q_T| \cdot |L_T : Q_S \cap L_T| \cdot |L_S|}.$$

Since by the assumption $P_T = G$, $K_S = 1$ and $K_T \leqslant P_S$ we deduce that $\frac{|G|}{|H|} \leqslant 1$. □

The following result collects relations between the left and the right invariants of the sections.

**Proposition 4.2.2.** *Let $G$, $H$ and $K$ be finite groups. Then the following hold for the sections $(T, S) \in \Sigma_{G \times H}$ and $(V, U) \in \Sigma_{H \times K}$.*

*(a)* $k_1(T) \le k_1(T{*}V) \le p_1(T{*}V) \le p_1(T)$ *and* $k_2(V) \le k_2(T{*}V) \le p_2(T{*}V) \le p_2(V)$ *similarly*
$k_1(S) \le k_1(S{*}U) \le p_1(S{*}U) \le p_1(S)$ *and* $k_2(U) \le k_2(S{*}U) \le p_2(S{*}U) \le p_2(U)$.

*(b) In particular,* $l_0(T, S) \preceq l_0(T * V, S * U)$ *and* $r_0(T, S) \preceq r_0(T * V, S * U)$.

(c) *Let* $\eta_T : p_2(T)/k_2(T) \to p_1(T)/k_1(T)$ *and* $\eta_S : p_2(S)/k_2(S) \to p_1(S)/k_1(S)$ *be the isomorphisms from Proposition 4.2.1. Then one has*

$$k_1(T * V)/k_1(T) \cong \eta_T((p_2(T) \cap k_1(V)k_2(T))/k_2(T)),$$

*similarly*

$$k_1(S * U)/k_1(S) \cong \eta_S((p_2(S) \cap k_1(U)k_2(S))/k_2(S)).$$

(d) *If* $r(T,S) = l(V,U)$ *then* $l(T * V, S * U) = l(T,S)$ *and* $r(T * V, S * U) = r(V,U)$.

*Proof.* The parts (a) and (c) are reformulations of [5, Lemma 2.3.22]. Part (b) follows from part (a) and the definition of the partial order and (d) is an easy calculation. □

**Definition 4.2.3.** *Given finite groups* $G$, $H$, $K$ *and Galois morphisms* $\left(\frac{G\times H}{X \xrightarrow{g} Y}\right)$ *and* $\left(\frac{H\times K}{X' \xrightarrow{f} Y'}\right)$ *we define their composition as the morphism*

$$\left(\frac{G \times K}{X \times_H X' \xrightarrow{g\times_H f} Y \times_H Y'}\right). \tag{4.14}$$

*Here* $X \times_H X'$ *and* $Y \times_H Y'$ *are the usual amalgamated products of bisets and* $(g \times_H f)(x,_H x') = \big(g(x),_H f(x')\big)$. *It is a Galois Morphism of* $G\times K$*-sets by Proposition 9.13 of [3]. Moreover the above composition induces a bilinear associative product*

$$\Gamma(G,H) \times \Gamma(H,K) \to \Gamma(G,K). \tag{4.15}$$

The following corollary gives an explicit formula for the composition of two bases elements of the section Burnside ring.

**Corollary 4.2.4** (Mackey Formula)**.** *Let* $(T,S) \in \Sigma_{G\times H}$ *and* $(V,U) \in \Sigma_{H\times K}$. *There exists an isomorphism of Galois morphisms*

$$\left(\frac{G\times H}{S \trianglelefteq T}\right) \times_H \left(\frac{H\times K}{U \trianglelefteq V}\right) \cong \bigsqcup_{t\in p_2(S)\backslash H/p_1(U)} \left(\frac{G\times K}{S *{}^{(t,1)}U \trianglelefteq T *{}^{(t,1)}V}\right). \tag{4.16}$$

*Proof.* By definition

$$\left(\frac{G\times H}{S\trianglelefteq T}\right)\times_H\left(\frac{H\times K}{U\trianglelefteq V}\right):=\left(\frac{G\times H}{S}\right)\times_H\left(\frac{H\times K}{U}\right)\to\left(\frac{G\times H}{T}\right)\times_H\left(\frac{H\times K}{V}\right). \tag{4.17}$$

Then applying the Mackey formula for bisets to the right hand side we get

$$\left(\frac{G\times H}{S\trianglelefteq T}\right)\times_H\left(\frac{H\times K}{U\trianglelefteq V}\right)\cong\bigsqcup_{t\in p_2(S)\backslash H/p_1(U)}\left(\frac{G\times K}{S*{}^{(t,1)}U}\right)\to\bigsqcup_{t\in p_2(T)\backslash H/p_1(V)}\left(\frac{G\times K}{T*{}^{(t,1)}V}\right). \tag{4.18}$$

So by Lemma 3.3 of [3] we have the isomorphism of the following Galois morphisms

$$\left(\frac{G\times H}{S\trianglelefteq T}\right)\times_H\left(\frac{H\times K}{U\trianglelefteq V}\right)\cong\bigsqcup_{t\in p_2(S)\backslash H/p_1(U)}\left(\frac{G\times K}{S*{}^{(t,1)}U}\right)\to\bigsqcup_{t\in p_2(S)\backslash H/p_1(U)}\left(\frac{G\times K}{T*{}^{(t,1)}V}\right). \tag{4.19}$$

Hence by the definition of composition

$$\left(\frac{G\times H}{S\trianglelefteq T}\right)\times_H\left(\frac{H\times K}{U\trianglelefteq V}\right)\cong\bigsqcup_{t\in p_2(S)\backslash H/p_1(U)}\left(\frac{G\times K}{S*{}^{(t,1)}U}\right)\to\left(\frac{G\times K}{T*{}^{(t,1)}V}\right) \tag{4.20}$$

$$\cong\bigsqcup_{t\in p_2(S)\backslash H/p_1(U)}\left(\frac{G\times K}{S*{}^{(t,1)}U\trianglelefteq T*{}^{(t,1)}V}\right). \tag{4.21}$$

□

**4.2.5. Basic bisets.** *Let $G$ be a finite group and $X$ be a $G$-set. The identity morphism $X\to X$ is a Galois morphism and induces a homomorphism from the Burnside ring $B(G)$ to $\Gamma(G)$. More explicitly if $X=G/H$ for some subgroup $H$ of $G$, then its image in $\Gamma(G)$ is $[H,H]_G$. In particular the basic bisets of induction, restriction, inflation and deflation have images in $\Gamma(G,G)$ as listed below.*

*Let $H \leqslant G$ and $N \trianglelefteq G$ and $\pi : G \to G/N$ be the natural homomorphism.*

$$\mathrm{Ind}_H^G := \Big(\frac{G \times H}{\Delta(H) \trianglelefteq \Delta(H)}\Big) \in \Gamma(G,H), \qquad \mathrm{Res}_H^G := \Big(\frac{H \times G}{\Delta(H) \trianglelefteq \Delta(H)}\Big) \in \Gamma(H,G),$$
$$\mathrm{Inf}_{G/N}^G := \Big(\frac{G \times G/N}{\Delta_\pi(G) \trianglelefteq \Delta_\pi(G)}\Big) \in \Gamma(G,G/N),\ \mathrm{Def}_{G/N}^G := \Big(\frac{G/N \times G}{{}_\pi\Delta(G) \trianglelefteq {}_\pi\Delta(G)}\Big) \in \Gamma(G/N,G).$$

*With these definitions we may decompose any section into a product as follows.*

**Proposition 4.2.6.** *Let $G$ and $H$ be finite groups and $(T,S)$ be a section of $G \times H$ with $l(T,S) = (P_T, K_T, P_S, K_S)$ and $r(T,S) = (Q_T, L_T, Q_S, L_S)$. Define $\bar{G} := P_T/K_S$, $\bar{H} := Q_T/L_S$, $\bar{S} := can(S/(K_S \times L_S)) \leq \bar{G} \times \bar{H}$ and $\bar{T} := can(T/(K_S \times L_S)) \leq \bar{G} \times \bar{H}$, where*

$$can : (P_T \times Q_T)/(K_S \times L_S) \to \bar{G} \times \bar{H}$$

*is the canonical homomorphism. Then*

$$\Big(\frac{G \times H}{S \trianglelefteq T}\Big) \cong \mathrm{Ind}_{P_T}^G \times_{P_T} \mathrm{Inf}_{\bar{G}}^{P_T} \times_{\bar{G}} \left(\frac{\bar{G} \times \bar{H}}{\bar{S} \trianglelefteq \bar{T}}\right) \times_{\bar{H}} \mathrm{Def}_{\bar{H}}^{Q_T} \times_{Q_T} \mathrm{Res}_{Q_T}^H. \tag{4.22}$$

*Furthermore we have*

$$k_1(\bar{S}) = 1,\ k_2(\bar{S}) = 1,\ \textit{and}\ p_1(\bar{T}) = \bar{G},\ p_2(\bar{T}) = \bar{H}.$$

*Proof.* Let denote the middle element in the equation 4.22 by $X$. Then, by definition 4.2.5 of induction, restriction, deflation and inflation we know that the product $\mathrm{Ind}_{P_T}^G \mathrm{Inf}_{P_T/K_S}^{P_T} X\ \mathrm{Def}_{Q_T/L_S}^{Q_T} \mathrm{Res}_{Q_T}^H$ is congruent to

$$\Big(\tfrac{G \times P_T}{\Delta(P_T) \trianglelefteq \Delta(P_T)}\Big)\Big(\tfrac{P_T \times P_T/K_S}{\Delta_\pi(P_T) \trianglelefteq \Delta_\pi(P_T)}\Big)\Big(\tfrac{P_T/K_S \times Q_T/L_S}{S/(K_S \times L_S) \trianglelefteq T/(K_S \times L_S)}\Big)\Big(\tfrac{Q_T/L_S \times Q_T}{{}_\pi\Delta(Q_T) \trianglelefteq {}_\pi\Delta(Q_T)}\Big)\Big(\tfrac{Q_T \times H}{\Delta(Q_T) \trianglelefteq \Delta(Q_T)}\Big).$$

Now using the Mackey formula 4.2.4 let first compose two paranthesis in the left hand side and then the morphisms in the right hand side of $X$.

$$\Big(\frac{G\times P_T}{\Delta(P_T)\trianglelefteq\Delta(P_T)}\Big)\Big(\frac{P_T\times P_T/K_S}{\Delta_\pi(P_T)\trianglelefteq\Delta_\pi(P_T)}\Big)\cong$$

$$\bigsqcup_{t\in p_2(\Delta(P_T))\backslash P_T/p_1(\Delta_\pi(P_T))}\Big(\frac{G\times P_T/K_S}{\Delta(P_T)*{}^{(t,1)}\Delta_\pi(P_T)\trianglelefteq\Delta(P_T)*{}^{(t,1)}\Delta_\pi(P_T)}\Big).$$

We have only the case $t=1$, since $p_2(\Delta(P_T))=P_T$. Thus the sum above

$$\cong\Big(\frac{G\times P_T/K_S}{\Delta(P_T)*\Delta_\pi(P_T)\trianglelefteq\Delta(P_T)*\Delta_\pi(P_T)}\Big)$$

and with an easy computation $\Delta(P_T)*\Delta_\pi(P_T)=\Delta_\pi(P_T)$. Hence

$$\Big(\frac{G\times P_T}{\Delta(P_T)\trianglelefteq\Delta(P_T)}\Big)\Big(\frac{P_T\times P_T/K_S}{\Delta_\pi(P_T)\trianglelefteq\Delta_\pi(P_T)}\Big)\cong\Big(\frac{G\times P_T/K_S}{\Delta_\pi(P_T)\trianglelefteq\Delta_\pi(P_T)}\Big).$$

For the composition of morphisms in the right hand side, note that similarly to the previous case $p_2(\Delta(Q_T))=Q_T$ and ${}_\pi\Delta(Q_T)*\Delta(Q_T)={}_\pi\Delta(Q_T)$. Thus by the Mackey formula

$$\Big(\frac{Q_T/L_S\times Q_T}{{}_\pi\Delta(Q_T)\trianglelefteq_\pi\Delta(Q_T)}\Big)\Big(\frac{Q_T\times H}{\Delta(Q_T)\trianglelefteq\Delta(Q_T)}\Big)\cong\Big(\frac{Q_T/L_S\times H}{{}_\pi\Delta(Q_T)\trianglelefteq_\pi\Delta(Q_T)}\Big).$$

Now again applying the Mackey formula it follows that

$$\cong\Big(\frac{G\times P_T/K_S}{\Delta_\pi(P_T)\trianglelefteq\Delta_\pi(P_T)}\Big)\Big(\frac{P_T/K_S\times Q_T/L_S}{S/(K_S\times L_S)\trianglelefteq T/(K_S\times L_S)}\Big)\Big(\frac{Q_T/L_S\times H}{{}_\pi\Delta(Q_T)\trianglelefteq_\pi\Delta(Q_T)}\Big)$$

$$\cong\Big(\frac{G\times Q_T/L_S}{P_S\times L_S/L_S\trianglelefteq P_T\times Q_T/L_S}\Big)\Big(\frac{Q_T/L_S\times H}{{}_\pi\Delta(Q_T)\trianglelefteq{}_\pi\Delta(Q_T)}\Big)\cong\Big(\frac{G\times H}{S\trianglelefteq T}\Big).$$

$\square$

# 5. THE ENDOMORPHISM RING

In chapter 3 we introduced a machinery for Green Biset Functors that helps us to study essential algebra more explicitly. In this chapter we apply this method to the section biset functor and demonstrate every detail how this machinery works. Let $k$ be a field of characteristic zero. We denote the endomorphism ring $k \otimes \Gamma(G \times G)$ by $E_G^k$ or $E_G$.

## 5.1. Idempotents

To apply Proposition 3.3.3 to our specific case we introduce the poset $\mathcal{G}_G$, which is already has been defined in the Chapter 2. Due to the poset structure on $\mathcal{G}_G$ the pair $(1, G)$ is the minimum element and the pair $(G, 1)$ is the maximum element of this set. In particular joins exists in $\mathcal{G}_G$ and given $(K, P), (L, Q) \in \mathcal{G}_G$, one has $(K, P) \vee (L, Q) = (KL, P \cap Q)$. So we define the map $e$ as the follows.

**Definition 5.1.1.** *Let $E_G^c$ be the covering algebra for $\Gamma$ and $G$ be a finite group. For each $(K, P) \in \mathcal{G}_G$ we define*

$$e : \mathcal{G}_G \to E_G^c \tag{5.1}$$

$$(K, P) \mapsto e_{(K,P)}. \tag{5.2}$$

*Note that, by definition $e_{(1,G)} = [E_{(1,G)}] = \left[\frac{G \times G}{\Delta(G) \trianglelefteq \Delta(G)}\right] = 1_{E_G^c}$.*

Our next proposition aims to show that those elements are idempotents of the covering algebra.

**Proposition 5.1.2.** *Let $G$ and $H$ be finite groups. If $(K,P),(K',P') \in \mathcal{G}_G$ and $(T,S) \in \Sigma_{G\times H}$ with $l(T,S) = (G,K',P',1)$ then*

*(a)* $E_{(K,P)} \times_G (\frac{G\times H}{S\trianglelefteq T}) \cong |P\backslash G/P'|(\frac{G\times H}{U\trianglelefteq V})$ *with $l(V,U) = (G,KK',P\cap P',1)$, In particular one has*

$$E_{(K,P)} \times_G E_{(K',P')} \cong |P\backslash G/P'| E_{(KK',P\cap P')}. \tag{5.3}$$

*(b) Assume that $(K,P) \preceq (K',P')$. Then*

$$E_{(K,P)} \times_G \Big(\frac{G\times H}{S\trianglelefteq T}\Big) \cong |G:P|\Big(\frac{G\times H}{S\trianglelefteq T}\Big). \tag{5.4}$$

*In particular*

$$E_{(K,P)} \times_G E_{(K',P')} \cong |G:P| E_{(K',P')}. \tag{5.5}$$

*(c) Assume that $r(T,S) = (H,L,Q,1)$ for some $(L,Q) \in \mathcal{G}_H$. Then*

$$\Big(\frac{G\times H}{S\trianglelefteq T}\Big) \times_H \Big(\frac{G\times H}{S\trianglelefteq T}\Big)^{op} \cong |H:Q| E_{(K',P')}. \tag{5.6}$$

*Proof.* (a) By definition $E_{(K,P)} = \pi(I_P^G \to J_K^G)$, and since $I_P^G = \Big(\frac{G\times G}{\Delta(P)}\Big)$, and $J_K^G = \Big(\frac{G\times G}{\Delta_K(G)}\Big)$ the given product is

$$E_{(K,P)} \times_G (\frac{G\times H}{S\trianglelefteq T}) \cong \Big(\frac{G\times H}{\Delta(P)\trianglelefteq \Delta_K(G)}\Big)\Big(\frac{G\times H}{S\trianglelefteq T}\Big), \tag{5.7}$$

and then using the Mackey formula

$$\cong \bigsqcup_{t\in P\backslash G/P'} \Big(\frac{G\times H}{\Delta(P) *^{(t,1)} S \trianglelefteq \Delta_K(G) *^{(t,1)} T}\Big) \tag{5.8}$$

$$\cong \bigsqcup_{t\in P\backslash G/P'} \Big(\frac{G\times H}{(\Delta(P) * S)^{(t,1)} \trianglelefteq (\Delta_K(G) * T)^{(t,1)}}\Big). \tag{5.9}$$

Now since the morphisms $G/S \to G/T$ and $G/{}^gS \to G/{}^gT$ are isomorphic, for any $g \in G$, and any section $(T, S)$ of $G$ the above sum isomorphic to the following one

$$\cong |P \setminus G/P'|\Big(\frac{G \times H}{\Delta(P) * S \trianglelefteq \Delta_K(G) * T}\Big). \tag{5.10}$$

Then it is an easy computation to show that the left invariant of the section above is equal to $(G, KK', P \cap P', 1)$.

Furthermore we also have,

$$E_{(K,P)} \times_G E_{(L,Q)} = \pi(I_P^G \to J_K^G) \times_G \pi(I_Q^G \to J_L^G) \tag{5.11}$$

$$\sum_{x \in P\setminus G/Q} \pi(I_{P\cap {}^xQ}^G \to J_{K^xL}^G) = |P \setminus G/Q| E_{(KL,P\cap Q)} \tag{5.12}$$

since $Q$ and $L$ are normal subgroups in G.

(b) Recall that $(K, P) \preceq (K', P')$ means $K \leqslant K'$ and $P \geqslant P'$. Hence we have $|P \setminus G/P'| = |G : P|$. If $(g, h) \in \Delta_K(G) * T$ then there exists $x \in G$ such that $(g, x) \in \Delta_K(G)$ and $(x, h) \in T$. But then $gK = xK$ and so $x^{-1}g \in K \leqslant K'$. Thus $(x^{-1}g, 1) \in T$, which implies that $(g, h) \in T$. Therefore $\Delta_K(G) * T = T$ and similarly $\Delta(P) * S = S$. So we are done by part (a).

(c) Observer that $S * S^{op} = \Delta(p_1(S)) = \Delta(P')$ and $T * T^{op} = \Delta_{K'}(G)$. Thus using Mackey formula we have the desired result.

□

**Proposition 5.1.3.** *The elements $e_{(K,P)}, (K, P) \in \mathcal{G}_G$ are idempotents in $E_G$ and we have*

$$e_{(K,P)} \cdot e_{(L,Q)} = e_{(K,P)\vee(L,Q)} \quad \text{and} \quad e_{(1,G)} = 1_{E_G}. \tag{5.13}$$

*Proof.* By definition $e_{(K,P)} := \frac{1}{|G:P|}\left[E_{(K,P)}\right]$. Hence by the Proposition 5.1.2

$$e_{(K,P)} \cdot e_{(L,Q)} = e_{(KL,P\cap Q)} = e_{(K,P)\vee(L,Q)}. \tag{5.14}$$

$\square$

In particular the conditions in Notation 3.3.2 are satisfied. Also setting

$$f_{(K,P)} := \sum_{(K,P)\preceq(L,Q)\in\mathcal{G}_G} \mu_{(K,P),(L,Q)} e_{(L,Q)}, \tag{5.15}$$

where $\mu_{?,?}$ is the Mobius function of the poset $\mathcal{G}_G$, we get

$$e_{(K,P)} = \sum_{(K,P)\preceq(L,Q)\in\mathcal{G}_G} f_{(L,Q)} \quad \text{and} \tag{5.16}$$

$$\sum_{(K,L)\in\mathcal{G}_G} f_{(K,L)} = e_{(1,G)} = 1 \in E_G \tag{5.17}$$

and the following corollary of Proposition 3.3.3.

**Proposition 5.1.4.** *For all $(K,P),(L,Q) \in \mathcal{G}_G$ we have*

$$e_{(K,P)} \cdot f_{(L,Q)} = f_{(L,Q)} \cdot e_{(K,P)} = \begin{cases} f_{(L,Q)}, & \textit{if} \quad (K,P) \preceq (L,Q), \\ 0, & \textit{otherwise}, \end{cases} \tag{5.18}$$

$$\textit{and} \quad f_{(K,P)} \cdot f_{(L,Q)} = \begin{cases} f_{(K,P)}, & \textit{if} \quad (K,P) = (L,Q), \\ 0, & \textit{otherwise}. \end{cases} \tag{5.19}$$

# 6. LINKAGE

In this chapter we introduce an equivalence relation on the poset $\mathcal{G}_G$, which we call Linkage system. Our aim is to determine the central idempotents of the covering algebra $E_G^c$. Moreover, we unearth the relation between crossed modules and the Linkage system.

## 6.1. Linked Pairs

**Definition 6.1.1.** *Let $G$ and $H$ be finite groups. For $(K, P) \in \mathcal{G}_G$ and $(L, Q) \in \mathcal{G}_H$, we say that the quadruples $(G, K, P, 1)$ and $(H, L, Q, 1)$ are linked if there exists $(T, S) \in \Sigma_{G\times H}$ with $l(T, S) = (G, K, P, 1)$ and $r(T, S) = (H, L, Q, 1)$. In this case, we write $(G, K, P, 1) \underset{(T,S)}{\sim} (H, L, Q, 1)$ or just $(G, K, P, 1) \sim (H, L, Q, 1)$. If $H = G$, and $(K, P), (L, Q) \in \mathcal{G}_G$ then we call it being $G$-linked. We write $(K, P) \sim_G (L, Q)$ or just $(K, P) \sim (L, Q)$. Being linked is clearly an equivalence relation and we write $\{K, P\}_G$ for the $G$-linkage class of $(K, P)$.*

**Definition 6.1.2.** *Let $(K, P) \in \mathcal{G}_G$. Define an action of $G/K$ on $P$ as follows*

$$gK \cdot p = gpg^{-1}. \tag{6.1}$$

*It is well defined since $P$ is normal in $G$ and $[K, P] = 1$. Let $i_P : P \to G/K$ be the restriction of the natural homomorphism $G \to G/K$.Then $(P, G/K, i_P)$ becomes a crossed module.*

**Proposition 6.1.3.** *Let $G$ and $H$ be finite groups and $(K, P) \in \mathcal{G}_G$ and $(L, Q) \in \mathcal{G}_H$. Then the following are equivalent:*

*(a)* $(G, K, P, 1) \sim (H, L, Q, 1)$.

*(b)* $(P, G/K, i_P)$ *isomorphic to* $(Q, H/L, i_Q)$ *as crossed modules.*

*Proof.* We first show that $(a)$ implies $(b)$. Since $(G, K, P, 1) \sim (H, L, Q, 1)$ there exists $(T, S) \in \Sigma_{G\times H}$ with left invariant $(G, K, P, 1)$ and right invariant $(H, L, Q, 1)$. Thus by the Goursat theorem there are group isomorphisms $\eta_1 : H/L \to G/K$ and $\eta_2 : Q \to P$. Now since $(T, S)$ is a section of $G \times H$, by the Theorem 4.1.1 we have $(\eta_2, \eta_1)$ is an isomorphism of crossed modules $(Q, H/L, i_Q)$ and $(P, G/K, i_P)$. For the converse, if $(\alpha, \beta) : (Q, H/L, i_Q) \to (P, G/K, i_P)$ is an isomorphism of crossed modules, then the pair $((G, K, \beta, L, H), (P, 1, \alpha, 1, Q))$ of Goursat quintuples satisfy the conditions of the Theorem 4.1.1 (b) and hence there exists a section $(T, S)$ with left and right invariants given by $(G, K, P, 1)$ and $(H, L, Q, 1)$, respectively. In particular $(G, K, P, 1)$ and $(H, L, Q, 1)$ are linked. $\square$

### 6.2. Central Idempotents

**Notation 6.2.1.** *Let $G$ be a finite group. For $(K, P), (L, Q) \in \mathcal{G}_G$, we write $\{K, P\}_G \preceq \{L, Q\}_G$ if and only if there exists $(K', P') \in \{K, P\}_G$ and $(L', Q') \in \{L, Q\}_G$ with $(K', P') \preceq (L', Q')$.*

**Proposition 6.2.2.** *Let $G$ be a finite group. Then $\{K, P\}_G \preceq \{L, Q\}_G$ is a partial order on the set $\mathcal{G}_G/\sim$ of linkage classes of $\mathcal{G}_G$.*

*Proof.* It is obviously reflexive since $(K, P) \preceq (K, P)$ implies $\{K, P\}_G \preceq \{K, P\}_G$ for every element $(K, P) \in \mathcal{G}_G$. To see that this relation is transitive, we show that $\{K, P\}_G \preceq \{L, Q\}_G$ if and only if for each $(K', P') \sim_G (K, P)$ there exists $(L', Q') \sim (L, Q)$ such that $(K', P') \preceq (L', Q')$. Clearly the converse follows from the definition. To prove the forward implication, suppose $(K, P) \preceq (L, Q)$ and $(K', P') \underset{(T,S)}{\sim} (K, P)$ and consider

$$E_{(L,Q)} \cdot \Big(\frac{G \times G}{S \trianglelefteq T}\Big) = \Big(\frac{G \times G}{S' \trianglelefteq T'}\Big). \tag{6.2}$$

By Proposition 5.1.2(a), we have $l_0(T', S') = (L, Q)$. Also putting $r_0(T', S') = (L', Q')$, we get $(L', Q') \sim (L, Q)$ and $(K', P') \preceq (L', Q')$.

Then transitivity and anstisymmetry properties easily follow from this fact. $\square$

Now we take the class sums of the idempotents defined in the previous chapter.

**Definition 6.2.3.** *We write*

$$e_{\{K,P\}_G} := \sum_{(K',P') \in \{K,P\}_G} e_{(K',P')} \in \Gamma(G,G) \tag{6.3}$$

$$\text{and} \quad f_{\{K,P\}_G} := \sum_{(K',P') \in \{K,P\}_G} f_{(K',P')} \in \Gamma(G,G) \tag{6.4}$$

*for* $(K,P) \in \mathcal{G}_G$.

Since the conditions of Notation 3.3.2 are also satisfied by the elements $e_{\{K,P\}_G}$ and $f_{\{K,P\}_G}$ as $(K,P)$ runs over linkage classes in $\mathcal{G}_G$, we obtain the following corollary.

**Proposition 6.2.4.** *Let* $(K,P),(L,Q) \in \mathcal{G}_G$*. Then*

$$e_{\{K,P\}_G} f_{\{L,Q\}_G} = f_{\{L,Q\}_G} e_{\{K,P\}_G} = 0 \quad \text{unless} \quad \{K,P\}_G \preceq \{L,Q\}_G, \tag{6.5}$$

$$e_{\{K,P\}_G} f_{\{K,P\}_G} = f_{\{K,P\}_G} e_{\{K,P\}_G} = f_{\{K,P\}_G} \tag{6.6}$$

$$\text{and} \quad f_{\{K,P\}_G} f_{\{L,Q\}_G} = \begin{cases} f_{\{K,P\}_G}, & \text{if} \quad \{K,P\}_G = \{L,Q\}_G, \\ 0, & \text{otherwise.} \end{cases} \tag{6.7}$$

# 7. THE COVERING ALGEBRA

## 7.1. Covering Pairs

As in the Section 3.3 we observed that to aproximate the essential algebra for a Green Biset Functor one can use the covering algebra we give an explicit definition of the covering algebra for section biset functors in this chapter. First we introduced some invariants for the basis elements of section Burnside ring, which we could not managed to do generally for any Green Biset Functor. Then we define covering elements in $E_G$, by putting some condition on the invariants. Then we shove that the subalgebra generated by this covering elements indeed is a covering algebra which we defined before more generally for any Green Biset Functor. Moreover we prove that it is Morita equivalent to the direct sum of group algebras over some groups, which we introduce here. So the intersection of the ideal $I_G$ with covering algebra can be easily calculated.

**Definition 7.1.1.** *Let $G$ and $H$ be finite groups. We call the pair $(T, S) \in \Sigma_{G\times H}$* **covering** *if $p_1(T) = G, p_2(T) = H$ and $k_1(S) = k_2(S) = 1$, and denote the set of all such pairs by $\Sigma^c_{G\times H}$. Notice that for a covering pair $(T, S)$, we have $l_0(T, S) \in \mathcal{G}_G$ and $r_0(T, S) \in \mathcal{G}_H$.*

**Proposition 7.1.2.** *Let $G$ be a finite group. The elements $[\frac{G\times G}{S\trianglelefteq T}]$ as $(T, S)$ runs over all covering pairs generates a subalgebra of $E_G$. Denote it by $E^c_G$. Then $E^c_G$ is a covering algebra in the sense it was defined in Section 3.3.*

*Proof.* By the Mackey formula and the Proposition 4.2.2 it is clear that $E^c_G$ is a subalgebra of $E_G$. Moreover, it is a covering algebra since by Proposition 4.2.6 any basis element outside of $E^c_G$ factors through a group of smaller order. $\square$

Next we define the groups that we indicated in the introduction of this chapter.

**Definition 7.1.3.** *Let $G$, $H$ be finite groups and let $(K,P) \in \mathcal{G}_G$ and $(L,Q) \in \mathcal{G}_H$ be linked pairs. Define*

$$\Gamma_{(G,K,P)} = \left\{ \frac{1}{|G:P|} \begin{bmatrix} G \times G \\ S \trianglelefteq T \end{bmatrix} \mid l(T,S) = (G,K,P,1) = r(T,S) \right\} \tag{7.1}$$

*and* $${}_{(G,K,P)}\Gamma_{(H,L,Q)} = \left\{ \begin{bmatrix} G \times H \\ U \trianglelefteq V \end{bmatrix} \mid l(V,U) = (G,K,P,1), r(V,U) = (H,L,Q,1) \right\} \tag{7.2}$$

**Theorem 7.1.4.** *(a) The set $\Gamma_{(G,K,P)}$ is a finite group under the multiplication induced by the product in $E_G$. The identity is $e_{(K,P)}$ and inverses are given by taking opposites.*

*(b) The set ${}_{(G,K,P)}\Gamma_{(H,L,Q)}$ is a $(\Gamma_{(G,K,P)}, \Gamma_{(H,L,Q)})$-biset which is both left and right transitive and left and right free.*

*(c) Any $\gamma \in {}_{(G,K,P)}\Gamma_{(H,L,Q)}$ induces a group isomorphism*

$$\gamma : \Gamma_{(H,L,Q)} \xrightarrow{\sim} \Gamma_{(G,K,P)}. \tag{7.3}$$

*(d) The functor*

$$k[{}_{(G,K,P)}\Gamma_{(H,L,Q)}] \otimes_{k\Gamma_{(H,L,Q)}} - : {}_{k\Gamma_{(H,L,Q)}}\mathrm{mod} \to {}_{\mathrm{k}\Gamma_{(\mathrm{G,K,P})}}\mathrm{mod} \tag{7.4}$$

*is an equivalence of categories. It induces a canonical bijection*

$$\mathrm{Irr}(k\Gamma_{(H,L,Q)}) \xrightarrow{\sim} \mathrm{Irr}(k\Gamma_{(G,K,P)}). \tag{7.5}$$

*(e) There is an isomorphism of groups*

$$\Gamma_{(G,K,P)} \cong \mathrm{Out}(P, G/K, i_P), \tag{7.6}$$

*where the right hand side is the group of outer automorphisms of the crossed module $(P, G/K, i_P)$.*

*Proof.* All except the last claim are easy justifications, we leave the details to the reader. We only prove the last part by constructing an isomorphism. Define

$$\Theta : \mathrm{Aut}(P, G/K, i_P) \to \Gamma_{(G,K,P)} \tag{7.7}$$

associating $(\alpha, \beta)$ to the element in $\Gamma_{(G,K,P)}$ corresponding to the section $(T, S)$, where

$$T = \{(x, y) \mid xK = \beta(yK) \,\mathrm{for}\, x, y \in G\} \tag{7.8}$$

$$\text{and} \quad S = \{(\alpha(p), p) \mid p \in P\}. \tag{7.9}$$

Note that $(T, S)$ is indeed a section by Goursat's theorem for sections. It is easy to check that $\Theta$ is a group homomorphism. Moreover if $(T, S)$ is a section of $G \times G$ with $l_0(T, S) = (K, P) = r_0(T, S)$ then by Goursat's Theorem for sections, we obtain an automorphism of the crossed module $(P, G/K, \iota_P)$. It remains to show that the kernel of $\Theta$ is the group of inner automorphisms. □

## 7.2. Technical Results

We need the following technical results for the proof of Theorem 7.3.1. The following proposition is the version of Lemma 6.3 and its corollary from [4].

**Proposition 7.2.1.** *With the above notation*

*(a) Let $(T, S)$ be covering and $(K, P), (L, Q) \in \mathcal{G}_G$. If $f_{\{K,P\}_G}[\frac{G\times G}{S \trianglelefteq T}]f_{\{L,Q\}_G}$ is non zero, then $\{K, P\}_G = \{L, Q\}_G$.*

*(b) The set $\{f_{\{K,P\}_G} \mid \{K, P\}_G \in \mathcal{G}_G/\sim\}$ is a set of mutually orthogonal central idempotents of $E^c_G$ and their sum is 1.*

*(c) The covering algebra $E^c_G$ decomposes into its two-sided ideals as*

$$E^c_G = \bigoplus_{\{K,P\}_G \in \mathcal{G}_G/\sim} f_{\{K,P\}_G} E^c_G. \tag{7.10}$$

*Proof.* (a) By definition $f_{\{K,P\}}$ is a sum of $f_{(K',P')}$ as $(K',P')$ runs through the linkage class of $\{K,P\}_G$. Hence for the product given in (a) being non-zero, there must exist at least a pair, say $(K',P')$, in the linkage class $\{K,P\}_G$ and a pair $(L',Q')$ in the linkage class of $\{L,Q\}_G$ such that the product $f_{(K',P')}[\frac{G\times G}{S\trianglelefteq T}]f_{(L',Q')}$ is non-zero . Moreover, $f_{(L',Q')}$ is the sum of $e_{(L'',Q'')}$ where $(L'',Q'')$ runs through all the poset elements of $\mathcal{G}_G$ which are bigger than $(L',Q')$. Then similarly $f_{(K',P')}[\frac{G\times G}{S\trianglelefteq T}]e_{(L'',Q'')}$ must be non-zero at least for some $(L',Q')\preceq(L'',Q'')$. Hence we have $[\frac{G\times G}{S\trianglelefteq T}]e_{(L'',Q'')}\neq 0$, and by the Proposition 5.1.2 it is in the form $[\frac{G\times G}{U\trianglelefteq V}]$ for some $(V,U)\in\Sigma_G^c$. Also by the same proposition, we know that the right middle invariant of the section $(V,U)$ is bigger than $(L'',Q'')$. Let $(K'',P'')$ be the left middle invariant of the section $(V,U)$. Then

$$f_{(K',P')}e_{(K'',P'')}\Big[\frac{G\times G}{S\trianglelefteq T}\Big]e_{(L'',Q'')}=f_{(K',P')}\Big[\frac{G\times G}{S\trianglelefteq T}\Big]e_{(L'',Q'')}\neq 0.$$

Hence $f_{(K',P')}e_{(K'',P'')}\neq 0$, which implies $(K'',P'')\preceq(K',P')$ by Proposition 5.1.4. If we gather it all together,

$$\{L,Q\}_G=\{L',Q'\}_G\preceq\{L'',Q''\}_G\preceq\{r_0(V,U)\}_G=\{l_0(V,U)\}_G=\{K'',P''\}_G\preceq \{K',P'\}_G=\{K,P\}_G.$$

Note that by the similar reasoning we also can derive $\{K,P\}_G\preceq\{L,Q\}_G$ and hence there must be an equality.

(b) By Proposition 6.2.4, for $\{K,P\}_G\in\mathcal{G}_G/\sim$, the elements $f_{\{K,P\}_G}$ are the mutually orthogonal idempotents of the covering algebra. Also their sum is equal to the sum of all $f_{(K,P)}$ where $(K,P)$ run through all the poset elements of $\mathcal{G}_G$ and this sum is equal to 1 by Equation (5.17). Hence for any $b\in E_G^c$ we have

$$\sum_{\{K,P\}_G\in\mathcal{G}_G/\sim} f_{\{K,P\}_G}b=b=\sum_{\{K,P\}_G\in\mathcal{G}_G/\sim} bf_{\{K,P\}_G},$$

and hence we obtain $f_{\{K,P\}_G}b=bf_{\{K,P\}_G}$ for every $\{K,P\}_G\in\mathcal{G}_G/\sim$.

(c) This is immediate from case $(b)$.

□

**7.2.2.** *We can also decompose $E_G^c$ into $k$-submodules via linkage classes. For any $\{K,P\}_G \in \mathcal{G}_G/\sim$, let $E_G^{c,\{K,P\}}$ be the submodule of $E_G^c$ generated by all $\left[\frac{G\times G}{S\trianglelefteq T}\right]$ with $(T,S) \in \Sigma^c_{G\times G}$ and $l_0(T,S) \sim_G (K,P)$ (or equivalently $r_0(T,S) \sim_G (K,P)$). Then*

$$E_G^c = \bigoplus_{\{K,P\}_G \in \mathcal{G}_G/\sim} E_G^{c,\{K,P\}}. \tag{7.11}$$

*This decomposition is related to the ideal decomposition in the following way.*

**Lemma 7.2.3.** *Let $(K,P) \in \mathcal{G}_G$.*

*(a) The following equality holds*

$$\bigoplus_{\{K,P\}_G \preceq \{L,Q\}_G \in \mathcal{G}_G/\sim} E_G^c f_{\{L,Q\}_G} = \bigoplus_{\{K,P\}_G \preceq \{L,Q\}_G \in \mathcal{G}_G/\sim} E_G^{c,\{L,Q\}}. \tag{7.12}$$

*(b) The projection*

$$\omega : E_G^{c,\{K,P\}} \to E_G^c f_{\{K,P\}_G},\ b \mapsto b f_{\{K,P\}_G} \tag{7.13}$$

*with respect to the ideal decomposition of $E_G^c$ is an isomorphism of $k$-modules. Its inverse is given by the projection with respect to the submodule decomposition of $E_G^c$.*

*(c) If $\{K,P\}_G = \{(K_1,P_1),(K_2,P_2),\ldots,(K_n,P_n)\}$ then $\omega$ becomes the direct sum, over $i$ and $j$, of the $k$-module isomorphisms*

$$\omega_{ij} : k[{}_{(G,K_i,P_i)}\Gamma_{(G,K_j,P_j)}] \to f_{(K_i,P_i)} E_G^c f_{(K_j,P_j)}, \tag{7.14}$$

*given by $b_{ij} \mapsto f_{(K_i,P_i)} b f_{(K_j,P_j)}$.*

*(d) The isomorphism $\omega_{ii}$ defined above induces an isomorphism of $k$-algebras*

$$k[\Gamma_{(G,K,P)}] \to f_{(K,P)} E_G^c f_{(K,P)} \quad \textit{given by} \quad a \mapsto f_{(K,P)} a f_{(K,P)}. \tag{7.15}$$

*Proof.* Using Proposition 5.1.4 one can derive that the product $[\frac{G\times H}{S\trianglelefteq T}]f_{(L,Q)}$ is zero for any section $(T,S)\in\Sigma^c_{G\times G}$ with $r_0(T,S)=(K,P)$ and $(K,P)\not\preceq(L,Q)\in\mathcal{G}_G$. Similarly one can get its left-sided version too.

(a) To prove the equality first note that by Proposition 7.2.1 the left hand side equals the annihilator of the set of all $f_{\{K',P'\}_G}$ with $\{K',P'\}_G\not\preceq\{K,P\}_G$. On the other hand by the observation above the right hand side also annihilates $f_{\{K',P'\}_G}$ for all $\{K',P'\}_G\not\preceq\{K,P\}_G$, which shows that the right hand side is contained in the left hand side. But right hand side obviously contains the left hand side since it contains $f_{\{L,Q\}_G}$ for every $\{L,Q\}_G\preceq\{K,P\}_G$.

(b) By the same way we proved previous case one can also show that

$$\bigoplus_{\{K,P\}_G\prec\{L,Q\}_G\in\mathcal{G}_G/\sim} E^c_G f_{\{L,Q\}_G}=\bigoplus_{\{K,P\}_G\prec\{L,Q\}_G\in\mathcal{G}_G/\sim} E^{c,\{L,Q\}}_G. \tag{7.16}$$

Hence $E^{c,\{K,P\}}_G$ and $E^c_G f_{\{K,P\}_G}$ are both complements to the same submodule given above. So $\omega$ is an isomorphism of $k$-modules.

(c) By definition, and since $f_{\{K,P\}_G}$ is central in $E^c_G$ we have a direct sum decomposition

$$E^{c,\{K,P\}}_G=\bigoplus_{i,j=1}^{n} k[_{(G,K_i,P_i)}\Gamma_{(G,K_j,P_j)}], \tag{7.17}$$

$$E^c_G f_{\{K,P\}_G}=f_{\{K,P\}_G}E^c_G f_{\{K,P\}_G}=\bigoplus_{i,j=1}^{n} f_{\{K_i,P_i\}_G}E^c_G f_{\{K_j,P_j\}_G} \tag{7.18}$$

into $k$-submodules. Moreover for $b_{ij}\in k[_{(G,K_i,P_i)}\Gamma_{(G,K_j,P_j)}]$ using the fact we observed in the beginning of the proof its clear that

$$\omega(b_{ij})=f_{\{K,P\}_G}b_{ij}f_{\{K,P\}_G}=f_{(K_i,P_i)}b_{ij}f_{(K_j,P_j)}\in f_{(K_i,P_i)}E^c_G f_{(K_j,P_j)}. \tag{7.19}$$

Then each $\omega_{ij}$ is an isomorphism of $k$-modules since $\omega$ is an isomorphism of $k$-modules.

(d) Let $a$ and $b$ be arbitrarily chosen elements in the $k$-algebra $k[\Gamma_{(G,K,P)}]$. Also enumerate $\{K,P\}_G$ as in the previous case, and choose $(K,P)=(K_i,P_i)$. Using the fact that $f_{\{K,P\}_G}$ is central and $f_{(K_i,P_i)}$'s are orthogonal idempotents in $E_G^c$ we have

$$\omega_{ii}(a)\omega_{ii}(b) = f_{(K,P)}af_{(K,P)}bf_{(K,P)} = f_{(K,P)}af_{\{K,P\}_G}bf_{(K,P)} \tag{7.20}$$

$$= f_{(K,P)}f_{\{K,P\}_G}abf_{(K,P)} = f_{(K,P)}abf_{(K,P)} = \omega_{ii}(ab). \tag{7.21}$$

□

## 7.3. Morita Equivalence

Now we are ready to state the main result of this Chapter.

**Theorem 7.3.1.** *There exists a $k$-algebra isomorphism*

$$E_G^c \xrightarrow{\sim} \bigoplus_{\{K,P\}_G\in\mathcal{G}_G/\sim} Mat_{|\{K,P\}_G|}k\Gamma_{(G,K,P)} \tag{7.22}$$

*with the following property:*
*For every $\{K,P\}_G = \{(K_1,P_1),\cdots,(K_n,P_n)\} \in \mathcal{G}_G/\sim$, the isomorphism maps $f_{(K_i,P_i)} \in E_G^c$ to the idempotent matrix $e_i = diag(0,\cdots,0,1,0,\cdots,0) \in Mat_{|\{K,P\}_G|}(k\Gamma_{(G,K,P)})$, for $i=1,\cdots,|\{K,P\}_G|$, in the $\{K,P\}_G$-component.*

*Proof.* In Proposition 7.2.1 we have given a decomposition of $E_G^c$ into two-sided ideals. So it suffices to show that there exists a $k$-algebra isomorphism between $E_G^c f_{\{K,P\}_G}$ and $\mathrm{Mat}_n(k\Gamma_{(G,K,P)})$. Aiming this we construct a map from $\mathrm{Mat}_n(k\Gamma_{(G,K,P)})$ to $E_G^{c,\{K,P\}}$ and since we already have a $k$-algebra isomorphisms $\omega$, composing these two maps we get the desired isomorphism.

Enumerate the elements of $\{K,P\}_G$ as in Lemma 7.2.3. Then chose an element $(T_i,S_i)\in\Sigma^c_{G\times G}$ with $l_0(T_i,S_i)=(K,P)$ and $r_0(T_i,S_i)=(K_i,P_i)$, and set $x_i:=[\frac{G\times G}{S\trianglelefteq T}]$. Denote by $y_i=\frac{1}{|G:P_i|}x_i$ and by $\bar{y}_i=\frac{1}{|G:P|}x_i^{op}$. Then its an easy calculation to show that $y_i\underset{G}{\cdot}\bar{y}_i=e_{(K,P)}$ and $\bar{y}_i\underset{G}{\cdot}y_i=e_{(K_i,P_i)}$. Hence the maps $\sigma_{ij}:k\Gamma_{(G,K,P)}\to k[_{(G,K_i,P_i)}\Gamma_{(G,K_j,P_j)}]$, $a\mapsto\bar{y}_i a y_j$ are $k$-module isomorphisms with the inverse maps $b\mapsto y_i b\bar{y}_j$. Taking the direct sum of the maps $\sigma_{ij}$ and using the direct sum decomposition in (7.17) we obtain a $k$-module isomorphism $\sigma:\mathrm{Mat}_n(k\Gamma_{(G,K,P)})\to E_G^{c,\{K,P\}}$. In fact it is a $k$-algebra isomorphism. So we have a desired isomorphism $\omega\circ\sigma$ of $k$-algebras, which maps diagonal idempotent matrix $e_i$ to $f_{(K_i,P_i)}$. $\square$

# 8. THE ESSENTIAL ALGEBRA

Let $k$ be a field of characteristic zero, and fix a finite group $G$. In this chapter we determine the essential algebra $\bar{E}_G$ and its simple modules. Once more the results in this section are similar to the results in [4, Section 8].

## 8.1. Reduced Pairs

As in the case of fibered biset functors, the essential algebra is isomorphic to a subalgebra of the covering algebra. To describe the generators of this subalgebra we introduce reduced element as follows.

**Definition 8.1.1.** *Let $(K, P) \in \mathcal{G}_G$. We call $(K, P)$ a* **reduced pair** *if $e_{K,P}$ is not contained in the essential ideal $I_G$. We denote the subset of $\mathcal{G}_G$ consisting of the reduced pairs by $\mathcal{R}_G = \mathcal{R}_k(G)$.*

It is easy to prove that being reduced is compatible with being linked, that is, if $(K, P), (K', P') \in \mathcal{G}_G$ are $G$-linked then $(K, P)$ is reduced if and only if $(K', P')$ is reduced. (cf. [4, Notation 8.1]) We write $\widetilde{\mathcal{R}}_G$ for the set of linkage classes of reduced pairs for $G$. Note also that $\mathcal{R}_G$ is a lower set, that is, if $(K, P)$ is reduced and $(K', P') \preceq (K, P)$, then $(K', P')$ is also reduced.

**Theorem 8.1.2.** *Let $(K, P) \in \mathcal{G}_G$.*

*(a) The pair $(K, P)$ is reduced if $K \leqslant P$.*
*(b) If $(K, P)$ is reduced, then for any non-trivial $N \trianglelefteq G$ with $N \leqslant K$, we have $P \cap N \neq 1$.*
*(c) The pair $(K, P)$ is not reduced if there is a group $H$ and a pair $(L, Q) \in \mathcal{G}_H$ such that $|H| < |G|$ and $(G, K, P, 1) \sim (H, L, Q, 1)$.*

*Proof.* (a) Assume for contradiction that $(K,P)$ is not reduced. Hence $e_{(K,P)} \in I_G$, that is there exist a finite group $H$ with order strictly less than the order of $G$ and sections $(T,S)$ and $(V,U)$ in $G\times H$ and $H \times G$ respectively such that $e_{(K,P)}$ occurs as a summand in $\left[\frac{G\times H}{S\trianglelefteq T}\right] \underset{H}{\cdot} \left[\frac{H\times G}{U\trianglelefteq V}\right]$. Then by the Mackey formula there is a section of $H\times G$, say $(V',U')$, such that $(S*U', T*V') = (\Delta_K(G), \Delta(P))$. Thus it follows that $l(T,S) = (G,K',P',1)$ for some $(K',P') \preceq (K,P)$. Now by Proposition 5.1.2(b), we have $e_{(K,P)} \underset{G}{\cdot} \left[\frac{G\times H}{S\trianglelefteq T}\right] = \left[\frac{G\times H}{S'\trianglelefteq T'}\right]$ with $(T',S') \in \Sigma_{G\times H}$ satisfying $l(T',S') = (G,K,P,1)$. Finally Proposition 4.2.1(c) applied to $(T',S')$ implies that $|G| \leqslant |H|$, a contradiction.

(b) Let $(K,P) \in \mathcal{R}_G$. Assume, for contradiction, that there exist $N \trianglelefteq G$ with $N \leqslant K$ such that $P \cap N = 1$. Let

$$x = \left[\mathrm{Inf}^G_{G/K}\mathrm{Iso}(\eta_T)\mathrm{Def}^{G/N}_{\frac{G/N}{K/N}}, \mathrm{Ind}^G_P\mathrm{Iso}(\eta_S)\mathrm{Res}^{G/N}_{PN/N}\right] \tag{8.1}$$

$$\text{and} \quad y = \left[\mathrm{Inf}^{G/N}_{\frac{G/N}{K/N}}\mathrm{Iso}(\eta_T^{-1})\mathrm{Def}^G_{G/K}, \mathrm{Ind}^{G/N}_{PN/N}\mathrm{Iso}(\eta_S^{-1})\mathrm{Res}^G_P\right], \tag{8.2}$$

with the canonical isomorphisms
$\eta_T : \frac{G/N}{K/N} \xrightarrow{\sim} G/K$ and $\eta_S : PN/N \xrightarrow{\sim} P$. Then $e_{(K,P)} = \frac{1}{|G:P|} x \cdot_{G/N} \frac{1}{|G:PN|} y \in I_G$, which is a contradiction since $|G/N| < |G|$.

(c) This part is straightforward by the definition of linkage.

$\square$

**Corollary 8.1.3.** *The pair $(K,P) \in \mathcal{G}_G$ is reduced in each of the following cases:*

*(a)* $K = 1$,

*(b)* $P = G$.

*In particular the essential algebra $\bar{E}_G$ is non-zero for any finite group $G$.*

**Corollary 8.1.4.** *The pair $(K, P) \in \mathcal{G}_G$ is not reduced in each of the following cases:*

*(a)* $P < K$,

*(b)* $PK = G$ *and* $K \nleqslant P$.

*Proof.* In both cases it is sufficient to find a triple $(H, L, Q)$ with $|H| < |G|$, which is linked to $(G, K, P)$.

(a) Let $G/K$ act on $P$ by conjugation and construct the semi-direct product $\bar{G} = P \rtimes G/K$. Write $\bar{P}$ for the image (under $\alpha^{-1} : x \mapsto (x, 1)$) of $P$ in $\bar{G}$ so that we have an isomorphism $\beta : \bar{G}/\bar{P} \xrightarrow{\sim} G/K$.
We define the subgroups $S = \{(x, (x, 1)) : x \in P\}$ and $T = \{(g, (x, yK)) : gK = \beta((x, yK)\bar{P})\}$ of $G \times \bar{G}$ and claim that $S \trianglelefteq T$. Indeed we clearly have $S \leqslant T$ and the normality follows by direct calculations. Also note that $P < K$ and $[K, P] = 1$, hence $P$ is abelian. The Goursat correspondents of $T$ and $S$ are $(G, K, \beta, \bar{P}, \bar{G})$ and $(P, 1, \alpha, 1, \bar{P})$ respectively. Hence the quadruples $(G, K, P, 1)$ and $(\bar{G}, \bar{P}, \bar{P}, 1)$ are linked, as required.

(b) Let $S = \Delta(P) \leqslant G \times P$ and $T = \{(g, h) \in G \times P | g^{-1}h \in K\}$. Clearly $S$ is a normal subgroup of $T$. Since $G = PK$, the pair $(T, S)$ is covering and hence $(G, K, P)$ is linked to $(P, K \cap P, P)$. Note that $K \nleqslant P$ is also necessary since otherwise $G = PK$ implies $G = P$ and the factorization is not over a group of smaller order.

$\square$

## 8.2. The Essential Algebra $\bar{E}_G$

The following lemma determines a basis for the essential ideal. Its proof is similar to the proof of [4, Lemma 8.2].

**Lemma 8.2.1.** *The ideal $I_G$ of $E_G$ is generated as a $k$-module by the standard basis elements $\left[\frac{G\times G}{S\trianglelefteq T}\right]$ with $(T,S)\in\Sigma_{G\times G}$ satisfying*

*(i)* $p_1(T)\neq G$ *and* $k_1(S)\neq 1$ *or*
*(ii)* $p_1(T)=G$, $k_1(S)=1$ *and* $l_0(T,S)\notin\mathcal{R}_G$.

*Equivalently, it is generated by the standard basis elements $\left[\frac{G\times G}{S\trianglelefteq T}\right]$ with $(T,S)\in\Sigma_{G\times G}$ satisfying*

*(i′)* $p_2(T)\neq G$ *and* $k_2(S)\neq 1$ *or*
*(ii′)* $p_2(T)=G$, $k_2(S)=1$ *and* $r_0(T,S)\notin\mathcal{R}_G$.

*Proof.* Let $(T,S)\in\Sigma_{G\times G}$. If $(T,S)$ satisfies $(i)$ then by the decomposition in Proposition 4.2.6 we have $[\frac{G\times G}{S\trianglelefteq T}]\in I_G$. If it satisfies $(ii)$ then $(K,P):=l_0(T,S)\notin\mathcal{R}_G$, which by definition means that $e_{(K,P)}\in I_G$, but then since $I_G$ is an ideal $[\frac{G\times G}{S\trianglelefteq T}]=e_{(K,P)}[\frac{G\times G}{S\trianglelefteq T}]\in I_G$.

Conversely, to prove that every element in $I_G$ can be written as a $k$-linear combination of elements as given in the lemma, it suffices to show that the product $[\frac{G\times H}{S\trianglelefteq T}]\cdot_H[\frac{H\times G}{U\trianglelefteq V}]$ can be written as such a linear combination for any group $|H|<|G|$ and $(T,S)\in\Sigma_{G\times H}$, $(V,U)\in\Sigma_{H\times G}$. By the Mackey formula this product consist of the elements of the form $[\frac{G\times G}{S*U'\trianglelefteq T*V'}]$ for some $(V',U')\in\Sigma_{H\times G}$. Assume that $(T*V',S*U')$ does not satisfy conditions $(i)$ and $(ii)$. So we have that the left invariant of the section $(T*V',S*U')$ is equal to $(G,K,P,1)$ for some $(K,P)\in\mathcal{G}_G$ and $(K,P)\in\mathcal{R}_G$. Then by the [4, Proposition 2.6], we have $l(T,S)=(G,K',P',1)$ with $(K',P')\preceq(K,P)$.

But then since $\mathcal{R}_G$ is a lower set the pair $(K', P')$ also is reduced and we obtain the contradiction

$$e_{(K',P')} = \left[\frac{G \times H}{S \trianglelefteq T}\right] \cdot_H \left[\frac{G \times H}{S \trianglelefteq T}\right]^{op} \in I_G. \tag{8.3}$$

□

Now we are ready to determine the intersection of the covering algebra and the ideal $I_G$. This also reveals the structure of the essential algebra.

**Proposition 8.2.2.** *(a) The algebra $E^c_G$ is a covering algebra for $E_G$, in the sense of Section 2.5.*

*(b) The equality*

$$E^c_G \cap I_G = \bigoplus_{\substack{\{K,P\}_G \in \mathcal{G}_G/\sim \\ (K,P) \notin \mathcal{R}_G}} f_{\{K,P\}_G} E^c_G \tag{8.4}$$

*holds.*

*(c) The canonical epimorphism $E_G \to \bar{E}_G$ maps the subalgebra*

$$\bigoplus_{\{K,P\}_G \in \widetilde{\mathcal{R}}_G} f_{\{K,P\}_G} E^c_G \tag{8.5}$$

*of $E^c_G$ isomorphically onto $\bar{E}_G$.*

*(d) For each $(K, P) \in \mathcal{R}_G$, the map*

$$k\Gamma_{(G,K,P)} \to \bar{f}_{(K,P)} \bar{E}_G \bar{f}_{(K,P)}, \quad a \mapsto \bar{f}_{(K,P)} \bar{a} \bar{f}_{(K,P)} \tag{8.6}$$

*is a $k$-algebra isomorphism.*

(e) *There is a bijective correspondence between the isomorphism classes of simple $\bar{E}_G$-modules and the set of triples $(K, P, [V])$ where $(K, P)$ runs over linkage classes of reduced pairs for $G$ and for a representative $(K, P)$ of the linkage class of $(K, P)$, $[V]$ runs over irreducible $k\Gamma_{(G,K,P)}$-modules. The correspondence is given by associating the triple $(K, P, [V])$ to the $\bar{E}_G$-module $\bar{E}_G\bar{f}_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V$.*

*Proof.* (a) By Lemma 8.2.1 a standart basis element $\left[\frac{G\times G}{S\trianglelefteq T}\right]$ of $E_G$ belongs to covering algebra if $(T, S) \in \Sigma^c_{G\times G}$ and it belongs to $I_G$ if $(T, S) \notin \Sigma^c_{G\times G}$ so their sum is a whole endomorphism ring. Thus $E^c_G$ is a covering algebra for $E_G$.

(b) The basis elements $\left[\frac{G\times G}{S\trianglelefteq T}\right]$ with $(T, S) \in \Sigma^c_{G\times G}$ such that $l_0(T, S) \notin \mathcal{R}_G$ generate the intersection of covering algebra and ideal $I_G$ as $k$-module. Hence we have an equality

$$E^c_G \cap I_G = \bigoplus_{\substack{\{K,P\}_G\in\mathcal{G}_G/\sim\\(K,P)\notin\mathcal{R}_G}} E^{c,\{K,P\}}_G. \tag{8.7}$$

Since the set of reduced elements is a lower set and due to Lemma 7.2.3 we have the equality

$$\bigoplus_{\substack{\{K,P\}_G\in\mathcal{G}_G/\sim\\(K,P)\notin\mathcal{R}_G}} E^{c,\{K,P\}}_G = \bigoplus_{\substack{\{K,P\}_G\in\mathcal{G}_G/\sim\\(K,P)\notin\mathcal{R}_G}} f_{\{K,P\}_G}E^c_G. \tag{8.8}$$

Thus we get the desired result.

(c) This follows from Proposition 7.2.1 and Part (b).

(d) This follows from Proposition 7.2.3(d) and Part (c).

(e) Let $[V]$ be the class of simple $k\Gamma_{(G,K,P)}$-module and $(K, P) \in \widetilde{\mathcal{R}}_G$ such that $(K, P) = (K_i, P_i)$ where the elements of $\{K, P\}_G$ are enumerated as in Lemma 7.2.3. Then by the Morita equivalence between the algebras $\mathrm{Mat}_n(k\Gamma_{(G,K,P)})$ and $k\Gamma_{(G,K,P)}$, the class of the simple module $[V]$ corresponds to the class of the simple module $\mathrm{Mat}_n(k\Gamma_{(G,K,P)})e_i \otimes_{k\Gamma_{(G,K,P)}} V$.

Then for $y_i = e_i = e_{(K,P)}$ the isomorphism $\omega \circ \sigma$ (from the proof of Theorem 7.3.1) transports this simple module to the irreducible module $f_{\{K,P\}_G} E^c_G f_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V = E^c_G f_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V$. The last equality is due to the fact that $f_{\{K,P\}_G}$ is central in the covering algebra, and $f_{(K,P)}$ are orthogonal idempotents. So by the isomorphism in Part (c) we are done.

□

**8.2.3.** *Finally we parameterize simple modules over $\bar{E}_G$. Let*

$$\mathcal{S}_G = \mathcal{S}_k(G) := \{((K,P),[V]) \mid (K,P) \in \mathcal{R}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,P)})\}. \tag{8.9}$$

*Define an equivalence relation on $\mathcal{S}_G$ by $((K,P),[V]) \sim ((K',P'),[V'])$ if $(K,P)$ and $(K',P')$ are $G$-linked and the canonical bijection $\mathrm{Irr}(k\Gamma_{(G,K',P')}) \xrightarrow{\sim} \mathrm{Irr}(k\Gamma_{(G,K,P)})$ from Theorem 7.1.4 maps $[V']$ to $[V]$. Also let $\tilde{\mathcal{S}}_G$ be a set of representatives of the equivalence classes of $\mathcal{S}_G$, that is,*

$$\tilde{\mathcal{S}}_G := \{((K,P),[V]) \mid (K,P) \in \tilde{\mathcal{R}}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,P)})\}. \tag{8.10}$$

*By the canonical isomorphism from Proposition 8.2.2(c), we can view each simple $k\Gamma_{(G,K,P)}$-module as a simple $\bar{f}_{(K,P)}\bar{E}_G\bar{f}_{(K,P)}$-module, and we can view $\bar{E}_G\bar{f}_{(K,P)}$ as $(\bar{E}_G, k\Gamma_{(G,K,P)})$-bimodule. Hence $\bar{E}_G\bar{f}_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V$ is a simple $\bar{E}_G$-module for each simple $k\Gamma_{(G,K,P)}$-module. This induces a bijection between the set $\tilde{\mathcal{S}}_G$ and Irr$(\bar{E}_G)$.*

# 9. MAIN RESULT

## 9.1. Simple Functors over $k\Gamma$

In this final section we parameterize simple section biset functors. In the previous section we have classified simple modules over the endomorphism ring $E_G$ for any finite group $G$. This gives us a map from seeds to isomorphism classes of simple section biset functors. In this section we introduce an equivalence relation on seeds to make this map bijective.
By the parametrization in Section 8.2.3 we may write the set of all seeds for $k\Gamma$ as

$$\mathrm{Seeds}(k\Gamma) = \{(G, K, P, [V]) | G \in Ob(\mathcal{P}_\Gamma), (K, P, [V]) \in \tilde{\mathcal{S}}_G\}. \tag{9.1}$$

**Definition 9.1.1.** *Given a seed $(G, K, P, [V])$, we construct a simple $E_G$-module $\widetilde{V}$ by*

$$\widetilde{V} = \bar{E}_G \bar{f}_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V. \tag{9.2}$$

Hence as explained in Section 3.2, we associate a simple section functor $S_{G,\widetilde{V}} = S_{G,K,P,[V]}$ to the seed $(G, K, P, [V])$. Clearly replacing $V$ with an isomorphic copy of $V$ does not change the isomorphism type of the corresponding simple section biset functor. In particular we obtain a function

$$\omega : \mathrm{Seeds}(k\Gamma) \to \mathrm{Irr}(k\Gamma), \tag{9.3}$$

where $\mathrm{Irr}(k\Gamma)$ denotes the set of isomorphism classes of simple section biset functors. Note that $\omega$ is surjective. Indeed if $S$ is a simple section biset functor, we let $G$ be a minimal group for $S$.

Then $S(G)$ is a simple module for the essential algebra $\bar{E}_G$, and hence there is a triple $(K, P, [V]) \in \tilde{\mathcal{S}}_G$ such that $S(G) \cong \bar{E}_G \bar{f}_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V$. Therefore $S \cong S_{G,K,P,[V]}$.

**Definition 9.1.2.** *As in [4], we define an equivalence relation on Seeds$(k\Gamma)$ by*

$$(G, K, P, [V]) \sim (H, L, Q, [W]) \tag{9.4}$$

*if and only if $(G, K, P, 1) \sim (H, L, Q, 1)$ and $V \cong k[{}_{(G,K,P)}\Gamma_{(H,L,Q)}] \otimes_{k\Gamma_{(H,L,Q)}} W$. If these conditions are satisfied, we say that the quadruples $(G, K, P, [V])$ and $(H, L, Q, [W])$ are* linked.

We claim that the function $\widetilde{\omega}$ induced by $\omega$ on the set of linkage classes Seeds$(k\Gamma)/\sim$ of seeds is bijective. Ideas used in the proof below can be found in [4, Section 9].

**Theorem 9.1.3.** *There is a bijective correspondence between*

*(a) the set Irr$(k\Gamma)$ of isomorphism classes of simple section biset functors,*

*(b) the set Seeds$(k\Gamma)/\sim$ of linkage classes of quadruples $(G, K, P, [V])$.*

**9.1.4.** *Let $M$ be a section biset functor and $G$ and $H$ be finite groups. We have decompositions*

$$M(G) = \bigoplus_{(K,P)\in\mathcal{G}_G} f_{(K,P)}M(G), \quad M(H) = \bigoplus_{(L,Q)\in\mathcal{G}_H} f_{(L,Q)}M(H). \tag{9.5}$$

*Although these decompositions may not be related to each other, in general, the terms corresponding to linked quadruples are isomorphic. Indeed suppose there are pairs $(K, P) \in \mathcal{G}_G$ and $(L, Q) \in \mathcal{G}_H$ such that the triples $(G, K, P)$ and $(H, L, Q)$ are linked, so that the set ${}_{(G,K,P)}\Gamma_{(H,L,Q)}$ is non-empty.*

*Then, for each* $x \in {}_{(G,K,P)}\Gamma_{(H,L,Q)}$ *the map*

$$f_{(K,P)}M(G) \to f_{(L,Q)}M(H), m \mapsto |G:P|^{-1}x \cdot m \tag{9.6}$$

*is an isomorphism of* $k$*-modules with the inverse given by multiplication by* $x^{\mathrm{op}}$ *(cf. [4, Lemma 9.3]). This follows easily since we have* $xx^{\mathrm{op}} = e_{(K,P)}$*,* $x^{\mathrm{op}}x = e_{(L,Q)}$*,* $e_{(K,P)}f_{(K,P)} = f_{(K,P)}$ *and* $e_{(L,Q)}f_{(L,Q)} = f_{(L,Q)}$*.*

We will also need the following lemma.

**Lemma 9.1.5.** *Let* $G$ *and* $H$ *be finite groups,* $(K,P) \in \mathcal{R}_G$ *and* $(L,Q) \in \mathcal{R}_H$ *be such that* $(G,K,P,1) \sim (H,L,Q,1)$*. Then* $G$ *and* $H$ *have the same order.*

*Proof.* Since $(G,K,P,1)$ and $(H,L,Q,1)$ are linked, there is a section $[\frac{G\times H}{S \trianglelefteq T}] \in {}_{(G,K,P)}\Gamma_{(H,L,Q)}$. Assume $|G| > |H|$. Then there is a factorization $|H:Q||G:P|e_{(K,P)} = [\frac{G\times H}{S\trianglelefteq T}] \times_H [\frac{G\times H}{S\trianglelefteq T}]^{op}$. This is impossible since $e_{(K,P)}$ is reduced. $\square$

**9.1.6.** *Let* $(H,L,Q,[W])$ *be a seed and* $S_{H,\widetilde{W}}$ *be the corresponding simple section biset functor so that*

$$S_{H,\widetilde{W}}(H) = \widetilde{W} = \bar{E}_H \bar{f}_{(L,Q)} \otimes_{k\Gamma_{(H,L,Q)}} W. \tag{9.7}$$

*Note that since there is an isomorphism*

$$k\Gamma_{(H,L,Q)} \cong f_{(L,Q)}E_H^c f_{(L,Q)} \tag{9.8}$$

*of* $k$*-algebras, by Lemma 7.2.3, we may regard* $f_{(L,Q)}S_{H,\widetilde{W}}(H)$ *as a* $k\Gamma_{(H,L,Q)}$*-module, and as such, it is isomorphic to* $W$*.*

*In order to determine other seeds which corresponds to the functor $S_{(H,\widetilde{W})}$, we need to know the evaluations $S_{H,\widetilde{W}}(G)$ for groups with $|G| = |H|$. Hence let $G$ be a group of order $|H|$. Clearly there is a seed of the form $(G, K, P, [V])$ corresponding to $S_{H,\widetilde{W}}$ only if $S_{H,\widetilde{W}}(G)$ is non-zero, which we assume from now on. Then since $H$ is a minimal group for $S_{H,\widetilde{W}}$, $G$ is also minimal. In particular $S_{H,\widetilde{W}}(G)$ is annihilated by $I_G$, and hence it is a simple $\bar{E}_G$-module. Put $\widetilde{V} = S_{H,\widetilde{W}}(G)$. By Section 8.2.3, there is a triple $(K, P, [V])$ such that $\widetilde{V} \cong \bar{E}_G \bar{f}_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V$. Hence $\bar{f}_{(K,P)} S_{H,\widetilde{W}}(G) \cong V$ and $S_{H,\widetilde{W}} \cong S_{G,\widetilde{V}}$.*

*In particular $S_{H,W}$ has a seed $(G, K, P, [V])$ if and only if $G$ is minimal for $S_{H,W}$ and as a simple $\bar{E}_G$-module, $S_{H,W}(G)$ corresponds to the triple $(K, P, [V])$. In this case, we additionally have the following result.*

**Lemma 9.1.7.** *Let $G$ and $H$ be finite groups of the same order, let $\Gamma_k^c(G, H)$ be the submodule of $\Gamma_k(G, H)$ spanned by all sections corresponding to covering pairs. Also let $(K, P) \in \mathcal{G}_G$ and $(L, Q) \in \mathcal{G}_H$. Then*

*(a) There is an isomorphism of $(k\Gamma_{(G,K,P)}, k\Gamma_{(H,L,Q)})$-bimodules*

$$k[{}_{(G,K,P)}\Gamma_{(H,L,Q)}] \to f_{(K,P)} \Gamma_k^c(G, H) f_{(L,Q)}, \tag{9.9}$$

*given by mapping $b$ to $f_{(K,P)} b f_{(L,Q)}$. Here the actions on the left and on the right of $f_{(K,P)} \Gamma_k^c(G, H) f_{(L,Q)}$ are given through the isomorphism in Theorem 7.1.4.*

*(b) Suppose $(L, Q) \in \mathcal{R}_H$ and $W$ is a simple $k\Gamma_{(H,L,Q)}$-module. Then there is an epimorphism*

$$k[{}_{(G,K,P)}\Gamma_{(H,L,Q)}] \otimes_{k\Gamma_{(H,L,Q)}} W \to f_{(K,P)} S_{H,\widetilde{W}}(G) \tag{9.10}$$

*of $k\Gamma_{(G,K,P)}$-modules.*

*Proof.* The proof is very similar to the proof of [4, Lemma 9.5]. We give a sketch since as the original version, it follows from previous results. For part (a), if the set ${}_{(G,K,P)}\Gamma_{(H,L,Q)}$ is non-empty, then by Section 7.4, there are isomorphisms

$$k[{}_{(G,K,P)}\Gamma_{(H,L,Q)}] \to k\Gamma_{(G,K,P)} \tag{9.11}$$

$$\textit{and} \quad f_{(K,P)}\Gamma_k^c(G,H)f_{(L,Q)} \to f_{(K,P)}\Gamma_G^c f_{(L,Q)} \tag{9.12}$$

of $k\Gamma_{(G,K,P)}$-modules. Also by Lemma 5.17 (d), there is an isomorphism

$$k\Gamma_{(G,K,P)} \to f_{(K,P)}\Gamma_G^c f_{(L,Q)}. \tag{9.13}$$

We get the result by composing these isomorphisms. The case that ${}_{(G,K,P)}\Gamma_{(H,L,Q)}$ is empty follows from basic properties of $e_{K,P}$ and $f_{K,P}$. We leave the justification to the reader.

Part (b) basically follows from part (a) and the general construction of simple functors we explained above. □

Now by Lemma 9.1.7, we must have a surjective homomorphism

$$k_{(G,K,P)}\Gamma_{(H,L,Q)} \otimes_{k\Gamma_{(H,L,Q)}} W \to V \tag{9.14}$$

of $k\Gamma_{(G,K,P)}$-modules.

In particular the set ${}_{(G,K,P)}\Gamma_{(H,L,Q)}$ is non-empty, which implies that the quadruples $(G, K, P, 1)$ and $(H, L, Q, 1)$ are linked. Moreover by Theorem 7.1.4, the $k\Gamma_{(G,K,P)}$-module $k_{(G,K,P)}\Gamma_{(H,L,Q)} \otimes_{k\Gamma_{(H,L,Q)}} W$ is simple. Hence we must have an isomorphism

$$k_{(G,K,P)}\Gamma_{(H,L,Q)} \otimes_{k\Gamma_{(H,L,Q)}} W \cong V \tag{9.15}$$

of $k\Gamma_{(G,K,P)}$-modules. As a result the seeds $(G, K, P, [V])$ and $(H, L, Q, [W])$ are linked, that is, seeds corresponding to the simple functor $S_{H,\widetilde{W}}$ are all linked to $(H, L, Q, [W])$.

**9.1.8.** *Finally we show that if $(G, K, P, [V])$ and $(H, L, Q, [W])$ are linked then $S_{G,\widetilde{V}} \cong S_{H,\widetilde{W}}$. First note that by Lemma 9.1.5, the groups $G$ and $H$ are of the same order. As discussed above, the evaluation $S_{H,\widetilde{W}}(G)$ is a simple $\bar{E}_G$-module. Also since $(G, K, P, 1) \sim (H, L, Q, 1)$, we get $\bar{f}_{(K,P)}S_{H,\widetilde{W}}(G) \cong \bar{f}_{(L,Q)}S_{H,\widetilde{W}}(H)$ as $k$-modules, by Section 9.1.4. In particular $\bar{f}_{(K,P)}S_{H,\widetilde{W}}(G)$ is non-zero. It remains to show that there is an isomorphism $S_{H,\widetilde{W}}(G) \cong \widetilde{V}$ of $\bar{E}_G$-modules. We already know that*

$$V \cong k_{(G,K,P)}\Gamma_{(H,L,Q)} \otimes_{k\Gamma_{(H,L,Q)}} W. \tag{9.16}$$

*Hence by Lemma 9.1.7, there is a homomorphism $V \to \bar{f}_{(K,P)}S_{H,\widetilde{W}}(G)$ of $k\Gamma_{(G,K,P)}$-modules. Now we have the following isomorphisms.*

$$\begin{aligned}
\{0\} &\neq \mathrm{Hom}_{k\Gamma_{(G,K,P)}}(V, \bar{f}_{(K,P)}S_{H,\widetilde{W}}(G)) && (9.17)\\
&\cong \mathrm{Hom}_{k\Gamma_{(G,K,P)}}(V, \mathrm{Hom}_{\bar{E}_G}(\bar{E}_G\bar{f}_{(K,P)}, S_{H,\widetilde{W}}(G))) && (9.18)\\
&\cong \mathrm{Hom}_{\bar{E}_G}(\bar{E}_G\bar{f}_{(K,P)} \otimes_{k\Gamma_{(G,K,P)}} V, S_{H,\widetilde{W}}(G)) \cong \mathrm{Hom}_{\bar{E}_G}(\widetilde{V}, S_{H,\widetilde{W}}(G)). && (9.19)
\end{aligned}$$

*In particular there is a non-zero homomorphism $\widetilde{V} \to S_{H,\widetilde{W}}(G)$. Since both of these modules are simple, they must be isomorphic. With this step, we have completed the proof of Theorem 9.1.3.*

**Remark 9.1.9.** *Let $G$ and $H$ be non-isomorphic finite groups. For reduced pairs $(K,P) \in \mathcal{R}_G$ and $(L,Q) \in \mathcal{R}_H$ if there exists a section $(T,S)$ such that $(G,K,P,1) \underset{(T,S)}{\sim} (H,L,Q,1)$ then we can take $V := k_{(G,K,P)}\Gamma_{(H,L,Q)} \otimes_{k\Gamma_{(H,L,Q)}} W$ to be the irreducible $k\Gamma_{(G,K,P)}$-module corresponding to $W \in \mathrm{Irr}(k\Gamma_{(H,L,Q)})$. Then $S_{G,\widetilde{V}} \cong S_{H,\widetilde{W}}$, which shows that there exists simple section biset functors with non-isomorphic minimal groups.*

*For example let $G$ be the group of quaternions, $\langle x, y \,|\, x^4 = 1,\, yxy^{-1} = x^{-1},\, x^2 = y^2\rangle$ and $H := \langle a, b \,|\, a^4 = b^2 = 1,\, bab^{-1} = a^{-1}\rangle$ the dihedral group of order 8. Consider the section $(T,S)$ of $G \times H$ given by*

$$T := \langle (x,a), (y,b)\rangle \,\mathrm{and\,S} := \langle (\mathrm{x}, \mathrm{a})\rangle. \tag{9.20}$$

*Clearly $S$ is a subgroup of $T$ and since ${}^{(y,b)}(x,a) = (x^{-1}, a^{-1})$ moreover $S$ is normal in $T$ and hence $(T,S)$ is a section of $G \times H$. The left and right invariant of $(T,S)$ are as follows*

$$l(T,S) = (G, \langle x^2\rangle, \langle x\rangle, 1), \quad r(T,S) = (H, \langle a^2\rangle, \langle a\rangle, 1). \tag{9.21}$$

*So $(K,P) := l_0(T,S) = (\langle x^2\rangle, \langle x\rangle) \in \mathcal{G}_G$ and $(L,Q) := r_0(T,S) = (\langle a^2\rangle, \langle a\rangle) \in \mathcal{G}_H$ are linked pairs. Moreover by Theorem 8.1.2 (a) both $(K,P)$ and $(L,Q)$ are reduced pairs.*

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