SMOOTHING PROPERTIES OF INITIAL-BOUNDARY VALUE PROBLEMS

by

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ABSTRACT

SMOOTHING PROPERTIES OF INITIAL-BOUNDARY VALUE PROBLEMS

This thesis discusses the smoothing properties of dispersive partial differential equations. In the first part of the thesis, we consider the Davey–Stewartson system on $\mathbb{R}^2$ and demonstrate that the nonlinear part of the solution flow is smoother than the initial data. As an application of the smoothing result, we address the dissipative Davey–Stewartson system and give a simplified proof of the existence of a global attractor for the system. In the next part, we study well-posedness and regularity properties of the biharmonic Schrödinger equation on the half-line. More precisely, we prove local existence and uniqueness and show that the data to solution map is continuous. Moreover, we establish global well-posedness and global smoothing for higher regular spaces by showing that the solution grows at most linearly. As regards to the smoothing result, the derivative gain we obtain for the nonlinear part of the solution is up to full derivative. The last part of the thesis addresses the Hirota–Satsuma system on the torus. The Hirota–Satsuma system is given by two Korteweg-de Vries equations exhibiting different dispersion relations which is due to the coupling coefficient a. The main result demonstrates the regularity level of the nonlinear part of the evolution compared to initial data. The gain in regularity depends very much on the arithmetic properties of the coefficient a. Then, we consider the forced and damped Hirota–Satsuma system and establish the analogous smoothing estimates. By the help of the smoothing estimates, we prove the existence and regularity of a global attractor in the energy space.

ÖZET

BAŞLANGIÇ-SINIR DEĞER PROBLEMLERİNİN YUMUŞATMA ÖZELLİKLERİ

Bu tezde dispersif kısmi türevli denklemlerin yumuşatma özellikleri ele alınmıştır. Tezin ilk kısmında, Davey–Stewartson sistemini $\mathbb{R}^2$ üstünde ele aldık ve çözümün doğrusal olmayan kısmının başlangıç verisinden daha yumuşak olduğunu gösterdik. Bu sonucun uygulaması olarak ise, sönümlemeli Davey–Stewarston sistemini göz önünde bulundurup, sistemin global çekerinin varlığına dair basitleştirilmiş bir ispat verdik. Bir sonraki kısımda, yarı doğru üstünde, çift-harmonik Schrödinger denkleminin iyi konulmuşluğunu ve düzgünlük özelliklerini çalıştık. Daha iyi bir ifadeyle anlatmak gerekirse, çözümün yerel varlığını ve tekliğini ispatladık, ayrıca veri-çözüm fonksiyonunun sürekli olduğunu gösterdik. Çözümün en fazla doğrusal büyüdüğünü göstererek global iyi konulmuşluğu ve yumuşatmayı daha yüksek mertebeli düzgün uzaylar için elde ettik. Yumuşatma sonucuna gelecek olursak, çözümün doğrusal olmayan kısmı için elde ettiğimiz türev kazancı en fazla tam türev oldu. Tezin son kısmı Hirota–Satsuma sistemini torus üstünde ele almıştır. Hirota–Satsuma sistemi bağlaşım katsayısı a'dan ötürü farklı yayılma ilişkileri sergileyen iki Korteweg-de Vries denklemi tarafından belirlenir. Ana sonuç, başlangıç verisine kıyasla, çözümün doğrusal olmayan kısmının düzgünlük seviyesini gösterir. Düzgünlükteki kazanç daha çok a katsayısının aritmetik özelliklerine bağlıdır. Daha sonra, zorlanmış ve sönümlenmiş Hirota–Satsuma sistemini ele alıp benzer yumuşatma kestirimleri elde ettik. Yumuşatma kestirimleri sayesinde, sistemin global çekerinin varlığını ve düzgünlüğünü enerji uzayında ispatladık.

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LIST OF SYMBOLS

$A+$	$A+\epsilon$
$A-$	$A-\epsilon$
$A \lesssim B$	$A \leq CB$ for some absolute constant $C>0$
$A \gtrsim B$	$A \geq CB$ for some absolute constant $C>0$
$A \approx B$	$A \lesssim B$ and $B \lesssim A$
$A \ll B$	$A \leq \frac{1}{C}B$ for some sufficiently large constant $C>0$
$A \gg B$	$A \geq \frac{1}{C}B$ for some sufficiently large constant $C>0$
$A \simeq B$	$\lvert A-B\rvert \leq \epsilon$ for some sufficiently small $\epsilon>0$
$\mathbb{C}$	The field of complex numbers
$C_t^0 H_x^s$	The Banach space of H^s valued continuous functions with the norm $\sup_t \lVert u(\cdot,t)\rVert_{H^s}$
$C^\infty(\mathbb{R}^n)$	The space of infinitely differentiable functions on $\mathbb{R}^n$
$\mathcal{D}(\Omega)$	The space of distributions on $\Omega \subset \mathbb{R}^n$
$\mathcal{F}$	The Fourier transform on $\mathbb{R}^d$ given by $\mathcal{F}u(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} u(x)e^{-i\xi\cdot x}\,\mathrm{d}x$
$f*g$	The convolution of f and g
$\langle f,g\rangle_{L^2(\mathbb{R}^n)}$	The inner product on the L^2 space defined as $\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,\mathrm{d}x$
$\lVert f\rVert_{H^s}$	The norm given by $\left\lVert \langle\xi\rangle^s \widehat{f}(\xi)\right\rVert_{L^2_\xi}$, $s\in\mathbb{R}$
$\lVert f\rVert_{\dot{H}^s}$	The norm given by $\left\lVert \lvert\xi\rvert^s \widehat{f}(\xi)\right\rVert_{L^2_\xi}$, $s\geq 0$
Im	Imaginary part
$L^p(\mathbb{R}^n)$	The Lebesgue spaces of measurable functions f with norm $\lVert f\rVert_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x < \infty$ for $p\in[1,\infty)$; $\lVert f\rVert_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup}\lvert f\rvert < \infty$ for $p=\infty$
ℓ^p	The space of functions $a:\mathbb{Z}\to\mathbb{C}$ satisfying $\lVert a\rVert_{\ell^p}^p = \sum_{k\in\mathbb{Z}} \lvert a_k\rvert^p < \infty$ for $p\in[1,\infty)$ or $\lVert a\rVert_{\ell^\infty} = \sup_{k\in\mathbb{Z}} \lvert a_k\rvert < \infty$ for $p=\infty$
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}^+$	The set of positive real numbers

$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathrm{supp}\varphi$	The support of the function φ
$\mathbb{T}$	The torus $\mathbb{R}/2\pi\mathbb{Z}$
$\overline{z}$	The conjugate of a complex number z
$\mathbb{Z}^*$	The set of non-zero integers
$\langle\cdot\rangle$	$=\sqrt{1+\lvert\cdot\rvert^2}$

LIST OF ACRONYMS/ABBREVIATIONS

1D	One Dimensional
2D	Two Dimensional
DS	Davey–Stewartson
HS	Hirota–Satsuma
IBVP	Initial-Boundary Value Problem
IVP	Initial Value Problem
KdV	Korteweg-de Vries
LHS	Left Hand Side
NLS	Nonlinear Schrödinger
PDE	Partial Differential Equation

1. INTRODUCTION

This thesis is devoted to study the smoothing properties of nonlinear dispersive equations on certain domains. In order to understand the dynamics of given nonlinear dispersive equation nicely, one has to pay close attention to the corresponding linear equation. The characterization of linear dispersive PDE is decided by its wave solutions' propagation in the medium in which large frequency components travel faster in contrast to the smaller ones giving rise to a dispersion. This behavior is more apparent for unbounded domains, whereas on bounded domains, e.g. torus, different frequency solution components cannot spread out, rather they rotate around the torus with distinct velocities. The dispersive character exhibited by the nonlinear PDEs in this study will shape the basis of our exploration of various frequency interactions stemming from nonlinear nature of these equations. Consider a linear dispersive PDE of the form

$$\partial_t u(x,t) = Lu(x,t),\ u(x,0) = u_0(x) \tag{1.1}$$

where $L = ih(D)$, $D := \frac{1}{i}\nabla = \frac{1}{i}(\partial_{x_1}, \cdots, \partial_{x_d})$ and h is a real-valued polynomial of order k:

$$h(\xi) = h(\xi_1, \cdots, \xi_d) = \sum_{|\alpha| \le k} c_\alpha \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d},$$

here, $k \ge 1$ is an integer, $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The explicit form of solutions to (1.1) can easily be found by taking spatial Fourier transform of the equation (1.1) by which we obtain $u(x,t) = e^{tL}u_0(x)$ where e^{tL} denotes the linear propagator

$$e^{tL}u_0(x) = \int_{\mathbb{R}^d} e^{ith(\xi)+ix\cdot\xi}\widehat{u_0}(\xi)\, \mathrm{d}\xi.$$

Note that such solution exists globally in time. Perturbing the linear equation (1.1) by the nonlinear term $N(u)$, we next consider the nonlinear equation

$$\partial_t u(x,t) = Lu(x,t) + N(u(x,t)),\ u(x,0) = u_0(x). \tag{1.2}$$

The Cauchy problem (1.2) is equivalent to integral equation

$$u(x,t) = e^{tL}u_0(x) + \int_0^t e^{(t-s)L}N(u(x,s))\,\mathrm{d}s$$

that is called the Duhamel's solution formula for the equation (1.2). In our discussion, we study distributional solutions of nonlinear PDEs which are constructed by a fixed point argument applied to Duhamel's formula. Via this formulation, our primary objective is to show that the nonlinear part $u(x,t) - e^{tL}u_0(x)$ of the Duhamel's formula lies in a more regular space than the initial data u_0 belongs to.

Previous methods for establishing well-posedness of dispersive PDEs such as energy method, oscillatory integral method have proven to be successful for higher regularity spaces. In his seminal papers [1, 2], Bourgain introduced new function spaces, $X^{s,b}$, that take dispersion relation of the given equation into consideration, although the similar weighted spaces related to wave equations were used in the works of Beals [3], Klainerman–Machedon [4] previously. Using these spaces together with Fourier restriction methods, Bourgain improved the previous well-posedness result. The $X^{s,b}$ space theory is of extreme help in establishing low-regularity well-posedness results for dispersive equations in today's research. Our work also uses $X^{s,b}$ spaces to capture the nonlinear smoothing effect for certain dispersive PDEs we next address.

The third chapter is concerned with the smoothing result of the Davey–Stewartson (DS) system. DS systems appear in the theory of water waves and describe the evolution of weakly nonlinear water waves [5]. It is remarkable to note that the first study of well-posedness for the system was initiated by Ghidaglia and Saut [6] in the spaces L^2, H^1, H^2. Since then considerable amount of work have still been dedicated to the DS systems. The approach we follow in this study to prove multilinear estimate for DS system is via Tao's $[k; Z]$ multiplier method [7], which is based on dyadic decomposition and induction on scales type techniques as in restriction theory. As an application of the smoothing result, we address the dissipative DS system and give a simplified proof of the existence of a global attractor for this system.

The next chapter deals with the biharmonic Schrödinger equation on the half-line and establishes well-posedness and smoothing results. To reach the results, after extending the initial data to the full line, we construct the nonlinear solution map via Duhamel's formula adapted to the boundary conditions. The explicit representation formula for the solution of the linear IBVP is derived by the use of Laplace transform, which is then used to establish bound for the boundary forcing term of the Duhamel's formula. To be able to run the fixed point argument on the nonlinear solution map successfully, we require the restricted norm method, so we prove a number of estimates on the terms of Duhamel's formula to close the argument.

In the last chapter, we obtain the smoothing estimate for the Hirota–Satsuma system on the torus. This is a system of coupled KdV type equations that models the interactions of two long waves with separate dispersion relations. With the help of normal form transformation, we rewrite the system in an equivalent form and make use of oscillatory effects to cope with the derivative in the nonlinearities. The disadvantage is that the transformation introduces trilinear terms rather than bilinear. The trilinear terms are dealt with the restricted norm method so as to prove smoothing. We also obtain the smoothing result for the forced and damped Hirota–Satsuma system. Using the smoothing effect, we prove the existence and regularity of the global attractors for the system.

2. OVERVIEW OF THE THEORY

Consider a linear partial differential equation of the form

$$iu_t + h\Big(\frac{1}{i}\nabla\Big)u = 0 \tag{2.1}$$

where h is a real-valued polynomial. We seek out plane wave solutions $u(x,t) = Ae^{i(\xi x - \omega t)}$ where A, ξ and ω represent the amplitude, the wave number and the frequency respectively. The equation (2.1) imposes relationship between ξ and ω, $\omega = \omega(\xi)$. This is called dispersion relation. The phase velocity is defined by $c_p(\xi) = \frac{\omega}{\xi}$ with which the solution can also be written as $u(x,t) = e^{i\xi(x - c_p(\xi)t)} = u(x - c_p(\xi)t, 0)$. We say that the wave travels with velocity $c_p(\xi)$. Also, the related notion, the group velocity is defined by $c_g = \frac{d\omega}{d\xi}$. If c_g is not a constant, that is $\frac{d^2\omega}{d\xi^2} \neq 0$, then the equation (2.1) is called dispersive. In the context of physics, this means that the wave solutions of different wavelength propagate at different phase velocities as time increases. Under this characterization, the transport equation $u_t = u_x$, with the dispersion relation $w(\xi) = \xi$, is not dispersive while the Schrödinger equation $iu_t + u_{xx} = 0$, with the respective dispersion relation $w(\xi) = -\xi^2$, is dispersive.

2.1. $X^{s,b}$ Spaces

The characterization of dispersive equations via Fourier transform motivates the definition of $X^{s,b}$ spaces. Taking the space-time Fourier transform of the linear equation (2.1) shows that the space-time Fourier transform $\widehat{u}(\xi,\tau)$ is supported on the characteristic surface $\{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : \tau = h(\xi)\}$ of the frequency space. This on the other hand is no true for the nonlinear perturbation $iu_t + h(\frac{1}{i}\nabla)u = N(u)$ of the linear equation (2.1). However, the support of the space-time Fourier transform of the localized solution concentrates near this surface. By this observation we introduce the $X^{s,b}$ space as the closure of the Schwartz functions under the norm

$$\|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}^d\times\mathbb{R})} = \left\|\langle\xi\rangle^s\langle\tau - h(\xi)\rangle^b\widehat{u}(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^d\times\mathbb{R})}.$$

For functions on $\mathbb{T}^d \times \mathbb{R}$, the above norm is replaced by

$$\|u\|_{X^{s,b}_{\tau=h(k)}(\mathbb{T}^d\times\mathbb{R})} = \left\|\langle k\rangle^s \langle \tau - h(k)\rangle^b \widehat{u}(k,\tau)\right\|_{\ell^2_k L^2_\tau(\mathbb{T}^d\times\mathbb{R})}.$$

In other words, differentiating functions in $X^{s,b}$ spaces s-times with respect to elliptic derivative $\langle\nabla\rangle$ and b-times using dispersive derivative $i\partial_t + h(\frac{1}{i}\nabla)$ will produce square-integrable functions. Alternatively, one can use yet another form of $X^{s,b}$ norm:

$$\|u\|_{X^{s,b}_{\tau=h(\xi)}} = \|W(-t)u\|_{H^s_x H^b_t}$$

where $W(t) = \exp(ith(\frac{1}{i}\nabla))$ is the unitary group corresponding to the linear equation (2.1) and $\|g\|_{H^s_x H^b_t} = \left\|\langle\xi\rangle^s\langle\tau\rangle^b\widehat{g}(\xi,\tau)\right\|_{L^2_{\xi,\tau}} = \left\|\left\|\langle\xi\rangle^s g(\widehat{\xi},t)\right\|_{H^b_t}\right\|_{L^2_\xi}$. In particular, for $b = 0$, the dispersion relation $\tau = h(\xi)$ is insignificant so that $X^{s,b} = H^s_x L^2_t$. The restricted $X^{s,b}$ space, $X^{s,b}_\delta$, is also defined via the norm

$$\|u\|_{X^{s,b}_\delta} = \inf_{\tilde{u}=u,|t|\le\delta} \|\tilde{u}\|_{X^{s,b}}.$$

We continue our discussion by introducing some properties of $X^{s,b}$ spaces. Taking advantage of Parseval's identity and Cauchy-Schwarz one can observe the duality relationship: $(X^{s,b}_{\tau=h(\xi)})^* = X^{-s,-b}_{\tau=-h(-\xi)}$. Also, the $X^{s,b}$ spaces interpolate very well in both indices s and b. Once $b > \frac{1}{2}$, $X^{s,b}$ spaces turn out to be very useful in proving well-posedness in the space $C^0_t H^s_x$ that the following lemma shows.

Lemma 2.1.1 (See [8])**.** *For any $b > \frac{1}{2}$, $s \in \mathbb{R}$, $X^{s,b}$ space given by a continous dispersion relation embeds into $C^0_t H^s_x$.*

In the following, we shall remove the dispersion relation subscript from the norm of $X^{s,b}$ functions. Let φ be a smooth function satisfying $\varphi(t) = 0$ if $|t| \ge 2$, and $\varphi(t) = 1$ for $|t| \le 1$. Also, let $\varphi_\delta(t) = \varphi(t/\delta)$ for $0 < \delta \le 1$. Consider the Cauchy problem

$$\begin{cases} iu_t + h(\frac{1}{i}\nabla)u = N(u), \ (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(0) = u_0. \end{cases} \tag{2.2}$$

In order to find a unique solution for (2.2) in some subset of $C_t^0 H_x^s$ within the time existence interval $[-\delta, \delta]$, the Banach fixed point argument is implemented to the Duhamel operator

$$\begin{aligned}\Gamma u(t) &= \varphi(t)W(t)u_0 - i\varphi_\delta(t)\int_0^t W(t-s)N(u(s))\,\mathrm{d}s \\ &=: \varphi(t)W(t)u_0 - i\varphi_\delta(t)[W *_t N(u)](t)\end{aligned}$$

on the ball $B_\delta = \{u \in X^{s,b} : \|u\|_{X^{s,b}} \le C\|u_0\|_{H^s}\}$. The localized linear solution is bounded in $X^{s,b}$:

Lemma 2.1.2 (See [9])**.** *For $s, b \in \mathbb{R}$, we have*

$$\|\varphi(t)W(t)g\|_{X^{s,b}} \lesssim \|g\|_{H^s}.$$

The remaining estimate for the integral part of the Duhamel formula is as follows:

Lemma 2.1.3 (See [10])**.** *Let $-\frac{1}{2} < b' \le 0 \le b \le b'+1$. Then,*

$$\|\varphi_\delta(t)[W *_t N(u)](t)\|_{X^{s,b}} \lesssim \delta^{1-b+b'}\|N(u)\|_{X_\delta^{s,b'}}.$$

Combining the above lemmas, we have

$$\|\Gamma u\|_{X^{s,b}} \lesssim \|u_0\|_{H^s} + \delta^{1+b'-b}\|N(u)\|_{X_\delta^{s,b'}}.$$

In order to close the contraction argument on the ball B_δ, one has to prove an estimate of the form $\|N(u)\|_{X_\delta^{s,b'}} \lesssim \|u\|_{X^{s,b}}^\gamma$ subjected to the nonlinearity and dispersion relation associated to the given equation and has to seek sufficiently small δ.

2.2. Differentiation by Parts On the Torus

The essence of the method is based on a normal form transformation introduced in [11]. In this paper, Shatah constructs a transformation that raises the degree of the nonlinearity of Klein–Gordon equation in order to be able to use direct perturbation methods in studying the equation on $\mathbb{R}^3$.

This procedure is known as Poincaré's theory of normal forms for ordinary differential equations, see [12,13]. The well-posedness for the periodic KdV equation was obtained by Babin–Ilyin–Titi [14] using the normal forms as an alternative method. In our work, the normal forms transform method employed to the periodic Hirota–Satsuma system is used as that in [15–17] to obtain smoothing. In the following we shall give the idea of the method. In our case, taking the Fourier transform of the system of equations leads to a system of differential equations for the Fourier sequence of solutions. Multiplying each Fourier coefficient by a modulation factor, the resulting equation for a particular Fourier coefficient u_k turns out roughly in the form: $\partial_t u_k = e^{i\Omega t} N(u_k)$, $\Omega = \Omega(k)$. Then,

$$\begin{aligned}\partial_t u_k &= \partial_t\Big(\frac{e^{i\Omega t}}{i\Omega} N(u_k)\Big) - \frac{e^{i\Omega t}}{i\Omega} N'(u_k)\partial_t u_k \\ &= \partial_t\Big(\frac{e^{i\Omega t}}{i\Omega} N(u_k)\Big) - \frac{e^{2i\Omega t}}{i\Omega} N'(u_k)N(u_k).\end{aligned}$$

Accordingly,

$$\partial_t\Big(u_k - \frac{e^{i\Omega t}}{i\Omega} N(u_k)\Big) = -\frac{e^{2i\Omega t}}{i\Omega} N'(u_k)N(u_k).$$

The conclusion is that the gain Ω in the denominator eliminates the derivative in the nonlinearity of the original equation, in return, the nonlinearity N changes to NN'. Nevertheless, the advantage is to gain large denominators. So if there are resonances (frequencies at which $\Omega = 0$), then each has to be treated separately.

2.3. Global Attrators

The long term dynamics of a given dissipative partial differential equation can be described by compact, invariant, attracting subsets (global attractors) of the phase space into which all trajectories converge as $t \to \infty$. Rather than working with infinite dimensional phase space, one can study the long time asymptotics of the solution flow via global attractors which may be finite dimensional, see for instance [18].

Let H be a phase space and $U(t) : H \to H$ denote the evolution operator, mapping data to solution. The family of operators $\{U(t)\}_{t\geq 0}$ enjoy the semigroup properties:

$$\begin{cases} U(t+s) = U(t)U(s), & \forall t, s \geq 0, \\ U(0) = I & \text{(Identity in } H). \end{cases}$$

In the following, we give some required definitions from [19].

Definition 2.3.1. *A set X is said to be invariant under the flow $U(t)$ if we have $U(t)X = X$ for all $t \geq 0$.*

Definition 2.3.2. *An attractor is a set $\mathcal{A} \subset H$ which is invariant under the flow and possesses an open neighbourhood $\mathcal{N}$ such that for every $u_0 \in \mathcal{N}$ it satisfies*

$$d(U(t)u_0, \mathcal{A}) \to 0 \ \text{ as } t \to \infty. \tag{2.3}$$

The distance in the definition 2.3.2 is in the sense of a distance of a point to a set: $d(a, B) = \inf_{b\in B} d(a, b)$ where $d(a, b)$ measures the distance from a to b in H. The largest open set $\mathcal{N}$ satisfying (2.3) is called the basin of attraction of $\mathcal{A}$. We say that $\mathcal{A}$ uniformly attracts a set $\mathcal{B} \subset \mathcal{N}$ if

$$d(U(t)\mathcal{B}, \mathcal{A}) \to 0 \qquad \text{as } t \to \infty, \tag{2.4}$$

where $d(S_1, S_2) = \sup_{x\in S_1} \inf_{y\in S_2} d(x, y)$ for the two sets S_1, S_2. We also say that $\mathcal{A}$ attracts the bounded sets of $\mathcal{N}$ if $\mathcal{A}$ uniformly attracts each bounded set of $\mathcal{N}$. Note that an attractor may or may not have such a property.

Definition 2.3.3. *The subset $\mathcal{A} \subset H$ is called a global attractor for the semigroup $\{U(t)\}_{t\geq 0}$ if $\mathcal{A}$ is a compact attractor that attracts the bounded sets of H (the basin of attraction of $\mathcal{A}$ is the whole phase space H then).*

Next definition is crucial in establishing the existence of a global attractor:

Definition 2.3.4. *A bounded subset $\mathcal{B}$ of a phase space H is called an absorbing set if for any bounded $\mathcal{S} \subset H$, there exists $T = T(\mathcal{S})$ such that $U(t)\mathcal{S} \subset \mathcal{B}$ for all $t \geq T$.*

Note that a global attractor for a semigroup, if exists, implies the existence of an absorbing set, to see this, use (2.4) for any bounded subset of H to conclude that for any $\epsilon > 0$, ϵ-neighbourhood of a global attractor $\mathcal{A}$ satisfies the requirement for an absorbing set. Once having the absorbing set for the semigroup $\{U(t)\}_{t\geq 0}$, the existence of global attractor is provided by the additional assumption for the semigroup:

Theorem 2.3.5 (See [19])**.** *Suppose that H is a metric space and $U(t) : H \to H$ is a continuous semigroup defined for all $t \geq 0$. Furthermore, suppose that there is an absorbing set $\mathcal{B}$. If the semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically compact, that is, for every bounded sequence $\{x_k\}$ in H and every sequence of times $t_k \to \infty$, the set $\{U(t_k)x_k\}_k$ is relatively compact in H, then the ω-limit set*

$$\omega(\mathcal{B}) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} U(t)\mathcal{B}}$$

is a global attractor, where the closure is taken on H.

Note that $\varphi \in \omega(\mathcal{B})$ if and only if there exists a sequence $\varphi_n \in \mathcal{B}$ and a sequence of times $t_n \to \infty$ such that $U(t_n)\varphi_n \to \varphi$ as $n \to \infty$. In our context, we address the problem of existence of global attractors for the Davey–Stewartson and the periodic Hirota–Satsuma systems. The proofs of the existence of global attractors will essentially be based on the smoothing estimates that will be obtained for the respective dissipative systems.

3. THE DAVEY–STEWARTSON SYSTEM

3.1. Introduction

The Davey–Stewartson equations in dimensionless are given by couple of equations of the form

$$\begin{cases} i\partial_t u + c_0\partial_x^2 u + \partial_y^2 u = c_1|u|^2 u + c_2 u\partial_x\phi \\ \partial_x^2\phi + c_3\partial_y^2\phi = \partial_x(|u|^2) \end{cases} \tag{3.1}$$

where $u = u(x, y, t)$ is complex-valued and $\phi = \phi(x, y, t)$ is real-valued functions that represent amplitude and mean velocity potential, respectively; here, the constants are real numbers and their signs determine the character of the equation. The results of this chapter have been announced in [20]. The systems (3.1) were first derived by Davey and Stewartson [5], Benney and Roskes [21], Djordjevic and Redekopp [22] and model the time evolution of 2D surface of water waves that propagate predominantly in one direction, but the wave amplitude is modulated slowly in the horizontal directions. In [22], Djordjevic and Redekopp showed that the parameter c_3 can be negative when capillary effects are important. According to the signs of c_0 and c_3 respectively, the system (3.1) is classified as follows

$$\begin{array}{cc} Elliptic - Elliptic & Elliptic - Hyperbolic \\ (+,+) & (+,-) \\ Hyperbolic - Elliptic & Hyperbolic - Hyperbolic \\ (-,+) & (-,-) \end{array}$$

DS systems are very well studied in terms of well-posedness and stability, blow-up profiles, existence of standing and travelling waves.

The investigation of the system (3.1) in terms of well-posedness was initiated by Ghidaglia and Saut [6] who established the local well-posedness in the elliptic–elliptic, elliptic–hyperbolic and hyperbolic–elliptic cases. More precisely, they studied the local and global properties of the elliptic–elliptic and the hyperbolic–elliptic cases in L^2, H^1, H^2; also in the elliptic–hyperbolic case, they obtained a global existence of weak solution of (3.1) under smallness assumption for data in L^2. Linares and Ponce [23] showed that under some smallness assumptions on the data, elliptic–hyperbolic and hyperbolic–hyperbolic cases of (3.1) are locally well-posed in the spaces $H^s(\mathbb{R}^2) \cap H^6(\mathbb{R}^2 : r^6 dxdy)$ for $s \geq 12$, and $H^s(\mathbb{R}^2) \cap H^3(\mathbb{R}^2 : r^2 dxdy)$ for $s \geq 6$ respectively. As regards to the initial value problem posed on the 2-torus, Godet [24] obtained a local well-posedness result for the hyperbolic–elliptic problem in $H^s(\mathbb{T}^2)$ for $s > 1/2$ as well as a blow-up rate for this equation. Concerning the half-plane problem, Fokas [25] studied the DS equation on the half-plane by using the inverse scattering transform techniques along with the formulation of a d-bar problem for a sectionally non-analytic function. As for the problem of global well-posedness, it is conjectured that the elliptic–elliptic type of (3.1) is globally well-posed in H^s for all $s \geq 0$. Toward this conjecture, Shen and Guo [26] proved that the initial value problem of (3.1) in the elliptic–elliptic case (with some assumptions on the constants) is globally well-posed for data in $H^s(\mathbb{R}^2)$, for $s > 4/7$. Thereafter, Yang et al. [27] improved this result by establishing global well-posedness in $H^s(\mathbb{R}^2)$ for $s > 2/5$ where they took advantage of the I-method. In particular, they obtained a polynomial in time bound for the H^s norm of the solution for $s > 2/5$. Some of the other results regarding the system (3.1) can be found in [28–32]. Upon considering elliptic–elliptic type of the system (3.1), we will study the initial value problem

$$\begin{cases} i\partial_t u + \Delta u = c_1|u|^2 u + c_2 u \partial_x \phi, \quad (x,y) \in \mathbb{R}^2, \ t \in \mathbb{R}, \\ \partial_x^2 \phi + \partial_y^2 \phi = \partial_x(|u|^2), \\ u(x,y,0) = u_0(x,y) \in H^s(\mathbb{R}^2). \end{cases} \tag{3.2}$$

To reformulate (3.2) in a better form we implement the Fourier transform in the spatial variable to the second equation of (3.2) so that the system reduces to a single equation

$$i\partial_t u + \Delta u = c_1|u|^2 u + c_2 K(|u|^2) u$$

where K is the pseudo-differential operator with symbol α given by

$$\widehat{K(f)}(\xi) = \alpha(\xi)\hat{f}(\xi) \text{ and } \alpha(\xi) = \frac{\xi_1^2}{|\xi|^2},\ \xi = (\xi_1, \xi_2) \neq (0,0).$$

Therefore, by the Duhamel formula, the equation (3.2) is equivalent to

$$u(x,y,t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}(c_1|u|^2u + c_2K(|u|^2)u)(\tau)d\tau \tag{3.3}$$

where $e^{it\Delta}$ denotes the free solution operator of the corresponding linear problem for the equation (3.2). In this chapter, our primary goal is to obtain smoothing properties of (3.2) globally in time; therefore, in order to take advantage of global well-posedness result of Theorem 3.2.1, we will assume the restriction that $c_1+c_2 > 0$. In the absence of this restriction, smoothing argument here works in the local sense only. The smoothing in this text means that the nonlinear part of the solution flow in relation to (3.3) lies in a more regular space than the initial data belong to. To make it rigorous, below we state our result:

Theorem 3.1.1. *Let $c_1 + c_2 > 0$. Fix $s > \frac{1}{2}$ and $a < \min(\frac{1}{2}, s - \frac{1}{2})$. Consider the solution to IVP (3.2) on $\mathbb{R}^2 \times \mathbb{R}$ with data $u_0 \in H^s(\mathbb{R}^2)$. Suppose that there is an a priori growth bound $\|u(t)\|_{H^s} \leq C(\|u_0\|_{H^s})\langle t\rangle^{\beta(s)}$ for some $\beta(s)$. Then,*

$$u(x,y,t) - e^{it\Delta}u_0 \in C_t^0 H_{x,y}^{s+a}; \tag{3.4}$$

furthermore, we have the growth bound

$$\left\|u(t) - e^{it\Delta}u_0\right\|_{H^{s+a}} \leq C(s, a, \|u_0\|_{H^s})\langle t\rangle^{1+\beta(s)(3+\frac{2}{s})}.$$

Let I denote the identity operator and K denote the multiplier operator introduced as above, hence to be able to prove Theorem 3.1.1 we will need the key trilinear estimate:

Proposition 3.1.2. *For $s > \frac{1}{2}$, $a < min(\frac{1}{2}, s - \frac{1}{2})$ and $b = \frac{1}{2} + \epsilon$ for $\epsilon > 0$ sufficiently small, we have*

$$\|(c_1I + c_2K)(u\overline{v})w\|_{X^{s+a,b-1}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{s,b}}. \tag{3.5}$$

Note that we can replace the $X^{s,b}$ norm in the above proposition with the time restricted version of this norm. $X^{s,b}$ spaces were introduced by Bourgain [1, 2] in order to capture the dispersive smoothing effect, that is intrinsic to the equation under consideration. Taking into account its diverse applications, smoothing estimates as such in Theorem 3.1.1 were established for numerous PDEs in the literature; for instance, Linares and Scialom obtained the smoothing estimate for the mKdV equation, [33], attaining one derivative gain for the nonlinear part of the solution with $H^s(\mathbb{R})$ initial data, for $s \geq 1$. Using this smoothing result, they gave a simplified proof of dispersive blow-up in solutions to the generalized KdV equation, which was studied previously by Bona and Saut, [34]. Also, smoothing estimates of nonlinear Schrödinger type equations on $\mathbb{R}^n$ were obtained in [35]; in particular, for $n = 2$ it was shown that the nonlinear part of the solution lies in $C_t^0 H_x^{s+a}([0,T] \times \mathbb{R}^2)$, $a = \frac{1}{2}, 1-$, for $H^s(\mathbb{R}^2)$ data when $s > \frac{3}{4}$. It is noted that this result extends to the elliptic–elliptic DS system. Therefore, Theorem 3.1.1 extends the smoothing argument in [35] to the range $s \in (\frac{1}{2}, \frac{3}{4}]$ by the gain of $s - \frac{1}{2}$, yet the gain achieved in [35] is still larger for $s > \frac{3}{4}$. Their arguments in the proof rely on $L^p L^q$-type estimates, whereas our proof makes use of the notion of Bourgain spaces $X^{s,b}$.

Smoothing estimates have many nice applications such as those appeared in the nonlinear Talbot effect, the bounds for higher order Sobolev norms and the existence of global attractors for dissipative and dispersive PDEs, see Chapter 5 of [9]. Hence in the remaining part of the chapter, our motivation in this regard is to present the simplified proof of the existence of a global attractor for the forced and damped Davey–Stewartson system in the energy space H^1 by making use of the smoothing estimate of Theorem 3.1.1. In [36], Wang and Guo obtained the existence of a global attractor in H^1 yet with a proof based on the splitting argument that requires more regular initial data to reach the compactness. The smoothing effect replaces the splitting method of [36]; as a result simplifying the proof and also as a byproduct gives us that the global attractor is indeed a compact subset of $H^{\frac{3}{2}-}$.

Thus, we consider the forced and damped Davey–Stewartson system

$$\begin{cases} iu_t + \Delta u + i\delta u = c_1|u|^2u + c_2 u\partial_x\phi + f, \ (x,y) \in \mathbb{R}^2 \\ \Delta\phi = \partial_x(|u|^2) \end{cases} \tag{3.6}$$

where the forcing term $f \in L^2(\mathbb{R}^2)$ is time independent, $\delta > 0$, $c_1 \geq 0$ and $c_1 + c_2 \geq 0$. The use of Theorem 2.3.5 which is provided by a smoothing estimate for the dissipative problem (3.6) yields the following result:

Theorem 3.1.3. *Consider the forced and weakly damped Davey–Stewartson system* (3.6) *on* $\mathbb{R}^2 \times [0,\infty)$ *with the initial data* $u(x,0) = u_0(x) \in H^1(\mathbb{R}^2)$*. Then, the equation* (3.6) *possesses a global attractor in* $H^1(\mathbb{R}^2)$*. Furthermore, for any* $a \in (0, \frac{1}{2})$ *the global attractor is a subset of* $H^{1+a}(\mathbb{R}^2)$*.*

We now briefly explain the organization of the chapter. In Section 3.2, we introduce function spaces and necessary tools for the proof of Theorem 3.1.1. In Section 3.3, we discuss Tao's $[k; Z]$ multiplier method in order to prove the key trilinear estimate (3.5). The proof of (3.5) will be given in Section 3.4. In Section 3.5, we prove Theorem 3.1.1 and finally Section 3.6 is devoted to prove the existence of a global attractor for the Davey–Stewartson system.

3.2. Notation and Preliminaries

For $s, b \in \mathbb{R}$, we require $X^{s,b}$ spaces corresponding to the evolution u of the DS system that is defined by means of the norm

$$\|u\|_{X^{s,b}} = \left\| \langle\xi\rangle^s \langle \tau + |\xi|^2 \rangle^b \widehat{u}(\xi,\tau) \right\|_{L^2_{\xi,\tau}} .$$

Localized $X^{s,b}$ space is also defined by

$$\|u\|_{X^{s,b}_\delta} = \inf_{\tilde{u}=u,\, |t|\leq\delta} \|\tilde{u}\|_{X^{s,b}} .$$

Consider the IVP (3.2) with the local existence time δ. We will quantify the dependence of δ to an initial data which we use in the proof of Theorem 3.1.1.

From the dilation symmetry of this equation, assuming that (u, ϕ) solve (3.2) with initial data u_0 on $[0, \lambda^{-2}]$, we come up with the symmetry solutions

$$u^\lambda(x, y, t) = \lambda^{-1} u(x/\lambda, y/\lambda, t/\lambda^2),\ \phi^\lambda(x, y, t) = \lambda^{-1}\phi(x/\lambda, y/\lambda, t/\lambda^2)$$

with data $u_0^\lambda(x, y) = \lambda^{-1} u_0(x/\lambda, y/\lambda)$ which solve the equation on $[0, 1]$. Thus for $\lambda > 1$, by comparing the H^s norms of u_0 and u_0^λ, the solution (u, ϕ) can be defined with respect to the local existence time

$$\delta \sim (C + \|u_0\|_{H^s})^{-\frac{2}{s}} \tag{3.7}$$

where $C = C(\|u_0\|_{L^2})$. Next we discuss a global well-posedness result for the elliptic–elliptic problem. We will exploit it once iterating our local result. Note that recalling the Sobolev index range $s > 1/2$ in our case, we indeed need a result that at least covers this range. Hence, rewriting the elliptic–elliptic IVP

$$\begin{cases} iu_t + \Delta u = c_1|u|^2 u + c_2 K(|u|^2)u \\ u(x, 0) = u_0(x) \in H_x^s(\mathbb{R}^2), \end{cases} \tag{3.8}$$

with a multiplier operator K given by

$$K(f) = \mathcal{F}^{-1} \frac{\xi_1^2}{|\xi|^2} \mathcal{F} f \tag{3.9}$$

for $\xi = (\xi_1, \xi_2) \neq 0$, the required global well-posedness result for (3.8) and (3.9) is stated as follows:

Theorem 3.2.1 (See [27])**.** *Let $c_1 + c_2 > 0$. For any $1 > s > \frac{2}{5}$, the initial value problem* (3.8) & (3.9) *is globally well-posed in $H^s(\mathbb{R}^2)$. Furthermore, there is a growth bound*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s(\mathbb{R}^2)} \leq C(1 + T)^{\frac{3s(1-s)}{2(5s-2)}+}$$

where the constant C depends only on the index s, $\|u_0\|_{L^2}$.

The essential ingredients of the proof of Theorem 3.2.1 are the interaction of Morawetz-type estimate and the almost conservation of the modified energy obtained from plugging the smoothing of order $1-s$ operator $\widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi)$,

$$m(\xi) := \begin{cases} 1, & |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^{s-1}, & |\xi| > 2N \end{cases}$$

in the usual Hamiltonian energy of the equation (3.8). The globalizing technique used in [27] is called the I-method, in view of the operator $I : H^s \to H^1$, which was introduced by Colliander–Keel–Staffilani–Takaoka–Tao, [37]. Next by virtue of Theorem 3.2.1, we reserve the following growth bound to be used later. For $s > \frac{1}{2}$, define $\beta(s) \geq \frac{3s(1-s)}{2(5s-2)}$ so that we have the a priori estimate

$$\|u(t)\|_{H^s} \lesssim \langle t\rangle^{\beta(s)} =: T(t) \tag{3.10}$$

for some non-decreasing function $T(t)$ (which we need in iterating the local result), and where the implicit constant depends on the Sobolev index and L^2 norm of the initial data.

The forced and weakly damped DS system (3.6) can be reduced to the single equation

$$iu_t + \Delta u + i\delta u = c_1|u|^2 u + c_2 K(|u|^2)u + f \tag{3.11}$$

with the same multiplier operator K as in (3.9). Below we list a few properties of K to be used in the energy calculations of (3.11):

(i) K is a bounded linear operator on L^p, $1 < p < \infty$,

(ii) $\overline{K(\psi)} = K(\overline{\psi})$,

(iii) $\int K(\psi)\varphi = \int \psi K(\varphi)$.

As a consequence of the smoothing estimate, we will study the existence of global attractors for the dissipative DS equation. The theory of existence of global attractors is only meaningful for systems as $t \to \infty$. Thus to see that the equation (3.6) is globally well-posed in the energy space, we can proceed in a similar fashion to the proof of the existence of an absorbing set in section 2 of [36] to obtain the following a priori estimate:

$$\|u(t)\|_{H^1} \leq Ae^{-Bt} + C, \ t > 0, \tag{3.12}$$

where $A = A(\|f\|_{L^2}, \|u_0\|_{H^1})$, $B = B(\delta) > 0$, $C = C(\delta, \|f\|_{L^2})$. Note that (3.12) implies the existence of an absorbing ball with radius $C = C(\delta, \|f\|_{L^2})$ as well.

3.3. Main Method

In this section, we discuss Tao's $[k; Z]$ multipliers method [7] so as to study the estimate of multilinear expression associated with the elliptic–elliptic type of Davey–Stewartson system. Suppose that Z is any additive abelian group with an invariant measure $d\xi$, as an example, one can take $Z = \mathbb{R}^{n+1}$ with Lebesgue measure or $\mathbb{Z}^n \times \mathbb{R}$ with the product of counting and Lebesgue measures. Let $k \geq 2$ be any integer, and let $\Gamma_k(Z) \subset Z^k$ denote the hyperplane

$$\Gamma_k(Z) := \{(\xi_1, \cdots, \xi_k) \in Z^k : \xi_1 + \cdots + \xi_k = 0\}$$

endowed with the measure

$$\int_{\Gamma_k(Z)} f = \int_{Z^{k-1}} f(\xi_1, \cdots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1})\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{k-1}.$$

Definition 3.3.1. *A complex-valued function $m : \Gamma_k(Z) \to \mathbb{C}$ is called the $[k; Z]$ multiplier if the inequality*

$$\Big| \int_{\Gamma_k(Z)} m(\xi_1, \cdots, \xi_k) \prod_{i=1}^{k} f_i(\xi_i) \Big| \leq \|m\|_{[k;Z]} \prod_{i=1}^{k} \|f_i\|_{L^2(Z)}$$

holds for all test functions f_i on Z and the best constant, denoted by $\|m\|_{[k;Z]}$.

Note that $\|\cdot\|_{[k;Z]}$ determines a norm on the $[k;Z]$ multipliers m. Since multilinear estimates might boil down to bilinear estimates of some sort (reason later), we emphasize the case $k=3$ which specifically associates with the bilinear estimates for our Schrödinger-type equation. So we write

$$\xi_1+\xi_2+\xi_3=0, \qquad \tau_1+\tau_2+\tau_3=0, \tag{3.13}$$

$$\lambda_j := \tau_j + h_j(\xi_j), \qquad h_j(\xi_j) = \pm|\xi_j|^2 \text{ for } 1 \le j \le 3. \tag{3.14}$$

Here, λ_j measures how close in frequency the j^{th} wave is to a free solution. It is remarkable to note that for $d \geq 2$, whenever the two of the frequencies ξ_1, ξ_2, ξ_3 form an orthogonal pair, λ_j's simultaneously vanish. Thus in order to take care of this situation, we introduce the function $h : \mathit{\Gamma}_3(\mathbb{R}^2) \to \mathbb{R}$ defined by

$$h(\xi_1,\xi_2,\xi_3) := h_1(\xi_1)+h_2(\xi_2)+h_3(\xi_3) = \lambda_1+\lambda_2+\lambda_3 \tag{3.15}$$

which measures to what extent the frequencies ξ_1, ξ_2, ξ_3 resonate with each other so let it be referred as a resonance function. The domain on which the resonance function may vanish depends on the sign of Schrödinger dispersion relation $h_j(\xi_j) = \pm|\xi_j|^2$. Up to symmetry, we have two possibilities: the $(+++)$ case

$$h_1(\xi) = h_2(\xi) = h_3(\xi) = |\xi|^2 \tag{3.16}$$

and the $(++-)$ case

$$h_1(\xi) = h_2(\xi) = |\xi|^2, h_3(\xi) = -|\xi|^2. \tag{3.17}$$

Comparing the two cases, analysis of the first one is rather easier since the resonance function

$$h(\xi_1,\xi_2,\xi_3) = |\xi_1|^2+|\xi_2|^2+|\xi_3|^2$$

vanishes only at the origin. As for the second case, the resonance function

$$h(\xi_1,\xi_2,\xi_3) = |\xi_1|^2+|\xi_2|^2-|\xi_3|^2$$

can vanish whenever the frequencies ξ_1 and ξ_2 become orthogonal.

More precisely, for $(\xi_1, \xi_2, \xi_3) \in \mathit{\Gamma}_3(\mathbb{R}^2)$, we write

$$|h(\xi_1, \xi_2, \xi_3)| = ||\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2| = 2|\xi_1 \cdot \xi_2| \sim |\xi_1||\xi_2||\pi/2 - \angle(\xi_1, \xi_2)|$$

where $\angle(\xi_1, \xi_2)$ denotes the angle between ξ_1 and ξ_2. So the extent to which ξ_1 and ξ_2 get closer to being orthogonal, the more rapidly resonance function tends to vanish. At this point, we assume that

$$|h(\xi_1, \xi_2, \xi_3)| \lesssim |\xi_1||\xi_2|, \tag{3.18}$$

and

$$\angle(\xi_1, \xi_2) = \pi/2 + O(\frac{|h(\xi_1, \xi_2, \xi_3)|}{|\xi_1||\xi_2|}). \tag{3.19}$$

We will estimate the $[3; \mathbb{R}^3]$ norm of multipliers by making use of the dyadic decomposition of the variables ξ_j, λ_j and the function $h(\xi_1, \xi_2, \xi_3)$. Thus, we use the capitalized variables N_j, L_j and H to denote the magnitude of the pieces into which the variables ξ_j, λ_j and the resonance function h decomposed, respectively. Here, these variables are assumed to be dyadic; that is, they range over the numbers of the form 2^k, $k \in \mathbb{Z}$. In terms of sizes of the variables $N_j > 0$, $j = 1, 2, 3$, we write $N_{\min} \leq N_{\text{med}} \leq N_{\max}$ to denote the minimum, median and maximum of N_1, N_2, N_3. This, in its own right, saves us from repetitive analysis and reduces the number of cases substantially. Likewise we define $L_{\min} \leq L_{\text{med}} \leq L_{\max}$ for $L_j > 0$, $j = 1, 2, 3$. Next we make some assumptions on the sizes of these variables. Before doing so, we need to state several lemmas from [7].

Lemma 3.3.2 (Comparison Principle)**.** *Let m and M be $[k; Z]$ multipliers, if for all $\xi \in \mathit{\Gamma}_k(Z)$ $|m(\xi)| \leq M(\xi)$, then $\|m\|_{[k;Z]} \leq \|M\|_{[k;Z]}$. Furthermore, if $a_1, ..., a_k$ are real-valued functions on Z and m is a $[k; Z]$ multiplier, then*

$$\left\| m(\xi) \prod_{i=1}^{k} a_i(\xi_i) \right\|_{[k;Z]} \leq \|m\|_{[k;Z]} \prod_{i=1}^{k} \|a_i\|_{L^\infty(Z)} .$$

Lemma 3.3.3. *For any $\xi_0 \in \mathit{\Gamma}_k(Z)$ and any $[k;Z]$ multiplier m, we have the translation invariance of the norm*

$$\|m(\xi+\xi_0)\|_{[k;Z]} = \|m(\xi)\|_{[k;Z]} \tag{3.20}$$

also we have the averaging estimate

$$\|m*\mu\|_{[k;Z]} \le \|m\|_{[k;Z]} \|\mu\|_{L^1(\mathit{\Gamma}_k(Z))} \tag{3.21}$$

for any finite measure μ on $\mathit{\Gamma}_k(Z)$.

It is remarkable that by the finiteness assumption of the measure μ, the mapping $m \to m*\mu$ can be regarded as an averaging of m. Therefore by an averaging over unit time scales, implementation of the lemmas above allows us to restrict the multiplier $m(\xi_1,\xi_2,\xi_3)$ to the region

$$|\lambda_j| \gtrsim 1, \quad j = 1,2,3.$$

Furthermore, since we deal with a multiplier with no singularities for $|\xi_j| \ll 1$, by using the similar reason we may assume that

$$\max(|\xi_1|,|\xi_2|,|\xi_3|) \gtrsim 1.$$

Therefore, through a decomposition of the variables we may assume without loss of generality that

$$N_{\max} \gtrsim 1, \quad L_{\min} \gtrsim 1.$$

We now fix some summation conventions in order to be used in the rest. Any summation of the form $L_{\max} \sim ...$ is a sum over the three dyadic variables L_1, L_2, L_3, for example,

$$\sum_{L_{\max}\sim H} := \sum_{L_1,L_2,L_3 \gtrsim 1 : L_{\max}\sim H} .$$

Furthermore, any summation of the form $N_{\max} \sim ...$ is a sum over the three dyadic variables $N_1, N_2, N_3 > 0$, for instance,

$$\sum_{N_{\max}\sim N_{\text{med}}\sim N} := \sum_{N_1,N_2,N_3>0: N_{\max}\sim N_{\text{med}}\sim N} .$$

Let m be a $[3;\mathbb{R}^2\times\mathbb{R}]$ multiplier. Next we intend to study the problem of controlling

$$\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3))\|_{[3;\mathbb{R}^2\times\mathbb{R}]} . \tag{3.22}$$

By a dyadic decomposition of the support of m in the variables ξ_j, λ_j together with a dyadic decomposition of the resonance function h, we write

$$(3.22) \lesssim \left\| \sum_{N_{\max}\gtrsim 1} \sum_{H} \sum_{L_1,L_2,L_3} m((N_1,L_1),(N_2,L_2),(N_3,L_3)) X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \right\|_{[3;\mathbb{R}^2\times\mathbb{R}]} \tag{3.23}$$

where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the multiplier

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3}(\xi,\tau) := \chi_{|h(\xi)|\sim H} \prod_{j=1}^{3} \chi_{|\xi_j|\sim N_j} \chi_{|\lambda_j|\sim L_j} .$$

Note that N_j and L_j, in turn, measure the size of the frequency of the j^{th} wave and how closely it approximates a free solution, whereas H measures the amount of resonance. From (3.13), (3.14) and (3.15), it can be deduced that $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{\max} \sim N_{\text{med}} \tag{3.24}$$

and

$$L_{\max} \sim \max(L_{\text{med}}, H). \tag{3.25}$$

Therefore, using (3.24), (3.25) and implementing Schur's test [7] (which enables us to replace sum with a supremum) to the sums in $N_{\max}$ and N_{med}, we obtain

$$(3.23) \lesssim \sup_{N\gtrsim 1} \Big\| \sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{H} \sum_{L_{max}\sim \max(L_{\text{med}},H)} m((N_1,L_1),(N_2,L_2),(N_3,L_3)) \times X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \Big\|_{[3;\mathbb{R}^2\times\mathbb{R}]} .$$

Therefore, by the triangle inequality and (3.25) it suffices to control

$$\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_2,L_3\gtrsim 1} m((N_1,L_1),(N_2,L_2),(N_3,L_3))\left\|X_{N_1,N_2,N_3;L_{\max};L_1,L_2,L_3}\right\|_{[3;\mathbb{R}^2\times\mathbb{R}]} \tag{3.26}$$

or

$$\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_{\max}\sim L_{\text{med}}}\ \sum_{H\ll L_{\max}} m((N_1,L_1),(N_2,L_2),(N_3,L_3)) \times \left\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\right\|_{[3;\mathbb{R}^2\times\mathbb{R}]} \tag{3.27}$$

for all $N \gtrsim 1$. The following lemma gives a sharp bound for the quantity

$$\left\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\right\|_{[3;\mathbb{R}^2\times\mathbb{R}]}. \tag{3.28}$$

Lemma 3.3.4 (See [7]). *Suppose that $N_1, N_2, N_3 > 0$, $L_1, L_2, L_3 > 0$ and $H > 0$ satisfy* (3.24) *and* (3.25).

(i) In the case $(+++)$, let the dispersion relations be given by (3.16) *so we may assume that $H \sim N_{\max}^2$. Then,*

$$(3.28) \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} N_{\min}^{1/2} \min(N_{\max}N_{\min}, L_{med})^{1/2} \tag{3.29}$$

(ii) In the case $(++-)$, let the dispersion relations be given by (3.17) *and from* (3.18) *$H \lesssim N_1N_2$. Then,*

- *($(++)$ case) If $N_1 \sim N_2 \gtrsim N_3$, then* (3.28) *vanishes unless $H \sim N_1^2$ in this case we have*

$$(3.28) \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} N_{\min}^{1/2} \min(N_{\max}N_{\min}, L_{med})^{1/2} \tag{3.30}$$

- *($(+-)$ coherence) If $N_1 \sim N_3 \gtrsim N_2$ and $H \sim L_2 \gg L_1, L_3, N_2^2$ then*

$$(3.28) \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} N_{\min}^{1/2} \min(H, \frac{H}{N_{\min}^2} L_{med})^{1/2} \tag{3.31}$$

The same estimate holds with the roles of 1 and 2 reversed.

- *In all other cases,*

$$(3.28) \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} N_{\min}^{1/2} \min(H, L_{med})^{1/2} \min(1, \frac{H}{N_{\min}^2})^{1/2}. \tag{3.32}$$

Also the lemma below demonstrates that higher-order multilinear estimates might be reduced to the lower-ordered ones and by this means the analysis of the whole multiplier splits up.

Lemma 3.3.5 (See [7]). *If $k_1, k_2 \geq 1$ and m_1 and m_2 are functions on Z^{k_1} and Z^{k_2} respectively, then we have the composition estimate*

$$\|m_1(\xi_1, ..., \xi_{k_1}) m_2(\xi_{k_1+1}, ..., \xi_{k_1+k_2})\|_{[k_1+k_2;Z]} \leq \|m_1(\xi_1, ..., \xi_{k_1})\|_{[k_1+1;Z]} \|m_2(\xi_1, ..., \xi_{k_2})\|_{[k_2+1;Z]}. \tag{3.33}$$

In particular, for every $m : Z^k \to \mathbb{R}$ we have the TT^* identity

$$\left\| m(\xi_1, ..., \xi_k)\overline{m(-\xi_{k+1}, ..., -\xi_{2k})} \right\|_{[2k;Z]} = \|m(\xi_1, ..., \xi_k)\|^2_{[k+1;Z]}. \tag{3.34}$$

3.4. Proof of Proposition 3.1.2: Trilinear $X^{s,b}$ Estimate

The required $X^{s,b}$ estimate amounts to showing that

$$\Big| \int_{\Gamma_4(\mathbb{R}^2\times\mathbb{R})} m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3), (\xi_4, \tau_4)) \prod_{j=1}^{4} f_j(\xi_j, \tau_j) \Big| \leq \|m\|_{[4;\mathbb{R}^2\times\mathbb{R}]} \prod_{j=1}^{4} \|f_j\|_{L^2(\mathbb{R}^2\times\mathbb{R})}$$

where

$$m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3), (\xi_4, \tau_4)) = \frac{[c_1 + c_2\alpha(\xi_1+\xi_2)]\langle\xi_4\rangle^{s+a}}{\prod_{j=1}^{3}\langle\xi_j\rangle^s \langle\tau_1+|\xi_1|^2\rangle^b \langle\tau_2-|\xi_2|^2\rangle^b \langle\tau_3+|\xi_3|^2\rangle^b \langle\tau_4-|\xi_4|^2\rangle^{1-b}}.$$

Thus, it suffices to show that

$$\|m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3), (\xi_4, \tau_4))\|_{[4;\mathbb{R}^2\times\mathbb{R}]} \lesssim 1.$$

We may suppose without loss of generality that

$$\xi_4| \approx \max_{1\le j\le 4} |\xi_j|$$

because other cases are easier and follow immediately. In this case, the structure of the hyperplane $\Gamma_4(\mathbb{R}^2 \times \mathbb{R})$ suggests three cases to consider:

- Case 1 $|\xi_4| \approx |\xi_1|$
- Case 2 $|\xi_4| \approx |\xi_2|$
- Case 3 $|\xi_4| \approx |\xi_3|$

We begin by examining the Case 2. In this case, the multiplier is estimated by

$$\begin{aligned}
&\big|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\big| \\
&\qquad\qquad \lesssim \frac{\langle\xi_1\rangle^{-s}}{\langle\tau_1+|\xi_1|^2\rangle^b\langle\tau_2-|\xi_2|^2\rangle^b} \times \frac{\langle\xi_4\rangle^a\langle\xi_3\rangle^{-s}}{\langle\tau_3+|\xi_3|^2\rangle^b\langle\tau_4-|\xi_4|^2\rangle^{1-b}} \\
&\qquad\qquad =: m_{1,2}((\xi_1,\tau_1),(\xi_2,\tau_2)) \times m_{3,4}((\xi_3,\tau_3),(\xi_4,\tau_4)).
\end{aligned}$$

Using Lemma 3.3.2 and Lemma 3.3.5, we have the bound

$$\begin{aligned}
\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4;\mathbb{R}^2\times\mathbb{R}]} &\lesssim \|m_{1,2}((\xi_1,\tau_1),(\xi_2,\tau_2))\|_{[3;\mathbb{R}^2\times\mathbb{R}]} \\
&\quad \times \|m_{3,4}((\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[3;\mathbb{R}^2\times\mathbb{R}]}\,.
\end{aligned}$$

We shall introduce the variables $\xi_{\mathrm{d}2}$, $\xi_{\mathrm{d}3}$, $\tau_{\mathrm{d}2}$, $\tau_{\mathrm{d}3}$ satisfying

$$(\xi_1,\tau_1)+(\xi_2,\tau_2)+(\xi_{\mathrm{d}3},\tau_{\mathrm{d}3}) = 0, \tag{3.35}$$

$$(\xi_{\mathrm{d}2},\tau_{\mathrm{d}2})+(\xi_3,\tau_3)+(\xi_4,\tau_4) = 0. \tag{3.36}$$

By the decomposition of the support of $m_{1,2}$, $m_{3,4}$ and the corresponding resonance functions, it suffices to estimate the related sums (3.26) or (3.27).

Then, we start with controlling $m_{3,4}$ and need to consider the following cases

(i) $N_{\mathrm{d2}} \sim N_3 \sim N_4$

(ii) $N_3 \sim N_4 \gg N_{\mathrm{d2}}$

(iii) $N_{\mathrm{d2}} \sim N_3 \gg N_4$

(iv) $N_{\mathrm{d2}} \sim N_4 \gg N_3$.

In the first of these cases $N_{\min} \sim N_{\max} \sim N \gtrsim 1$, so by the estimate (3.32) of Lemma 3.3.4 we obtain

$$\frac{\langle N_4\rangle^a \langle N_3\rangle^{-s}}{L_3^b L_4^{1-b}} \|X_{N_{\mathrm{d2}},N_3,N_4;H;L_{\mathrm{d2}},L_3,L_4}\|_{[3;\mathbb{R}^2\times\mathbb{R}]} \lesssim \frac{N^{-s+a}H^{2\epsilon}}{L_{\min}^{\epsilon}L_{\mathrm{med}}^{\epsilon}}.$$

It follows that if $H \sim L_{\max}$, then since $H \lesssim N_{\max}^2$, we have that

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{\mathrm{d2}},L_3,L_4\gtrsim 1} \frac{N^{-s+a}H^{2\epsilon}}{L_{\min}^{\epsilon}L_{\mathrm{med}}^{\epsilon}} \lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{\mathrm{d2}},L_3,L_4\gtrsim 1} \frac{N^{-s+a+6\epsilon}}{L_{\min}^{\epsilon}L_{\mathrm{med}}^{\epsilon}L_{\max}^{\epsilon}}$$

which is finite provided that $a < s$. When $H \ll L_{\max}$, then $L_{\mathrm{med}} \sim L_{\max}$; summing in H first, the sum is finite for $a < s$ as well. For the second case, we consider the estimates (3.31) and (3.32) in Lemma 3.3.4. As $N_3 \sim N_4 \sim N \gtrsim 1$, we may establish the estimates corresponding to the estimate (3.32) by following just the same lines of the previous case. So it suffices to make use of the estimate (3.31) merely, in which case we have $H \sim L_{\mathrm{d2}} \gg L_3, L_4, N_{\mathrm{d2}}^2 \sim N_{\min}^2$. Consequently, bearing in mind that $L_{\max} \sim H \lesssim N^2$, the sum is controlled by

$$\begin{aligned}
&\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{\mathrm{d2}},L_3,L_4\gtrsim 1} \frac{\langle N_4\rangle^a \langle N_3\rangle^{-s}}{L_3^b L_4^{1-b}} \|X_{N_{\mathrm{d2}},N_3,N_4;L_{\max};L_{\mathrm{d2}},L_3,L_4}\|_{[3;\mathbb{R}^2\times\mathbb{R}]} \\
&\qquad \lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{\mathrm{d2}},L_3,L_4\gtrsim 1} \frac{N^{-s+a-1/2}N_{\min}^{1/2}\big(H^{1/2+\epsilon}(\frac{HL_{\mathrm{med}}}{N_{\min}^2})^{1/2-\epsilon}\big)^{1/2}}{L_{\min}^{\epsilon}L_{\mathrm{med}}^{1/2-\epsilon}} \\
&\qquad \lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{\mathrm{d2}},L_3,L_4\gtrsim 1} \frac{N^{-s+a+1/2+3\epsilon}N_{\min}^{\epsilon}}{L_{\min}^{\epsilon}L_{\mathrm{med}}^{1/4-\epsilon/2}L_{\max}^{\epsilon/2}\langle N_{\min}\rangle^{2\epsilon}} \lesssim 1
\end{aligned}$$

provided that $a < s-1/2$. Note at this point that for a non-trivial smoothing argument it is necessary to make the assumption that $s > 1/2$.

In the third and fourth cases, due to its effect in the use of Lemma 3.3.4, we have to decide the sign in the quantity $\lambda_{d2} = \tau_{d2} \pm |\xi_{d2}|^2$. No selection would lose the generality though, we prefer to set $\lambda_{d2} = \tau_{d2} + |\xi_{d2}|^2$. Hence for the third case, by this choice of the dummy modulation variable, we fall under the (++) case which leads to $H \sim N_{\max}^2$. Thus, the estimate (3.30) in Lemma 3.3.4 is to be used. Once $H \sim L_{\max}$, the sum is bounded by

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{d2},L_3,L_4\gtrsim 1} \frac{N^{-s+a+3\epsilon/2} N_{\min}^{3\epsilon/2}}{L_{min}^{\epsilon} L_{\text{med}}^{\epsilon/2}} \lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{d2},L_3,L_4\gtrsim 1} \frac{N^{-s+a+9\epsilon/2} N_{\min}^{3\epsilon/2}}{L_{min}^{\epsilon} L_{\text{med}}^{\epsilon/2} L_{\max}^{\epsilon/2} \langle N_{\min}\rangle^{2\epsilon}}$$

this is finite as long as $a < s$. Also, if $H \ll L_{\max}$, then as $H \sim N^2$ and $L_{\text{med}} \sim L_{\max}$, we have the bound

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \sum_{H \ll L_{max}} \frac{N^{-s+a+\epsilon} H^{\epsilon/4} N_{\min}^{3\epsilon/2}}{L_{\min}^{\epsilon} L_{\text{med}}^{\epsilon/8} L_{\max}^{3\epsilon/8}} \lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \frac{N^{-s+a+\epsilon} N_{\min}^{3\epsilon/2}}{L_{\min}^{\epsilon} L_{\text{med}}^{\epsilon/8} L_{\max}^{\epsilon/8}}$$

which is finite for $a < s$. In the last case, sign analysis of the modulations λ_{d2} (which is set in the previous case) and λ_4 suggests utilizing (3.31) and (3.32) in Lemma 3.3.4. In the separate case $H \sim L_{\max} \sim L_3 \gg L_{d2}, L_4, N_3^2$, the bound (3.31) gives rise to

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{d2},L_3,L_4\gtrsim 1} \frac{\langle N_4\rangle^a \langle N_3\rangle^{-s}}{L_3^b L_4^{1-b}} \left\| X_{N_{d2},N_3,N_4;L_{\max};L_{d2},L_3,L_4} \right\|_{[3;\mathbb{R}^2\times\mathbb{R}]} \lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{d2},L_3,L_4\gtrsim 1} \frac{N^{a-1/2} N_{\min}^{1/2} H^{1/2} L_{\min}^{\epsilon}}{\langle N_{\min}\rangle^s L_{\max}^{1/2+\epsilon}}.$$

Using the inequality $H \lesssim L_{\max}^{1-\epsilon} N^{2\epsilon}$, the above sum can be controlled by

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{d2},L_3,L_4\gtrsim 1} \frac{N^{a-1/2+\epsilon} N_{\min}^{\epsilon}}{\langle N_{\min}\rangle^{s-1/2+\epsilon} L_{\max}^{\epsilon/6} L_{\text{med}}^{\epsilon/6} L_{\min}^{\epsilon/6}}$$

which is summable provided that $a < 1/2$.

For the situations with $L_{\max} \gg H$, where the estimate (3.32) is available, we have the bound

$$\begin{aligned}
\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \sum_{H\ll L_{max}} & \frac{N^{a-1/2} N_{\min}^{\epsilon} H^{2\epsilon} L_{\text{med}}^{1/2-2\epsilon}}{\langle N_{\min}\rangle^{s-1/2+\epsilon} L_{\min}^{\epsilon} L_{\text{med}}^{1/2-\epsilon}} \\
&\lesssim \sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \frac{N^{a-1/2+4\epsilon} N_{\min}^{\epsilon}}{\langle N_{\min}\rangle^{s-1/2+\epsilon} L_{\min}^{\epsilon} L_{\text{med}}^{\epsilon/2} L_{\max}^{\epsilon/2}} \\
&\lesssim 1
\end{aligned}$$

provided that $a < 1/2$. Also for the instances with $L_{\max} \sim H$ where the estimate (3.32) is available, we proceed similarly as above to show that the sum is finite for $a < 1/2$:

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{\text{d2}},L_3,L_4\gtrsim 1} \frac{N^{a-1/2+5\epsilon} N_{\min}^{\epsilon}}{\langle N_{\min}\rangle^{s-1/2+\epsilon} L_{\min}^{\epsilon} L_{\text{med}}^{\epsilon/2} L_{\max}^{\epsilon}} \lesssim 1.$$

This completes controlling $m_{3,4}$. The proof with regard to $m_{1,2}$ is the repetition of performed for $m_{3,4}$ without the multiplier $\langle \xi_4 \rangle^a$, and it follows by assuming (3.35) and $s > 1/2$. As for the Case 1, we write

$$\begin{aligned}
|m((\xi_1,\tau_1), (\xi_2,\tau_2),&(\xi_3,\tau_3),(\xi_4,\tau_4))| \\
&\lesssim \frac{\langle \xi_3 \rangle^{-s}}{\langle \tau_1 + |\xi_1|^2\rangle^b \langle \tau_3 + |\xi_3|^2\rangle^b} \times \frac{\langle \xi_4 \rangle^{a} \langle \xi_2 \rangle^{-s}}{\langle \tau_2 - |\xi_2|^2\rangle^b \langle \tau_4 - |\xi_4|^2\rangle^{1-b}} \times \\
&=: m_{1,3}((\xi_1,\tau_1),(\xi_3,\tau_3)) \times m_{2,4}((\xi_2,\tau_2),(\xi_4,\tau_4)).
\end{aligned}$$

As before pick dummy variables $\xi_{\text{d}j}$, $\tau_{\text{d}j}$ for $j = 2, 3$ satisfying

$$\begin{aligned}
(\xi_2,\tau_2) + (\xi_{\text{d3}},\tau_{\text{d3}}) + (\xi_4,\tau_4) = 0, \\
(\xi_1,\tau_1) + (\xi_{\text{d2}},\tau_{\text{d2}}) + (\xi_3,\tau_3) = 0
\end{aligned}$$

and $\lambda_{\text{d2}} = \tau_{\text{d2}} + |\xi_{\text{d2}}|^2$, $\lambda_{\text{d3}} = \tau_{\text{d3}} - |\xi_{\text{d3}}|^2$. Then, the analysis of these multipliers falls into $(+++)$ case which is substantially easier to handle, and only the estimate (3.29) for (3.28) is taken into consideration.

In the spirit of the analysis of the subcase (iii), $[3;\mathbb{R}^3]$ norms of the two multipliers $m_{1,3}$ and $m_{2,4}$ can be shown to be finite provided that $a < 1/2$ and $s > 1/2$. Lastly, Case 3 immediately follows from Case 1 because the variables ξ_1 and ξ_3 appear to be symmetric.

Remark 3.4.1. To see that the trilinear estimate (3.5) fails for $a > 1/2$, set

$$\widehat{u} = \chi_{Q_1},\ \widehat{v} = \chi_{Q_2},\ \widehat{w} = \chi_{Q_2}$$

where χ_S is the characteristic function of the set S and

$$\begin{aligned}
Q_1 &= \{(\xi_1, \xi_2, \tau) \in \mathbb{R}^3 : |\xi_1| \le 1, |\xi_2 - N| \le 1/N, |\tau + N^2| \le 1\}\\
Q_2 &= \{(\xi_1, \xi_2, \tau) \in \mathbb{R}^3 : |\xi_1| \le 1, |\xi_2| \le 1/\sqrt{N}, |\tau| \le 1\}
\end{aligned}$$

for $N \in \mathbb{N}$ large. On the one hand, since the volume of the former set $\approx 1/N$ and of the latter $\approx N^{-1/2}$, we have $\|u\|_{X^{s,b}} \approx N^{s-1/2}$ and $\|v\|_{X^{s,b}} = \|w\|_{X^{s,b}} \approx N^{-1/4}$, on the other

$$\begin{aligned}
||u\bar{v}w||_{X^{s+a,b-1}} &\ge CN^{-1}\Big(\int_{Q_1} \langle \tau + |\xi|^2\rangle^{2(b-1)} \langle \xi\rangle^{2(s+a)}\, \mathrm{d}\xi\, \mathrm{d}\tau\Big)^{1/2}\\
&\approx CN^{s+a-1}\Big(\int_{Q_1} \mathrm{d}\xi\, \mathrm{d}\tau\Big)^{1/2} \ge CN^{s+a-3/2}.
\end{aligned}$$

Thus for $a > 1/2$, Proposition 3.1.2 implies that $CN^{s+a-3/2} \lesssim N^{s-1/2}N^{-1/4}N^{-1/4} = N^{s-1} \implies C \lesssim N^{1/2-a}$.

In the sequel, we make use of the ideas in [16] and [38] so as to finish the proof of Theorem 3.1.1.

3.5. Proof of Theorem 3.1.1

Let δ be the local existence time given by the local theory. Since $b > \frac{1}{2}$ we use the embedding $X^{s+a,b} \hookrightarrow C_t^0 H_x^{s+a}$ along with Lemma 2.1.3, Proposition 3.1.2 and local theory bound to obtain

$$\begin{aligned} \left\|u(t) - e^{it\Delta}u(0)\right\|_{C_t^0 H_x^{s+a}([-\delta,\delta]\times\mathbb{R}^2)} &\lesssim \left\| \int_0^t e^{i(t-\tau)\Delta}[c_1|u|^2 u + c_2 K(|u|^2)u](\tau)d\tau \right\|_{X_\delta^{s+a,b}} \\ &\lesssim \left\|c_1|u|^2 u + c_2 K(|u|^2)u\right\|_{X_\delta^{s+a,b-1}} \\ &\lesssim \|u\|^3_{X_\delta^{s,b}} \lesssim \|u_0\|^3_{H^s}. \end{aligned} \tag{3.37}$$

Therefore, (3.37) proves (3.4) in the local time interval $[-\delta, \delta]$, in order to prove it for all times we have to iterate this result and then prove the continuity argument. To do so, fix t large, then for if $r \leq t$, from (3.10) with the choice of $\beta(s)$ (remember that for $s > 1/2$ this choice makes $T(t)$ non-decreasing function of time) we have

$$\|u(r)\|_{H^s} \lesssim T(r) \leq T(t).$$

Thus, under favor of global well-posedness result of Theorem 3.2.1, considering the initial value problem (3.2) with $u((j-1)\delta)$ being the initial data, and implementing (3.37) to this local problem, we obtain, for any j with $j\delta \leq t$, that

$$\left\|u(j\delta) - e^{i\delta\Delta}u((j-1)\delta)\right\|_{H^{s+a}} \lesssim \|u((j-1)\delta)\|^3_{H^s} \lesssim T(t)^3.$$

By the local theory, we pick $\delta \sim T(t)^{-\frac{2}{s}}$ so that, for $J = t/\delta \sim tT(t)^{\frac{2}{s}}$, we get

$$\begin{aligned} \left\|u(t) - e^{it\Delta}u_0\right\|_{H^{s+a}} &\leq \sum_{j=1}^{J} \left\|e^{i\delta(J-j)\Delta}u(j\delta) - e^{i\delta(J-j+1)\Delta}u((j-1)\delta)\right\|_{H^{s+a}} \\ &= \sum_{j=1}^{J} \left\|u(j\delta) - e^{i\delta\Delta}u((j-1)\delta)\right\|_{H^{s+a}} \lesssim JT(t)^3 \lesssim \langle t\rangle T(t)^{3+\frac{2}{s}}. \end{aligned}$$

This finishes the iteration argument. In order for (3.4) to hold, we are left to show that the difference $D(t) - D(r)$, where

$$D(t) := u(t) - e^{it\Delta}u_0 = -i\int_0^t e^{i(t-\tau)\Delta}(c_1|u|^2 u + c_2 K(|u|^2)u)(\tau)d\tau,$$

is continuous in H^{s+a}. Assume that r is fixed and that $t > r$, then

$$\begin{aligned}
&\|D(t) - D(r)\|_{H^{s+a}} \\
&\lesssim \left\| \langle\xi\rangle^{s+a} \mathcal{F}\Big((e^{i(t-r)\Delta} - \mathrm{Id}) \int_0^r e^{i(r-\tau)\Delta} (c_1|u|^2 u + c_2 K(|u|^2)u)(\tau) d\tau \Big) \right\|_{L^2_\xi} \\
&+ \left\| \langle\xi\rangle^{s+a} \mathcal{F}\Big(\int_r^t e^{i(t-\tau)\Delta} (c_1|u|^2 u + c_2 K(|u|^2)u)(\tau) d\tau \Big) \right\|_{L^2_\xi} =: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

Using the inequality $|e^{i(r-t)|\xi|^2} - 1| \lesssim \min(1, |\xi|^2|t-r|) \le (|\xi|^2|t-r|)^\epsilon$ in the subsequent calculation (for a sufficiently small $\epsilon > 0$), we obtain

$$\mathrm{I}_1 \lesssim \sum_{j=1}^{r/\delta} \left\| \int_{(j-1)\delta}^{j\delta} e^{i(r-\tau)\Delta} (c_1|u|^2 u + c_2 K(|u|^2)u)(\tau) d\tau \right\|_{H^{s+2\epsilon+a}} \tag{3.38}$$

where we pick the same δ given by the local existence time of the solution. Dependence of δ on $\sup_{t\in[0,r]} \|u(t)\|_{H^s(\mathbb{R}^2)}$ implies that this is a finite sum. As the length of each interval of integration is δ, by the time translation and time reversal symmetries of the solution, it suffices just to estimate the following integral for $t \in [-\delta/2, \delta/2]$:

$$\begin{aligned}
&\left\| \int_0^t e^{i(t-\tau)\Delta} (c_1|u|^2 u + c_2 K(|u|^2)u)(\tau) d\tau \right\|_{H^{s+2\epsilon+a}} \\
&\le \left\| \eta(t/\delta) \int_0^t e^{i(t-\tau)\Delta} (c_1|u|^2 u + c_2 K(|u|^2)u)(\tau) d\tau \right\|_{L^\infty_t H^{s+2\epsilon+a}_{x,y}} \\
&\lesssim \left\| \eta(t/\delta) \int_0^t e^{i(t-\tau)\Delta} (c_1|u|^2 u + c_2 K(|u|^2)u)(\tau) d\tau \right\|_{X^{s+2\epsilon+a,\frac{1}{2}+\epsilon}} \\
&\lesssim \delta^\epsilon \left\| c_1|u|^2 u + c_2 K(|u|^2)u \right\|_{X_\delta^{s+2\epsilon+a,b-1}} \\
&\lesssim \delta^\epsilon \|u\|^3_{X_\delta^{s+2\epsilon,b}} \lesssim \delta^\epsilon \|u_0\|^3_{H^{s+2\epsilon}}
\end{aligned}$$

where we have used Lemma 2.1.3 with $b = \frac{1}{2} + 2\epsilon$, Proposition 3.1.2 with $s + 2\epsilon$ and finally the local theory bound. As a result, the sum in (3.38) is bounded and hence I_1 converges to 0 as $t \to r$. Also, the same result follows for I_2 by using Proposition 3.1.2 and Lemma 2.1.3.

3.6. Existence of a Global Attractor: Proof of Theorem 3.1.3

This section is devoted to give an alternative proof of the existence of a global attractor for the forced and weakly damped DS by using the smoothing estimates. Firstly, we show that the evolution operator is weakly continuous and then we exploit this in handling the corresponding energy equation to upgrade the weak convergence, resulting from boundedness of the flow, to strong convergence giving rise to the asymptotic compactness of the flow. Throughout this section, $\xrightarrow{w}$ and $\xrightarrow{w^*}$ will denote the weak and the weak* convergences, respectively.

Lemma 3.6.1. *If $u_0^k \xrightarrow{w} u_0$ in H^1, then for all $T > 0$ and $a \in (0, \frac{1}{2})$, the linear and the nonlinear parts of the semigroup operator $U(t)$ of (3.6) satisfy*

$$L^\delta(t)u_0^k \xrightarrow{w} L^\delta(t)u_0 \text{ in } L^2([0,T];H^1)$$
$$N(t)u_0^k \xrightarrow{w} N(t)u_0 \text{ in } L^2([0,T];H^{1+a}).$$

Moreover, for $t \in [0,T]$,

$$L^\delta(t)u_0^k \xrightarrow{w} L^\delta(t)u_0 \text{ in } H^1$$
$$N(t)u_0^k \xrightarrow{w} N(t)u_0 \text{ in } H^{1+a}.$$

Proof. We just verify the assertions concerning the nonlinear part as the ones for the linear part will follow from the Fourier representation of the linear flow at once. To make use of the smoothing result for the forced problem (3.6), we shall transform the equation (3.6) with data u_0 by setting

$$g = \left(\widehat{f}/\langle\xi\rangle^2\right)^\vee = (1-\Delta)^{-1}f \in H^2,$$

and $v = u + g$. As a result we obtain the equation

$$iv_t + \Delta v + i\delta v = c_1|v-g|^2(v-g) + c_2(v-g)\phi_x + (1+i\delta g), \tag{3.39}$$

with $\phi_x = K(|v-g|^2)$ and data $v(\cdot,0) = u_0(\cdot) + g(\cdot)$.

Let L^δ denote the semigroup for the corresponding homogeneous linear equation

$$iw_t + \Delta w + i\delta w = 0. \tag{3.40}$$

Then, the nonlinear part $n = v - w$ satisfies the equation

$$\begin{aligned} in_t + \Delta n + i\delta n = c_1|n+w-g|^2(n+w-g) \\ + c_2(n+w-g)K(|n+w-g|^2) + (1+i\delta g) \end{aligned} \tag{3.41}$$

with data $n(\cdot,0) = 0$. Proceeding as in Section 3.5, notably using the trilinear estimate (3.5) with $s = 1$, $a \in (0, \frac{1}{2})$, for $T \leq 1$, we get

$$\|n\|_{C_t^0 H^{1+a}([0,T]\times\mathbb{R}^2)} \lesssim \|u_0\|_{H^1}^3 + \|g\|_{H^{a-1}} \leq \|u_0\|_{H^1}^3 + \|f\|_{L^2}. \tag{3.42}$$

Note that in the above estimate we use a variant of Lemma 2.1.3 that replaces the linear group for the non-dissipative equation by the dissipative group $L^\delta(t) = e^{it\Delta - t\delta}$. For a proof of this, see [39]. Using the H^1 global well-posedness (a priori bound (3.12)) in iterating the local result above, we obtain the global bound

$$\|n(t)\|_{H^{1+a}} \leq C(a, \delta, \|f\|_{L^2}\, \|u_0\|_{H^1}); \tag{3.43}$$

for the details, see the section 3 of [39], or section 6 of [17]. Lastly, the continuity in H^{1+a} follows as in Section 3.5, so for any $T > 0$ and initial data $u_0 \in H^1$, we have

$$n \in C_t^0 H^{1+a}([0,T]\times\mathbb{R}^2). \tag{3.44}$$

In order to show that $N(t)u_0^k \xrightarrow{w} N(t)u_0$ in spaces given by the statement of the lemma, it suffices to show that every subsequence of $N(t)u_0^k$ has a further subsequence which converges weakly to the same limit. Let w^k denote the solution to (3.40) with data u_0^k. In relation to this denote by n^k the nonlinear part. Since weak convergence $u_0^k \xrightarrow{w} u_0$ in H^1 implies that $\sup_k \left\|u_0^k\right\|_{H^1} \leq M$ for some $M > 0$, using this in (3.42) and (3.43), we infer that, for every $T > 0$, $\{n^k\}_k$ is bounded in

$$C([0,T]; H^{1+a}) \cap C^1([0,T]; H^{a-1}) \tag{3.45}$$

with a uniform bound of (3.43). Firstly, in conjunction with this boundedness we infer, by Arzelà–Ascoli theorem, that $\{n^k\}_k$ is relatively compact in $C([0,T]; H^{a-1}_{loc})$ thanks to the uniform boundedness of the derivatives which implies equicontinuity.

Therefore, interpolating between this and (3.45) gives us a subsequence of $\{n^k\}_k$ that converges strongly in $C([0,T]; H^{1+a}_{loc})$. Secondly, by the Banach–Alaoglu theorem, boundedness of $\{n^k\}_k$ in (3.45) yields a weak* convergent subsequence in $L^\infty([0,T]; H^{1+a})$. Therefore combining these two, we reach a further subsequence, denoted also by $\{n^k\}_k$, convergent in the above spaces with the corresponding type of convergences. We shall write $n^k \xrightarrow{w^*} n$ in $L^\infty([0,T]; H^{1+a})$, and $n^k \to n$ strongly in $C([0,T]; H^{1+a}_{loc})$ (weak* limits are unique). The analogous arguments for the linear parts w^k hold in H^1 as well, so denote the corresponding limit by w. Later we will see that the limit n is indeed the weak limit in the spaces dictated by the lemma. Using local strong convergence above, next, we will show that n is a distributional solution. Note that n^k satisfies the equation (3.41) and let $F(p,q) = c_1|p+q-g|^2(p+q-g) + c_2(p+q-g)K(|p+q-g|^2)$. Thus for any $\varphi \in C_c^\infty([0,T]; \mathbb{R}^2)$,

$$
\begin{aligned}
&\iint \Big(in_t + \Delta n + i\delta n - F(n,w) - (1+i\delta)g\Big)\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \iint \Big(\big[-i\varphi_t + \Delta\varphi + i\delta\varphi\big]n - \big[(1+i\delta g) + F(n,w)\big]\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty} \iint \Big(\big[-i\varphi_t + \Delta\varphi + i\delta\varphi\big]n^k - \big[(1+i\delta g) + F(n^k,w^k)\big]\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{k\to\infty} \iint \Big(in^k_t + \Delta n^k + i\delta n^k - F(n^k,w^k) - (1+i\delta)g\Big)\varphi \,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}
$$

which proves that n is a distributional solution. It is just left to verify the second equality above. It suffices to show that the following identity

$$
\begin{aligned}
&\Big|\iint \Big(\big[-i\varphi_t + \Delta\varphi + i\delta\varphi\big](n^k - n) - \big[F(n^k,w^k) - F(n,w)\big]\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \Big|\iint \big[-i\varphi_t + \Delta\varphi + i\delta\varphi\big](n^k - n)\mathrm{d}x\,\mathrm{d}t\Big| + \|\varphi\|_{L^\infty_{x,t}} \Big|\iint_{\mathrm{supp}\varphi} \big[F(n^k,w^k) - F(n,w)\big]\,\mathrm{d}x\,\mathrm{d}t\Big|
\end{aligned}
$$

eventually decreases to zero for increasing k.

The first integral is clearly decaying due to strong local convergence of n^k, whereas the integral part of the second summand is majorized by

$$\begin{aligned}
&\Big|\iint\limits_{\mathrm{supp}\varphi} \big(n^k - n + w^k - w\big)\big[c_1 I + c_2 K\big](|n^k + w^k - g|^2)\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&+\Big|\iint\limits_{\mathrm{supp}\varphi} (\overline{n^k - n + w^k - w})(n^k + w^k - g)\big[c_1 I + c_2 K\big](n + w - g)\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&+\Big|\iint\limits_{\mathrm{supp}\varphi} \big(n^k - n + w^k - w\big)\big(\overline{n + w - g}\big)\big[c_1 I + c_2 K\big](n + w - g)\,\mathrm{d}x\,\mathrm{d}t\Big|
\end{aligned}$$

where we have used the property (iii) of the operator K given in Section 3.2 in the computation above. Owing to the strong local convergences of n^k and w^k along with the boundedness of K, we conclude that these sums vanish in the limit. Moreover, by the uniqueness of the weak* limits, n is a unique distribution belonging to $C([0,T];H^{1+a})$ yielding that $n = N(t)u_0$ (since n has shown to be the distributional solution of (3.41)). Therefore, using the fact that $L^2([0,T];H^{1+a})$ embeds in the dual of $L^\infty([0,T];H^{1+a})$, weak* convergence in $L^\infty([0,T];H^{1+a})$ implies the weak convergence in $L^2([0,T];H^{1+a})$. This finishes the proof of the first assertion in the lemma. To prove the second argument, fix a $\tilde{t} \in [0,T]$. As before smoothing estimate together with the boundedness of the initial data u_0^k imply that $\{N(\tilde{t})u_0^k\}_k$ is bounded in H^{1+a}. Thus, there exists a weakly convergent subsequence, still denoted by $\{N(\tilde{t})u_0^k\}$, that converges in H^{1+a}, say to $\widetilde{n}$. But as we know, from the previous discussion above, that $N(t)u_0^k \xrightarrow{w^*} N(t)u_0$ in $C([0,T];H^{1+a})$. So the uniqueness entails that $\widetilde{n} = N(\tilde{t})u_0$. □

Proof of Theorem 3.1.3. To begin with, we note that the existence of an absorbing ball $\mathcal{B}$ of the evolution follows from (3.12); indeed the detailed proof was given in [36]. Hence to attain a global attractor, it is just left to affirm that the propagator $U(t)$ is asymptotically compact.

Therefore, it is sufficient to show that for any sequence of initial data $\{u_{0,k}\}_k$ in absorbing ball and any sequence of times $t_k \to \infty$, the sequence $\{U(t_k)u_{0,k}\}_k$ has a convergent subsequence in H^1. Note that for $u_0 \in \mathcal{B}$, (3.43) implies that the nonlinear part $N(t)u_0$ of

$$U(t)u_0 = L^\delta(t)u_0 + N(t)u_0$$

is contained in a ball B_R in H^{1+a} with radius $R = R(a, \delta, \|f\|_{L^2})$, $a \in (0, \frac{1}{2})$. As a result, $\{N(t_k)u_{0,k}\}_k \subset B_R$. Therefore, we can find a subsequence, still denoted by $N(t_k)u_{0,k}$, that converges weakly in H^{1+a}. Moreover, since the weak and weak* topologies agree on a reflexive spaces, the Banach–Alaoglu theorem yields that, up to a subsequence, $U(t_k)u_{0,k}$ converges weakly in H^1. As $L^\delta(t_k)u_{0,k} \to 0$ strongly in H^1 as $t_k \to \infty$, $N(t_k)u_{0,k}$ and $U(t_k)u_{0,k}$ converge to the same limit, say to u. Furthermore, for every $T > 0$, we can find a further subsequence so that $N(t_k - T)u_{0,k}$ and $U(t_k - T)u_{0,k}$ converge weakly in H^{1+a} and H^1, respectively. As above, by the decay of the linear part, the limits are the same, so denote it by u_T. By Lemma 3.6.1,

$$U(t_k - T)u_{0,k} \xrightarrow{w} u_T \text{ in } H^1 \implies U(t_k)u_{0,k} = U(T)\big(U(t_k - T)u_{0,k}\big) \xrightarrow{w} U(T)u_T \text{ in } H^1.$$

Therefore, by the uniqueness of a weak limit, $U(T)u_T = u$. In a subsequent discussion, sometimes we need to take $T \to \infty$ in order to obtain strong convergences. So to make sense of this, we may implement a diagonalization argument for a countable set $\{T \in \mathbb{N}\}$ so that, up to a same subsequence for all T, $U(t_k - \cdot)u_{0,k}$ and $N(t_k - \cdot)u_{0,k}$ converge weakly at each T in the corresponding spaces above. Next we want to upgrade the weak H^1 convergence of the solution flow $U(t_k)u_{0,k}$ to a strong H^1 convergence. Firstly using the equation (3.11), we can obtain that $\frac{d}{dt}\|u\|_{L^2} + 2\,\delta\,\|u\|_{L^2} = 2\,\mathrm{Im}\langle f, u\rangle_{L^2}$, and then application of the Gronwall lemma for the evolution $U(t)$ gives that

$$\begin{aligned}
\|U(t_k)u_{0,k}\|_{L^2}^2 &= e^{-2\delta T}\,\|U(t_k - T)u_{0,k}\|_{L^2}^2 \\
&\qquad + 2\,\mathrm{Im}\int_0^T e^{-2\delta(T-\tau)}\langle f, U(t_k - T + \tau)u_{0,k}\rangle_{L^2_x}\,\mathrm{d}\,\tau \\
\|U(T)u_T\|_{L^2}^2 &= e^{-2\delta T}\,\|u_T\|_{L^2}^2 + 2\,\mathrm{Im}\int_0^T e^{-2\delta(T-\tau)}\langle f, U(\tau)u_T\rangle_{L^2_x}\,\mathrm{d}\,\tau,
\end{aligned}$$

which yields

$$\|U(t_k)u_{0,k}\|_{L^2}^2 - \|U(T)u_T\|_{L^2}^2 = e^{-2\delta T}\big(\,\|U(t_k - T)u_{0,k}\|_{L^2}^2 - \|u_T\|_{L^2}^2\,\big)$$
$$+ 2\,\mathrm{Im}\int_0^T e^{-2\delta(T-\tau)}\langle f, U(t_k - T + \tau)u_{0,k} - U(\tau)u_T\rangle_{L^2_x}\,\mathrm{d}\,\tau.$$

The first summand becomes negligible by taking sufficiently large T since letting $k \to \infty$ and using the fact that $u_{0,k} \in \mathcal{B}$, we ensure that the norms in the parentheses are finite (weak convergence in H^1 implies that $\|u_T\|_{L^2} \leq \liminf_k \|U(t_k - T)u_{0,k}\|_{L^2}$). The second summand vanishes in the limit by Lemma 3.6.1 because, with $U(T)u_T = u$, we have

$$U(t_k)u_{0,k} \xrightarrow{w} u \text{ in } H^1 \implies$$
$$U(t_k - T + \tau)u_{0,k} = U(\tau - T)\big(U(t_k)u_{0,k}\big) \xrightarrow{w} U(\tau)u_T \text{ in } L^2([0,T]; H^1).$$

As a consequence we get that

$$\limsup_k \big(\,\|U(t_k)u_{0,k}\|_{L^2}^2 - \|U(T)u_T\|_{L^2}^2\,\big) = \limsup_k \big(\,\|U(t_k)u_{0,k}\|_{L^2}^2 - \|u\|_{L^2}^2\,\big) \leq 0,$$

which, along with $U(t_k)u_{0,k} \xrightarrow{w} u$ in H^1, implies that $U(t_k)u_{0,k} \to u$ strongly in L^2. This strong L^2 convergence will be important in the upcoming energy calculations. So define the functional E by

$$E(u_0)(t) = \|\nabla U(t)u_0\|_{L^2}^2 + \frac{c_1}{2}\,\|U(t)u_0\|_{L^4}^4 + \frac{c_2}{2}\int K(|U(t)u_0|^2)|U(t)u_0|^2\,\mathrm{d}x$$
$$+ 2\,\mathrm{Re}\int f\overline{U(t)u_0}\,\mathrm{d}x,$$

and the time derivative is as follows

$$\frac{d}{dt}E(u_0)(t) = -2\delta E(u_0)(t) + F(u_0)(t)$$

where

$$F(u_0)(t) = -\delta c_1\,\|U(t)u_0\|_{L^4}^4 - \delta c_2\int K(|U(t)u_0|^2)|U(t)u_0|^2\,\mathrm{d}x + 2\delta\,\mathrm{Re}\int f\overline{U(t)u_0}\,\mathrm{d}x.$$

Gronwall lemma implies that

$$E(u_{0,k})(t_k) - E(u_T)(T) = \sum_{j=1}^{4} I_j,$$

where

$$\begin{aligned}
I_1 &= e^{-2\delta T}\big(E(u_{0,k})(t_k - T) - E(u_T)(0)\big)\\
I_2 &= -\delta c_1 \int_0^T e^{-2\delta(T-\tau)}\big(\,\|U(t_k - T + \tau)u_{0,k}\|_{L^4}^4 - \|U(\tau)u_T\|_{L^4}^4\,\big)\mathrm{d}\tau\\
I_3 &= -\delta c_2 \int_0^T \int e^{-2\delta(T-\tau)}\big(K(|U(t_k - T + \tau)u_{0,k}|^2)|U(t_k - T + \tau)u_{0,k}|^2\\
&\qquad\qquad\qquad\qquad\qquad - K(|U(\tau)u_T|^2)|U(\tau)u_T|^2\big)\mathrm{d}x\,\mathrm{d}\tau\\
I_4 &= 2\delta\,\mathrm{Re}\int_0^T e^{-2\delta(T-\tau)}\langle f, U(t_k - T + \tau)u_{0,k} - U(\tau)u_{0,k}\rangle_{L^2_x}\mathrm{d}\tau.
\end{aligned}$$

I_1 gets arbitrarily small by increasing T, and I_2 can be majorized by

$$\begin{aligned}
\delta c_1 \int_0^T & e^{-2\delta(T-\tau)}\,\|U(t_k - T + \tau)u_{0,k} - U(\tau)u_T\|_{L^4}\\
&\times\big(\,\|U(t_k - T + \tau)u_{0,k}\|_{L^4} + \|U(\tau)u_T\|_{L^4}\,\big)^3\mathrm{d}\tau.
\end{aligned}$$

Note that by L^4 Gagliardo–Nirenberg inequality

$$\begin{aligned}
&\|U(t_k - T + \tau)u_{0,k} - U(\tau)u_T\|_{L^4}\\
&\quad\lesssim \big\|U(\tau - T)\big(U(t_k)u_{0,k} - U(T)u_T\big)\big\|_{L^2}^{1/2}\,\|U(t_k - T + \tau)u_{0,k} - U(\tau)u_T\|_{H^1}^{1/2}.
\end{aligned}$$

Then, by the strong continuity of $U(\tau - T)$ and the strong L^2 convergence $U(t_k)u_{0,k} \to U(T)u_T$, the majorant of I_2 above vanishes in the limit as weak H^1 convergence yields the H^1 boundedness of the norms. Write the third term as

$$\begin{aligned}
I_3 = -\delta c_2 \int_0^T \int & e^{-2\delta(T-\tau)}\big[K(|U(t_k - T + \tau)u_{0,k}|^2) - K(|U(\tau)u_T|^2)\big]\\
&\times|U(t_k - T + \tau)u_{0,k}|^2\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}$$

$$- \delta c_2 \int_0^T \int e^{-2\delta(T-\tau)} K(|U(\tau)u_T|^2)\big(|U(t_k - T + \tau)u_{0,k}|^2 - |U(\tau)u_T|^2\big) \mathrm{d}x \, \mathrm{d}\tau;$$

first use the linearity and the L^p boundedness of the operator K, and then proceed by using the same reasoning as above to conclude that I_3 decays to zero. Finally, by using the weak continuity (Lemma 3.6.1), I_4 vanishes in the limit. Therefore, as $U(T)u_T = u$, we get that

$$\limsup_{k\to\infty} \big(E(u_{0,k})(t_k) - E(u)(0)\big) = \limsup_{k\to\infty} \big(E(u_{0,k})(t_k) - E(u_T)(T)\big) \leq 0.$$

This inequality together with the definition of E and taking limits as above implies that

$$\limsup_{k\to\infty} \|\nabla U(t_k)u_{0,k}\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 .$$

Therefore, this with $U(t_k)u_{0,k} \xrightarrow{w} u$ in H^1 leads to the L^2 strong convergence of $\nabla U(t_k)u_{0,k}$ to ∇u. Consequently, $U(t_k)u_{0,k} \to u$ strongly in H^1. This finishes the proof of asymptotic compactness, so the proof is complete. $\square$

4. BIHARMONIC SCHRÖDINGER EQUATION ON $\mathbb{R}^+$

4.1. Introduction

This chapter is devoted to study the initial boundary value problem (IBVP) for the cubic biharmonic nonlinear Schrödinger equation (biharmonic NLS) on the half line

$$\begin{cases} iu_t + \partial_x^4 u + \mu|u|^2 u = 0, \quad x \in \mathbb{R}^+,\, t \in \mathbb{R}^+, \\ u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), \\ u(x,0) = g(x). \end{cases} \tag{4.1}$$

The results of this work have appeared in [40]. Here, $\mu = \pm 1$ and the data (g, h_1, h_2) are taken in the space $H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ with the compatibility conditions $g(0) = h_1(0)$ when $\frac{1}{2} < s \le \frac{3}{2}$, and $g(0) = h_1(0)$, $g'(0) = h_2(0)$ when $\frac{3}{2} < s \le \frac{9}{2}$. These compatibility conditions are necessary since the solutions we are concerned with have continuous L_x^2 traces for $s > \frac{1}{2}$. For the notion of traces of functions in $H^s(\mathbb{R})$, we assume, for our throughout discussion, that $s \neq n + \frac{1}{2}$ for $n = 0, 1, 2, \cdots$. Note that choosing data triples $(g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ is due to the local smoothing inequalities of [41], [42]: $\left\|\partial_x^k e^{it\partial_x^4} g\right\|_{L_x^\infty H_{t\in(0,T)}^{\frac{2s+3-2k}{8}}} \lesssim \|g\|_{H^s}$, for $k = 0, 1$ and these inequalities are sharp in the sense that the numbers $\frac{2s+3}{8}$ and $\frac{2s+1}{8}$ cannot be replaced by any bigger number and hence taking such data makes sense. We also verify the appropriateness of the selected spaces in our computations. Fourth order NLS with power-type nonlinearity

$$iu_t + \Delta u + \lambda \Delta^2 u + |u|^p u = 0, \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}$$

was introduced by Karpman and Shagalov [43, 44] to consider the effect of the small fourth order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Indeed, when $\lambda < 0$, they studied the stability/instability of solutions depending on certain restrictions on the parameters λ, p.

When Laplacian is removed, the equation

$$iu_t + \lambda \Delta^2 u + \mu |u|^p u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \tag{4.2}$$

is called biharmonic NLS, in addition it is said to be defocusing if $\lambda\mu > 0$, and focusing if $\lambda\mu < 0$. From a physical point of view, as a model equation, biharmonic NLS arises in many context such as deep water wave dynamics [45], vortex filaments [46], solitary waves [43, 44]. Furthermore it was used as a model equation in [47], [48] to study the stability of solitons in magnetic materials once the effective quasi particle mass becomes infinite. Fourth order NLS with various nonlinearities have been extensively studied on the well-posedness in the periodic and non-periodic settings. As half line problems are relavent to the initial value problems posed in the non-periodic setting, here it is better to review some of those posed on $\mathbb{R}^d$. So we write

$$\begin{cases} iu_t + \kappa \Delta u + \lambda \Delta^2 u + F(u) = 0, \\ u(x,0) = g(x). \end{cases} \tag{4.3}$$

The initial value problem (IVP) (4.3) on $\mathbb{R}^n \times (0,\infty)$ with $\kappa = 0$, $\lambda = 1$ and nonlinearities $F(u) = \partial_x(|u|^{p-1}u)$, $2 \leq p \in \mathbb{N}$ have been studied in [49] in terms of well-posedness and scattering of the solution. In particular, it turns out that when $n = 1$ and $p = 3$, the authors obtained the local well-posedness of (4.3) in the Sobolev spaces $H^s(\mathbb{R})$ for $s \geq 0$. Furthermore this result is almost sharp in the sense that the flow map from $H^s(\mathbb{R})$ to $C(\mathbb{R}, H^s(\mathbb{R}))$ is not C^3. The local and global well-posedness for the IVP (4.3) on $\mathbb{R} \times \mathbb{R}$ with $\kappa = 0$, $\lambda = -1$ and $F(u) = \pm |u|^2 u$, were established in [50] for data $g \in H^s(\mathbb{R})$ with $s \geq -\frac{1}{2}$, also the equation was shown to be ill-posed below this range ($s < -\frac{1}{2}$), by proving that the flow map is not uniformly continuous. In [51], the IVP (4.3) on $\mathbb{R} \times \mathbb{R}$ with $\kappa = 1$, $\lambda \neq 0$ and the nonlinearity

$$F(u) = -\frac{1}{2}|u|^2 u + c_1 |u|^4 u + c_2 (\partial_x u)^2 \bar{u} + c_3 |\partial_x u|^2 + c_4 u^2 \partial_x^2 \bar{u} + c_5 |u|^2 \partial_x^2 u \tag{4.4}$$

(with certain restrictions on the constants) was proved to be locally well-posed in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$ by the restricted norm method. For higher dimensions, Pausader [52] showed that the equation (4.3) with $\kappa = 0$, $\lambda = 1$ and $F(u) = |u|^2 u$ is globally well-posed for $n \leq 8$, and ill-posed for $n \geq 9$.

For the other well-posedness results related to the equation (4.3) see for instance [53–59]. Initial boundary value problems for the fourth order NLS have been recently started to be addressed. In the case of the half line, Hu etal [60] obtained a solution of some form of the equation (4.3) in the IBVP setting (with a similar nonlinearity as in (4.4)) after reformulating the problem as a Riemann-Hilbert problem. Özsarı-Yolcu [42] studied the IBVP of the equation (4.2) with $\lambda = 1$, $\mu \in \mathbb{C}$ and the inhomogeneous Dirichlet-Neumann boundary data on the half line where they make use of the unified transform method in obtaining the solution. By making some assumptions on the relation of s and p, the authors obtained the local well-posedness in $H^s(\mathbb{R}^+)$ for $s \in (\frac{1}{2}, \frac{9}{2})$, $s \neq \frac{3}{2}$, and $s \in [0, \frac{1}{2})$ separately. Moreover, for the defocusing problem they established the global well-posedness in the energy space $H^2(\mathbb{R}^+)$. It is remarkable to note that [42] is the first treatment of the fourth order Schrödinger equations on a half line subject to the inhomogeneous boundary conditions. Lastly, more recently Filho-Cavalcante-Gallego [61] addressed the IBVP of the cubic biharmonic NLS (4.2) when $\lambda = -1$ with the same set of initial-boundary data as in [42]. The authors proved the local well-posedness in $H^s(\mathbb{R}^+)$ for $0 \leq s < \frac{1}{2}$ by the Fourier restriction norm method and using the Duhamel boundary forcing operator for the corresponding linear equation.

In this chapter, we continue the program initiated in [62] that establishes the regularity properties of cubic NLS on a half line using the tools available in the case of the full line. Biharmonic cubic NLS is higher order dispersive PDE version of cubic NLS, so as expected, we obtain well-posedness in a less regular space by adapting the estimates of [62]. We will use Laplace transform method proposed by Bona-Sun-Zhang [63] to divide the problem into a linear IBVP on the half line and nonlinear IVP on the full line after extending the data into $\mathbb{R}$. By this method we can write the explicit solution for a linear IBVP and then using it, we set up an equivalent integral equation on $\mathbb{R} \times \mathbb{R}$ for the full solution. We then examine the integral equation with the $X^{s,b}$ method, see [1, 2]. To state our theorems we begin with a definition.

Definition 4.1.1. *We say that the biharmonic NLS equation* (4.1) *is locally well-posed in* $H^s(\mathbb{R}^+)$ *if for any data* $(g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ *with the additional compatibility conditions discussed above, the integral equation* (4.8) *has a unique solution in*

$$X^{s,b}(\mathbb{R} \times [0,T]) \cap C_t^0 H_x^s([0,T] \times \mathbb{R}) \cap C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R} \times [0,T])$$

for some $b < \frac{1}{2}$ *and sufficiently small* $T = T(\|g\|_{H^s(\mathbb{R}^+)}, \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}, \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)})$. *Furthermore, if* u_1 *and* u_2 *are two such solutions coming from different extensions* g_{e1} *and* g_{e2}*, then their restriction to* $\mathbb{R}^+ \times [0,T]$ *are the same. In addition, if* $g_n \to g$ *in* $H^s(\mathbb{R}^+)$*,* $h_{n1} \to h_1$ *in* $H^{\frac{2s+3}{8}}(\mathbb{R}^+)$ *and* $h_{n2} \to h_2$ *in* $H^{\frac{2s+1}{8}}(\mathbb{R}^+)$*, then* $u_n \to u$ *in the space above.*

We state our local result below and note that it improves the result for the cubic biharmonic NLS in [42] which establishes the well-posedness for $s \geq 0$. As already mentioned [42] utilizes the uniform transform method of Fokas to obtain the local well-posedness for the biharmonic NLS with power nonlinearities. The method is based on inverse-scattering techniques and used to obtain representation formula for the solution of the linear biharmonic Schrödinger equation. In order to establish the local theory we will need to obtain some essential estimates regarding the linear and nonlinear terms of the integral equation representation for the solution in Section 4.4 below.

Theorem 4.1.2. *For any* $s \in (-\frac{1}{3}, \frac{9}{2})$*,* $s \neq \frac{1}{2}, \frac{3}{2}$*, the equation* (4.1) *is locally well-posed in* $H^s(\mathbb{R}^+)$ *with the local existence time* T *satisfying* $T \approx \left(C + \|g\|_{H^s(\mathbb{R}^+)}\right)^{-\frac{8}{2s+3}}$ *where the constant* C *depends on* $\|g\|_{L^2} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$.

Next theorem is concerned with the smoothing result of the equation (4.1) that is, it demonstrates that the nonlinear part of the solution is smoother than the initial data. It reads that smoothing vanishes at the upper end point $s = \frac{9}{2}$, nevertheless, the gain of a derivative at the lower end point $s = -\frac{1}{3}$ is still $\frac{1}{3}$.

The proof of the smoothing theorem below will be based on the restricted norm method of Bourgain [1, 2] and in the sequel, we will denote the operator W_0^t as the linear part of the solution of the equation (4.1).

Theorem 4.1.3. *Fix* $s \in (-\frac{1}{3}, \frac{9}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$, $(g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ *with the compatibility conditions given for the equation* (4.1). *Then for* $a < \min(1, 2s+1, \frac{9}{2} - s)$ *and* t *in the local existence interval* $[0, T]$, *we have*

$$u(x,t) - W_0^t(g, h_1, h_2) \in C_t^0 H_x^{s+a}([0,T] \times \mathbb{R}^+).$$

The smoothing estimates of this sort were obtained for NLS in certain papers in the periodic, see [38, 64, 65] and non-periodic cases, see [66, 67]. The first smoothing result related to the initial boundary value problem is established for cubic NLS, [62]. Also using the same approach as in [62], the papers [68], [69] establish the regularity properties of the Boussinesq equation and the Zakharov system on the half line respectively. In order to prove the above theorems we take advantage of the Duhamel formulation by which we run a fixed point argument. With this formulation we express the solution as a superposition of the linear evolutions which incorporate the boundary term and the initial data with the nonlinearity. Also to estimate the terms coming from Duhamel formula, we first solve the corresponding linear problem by taking Laplace transform of the equation in the temporal variable and inverting back by the Mellin transform so that we obtain an explicit formula for the linear evolution after extending the initial data to the whole line. Afterwards the nonlinear part of the formula will be treated by the $X^{s,b}$ method. Note that in the boundary value problems $b < \frac{1}{2}$ is necessary in order to carry out the contraction argument, while $b > \frac{1}{2}$ is required on the full line. As for the uniqueness, the solution we constructed is the unique fixed point of the Duhamel operator (4.18) by the contraction argument, yet it is not clear if the restriction of the fixed point of (4.18) to the half line is independent of the different extensions of the initial data.

In this regard, the proof of uniqueness in our case proceeds in two steps: one is for the case $s > \frac{1}{2}$ where we exploit the Sobolev embedding and well-known Gronwall's inequality on $\mathbb{R}^+$, and the other is for the low regularity case $-\frac{1}{3} < s < \frac{1}{2}$ where we make use of the uniqueness obtained for $s > \frac{1}{2}$ and the smoothing estimate of Theorem 4.1.3 to establish the uniqueness in this range, also in contrast to the case $s > \frac{1}{2}$, it is not immediate to exhibit that different extensions produce the same solution. In particular, in order to establish uniqueness down to the local theory threshold $H^{-\frac{1}{3}}(\mathbb{R}^+)$, we require smoothing estimate of Theorem 4.1.3.

When $\mu = 1$ in (4.1) (the defocusing case), the following theorem provides bounds for higher order Sobolev norms. This is based on smoothing result obtained in Theorem 4.1.3 and a priori estimate at the energy level, Lemma A.0.1.

Theorem 4.1.4. *Let* $\mu = 1$ *in the equation* (4.1). *In the case* $s \in [2, \frac{5}{2})$, $g \in H^s(\mathbb{R}^+)$, $h_1 \in H^{\frac{2s+3}{8}}(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ *and* $h_2 \in H^{\frac{2s+1}{8}}(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$, *the associated local solution is global and the smoothing result holds globally. Furthermore, for* $2 < s < \frac{5}{2}$ *the solution has the growth bound*

$$\|u(t)\|_{H^s(\mathbb{R}^+)} \lesssim \langle T \rangle.$$

Here we note that the equation (4.1) does not satisfy the mass and energy conservations once the boundary data h_1 and h_2 are nonzero. Hence the global well-posedness at the energy level, H^2, for the equation (4.1) is a nontrivial problem in the presence of inhomogeneous boundary conditions, see Theorem 1.3 of [42]. The Lemma A.0.1, which is the key to obtaining the growth bound in Theorem 4.1.4, results from the proof of Theorem 1.3 of [42].

As far as we know this work is the first treatment of the fourth order biharmonic Schrödinger equation subject to the inhomogeneous boundary conditions where well-posed solutions are constructed below the L^2 space.

Now we outline the organization of the chapter. In Section 4.2, we define the notion of a solution. To be more precise we reformulate (4.1) as an integral equation (Duhamel's formula) and set this to be a solution map which we then show is a contraction in a suitable metric space. Thus by using the Duhamel's formula, the solution we constructed is a superposition of a linear and a nonlinear evolutions. We also introduce the space $H^s(\mathbb{R}^+)$ and discuss whenever one can extend the initial and boundary data. In Section 4.3 we illustrate, by an application of the Laplace transform on the half line, how to find the explicit solution formula for the linear problem with zero initial data. In Section 4.4, we state and prove linear and nonlinear a priori estimates. Linear estimates relate to two separate processes one is for a solution to a free fourth order Schrödinger equation and the other is for a solution to IBVP subject to the inhomogeneous boundary data. The estimates for the latter also clarify the regularity level of the boundary data h_1, h_2 and the selection of the spaces they are taken. In the remaining part of the Section 4.4, we prove the multilinear estimates associated to the nonlinear term coming from the integral part of the solution representation. In Section 4.5, we prove Theorem 4.1.2 by establishing the local well-posedness theory via the contraction argument and argue the dependence of the local existence time to the initial and boundary data. Theorems 4.1.3 and 4.1.4 are proved in Section 4.6 and the uniqueness is proved in Section 4.5.1.

4.1.1. Notation

We define the space time Fourier transform as

$$\widehat{f}(\xi,\tau) = \mathcal{F}f(\xi,\tau) = \int_{\mathbb{R}^2} e^{-ix\xi - it\tau} f(x,t)\,dxdt.$$

For $s > -\frac{1}{2}$, Sobolev spaces $H^s(\mathbb{R}^+)$ on the half line are defined as

$$H^s(\mathbb{R}^+) = \big\{ g \in \mathcal{D}(\mathbb{R}^+) \,:\, \exists\, \widetilde{g} \in H^s(\mathbb{R}) \text{ such that } \widetilde{g}\chi_{(0,\infty)} = g \big\}$$

with the norm

$$\|g\|_{H^s(\mathbb{R}^+)} = \inf \big\{ \, \|\widetilde{g}\|_{H^s(\mathbb{R})} \,:\, \widetilde{g}\chi_{(0,\infty)} = g \big\}.$$

The restriction $s > -\frac{1}{2}$ is necessary because multiplication with the characteristic function $\chi_{(0,\infty)}$ is not well-defined for H^s distributions when $s \leq -\frac{1}{2}$. Moreover we write $W^t g$ for the linear biharmonic Scrödinger propagator

$$W^t g(x,t) = e^{it\Delta^2} g = \int e^{ix\xi + it\xi^4} \widehat{g}(\xi,t) d\xi.$$

For a space time function f, the notation D_0 means the evaluation at the boundary $x = 0$, that is

$$D_0\big(f(x,t)\big) = f(0,t).$$

Throughout we write η for a smooth compactly supported function that is equal to 1 on $[-1,1]$ and $\operatorname{supp} \eta \subset [-2,2]$. Also let $\rho \in C^\infty$ be a cut-off function satisfying $\rho = 1$ on $[0,\infty)$ and $\operatorname{supp} \rho \subset [-1,\infty)$.

4.2. Notion of a Solution

In order to find solutions of (4.1) we start with constructing the solution of the linear IBVP

$$\begin{cases} iu_t + u_{xxxx} = 0 \\ u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), \\ u(x,0) = g(x), \end{cases} \tag{4.5}$$

with the compatibility conditions $g(0) = h_1(0)$ for $\frac{1}{2} < s \leq \frac{3}{2}$ and $g(0) = h_1(0)$, $g'(0) = h_2(0)$ for $\frac{3}{2} < s \leq \frac{9}{2}$. We shall denote the solution of (4.5) by $W_0^t(g, h_1, h_2)$. This solution can be written as

$$W_0^t(g,h_1,h_2) = W_0^t(0, h_1 - p_1, h_2 - p_2) + W^t g_e$$

where g_e is an extension of g to the full line $\mathbb{R}$ such that $\|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}$ and the traces $p_1(t) = \eta(t) D_0(W^t g_e)$, $p_2(t) = \eta(t) D_0\big(\partial_x\big[W^t g_e\big]\big)$ are well well-defined and belong to the spaces $H^{\frac{2s+3}{8}}(\mathbb{R}^+)$, $H^{\frac{2s+1}{8}}(\mathbb{R}^+)$ respectively, by Lemma 4.4.1 below.

As a result we decomposed the solution operator as a sum of free biharmonic Scrödinger evolution and the boundary operator corresponding to the zero initial data. Therefore we consider

$$\begin{cases} iu_t + u_{xxxx} = 0, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), \\ u(x,0) = 0 \end{cases} \tag{4.6}$$

where $W_0^t(0, h_1, h_2)$ denotes the solution to this problem. By an application of the Laplace transform described in the next section, we obtain explicit representation for $W_0^t(0, h_1, h_2)$.

Lemma 4.2.1. *Assume that h_1 and h_2 are Schwartz functions. The solution of* (4.6) *can explicitly be written in the form*

$$\begin{aligned} u(x,t) = & \frac{-1+i}{\pi}\Big[W_1h_2 - iW_2h_1 - W_3h_1 - W_4h_2\Big] \\ & - \frac{\sqrt{2}i}{\pi}\Big[W_5h_2 + \Big(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)W_6h_1 + \Big(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)W_7h_1 - W_8h_2\Big] \end{aligned}$$

where

$$\begin{aligned} W_1h_2(x,t) &= \int_0^\infty e^{i\beta^4 t - \beta x}\beta^2\,\widehat{h}_2(\beta^4)\rho(\beta x)d\beta, \\ W_2h_1(x,t) &= \int_0^\infty e^{i\beta^4 t - \beta x}\beta^3\,\widehat{h}_1(\beta^4)\rho(\beta x)d\beta, \\ W_3h_1(x,t) &= \int_0^\infty e^{i\beta^4 t + i\beta x}\beta^3\,\widehat{h}_1(\beta^4)d\beta, \\ W_4h_2(x,t) &= \int_0^\infty e^{i\beta^4 t + i\beta x}\beta^2\,\widehat{h}_2(\beta^4)d\beta, \\ W_5h_2(x,t) &= \int_0^\infty e^{-i\beta^4 t}e^{[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}]\beta x}\beta^2\,\widehat{h}_2(-\beta^4)\rho(\beta x)d\beta, \\ W_6h_1(x,t) &= \int_0^\infty e^{-i\beta^4 t}e^{[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}]\beta x}\beta^3\,\widehat{h}_1(-\beta^4)\rho(\beta x)d\beta, \\ W_7h_1(x,t) &= \int_0^\infty e^{-i\beta^4 t}e^{[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}]\beta x}\beta^3\,\widehat{h}_1(-\beta^4)\rho(\beta x)d\beta, \end{aligned}$$

$$W_8 h_2(x,t) = \int_0^\infty e^{-i\beta^4 t} e^{[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}]\beta x} \beta^2 \widehat{h}_2(-\beta^4)\rho(\beta x) d\beta.$$

Here by an abuse of notation we take

$$\widehat{h}_j(\xi) = \mathcal{F}\big(\chi_{(0,\infty)} h_j\big)(\xi) = \int_0^\infty e^{-i\xi t} h_j(t) dt. \tag{4.7}$$

We use this explicit form to obtain bounds on $W_0^t(0, h_1, h_2)$ in Section 4.4 below. Next, by the Duhamel formulation, we consider the integral equation equivalent to (4.1) on $[0, T]$, $t \leq T < 1$:

$$u(t) = \eta(t) W^t g_e + \eta(t) \int_0^t W^{t-t'} F(u) dt' + \eta(t) W_0^t\big(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2\big)(t), \tag{4.8}$$

where

$$F(u) = \eta(t/T)|u|^2 u, \qquad p_1(t) = \eta(t) D_0(W^t g_e), \qquad p_2(t) = \eta(t) D_0\big(\partial_x [W^t g_e]\big),$$
$$q_1(t) = \eta(t) D_0\Big(\int_0^t W^{t-t'} F(u) dt'\Big), \qquad q_2(t) = \eta(t) D_0\Big(\partial_x\Big[\int_0^t W^{t-t'} F(u) dt'\Big]\Big).$$

In the following, we want to prove that the integral equation (4.8) has a unique solution in a suitable function space (given by definition 4.1.1) on $\mathbb{R} \times \mathbb{R}$ for sufficiently small T. Note that the restriction of u to $\mathbb{R}^+ \times [0, T]$ is a distributional solution of (4.1) whereas smooth solutions of the equation (4.8) are classical solutions of (4.1).

We implement contraction argument in $X^{s,b}(\mathbb{R} \times \mathbb{R})$ spaces:

$$\|u\|_{X^{s,b}} = \big\|\langle\xi\rangle^s \langle\tau - \xi^4\rangle^b \widehat{u}(\xi, \tau)\big\|_{L^2_\tau L^2_\xi}. \tag{4.9}$$

In order to carry out the contraction argument in the local theory we will need the following standard results from [8]

$$\text{for any } s \in \mathbb{R} \text{ and } b > \frac{1}{2}, \text{ we have } X^{s,b} \subset C^t_0 H^s_x. \tag{4.10}$$

For any s, $b \in \mathbb{R}$,

$$\big\|\eta(t) W^t g\big\|_{X^{s,b}} \lesssim \|g\|_{H^s}. \tag{4.11}$$

For $T<1$ and $-\frac{1}{2}<b_1<b_2<\frac{1}{2}$ we have

$$\|\eta(t/T)F\|_{X^{s,b_1}} \lesssim T^{b_2-b_1}\|F\|_{X^{s,b_2}}. \tag{4.12}$$

We also need the following estimate whose proof can be obtained by adapting the proof of Lemma 3.12 in [9]. For any $s\in\mathbb{R}$, $0\leq b_1<\frac{1}{2}$ and $b_2=1-b_1$, we have

$$\left\|\eta(t)\int_0^t W^{t-t'}F dt'\right\|_{X^{s,b_2}} \lesssim \|F\|_{X^{s,-b_1}}. \tag{4.13}$$

Next for the boundary data h_1 and h_2, we need estimates on the sizes of the norms $\left\|\chi_{(0,\infty)}h_1\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})}$ and $\left\|\chi_{(0,\infty)}h_2\right\|_{H^{\frac{2s+1}{8}}(\mathbb{R})}$ which is the content of the next lemma.

Lemma 4.2.2 (See [62])**.** *Assume $h\in H^s(\mathbb{R}^+)$ for some $s\in(-\frac{1}{2},\frac{5}{2})$.*

1. *If $-\frac{1}{2}<s<\frac{1}{2}$, then $\left\|\chi_{(0,\infty)}h\right\|_{H^s(\mathbb{R})}\lesssim\|h\|_{H^s(\mathbb{R}^+)}$.*
2. *If $\frac{1}{2}<s<\frac{3}{2}$ and $h(0)=0$, then $\left\|\chi_{(0,\infty)}h\right\|_{H^s(\mathbb{R})}\lesssim\|h\|_{H^s(\mathbb{R}^+)}$.*
3. *If $\frac{1}{2}<s<\frac{3}{2}$, then $\|h_{even}\|_{H^s(\mathbb{R})}\lesssim\|h\|_{H^s(\mathbb{R}^+)}$.*
4. *If $\frac{1}{2}<s<\frac{5}{2}$, $s\neq\frac{3}{2}$ and $h(0)=0$, then $\|h_{odd}\|_{H^s(\mathbb{R})}\lesssim\|h\|_{H^s(\mathbb{R}^+)}$.*

where $h_{even}(x)=h(|x|)$ *and* $h_{odd}(x)=\begin{cases} h(|x|) & \text{if } x\geq 0\\ -h(|x|) & \text{if } x\leq 0.\end{cases}$

As a final note following will be useful in establishing the Theorem 4.1.4.

Remark 4.2.3. By the definition of linear flow W^t and the Lemma 4.2.1 we may write

$$W_0^t(g,h_1,h_2)-W_0^t(\tilde{g},h_1,h_2)=W_0^t(g-\tilde{g},0,0).$$

Moreover, by writing $W_0^t(g,0,0)$ with the method of odd extension and then utilizing Lemma 4.4.3, Lemma 4.4.1 below and 4. of Lemma 4.2.2 we obtain the bound

$$\left\|W_0^t(g,0,0)\right\|_{H^s(\mathbb{R}^+)}\lesssim\left\|W^t g_{\text{odd}}\right\|_{H^s(\mathbb{R})}=\|g_{\text{odd}}\|_{H^s(\mathbb{R})}\lesssim\|g\|_{H^s(\mathbb{R}^+)}.$$

4.3. Proof of Lemma 4.2.1: Boundary Term

In this section we obtain explicit solution formula for the linear problem (4.6) by the application of the Laplace transform. So taking the Laplace transform of the equation (4.6) in t leads to the initial value problem in the spatial variable x

$$\begin{cases} \widetilde{u}_{xxxx} + i\lambda\widetilde{u} = 0 \\ \widetilde{u}(0,\lambda) = \widetilde{h}_1(\lambda), \quad \widetilde{u}_x(0,\lambda) = \widetilde{h}_2(\lambda) \end{cases} \tag{4.14}$$

where

$$\widetilde{u}(x,\lambda) = \int_0^\infty e^{-\lambda t} u(x,t)dt, \qquad \widetilde{h}_j(\lambda) = \int_0^\infty e^{-\lambda t} h_j(t)dt, \quad j = 1,2.$$

The solution of (4.14) can be written as follows

$$\widetilde{u}(x,\lambda) = c_1(\lambda)e^{r_1(\lambda)x} + c_2(\lambda)e^{r_2(\lambda)x}$$

where $r_1(\lambda)$ and $r_2(\lambda)$ are solutions of the characteristic equation $r^4(\lambda) + i\lambda = 0$ for which $\operatorname{Re} r_1 < 0$, $\operatorname{Re} r_2 < 0$. Employing the initial conditions and suppressing the λ dependence of $r_1(\lambda)$ and $r_2(\lambda)$, we have

$$c_1(\lambda) = \frac{\widetilde{h}_2(\lambda) - r_2\widetilde{h}_1(\lambda)}{r_1 - r_2}, \ c_2(\lambda) = \frac{r_1\widetilde{h}_1(\lambda) - \widetilde{h}_2(\lambda)}{r_1 - r_2}.$$

Then by Mellin inversion we can express the solution as

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{r_1 - r_2}\Big[\big(\widetilde{h}_2(\lambda) - r_2\widetilde{h}_1(\lambda)\big)e^{r_1 x} + \big(r_1\widetilde{h}_1(\lambda) - \widetilde{h}_2(\lambda)\big)e^{r_2 x}\Big]d\lambda$$

for $x, t > 0$ and where $\gamma > 0$ is fixed. Letting $\gamma \to 0$, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\beta t}}{r_1 - r_2}\Big[\big(\widetilde{h}_2(i\beta) - r_2\widetilde{h}_1(i\beta)\big)e^{r_1(i\beta)x} + \big(r_1\widetilde{h}_1(i\beta) - \widetilde{h}_2(i\beta)\big)e^{r_2(i\beta)x}\Big]d\beta \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} \frac{e^{i\beta t}}{i\sqrt{2}\sqrt[4]{-\beta}}\Big[\widetilde{h}_2(i\beta) + \Big(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)\sqrt[4]{-\beta}\,\widetilde{h}_1(i\beta)\Big]e^{\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\sqrt[4]{-\beta}x}d\beta \\
&+ \frac{1}{2\pi}\int_{-\infty}^{0} \frac{e^{i\beta t}}{i\sqrt{2}\sqrt[4]{-\beta}}\Big[\Big(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)\sqrt[4]{-\beta}\,\widetilde{h}_1(i\beta) - \widetilde{h}_2(i\beta)\Big]e^{\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\sqrt[4]{-\beta}x}d\beta \\
&+ \frac{1}{2\pi}\int_{0}^{\infty} \frac{e^{i\beta t}}{-(1+i)\sqrt[4]{\beta}}\big[\widetilde{h}_2(i\beta) - i\sqrt[4]{\beta}\,\widetilde{h}_1(i\beta)\big]e^{-\sqrt[4]{\beta}x}d\beta \\
&+ \frac{1}{2\pi}\int_{0}^{\infty} \frac{e^{i\beta t}}{-(1+i)\sqrt[4]{\beta}}\Big[-\sqrt[4]{\beta}\,\widetilde{h}_1(i\beta) - \widetilde{h}_2(i\beta)\Big]e^{i\sqrt[4]{\beta}x}d\beta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^\infty \frac{e^{-i\beta^4 t}}{i\sqrt{2}\beta}\Big[\widetilde{h}_2(-i\beta^4) + \Big(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)\beta\,\widetilde{h}_1(-i\beta^4)\Big]e^{\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\beta x}4\beta^3 d\beta \\
&+ \frac{1}{2\pi}\int_0^\infty \frac{e^{-i\beta^4 t}}{i\sqrt{2}\beta}\Big[\Big(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)\beta\,\widetilde{h}_1(-i\beta^4) - \widetilde{h}_2(-i\beta^4)\Big]e^{\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\beta x}4\beta^3 d\beta \\
&+ \frac{1}{2\pi}\int_0^\infty \frac{e^{i\beta^4 t}}{-(1+i)\beta}\big[\widetilde{h}_2(i\beta^4) - i\beta\widetilde{h}_1(i\beta^4)\big]e^{-\beta x}4\beta^3 d\beta \\
&+ \frac{1}{2\pi}\int_0^\infty \frac{e^{i\beta^4 t}}{-(1+i)\beta}\Big[-\beta\,\widetilde{h}_1(i\beta^4) - \widetilde{h}_2(i\beta^4)\Big]e^{i\beta x}4\beta^3 d\beta.
\end{aligned}
$$

By a slight abuse of notation after writing $\widehat{h}_j$ instead of $\widetilde{h}_j$ to denote the Fourier transform of $\chi_{(0,\infty)}h_j$, $j = 1, 2$, we obtain

$$
\begin{aligned}
u(x,t) &= -\frac{\sqrt{2}}{\pi}i\int_0^\infty e^{-i\beta^4 t}e^{\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\beta x}\Big[\widehat{h}_2(-\beta^4) + \Big(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)\beta\,\widehat{h}_1(-\beta^4)\Big]\beta^2 d\beta \\
&- \frac{\sqrt{2}}{\pi}i\int_0^\infty e^{-i\beta^4 t}e^{\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\beta x}\Big[\Big(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)\beta\,\widehat{h}_1(-\beta^4) - \widehat{h}_2(-\beta^4)\Big]\beta^2 d\beta \\
&+ \frac{-1+i}{\pi}\int_0^\infty e^{i\beta^4 t - \beta x}\big[\widehat{h}_2(\beta^4) - i\beta\widehat{h}_1(\beta^4)\big]\beta^2 d\beta \\
&+ \frac{-1+i}{\pi}\int_0^\infty e^{i\beta^4 t + i\beta x}\big[-\beta\,\widehat{h}_1(\beta^4) - \widehat{h}_2(\beta^4)\big]\beta^2 d\beta.
\end{aligned}
$$

Finally, we add the cut-off function ρ in the above integrals except the last one to extend the solution to all x. Note that with this choice the integrals converge for all x.

4.4. A Priori Estimates

4.4.1. Estimates for the Linear Terms

In this section we justify that the linear terms in (4.8) stay in the function space given in the definition 4.1.1. First we begin with the Kato smoothing inequality depicting interaction between the space and time derivatives. Note that this affirms the selection of the spaces that the data g, h_1 and h_2 reside in.

Lemma 4.4.1 (Kato smoothing inequality). *For any $s \in \mathbb{R}$, $g \in H^s(\mathbb{R})$, we have $\eta(t)W^t g \in C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R})$ and $\eta(t)\partial_x[W^t g] \in C_x^0 H_t^{\frac{2s+1}{8}}(\mathbb{R}\times\mathbb{R})$. Moreover,*

$$\left\|\eta(t)W^t g\right\|_{L_x^\infty H_t^{\frac{2s+3}{8}}} \lesssim \|g\|_{H_x^s} \text{ and } \left\|\eta(t)\partial_x[W^t g]\right\|_{L_x^\infty H_t^{\frac{2s+1}{8}}} \lesssim \|g\|_{H_x^s}.$$

Proof. We start by writing that

$$\begin{aligned}\mathcal{F}_t(\eta W^t g)(\tau) &= \int e^{ix\xi}\widehat{\eta}(\tau-\xi^4)\widehat{g}(\xi)d\xi \\ &= \int_{|\xi|<1} e^{ix\xi}\widehat{\eta}(\tau-\xi^4)\widehat{g}(\xi)d\xi + \int_{|\xi|\geq 1} e^{ix\xi}\widehat{\eta}(\tau-\xi^4)\widehat{g}(\xi)d\xi.\end{aligned}$$

Using the fact that η is a Schwarz function, the contribution of the $H_t^{\frac{2s+3}{8}}$ norm of the first term above is bounded by

$$\begin{aligned}\int_{|\xi|<1}\left\|\langle\tau\rangle^{\frac{2s+3}{8}}\widehat{\eta}(\tau-\xi^4)\right\|_{L_\tau^2}|\widehat{g}(\xi)|d\xi &\lesssim \int_{|\xi|<1}\left\|\langle\tau\rangle^{\frac{2s+3}{8}}\widehat{\eta}(\tau-\xi^4)\right\|_{L_\tau^2}\langle\xi\rangle^s|\widehat{g}(\xi)|d\xi \\ &\lesssim \int_{|\xi|<1}\langle\xi\rangle^s|\widehat{g}(\xi)|d\xi \lesssim \|g\|_{H^s}.\end{aligned}$$

Next by the inequality $\langle x+y\rangle^r \lesssim \langle x\rangle^{|r|}\langle y\rangle^r$ for any $r\in\mathbb{R}$, and a change of variable, the contribution for the second term is estimated by

$$\begin{aligned}&\left\|\int_{|\xi|\geq 1}\langle\tau\rangle^{\frac{2s+3}{8}}|\widehat{\eta}(\tau-\xi^4)||\widehat{g}(\xi)|d\xi\right\|_{L_\tau^2} \\ &\lesssim \left\|\int_{|\xi|\geq 1}\langle\tau-\xi^4\rangle^{\frac{|2s+3|}{8}}\langle\xi\rangle^{\frac{2s+3}{2}}|\widehat{\eta}(\tau-\xi^4)||\widehat{g}(\xi)|d\xi\right\|_{L_\tau^2} \\ &\lesssim \left\|\int_{|\rho|\geq 1}\langle\tau-\rho\rangle^{\frac{|2s+3|}{8}}\langle\rho\rangle^{\frac{2s-3}{8}}|\widehat{\eta}(\tau-\rho)||\widehat{g}(\pm\rho^{\frac14})|d\rho\right\|_{L_\tau^2} \\ &\lesssim \left\|\langle\cdot\rangle^{\frac{|2s+3|}{8}}\widehat{\eta}(\cdot)\right\|_{L^1}\left\|\rho^{\frac{2s-3}{8}}\widehat{g}(\pm\rho^{\frac14})\right\|_{L^2_{\rho\geq 1}} \\ &\lesssim \left\|\rho^{\frac{2s-3}{8}}\widehat{g}(\pm\rho^{\frac14})\right\|_{L^2_{\rho\geq 1}}\end{aligned}$$

where we have used Young's inequality in the third inequality.

Changing variable back to ξ this is bounded by

$$\Big(\int_1^\infty \langle\xi\rangle^{2s-3}|\widehat{g}(\pm\xi)|^2\xi^3 d\xi\Big)^{\frac{1}{2}} \lesssim \Big(\int \langle\xi\rangle^{2s}|\widehat{g}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} = \|g\|_{H^s}.$$

From these and the dominated convergence theorem continuity statement follows. Using the same argument we estimate $\|\eta(t)\partial_x[W^t g]\|_{H_t^{\frac{2s+1}{8}}}$ likewise. □

Proposition 4.4.2, Lemma 4.4.3, and Lemma 4.4.4 below verify that the boundary operator belongs to the space from definition 4.1.1.

Proposition 4.4.2. *For any $s \geq -\frac{1}{2}$, $b \leq \frac{1}{2}$ and h_1, h_2 satisfying $\chi_{(0,\infty)}h_1 \in H^{\frac{2s+3}{8}}$, $\chi_{(0,\infty)}h_2 \in H^{\frac{2s+1}{8}}$, we have*

$$\big\|\eta(t)W_0^t(0,h_1,h_2)\big\|_{X^{s,b}} \lesssim \big\|\chi_{(0,\infty)}h_1\big\|_{H_t^{\frac{2s+3}{8}}} + \big\|\chi_{(0,\infty)}h_2\big\|_{H_t^{\frac{2s+1}{8}}}.$$

Proof. First recall that

$$\begin{aligned} W_0^t(0,h_1,h_2) = & \frac{-1+i}{\pi}\Big[W_1h_2 - iW_2h_1 - W_3h_1 - W_4h_2\Big] \\ & - \frac{\sqrt{2}i}{\pi}\Big[W_5h_2 + \Big(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)W_6h_1 + \Big(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)W_7h_1 - W_8h_2\Big] \end{aligned}$$

where the terms W_1h_2, W_2h_1, W_3h_1, W_4h_2, W_5h_2, W_6h_1, W_7h_1 and W_8h_2 are given in Lemma 4.2.1, also recall the notation of expressing $\widehat{h}$ as $\mathcal{F}_t(\chi_{(0,\infty)}h)$. Note that

$$W_3h_1 = W^t\psi_3 \text{ and } W_4h_2 = W^t\psi_2$$

where

$$\widehat{\psi}_3(\beta) = \beta^3\widehat{h}_1(\beta^4)\chi_{(0,\infty)}(\beta) \text{ and } \widehat{\psi}_4(\beta) = \beta^2\widehat{h}_2(\beta^4)\chi_{(0,\infty)}(\beta). \tag{4.15}$$

By change of variables we have

$$\begin{aligned} \|\psi_3\|_{H^s} = \Big\|\langle\beta\rangle^s\widehat{\psi}_3(\beta)\Big\|_{L^2_\beta} &= \Big(\int_0^\infty \langle\beta\rangle^{2s}\beta^6|\widehat{h}_1(\beta^4)|^2 d\beta\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\int_0^\infty \langle\rho\rangle^{\frac{2s+3}{4}}|\widehat{h}_1(\rho)|^2 d\rho\Big)^{\frac{1}{2}} \lesssim \big\|\chi_{(0,\infty)}h_1\big\|_{H^{\frac{2s+3}{8}}} \end{aligned} \tag{4.16}$$

and similarly

$$\begin{aligned}\|\psi_4\|_{H^s} &= \left\|\langle\beta\rangle^s\widehat{\psi}_4(\beta)\right\|_{L^2_\beta} = \Big(\int_0^\infty \langle\beta\rangle^{2s}\beta^4|\widehat{h}_2(\beta^4)|^2 d\beta\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\int_0^\infty \langle\rho\rangle^{\frac{2s+1}{4}}|\widehat{h}_2(\rho)|^2 d\rho\Big)^{\frac{1}{2}} \lesssim \left\|\chi_{(0,\infty)}h_2\right\|_{H^{\frac{2s+1}{8}}}. \end{aligned} \tag{4.17}$$

Then using (4.11) together with the bounds (4.16) and (4.17), we have

$$\begin{aligned}\|\eta(t)W_3h_1\|_{X^{s,b}} &= \left\|\eta W^t\psi_3\right\|_{X^{s,b}} \\ &\lesssim \|\psi_3\|_{H^s} \lesssim \left\|\chi_{(0,\infty)}h_1\right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})}\end{aligned}$$

and

$$\begin{aligned}\|\eta(t)W_4h_2\|_{X^{s,b}} &= \left\|\eta W^t\psi_4\right\|_{X^{s,b}} \\ &\lesssim \|\psi_4\|_{H^s} \lesssim \left\|\chi_{(0,\infty)}h_2\right\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}.\end{aligned}$$

For W_1h_2 and W_2h_1, set $f(x) = e^{-x}\rho(x)$. Note that f is a Schwarz function. Assume $s \in 4\mathbb{N}$, we can write

$$\begin{aligned}\partial_x^s W_1h_2 &= \eta\int_0^\infty e^{i\beta^4 t} f^{(s)}(\beta x)\beta^{s+2}\widehat{h}_2(\beta^4)d\beta \\ &= (-i)^{s/4}\eta\int_0^\infty e^{i\beta^4 t} f^{(s)}(\beta x)\beta^2\mathcal{F}_t[\chi_{(0,\infty)}\partial_t^{(s/4)}h_2](\beta^4)d\beta\end{aligned}$$

and

$$\partial_x^s W_2h_1 = (-i)^{s/4}\eta\int_0^\infty e^{i\beta^4 t} f^{(s)}(\beta x)\beta^3\mathcal{F}_t[\chi_{(0,\infty)}\partial_t^{(s/4)}h_1](\beta^4)d\beta.$$

Then using these with the interpolation it suffices to prove the bounds for $s = 0$. We have

$$\widehat{\eta W_1 h_2}(\xi,\tau) = \mathcal{F}_t\Big(\eta(t)\int_0^\infty e^{i\beta^4 t}\beta^2\widehat{h}_2(\beta^4)\mathcal{F}_x(f(\beta x))d\beta\Big)(\tau)$$

$$= \int_0^\infty \widehat{\eta}(\tau - \beta^4)\beta\widehat{h}_2(\beta^4)\widehat{f}(\xi/\beta)d\beta$$

and

$$\widehat{\eta W_2 h_1}(\xi,\tau) = \int_0^\infty \widehat{\eta}(\tau - \beta^4)\beta^2\widehat{h}_1(\beta^4)\widehat{f}(\xi/\beta)d\beta.$$

Since f is a Schwarz function,

$$\begin{aligned} |\widehat{f}(\xi/\beta)| &\lesssim \frac{1}{\langle \xi/\beta\rangle^4} \\ &\lesssim \frac{1}{1+(\xi/\beta)^4} \\ &= \frac{\beta^4}{\beta^4+\xi^4}. \end{aligned}$$

Also, as η is a compact supported C^∞ function, we may write

$$|\widehat{\eta}(\tau-\beta^4)| \lesssim \langle \tau - \beta^4\rangle^{-3}$$

as well. Therefore,

$$\|\eta W_1 h_2\|_{X^{0,b}} \lesssim \left\| \langle \tau - \xi^4\rangle^b \int_0^\infty \langle \tau-\beta^4\rangle^{-3}\frac{\beta^5}{\beta^4+\xi^4}|\widehat{h}_2(\beta^4)|d\beta \right\|_{L^2_{\xi,\tau}}.$$

We separate the integral into regions where

$$\beta^4+\xi^4 \le 1 \text{ and } \beta^4+\xi^4 > 1.$$

In the first case, we have

$$\begin{aligned} \left\| \int_0^1 \langle\tau\rangle^{b-3}\frac{\beta^5}{\beta^4+\xi^4}|\widehat{h}_2(\beta^4)|d\beta\right\|_{L^2_{|\xi|\le 1}L^2_\tau} &\lesssim \|\langle\tau\rangle^{b-3}\|_{L^2_\tau}\int_0^1 \left\|\frac{\beta^5}{\beta^4+\xi^4}\right\|_{L^2_{|\xi|\le 1}} |\widehat{h}_2(\beta^4)|d\beta \\ &\lesssim \int_0^1 \beta^{\frac{3}{2}}|\widehat{h}_2(\beta^4)|d\beta \\ &\approx \int_0^1 \rho^{-\frac{3}{8}}|\widehat{h}_2(\rho)|d\rho \\ &\lesssim \left\|\chi_{(0,\infty)}h_2\right\|_{L^2(\mathbb{R})} \\ &\le \left\|\chi_{(0,\infty)}h_2\right\|_{H^{\frac{1}{8}}(\mathbb{R})} \end{aligned}$$

where we have used Minkowski's and Cauchy-Schwarz inequalities in the first and third bounds respectively. For the other case where $\beta^4+\xi^4>1$, making use of relations $\langle \tau-\xi^4\rangle \lesssim \langle \tau-\beta^4\rangle\langle \beta^4+\xi^4\rangle$ and $\beta^4+\xi^4\sim\langle\beta^4+\xi^4\rangle$ we have the bound

$$\begin{aligned}
&\left\|\int_0^\infty \langle \tau-\beta^4\rangle^{b-3}\frac{\beta^5}{(\beta^4+\xi^4)^{1-b}}|\widehat{h}_2(\beta^4)|d\beta\right\|_{L^2_{\xi,\tau}} \\
&\qquad\lesssim \left\|\int_0^\infty \langle \tau-\beta^4\rangle^{b-3}\left\|\frac{\beta^5}{(\beta^4+\xi^4)^{1-b}}\right\|_{L^2_\xi}|\widehat{h}_2(\beta^4)|d\beta\right\|_{L^2_\tau} \\
&\qquad\lesssim \left\|\int_0^\infty \langle \tau-\beta^4\rangle^{b-3}\beta^{\frac{3}{2}+4b}|\widehat{h}_2(\beta^4)|d\beta\right\|_{L^2_\tau} \\
&\qquad\lesssim \left\|\int_0^\infty \langle \tau-\rho\rangle^{b-3}\rho^{b-\frac{3}{8}}|\widehat{h}_2(\rho)|d\rho\right\|_{L^2_\tau} \\
&\qquad\lesssim \left\|\langle\tau\rangle^{b-3}\right\|_{L^1_\tau}\left\|\langle\rho\rangle^{\frac{1}{8}}\widehat{h}_2(\rho)\right\|_{L^2_\rho}\lesssim \left\|\chi_{(0,\infty)}h_2\right\|_{H^{\frac{1}{8}}(\mathbb{R})}
\end{aligned}$$

where we have used Minkowski's and Young inequalities and note that we require $b\le\frac{1}{2}$ in the fourth inequality so that $b-\frac{3}{8}\le\frac{1}{8}$. Accordingly, using the similar arguments, we have

$$\begin{aligned}
\|\eta W_2h_1\|_{X^{0,b}} &\lesssim \int_0^1 \beta^{\frac{5}{2}}|\widehat{h}_1(\beta^4)|d\beta+\left\|\int_0^\infty \langle \tau-\beta^4\rangle^{b-3}\beta^{\frac{5}{2}+4b}|\widehat{h}_1(\beta^4)|d\beta\right\|_{L^2_\tau} \\
&\lesssim \int_0^1 \rho^{-\frac{1}{8}}|\widehat{h}_1(\rho)|d\rho+\left\|\int_0^\infty \langle \tau-\rho\rangle^{b-3}\rho^{b-\frac{1}{8}}|\widehat{h}_1(\rho)|d\rho\right\|_{L^2_\tau} \\
&\lesssim \left\|\chi_{(0,\infty)}h_1\right\|_{L^2_\rho}+\left\|\langle\tau\rangle^{b-3}\right\|_{L^1_\tau}\left\|\langle\rho\rangle^{b-\frac{1}{8}}\widehat{h}_1(\rho)\right\|_{L^2_\rho} \\
&\lesssim \left\|\chi_{(0,\infty)}h_1\right\|_{H^{\frac{3}{8}}(\mathbb{R})}+\left\|\langle\rho\rangle^{\frac{3}{8}}\widehat{h}_1(\rho)\right\|_{L^2_\rho}\lesssim \left\|\chi_{(0,\infty)}h_1\right\|_{H^{\frac{3}{8}}(\mathbb{R})}.
\end{aligned}$$

For the remaining terms of $W_0^t(0,h_1,h_2)$, estimates are similar; for W_5h_2 and W_6h_1 we let $f_1(x)=e^{(-\sqrt{2}/2+i\sqrt{2}/2)x}\rho(x)$ and for W_7h_1 and W_8h_2 set $f_2(x)=e^{(-\sqrt{2}/2-i\sqrt{2}/2)x}\rho(x)$ both of which are clearly Schwarz functions. So we adapt the previous estimates by swapping f with f_1 and f_2 for the terms W_5h_2, W_6h_1 and W_7h_1, W_8h_2 respectively. Eventually we have the bounds

$$\|\eta W_jh_2\|_{X^{0,b}}\lesssim \left\|\chi_{(0,\infty)}h_2\right\|_{H^{\frac{1}{8}}(\mathbb{R})}\quad\text{for j} = 5,8,$$

$$\|\eta W_j h_1\|_{X^{0,b}} \lesssim \left\|\chi_{(0,\infty)} h_1\right\|_{H^{\frac{3}{8}}(\mathbb{R})} \quad \text{for j} = 6, 7.$$

As before interpolating between the integers $s \in 4\mathbb{N}$ we obtain the bounds for any $s \geq 0$. To treat the $s < 0$ case we define the Fourier multiplier operator $\langle D\rangle_x^{-\frac{1}{2}}$ given by $\langle \xi\rangle^{-\frac{1}{2}}$ on the Fourier side. In this case,

$$\langle D\rangle_x^{-\frac{1}{2}}\big[\eta W_1 h_2\big](x,t) = \eta(t)\int_0^\infty e^{i\beta^4 t}\beta^2 \widehat{h}_2(\beta^4)\langle D\rangle_x^{-\frac{1}{2}}\big[f(\beta x)\big]\,d\beta$$

with similar formulas for the other terms of $W_0^t(0,h_1,h_2)$ other than W_3h_1 and W_4h_2. Note that

$$\begin{aligned}\mathcal{F}_{x,t}\Big(\langle D\rangle_x^{-\frac{1}{2}}\big[\eta W_1 h_2\big]\Big)(\xi,\tau)&\\ &= \int_0^\infty \widehat{\eta}(\tau-\beta^4)\beta^2\widehat{h}_2(\beta^4)\mathcal{F}_x\big(\langle D\rangle_x^{-\frac{1}{2}}\big[f(\beta x)\big]\big)\,d\beta\\ &= \int_0^\infty \widehat{\eta}(\tau-\beta^4)\beta^2\widehat{h}_2(\beta^4)\mathcal{F}_x\big(\big[\langle D\rangle_x^{-\frac{1}{2}}f\big](\beta x)\big)(\xi)\langle \xi/\beta\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}}\,d\beta.\end{aligned}$$

As $\langle D\rangle_x^{-\frac{1}{2}}f$ is a Schwarz function, we are free to establish the bounds

$$\Big|\mathcal{F}_x\big(\big[\langle D\rangle_x^{-\frac{1}{2}}f\big](\beta x)\big)(\xi)\Big| = \Big|\frac{1}{\beta}\widehat{\langle D\rangle_x^{-\frac{1}{2}}f}(\xi/\beta)\Big| \lesssim \frac{1}{|\beta|}\langle \xi/\beta\rangle^{-\frac{9}{2}}$$

and

$$|\widehat{\eta}(\tau-\beta^4)| \lesssim \langle \tau-\beta^4\rangle^{-3}.$$

This leads to the bound

$$\begin{aligned}\Big\|\langle D\rangle_x^{-\frac{1}{2}}\eta W_1 h_2\Big\|_{X^{0,b}} &\lesssim \Big\|\langle\tau-\xi^4\rangle^b\int_0^\infty \langle\tau-\beta^4\rangle^{-3}\beta\langle\xi/\beta\rangle^{-4}\langle\xi\rangle^{-\frac{1}{2}}|\widehat{h}_2(\beta^4)|\,d\beta\Big\|_{L^2_{\xi,\tau}}\\ &\lesssim \Big\|\langle\tau-\xi^4\rangle^b\int_0^\infty \langle\tau-\beta^4\rangle^{-3}\frac{\beta^5}{\beta^4+\xi^4}|\widehat{h}_2(\beta^4)|\,d\beta\Big\|_{L^2_{\xi,\tau}}\end{aligned}$$

which has been treated above. Thus interpolation between $s = -\frac{1}{2}$ and $s = 0$ yields the result. Other terms are handled similarly. $\square$

Lemma 4.4.3. *For $s \geq -1$ and boundary data (h_1, h_2) satisfying $(\chi_{(0,\infty)}h_1, \chi_{(0,\infty)}h_2) \in H^{\frac{2s+3}{8}}(\mathbb{R}) \times H^{\frac{2s+1}{8}}(\mathbb{R})$, we have*

$$W_0^t(0, h_1, h_2) \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R}).$$

Proof. We begin by showing that $W_3 h_1$ and $W_4 h_2$ belong to $C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})$. Since $W^t = e^{it\Delta^2}$ is unitary in H^s, we have

$$\begin{aligned}
\|W_3 h_1\|_{H_x^s} &= \left\|W^t \psi_3\right\|_{H_x^s} = \|\psi_3\|_{H_x^s} \lesssim \left\|\chi_{(0,\infty)} h_1\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \\
\|W_4 h_2\|_{H_x^s} &= \left\|W^t \psi_4\right\|_{H_x^s} = \|\psi_4\|_{H_x^s} \lesssim \left\|\chi_{(0,\infty)} h_2\right\|_{H^{\frac{2s+1}{8}}(\mathbb{R})}
\end{aligned}$$

where we have used (4.16) and (4.17) in the above inequalities respectively, and ψ_3, ψ_4 are defined as in (4.15). Continuity in the temporal variable follows from these bounds and the continuity of the linear group W^t in H^s. To show that the remaining terms of $W_t^0(0, h_1, h_2)$ lie in $C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})$, recalling the explicit form of the boundary operator from Lemma 4.2.1, we rewrite the remaining terms as follows

$$\begin{aligned}
W_1 h_2(x,t) &= \int_{\mathbb{R}} f(\beta x) \mathcal{F}_x(e^{it\Delta^2} \psi_1)(\beta) d\beta, \ \widehat{\psi_1}(\beta) = \beta^2 \widehat{h}_2(\beta^4) \chi_{(0,\infty)}(\beta), \\
W_2 h_1(x,t) &= \int_{\mathbb{R}} f(\beta x) \mathcal{F}_x(e^{it\Delta^2} \psi_2)(\beta) d\beta, \ \widehat{\psi_2}(\beta) = \beta^3 \widehat{h}_1(\beta^4) \chi_{(0,\infty)}(\beta), \\
W_5 h_2(x,t) &= \int_{\mathbb{R}} f_1(\beta x) \mathcal{F}_x(e^{-it\Delta^2} \psi_5)(\beta) d\beta, \ \widehat{\psi_5}(\beta) = \beta^2 \widehat{h}_2(-\beta^4) \chi_{(0,\infty)}(\beta), \\
W_6 h_1(x,t) &= \int_{\mathbb{R}} f_1(\beta x) \mathcal{F}_x(e^{-it\Delta^2} \psi_6)(\beta) d\beta, \ \widehat{\psi_6}(\beta) = \beta^3 \widehat{h}_1(-\beta^4) \chi_{(0,\infty)}(\beta), \\
W_7 h_1(x,t) &= \int_{\mathbb{R}} f_2(\beta x) \mathcal{F}_x(e^{-it\Delta^2} \psi_7)(\beta) d\beta, \ \widehat{\psi_7}(\beta) = \beta^3 \widehat{h}_1(-\beta^4) \chi_{(0,\infty)}(\beta), \\
W_8 h_2(x,t) &= \int_{\mathbb{R}} f_2(\beta x) \mathcal{F}_x(e^{-it\Delta^2} \psi_8)(\beta) d\beta, \ \widehat{\psi_8}(\beta) = \beta^2 \widehat{h}_2(-\beta^4) \chi_{(0,\infty)}(\beta),
\end{aligned}$$

where $f(x) = e^{-x}\rho(x)$, $f_1(x) = e^{(-\sqrt{2}/2 + i\sqrt{2}/2)x}\rho(x)$ and $f_2(x) = e^{(-\sqrt{2}/2 - i\sqrt{2}/2)x}\rho(x)$. Note that following the same computations done in (4.16) and (4.17), we have

$$\|\psi_j\|_{H_x^s} \lesssim \left\|\chi_{(0,\infty)} h_1\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \text{ for } j = 2, 6, 7$$

$$\|\psi_j\|_{H^s_x} \lesssim \left\|\chi_{(0,\infty)} h_2\right\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \text{ for } j = 1, 5, 8.$$

Using these and the continuity of the group $e^{\pm it\Delta^2}$ on H^s it suffices to show that the maps

$$g \mapsto Tg = \int_{\mathbb{R}} f(\beta x)\widehat{g}(\beta)d\beta, \; g \mapsto T_1 g = \int_{\mathbb{R}} f_1(\beta x)\widehat{g}(\beta)d\beta, \; g \mapsto T_2 g = \int_{\mathbb{R}} f_2(\beta x)\widehat{g}(\beta)d\beta$$

are bounded in H^s. We show this for the map $g \mapsto Tg$ only as each f, f_1 and f_2 are Schwarz functions leading to the same result. Consider first $s = 0$, we rewrite $Tg(x)$ by using the change of variable $\beta x \to \beta$ as follows

$$Tg(x) = \int_{\mathbb{R}} f(\beta)\widehat{g}(x^{-1}\beta)x^{-1}d\beta.$$

Therefore,

$$\begin{aligned}
\|Tg\|_{L^2_x} &\leq \int_{\mathbb{R}} |f(\beta)| \left\|x^{-1}\widehat{g}(x^{-1}\beta)\right\|_{L^2_x} d\beta \\
&= \int_{\mathbb{R}} |f(\beta)| \Big(\int x^{-2} |\widehat{g}(x^{-1}\beta)|^2 dx \Big)^{\frac{1}{2}} d\beta \\
&= \int_{\mathbb{R}} |f(\beta)| \Big(\int \beta^{-1} |\widehat{g}(z)|^2 dz \Big)^{\frac{1}{2}} d\beta \\
&= \|g\|_{L^2} \int_{\mathbb{R}} \frac{|f(\beta)|}{\sqrt{\beta}} d\beta \\
&\lesssim \|g\|_{L^2}
\end{aligned}$$

where the validity of the final inequality is due to the fact that f is a Schwarz function. Note that for any $s \in \mathbb{N}$ we write

$$\partial_x^s Tg(x) = \int_{\mathbb{R}} f^{(s)}(\beta x)\beta^s \widehat{g}(\beta)d\beta.$$

This with $s = 0$ result implies that $\|Tg\|_{H^s} \lesssim \|g\|_{H^s}$, $s \in \mathbb{N}$. Hence by interpolation, $s \geq 0$ case follows. As for $s = -1$, we pick ρ such that $\int f dx = 0$ so that $\partial_x^{-1} f$ belongs to the Schwarz space. Then we write

$$\partial_x^{-1} Tg(x) = \int_{\mathbb{R}} \partial_x^{-1}\big(f(\beta x)\big)\widehat{g}(\beta)d\beta = \int_{\mathbb{R}} \partial_x^{-1} f(\beta x)\beta^{-1}\widehat{g}(\beta)d\beta.$$

Combining this with $s = 0$ result and then applying the interpolation argument we get the bound for $s \geq -1$. $\square$

Lemma 4.4.4. *For $s \geq -1$ and boundary data (h_1, h_2) satisfying $(\chi_{(0,\infty)}h_1, \chi_{(0,\infty)}h_2) \in H^{\frac{2s+3}{8}}(\mathbb{R}) \times H^{\frac{2s+1}{8}}(\mathbb{R})$, we have*

$$\eta(t)W_0^t(0, h_1, h_2) \in C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R} \times \mathbb{R}).$$

Proof. Recalling $W_3 h_2 = W^t \psi_3$ and $W_4 h_1 = W^t \psi_4$, the claim for these terms follows by (4.7), (4.16), (4.17), the continuity of W^t and the Kato smoothing inequality (Lemma 4.4.1). With the notations of the previous lemma we rewrite the remaining terms of $W_0^t(0, h_1, h_2)$ as follows

$$\begin{aligned}
W_1 h_2(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f(\beta x)\big)(z) W^t \psi_1(z) dz, \\
W_2 h_1(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f(\beta x)\big)(z) W^t \psi_2(z) dz, \\
W_5 h_2(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_1(\beta x)\big)(z) W^{-t} \psi_5(z) dz, \\
W_6 h_1(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_1(\beta x)\big)(z) W^{-t} \psi_6(z) dz, \\
W_7 h_1(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_2(\beta x)\big)(z) W^{-t} \psi_7(z) dz, \\
W_8 h_2(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_2(\beta x)\big)(z) W^{-t} \psi_8(z) dz.
\end{aligned}$$

We show only $\eta(t)W_1 h_2 \in C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R} \times \mathbb{R})$ since the estimates for the other terms follow by the same arguments. Hence

$$\begin{aligned}
W_1 h_2(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f(\beta x)\big)(z) W^t \psi_1(z) dz \\
&= \int_{\mathbb{R}} \frac{1}{x} \widehat{f}\Big(\frac{z}{x}\Big) W^t \psi_1(z) dz \\
&= \int_{\mathbb{R}} \widehat{f}(z) W^t \psi_1(xz) dz.
\end{aligned}$$

Then Minkowski's and Kato smoothing inequalities lead to the bound

$$\|\eta W_1 h_2\|_{H_t^{\frac{2s+3}{8}}} \leq \int_{\mathbb{R}} |\widehat{f}(z)| \left\|\eta W^t \psi_1(xz)\right\|_{H_t^{\frac{2s+3}{8}}} dz$$

$$\leq \|\widehat{f}\|_{L^1} \left\|\eta W^t \psi_1(xz)\right\|_{H_t^{\frac{2s+3}{8}} L_z^\infty}$$
$$\lesssim \|\psi_1\|_{H_z^s} \lesssim \left\|\chi_{(0,\infty)} h_2\right\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}$$

since $\widehat{f} \in L^1$. Finally, continuity in the spatial variable follows from the dominated convergence theorem. □

4.4.2. Estimates for the Nonlinear Term

This section discusses the estimates for the nonlinear term in (4.8). These estimates will play crucial role in establishing the smoothing theorem and closing the fixed point argument.

Proposition 4.4.5. *For any compactly supported smooth function η and $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\left\|\eta(t) \int_0^t W^{t-t'} F dt'\right\|_{C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R})} + \left\|\eta(t) \partial_x \Big(\int_0^t W^{t-t'} F dt' \Big)\right\|_{C_x^0 H_t^{\frac{2s+1}{8}}(\mathbb{R}\times\mathbb{R})}$$
$$\lesssim \begin{cases} \|F\|_{X^{s,-b}} & \text{if } -\frac{1}{2} \leq s \leq \frac{1}{2}, \\ \|F\|_{X^{s,-b}} + \|F\|_{X^{\frac{1}{2}+, \frac{2s-5}{8}}} & \text{if } s > \frac{1}{2}. \end{cases}$$

Proof. Assume first that $-\frac{1}{2} \leq s \leq \frac{1}{2}$, then

$$\begin{aligned} \int_0^t W^{t-t'} F dt' &= \int_0^t \int_{\mathbb{R}} e^{ix\xi} e^{i(t-t')\xi^4} \widehat{F}(\xi, t') d\xi dt' \\ &= \int_{\mathbb{R}} \int_0^t e^{ix\xi} e^{i(t-t')\xi^4} \Big(\int_{\mathbb{R}} e^{it'\tau} \widehat{F}(\xi, \tau) d\tau \Big) dt' d\xi \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} e^{it\xi^4} \Big(\int_0^t e^{it'(\tau-\xi^4)} dt' \Big) \widehat{F}(\xi, \tau) d\xi d\tau \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\tau} - e^{it\xi^4}}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau. \end{aligned}$$

First, we wish to bound

$$\left\| \eta \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\tau} - e^{it\xi^4}}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \right\|_{H_t^{\frac{2s+3}{8}}}.$$

Let φ be a smooth cut-off function such that $\varphi = 1$ on $[-1, 1]$ and $\mathrm{supp}\varphi \subset \{x : |x| \leq 2\}$ and let $\varphi^c = 1 - \varphi$. We will proceed by writing

$$\begin{aligned} \eta(t) \int_0^t W^{t-t'} F dt' &= \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{\left(e^{it\tau} - e^{it\xi^4}\right)\varphi(\tau - \xi^4)}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \\ &+ \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\tau} \varphi^c(\tau - \xi^4)}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \\ &- \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\xi^4} \varphi^c(\tau - \xi^4)}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

By Taylor expansion, we write

$$\frac{e^{it\tau} - e^{it\xi^4}}{i(\tau - \xi^4)} = i e^{it\tau} \frac{1}{\tau - \xi^4} \left(e^{-it(\tau - \xi^4)} - 1\right) = i e^{it\tau} \sum_{k=1}^{\infty} \frac{(-it)^k}{k!} (\tau - \xi^4)^{k-1}.$$

For I, using Lemma A.0.3, we have the bound

$$\|\mathrm{I}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \lesssim \sum_{k=1}^{\infty} \frac{\left\|t^k \eta\right\|_{H^1}}{k!} \left\| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} e^{it\tau} (\tau - \xi^4)^{k-1} \varphi(\tau - \xi^4) \widehat{F}(\xi, \tau) d\xi d\tau \right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})}$$

which is bounded by

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left\| \langle \tau \rangle^{\frac{2s+3}{8}} \int_{\mathbb{R}} e^{ix\xi} (\tau - \xi^4)^{k-1} \varphi(\tau - \xi^4) \widehat{F}(\xi, \tau) d\xi d\tau \right\|_{L^2_\tau} \\ &\lesssim \left\| \langle \tau \rangle^{\frac{2s+3}{8}} \int_{|\tau - \xi^4| < 1} \widehat{F}(\xi, \tau) d\xi \right\|_{L^2_\tau} \end{aligned}$$

where we have used

$$\begin{aligned} \left\|t^k \eta\right\|_{H^1} &\approx \left\|t^k \eta\right\|_{L^2} + \left\|\partial_t (t^k \eta)\right\|_{L^2} \\ &\lesssim k \left\|t^{k-1} \eta\right\|_{L^2} + \left\|t^k \eta'\right\|_{L^2} \lesssim k. \end{aligned}$$

Using Cauchy-Schwarz inequality in ξ, this is bounded by

$$\begin{aligned}\Big[\int_{\mathbb{R}}\langle\tau\rangle^{\frac{2s+3}{4}}\Big(\int_{|\tau-\xi^4|<1}\langle\xi\rangle^{-2s}d\xi\Big)\Big(\int_{|\tau-\xi^4|<1}\langle\xi\rangle^{2s}|\widehat{F}(\xi,\tau)|^2d\xi\Big)d\tau\Big]^{\frac{1}{2}}\\ \lesssim \sup_{\tau}\Big(\langle\tau\rangle^{\frac{2s+3}{4}}\int_{|\tau-\xi^4|<1}\langle\xi\rangle^{-2s}d\xi\Big)^{\frac{1}{2}}\|F\|_{X^{s,-b}}\\ \lesssim \|F\|_{X^{s,-b}}.\end{aligned}$$

For $|\tau|\lesssim 1$, the supremum is apparently bounded whereas for $|\tau|\gg 1$, by the change of variable $\rho=\xi^4$, it is bounded by

$$\langle\tau\rangle^{\frac{2s+3}{4}}\int_{|\tau|-1}^{|\tau|+1}\langle\rho\rangle^{\frac{-s}{2}}\frac{1}{|\rho|^{\frac{3}{4}}}d\rho\lesssim\langle\tau\rangle^{\frac{2s+3}{4}}\int_{|\tau|-1}^{|\tau|+1}\langle\rho\rangle^{\frac{-2s-3}{4}}d\rho\lesssim 1$$

since $|\rho|\sim|\tau|\gg 1$. Next we consider II. By using Lemma A.0.3 we have

$$\begin{aligned}\|\mathrm{II}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} &\lesssim \|\eta\|_{H^1}\left\|\langle\tau\rangle^{\frac{2s+3}{8}}\int_{|\tau-\xi^4|\geq 1}\frac{|\widehat{F}(\xi,\tau)|}{\langle\tau-\xi^4\rangle}d\xi\right\|_{L^2_\tau}\\ &\lesssim \Big[\int_{\mathbb{R}}\langle\tau\rangle^{\frac{2s+3}{4}}\Big(\int\frac{d\xi}{\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{2-2b}}\Big)\Big(\int\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{-2b}|\widehat{F}(\xi,\tau)|^2d\xi\Big)d\tau\Big]^{\frac{1}{2}}\\ &\lesssim \sup_{\tau}\Big[\langle\tau\rangle^{\frac{2s+3}{4}}\int\frac{d\xi}{\langle\tau-\xi^4\rangle^{2-2b}\langle\xi\rangle^{2s}}\Big]^{\frac{1}{2}}\|F\|_{X^{s,-b}}\\ &\lesssim \|F\|_{X^{s,-b}}\end{aligned}$$

we have applied Cauchy-Schwarz inequality in the second line. To see that the supremum above is finite we write

$$\begin{aligned}&\langle\tau\rangle^{\frac{2s+3}{4}}\Big[\int_{|\xi|<1}\frac{d\xi}{\langle\tau-\xi^4\rangle^{2-2b}\langle\xi\rangle^{2s}}+\int_{|\xi|\geq 1}\frac{d\xi}{\langle\tau-\xi^4\rangle^{2-2b}\langle\xi\rangle^{2s}}\Big]\\ &\quad\lesssim\langle\tau\rangle^{\frac{2s+3}{4}}\Big[\langle\tau\rangle^{2b-2}\int_{|\xi|<1}\frac{d\xi}{\langle\xi\rangle^{2s}}+\int_{|\rho|\geq 1}\frac{d\rho}{\langle\tau-\rho\rangle^{2-2b}\langle\rho\rangle^{\frac{2s+3}{4}}}\Big]\\ &\quad\lesssim\langle\tau\rangle^{\frac{2s+3}{4}+2b-2}+\langle\tau\rangle^{\frac{2s+3}{4}}\langle\tau\rangle^{\frac{-2s-3}{4}}\lesssim 1\end{aligned}$$

where we have used Lemma A.0.5 in the $\rho-$integral and $\frac{1}{2}\leq\frac{2s+3}{4}\leq 1$ with $b<\frac{1}{2}$. Next for III, we divide the region of integration into two pieces $|\xi|<1$ and $|\xi|\geq 1$.

For $|\xi| < 1$ using Minkowski's inequality and then Cauchy-Schwarz inequality we have

$$
\begin{aligned}
\left\|\mathrm{III}_{|\xi|<1}\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} &:= \left\|\eta(t)\int\int_{|\xi|<1}\frac{e^{ix\xi}e^{it\xi^4}}{i(\tau-\xi^4)}\varphi^c(\tau-\xi^4)|\widehat{F}(\xi,\tau)|d\xi d\tau\right\|_{H_t^{\frac{2s+3}{8}}} \\
&\leq \int\int_{|\xi|<1}\left\|\eta(t)e^{it\xi^4}\right\|_{H_t^{\frac{2s+3}{8}}}\frac{\varphi^c(\tau-\xi^4)}{|\tau-\xi^4|}|\widehat{F}(\xi,\tau)|d\xi d\tau \\
&\lesssim \int\int_{|\xi|<1}\frac{|\widehat{F}(\xi,\tau)|}{\langle\tau-\xi^4\rangle}d\xi d\tau \\
&\lesssim \Big[\int\int_{|\xi|<1}\langle\tau\rangle^{2b-2}d\xi d\tau\Big]^{\frac{1}{2}}\Big[\iint\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{-2b}|\widehat{F}(\xi,\tau)|d\xi d\tau\Big]^{\frac{1}{2}} \\
&\lesssim \|F\|_{X^{s,-b}}
\end{aligned}
$$

since $2b-2 < -1$ for $b < \frac{1}{2}$. To treat the case regarding the region $|\xi| \geq 1$, we use change of variable $\rho = \xi^4$ as before to get

$$
\begin{aligned}
\left\|\mathrm{III}_{|\xi|\geq 1}\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} &:= \left\|\eta(t)\int\int_{|\xi|\geq 1}\frac{e^{ix\xi}e^{it\xi^4}}{i(\tau-\xi^4)}\varphi^c(\tau-\xi^4)|\widehat{F}(\xi,\tau)|d\xi d\tau\right\|_{H_t^{\frac{2s+3}{8}}} \\
&\lesssim \|\eta\|_{H^1}\left\|\int_{|\tau-\rho|>1}\int_{|\rho|>1}\frac{e^{ix\sqrt[4]{\rho}}e^{it\rho}}{|\tau-\rho|}\widehat{F}(\sqrt[4]{\rho},\tau)\frac{1}{|\rho|^{\frac{3}{4}}}d\rho d\tau\right\|_{H_t^{\frac{2s+3}{8}}} \\
&\lesssim \left\|\langle\rho\rangle^{\frac{2s+3}{8}}\mathcal{F}_t\circ\mathcal{F}_\rho^{-1}\Big(\int_{|\tau-\rho|>1,\,|\rho|>1}\frac{e^{ix\sqrt[4]{\rho}}}{|\tau-\rho|}\widehat{F}(\sqrt[4]{\rho},\tau)\frac{1}{|\rho|^{\frac{3}{4}}}d\tau\Big)(\rho)\right\|_{L^2_\rho} \\
&\lesssim \left\|\langle\rho\rangle^{\frac{2s+3}{8}}\int\frac{\widehat{F}(\sqrt[4]{\rho},\tau)}{\langle\tau-\rho\rangle|\rho|^{\frac{3}{4}}}d\tau\right\|_{L^2_{|\rho|\geq 1}} \\
&\lesssim \Big[\int\langle\rho\rangle^{\frac{2s+3}{4}}\langle\rho\rangle^{\frac{-3}{4}}\Big(\int\frac{d\tau}{\langle\tau-\rho\rangle^{2-2b}}\Big)\Big(\int\frac{|\widehat{F}(\sqrt[4]{\rho},\tau)|^2}{\langle\tau-\rho\rangle^{2b}}d\tau\Big)\frac{1}{|\rho|^{\frac{3}{4}}}d\rho\Big]^{\frac{1}{2}} \\
&\lesssim \Big[\iint\langle\rho\rangle^{\frac{s}{2}}\langle\tau-\rho\rangle^{-2b}|\widehat{F}(\sqrt[4]{\rho},\tau)|^2\frac{1}{|\rho|^{\frac{3}{4}}}d\rho d\tau\Big]^{\frac{1}{2}} \\
&\lesssim \|F\|_{X^{s,-b}}
\end{aligned}
$$

where we used Cauchy-Schwarz inequality in the fifth line and changed variables back to ξ in the last line. This finishes the proof for $-\frac{1}{2} \leq s \leq \frac{1}{2}$.

Next we consider $s > \frac{1}{2}$, in which case, instead of Lemma A.0.3, proof makes use of algebra property of Sobolev spaces

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}$$

in order to extract the Sobolev norm of η. As η is a smooth compactly supported function, the proof proceeds along the same lines as with the case $-\frac{1}{2} \leq s \leq \frac{1}{2}$ except for the one for II just because we needed $s \leq \frac{1}{2}$ to obtain the bound $\|\mathrm{II}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \lesssim \|F\|_{X^{s,-b}}$. Thus to estimate II, we use the identity

$$\langle\tau\rangle^{\frac{2s+3}{8}} \lesssim \langle\tau - \xi^4\rangle^{\frac{2s+3}{8}} + |\xi|^{\frac{2s+3}{2}}$$

to write

$$\begin{aligned} \|\mathrm{II}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} &\lesssim \|\eta\|_{H^1} \left\| \langle\tau\rangle^{\frac{2s+3}{8}} \int \frac{|\widehat{F}(\xi,\tau)|}{\langle\tau-\xi^4\rangle} d\xi \right\|_{L^2_\tau} \\ &\lesssim \left\| \int \langle\tau-\xi^4\rangle^{\frac{2s-5}{8}} |\widehat{F}(\xi,\tau)| d\xi \right\|_{L^2_\tau} + \left\| \int \frac{|\xi|^{\frac{2s+3}{2}}}{\langle\tau-\xi^4\rangle} |\widehat{F}(\xi,\tau)| d\xi \right\|_{L^2_\tau}. \end{aligned}$$

Using the Cauchy-Schwarz inequality in the ξ–integral, the second term is bounded by

$$\begin{aligned} &\left\| \int \frac{|\xi|^{\frac{2s+3}{2}}}{\langle\tau-\xi^4\rangle} |\widehat{F}(\xi,\tau)| d\xi \right\|_{L^2_\tau} \\ &\qquad \lesssim \Big[\int \Big(\int \frac{|\xi|^3}{\langle\tau-\xi^4\rangle^{2-2b}} d\xi \Big) \Big(\int \frac{|\xi|^{2s}}{\langle\tau-\xi^4\rangle^{2b}} |\widehat{F}(\xi,\tau)|^2 d\xi \Big) d\tau \Big]^{\frac{1}{2}} \\ &\qquad \lesssim \sup_\tau \Big[\int \frac{|\xi|^3}{\langle\tau-\xi^4\rangle^{2-2b}} d\xi \Big]^{\frac{1}{2}} \|F\|_{X^{s,-b}} \\ &\qquad \lesssim \sup_\tau \Big[\int \frac{1}{\langle\tau-\rho\rangle^{2-2b}} d\rho \Big]^{\frac{1}{2}} \|F\|_{X^{s,-b}} \lesssim \|F\|_{X^{s,-b}}. \end{aligned}$$

since $2-2b > 1$. Applying the Cauchy-Schwarz inequality in the ξ–integral for the first term in this case

$$\begin{aligned} &\left\| \int \langle\tau-\xi^4\rangle^{\frac{2s-5}{8}} |\widehat{F}(\xi,\tau)| d\xi \right\|_{L^2_\tau} \\ &\qquad \lesssim \Big[\int \Big(\int \frac{d\xi}{\langle\xi\rangle^{1+}} \Big) \Big(\int \langle\xi\rangle^{1+} \langle\tau-\xi^4\rangle^{\frac{2s-5}{4}} |\widehat{F}(\xi,\tau)|^2 d\xi \Big) d\tau \Big]^{\frac{1}{2}} \lesssim \|F\|_{X^{\frac{1}{2}+,\frac{2s-5}{8}}}. \end{aligned}$$

As a result we have obtained

$$\left\|\eta(t)\int_0^t W^{t-t'}F dt'\right\|_{C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R})} \lesssim \begin{cases} \|F\|_{X^{s,-b}} & \text{if } -\frac{1}{2}\leq s\leq \frac{1}{2} \\ \|F\|_{X^{s,-b}} + \|F\|_{X^{\frac{1}{2}+,\frac{2s-5}{8}}} & \text{if } s>\frac{1}{2}. \end{cases}$$

Now we move to the estimate on the derivative term where we take less time derivatives $\frac{2s+1}{8}$ while we have additional $i\xi$ factor coming from the spatial derivative. As before we divide the Duhamel integral into three pieces as follows

$$\begin{aligned} \eta(t)\partial_x\Big(\int_0^t W^{t-t'}F dt'\Big) &= \eta(t)\int_0^t \partial_x\big(W^{t-t'}F\big)dt' \\ &= \eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \xi e^{ix\xi}\frac{\big(e^{it\tau}-e^{it\xi^4}\big)\varphi(\tau-\xi^4)}{\tau-\xi^4}\widehat{F}(\xi,\tau)d\xi d\tau \\ &+ \eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \xi e^{ix\xi}\frac{e^{it\tau}\varphi^c(\tau-\xi^4)}{\tau-\xi^4}\widehat{F}(\xi,\tau)d\xi d\tau \\ &- \eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \xi e^{ix\xi}\frac{e^{it\xi^4}\varphi^c(\tau-\xi^4)}{\tau-\xi^4}\widehat{F}(\xi,\tau)d\xi d\tau \\ &=: \mathrm{I}^x + \mathrm{II}^x + \mathrm{III}^x. \end{aligned}$$

To bound I^x, note that on the region of integration we have $|\tau|\approx\xi^4$ hence the additional factor ξ leads to the situation $\langle\tau\rangle^{\frac{2s+1}{8}}|\xi|\lesssim\langle\tau\rangle^{\frac{2s+3}{8}}$ which was examined before for the integral I. In order to estimate III^x we divide the region of integration as before into pieces $|\xi|<1$ and $|\xi|\geq 1$. For the former case, the bounds are identical to those obtained for III, for the latter case, we make the same change of variable $\rho=\xi^4$ as done for III so that the additional factor of ξ contributes the additional factor of $|\rho|^{\frac{1}{4}}$ to the integral III^x that brings us back to the situation handled in bounding III. Nevertheless estimation for the term II^x needs verification. When $-\frac{1}{2}\leq s\leq\frac{1}{2}$, using Cauchy-Schwarz inequality we have the bound:

$$\begin{aligned} &\|\mathrm{II}^x\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \\ &\quad\lesssim \|\eta\|_{H^1}\left\|\langle\tau\rangle^{\frac{2s+1}{8}}\int_{|\tau-\xi^4|\geq 1}\frac{\xi}{\langle\tau-\xi^4\rangle}|\widehat{F}(\xi,\tau)|d\xi\right\|_{L^2_\tau} \\ &\quad\lesssim \Big[\int\langle\tau\rangle^{\frac{2s+1}{4}}\Big(\int\frac{\xi^2}{\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{2-2b}}d\xi\Big)\Big(\int\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{-2b}|\widehat{F}(\xi,\tau)|^2d\xi\Big)d\tau\Big]^{\frac{1}{2}} \end{aligned}$$

$$\lesssim \sup_{\tau} \left(\langle \tau \rangle^{\frac{2s+1}{4}} \int \frac{\xi^2}{\langle \xi \rangle^{2s} \langle \tau - \xi^4 \rangle^{2-2b}} d\xi \right) \|F\|_{X^{s,-b}} .$$

To see that the supremum above is bounded, we proceed as follows

$$\begin{aligned}
&\langle \tau \rangle^{\frac{2s+1}{4}} \Big[\int_{|\xi|<1} \frac{\xi^2}{\langle \tau - \xi^4 \rangle^{2-2b} \langle \xi \rangle^{2s}} d\xi + \int_{|\xi|\geq 1} \frac{\xi^2}{\langle \tau - \xi^4 \rangle^{2-2b} \langle \xi \rangle^{2s}} d\xi \Big] \\
&\qquad \lesssim \langle \tau \rangle^{\frac{2s+1}{4}} \Big[\langle \tau \rangle^{2b-2} + \int_{|\rho|\geq 1} \frac{|\rho|^{\frac{1}{2}}}{\langle \tau - \rho \rangle^{2-2b} \langle \rho \rangle^{\frac{2s+3}{4}}} d\rho \Big] \\
&\qquad \lesssim \langle \tau \rangle^{\frac{2s+1}{4}+2b-2} + \langle \tau \rangle^{\frac{2s+1}{4}} \langle \tau \rangle^{\frac{-2s-1}{4}} \lesssim 1
\end{aligned}$$

where we have used Lemma A.0.5 in the ρ-integral and the fact that $0 \leq \frac{2s+1}{4} \leq \frac{1}{2}$ with $b < \frac{1}{2}$. In the case $s > \frac{1}{2}$, we estimate II^x by

$$\|\mathrm{II}^x\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \left\| \langle \tau \rangle^{\frac{2s+1}{8}} \int \frac{|\xi|}{\langle \tau - \xi^4 \rangle} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau} .$$

We consider the cases $|\tau| \gtrsim \xi^4$ and $|\tau| \ll \xi^4$, in the first case, the above integral is bounded by

$$\left\| \langle \tau \rangle^{\frac{2s+3}{8}} \int \frac{|\widehat{F}(\xi, \tau)|}{\langle \tau - \xi^4 \rangle} d\xi \right\|_{L^2_\tau}$$

which was addressed before for II. For the second case notice that $|\tau - \xi^4| \approx \xi^4$ with which one has $|\xi| \lesssim \langle \tau - \xi^4 \rangle^{\frac{1}{4}}$. Thus we bound the integral by

$$\left\| \langle \tau \rangle^{\frac{2s+1}{8}} \int_{|\tau|\ll\xi^4} \frac{|\widehat{F}(\xi, \tau)|}{\langle \tau - \xi^4 \rangle^{\frac{3}{4}}} d\xi \right\|_{L^2_\tau} .$$

On the region where $|\tau| \ll \xi^4$, we have the relation

$$\langle \tau \rangle^{\frac{2s+1}{8}} \lesssim \langle \tau - \xi^4 \rangle^{\frac{2s+1}{8}} + |\xi|^{\frac{2s+1}{2}} \lesssim \langle \tau - \xi^4 \rangle^{\frac{2s+1}{8}}$$

through which we bound the above integral by

$$\left\| \int \langle \tau - \xi^4 \rangle^{\frac{2s-5}{8}} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau}$$

which was handled in bounding II. $\square$

Proposition 4.4.6. *For fixed $s > -\frac{1}{3}$ with $a < \min\{2s+1, 1\}$ and $\frac{1}{2}-b > 0$ sufficiently small, we have*

$$\|u_1\overline{u}_2u_3\|_{X^{s+a,-b}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}}.$$

Proof. Expressing the space time Fourier transform of $u_1\overline{u}_2u_3$ as a convolution

$$\mathcal{F}_{x,t}(u_1\overline{u}_2u_3)(\xi,\tau) = \int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} \widehat{u}_1(\xi_1,\tau_1)\overline{\widehat{u}_2(\xi_2,\tau_2)}\widehat{u}_3(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)$$

and then using the definition of $X^{s,b}$ norm we write

$$\|u_1\overline{u}_2u_3\|^2_{X^{s+a,-b}} = \left\|\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} \frac{\langle\xi\rangle^{s+a}\widehat{u}_1(\xi_1,\tau_1)\overline{\widehat{u}_2(\xi_2,\tau_2)}\widehat{u}_3(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)}{\langle\tau-\xi_4\rangle^b}\right\|^2_{L^2_{\xi,\tau}}$$

and define

$$f_j(\xi,\tau) = \langle\xi\rangle^s\langle\tau-\xi^4\rangle^b|\widehat{u}_j(\xi,\tau)| \text{ for } j=1,2,3.$$

Thus the desired bound is equivalent to showing that

$$\left\|\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)f_1(\xi_1,\tau_1)f_2(\xi_2,\tau_2)f_3(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\right\|^2_{L^2_{\xi,\tau}}$$
$$\lesssim \prod_{j=1}^{3}\|f_j\|^2_{L^2} = \prod_{j=1}^{3}\|u_j\|^2_{X^{s,b}}$$

where

$$M(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2) = \frac{\langle\xi\rangle^{s+a}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi-\xi_1+\xi_2\rangle^{-s}}{\langle\tau-\xi^4\rangle^b\langle\tau_1-\xi_1^4\rangle^b\langle\tau_2-\xi_2^4\rangle^b\langle\tau-\tau_1+\tau_2-(\xi-\xi_1+\xi_2)^4\rangle^b}.$$

By an application of the Cauchy-Schwarz inequality in the $\xi_1,\xi_2,\tau_1,\tau_2$ integrals and then using Hölder's and Young's inequalities respectively the norm above is majorized by

$$\left\|\Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\Big)^{\frac{1}{2}}\Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} f_1^2(\xi_1,\tau_1)f_2^2(\xi_2,\tau_2)f_3^2(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\Big)^{\frac{1}{2}}\right\|^2_{L^2_{\xi,\tau}}$$

$$= \left\| \Big(\int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} M^2 \Big) \Big(\int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} f_1^2(\xi_1,\tau_1) f_2^2(\xi_2,\tau_2) f_3^2(\xi-\xi_1+\xi_2, \tau-\tau_1+\tau_2) \Big) \right\|_{L^1_{\xi,\tau}}$$

$$\leq \sup_{\xi,\tau} \Big(\int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} M^2 \Big) \left\| \int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} f_1^2(\xi_1,\tau_1) f_2^2(\xi_2,\tau_2) f_3^2(\xi-\xi_1+\xi_2, \tau-\tau_1+\tau_2) \right\|_{L^1_{\xi,\tau}}$$

$$= \sup_{\xi,\tau} \Big(\int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} M^2 \Big) \| f_1^2 * f_2^2 * f_3^2 \|_{L^1_{\xi,\tau}} \lesssim \sup_{\xi,\tau} \Big(\int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} M^2 \Big) \prod_{j=1}^{3} \|f_j\|_{L^2}^2 .$$

Therefore, it suffices to show that the supremum above is finite. Application of Lemma A.0.5 in the τ_1, τ_2 integrals bounds the supremum by

$$\sup_{\xi,\tau} \int \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \tau - \xi^4 \rangle^{2b} \langle \tau - \xi_1^4 + \xi_2^4 - (\xi-\xi_1+\xi_2)^4 \rangle^{6b-2}} d\xi_1 d\xi_2.$$

Implementing the identity $\langle \alpha - \beta \rangle \lesssim \langle \tau - \alpha \rangle \langle \tau - \beta \rangle$ and then using Lemma A.0.2, this is bounded by

$$\sup_{\xi} \int \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \xi^4 - \xi_1^4 + \xi_2^4 - (\xi-\xi_1+\xi_2)^4 \rangle^{6b-2}} d\xi_1 d\xi_2$$
$$\lesssim \sup_{\xi} \int \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{1-}} d\xi_1 d\xi_2.$$

We divide the integration region into two pieces

$$R_1 = \{(\xi_1,\xi_2) : |\xi_1 - \xi| \ll 1 \,\text{or}\, |\xi_1 - \xi_2| \ll 1\}$$
$$R_2 = \{(\xi_1,\xi_2) : |\xi_1 - \xi| \gtrsim 1 \,\text{and}\, |\xi_1 - \xi_2| \gtrsim 1\}$$

to control the supremum. Clearly we have $\xi_1^2 + \xi_2^2 + \xi^2 \gtrsim 1$ on R_2, so the supremum on this region is estimated by

$$\int_{R_2} \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \xi_1^2 + \xi_2^2 + \xi^2 \rangle^{1-} \langle \xi_1 - \xi \rangle^{1-} \langle \xi_1 - \xi_2 \rangle^{1-}} d\xi_1 d\xi_2.$$

Since the sign of the Sobolev index s affects the way we follow in the proof, we begin with considering the case $s > 0$ first.

In this case, there are three separate cases to examine:

i) $|\xi - \xi_1 + \xi_2| \gtrsim |\xi|$

$$\int \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\xi_1^2+\xi_2^2+\xi^2\rangle^{1-}\langle\xi_1-\xi\rangle^{1-}\langle\xi_1-\xi_2\rangle^{1-}}d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle\xi\rangle^{2a-2+}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}}{\langle\xi_1-\xi\rangle^{1-}\langle\xi_2-\xi_1\rangle^{1-}}d\xi_2 d\xi_1$$
$$\lesssim \langle\xi\rangle^{2a-2+}\int \frac{\phi_{\max(2s,1-)}(\xi_1)}{\langle\xi_1-\xi\rangle^{1-}\langle\xi_1\rangle^{2s+\min(2s,1-)}}d\xi_1$$

where we have used the Lemma A.0.5 in the last line above. For $s \geq \frac{1}{2}$, using the Lemma A.0.5, this is bounded by

$$\langle\xi\rangle^{2a-2+}\int \frac{\log(1+\langle\xi_1\rangle)}{\langle\xi_1-\xi\rangle^{1-}\langle\xi_1\rangle^{2s+1-}}d\xi_1 \lesssim \langle\xi\rangle^{2a-3+} \lesssim 1$$

provided that $a < \frac{3}{2}$. As for $0 < s < \frac{1}{2}$, the Lemma A.0.5 yields the bound

$$\langle\xi\rangle^{2a-2+}\int \frac{d\xi_1}{\langle\xi_1-\xi\rangle^{1-}\langle\xi_1\rangle^{4s-}} \lesssim \begin{cases} \langle\xi\rangle^{2a-2-4s+} & \text{for } 0 < s \leq \frac{1}{4} \\ \langle\xi\rangle^{2a-3+} & \text{for } \frac{1}{4} < s < \frac{1}{2} \end{cases}$$

which is finite as long as $a < \min\{2s+1, \frac{3}{2}\}$.

ii) $|\xi_1| \gtrsim |\xi|$

$$\int \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\xi_1^2+\xi_2^2+\xi^2\rangle^{1-}\langle\xi_1-\xi\rangle^{1-}\langle\xi_1-\xi_2\rangle^{1-}}d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle\xi\rangle^{2a-2+}\langle\xi-\xi_1+\xi_2\rangle^{-2s}\langle\xi_2\rangle^{-2s}}{\langle\xi_1-\xi\rangle^{1-}\langle\xi_1-\xi_2\rangle^{1-}}d\xi_2 d\xi_1.$$

From the substitutions $x_1 = \xi - \xi_1 + \xi_2$ and $x_2 = \xi_2$ the integral above is replaced by

$$\int \frac{\langle\xi\rangle^{2a-2+}\langle x_1\rangle^{-2s}\langle x_2\rangle^{-2s}}{\langle x_1-\xi\rangle^{1-}\langle x_2-x_1\rangle^{1-}}dx_2 dx_1$$

which is identical to the integral estimated in the previous case.

iii) $|\xi_2| \gtrsim |\xi|$

$$\int \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\xi_1^2+\xi_2^2+\xi^2\rangle^{1-}\langle\xi_1-\xi\rangle^{1-}\langle\xi_1-\xi_2\rangle^{1-}}d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle\xi\rangle^{2a-2+}\langle\xi-\xi_1+\xi_2\rangle^{-2s}\langle\xi_1\rangle^{-2s}}{\langle\xi_1-\xi\rangle^{1-}\langle\xi_1-\xi_2\rangle^{1-}}d\xi_2 d\xi_1.$$

In this case after making change of variables $x_1 = \xi_1$ and $x_2 = \xi - \xi_1 + \xi_2$ in the above integral and then applying the Lemma A.0.5 we have the bound

$$\int \frac{\langle\xi\rangle^{2a-2+}\langle x_1\rangle^{-2s}\langle x_2\rangle^{-2s}}{\langle x_1-\xi\rangle^{1-}\langle x_2-\xi\rangle^{1-}}dx_1dx_2 = \langle\xi\rangle^{2a-2+}\Big(\int \frac{dx}{\langle x-\xi\rangle^{1-}\langle x\rangle^{2s}}\Big)^2$$
$$\lesssim \langle\xi\rangle^{2a-2-2\min(2s,1-)+}\phi^2_{\max(2s,1-)}(\xi) \lesssim \begin{cases} \langle\xi\rangle^{2a-2-4s+} & \text{for } 0 < s < \frac{1}{2} \\ \langle\xi\rangle^{2a-4+} & \text{for } s \geq \frac{1}{2} \end{cases}$$

which is bounded provided that $a < \min\{2s+1, 2\}$. Next we focus on the case $-\frac{1}{3} < s \leq 0$. In this case, since $\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s} \lesssim \langle\xi_1^2+\xi_2^2+\xi^2\rangle^{-3s}$

$$\int_{R_2} \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle(\xi_1^2+\xi_2^2+\xi^2)(\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}}d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle\xi\rangle^{2s+2a}}{\langle\xi_1^2+\xi_2^2+\xi^2\rangle^{1+3s-}\langle\xi_1-\xi\rangle^{1-}\langle\xi_1-\xi_2\rangle^{1-}}d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle\xi\rangle^{2s+2a}}{\langle\xi_2+\xi^2\rangle^{1+3s-}\langle\xi_1-\xi\rangle^{1-}\langle\xi_1-\xi_2\rangle^{1-}}d\xi_1 d\xi_2.$$

Since $\frac{1}{2} - b > 0$ was taken sufficiently small, using Lemma A.0.5 twice this integral is bounded by

$$\langle\xi\rangle^{2s+2a}\int \frac{d\xi_1}{\langle\xi_1+\xi^2\rangle^{1+3s-}\langle\xi_1-\xi\rangle^{1-}} \lesssim \langle\xi\rangle^{2a-4s-2+} \lesssim 1$$

provided that $a < 2s+1$. Next we move on estimating the supremum on the region R_1. In this region notice that

$$\langle\xi-\xi_1+\xi_2\rangle\langle\xi_1\rangle \approx \langle\xi_2\rangle\langle\xi\rangle.$$

Thus

$$\int_{R_1} \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle(\xi_1^2+\xi_2^2+\xi^2)(\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}}d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle\xi\rangle^{2a}\langle\xi_2\rangle^{-4s}}{\langle(\xi_2^2+\xi^2)(\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}}d\xi_1 d\xi_2.$$

Note that on R_1 the relation $\xi_2^2+\xi^2 \lesssim 1$ implies that $|\xi| \lesssim 1$, $|\xi_j| \lesssim 1$ for $j = 1, 2$ in which case the integral above turns out to be finite at once. So for a nontrivial situation we assume that $\xi_2^2+\xi^2 \gtrsim 1$. Then making the substitution $x = (\xi_2^2+\xi^2)(\xi_1-\xi_2)(\xi_1-\xi)$ in the ξ_1 integral and using the relations

$$2\xi_1 = \xi + \xi_2 \pm (\xi_2^2 + \xi^2)^{-\frac{1}{2}}\sqrt{4x + (\xi - \xi_2)^2(\xi_2^2 + \xi^2)} \text{ and } \frac{dx}{\xi_2^2 + \xi^2} = (2\xi_1 - \xi_2 - \xi)d\xi_1$$

along with the Lemma A.0.7, we have the bound

$$\int \frac{\langle\xi\rangle^{2a}\langle\xi_2\rangle^{-4s}}{\langle\xi_2^2 + \xi^2\rangle^{\frac{1}{2}}\langle x\rangle^{1-}\sqrt{|4x + (\xi_2 - \xi)^2(\xi_2^2 + \xi^2)|}}dxd\xi_2$$
$$\lesssim \int \frac{\langle\xi\rangle^{2a}\langle\xi_2\rangle^{-4s}}{\langle\xi_2^2 + \xi^2\rangle^{\frac{1}{2}}\langle(\xi_2 - \xi)^2(\xi_2^2 + \xi^2)\rangle^{\frac{1}{2}-}}d\xi_2.$$

We estimate the integral in the separate regions $|\xi_2| \lesssim 1$ and $|\xi_2| \gg 1$. In the former region this is bounded by

$$\int_{|\xi_2|\lesssim 1} \frac{\langle\xi\rangle^{2a}\langle\xi_2\rangle^{-4s}}{\langle\xi_2^2 + \xi^2\rangle^{\frac{1}{2}}\langle(\xi_2 - \xi)^2(\xi_2^2 + \xi^2)\rangle^{\frac{1}{2}-}}d\xi_2 \lesssim \langle\xi\rangle^{2a-3+} \lesssim 1$$

provided that $a < \frac{3}{2}$. As regards to the latter region, using the relation

$$(\xi_2 - \xi)^2(\xi_2^2 + \xi^2) \gtrsim (\xi_2^2 - \xi^2)^2$$

and then making the change of variable $x = \xi_2^2$ entails the bound

$$\begin{aligned}
\int_{|\xi_2|\gg 1} \frac{\langle\xi\rangle^{2a}\langle\xi_2\rangle^{-4s}}{\langle\xi_2^2 + \xi^2\rangle^{\frac{1}{2}}\langle(\xi_2 - \xi)^2(\xi_2^2 + \xi^2)\rangle^{\frac{1}{2}-}}d\xi_2 &\lesssim \int_{|\xi_2|\gg 1} \frac{\langle\xi\rangle^{2a}}{\langle\xi_2^2\rangle^{2s+\frac{1}{2}}\langle\xi_2^2 - \xi^2\rangle^{1-}}d\xi_2 \\
&\approx \int_{|x|\gg 1} \frac{\langle\xi\rangle^{2a}}{\langle x\rangle^{2s+\frac{1}{2}}\langle x - \xi^2\rangle^{1-}|x|^{\frac{1}{2}}}dx \\
&\lesssim \int \frac{\langle\xi\rangle^{2a}}{\langle x\rangle^{2s+1}\langle x - \xi^2\rangle^{1-}}dx \\
&\lesssim \begin{cases} \langle\xi\rangle^{2a-2-4s+} & \text{for } -\frac{1}{2} < s < 0 \\ \langle\xi\rangle^{2a-2+} & \text{for } s \geq 0 \end{cases}
\end{aligned}$$

which is finite provided that $a < \min\{1, 2s + 1\}$. $\square$

We take $\frac{2s+2a-1}{8} - b = \frac{2s+2a-5}{8} + (\frac{1}{2} - b)$ rather than $\frac{2s+2a-5}{8}$ in the Proposition 4.4.7 so as to extract a positive power of T in the contraction argument below in the local theory.

Proposition 4.4.7. *For fixed* $-\frac{1}{3} < s < \frac{9}{2}$, $0 \le a < \min\{1, 2s+1, \frac{9}{2} - s\}$, *and* $\frac{1}{2} - b > 0$ *sufficiently small, we have*

$$\text{for } -\frac{1}{3} < s + a \le \frac{1}{2},\ \|u_1\overline{u}_2u_3\|_{X^{s+a,-b}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}},$$

$$\text{for } \frac{1}{2} < s + a < \frac{9}{2},\ \|u_1\overline{u}_2u_3\|_{X^{\frac{1}{2}+,\frac{2s+2a-1}{8}-b}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}}.$$

Proof. When $-\frac{1}{3} < s + a \le \frac{1}{2}$, given statement follows from Proposition 4.4.6. So we only take account of the case $\frac{1}{2} < s + a \le \frac{9}{2}$ here. In this case, using the fact that $a < 2s + 1$ we take $s > -\frac{1}{6}$ all along. Next let

$$\mathrm{I} := \int \frac{\langle \tau - \xi^4 \rangle^{\frac{2s+2a-8b-1}{4}} \langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \tau - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 \rangle^{6b-2}} d\xi_1 d\xi_2.$$

Thus using the arguments of Proposition 4.4.6 we are required to show that

$$\sup_{\xi,\tau} \mathrm{I} < \infty.$$

We will demonstrate this in the separate cases $\frac{1}{2} < s + a < \frac{5}{2}$ and $\frac{5}{2} \le s + a < \frac{9}{2}$.

Case 1) $\frac{1}{2} < s + a < \frac{5}{2}$

Note that taking $\frac{1}{2} - b > 0$ sufficiently small we infer that $\frac{1}{2}(s + a) - 2b - \frac{1}{4} < 0$, also $s + a > \frac{1}{2}$ implies that $2b + \frac{1}{4} - \frac{1}{2}(s + a) < 6b - 2$. Hence using these, the identity $\langle \tau - a \rangle \langle \tau - b \rangle \gtrsim \langle a - b \rangle$ and then Lemma A.0.2 we have

$$\begin{aligned} \mathrm{I} &\lesssim \int \frac{\langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \xi^4 - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}} d\xi_1 d\xi_2 \\ &\lesssim \int \frac{\langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}} d\xi_1 d\xi_2 \end{aligned}$$

which is easily estimated, for $s > \frac{1}{2}$, by

$$\int \frac{\langle \xi \rangle^{1+}}{\langle \xi_1 \rangle^{2s} \langle \xi_2 \rangle^{2s} \langle \xi - \xi_1 + \xi_2 \rangle^{2s}} d\xi_1 d\xi_2 \lesssim \int \frac{\langle \xi \rangle^{1+}}{\langle \xi_2 \rangle^{2s} \langle \xi + \xi_2 \rangle^{2s}} d\xi_2 \lesssim \langle \xi \rangle^{1-2s+} \lesssim 1$$

by using Lemma A.0.5 twice. It is left to treat the case $-\frac{1}{6} < s \le \frac{1}{2}$. For this case, we will analyze the integral on the sets $\mathrm{R}_1 = \{(\xi_1, \xi_2) : |\xi_1 - \xi| \ll 1 \,\text{or}\, |\xi_1 - \xi_2| \ll 1\}$ and $\mathrm{R}_2 = \{(\xi_1, \xi_2) : |\xi_1 - \xi| \gtrsim 1 \,\text{and}\, |\xi_1 - \xi_2| \gtrsim 1\}$ as before. Recalling the identity $\langle \xi - \xi_1 + \xi_2 \rangle \langle \xi_1 \rangle \approx \langle \xi \rangle \langle \xi_2 \rangle$ that holds on the set R_1, we have the bound

$$\int_{R_1} \frac{\langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_1 d\xi_2 \lesssim \int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle (\xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2)) \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_1 d\xi_2.$$

Making substitution $x = (\xi^2 + \xi_2^2)(\xi_1 - \xi)(\xi_1 - \xi_2)$ in the ξ_1 integral and assuming $\xi^2 + \xi_2^2 \gtrsim 1$ as in the Proposition 4.4.6, the integral above is bounded by

$$\int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle x \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)} \sqrt{|4x + (\xi - \xi_2)^2 (\xi^2 + \xi_2^2)|}} dx d\xi_2 \lesssim \int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle (\xi - \xi_2)^2 (\xi^2 + \xi_2^2) \rangle^{2b - \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_2$$

where we have used Lemma A.0.5 which is applicable due to the fact that $\frac{1}{2} - b > 0$ is sufficiently small, and $a < \min\{2s+1, 1\}$. So we estimate this by

$$\int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle (\xi - \xi_2)^2 (\xi + \xi_2)^2 \rangle^{2b - \frac{1}{4} - \frac{1}{2}(s+a)}} \mathrm{d}\xi_2 \lesssim \int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2^2 \rangle^{-2s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle \xi_2^2 - \xi^2 \rangle^{4b - \frac{1}{2} - s - a}} \mathrm{d}\xi_2.$$

In the case $|\xi_2| \lesssim 1$, the integral is bounded by

$$\langle \xi \rangle^{1+2a-8b+} \lesssim 1$$

provided that $a < \frac{3}{2}$, whereas for the other case $|\xi_2| \gg 1$, we change variable $x = \xi_2^2$ to estimate the integral, using Lemma A.0.5, by

$$\int \frac{\langle \xi \rangle^{1-2s+}}{\langle x \rangle^{1+2s} \langle x - \xi^2 \rangle^{4b - \frac{1}{2} - s - a}} dx \lesssim \begin{cases} \langle \xi \rangle^{2+2a-8b+} & \text{for } 0 \le s \le \frac{1}{2} \\ \langle \xi \rangle^{2-4s+2a-8b+} & \text{for } -\frac{1}{6} < s < 0 \end{cases}$$

which is bounded since $a < \min\{2s+1, 1\}$ and $\frac{1}{2} - b > 0$ is sufficiently small.

Next we estimate the integral on the set R_2 by

$$\int \frac{\langle\xi\rangle^{1+}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\xi_1^2+\xi_2^2+\xi^2\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}\langle\xi_1-\xi\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}\langle\xi_1-\xi_2\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}}d\xi_1 d\xi_2.$$

We bound this in the separate cases $-\frac{1}{6}<s\leq 0$ and $0<s\leq\frac{1}{2}$. In the former case, using the identity

$$\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}\lesssim\langle\xi_1^2+\xi_2^2+\xi^2\rangle^{-3s}$$

we obtain the bound

$$\int \frac{\langle\xi\rangle^{1+}\langle\xi_1-\xi\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}\langle\xi_1-\xi_2\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}}{\langle\xi_1\rangle^{2b+\frac{1}{4}+\frac{1}{2}(5s-a)}\langle\xi-\xi_1+\xi_2\rangle^{2b+\frac{1}{4}+\frac{1}{2}(5s-a)}}d\xi_1 d\xi_2.$$

By a change of variable $\xi_2\to\xi_1+\xi_2$, $\xi_1\to\xi_1+\xi$, it suffices to estimate

$$\langle\xi\rangle^{1+}\Big(\int\frac{d\xi_1}{\langle\xi_1+\xi\rangle^{2b+\frac{1}{4}+\frac{1}{2}(5s-a)}\langle\xi_1\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}}\Big)^2.$$

Nothing that $2b+\frac{1}{4}-\frac{1}{2}(s+a)<1$ and $2b+\frac{1}{4}+\frac{1}{2}(5s-a)<1$ and then applying Lemma A.0.5 this is bounded by

$$\langle\xi\rangle^{2-8b-4s+2a+},$$

which is finite for $a<\min\{2s+1,1\}$ and $\frac{1}{2}-b>0$ sufficiently small. Now for the latter case $0<s\leq\frac{1}{2}$, after using the bound $\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}\lesssim\langle\xi\rangle^{-2s}$ and applying the same change of variables as above, the integral is bounded by

$$\int\frac{\langle\xi\rangle^{1-2s+}}{\langle\xi_1\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}\langle\xi_1+\xi\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}\langle\xi_2\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}\langle\xi_2+\xi\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}}d\xi_1 d\xi_2$$
$$\lesssim\langle\xi\rangle^{1-2s+}\Big(\int\frac{d\xi_1}{\langle\xi_1\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}\langle\xi_1+\xi\rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}}\Big)^2\lesssim\langle\xi\rangle^{2-8b+2a+}\lesssim 1$$

by using the Lemma A.0.5, $a<\min\{2s+1,1\}$ and the assumption that $\frac{1}{2}-b>0$ is sufficiently small.

Case 2) $\frac{5}{2}\leq s+a<\frac{9}{2}$

Note in this case $0\leq\frac{1}{2}(s+a)-2b-\frac{1}{4}<6b-2$. Making use of the proof of Lemma A.0.2, we write

$$\xi^4-\xi_1^4+\xi_2^4-(\xi-\xi_1+\xi_2)^4$$

$$= (\xi-\xi_1)(\xi_1-\xi_2)\Big[\frac{5}{2}(\xi+\xi_2)^2+\xi^2+\xi_2^2+2(\xi_1-\frac{1}{2}\xi-\frac{1}{2}\xi_2)^2\Big]$$
$$=: g(\xi,\xi_1,\xi_2).$$

Therefore, we have

$$\begin{aligned}\langle\tau-\xi^4\rangle &= \langle\tau-\xi_1^4+\xi_2^4-(\xi-\xi_1+\xi_2)^4-g(\xi,\xi_1,\xi_2)\rangle\\ &\lesssim \langle\tau-\xi_1^4+\xi_2^4-(\xi-\xi_1+\xi_2)^4\rangle+\langle g(\xi,\xi_1,\xi_2)\rangle\\ &\lesssim \langle\tau-\xi_1^4+\xi_2^4-(\xi-\xi_1+\xi_2)^4\rangle+\langle\xi_1\rangle^2\langle\xi_2\rangle^2\langle\xi\rangle^2\langle\xi_1-\xi\rangle\langle\xi_1-\xi_2\rangle.\end{aligned}$$

From this identity we obtain

$$\mathrm{I}\lesssim\int\frac{\langle\xi_1-\xi\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}\langle\xi_1-\xi_2\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}\langle\xi\rangle^{\frac{1}{2}+s+a-4b+}}{\langle\xi_1\rangle^{s-a+\frac{1}{2}+4b}\langle\xi_2\rangle^{s-a+\frac{1}{2}+4b}\langle\xi-\xi_1+\xi_2\rangle^{2s}}d\xi_1d\xi_2.$$

Substitutions $\xi_2\to\xi_2+\xi_1$ and $\xi_1\to\xi_1+\xi$ in the above integral lead to

$$\begin{aligned}&\langle\xi\rangle^{\frac{1}{2}+s+a-4b+}\int\frac{\langle\xi_1\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}\langle\xi_2\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}}{\langle\xi_1+\xi\rangle^{s-a+\frac{1}{2}+4b}\langle\xi_1+\xi_2+\xi\rangle^{s-a+\frac{1}{2}+4b}\langle\xi+\xi_2\rangle^{2s}}d\xi_1d\xi_2\\ &\lesssim\langle\xi\rangle^{\frac{1}{2}+s+a-4b+}\int\frac{\langle\xi_1\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}\langle\xi_2\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}}{\langle\xi_1+\xi\rangle^{s-a+\frac{1}{2}+4b}\langle\xi_1+\xi_2+\xi\rangle^{s-a+\frac{1}{2}+4b}\langle\xi+\xi_2\rangle^{s-a+\frac{1}{2}+4b}}d\xi_1d\xi_2\end{aligned}$$

where we have used $s+a-4b-\frac{1}{2}\geq 0$ in the last line above. Since $a<\min\{2s+1,1\}$, we note that $s>\frac{3}{2}$. Now by symmetry we have two cases $|\xi+\xi_1+\xi_2|\gtrsim|\xi|$ and $|\xi+\xi_1|\gtrsim|\xi|$ to consider. For the first one, using $\langle\xi_1\rangle\lesssim\langle\xi_1+\xi\rangle\langle\xi\rangle$ we have the bound

$$\langle\xi\rangle^{2a-8b+}\Big(\int\langle\xi_1\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}\langle\xi_1+\xi\rangle^{-s+a-\frac{1}{2}-4b}d\xi_1\Big)^2\lesssim\langle\xi\rangle^{3a+s-\frac{1}{2}-12b+}\lesssim 1$$

owing to the the restrictions on a, b and s. For the second one, the integral is bounded by

$$\langle\xi\rangle^{2a-8b+}\int\frac{\langle\xi_1\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}\langle\xi_2\rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}}{\langle\xi_2+\xi\rangle^{s-a+\frac{1}{2}+4b}\langle\xi_1+\xi_2+\xi\rangle^{s-a+\frac{1}{2}+4b}}d\xi_1d\xi_2.$$

The inequalities $\langle\xi_1\rangle\lesssim\langle\xi_1+\xi_2+\xi\rangle\langle\xi_2+\xi\rangle$ and $\langle\xi_2\rangle\lesssim\langle\xi_2+\xi\rangle\langle\xi\rangle$ give rise to the bound

$$\langle\xi\rangle^{\frac{s}{2}+\frac{5a}{2}-\frac{1}{4}-10b+}\int\langle\xi_2+\xi\rangle^{2a-8b-1}\langle\xi_1+\xi_2+\xi\rangle^{-\frac{s}{2}+\frac{3a}{2}-\frac{3}{4}-6b}d\xi_1d\xi_2$$

which can be easily verified to be finite by the restrictions on a, b and s. $\square$

4.5. Local Theory: The Proof of Theorem 4.1.2

In this section, we establish the local existence of solutions to (4.18). Firstly we aim to show that Γ defined by

$$\Gamma u(t) = \eta(t)W^t g_e + \eta(t)\int_0^t W^{t-s}F(u)ds + \eta(t)W_0^t(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2) \quad (4.18)$$

has a fixed point in the space $X^{s,b}$, and recall where $g_e \in H^s(\mathbb{R})$ is the extension of g such that $\|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}$ and

$$F(u) = \eta(t/T)|u|^2u, \qquad p_1(t) = \eta(t)D_0(W^t g_e), \qquad p_2(t) = \eta(t)D_0\big(\partial_x\big(W^t g_e\big)\big),$$
$$q_1(t) = \eta(t)D_0\Big(\int_0^t W^{t-t'}F(u)dt'\Big), \qquad q_2(t) = \eta(t)D_0\Big(\partial_x\Big[\int_0^t W^{t-t'}F(u)dt'\Big]\Big).$$

We also recall that $s \in (-\frac{1}{3}, \frac{9}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$ and $\frac{1}{2} - b > 0$ is sufficiently small. We start with showing that Γ is a bounded operator on $X^{s,b}$. To do so, we gather necessary bounds we have so far. Using (4.11) we have

$$\left\|\eta(t)W^t g_e\right\|_{X^{s,b}} \lesssim \|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}.$$

Next by (4.13), (4.12) followed by Proposition 4.4.6 we obtain

$$\begin{aligned}\left\|\eta\int_0^t W^{t-s}F(u)ds\right\|_{X^{s,b}} &\leq \left\|\eta\int_0^t W^{t-s}F(u)ds\right\|_{X^{s,\frac{1}{2}+}} \\ &\lesssim \|F(u)\|_{X^{s,-\frac{1}{2}+}} \\ &\lesssim T^{\frac{1}{2}-b-}\left\||u|^2u\right\|_{X^{s,-b}} \\ &\lesssim T^{\frac{1}{2}-b-}\|u\|^3_{X^{s,b}}.\end{aligned}$$

By using Proposition 4.4.2 and Lemma 4.2.2, we have

$$\begin{aligned}&\|\eta(t)W_0^t(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2)\|_{X^{s,b}} \\ &\qquad\lesssim \left\|\chi_{(0,\infty)}(h_1 - p_1 - q_1)\right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \left\|\chi_{(0,\infty)}(h_2 - p_2 - q_2)\right\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}\end{aligned}$$

$$\lesssim \|h_1 - p_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2 - p_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + \|q_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|q_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)}$$

$$\lesssim \|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + \|p_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \|p_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}$$

$$+ \|q_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \|q_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} .$$

By the Kato smoothing estimate

$$\|p_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \|p_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)} .$$

Moreover, to bound the q_i norms, we use Proposition 4.4.5, (4.12) and Proposition 4.4.7 to get

$$\begin{aligned}
&\|q_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \|q_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} \\
&\qquad \lesssim \begin{cases} \|F\|_{X^{s,-\frac{1}{2}+}} & \text{for } -\frac{1}{3} < s \leq \frac{1}{2} \\ \|F\|_{X^{s,-\frac{1}{2}+}} + \|F\|_{X^{\frac{1}{2}+,\frac{2s-5}{8}+}} & \text{for } \frac{1}{2} < s < \frac{9}{2} \end{cases} \\
&\qquad \lesssim T^{\frac{1}{2}-b-} \begin{cases} \||u|^2u\|_{X^{s,-b}} & \text{for } -\frac{1}{3} < s \leq \frac{1}{2} \\ \||u|^2u\|_{X^{s,-b}} + \||u|^2u\|_{X^{\frac{1}{2}+,\frac{2s-1}{8}-b}} & \text{for } \frac{1}{2} < s < \frac{9}{2} \end{cases} \\
&\qquad \lesssim T^{\frac{1}{2}-b-} \|u\|_{X^{s,b}}^3 .
\end{aligned}$$

Putting these estimates together for (4.18), we arrive at

$$\|\Gamma u\|_{X^{s,b}} \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|u\|_{X^{s,b}}^3 .$$

Having shown that Γ is bounded, our next objective is to reveal that Γ is indeed a contraction. To achieve this, we implement the similar calculations for the difference $\Gamma u - \Gamma\tilde{u}$ as follows:

$$\begin{aligned}
&\|\Gamma u - \Gamma\tilde{u}\|_{X^{s,b}} \\
&\leq \left\|\eta \int_0^t W^{t-s}[F(u) - F(\widetilde{u})]ds\right\|_{X^{s,b}} + \left\|\eta W_0^t(0, \widetilde{q}_1 - q_1, \widetilde{q}_2 - q_2)\right\|_{X^{s,b}} \\
&\lesssim \left\|\eta \int_0^t W^{t-s}[F(u) - F(\widetilde{u})]ds\right\|_{X^{s,\frac{1}{2}+}} + \left\|\eta \int_0^t W^{t-s}[F(u) - F(\widetilde{u})]ds\right\|_{L_x^\infty H_t^{\frac{2s+3}{8}}}
\end{aligned}$$

$$+\left\|\eta\partial_x\Big(\int_0^t W^{t-s}[F(u)-F(\widetilde{u})]ds\Big)\right\|_{L_x^\infty H_t^{\frac{2s+1}{8}}}$$
$$\lesssim T^{\frac{1}{2}-b-}\big(\left\||u|^2u-|\widetilde{u}|^2\widetilde{u}\right\|_{X^{s,-b}}+\chi_{(\frac{1}{2},\frac{9}{2})}(s)\left\||u|^2u-|\widetilde{u}|^2\widetilde{u}\right\|_{X^{\frac{1}{2}+,\frac{2s-1}{8}-b}}\big)$$
$$\lesssim T^{\frac{1}{2}-b-}\big(\|u\|_{X^{s,b}}^2+\|\widetilde{u}\|_{X^{s,b}}^2\big)\|u-\widetilde{u}\|_{X^{s,b}}.$$

In the last line, we have used Proposition 4.4.7 along with the inequality

$$||f|^\alpha f-|g|^\alpha g|\leq C(|f|^\alpha+|g|^\alpha)|f-g|$$

for some absolute constant C and $\alpha\geq 0$. Therefore, taking $0<T<1$ sufficiently small, Γ is a contraction on the ball

$$B=\big\{u\in X^{s,b}:\|u\|_{X^{s,b}}\leq C\big(\|g\|_{H^s(\mathbb{R}^+)}+\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}+\|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}\big)\big\}$$

with radius depending on the initial and boundary data. Hence by the Banach fixed point theorem, this ensures the existence of a solution to (4.1) in $X^{s,b}$ spaces. Next we establish that the fixed point of Γ lies in $C_t^0H_x^s([0,T]\times\mathbb{R})$. Since the operator $W^t=e^{it\Delta^2}$ is unitary on $H^s(\mathbb{R})$ we have

$$\left\|\eta W^tg_e\right\|_{C_t^0H_x^s}\lesssim\|g_e\|_{H^s(\mathbb{R})}\lesssim\|g\|_{H^s(\mathbb{R}^+)}.$$

By the embedding (4.10) and the contraction argument

$$\left\|\eta\int_0^t W^{t-s}F(u)ds\right\|_{C_t^0H_x^s}\lesssim\left\|\eta\int_0^t W^{t-s}F(u)ds\right\|_{X^{s,\frac{1}{2}+}}\lesssim\cdots\lesssim T^{\frac{1}{2}-b-}\|u\|_{X^{s,b}}^3.$$

Next from Lemma 4.4.3 and the previous estimates in the contraction argument

$$\left\|\eta W_0^t(0,h_1-p_1-q_1,h_2-p_2-q_2)\right\|_{C_t^0H_x^s}$$
$$\lesssim\left\|\chi_{(0,\infty)}(h_1-p_1-q_1)\right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})}+\left\|\chi_{(0,\infty)}(h_2-p_2-q_2)\right\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}$$
$$\lesssim\cdots\lesssim\|g\|_{H^s(\mathbb{R}^+)}+\|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)}+\|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)}+T^{\frac{1}{2}-b-}\|u\|_{X^{s,b}}^3.$$

We also show that $u=\Gamma u$ belongs to the space $C_x^0H_t^{\frac{2s+3}{8}}([0,T]\times\mathbb{R})$. We have already obtained the following bounds in the contraction argument

$$\left\|\eta W^tg_e\right\|_{C_x^0H_t^{\frac{2s+3}{8}}}\lesssim\|g_e\|_{H^s(\mathbb{R})}\lesssim\|g\|_{H^s(\mathbb{R}^+)},$$

$$\left\| \eta \int_0^t W^{t-s} F(u) ds \right\|_{C_x^0 H_t^{\frac{2s+3}{8}}} \lesssim T^{\frac{1}{2}-b-} \|u\|_{X^{s,b}}^3 .$$

For the remaining term of Γ we exploit Lemma 4.4.4 and the contraction argument to get

$$\begin{aligned} &\|\eta W_0^t(0,h_1 - p_1 - q_1, h_2 - p_2 - q_2)\|_{C_x^0 H_t^{\frac{2s+3}{8}}} \\ &\qquad \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|u\|_{X^{s,b}}^3 . \end{aligned}$$

As a result we have established that $u = \Gamma u$ lies in the Banach space of the definition 4.1.1. Therefore this finishes the proof of local existence of solutions to (4.1). The uniqueness of these solutions will be treated in the subsequent Section 4.5.1 below. The continuous dependence of these local solutions on the initial and boundary data follows from the fixed point argument and the a priori estimates as well. To see this let u and u_n be solutions of (4.1) with initial and boundary data g, h_1, h_2 and g_n, h_{n1}, h_{n2} respectively. Then from what we have already shown in the contraction argument, we have

$$\begin{aligned} \|u - u_n\|_{X^{s,b}} \leq C_0\big(&\|g - g_n\|_{H^s(\mathbb{R}^+)} + \|h_1 - h_{n1}\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2 - h_{n2}\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big) \\ &+ C_1 T^{\frac{1}{2}-b-} \|u - u_n\|_{X^{s,b}} \end{aligned}$$

where $C_0 > 0$ is a positive constant and C_1 depends on the radius of the ball in the fixed point argument and hence on the initial and boundary data. By means of contraction argument, we may take existence time $T < 1$ so that $C_1 T^{\frac{1}{2}-b-} < 1$. So by the inequality

$$\begin{aligned} \|u - u_n\|_{X^{s,b}} \leq \frac{C_0}{\big(1 - C_1 T^{\frac{1}{2}-b-}\big)} \big(&\|g - g_n\|_{H^s(\mathbb{R}^+)} + \|h_1 - h_{n1}\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} \\ &+ \|h_2 - h_{n2}\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big) \end{aligned}$$

the continuous dependence in $X^{s,b}$ follows. In a similar manner, we prove the continuous dependence in the spaces $C_t^0 H_x^s$ and $C_x^0 H_t^{\frac{2s+3}{8}}$ as well. In order to complete the proof of the Theorem 4.1.2, it is left to establish the quantification of the dependence of existence time T to the initial and boundary data. By a scaling argument, we easily see that if u solves the equation (4.1) with data g, h_1 and h_2 on $[0, \lambda^{-4}]$, then $u^\lambda(x,t) = \lambda^{-2}u(\lambda^{-1}x, \lambda^{-4}t)$ solves the equation (4.1) with data $g^\lambda(x) = \lambda^{-2}g(\lambda^{-1}x)$, $h_1^\lambda(t) = \lambda^{-2}h_1(\lambda^{-4}t)$ and $h_2^\lambda(t) = \lambda^{-3}h_1(\lambda^{-4}t)$ on $[0,1]$. Therefore, for $\lambda > 1$,

$$\begin{aligned}
\left\|h_1^\lambda\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} &\lesssim \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}, \\
\left\|h_2^\lambda\right\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} &\lesssim \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}.
\end{aligned}$$

Furthermore, (for $s \geq 0$)

$$\begin{aligned}
\left\|g^\lambda\right\|_{H^s(\mathbb{R}^+)} &\leq \left\|g^\lambda\right\|_{L^2(\mathbb{R}^+)} + \left\|g^\lambda\right\|_{\dot{H}^s(\mathbb{R}^+)} \\
&\leq \lambda^{-\frac{3}{2}} \|g\|_{L^2(\mathbb{R}^+)} + \lambda^{-\frac{3}{2}-s} \|g\|_{\dot{H}^s(\mathbb{R}^+)} \\
&\leq \|g\|_{L^2} + \lambda^{-\frac{3}{2}-s} \|g\|_{H^s(\mathbb{R}^+)}.
\end{aligned}$$

Then for $\lambda^{-\frac{3}{2}-s} \|g\|_{H^s(\mathbb{R}^+)} \approx 1$, the solution is defined up to the local existence time

$$T \approx \left(C + \|g\|_{H^s(\mathbb{R}^+)}\right)^{-\frac{8}{2s+3}}$$

where the constant C depends on $\|g\|_{L^2} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$. Moreover, in order to have local existence interval without implicit dependence on $\|g\|_{L^2}$ (to be used later in Section 4.6), we make use of the following bound

$$\left\|g^\lambda\right\|_{H^s} \leq \left\|g^\lambda\right\|_{L^2} + \left\|g^\lambda\right\|_{\dot{H}^s} \leq \lambda^{-\frac{3}{2}} \|g\|_{L^2} + \lambda^{-\frac{3}{2}-s} \|g\|_{\dot{H}^s} \leq \lambda^{-\frac{3}{2}} \|g\|_{H^s}$$

that gives rise to the local existence time $T \approx \left(C + \|g\|_{H^s}\right)^{-\frac{8}{3}}$, in this case, with constant C dependent to $\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$.

4.5.1. Uniqueness of Solutions

In this section, we exhibit that the solutions to the equation (4.1) constructed above are unique. The uniqueness statement of the Theorem 4.1.2 for $s > \frac{1}{2}$ follows from an energy argument which we want to illustrate next, and then using the smoothing theorem we will extend the uniqueness argument to the whole well-posedness range. Hence first consider the smooth solutions u and v of (4.1) with sufficient decay. Then using $u(x,0) = v(x,0)$, $u(0,t) = v(0,t)$ and $u_x(0,t) = v_x(0,t)$, we compute

$$\partial_t \|u - v\|^2_{L^2(\mathbb{R}^+)} = 2\,\mathrm{Re}\, i\mu \int_0^\infty \big(|u|^2 u - |v|^2 v\big)\overline{(u-v)}dx,$$

hence, for any $t > 0$, integrating this and then using Sobolev embedding $H^s(\mathbb{R}^+) \subset L^\infty(\mathbb{R}^+)$, $s > 1/2$, we obtain

$$\begin{aligned}\|(u-v)(t)\|^2_{L^2_x(\mathbb{R}^+)} &\leq 2\big(\, \|u\|^2_{L^\infty_{t\in[0,T]} L^\infty_x(\mathbb{R}^+)} + \|v\|^2_{L^\infty_{t\in[0,T]} L^\infty_x(\mathbb{R}^+)}\,\big) \int_0^t \|(u-v)(s)\|^2_{L^2_x(\mathbb{R}^+)}\, ds \\ &\lesssim \big(\, \|u\|^2_{L^\infty_{t\in[0,T]} H^s_x(\mathbb{R}^+)} + \|v\|^2_{L^\infty_{t\in[0,T]} H^s_x(\mathbb{R}^+)}\,\big) \int_0^t \|(u-v)(s)\|^2_{L^2_x(\mathbb{R}^+)}\, ds.\end{aligned}$$

Since, by the local theory, the solutions u and v belong to $C^0_t H^s_x([0,T] \times \mathbb{R}^+)$, this with the Gronwall's inequality imply that $u = v$. The uniqueness of rougher solutions follows from taking convolution of $u - v$ with smooth approximate identities and then carrying out a limiting argument as usual, see for instance [70]. Also since the norms are taken on $\mathbb{R}^+$ in the energy estimate above, the restriction of solution to the right half line is independent of the choice of extension of the initial data. Next we will prove the uniqueness of the local solutions in the case $s \in (-\frac{1}{3}, \frac{1}{2})$ by utilizing the uniqueness obtained above for $s > \frac{1}{2}$ and the smoothing estimate from Theorem 4.1.3. Here we follow the arguments of [68]. We get started by considering data (g, h_1, h_2) in $H^s_x(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ for $s \in (0, \frac{1}{2})$. Let g_e and $\widetilde{g_e}$ be two $H^s(\mathbb{R})$ extensions of $g \in H^s(\mathbb{R}^+)$. Associated to these extensions let u and $\widetilde{u}$ be the fixed points of Γ defined in (4.18). Next pick a sequence $g^k \in H^{\frac{1}{2}+}(\mathbb{R}^+)$ converging to g in $H^s(\mathbb{R}^+)$.

Then, by Lemma 4.5.1 below, we may assume that g_e^k and $\widetilde{g_e}^k$ are $H^{\frac{1}{2}+}(\mathbb{R})$ extensions of g^k that converge respectively to g_e and $\widetilde{g_e}$ in $H^r(\mathbb{R})$ for $r < s < \frac{1}{2}$. Running a contraction argument on the set $B_1 \cap B_2$ where

$$B_1 = \left\{u : \|u\|_{X^{\frac{1}{2}+,b}} \le C\big(\left\|g^k\right\|_{H^{\frac{1}{2}+}(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{1}{2}+}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{1}{4}+}(\mathbb{R}^+)} \big)\right\}$$
$$B_2 = \left\{u : \|u\|_{X^{s,b}} \le C\big(\|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big)\right\}$$

we construct $H^{\frac{1}{2}+}(\mathbb{R})$ solutions u^k and $\widetilde{u}^k$ to the equation (4.1) associated to the extensions g_e^k and $\widetilde{g_e}^k$ respectively. At this juncture we make use of the smoothing estimate of Theorem 4.1.3 to obtain local existence time

$$T = T\big(\|g\|_{H^s(\mathbb{R}^+)}, \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}, \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big)$$

for $s < \frac{1}{2}$. By the uniqueness of $H^{\frac{1}{2}+}$ solutions obtained above, the restrictions of solutions u^k and $\widetilde{u}^k$ to $\mathbb{R}^+$ are the same. Since, by the fixed point argument, $u^k \to u$ and $\widetilde{u}^k \to \widetilde{u}$ in $H^{s-}(\mathbb{R})$, we then have $u|_{\mathbb{R}^+} = \widetilde{u}|_{\mathbb{R}^+}$. Iterating this argument the uniqueness for $s > -\frac{1}{3}$ follows.

Lemma 4.5.1 (See [69])**.** *Fix $-\frac{1}{2} < s < \frac{1}{2}$ and $k > s$. Let $f \in H^s(\mathbb{R}^+)$ and $g \in H^k(\mathbb{R}^+)$. Let f^e be an H^s extension of f to $\mathbb{R}$. Then there is an H^k extension g^e of g to $\mathbb{R}$ such that*

$$\|f^e - g^e\|_{H^r(\mathbb{R})} \lesssim \|f - g\|_{H^s(\mathbb{R}^+)} \qquad \text{for } r < s.$$

4.6. Proofs of Theorem 4.1.3 and Theorem 4.1.4

Proof of Theorem 4.1.3. By (4.18), for $t \in [0,T]$ we write the difference of nonlinear and linear solutions as

$$u(t) - W_0^t(0, h_1 - p_1, h_2 - p_2)(t) = \eta(t) \int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt' - \eta(t) W_0^t(0, q_1, q_2)(t)$$

where

$$q_1(t) = \eta(t) D_0\Big(\int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt' \Big),$$
$$q_2(t) = \eta(t) D_0\Big(\partial_x \Big[\int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt' \Big] \Big).$$

Therefore using the embedding $X^{s,\frac{1}{2}+} \subset C_t^0 H_x^s$ in (4.10), (4.13), Lemma 4.4.3 and then Proposition 4.4.5, we have

$$\begin{aligned}
&\|u - W_0^t(0, h_1 - p_1, h_2 - p_2)\|_{C^0_{t\in[0,T]} H^{s+a}_{x\in\mathbb{R}^+}} \\
&\lesssim \Big\| \eta \int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt' \Big\|_{X^{s+a,\frac{1}{2}+}} + \big\| W_0^t(0, q_1, q_2) \big\|_{C_t^0 H_x^{s+a}} \\
&\lesssim \big\| \eta |u|^2 u \big\|_{X^{s+a,-\frac{1}{2}+}} + \|q_1\|_{H_t^{\frac{2s+2a+3}{8}}} + \|q_2\|_{H_t^{\frac{2s+2a+1}{8}}} \\
&\lesssim \big\| \eta |u|^2 u \big\|_{X^{s+a,-\frac{1}{2}+}} + \begin{cases} \big\| \eta |u|^2 u \big\|_{X^{s+a,-\frac{1}{2}+}} & \text{for } -\frac{1}{3} < s+a \le \frac{1}{2} \\ \big\| \eta |u|^2 u \big\|_{X^{s+a,-\frac{1}{2}+}} + \big\| \eta |u|^2 u \big\|_{X^{\frac{1}{2}+,\frac{2s+2a-5}{8}}} & \text{for } \frac{1}{2} < s+a < \frac{9}{2}. \end{cases}
\end{aligned}$$

By Propositions 4.4.6, Proposition 4.4.7 and Theorem 4.1.2 along with the local theory, this is bounded by

$$\|u\|^3_{X^{s,b}} \lesssim \big(\|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big)^3,$$

so the claim follows. □

Proof of Theorem 4.1.4. Fix $T > 0$ and assume the growth bound $\|u\|_{H^s(\mathbb{R}^+)} \le f(T)$ for f depending on $\|g\|_{H^s(\mathbb{R}^+)}$, $\|h_1\|_{H^{s_1}(\mathbb{R}^+)}$ and $\|h_2\|_{H^{s_2}(\mathbb{R}^+)}$, for some $s_1 \ge \frac{2s+3}{8}$, $s_2 \ge \frac{2s+1}{8}$. Using the final claim of the proof of Theorem 4.1.2, we may pick the local existence time based on $f(T)$: $\delta \approx (C + f(T))^{-\frac{8}{3}}$ where C is a constant proportional to $\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$. Therefore for $J \approx T/\delta$

$$\big\| u(J\delta) - W_0^{J\delta}(g, h_1, h_2) \big\|_{H^{s+a}_{x\in\mathbb{R}^+}}$$

$$
\begin{aligned}
&= \left\| \sum_{k=1}^{J} W_{k\delta}^{J\delta}(u(k\delta), h_1, h_2) - W_{(k-1)\delta}^{J\delta}(u((k-1)\delta), h_1, h_2) \right\|_{H^{s+a}_{x\in\mathbb{R}^+}} \\
&\le \sum_{k=1}^{J} \left\| W_{k\delta}^{J\delta}(u(k\delta), h_1, h_2) - W_{(k-1)\delta}^{J\delta}(u((k-1)\delta), h_1, h_2) \right\|_{H^{s+a}_{x\in\mathbb{R}^+}} \\
&\le \sum_{k=1}^{J} \left\| W_{k\delta}^{J\delta}\big([u(k\delta) - W_{(k-1)\delta}^{k\delta}(u((k-1)\delta), h_1, h_2)], 0, 0\big) \right\|_{H^{s+a}_{x\in\mathbb{R}^+}} \\
&\le \sum_{k=1}^{J} \left\| u(k\delta) - W_{(k-1)\delta}^{k\delta}(u((k-1)\delta), h_1, h_2) \right\|_{H^{s+a}_{x\in\mathbb{R}^+}} \lesssim J f(T)^3 \lesssim \langle T\rangle f(T)^{\frac{17}{3}}
\end{aligned}
$$

where we have used Remark 4.2.3 in the second and third inequalities. Also the implicit constants just depend on $\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}$, $\|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$. Then we have

$$
\|u(T)\|_{H^{s+a}(\mathbb{R}^+)} \lesssim \langle T\rangle f(T)^{\frac{17}{3}} + \left\| W_0^T(g, h_1, h_2) \right\|_{H^{s+a}(\mathbb{R}^+)}.
$$

To bound this, first recall that

$$
W_0^T(g, h_1, h_2) = W^T g_e + W_0^T(0, h_1 - p_1, h_2 - p_2)
$$

where $p_1(t) = \eta(t/\langle T\rangle)D_0(W^t g_e)$, $p_2(t) = \eta(t/\langle T\rangle)D_0(\partial_x[W^t g_e])$. Then by Lemma 4.2.2

$$
\begin{aligned}
&\|W_0^T(g, h_1, h_2)\|_{H^s(\mathbb{R})} \\
&\qquad \lesssim \|g_e\|_{H^s(\mathbb{R})} + \left\|\chi_{(0,\infty)}(h_1 - p_1)\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} + \left\|\chi_{(0,\infty)}(h_2 - p_2)\right\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \\
&\qquad \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} + \|p_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} + \|p_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R})}.
\end{aligned}
$$

We estimate p_1 and p_2 by writing $\eta(t/\langle T\rangle) = \sum_{j=1}^{\langle T\rangle} \eta_j(t)$ and then using Kato smoothing inequality (Lemma 4.4.1) as follows

$$
\|p_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} + \|p_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \langle T\rangle \|g_e\|_{H^s(\mathbb{R})} \lesssim \langle T\rangle \|g\|_{H^s(\mathbb{R}^+)}.
$$

So then we have

$$
\left\|W_0^T(g, h_1, h_2)\right\|_{H^s(\mathbb{R}^+)} \lesssim \langle T\rangle \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}
$$

which leads to the bound

$$
\|u(T)\|_{H^{s+a}(\mathbb{R}^+)} \lesssim \langle T\rangle \big[f(T)^{\frac{17}{3}} + \|g\|_{H^{s+a}(\mathbb{R}^+)}\big] + \|h_1\|_{H^{\frac{2s+2a+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+2a+1}{8}}(\mathbb{R}^+)}.
$$

When $s = 2$ and $s_1 = s_2 = 1$, Lemma A.0.1 implies that $f(t) \approx 1$. As a result this and above bound yield that $\|u(t)\|_{H^s(\mathbb{R}^+)} \lesssim \langle T \rangle$ for $2 < s < \frac{5}{2}$. $\square$

5. THE PERIODIC HIROTA-SATSUMA SYSTEM

5.1. Introduction

The Hirota-Satsuma system is a system of coupled KdV equations, introduced by Hirota and Satsuma in 1981, [71]. In this chapter, we consider the Hirota-Satsuma system with periodic boundary conditions

$$\begin{cases} u_t + au_{xxx} + 3a(u^2)_x + \beta(v^2)_x = 0, \quad & x \in \mathbb{T} \\ v_t + v_{xxx} + 3uv_x = 0, \\ (u,v)|_{t=0} = (u_0, v_0) \in \dot{H}^s(\mathbb{T}) \times H^s(\mathbb{T}) \end{cases} \tag{5.1}$$

where $a \in (\frac{1}{4}, 1)$, $\beta \in \mathbb{R}$ and $\dot{H}^s(\mathbb{T}) = \{f \in H^s(\mathbb{T}) : \int_0^{2\pi} f(x)\,\mathrm{d}x = 0\}$. The results of this chapter have appeared in [72]. Here the choice of the parameter a is related to the resonance equations coming from after applying the normal form transformation to the system (5.1). The system (5.1) is a generalization of the KdV equation (when $v = 0$) and describes the interplay of two long waves evolving with different dispersion relations. Note that the mean zero condition on v cannot be applicable since the system (5.1) does not preserve the mean value of v, and that the momentum conservation holds for u only:

$$\int u(x,t)\,\mathrm{d}x = \int u_0(x)\,\mathrm{d}x.$$

The system (5.1) also satisfies the following conservation laws [71]:

$$\begin{aligned} E_1(u,v) &= \int u^2 - \frac{2\beta}{3}v^2\,\mathrm{d}x, \\ E_2(u,v) &= \int (1-a)u_x^2 - 2\beta v_x^2 - 2(1-a)u^3 + 2\beta uv^2\,\mathrm{d}x. \end{aligned} \tag{5.2}$$

As a consequence of these conserved energies, it turns out that the energy space for the system is $H^1 \times H^1$. No other conserved quantities seem to exist for (5.1) that holds for any a and β; nevertheless for $a = -\frac{1}{2}$, the system is known to be completely integrable, [71, 73]. Before discussing the literature of the coupled KdV type systems, it makes sense to review more recent well-posedness results of the KdV equation

$$\begin{cases} u_t + u_{xxx} + uu_x = 0 \\ u(x,0) = u_0(x) \in H^s(K) \text{ for } K = \mathbb{R} \text{ or } \mathbb{T}. \end{cases} \tag{5.3}$$

Introducing the Fourier restriction spaces Bourgain extended the previous local well-posedness results of the KdV equation to the L^2 level on $\mathbb{R}$ and $\mathbb{T}$, [2]. Later in [74], Kenig, Ponce and Vega proved the local well-posedness in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$ and in $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. The local well-posedness in $H^{-\frac{3}{4}}(\mathbb{R})$ was established by Christ-Colliander-Tao in [75]. In [76], Colliander-Keel-Staffilani-Takaoka-Tao obtained the local and global well-posedness in $H^{-\frac{1}{2}}(\mathbb{T})$. The global well-posedness for $\mathbb{R}$ at the endpoint $s = -\frac{3}{4}$ was proved by Guo [77]. By using the integrability properties of the KdV equation, Kappeler and Topalov established the local and global well-posedness in $H^{-1}(\mathbb{T})$, [78]. Later, in [79, 80], Molinet showed that the KdV equation is ill-posed in $H^s(K)$ for $K = \mathbb{R}, \mathbb{T}$, $s < -1$. Finally, the global well-posedness in $H^{-1}(\mathbb{R})$ has recently been obtained by Killip and Visan [81], and the study of the well-posedness of (5.3) has been brought to a satisfactory conclusion.

The well-posedness theory of the Hirota-Satsuma system began with the work of He [82], with the assumptions $0 < a < 1$ and $\beta < 0$ on the coefficients, He obtained the existence and uniqueness of the global solutions in $L^\infty([0,T]; H^3(K) \times H^3(K))$ for $K = \mathbb{T}, \mathbb{R}$. In the real case, Feng [83] improved this result to the range $s \geq 1$ by considering slightly general coupled KdV-KdV system. In particular, it was shown that for $a \neq 1$ and $\beta < 0$, the system is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 1$. Also with the additional assumption $0 < a < 1$, the system was shown to be globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 1$. Later, Alvarez and Carvajal [84] pushed the local result down to $s > \frac{3}{4}$ for the real case. They also showed that the system with $a \neq 0$ is ill-posed in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$ for $s \in [-1, -3/4)$ and $s' \in \mathbb{R}$.

Recently, Yang and Zhang [85] have studied the well-posedness of the Cauchy problem for a class of coupled KdV-KdV (cKdV) systems in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, those including Gear-Grimshaw system, Hirota-Satsuma system, the Majdo-Biello system etc. In particular, regarding the Hirota-Satsuma system, they have given critical index $s^* \in \{-\frac{3}{4}, 0, \frac{3}{4}\}$ depending on the numeric value of the coefficient a for which the Hirota-Satsuma Cauchy problem is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ when $s > s^*$. As regards to the periodic case, Angulo [86] showed that for $a \neq 0, 1$ the Hirota-Satsuma system is locally well-posed in $\dot{H}^s(\mathbb{T}) \times \dot{H}^s(\mathbb{T})$ for $s \geq 1$. With the additional assumptions $\beta < 0$ and $a < 1$, Angulo further obtained the global well-posedness in the same space within the same Sobolev index range that follows from the conservation of the energy. Finally, in [87], Yang and Zhang have recently obtained the well-posedness results of the cKdV systems on the periodic domain $\mathbb{T}$ as a follow up of their corresponding work [85]. Here we shall merely summarize the well-posedness results of [87] concerning the Cauchy problem (5.1). The results depend on the arithmetic properties of the coefficients a and β. When $a = 1$ and $\beta = 0$, the system (5.1) is locally well-posed in $\dot{H}^s(\mathbb{T}) \times H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$. In the case $a \in (-\infty, \frac{1}{4}) \setminus \{0\}$, as the resonace interactions are relatively easier to control, the local well-posedness is established in $\dot{H}^s(\mathbb{T}) \times H^s(\mathbb{T})$ for $s \geq -\frac{1}{4}$; whereas in the remaining regime, $a \in [\frac{1}{4}, \infty) \setminus \{1\}$, the resonances raise special difficulties in which case one needs to know how well a given number can be approximated by rational numbers (Diophantine approximation). The idea of controlling resonances via the Diophantine approximation was initially implemented by Oh [88] to the Majdo-Biello system on the torus to establish the well-posedness. Using this approach, Yang and Zhang proved the local well-posedness in $\dot{H}^s(\mathbb{T}) \times H^s(\mathbb{T})$ for $s \geq \min\{1, s_a+\}$ with the mean zero assumption on the initial data u_0. Here s_a is defined by means of a number theoretic parameter based on the arithmetic properties of a. On account of the conserved energies (5.2) for the system (5.1), when $\frac{1}{4} \leq a < 1$ and $\beta < 0$, the local well-posedness can be upgraded to global well-posedness for $s \geq 1$. Also when $a \in (-\infty, \frac{1}{4}) \setminus \{0\}$ and $\beta < 0$, the direct application of the conservation of $E_1(u, v)$ and the corresponding local result yield the global well-posedness for $s \geq 0$.

In the first part of the chapter, we study the smoothing property of the Hirota-Satsuma system, in other words, we prove that the difference between the nonlinear evolution and the linear evolution lies in a more regular space than the inital data under consideration. The proof is based on the method of normal forms through differentiation by parts introduced by Babin-Ilyin-Titi [14] and the Bourgain's Fourier restricted norm method, [2]. The idea of using combination of these methods in proving nonlinear smoothing effect on a bounded domain was first used by Erdoğan and Tzirakis for the KdV equation [16] and the Zakharov system [17]. The result for the KdV equation is somewhat surprising since the KdV equation is known to have no smoothing estimate on the real line. Recently, Compaan [15] studied the smoothing properties of the Majdo-Biello system on the torus and proved the existence of global attractors in the sense of arguments in [16, 17]. Here we continue the program initiated by these papers. In our proof, via the normal form reduction, the derivatives in the nonlinearities can be eliminated, and in return for this, the orders of the nonlinearities increase (from quadratic to cubic) and many resonant terms come into play based on the arithmetic properties of the coupling parameter a. In order to control the new trilinear nonlinearities we rely on the $X^{s,b}$ estimates.

In the second part, we concentrate on the description of long time dynamics of the forced and weakly damped system:

$$\begin{cases} u_t + au_{xxx} + \gamma_1 u + 3a(u^2)_x + \beta(v^2)_x = f \\ v_t + v_{xxx} + \gamma_2 v + 3uv_x = g \\ (u, v)|_{t=0} = (u_0, v_0) \in \dot{H}^1(\mathbb{T}) \times H^1(\mathbb{T}) \end{cases} \tag{5.4}$$

where the damping coefficients γ_1, γ_2 are positive, $\beta < 0$, f, g are time independent, and $f \in H^1$ is mean zero, $g \in H^1$. To simplify the calculations, we will take $\gamma_1 = \gamma_2$; the general case follows from minor modifications in the computations. The smoothing estimates obtained in the first part will play an essential role in demonstrating that the system (5.4) possesses a global attractor, also the result here answers the regularity of the attractor above the energy level.

Using the method of [15,17,39], roughly, the idea is to write the solution in terms of linear and nonlinear parts and then to implement the smoothing estimates to the nonlinear part.

5.1.1. Notation

In order to simplify the calculations, the notation $\mathcal{O}(\delta)$ is to be used in place of a constant $C\delta$ where C might be dependent on the coupling parameter a yet not on the variables involved in the calculations. The Fourier sequence of a 2π periodic L^2 function u is defined as

$$u_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}\,\mathrm{d}x, \qquad k \in \mathbb{Z}.$$

By recalling the notation $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$, we define the Sobolev norms of $u \in H^s(\mathbb{T})$ as follows

$$\|u\|_{H^s} = \|\langle k\rangle^s \widehat{u}(k)\|_{\ell^2_k}.$$

Note that for a mean zero function $u \in \dot{H}^s$,

$$\|u\|_{H^s} \approx \||k|^s \widehat{u}(k)\|_{\ell^2_k}.$$

The spaces $X_a^{s,b}$ and $X_1^{s,b}$ associated to the system (5.1) are defined by the norms

$$\|u\|_{X_a^{s,b}} = \left\|\langle k\rangle^s \langle \tau - ak^3\rangle^b \widehat{u}(k,\tau)\right\|_{\ell^2_k L^2_\tau}$$
$$\|v\|_{X_1^{s,b}} = \left\|\langle k\rangle^s \langle \tau - k^3\rangle^b \widehat{v}(k,\tau)\right\|_{\ell^2_k L^2_\tau}.$$

The restricted norms are also defined by

$$\|u\|_{X^{s,b}_{a,\delta}} = \inf_{\tilde{u}=u \text{ on } [-\delta,\delta]} \|\tilde{u}\|_{X_a^{s,b}}$$
$$\|v\|_{X^{s,b}_{1,\delta}} = \inf_{\tilde{v}=v \text{ on } [-\delta,\delta]} \|\tilde{v}\|_{X_1^{s,b}}.$$

5.2. Statement of Results

5.2.1. Preliminaries

In order to state well-posedness results and hence to study the smoothing properties of the system (5.1) on the torus, we require special parameters related to given numbers, called "irrationality exponent", which might be used to learn how close given numbers can be approximated by rational numbers. Irrationality exponent of a real number arises in the study of Diophantine approximation theory. In our discussion, we utilize these quantities in controlling resonances.

Definition 5.2.1 (See [89])**.** *A number $r \in \mathbb{R}$ is said to be approximable with power μ, if the inequality*

$$0 < \left| r - \frac{n}{k} \right| < \frac{1}{|k|^{\mu}}$$

holds for infinitely many $(n, k) \in \mathbb{Z} \times \mathbb{Z}^*$, and

$$\mu(r) = \sup\{\mu \in \mathbb{R} : r \textit{ is approximable with power } \mu\} \tag{5.5}$$

is called the irrationality exponent of r.

We now review some properties of the irrationality exponent. Irrationality exponent maps the set of real numbers onto the set $\{1\} \cup [2, \infty]$, see [90, 91]. In particular, for $r \in \mathbb{Q}$, $\mu(r) = 1$ whereas for irrational number r we have $\mu(r) \geq 2$. By the Thue-Siegel-Roth theorem [92–94], for an irrational algebraic number r, $\mu(r) = 2$, also by the Khintchine theorem [95], for almost every $r \in \mathbb{R}$, $\mu(r) = 2$. The local theory for the Cauchy problem (5.1) is based on the critical index s_r for $r \geq \frac{1}{4}$ defined by

$$s_r = \begin{cases} 1 & \text{if } \mu(\rho_r) = 1 \text{ or } \mu(\rho_r) \geq 3 \\ \frac{\mu(\rho_r) - 1}{2} & \text{if } 2 \leq \mu(\rho_r) < 3 \end{cases}$$

where $\rho_r = \sqrt{12r - 3}$. In connection with the well-posedness of the system (5.1), we state some of the results of [87].

Theorem 5.2.2 (See [87]). *For $a \in [\frac{1}{4}, \infty) \setminus \{1\}$ and $s \geq \min\{1, s_a+\}$, the Hirota-Satsuma system* (5.1) *is locally well-posed in $\dot{H}^s \times H^s$. In particular, given any initial data $(u_0, v_0) \in \dot{H}^s \times H^s$ there exists $\delta \approx (\|u_0\|_{H^s} + \|v_0\|_{H^s})^{-\frac{3}{2}}$ and a unique solution*

$$(u, v) \in C([-\delta, \delta]; H^s \times H^s)$$

satisfying $\|u\|_{X^{s,1/2}_{a,\delta}} + \|v\|_{X^{s,1/2}_{1,\delta}} \lesssim \|u_0\|_{H^s} + \|v_0\|_{H^s}$.

Theorem 5.2.3 (See [87]). *Let $\beta < 0$ and $a \in [\frac{1}{4}, 1)$. Then for $s \geq 1$, the Hirota-Satsuma initial value problem* (5.1) *is globally well-posed in $\dot{H}^s \times H^s$.*

5.2.2. Smoothing Estimates for the Hirota-Satsuma System

Applying normal form transformation to the system (5.1) leads to the resonance equations (for $k_1 + k_2 = k$)

$$ak^3 - k_1^3 - k_2^3 = -3k(k_1 - r_1 k)(k_1 - r_2 k)$$
$$k^3 - ak_1^3 - k_2^3 = (1 - a)k_1(k_1 - \widetilde{r}_1 k)(k_1 - \widetilde{r}_2 k)$$

where

$$r_1 = \frac{1}{2} + \frac{\sqrt{12a - 3}}{6}, \quad r_2 = \frac{1}{2} - \frac{\sqrt{12a - 3}}{6} \text{ and } \widetilde{r}_1 = 1/r_1, \quad \widetilde{r}_2 = 1/r_2. \tag{5.6}$$

Note that r_1, r_2 are the roots of the equation $3x^2 - 3x + (1 - a)$, which immediately implies that $\widetilde{r}_1$, $\widetilde{r}_2$ are the roots of the equation $(1 - a)x^2 - 3x + 3$. Therefore, r_j and $\widetilde{r}_j$ are algebraic only when $a \in \mathbb{Q}$. By (5.6), we notice that $r_1, r_2 \in \mathbb{R}$ if and only if $a \geq \frac{1}{4}$, and that $\widetilde{r}_1, \widetilde{r}_2 \in \mathbb{R}$ if and only if $a \in [\frac{1}{4}, 1) \cup (1, \infty)$. In this chapter, we consider the problem (5.1) for $a \in (\frac{1}{4}, 1)$, also the problem for interval $(1, \infty)$ can be handled in a similar vein. As performing smoothing estimates, we will be dealing with many expressions such as $ak^3 - k_1^3 - k_2^3$ and $k^3 - ak_1^3 - k_2^3$ that appear in the denominators. Controlling such expressions in the case they get close to zero relies on the following lemma.

Lemma 5.2.4 (See [87]). *If $r \in \mathbb{R} \setminus \mathbb{Q}$ with $\mu(r) < \infty$, then for any $\epsilon > 0$, there exists a constant $K = K(r, \epsilon) > 0$ such that the inequality*

$$\left| r - \frac{n}{k} \right| \geq \frac{K}{|k|^{\mu(r)+\epsilon}} \tag{5.7}$$

holds for any $(n, k) \in \mathbb{Z} \times \mathbb{Z}^*$.

Note that when r_j, $\widetilde{r}_j$, as introduced in (5.6), are rational numbers, by the Proposition 5.3.1 below, we will still be able to use the inequality (5.7) for these numbers with $\mu(r_j) = \mu(\widetilde{r}_j) = 1$. Next lemma collects some invariance properties of the irrationality exponent, which is proved in [87].

Lemma 5.2.5. *The irrationality exponent defined by* (5.5) *satisfies*

(i) For any $r \in \mathbb{R}$ and $q \in \mathbb{Q}$, $\mu(q + r) = \mu(r)$,

(ii) For any $r \in \mathbb{R}$ and $q \in \mathbb{Q} \setminus \{0\}$, $\mu(qr) = \mu(r)$,

(iii) For any $r \in \mathbb{R} \setminus \{0\}$, $\mu(\frac{1}{r}) = \mu(r)$.

Using the lemma above we may write

$$\mu(\widetilde{r}_2) = \mu(\widetilde{r}_1) = \mu(r_1) = \mu(r_2) = \mu(\sqrt{12a - 3}) =: \mu(\rho_a). \tag{5.8}$$

With the notations introduced above we state our smoothing result as follows:

Theorem 5.2.6. *Fix $s > \frac{1}{2}$ and $a \in (\frac{1}{4}, 1)$. Consider the solution of* (5.1) *with initial data $(u(0, x), v(0, x)) = (u_0(x), v_0(x)) \in \dot{H}^s(\mathbb{T}) \times H^s(\mathbb{T})$. Let*

$$s_1 - s < \min\{1, s - \frac{1}{2}, s + 2 - \mu(\rho_a), 2s + 1 - \mu(\rho_a)\}.$$

Then, we have

$$u(x, t) - e^{-ta\partial_x^3} u_0 \in C_t^0 H_x^{s_1}, \tag{5.9}$$

$$v(x, t) - e^{-t\partial_x^3} v_0 \in C_t^0 H_x^{s_1}. \tag{5.10}$$

For almost every a, (5.9) *and* (5.10) *hold with $s_1 - s < \min\{1, s - \frac{1}{2}\}$.*

When $a = \frac{3p(p-q)+q^2}{q^2}$*,* $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ *with* $\frac{q}{2} < p < q$*, fix* $s > 1$*, then the smoothing statements* (5.9) *and* (5.10) *are valid for* $s_1 - s \leq \min\{1-, s-1\}$ *instead. Assume that we have a growth bound* $\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \lesssim \langle t\rangle^{\beta(s)}$*, for some* $\beta(s) > 0$*. Then*

$$\left\|u(t) - e^{-ta\partial_x^3}u_0\right\|_{H^{s_1}} + \left\|v(t) - e^{-t\partial_x^3}v_0\right\|_{H^{s_1}} \leq C(a, s, s_1, \|u_0\|_{H^s}, \|u_0\|_{H^s})\langle t\rangle^{1+\frac{9}{2}\beta(s)}.$$

Remark 5.2.7. In the case the coefficients r_1 and r_2 in (5.6) are rational numbers, which is the case when a is a rational of the form $\frac{3p(p-q)+q^2}{q^2}$ as stated in the above theorem, we have to control some additional terms due to the resonances, in which case smoothing is attained solely for $s > 1$. On the other side, if the coefficients r_1, r_2 are irrational algebraic numbers, which is the case when a is a rational number such that $a \neq \frac{3p(p-q)+q^2}{q^2}$ with $\frac{q}{2} < p < q$, then by (5.8), $\mu(\rho_a) = 2$ in Theorem 5.2.6 yields the best possible smoothing. Indeed, the best regularity gain given by the Theorem 5.2.6 is reached for almost every $a \in (\frac{1}{4}, 1)$. As a consequence, with regards to smoothing, the above discussion shows how the regularity level is unstable under a slight perturbation of a within $(\frac{1}{4}, 1)$.

Using smoothing estimates one can obtain growth bounds for higher order Sobolev norms in the lack of complete integrability. As a corollory of the smoothing theorem above, we obtain the following result.

Corollary 5.2.8. *For any* $s \geq 1$ *and almost every* $a \in (\frac{1}{4}, 1)$ *for which* $\mu(\rho_a) = 2$*, the global solution of* (5.1) *with* $\dot{H}^s \times H^s$ *data satisfies the growth bound*

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \leq C_0(1 + |t|)^{C_1}$$

where C_0 *depends on* $a, s, \|u_0\|_{H^s}, \|v_0\|_{H^s}$ *and* C_1 *depends on* s*.*

Proof. Due to the conserved energies, H^1 norms of u and v are bounded for all times. The idea is to use the result of Theorem 5.2.6 repeatedly to obtain the growth bound for the Sobolev norms above the energy level. To use this, suppose that the claim of the corollory holds for $s \geq 1$.

Hence for $s_1 \in (s, s + \min\{1, s - \frac{1}{2}\})$, Theorem 5.2.6 leads to

$$\left\| u(t) - e^{-at\partial_x^3} u_0 \right\|_{H^{s_1}} + \left\| v(t) - e^{-t\partial_x^3} v_0 \right\|_{H^{s_1}} \leq C(1 + |t|)^{1+\frac{9}{2}\beta(s)},$$

from this and the fact that linear groups are unitary, we get

$$\|u(t)\|_{H^{s_1}} + \|v(t)\|_{H^{s_1}} \leq \|u_0\|_{H^{s_1}} + \|v_0\|_{H^{s_1}} + C_0(1 + |t|)^{C_1}.$$

Continuing iteratively in this way, we reach any index s_1. □

5.2.3. Existence of a Global Attractor for the Hirota-Satsuma System

The essential tool in establishing the existence of a global atractor for the dissipative Hirota-Satsuma system (5.4) in this context will be the smoothing estimates which are to be discussed later. The existence of global attractors makes sense only for the globally well-posed problems, in this regard we note that the initial value problem (5.4) is locally and globally well-posed in $\dot{H}^1 \times H^1$. The local well-posedness follows by using the estimates of [87] and the global well-posedness follows from the energy estimate of Lemma 5.5.1 which also implies the existence of an absorbing ball. To set the stage for the description of the problem, let $U(t)$ be the semigroup operator mapping data to solution in the phase space and recall the definitions 2.3.1–2.3.4. Therefore, by Theorem 2.3.5, in the presence of an absorbing set, the proof of existence of a global attractor reduces to proving asymptotic compactness of the evolution operator. In our discussion, the proof of the asymptotic compactness of the flow depends very much on the smoothing estimate for the forced and weakly damped system that we establish later. Here we have the result:

Theorem 5.2.9. *Consider the forced and weakly damped Hirota-Satsuma system* (5.4) *on* $\mathbb{T} \times [0, \infty)$ *with* $(u_0, v_0) \in \dot{H}^1 \times H^1$. *Then, for almost every* $a \in (\frac{1}{4}, 1)$, *the equation has a global attractor in* $\dot{H}^1 \times H^1$. *Moreover, for any* $\alpha < \frac{1}{2}$, *the global attractor is a subset of* $H^{1+\alpha} \times H^{1+\alpha}$.

5.3. Proof of Theorem 5.2.6

We start by writing (5.1) in an equivalent form through differentiation parts. Below $\sum^*$ denotes the summation over all terms for which the corresponding denominator is never zero. The requirement of the nonzero denominators for the sums not written by this notation is provided by the mean zero assumption on u_0.

Proposition 5.3.1. *Let $u_0 \in \dot{H}^s$. The system* (5.1) *can be written as*

$$\begin{cases} \partial_t\big(e^{-iak^3t}[u_k + B_1(u,u)_k + B_2(v,v)_k]\big) = e^{-iak^3t}\big[R_1(u,v,v)_k + R_2(u,u,u)_k \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + R_3(u,v,v)_k + \rho_1(u,u)_k + \rho_2(v,v)_k\big] \\ \partial_t\big(e^{-ik^3t}[v_k + B_3(u,v)_k]\big) = e^{-ik^3t}\big[R_4(u,u,v)_k + \frac{\beta}{3a}R_4(v,v,v)_k \\ \qquad\qquad\qquad\qquad\qquad\qquad + R_5(u,u,v)_k + \rho_3(u,v)_k\big] \end{cases}$$

where the terms are defined as follows

$$B_1(f,g)_k = -\sum_{k_1+k_2=k} \frac{f_{k_1}g_{k_2}}{k_1k_2}, \quad B_2(f,g)_k = -\beta k \sum^*_{k_1+k_2=k} \frac{f_{k_1}g_{k_2}}{ak^3 - k_1^3 - k_2^3}$$

$$B_3(f,g)_k = -3 \sum^*_{k_1+k_2=k} \frac{k_2 f_{k_1}g_{k_2}}{k^3 - ak_1^3 - k_2^3}$$

$$R_1(f,g,h)_k = -2i\beta \sum^*_{k_1+k_2+k_3=k} \frac{k_2 f_{k_1}g_{k_2}h_{k_3}}{(k_1+k_2-r_1k)(k_1+k_2-r_2k)}$$

$$R_2(f,g,h)_k = 6ia \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \frac{f_{k_1}g_{k_2}h_{k_3}}{k_1}$$

$$R_3(f,g,h)_k = 2i\beta \sum_{k_1+k_2+k_3=k} \frac{f_{k_1}g_{k_2}h_{k_3}}{k_1}$$

$$R_4(f,g,h)_k = 9ia \sum^*_{k_1+k_2+k_3=k} \frac{k_3(k_1+k_2)f_{k_1}g_{k_2}h_{k_3}}{k^3 - a(k_1+k_2)^3 - k_3^3}$$

$$R_5(f,g,h)_k = 9i \sum^*_{k_1+k_2+k_3=k} \frac{k_3(k_2+k_3)f_{k_1}g_{k_2}h_{k_3}}{k^3 - ak_1^3 - (k_2+k_3)^3}$$

$$\rho_1(f,g)_k = -\frac{6ia}{k}|f_k|^2 g_k, \qquad \rho_2(f,g)_k = -2i\beta k f_{r_1k}g_{r_2k}$$

$$\rho_3(f,g)_k = -3ik\big[(1-\tilde{r}_1)f_{\tilde{r}_1k}g_{(1-\tilde{r}_1)k} + (1-\tilde{r}_2)f_{\tilde{r}_2k}g_{(1-\tilde{r}_2)k}\big].$$

Proof. Writing (5.1) on the Fourier side we have

$$\begin{cases} \partial_t u_k - iak^3 u_k + 3iak \sum\limits_{k_1+k_2=k} u_{k_1} u_{k_2} + i\beta k \sum\limits_{k_1+k_2=k} v_{k_1} v_{k_2} = 0 \\ \partial_t v_k - ik^3 v_k + 3i \sum\limits_{k_1+k_2=k} k_2 u_{k_1} v_{k_2} = 0. \end{cases}$$

Via the substitution $f_k(t) = e^{-iak^3t} u_k(t)$ and $g_k(t) = e^{-ik^3t} v_k(t)$, the above system transforms to

$$\begin{cases} \partial_t f_k = -3iak \sum\limits_{k_1+k_2=k} e^{-iat(k^3-k_1^3-k_2^3)} f_{k_1} f_{k_2} - i\beta k \sum\limits_{k_1+k_2=k} e^{-it(ak^3-k_1^3-k_2^3)} g_{k_1} g_{k_2} \\ \partial_t g_k = -3i \sum\limits_{k_1+k_2=k} k_2 e^{-it(k^3-ak_1^3-k_2^3)} f_{k_1} g_{k_2}. \end{cases} \tag{5.11}$$

Implementing differentiation by parts to the first equation in (5.11) we get

$$\begin{aligned} \partial_t f_k = &\sum_{k_1+k_2=k} \frac{\partial_t (e^{-3ikk_1k_2t} f_{k_1} f_{k_2})}{k_1 k_2} - \sum_{k_1+k_2=k} \frac{e^{-3iakk_1k_2t} \partial_t (f_{k_1} f_{k_2})}{k_1 k_2} \\ &+ \beta k \sideset{}{^*}\sum_{k_1+k_2=k} \frac{\partial_t (e^{-it(ak^3-k_1^3-k_2^3)} g_{k_1} g_{k_2})}{ak^3-k_1^3-k_2^3} - \beta k \sideset{}{^*}\sum_{k_1+k_2=k} \frac{e^{-it(ak^3-k_1^3-k_2^3)} \partial_t (g_{k_1} g_{k_2})}{ak^3-k_1^3-k_2^3} \\ &- 2i\beta k g_{r_1k} g_{r_2k}. \end{aligned}$$

Note here that $ak^3 - k_1^3 - k_2^3 = -3k(k_1 - r_1k)(k_1 - r_2k)$ where r_1 and r_2 are the roots of the quadratic equation $3x^2 - 3x + (1-a)$. So the resonant term corresponding to the second sum of the first equation in (5.11) come up when $r_1k \in \mathbb{Z}$, in which case we would have $r_1, r_2 \in \mathbb{Q}$. There is no contribution from $k = 0$ solution to the resonant term owing to the mean zero assumption on u_0. Following the arguments of [14], [16] and using the first line of (5.11), we have

$$\begin{aligned} &\sum_{k_1+k_2=k} \frac{e^{-3iakk_1k_2t} \partial_t (f_{k_1} f_{k_2})}{k_1 k_2} \\ &\qquad = -6ia \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \frac{e^{-3ia(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} f_{k_1} f_{k_2} f_{k_3} \\ &\qquad - 2i\beta \sum_{k_1+k_2+k_3=k} \frac{e^{-it(ak^3-ak_1^3-k_2^3-k_3^3)}}{k_1} f_{k_1} g_{k_2} g_{k_3} + \frac{6ia}{k} |f_k|^2 f_k. \end{aligned}$$

Likewise using the second line of (5.11), the fourth sum in $\partial_t f_k$ can be written as

$$-2i\beta \sum_{k_1+k_2+k_3=k}^{*} \frac{k_2 e^{-it(ak^3-ak_1^3-k_2^3-k_3^3)}}{(k_1+k_2-r_1k)(k_1+k_2-r_2k)} f_{k_1} g_{k_2} g_{k_3}.$$

As for the second equation in (5.11), again we use differentiation by parts to get

$$\begin{aligned}\partial_t g_k = {} & 3 \sum_{k_1+k_2=k}^{*} \frac{k_2 \partial_t (e^{-it(k^3-ak_1^3-k_2^3)} f_{k_1} g_{k_2})}{k^3-ak_1^3-k_2^3} - 3 \sum_{k_1+k_2=k}^{*} \frac{k_2 e^{-it(k^3-ak_1^3-k_2^3)}}{k^3-ak_1^3-k_2^3} (\partial_t f_{k_1} g_{k_2}) \\ & - 3ik\big[(1-\tilde r_1) f_{\tilde r_1 k} g_{(1-\tilde r_1)k} + (1-\tilde r_2) f_{\tilde r_2 k} g_{(1-\tilde r_2)k}\big].\end{aligned}$$

Note that in obtaining the resonant term we use the identity $k^3 - ak_1^3 - k_2^3 = (1-a)k_1(k_1-\tilde r_1 k)(k_1 - \tilde r_2 k)$ where $\tilde r_j = 1/r_j$, $j = 1,2$. By the mean zero assumption on u_0, the only contribution comes just when $\tilde r_1 k,\ \tilde r_2 k \in \mathbb{Z}$ in which case we need to have $\tilde r_1, \tilde r_2 \in \mathbb{Q}$. Using (5.11), we rewrite the second sum above as follows

$$\begin{aligned} & 3 \sum_{k_1+k_2=k}^{*} \frac{e^{-it(k^3-ak_1^3-k_2^3)} k_2}{k^3-ak_1^3-k_2^3} (\partial_t f_{k_1} g_{k_2}) \\ & \quad = -9ia \sum_{k_1+k_2+k_3=k}^{*} \frac{e^{-it(k^3-ak_1^3-ak_2^3-k_3^3)} (k_1+k_2) k_3}{k^3-a(k_1+k_2)^3-k_3^3} f_{k_1} f_{k_2} g_{k_3} \\ & \quad - 3i\beta \sum_{k_1+k_2+k_3=k}^{*} \frac{e^{-it(k^3-k_1^3-k_2^3-k_3^3)} (k_1+k_2) k_3}{k^3-a(k_1+k_2)^3-k_3^3} g_{k_1} g_{k_2} g_{k_3} \\ & \quad - 9i \sum_{k_1+k_2+k_3=k}^{*} \frac{e^{-it(k^3-ak_1^3-ak_2^3-k_3^3)} (k_2+k_3) k_3}{k^3-ak_1^3-(k_2+k_3)^3} f_{k_1} f_{k_2} g_{k_3}. \end{aligned}$$

Bringing all the terms together and reinstating the u and v variables yield the assertion.

$\square$

We integrate the system in Proposition 5.3.1 from 0 to t to get

$$\begin{cases} u_k(t) - e^{iak^3t}u_k(0) = -B_1(u,u)_k(t) - B_2(v,v)_k(t) + e^{iak^3t}B_1(u,u)_k(0) \\ \qquad +e^{iak^3t}B_2(v,v)_k(0) + \displaystyle\int_0^t e^{iak^3(t-s)}\big[R_1(u,v,v)_k(s) \\ \qquad +R_2(u,u,u)_k(s) + R_3(u,v,v)_k(s) + \rho_1(u,u)_k(s) + \rho_2(v,v)_k(s)\big]\mathrm{d}s \\ v_k(t) - e^{ik^3t}v_k(0) = -B_3(u,v)_k(t) + e^{ik^3t}B_3(u,v)_k(0) + \displaystyle\int_0^t e^{ik^3(t-s)}\big[R_4(u,u,v)_k(s) \\ \qquad +\tfrac{\beta}{3a}R_4(v,v,v)_k(s) + R_5(u,u,v)_k(s) + \rho_3(u,v)_k(s)\big]\mathrm{d}s \end{cases} \tag{5.12}$$

In the following we give the proofs of a priori estimates for ρ_j and B_j, $j = 1,2,3$.

Proposition 5.3.2. *For $s_1 - s \leq 2s+1$, we have*

$$\|\rho_1(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s}^2 \|v\|_{H_x^s}.$$

For $s_1 - s \leq s-1$,

$$\|\rho_j(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}, \; j = 2,3.$$

Also for $s_1 - s \leq 1$,

$$\|B_1(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.$$

Proof. The proofs of the first and third inequalities were given in [16]. As for the other terms, for $s_1 - s \leq s-1$, we have

$$\begin{aligned} \|\rho_2(u,v)\|_{H_x^{s_1}} &\lesssim \left\| \langle k\rangle^{s_1-2s+1} \langle r_1 k\rangle^s u_{r_1 k} \langle r_2 k\rangle^s v_{r_2 k} \right\|_{\ell_k^2} \\ &\lesssim \|\langle k\rangle^s u_k\|_{\ell_k^\infty} \|\langle k\rangle^s v_k\|_{\ell_k^2} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}, \end{aligned}$$

where the last inequality is due to $\ell^2 \hookrightarrow \ell^\infty$. ρ_3 estimate follows by the same argument as well. □

Proposition 5.3.3. *For $s > \frac{1}{2}$ and $s_1 - s < \min\{1, s+2-\mu(\rho_a)\}$, we have*

$$\|B_2(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.$$

When $a = \frac{3p(p-q)+q^2}{q^2}$, $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ *with* $\frac{q}{2} < p < q$, *we replace the above requirement by* $s_1 - s \leq 1$.

Proof. It is enough solely to consider the case $|k_1| \gtrsim |k_2|$ by the symmetry.

Case A. $|k_1 - r_1 k| \geq \frac{1}{3}$, $|k_1 - r_2 k| \geq \frac{1}{3}$

In this region,

$$|ak^3 - k_1^3 - k_2^3| = |3k(k_1 - r_1 k)(k_1 - r_2 k)| \geq |k| \max\{|k_1 - r_1 k|, |k_1 - r_2 k\} \gtrsim (r_1 - r_2)|k|^2.$$

Accordingly, since $|k_1| \gtrsim |k|$, we have the estimate

$$\|B_2(u,v)\|_{H_x^{s_1}} \lesssim \left\| \sum_{k_1+k_2=k} \langle k\rangle^{s_1-1} |u_{k_1}||v_{k_2}| \right\|_{l_k^2} \lesssim \left\| \langle k\rangle^{s_1-s-1} \Big(|u_k|\langle k\rangle^s * |v_k| \frac{\langle k\rangle^s}{\langle k\rangle^s} \Big) \right\|_{l_k^2}.$$

Assuming that $s_1 - s \leq 1$ and using Young's and Hölder's inequalities successively, the last norm above is majorized by

$$\|\langle k\rangle^s u_k\|_{l_k^2} \left\|\langle k\rangle^s v_k \langle k\rangle^{-s}\right\|_{l_k^1} \lesssim \|u\|_{H^s} \|\langle k\rangle^s u_k\|_{l_k^2} \left\|\langle k\rangle^{-s}\right\|_{l_k^2} \lesssim \|u\|_{H^s} \|v\|_{H^s}.$$

Case B. $|k_1 - r_1 k| < \frac{1}{3}$ or $|k_1 - r_2 k| < \frac{1}{3}$

We consider the case $|k_1 - r_1 k| < \frac{1}{3}$ only, the other one is similar. Then as $r_1 + r_2 = 1$, $k_1 \simeq r_1 k$ and $k_2 \simeq r_2 k$. This means that the sum under consideration consists of a single term of order $\approx k$. We next make use of the estimate due to irrationality exponent of r_1:

$$|k_1 - r_1 k| = |k| \left| r_1 - \frac{k_1}{k} \right| \geq |k| \frac{K(r_1,\epsilon)}{|k|^{\mu(r_1)+\epsilon}}$$

for any $\epsilon > 0$. This allows us to estimate the multiplier in the definition of $B_2(u,v)$ as follows:

$$\begin{aligned} |ak^3 - k_1^3 - k_2^3| &= 3|k||k_1 - r_1 k||k_1 - r_2 k| \\ &\geq 3K(r_1,\epsilon)|k|^{2-\mu(r_1)-\epsilon}\big[(r_1 - r_2)|k| - 1/3\big] \gtrsim |k|^{3-\mu(r_1)-\epsilon}. \end{aligned}$$

Therefore, using Hölder's inequality and the embedding $\ell^2 \hookrightarrow \ell^4$, we obtain that

$$\begin{aligned}
\|B_2(u,v)\|_{H_x^{s_1}} &\lesssim \left\|\langle k\rangle^{(s_1+\mu(r_1)-2+\epsilon)/2} u_k\right\|_{\ell_k^4} \left\|\langle k\rangle^{(s_1+\mu(r_1)-2+\epsilon)/2} v_k\right\|_{\ell_k^4} \\
&\lesssim \left\|\langle k\rangle^{(s_1-2s+\mu(r_1)-2+\epsilon)/2}\langle k\rangle^s u_k\right\|_{\ell_k^2} \left\|\langle k\rangle^{(s_1-2s+\mu(r_1)-2+\epsilon)/2}\langle k\rangle^s v_k\right\|_{\ell_k^2} \\
&\lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}
\end{aligned}$$

where the last inequality stems from the assumption that $s_1 - s < s + 2 - \mu(r_1)$. $\square$

Proposition 5.3.4. *Assume that* $u \in \dot{H}^s$. *For* $s > \frac{1}{2}$ *and* $s_1 - s < \min\{1, s+2-\mu(\rho_a)\}$, *we have*

$$\|B_3(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.$$

When $a = \frac{3p(p-q)+q^2}{q^2}$, $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ *with* $\frac{q}{2} < p < q$, *we replace the above requirement by* $s_1 - s \leq 1$.

Proof. Since there is no symmetry in this case, we consider the two cases:
Case A. $|k_1| \geq |k|$
In this region, $|k_1| \gtrsim |k_2|$, so it suffices to show that

$$\left\| \sum_{k_1+k_2=k}^{*} \frac{\langle k\rangle^{s_1} u_{k_1} v_{k_2}}{(k_1-\widetilde{r}_1 k)(k_1-\widetilde{r}_2 k)} \right\|_{\ell_k^2} \lesssim \|u\|_{H_x^s}\|v\|_{H_x^s}.$$

Case A.1. $|k_1 - \widetilde{r}_1 k| \geq \delta$, $|k_1 - \widetilde{r}_2 k| \geq \delta$
Notice that

$$|(k_1-\widetilde{r}_1 k)(k_1-\widetilde{r}_2 k)| \geq \delta \max\{|k_1-\widetilde{r}_1 k|, |k_1-\widetilde{r}_2 k|\} \geq \delta(\widetilde{r}_2-\widetilde{r}_1)|k| \gtrsim |k|$$

which yields the estimate:

$$\text{LHS of (5.13)} \lesssim \left\| \sum_{k_1+k_2=k} \langle k\rangle^{s_1-1}|u_{k_1}||v_{k_2}| \right\|_{l_k^2}$$

but this has already been handled in the proof of the previous proposition, thus the estimate in (5.13) holds when $s_1 - s \leq 1$.

Case A.2. $|k_1 - \widetilde{r}_1 k| < \delta$ or $|k_1 - \widetilde{r}_2 k| < \delta$

We consider the first case $|k_1 - \widetilde{r}_1 k| < \delta$. Second case is analogous. In this case, we have $k_1 \simeq \widetilde{r}_1 k$ and $k_2 = k - k_1 \simeq (1 - \widetilde{r}_1)k$, and hence $|k| \approx |k_1| \approx |k_2|$. As a result, the values of k_1 and k_2 in the sum are dependent on k. Therefore, the bound

$$|(k_1 - \widetilde{r}_1 k)(k_1 - \widetilde{r}_2 k)| \gtrsim |k|^{1-\mu(\widetilde{r}_1)+\epsilon} |k_1 - \widetilde{r}_2 k| \gtrsim |k|^{2-\mu(\widetilde{r}_1)-\epsilon}$$

implies that

$$\begin{aligned}\text{LHS of (5.13)} &\lesssim \left\| \langle k \rangle^{s_1 - 2 + \mu(\widetilde{r}_1)+\epsilon} u_{k_1} v_{k_2} \right\|_{\ell^2_k} \\ &\lesssim \left\| \langle k \rangle^{(s_1 - 2 + \mu(\widetilde{r}_1)+\epsilon)/2} u_k \right\|_{\ell^4_k} \left\| \langle k \rangle^{(s_1 - 2 + \mu(\widetilde{r}_1)+\epsilon)/2} v_k \right\|_{\ell^4_k} \lesssim \|u\|_{H^s_x} \|v\|_{H^s_x}\end{aligned}$$

provided that $s_1 - s < s + 2 - \mu(\widetilde{r}_1)$.

Case B. $|k_1| < |k|$

In this region $|k_2| \lesssim |k|$. Thus

$$\|B_3(u,v)\|_{H^{s_1}_x} \lesssim \left\| \sum_{k_1+k_2=k}^{*} \frac{\langle k \rangle^{s_1+1} u_{k_1} v_{k_2}}{k^3 - a k_1^3 - k_2^3} \right\|_{l^2_k}. \tag{5.13}$$

By the mean zero presumption on u, $k_1 \neq 0$, thus we may write $k_1 = \eta k$ for some $|k|^{-1} \leq |\eta| < 1$. It follows that

$$|k^3 - a k_1^3 - k_2^3| = |\eta k^3| |(1-a)\eta^2 + 3 - 3\eta| \geq |k|^2 |(1-a)\eta^2 + 3 - 3\eta| \gtrsim |k|^2.$$

Then the right side of (5.13) is bounded by

$$\left\| \sum_{k_1+k_2=k} \langle k \rangle^{s_1 - 1} |u_{k_1}| |v_{k_2}| \right\|_{l^2_k} \lesssim \|u\|_{H^s_x} \|v\|_{H^s_x}$$

as long as $s_1 - s \leq 1$. □

Writing the equations in (5.12) in the space side and then using the estimates in Propositions 5.3.2–5.3.4, we arrive at

$$\left\| u(t) - e^{-at\partial_x^3} u_0 \right\|_{H^{s_1}} \lesssim \|u_0\|^2_{H^s} + \|v_0\|^2_{H^s} + \|u\|^2_{H^s} + \|v\|^2_{H^s} + \int_0^t \|u(r)\|^2_{H^s} + \|v(r)\|^2_{H^s}\, \mathrm{d}r$$

$$+\left\|\int_0^t e^{-a(t-r)\partial_x^3}\big[R_1(u,v,v)(r)+R_2(u,u,u)(r)+R_3(u,v,v)(r)\big]\mathrm{d}r\right\|_{H^{s_1}} \tag{5.14}$$

and

$$\begin{aligned}\left\|v(t)-e^{-t\partial_x^3}v_0\right\|_{H^{s_1}} &\lesssim \|u_0\|_{H^s}\|v_0\|_{H^s}+\|u\|_{H^s}\|v\|_{H^s}+\int_0^t\|u(r)\|_{H^s}\|v(r)\|_{H^s}\,\mathrm{d}r\\ &+\left\|\int_0^t e^{-(t-r)\partial_x^3}\big[R_4(u,u,v)(r)+\frac{\beta}{3a}R_4(v,v,v)(r)+R_5(u,u,v)(r)\big]\mathrm{d}r\right\|_{H^{s_1}}\end{aligned} \tag{5.15}$$

Let δ be the local existence time coming from the local existence theory for the Hirota-Satsuma system. Let $\psi_\delta(t)=\psi(t/\delta)$ where ψ is a compactly supported function supported on $[-2,2]$ and $\psi=1$ on $[-1,1]$. For $t\in[-\delta,\delta]$, to estimate the H^{s_1} norms of the integral parts in (5.14), (5.15), we need the following standard lemma, see [10].

Lemma 5.3.5. *For $\frac{1}{2}<b\le 1$, and $\alpha\neq 0$*

$$\left\|\psi_\delta(t)\int_0^t e^{-\alpha\partial_x^3(t-r)}F(r)\,dr\right\|_{X_\alpha^{s,b}}\lesssim\|F\|_{X_{\alpha,\delta}^{s,b-1}}.$$

Therefore by the Lemma 5.3.5 and the embedding $X_{\alpha,\delta}^{s_1,b}\hookrightarrow L^\infty_{t\in[-\delta,\delta]}H_x^{s_1}$ for $b>\frac{1}{2}$, $\alpha\neq 0$, we have

$$\begin{aligned}&\left\|\int_0^t e^{-a(t-r)\partial_x^3}\big[R_1(u,v,v)(r)+R_2(u,u,u)(r)+R_3(u,v,v)(r)\big]\mathrm{d}r\right\|_{L^\infty_{t\in[-\delta,\delta]}H_x^{s_1}}\\ &\lesssim\left\|\psi_\delta(t)\int_0^t e^{-a(t-r)\partial_x^3}\big[R_1(u,v,v)(r)+R_2(u,u,u)(r)+R_3(u,v,v)(r)\big]\mathrm{d}r\right\|_{X_a^{s_1,b}}\\ &\lesssim\|R_1(u,v,v)\|_{X_{a,\delta}^{s_1,b-1}}+\|R_2(u,u,u)\|_{X_{a,\delta}^{s_1,b-1}}+\|R_3(u,v,v)\|_{X_{a,\delta}^{s_1,b-1}}\end{aligned} \tag{5.16}$$

and similarly

$$\begin{aligned}&\left\|\int_0^t e^{-(t-r)\partial_x^3}\big[R_4(u,u,v)(r)+\frac{\beta}{3a}R_4(v,v,v)(r)+R_5(u,u,v)(r)\big]\mathrm{d}r\right\|_{L^\infty_{t\in[-\delta,\delta]}H_x^{s_1}}\\ &\lesssim\|R_4(u,u,v)\|_{X_{1,\delta}^{s_1,b-1}}+\|R_4(v,v,v)\|_{X_{1,\delta}^{s_1,b-1}}+\|R_5(u,u,v)\|_{X_{1,\delta}^{s_1,b-1}}.\end{aligned} \tag{5.17}$$

The following estimates for $R_j,\ j=1,2,3,4,5$ are necessary so as to close the argument. Their proofs will be given later on.

Proposition 5.3.6. *Assume that $u \in \dot{H}^s$. For $s > \frac{1}{2}$, $b - \frac{1}{2} > 0$ sufficiently small and $s_1 - s < \min\{1, s-\frac{1}{2}, s+2-\mu(\rho_a), 2s+1-\mu(\rho_a)\}$, we have*

$$\|R_1(u,v,w)\|_{X_a^{s_1,b-1}} \lesssim \|u\|_{X_a^{s,1/2}} \|v\|_{X_1^{s,1/2}} \|w\|_{X_1^{s,1/2}}. \tag{5.18}$$

When $a = \frac{3p(p-q)+q^2}{q^2}$, $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ with $\frac{q}{2} < p < q$, we replace the above requirement by $s_1 - s \le \min\{1, s-\frac{1}{2}\}$.

Proposition 5.3.7. *Assume that $u \in \dot{H}^s$. For $s > \frac{1}{2}$, $b - \frac{1}{2} > 0$ sufficiently small and $s_1 - s \le 1-$, we have*

$$\|R_2(u,v,w)\|_{X_a^{s_1,b-1}} \lesssim \|u\|_{X_a^{s,1/2}} \|v\|_{X_a^{s,1/2}} \|w\|_{X_a^{s,1/2}}.$$

Proposition 5.3.8. *Assume that $u \in \dot{H}^s$. For $s > \frac{1}{2}$, $b - \frac{1}{2} > 0$ sufficiently small and $s_1 - s \le 1$, we have*

$$\|R_3(u,v,w)\|_{X_a^{s_1,b-1}} \lesssim \|u\|_{X_a^{s,1/2}} \|v\|_{X_1^{s,1/2}} \|w\|_{X_1^{s,1/2}}.$$

Proposition 5.3.9. *Assume that $u \in \dot{H}^s$. For $s > \frac{1}{2}$, $b - \frac{1}{2} > 0$ sufficiently small and $s_1 - s < \min\{1, s+2-\mu(\rho_a), 2s+1-\mu(\rho_a)\}$, we have*

$$\|R_4(u,u,v)\|_{X_1^{s_1,b-1}} \lesssim \|u\|^2_{X_a^{s,1/2}} \|v\|_{X_1^{s,1/2}}.$$

When $a = \frac{3p(p-q)+q^2}{q^2}$, $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ with $\frac{q}{2} < p < q$, we replace the above requirement by $s_1 - s \le 1$.

Proposition 5.3.10. *For $s > \frac{1}{2}$, $b - \frac{1}{2} > 0$ sufficiently small and $s_1 - s < \min\{1, s+\frac{5}{2}-\mu(\rho_a), 2s+1-\mu(\rho_a)\}$, we have*

$$\|R_4(u,v,w)\|_{X_1^{s_1,b-1}} \lesssim \|u\|_{X_1^{s,1/2}} \|v\|_{X_1^{s,1/2}} \|w\|_{X_1^{s,1/2}}.$$

When $a = \frac{3p(p-q)+q^2}{q^2}$, $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ with $\frac{q}{2} < p < q$, we replace the above requirement by $s_1 - s \le 1$.

Proposition 5.3.11. *Assume that $u \in \dot{H}^s$. For $s > \frac{1}{2}$, $b - \frac{1}{2} > 0$ sufficiently small and $s_1 - s < \min\{1, s-\frac{1}{2}, s+\frac{5}{2}-\mu(\rho_a), 2s+1-\mu(\rho_a)\}$, we have*

$$\|R_5(u,u,v)\|_{X_1^{s_1,b-1}} \lesssim \|u\|^2_{X_a^{s,1/2}} \|v\|_{X_1^{s,1/2}}.$$

When $a = \frac{3p(p-q)+q^2}{q^2}$, $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ *with* $\frac{q}{2} < p < q$, *we replace the above requirement by* $s_1 - s \le \min\{1-, s - \frac{1}{2}\}$.

Using (5.16) and (5.17) together with the Propositions 5.3.6–5.3.11 in (5.14) and (5.15), we have

$$\left\| u(t) - e^{-at\partial_x^3} u_0 \right\|_{H^{s_1}} + \left\| v(t) - e^{-t\partial_x^3} v_0 \right\|_{H^{s_1}} \lesssim \big(\|u_0\|_{H^s} + \|v_0\|_{H^s} \big)^2$$
$$+ \big(\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \big)^2 + \int_0^t \big(\|u(r)\|_{H^s} + \|v(r)\|_{H^s} \big)^2 \, dr + \big(\|u\|_{X^{s,1/2}_{a,\delta}} + \|v\|_{X^{s,1/2}_{1,\delta}} \big)^3.$$

Next we shall obtain the polynomial growth bound stated in the theorem. To do so fix t large. Let $T(r) = \langle r \rangle^{\beta(s)}$. For $r \le t$, we have that

$$\|u(r)\|_{H^s} + \|v(r)\|_{H^s} \lesssim T(t).$$

Therefore, for $\delta \approx T(t)^{-\frac{3}{2}}$ and any $j \le t/\delta \approx tT(t)^{\frac{3}{2}}$,

$$\left\| u(j\delta) - e^{-\delta a\partial_x^3} u((j-1)\delta) \right\|_{H^{s_1}} + \left\| v(j\delta) - e^{-\delta\partial_x^3} v((j-1)\delta) \right\|_{H^{s_1}} \lesssim T(t)^3$$

where we have used the local theory bound

$$\|u\|_{X^{s,1/2}_{a,[(j-1)\delta,j\delta]}} + \|v\|_{X^{s,1/2}_{1,[(j-1)\delta,j\delta]}} \lesssim \|u((j-1)\delta)\|_{H^s} \lesssim T(t).$$

Letting $J = t/\delta \approx tT(t)^{\frac{3}{2}}$ yields that

$$\begin{aligned}\left\| u(J\delta) - e^{-J\delta a\partial_x^3} u_0 \right\|_{H^{s_1}} &\le \sum_{j=1}^{J} \left\| e^{-(J-j)\delta a\partial_x^3} u(j\delta) - e^{-(J-j+1)\delta a\partial_x^3} u((j-1)\delta) \right\|_{H^{s_1}} \\ &= \sum_{j=1}^{J} \left\| u(j\delta) - e^{-\delta a\partial_x^3} u((j-1)\delta) \right\|_{H^{s_1}} \lesssim JT(t)^3 \approx tT(t)^{9/2}.\end{aligned}$$

The similar estimate gives the same bound for v completing the demonstration of the growth bound. The continuity in $H^{s_1} \times H^{s_1}$ follows from the continuity of u and v in H^s, the embedding $X^{s,b}_a, X^{s,b}_1 \hookrightarrow C^0_t H^s_x$, and the estimates stated above, see [16].

5.4. Proofs of Smoothing Estimates

5.4.1. Proof of Proposition 5.3.6

We start by defining the functions:

$$f_1(k,\tau) = \langle k\rangle^s \langle \tau - ak^3\rangle^{\frac{1}{2}} |\widehat{u}_k(\tau)|,$$
$$f_2(k,\tau) = \langle k\rangle^s \langle \tau - k^3\rangle^{\frac{1}{2}} |\widehat{v}_k(\tau)|,$$
$$f_3(k,\tau) = \langle k\rangle^s \langle \tau - k^3\rangle^{\frac{1}{2}} |\widehat{w}_k(\tau)|.$$

Therefore using these functions, the convolution structure suggest to prove that

$$\left\| \int_{\sum \tau_j=\tau} \sum_{\sum k_j=k}^{*} M f_1(k_1,\tau_1) f_2(k_2,\tau_2) f_3(k_3,\tau_3) \right\|^2_{\ell^2_k L^2_\tau} \lesssim \prod_{j=1}^{3} \|f_j\|^2_{\ell^2_k L^2_\tau}, \tag{5.19}$$

where

$$M = \frac{|k_2| \langle k\rangle^{s_1} \langle k_1\rangle^{-s} \langle k_2\rangle^{-s} \langle k_3\rangle^{-s}}{|k_1+k_2-r_1k||k_1+k_2-r_2k| \langle \tau - ak^3\rangle^{1-b} \langle \tau_1 - ak_1^3\rangle^{1/2} \langle \tau_2 - k_2^3\rangle^{1/2} \langle \tau_3 - k_3^3\rangle^{1/2}}.$$

By the Cauchy-Schwarz inequality in τ_1, τ_2, k_1, k_2 variables, and the application of Young's inequality, the norm in the left hand side of (5.19) is estimated by

$$\sup_{k,\tau} \Big(\int_{\sum \tau_j=\tau} \sum_{\sum k_j=k}^{*} M^2 \Big) \left\| f_1^2 * f_2^2 * f_3^2 \right\|_{\ell^1_k L^1_\tau} \lesssim \sup_{k,\tau} \Big(\int_{\sum \tau_j=\tau} \sum_{\sum k_j=k}^{*} M^2 \Big) \prod_{j=1}^{3} \|f_j\|^2_{\ell^2_k L^2_\tau}.$$

Accordingly it suffices to demonstrate that the supremum above is finite. The implementation of the Lemma A.0.6 in the τ_1 and τ_2 integrals remove the τ dependence in the supremum and yields a bound

$$\sup_k \langle k\rangle^{2s_1} \sum_{k_1,k_2}^{*} \frac{|k_2|^2 \langle k_1\rangle^{-2s} \langle k_2\rangle^{-2s} \langle k-k_1-k_2\rangle^{-2s}}{(k_1+k_2-r_1k)^2 (k_1+k_2-r_2k)^2 \langle ak^3 - ak_1^3 - k_2^3 - (k-k_1-k_2)^3\rangle^{2-2b}}.$$

By a change of variable $k_2 \mapsto n - k_1$, it suffices to estimate

$$\sup_k \langle k\rangle^{2s_1} \sum_{k_1,n}^{*} \frac{\langle k_1\rangle^{-2s} \langle n-k_1\rangle^{-2s} \langle n-k\rangle^{-2s} |n-k_1|^2}{(n-r_1k)^2 (n-r_2k)^2 \langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2-2b}}. \tag{5.20}$$

Case A. $k_1 = k$

In this case, the supremum in (5.20) is replaced by

$$\sup_k \langle k\rangle^{2s_1-2s} \sum_n^* \frac{\langle n-k\rangle^{-4s}|n-k|^2}{(n-r_1k)^2(n-r_2k)^2}.$$

Case A.1. $|n-r_1k| \geq \delta|k|$, $|n-r_2k| \geq \delta|k|$

Note that $|n| \leq \left[\frac{r_j+\delta}{\delta}\right]|n-r_jk|$, $j=1,2$. Therefore, in this region $|n-k| \lesssim |n-r_jk|$, $j=1,2$. Then the supremum is bounded by

$$\sup_k \langle k\rangle^{2s_1-2s-2} \sum_n^* \langle n-k\rangle^{-4s} \lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \lesssim 1,$$

for $s_1 - s \leq 1$.

Case A.2. $\delta \leq |n-r_1k| < \delta|k|$ or $\delta \leq |n-r_2k| < \delta|k|$

Assume that $\delta \leq |n-r_1k| < \delta|k|$, the other case is similar. Notice that since $|n-k| < (1-r_1)|k| + \delta|k|$, we have $|n-k| \lesssim |k|$. Also the estimate $|n-r_2k| \gtrsim |k|$ follows from $|n-r_2k| \geq (r_1-r_2)|k| - |n-r_1k| \geq (r_1-r_2)|k| - \delta$. As a result, for small but fixed $\delta > 0$, using the Lemma A.0.6, the supremum is majorized by

$$\sup_k \langle k\rangle^{2s_1-2s} \sum_n^* \frac{\langle n-k\rangle^{-4s}}{\langle n-r_1k\rangle^2} \lesssim \sup_k \langle k\rangle^{2s_1-2s} \langle (1-r_1)k\rangle^{-2} \lesssim \langle k\rangle^{2s_1-2s-2} \lesssim 1$$

provided that $s_1 - s \leq 1$.

Case A.3. $|n-r_1k| < \delta$ or $|n-r_2k| < \delta$

Suppose that $|n-r_1k| < \delta$, the other case can be dealt in the same way. We note that $|n-k| \gtrsim |k|$, $|n-r_2k| \gtrsim |k|$ and $|n-r_1k| \gtrsim |k|^{1-\mu(r_1)-\epsilon}$. These estimates imply that the supremum above is bounded by $\sup_k \langle k\rangle^{2s_1-6s-2+2\mu(r_1)+2\epsilon} \lesssim 1$ whenever $s_1 - s < 2s + 1 - \mu(r_1)$.

Case B. $k_1 \neq k$

In this case we consider the following cases to show that the supremum (5.20) is finite.

Case B.1. $|n-r_1k| < \delta$ or $|n-r_2k| < \delta$

Assume the first case $|n-r_1k| < \delta$, the other case follows from a similar treatment. In this region, $|n-k| \gtrsim |k|$, $|n-r_2k| \gtrsim |k|$. Via these estimates, the resulting bound for (5.20) is as follows

$$\sup_k \langle k\rangle^{2s_1-2s-4+2\mu(r_1)+2\epsilon} \sum_{\substack{n\simeq r_1k \\ k_1}} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}(n-k_1)^2}{\langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2-2b}}. \tag{5.21}$$

Case B.1.1. $|k_1| < \delta|k|$

In this region, $(r_1-\delta)|k|-\delta < |n-r_1k| < (r_1+\delta)|k|+\delta$. As u is mean zero, $k_1 \neq 0$, hence we may write $k_1 = \eta_1 k$ for some $|k|^{-1} \leq |\eta_1| < \delta$. Also $n = r_1k+\eta_2$ for some $|\eta_2| < \delta$. Using these we obtain

$$
\begin{aligned}
|ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3|& \\
&= \Big|(k_1-k)\Big(\eta_1k^2\big((1-a)(1+\eta_1)-3r_1\big)+3\eta_2(2r_1-1-\eta_1)k+3\eta_2^2\Big)\Big| \\
&\geq |k_1-k|\Big(|k|\big(3r_1-(1-a)(1+\delta)-3\delta(2r_1-1+\delta)\big)-3\delta^2\Big) \\
&\gtrsim |k||k_1-k|.
\end{aligned}
$$

Therefore,

$$(5.21) \lesssim \sup_k \langle k\rangle^{2s_1-4s-4+2b+2\mu(r_1)+2\epsilon}\sum_{k_1}\langle k_1\rangle^{-2s}\langle k_1-k\rangle^{2b-2} \lesssim \sup_k \langle k\rangle^{2s_1-4s-6+4b+2\mu(r_1)+2\epsilon}$$

which is finite so long as $s_1 - s < s+2-\mu(r_1)$.

Case B.1.2. $|n-k_1| < \delta|k|$

Firstly note that $(r_1-\delta)|k|-\delta < |k_1| < (r_1+\delta)|k|+\delta$. We need to bound:

$$\sup_k \langle k\rangle^{2s_1-4s-2+2\mu(r_1)+2\epsilon}\sum_{\substack{k_1\\ n\simeq r_1k}}\frac{\langle n-k_1\rangle^{-2s}}{\langle ak^3-ak_1^3-(n-k_1)^3-(k-n)^3\rangle^{2-2b}}. \tag{5.22}$$

In the case $kk_1 < 0$, we write $n-k_1 = \eta_1k$ and $n = r_1k+\eta_2$ for some $|\eta_1| < \delta$, $|\eta_2| < \delta$ to get

$$
\begin{aligned}
|ak^3 - ak_1^3 &- (n-k_1)^3 - (k-n)^3| \\
&= |(r_1^3-\eta_1^3)k^3 - ak_1^3 + 3\eta_2(1-r_1)^2k^2 - 3\eta_2^2(1-r_1)k+\eta_2^3| \\
&\geq |(r_1^3-\eta_1^3)k^3 - ak_1^3| - 3\delta(1-r_1)^2k^2 - 3\delta^2(1-r_1)|k| - \delta^3 \\
&\geq (r_1^3-\delta^3)|k|^3 - 3\delta(1-r_1)^2k^2 - 3\delta^2(1-r_1)|k| - \delta^3 \\
&\gtrsim |k|^3
\end{aligned}
$$

by taking sufficiently small δ.

This yields that the supremum is bounded when $s_1 - s < s + \frac{5}{2} - \mu(r_1)$:

$$(5.22) \lesssim \sup_k \langle k\rangle^{2s_1-4s-8+6b+2\mu(r_1)+2\epsilon} \sum_{\substack{k_1 \\ n\simeq r_1 k}} \langle n-k_1\rangle^{-2s}$$
$$\lesssim \sup_k \langle k\rangle^{2s_1-4s-8+6b+2\mu(r_1)+2\epsilon} \lesssim 1.$$

Next we consider the case in which $kk_1 > 0$. Observing that $(n,k,k_1) \mapsto (-n,-k,-k_1)$ is a symmetry for (5.22), we may assume that $k, k_1 > 0$. By this assumption and the inequality $|n - r_1 k| < \delta$, we must have $n > 0$ as well, otherwise we would have $|n-k_1| \geq |n| \simeq r_1|k|$. For this case we just write $n = r_1 k + \eta$ for some $|\eta| < \delta$ to obtain

$$\begin{aligned}
&|ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3| \\
&\qquad = |k_1 - k||(1-a)(k^2 + kk_1 + k_1^2) + 3(r_1 k + \eta)^2 - (r_1 k + \eta)(k_1 + k)| \\
&\qquad = |k_1 - k||(1-a)k_1^2 + (-3r_1 + 1 - a)kk_1 + \mathcal{O}(\delta)(k + k_1) + \mathcal{O}(\delta^2)| \\
&\qquad \gtrsim |k - k_1|k^2
\end{aligned}$$

where the last inequality is always valid for k_1 satisfying $(r_1 - \delta)k - \delta < k_1 < (r_1+\delta)k + \delta$ with sufficiently small δ. Since $|k - k_1| \gtrsim 1$,

$$(5.22) \lesssim \sup_k \langle k\rangle^{2s_1-4s-2+2\mu(r_1)+2\epsilon} \sum_{\substack{k_1 \\ n\simeq r_1 k}} \frac{\langle n-k_1\rangle^{-2s}}{\langle (k-k_1)k^2\rangle^{2-2b}}$$
$$\lesssim \sup_k \langle k\rangle^{2s_1-4s-6+4b+2\mu(r_1)+2\epsilon} \lesssim \sum_{k_1} \frac{\langle k_1 - r_1 k\rangle^{-2s}}{\langle k_1 - k\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1-4s-8+6b+2\mu(r_1)+2\epsilon}$$

is finite provided that $s_1 - s < s + \frac{5}{2} - \mu(r_1)$.

Case B.1.3. $|n - k_1| \geq \delta|k|$, $|k_1| \geq \delta|k|$

Note that $|n - k_1| \leq \left(\frac{2-r_1}{\delta} + 1\right)|k_1| + \delta$. Since $s > 1/2$, we have

$$(5.21) \lesssim \sup_k \langle k\rangle^{2s_1-2s-4+2\mu(r_1)+2\epsilon} \sum_{\substack{n\simeq r_1 k \\ k_1}} \frac{\langle k_1\rangle^{-2s+1}\langle n-k_1\rangle^{-2s+1}}{\langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-6s-2+2\mu(r_1)+2\epsilon} \sum_{\substack{n\simeq r_1 k\\ k_1}} \langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2b-2}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-6s-2+2\mu(r_1)+2\epsilon} \lesssim 1$$

whenever $s_1 - s < 2s + 1 - \mu(r_1)$.

Case B.2. $\delta \leq |n - r_1 k| < \delta|k|$ or $\delta \leq |n - r_2 k| < \delta|k|$

Assume that $\delta \leq |n - r_1 k| < \delta|k|$, the other case can be treated in a similar fashion. In this region we note that $|n - r_2 k| > (r_1 - r_2 - \delta)|k|$ which implies $|n - r_2 k| \gtrsim |k|$. The other required estimates are $(r_1 - \delta)|k| < |n| < (r_1 + \delta)|k|$, $|n - k| \gtrsim |k|$. Accordingly we need to bound:

$$(5.20) \lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \sum_{\substack{k_1\\ |k|/4\leq|n|\leq 2|k|}} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}(n-k_1)^2}{\langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2-2b}}.$$

Case B.2.1. $|k_1| < \delta|k|$

Notice in this case that $|n - k_1| \lesssim |k|$. Hence the supremum above is bounded by

$$\sup_k \langle k\rangle^{2s_1-2s} \sum_{\substack{k_1\\ |n|\geq|k|/4}} \langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-2s} \sum_{|n|\geq|k|/4} \langle n\rangle^{-2s}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-4s+1} \lesssim 1$$

provided that $s_1 - s \leq s - 1/2$.

Case B.2.2. $|n - k_1| < \delta|k|$

The computation for this case is the same as that in the previous case.

Case B.2.3. $|k_1| \geq \delta|k|$, $|n - k_1| \geq \delta|k|$

Here $|n - k_1| \lesssim |k_1|$ which leads to the bound

$$\sup_k \langle k\rangle^{2s_1-2s-2} \sum_{\substack{k_1\\ |n|\leq 2|k|}} \frac{\langle k_1\rangle^{-2s+1}\langle n-k_1\rangle^{-2s+1}}{\langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-6s} \sum_{\substack{k_1\\ |n|\leq 2|k|}} \langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2b-2} \lesssim \sup_k \langle k\rangle^{2s_1-6s+1} \lesssim 1$$

for $s_1 - s \leq 2s - \frac{1}{2}$.

Case B.3. $|n - r_1 k| \geq \delta|k|$, $|n - r_2 k| \geq \delta|k|$

In this case, we make use of the inequality $|n - k_1| \leq \left(\frac{r_1+\delta}{\delta}\right)|n - r_1 k| + |k_1|$ so as to have $|n - k_1|^2 \lesssim \langle n - k_1\rangle\big(|k_1| + |n - r_1 k|\big)$.

Case B.3.1. $|k_1| \geq \delta|k|$, $|n - k_1| \geq \delta|k|$

In this region, using the inequality above, the supremum (5.20) can be bounded by

$$\begin{aligned}
\sup_k \langle k\rangle^{2s_1-4s-2} \sum_{k_1,n} & \frac{\langle n-k\rangle^{-2s}}{\langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \\
& \lesssim \sup_k \langle k\rangle^{2s_1-4s-2} \sum_n \langle n-k\rangle^{-2s} \\
& \lesssim \sup_k \langle k\rangle^{2s_1-4s-2} \lesssim 1
\end{aligned}$$

as long as $s_1 - s \leq s + 1$.

Case B.3.2. $|k_1| < \delta|k|$

In this case, the inequality $|n - k_1| < \left(\frac{r_1+2\delta}{\delta}\right)|n - r_1 k|$ gives rise to the bound

$$\begin{aligned}
(5.20) & \lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{k_1,n} \langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s} \\
& \lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_n \langle n\rangle^{-2s}\langle n-k\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \lesssim 1
\end{aligned}$$

for $s_1 - s \leq 1$.

Case B.3.3. $|n - k_1| < \delta|k|$

The computation in the preceding case works for this case as well.

5.4.2. Proof of Proposition 5.3.7

Following the argument in the proof of Proposition 5.3.6, we need to show that the supremum

$$\sup_k \langle k\rangle^{2s_1} \sum_{\substack{k_1\neq 0\\ k_2\\ (k_1+k_2)(k-k_1)(k-k_2)\neq 0}} \frac{\langle k_1\rangle^{-2s}\langle k_2\rangle^{-2s}\langle k-k_1-k_2\rangle^{-2s}}{|k_1|^2\langle (k-k_1)(k-k_2)(k_1+k_2)\rangle^{2-2b}}$$

is finite. By the change of variable $k_2 \mapsto n - k_1$, it suffices to show that

$$\sup_k \langle k\rangle^{2s_1} \sum_{k_1,n} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle n\rangle^{2-2b}\langle k_1-k\rangle^{2-2b}\langle n-k-k_1\rangle^{2-2b}}. \tag{5.23}$$

is finite.

Case A. $|k_1| \gtrsim |k|$

In this region,

$$(5.23) \lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{k_1,n} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle k\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1-2s-4+2b}$$

which is finite provided that $s_1 - s \leq 2 - b$.

Case B. $|k_1| \ll |k|$

In this case, the spremum is finite for $s_1 - s \leq 2 - 2b$:

$$\begin{aligned}
(5.23) &\lesssim \sup_k \langle k\rangle^{2s_1-2+2b} \sum_{k_1,n} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle n\rangle^{2-2b}\langle n-k-k_1\rangle^{2-2b}} \\
&\lesssim \sup_k \langle k\rangle^{2s_1-2+2b} \sum_{k_1,n} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle k+k_1\rangle^{2-2b}} \\
&\lesssim \sup_k \langle k\rangle^{2s_1-2s-4+4b} \lesssim 1.
\end{aligned}$$

5.4.3. Proof of Proposition 5.3.8

Proceeding as in the proof of Proposition 5.3.6, it suffices to show that the supremum

$$\sup_k \langle k\rangle^{2s_1} \sum_{\substack{k_1\neq 0\\ k_2}} \frac{\langle k_1\rangle^{-2s-2}\langle k_2\rangle^{-2s}\langle k-k_1-k_2\rangle^{-2s}}{\langle ak^3 - ak_1^3 - k_2^3 - (k-k_1-k_2)^3\rangle^{2-2b}}$$

is finite, or equivalently, by the change of variable $k_2 \mapsto n - k_1$, we shall show that the supremum

$$\sup_k \langle k\rangle^{2s_1} \sum_{\substack{k_1\neq 0\\ n}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \tag{5.24}$$

is finite.

Case A. $k_1 = k$

In this case, we have

$$(5.24) \lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \sum_n \langle n-k\rangle^{-4s} \lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \lesssim 1$$

for $s_1 - s \leq 1$.

Case B. $k_1 \neq k$

Case B.1. $|k_1| > \delta|k|$

In this case, (5.24) is finite provide that $s_1 - s \leq 1$:

$$\sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{k_1\neq 0\\ n}} \langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \lesssim 1.$$

Case B.2. $|k_1| \leq \delta|k|$

Case B.2.1. $|n-k_1| \leq \delta|k|$

In this case, we have $|n-k| \geq |k_1-k| - |k_1-n| \geq (1-2\delta)|k|$. By writing $k_1 = \eta_1 k$, $n - k_1 = \eta_2 k$ for some $|k|^{-1} \leq |\eta_1| \leq \delta$ and $0 \leq |\eta_2| \leq \delta$, we obtain

$$\begin{aligned} |ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3| &= \left| k^3\Big((1-\eta_1-\eta_2)^3 - a + a\eta_1^3 + \eta_2^3\Big)\right| \\ &= |k^3[1 - a + \mathcal{O}(\delta)]| \gtrsim |k|^3. \end{aligned}$$

Using the bound above we get

$$\begin{aligned} (5.24) &\lesssim \sup_k \langle k\rangle^{2s_1-2s} \sum_{\substack{k_1\neq 0\\ |n|\lesssim|k|}} \frac{\langle k_1\rangle^{-2s-2}\langle k_1-n\rangle^{-2s}}{\langle k^3\rangle^{2-2b}} \\ &\lesssim \sup_k \langle k\rangle^{2s_1-2s-6+6b} \sum_{k_1,n} \langle k_1\rangle^{-2s-2}\langle k_1-n\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-2s-6+6b} \lesssim 1 \end{aligned}$$

as long as $s_1 - s \le 3 - 3b$.

Case B.2.2. $|n-k| \le \delta |k|$

In this case, we have $|n - k_1| \gtrsim |k|$. Writing $k_1 = \eta_1 k$, $n - k = \eta_2 k$ for some $|k|^{-1} \le |\eta_1| \le \delta$ and $0 \le |\eta_2| \le \delta$, we get

$$\begin{aligned} |ak^3 - ak_1^3 - (n-k_1)^3 - (k-n)^3| &= \left| k^3 \Big((1 - \eta_1 + \eta_2)^3 - a + a\eta_1^3 - \eta_2^3 \Big) \right| \\ &= |k^3[1 - a + \mathcal{O}(\delta)]| \gtrsim |k|^3. \end{aligned}$$

Proceeding as in the previous case the supremum (5.24) can be shown to be finite in this region if $s_1 - s \le 3 - 3b$.

5.4.4. Proof of Proposition 5.3.9

Using the arguments of the proof of Proposition 5.3.6 we are left with a supremum

$$\begin{aligned} \sup_k \langle k \rangle^{2s_1} \sum_{k_1, k_2 \neq 0}^{*} & \frac{\langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k - k_1 - k_2 \rangle^{-2s}}{\big(k^3 - a(k_1+k_2)^3 - (k-k_1-k_2)^3\big)^2} \\ & \times \frac{|k-k_1-k_2|^2 |k_1+k_2|^2}{\langle k^3 - (k-k_1-k_2)^3 - ak_1^3 - ak_2^3 \rangle^{2-2b}}. \end{aligned}$$

By a change of variable $k_2 \mapsto n - k_1$, the supremum above takes the form

$$\sup_k \langle k \rangle^{2s_1} \sum_{\substack{n \neq 0 \\ k_1 \neq 0}}^{*} \frac{\langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s} |n-k|^2}{(n - \widetilde{r}_1 k)^2 (n - \widetilde{r}_2 k)^2 \langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3 \rangle^{2-2b}}. \tag{5.25}$$

Note here that the condition $n \neq 0$ results from the factor n^2 appearing in the denominator of the prior sum that is reduced to the one in (5.25).

Case A. $|n - \widetilde{r}_1 k| < \delta$ or $|n - \widetilde{r}_2 k| < \delta$

Assume the first case $|n - \widetilde{r}_1 k| < \delta$. Handling the other one is similar. Note that

$$\begin{aligned} & |n - \widetilde{r}_2 k| \ge (\widetilde{r}_2 - \widetilde{r}_1)|k| - |n - \widetilde{r}_1 k| > (\widetilde{r}_2 - \widetilde{r}_1)|k| - \delta, \\ & (\widetilde{r}_1 - 1)|k| - \delta < |n - k| < (\widetilde{r}_1 - 1)|k| + \delta \end{aligned}$$

and $|n-\widetilde{r}_1 k| \geq |k|\frac{K(\widetilde{r}_1,\epsilon)}{|k|^{\mu(\widetilde{r}_1)+\epsilon}} \gtrsim |k|^{1-\mu(\widetilde{r}_1)-\epsilon}$. These estimates imply that

$$(5.25) \lesssim \sup_k \langle k\rangle^{2s_1-2s-2+2\mu(\widetilde{r}_1)+2\epsilon} \sum_{\substack{n\simeq\widetilde{r}_1 k\\ k_1\neq 0}}^{*} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}}{\langle k^3-ak_1^3-a(n-k_1)^3-(k-n)^3\rangle^{2-2b}}$$

Case A.1. $|k_1| \geq \delta|k|$, $|n-k_1| \geq \delta|k|$

The supremum above in this case is bounded by

$$\sup_k \langle k\rangle^{2s_1-6s-2+2\mu(\widetilde{r}_1)+2\epsilon} \sum_{\substack{n\simeq\widetilde{r}_1 k\\ k_1\neq 0}}^{*} \langle k^3-ak_1^3-a(n-k_1)^3-(k-n)^3\rangle^{2b-2}.$$

Write $n=\widetilde{r}_1 k+\eta$ for some $|\eta|<\delta$ to get

$$\begin{aligned}|k^3-ak_1^3-&a(n-k_1)^3-(k-n)^3| \\ &= |(\widetilde{r}_1 k+\eta)[3ak_1^2-3a\widetilde{r}_1 k_1 k+\mathcal{O}(\delta)(k_1+k)+\mathcal{O}(\delta^2)]| \\ &\gtrsim |3ak_1^2-3a\widetilde{r}_1 k_1 k+\mathcal{O}(\delta)(k_1+k)+\mathcal{O}(\delta^2)|.\end{aligned}$$

Use this estimate along with the second claim of Lemma A.0.6 for the sum in k_1 to conclude that the supremum is finite whenever $s_1-s<2s+1-\mu(\widetilde{r}_1)$.

Case A.2. $|k_1|<\delta|k|$

Here $|n-k_1| \gtrsim |k|$ since $|n-k_1|>(\widetilde{r}_1-\delta)|k|-\delta$. As above we write $n=\widetilde{r}_1 k+\eta$ for some $|\eta|<\delta$ to obtain

$$\begin{aligned}|k^3-ak_1^3-&a(n-k_1)^3-(k-n)^3| \\ &= |(\widetilde{r}_1 k+\eta)[3ak_1(\widetilde{r}_1 k-k_1)+\mathcal{O}(\delta)(k_1+k)+\mathcal{O}(\delta^2)]| \gtrsim |k_1||k|^2.\end{aligned}$$

It follows that the supremum is bounded by

$$\sup_k \langle k\rangle^{2s_1-4s-2+2\mu(\widetilde{r}_1)+2\epsilon} \sum_{k_1\neq 0} \frac{\langle k_1\rangle^{-2s-1}}{\langle k\rangle^{4-4b}} \lesssim \sup_k \langle k\rangle^{2s_1-4s-6+4b+2\mu(\widetilde{r}_1)+2\epsilon}$$

that is finite only if $s_1-s<s+2-\mu(\widetilde{r}_1)$.

Case A.3. $|n-k_1|<\delta|k|$

Clearly $|k_1|\gtrsim|k|$. Moreover, first writing $n=\widetilde{r}_1k+\eta$ for some $|\eta|<\delta$, and then reinstating the variable n we get

$$|k^3-ak_1^3-a(n-k_1)^3-(k-n)^3|=|n||3ak_1(n-k_1)+\mathcal{O}(\delta)k+\mathcal{O}(\delta^2)|\gtrsim|k|^2|n-k_1|.$$

Therefore, by the mean zero assumption on u, $n-k_1\neq0$, we have

$$\sup_k\langle k\rangle^{2s_1-4s-2+2\mu(\widetilde{r}_1)+2\epsilon}\sum_{\substack{n\simeq\widetilde{r}_1k\\|k_1|\gtrsim|k|}}\frac{\langle k_1-n\rangle^{-2s-1}}{\langle k\rangle^{4-4b}}\lesssim\sup_k\langle k\rangle^{2s_1-4s-6+4b+2\mu(\widetilde{r}_1)+2\epsilon}\lesssim1$$

provided that $s_1-s<s+2-\mu(\widetilde{r}_1)$.

Case B. $\delta\leq|n-\widetilde{r}_1k|<\delta|k|$ or $\delta\leq|n-\widetilde{r}_2k|<\delta|k|$

Suppose that $\delta\leq|n-\widetilde{r}_1k|<\delta|k|$, the other case is analogous. Notice in this case that $|n-\widetilde{r}_2k|\gtrsim|k|$ since $|n-\widetilde{r}_2k|\geq(\widetilde{r}_2-\widetilde{r}_1)|k|-|n-\widetilde{r}_1k|>(\widetilde{r}_2-\widetilde{r}_1-\delta)|k|$. Furthermore, $(\widetilde{r}_1-1-\delta)|k|<|n-k|<(\widetilde{r}_1-1+\delta)|k|$. So we have

$$(5.25)\lesssim\sup_k\langle k\rangle^{2s_1-2s}\sum_{\substack{n\neq0\\k_1\neq0}}\frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}}{\langle k^3-ak_1^3-a(n-k_1)^3-(k-n)^3\rangle^{2-2b}}.$$

Case B.1. $|k_1|\geq\delta|k|$, $|n-k_1|\geq\delta|k|$

In this region, using Lemma A.0.6 the supremum is bounded by

$$\begin{aligned}&\sup_k\langle k\rangle^{2s_1-4s}\sum_{\substack{n\neq0\\|k_1|\gtrsim|k|}}\frac{\langle k_1\rangle^{-2s}}{\langle k^3-ak_1^3-a(n-k_1)^3-(k-n)^3\rangle^{2-2b}}\\&\lesssim\sup_k\langle k\rangle^{2s_1-4s}\sum_{|k_1|\gtrsim|k|}\langle k_1\rangle^{-2s}\\&\lesssim\sup_k\langle k\rangle^{2s_1-6s+1}\lesssim1,\end{aligned}$$

when $s_1-s\leq2s-\frac{1}{2}$.

Case B.2. $|k_1|<\delta|k|$

Notice that $|n-k_1|\gtrsim|k|$ because $|n-k_1|>(\widetilde{r}_1-2\delta)|k|$. We write $n=(\widetilde{r}_1+\eta)k$ for some $|\eta|<\delta$ to attain

$$|k^3-ak_1^3-a(n-k_1)^3-(k-n)^3|=|(\widetilde{r}_1+\eta)k[3ak_1((\widetilde{r}_1+\eta)k-k_1)+\mathcal{O}(\delta)k^2]|$$

$$\geq 3a(\widetilde{r}_1 - 2\delta)^2|k_1||k|^2 \gtrsim |k|^2|k_1|.$$

Thus the supremum is finite when $s_1 - s < s + \frac{1}{2}$:

$$\sup_k \langle k\rangle^{2s_1-4s} \sum_{\substack{|n|\lesssim|k| \\ k_1\neq 0}} \frac{\langle k_1\rangle^{-2s}}{\langle k^2 k_1\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1-4s-3+4b} \sum_{|k_1|\lesssim|k|} \langle k_1\rangle^{-2s-2+2b} \lesssim \sup_k \langle k\rangle^{2s_1-4s-3+4b}.$$

Case B.3. $|n - k_1| < \delta|k|$

In this case $|k_1| \gtrsim |k|$ due to $|k_1| > (\widetilde{r}_1 - 2\delta)|k|$. Accordingly, as in the previous case, first writing $n = (\widetilde{r}_1 + \eta)k$ for some $|\eta| < \delta$ and then reinstating the variable n, we have

$$\begin{aligned} |k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3| &= \big|(\widetilde{r}_1 + \eta)k\big(3ak_1((\widetilde{r}_1 + \eta)k - k_1) + \mathcal{O}(\delta)k^2\big)\big| \\ &\gtrsim |k|^2|n - k_1|. \end{aligned}$$

This, recalling the mean zero assumption on u, gives rise to the bound for the supremum

$$\sup_k \langle k\rangle^{2s_1-4s} \sum_{\substack{|n|\lesssim|k| \\ |k_1|\gtrsim|k|}} \frac{\langle n-k_1\rangle^{-2s-1}}{\langle k^2\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1-4s-3+4b} \lesssim 1$$

on the condition that $s_1 - s < s + \frac{1}{2}$.

Case C. $|n - \widetilde{r}_1 k| \geq \delta|k|$, $|n - \widetilde{r}_2 k| \geq \delta|k|$

We note that $|n| \leq \left[\frac{\widetilde{r}_j + \delta}{\delta}\right]|n - \widetilde{r}_j k|$, $j = 1, 2$. In this region, this implies that $|n - k| \lesssim |n - \widetilde{r}_1 k|$. Hence the supremum is finite if $s_1 - s \leq 1$:

$$\begin{aligned} (5.25) &\lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{n\neq 0 \\ k_1\neq 0}} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \\ &\lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{n\neq 0 \\ k_1\neq 0}} \langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s} \\ &\lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_n \langle n\rangle^{-2s}\langle n-k\rangle^{-2s} \\ &\lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \lesssim 1. \end{aligned}$$

5.4.5. Proof of Proposition 5.3.10

We are to handle the supremum

$$\sup_k \langle k\rangle^{2s_1} \sum_{k_1,k_2}^{*} \frac{\langle k_1\rangle^{-2s}\langle k_2\rangle^{-2s}\langle k-k_1-k_2\rangle^{-2s}}{(k_1+k_2)^2(k_1+k_2-\widetilde{r}_1k)^2(k_1+k_2-\widetilde{r}_2k)^2} \times \frac{|k-k_1-k_2|^2|k_1+k_2|^2}{\langle (k-k_1)(k-k_2)(k_1+k_2)\rangle^{2-2b}}$$

which is equivalent, by a change of variable $k_2 \mapsto n-k_1$, to

$$\sup_k \langle k\rangle^{2s_1} \sum_{\substack{n\neq 0\\ k_1}}^{*} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}|n-k|^2}{(n-\widetilde{r}_1k)^2(n-\widetilde{r}_2k)^2\langle n(k-k_1)(k+k_1-n)\rangle^{2-2b}}. \tag{5.26}$$

In the case $n(k-k_1)(k+k_1-n)=0$, that is either $k_1=k$ or $k_1=n-k$, (5.26) boils down to

$$\sup_k \langle k\rangle^{2s_1-2s} \sum_{n\neq 0}^{*} \frac{\langle n-k\rangle^{-4s}|n-k|^2}{(n-\widetilde{r}_1k)^2(n-\widetilde{r}_2k)^2}$$

which essentially can be treated as that in the Case A. of the proof of Proposition 5.3.6. Hence the supremum is finite if $s_1-s\le 1$ and $s_1-s<2s+1-\mu(\widetilde{r}_j)$. Next we move to the complementary case:

Case A. $n(k-k_1)(k+k_1-n)\neq 0$

In this case,

$$(5.26) \lesssim \sup_k \langle k\rangle^{2s_1} \sum_{\substack{n\neq 0\\ k_1}}^{*} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}|n-k|^2}{(n-\widetilde{r}_1k)^2(n-\widetilde{r}_2k)^2\langle n\rangle^{2-2b}\langle n-k-k_1\rangle^{2-2b}\langle k_1-k\rangle^{2-2b}}.$$

Case A.1. $|n-\widetilde{r}_1k|\ge \delta|k|$, $|n-\widetilde{r}_2k|\ge\delta|k|$

In this region, $|n|\le \left(\frac{\widetilde{r}_j+\delta}{\delta}\right)|n-\widetilde{r}_jk|$, $j=1,2$. Thus, $|n-k|\lesssim|n-\widetilde{r}_1k|$, by which the supremum above is estimated by

$$\sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{n\neq 0\\ k_1}} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle n\rangle^{2-2b}\langle n-k-k_1\rangle^{2-2b}\langle k_1-k\rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{n\neq 0\\ k_1}} \langle k_1\rangle^{-2s} \langle n-k_1\rangle^{-2s} \langle n-k\rangle^{-2s}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \lesssim 1$$

for $s_1 - s \leq 1$.

Case A.2. $\delta \leq |n-\widetilde{r}_1 k| < \delta|k|$ or $\delta \leq |n-\widetilde{r}_2 k| < \delta|k|$

Assume the case $\delta \leq |n-\widetilde{r}_1 k| < \delta|k|$, the other one is treated similarly. In this case, the estimates

$$|n-\widetilde{r}_2 k| \geq (\widetilde{r}_2-\widetilde{r}_1)|k| - |n-\widetilde{r}_1 k| > (\widetilde{r}_2-\widetilde{r}_1-\delta)|k|,$$

$(\widetilde{r}_1-\delta)|k| < |n| < (\widetilde{r}_1+\delta)|k|$, and $(\widetilde{r}_1-1-\delta)|k| < |n-k| < (\widetilde{r}_1-1+\delta)|k|$ lead to the bound

$$\sup_k \langle k\rangle^{2s_1-2s-2+2b} \sum_{\substack{n\neq 0\\ k_1}} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}}{\langle k_1-k\rangle^{2-2b}\langle k_1+k-n\rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-2s-4+4b} \sum_{k_1} \frac{\langle k_1\rangle^{-2s}}{\langle k_1-k\rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-2s-6+6b} \lesssim 1$$

provided that $s_1 - s \leq 3-3b$.

Case A.3. $|n-\widetilde{r}_1 k| < \delta$ or $|n-\widetilde{r}_2 k| < \delta$

Assume that $|n-\widetilde{r}_1 k| < \delta$, the treatment of the other case is similar. Note that

$$(\widetilde{r}_1-1)|k| - \delta < |n-k| < (\widetilde{r}_1-1)|k| + \delta, \tag{5.27}$$

$$|n-\widetilde{r}_2 k| \geq (\widetilde{r}_2-\widetilde{r}_1)|k| - |n-\widetilde{r}_1 k| > (\widetilde{r}_2-\widetilde{r}_1)|k| - \delta. \tag{5.28}$$

Therefore the supremum is majorized by

$$\sup_k \langle k\rangle^{2s_1-2s-4+2b+2\mu(\widetilde{r}_1)+2\epsilon} \sum_{\substack{n\simeq\widetilde{r}_1 k\\ k_1}} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}}{\langle k-k_1\rangle^{2-2b}\langle n-k-k_1\rangle^{2-2b}}. \tag{5.29}$$

There is merely a single term for the sum in n, that is the one with $n \simeq \widetilde{r}_1 k$. So only the estimate regarding the sum in k_1 matters here. If $|k_1| \ll |k|$ then all the other factors in the sum in (5.27) are of order $\gtrsim |k|$; likewise if $|n-k_1| \ll |k|$ then the remaining factors are again of order $\gtrsim |k|$. Thus in these cases, the sum in (5.27) $\lesssim \langle k\rangle^{-4-2s+4b}$ entailing (5.27) $\lesssim \langle k\rangle^{2s_1-4s-8+6b+2\mu(\widetilde{r}_1)+2\epsilon}$ which is finite as long as $2s_1-4s-8+6b+2\mu(\widetilde{r}_1)+2\epsilon \leq 0$ or equivalently $s_1 - s < s+5/2-\mu(\widetilde{r}_1)$. If $|k_1-k| \ll |k|$ then the factors with exponent $-2s$ in the numerator are of order $\gtrsim |k|$; likewise if $|n-k-k_1| \ll |k|$ then the factors in the numerator are of order $\gtrsim |k|$. In the either case, the sum in (5.27) $\lesssim \langle k\rangle^{-4s-3+4b}$ giving rise to (5.27) $\lesssim \langle k\rangle^{2s_1-6s-7+6b+2\mu(\widetilde{r}_1)+2\epsilon} \lesssim 1$ provided that $s_1 - s < 2s+2-\mu(\widetilde{r}_1)$.

5.4.6. Proof of Proposition 5.3.11

In order to handle R_5, we need to divide the sum into pieces where $k_1+k_2 \neq 0$ and $k_1+k_2 = 0$.

$$
\begin{aligned}
R_5(u,u,v)_k &= -9kv_k \sum_{k_1\neq 0}^{*} \frac{(k-k_1)|u_{k_1}|^2}{k^3-ak_1^3-(k-k_1)^3} + 9i \sum_{\substack{k_1+k_2+k_3=k\\ k_1+k_2\neq 0\\ k_1\neq 0}}^{*} \frac{k_3(k_2+k_3)u_{k_1}u_{k_2}v_{k_3}}{k^3-ak_1^3-(k_2+k_3)^3} \\
&= -9kv_k \sum_{k_1>0}^{*} \left(\frac{k+k_1}{k^3+ak_1^3-(k+k_1)^3} + \frac{k-k_1}{k^3-ak_1^3-(k-k_1)^3} \right) |u_{k_1}|^2 \\
&\quad + 9i \sum_{\substack{k_1+k_2+k_3=k\\ k_1+k_2\neq 0\\ k_1\neq 0}}^{*} \frac{k_3(k_2+k_3)u_{k_1}u_{k_2}v_{k_3}}{k^3-ak_1^3-(k_2+k_3)^3} \\
&= 18kv_k \sum_{k_1>0}^{*} \frac{k_1^2|u_{k_1}|^2}{(1-a)(k_1-\widetilde{r}_1 k)(k_1+\widetilde{r}_1 k)(k_1-\widetilde{r}_2 k)(k_1+\widetilde{r}_2 k)} \\
&\qquad + 9i \sum_{\substack{k_1+k_2+k_3=k\\ k_1+k_2\neq 0\\ k_1\neq 0}}^{*} \frac{k_3(k_2+k_3)u_{k_1}u_{k_2}v_{k_3}}{k^3-ak_1^3-(k_2+k_3)^3} =: S_1 + S_2.
\end{aligned}
$$

For the first sum, using Cauchy-Schwarz and Young inequalities and the Lemma A.0.6 yields that,

$$\|S_1\|_{X_1^{s_1,b-1}} \lesssim \sup_k \langle k\rangle^{2s_1+2-2s} \sum_{k_1>0}^{*} \frac{\langle k_1\rangle^{4-4s}}{(k_1-\widetilde{r}_1k)^2(k_1+\widetilde{r}_1k)^2(k_1-\widetilde{r}_2k)^2(k_1+\widetilde{r}_2k)^2} \times \|u\|^2_{X_a^{s,1/2}} \|v\|_{X_1^{s,1/2}}.$$

Thus it is required to show that the supremum above is finite. Since $k_1 > 0$ and $\widetilde{r}_1, \widetilde{r}_2 > 0$, to take advantage of the multipliers in the denominator of the sum in the supremum, we consider the cases in which $k < 0$ and $k > 0$. We just examine the $k < 0$ case as the other case can be treated similarly. Thus, by the sign considerations, both $|k_1 - \widetilde{r}_1k|$ and $|k_1 - \widetilde{r}_2k|$ are of order $\gtrsim |k|, k_1$ by which the supremum is replaced by the bound

$$\sup_k \langle k\rangle^{2s_1-2s} \sum_{k_1>0}^{*} \frac{\langle k_1\rangle^{2-4s}}{(k_1+\widetilde{r}_1k)^2(k_1+\widetilde{r}_2k)^2}. \tag{5.30}$$

First we observe that the case $|k_1 + \widetilde{r}_1k|, |k_1 + \widetilde{r}_2k| \le \delta|k|$ cannot arise concurrently, because choosing $\delta < \frac{\widetilde{r}_2-\widetilde{r}_1}{2}$ entails that

$$(\widetilde{r}_2 - \widetilde{r}_1)|k| \le |k_1 + \widetilde{r}_1k| + |k_1 + \widetilde{r}_2k| \le 2\delta|k| < (\widetilde{r}_2 - \widetilde{r}_1)|k|.$$

We consider the following cases:

Case A. $|k_1 + \widetilde{r}_1k|, |k_1 + \widetilde{r}_2k| \ge \delta|k|$

In this case, $|k_1 + \widetilde{r}_jk| \ge \left(\frac{\delta}{\delta+\widetilde{r}_j}\right)k_1$, $j = 1, 2$, that implies

$$(5.30) \lesssim \sup_k \langle k\rangle^{2s_1-2s-2} \sum_{k_1>0} \langle k_1\rangle^{-4s}$$

which is finite provided that $s_1 - s \le 1$.

Case B. $|k_1 + \widetilde{r}_1k| \ge \delta|k|$, $\delta \le |k_1 + \widetilde{r}_2k| < \delta|k|$ (or with the roles of $\widetilde{r}_1$ and $\widetilde{r}_2$ are switched)

Note that $(\widetilde{r}_2 - \delta)|k| < k_1 < (\widetilde{r}_2 + \delta)|k|$. Then the supremum is bounded by

$$\sup_k \langle k\rangle^{2s_1-2s-1} \sum_{k_1\ge|k|} \langle k_1\rangle^{-4s+1}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-6s+1} \lesssim 1$$

for $s_1 - s \leq 2s - \frac{1}{2}$.

Case C. $|k_1 + \widetilde{r}_1 k| \geq \delta|k|$, $|k_1 + \widetilde{r}_2 k| < \delta$ (or with the roles of $\widetilde{r}_1$ and $\widetilde{r}_2$ are switched)

Using the bound $|k_1 + \widetilde{r}_2 k| \gtrsim |k|^{1-\mu(\widetilde{r}_2)-\epsilon}$ and $k_1 \simeq -\widetilde{r}_2 k$,

$$(5.30) \lesssim \sup_k \langle k\rangle^{2s_1-6s-2+2\mu(\widetilde{r}_2)+2\epsilon} \lesssim 1$$

as long as $s_1 - s < 2s + 1 - \mu(\widetilde{r}_2)$. As for the $X_1^{s_1,b-1}$ norm of the sum S_2, proceeding as before, we need to show that

$$\sup_k \langle k\rangle^{2s_1} \sum_{\substack{k_1\neq 0, k_2\\ k_1+k_2\neq 0}} \frac{\langle k_1\rangle^{-2s}\langle k_2\rangle^{-2s}\langle k-k_1-k_2\rangle^{-2s}|k-k_1-k_2|^2|k-k_1|^2}{\left[k^3 - ak_1^3 - (k_2+k_3)^3\right]^2 \langle k^3 - ak_1^3 - ak_2^3 - (k-k_1-k_2)^3\rangle^{2-2b}} \lesssim 1.$$

This, by the change of variable $k_2 \mapsto n - k_1$, is equivalent to estimate

$$\sup_k \langle k\rangle^{2s_1} \sum_{\substack{k_1\neq 0\\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}|n-k|^2|k-k_1|^2}{(k_1-\widetilde{r}_1 k)^2(k_1-\widetilde{r}_2 k)^2\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}. \tag{5.31}$$

Case A. $|k_1 - \widetilde{r}_1 k| < \delta$ or $|k_1 - \widetilde{r}_2 k| < \delta$

The treatment of the both cases are similar, so assume that $|k_1 - \widetilde{r}_1 k| < \delta$. We have the following estimates

$$|k_1 - k| \leq (\widetilde{r}_1 - 1)|k| + |k_1 - \widetilde{r}_1 k| < (\widetilde{r}_1 - 1)|k| + \delta,$$
$$|k_1 - \widetilde{r}_2 k| \geq (\widetilde{r}_2 - \widetilde{r}_1)|k| - |k_1 - \widetilde{r}_1 k| > (\widetilde{r}_2 - \widetilde{r}_1)|k| - \delta.$$

Case A.1. $|n - k_1| \geq \delta|k|$, $|n-k| \geq \delta|k|$

Using the inequality $|n-k| \leq |n-k_1| + |k_1 - k|$, the relation $-2s+1 < 0$ and the above estimates,

$$\begin{aligned}(5.31) &\lesssim \sup_k \langle k\rangle^{2s_1} \sum_{\substack{k_1\simeq\widetilde{r}_1 k\\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s+1}\langle n-k\rangle^{-2s+1}}{(k_1-\widetilde{r}_1 k)^2\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}\\
&\quad + \sup_k \langle k\rangle^{2s_1+1} \sum_{\substack{k_1\simeq\widetilde{r}_1 k\\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s+1}}{(k_1-\widetilde{r}_1 k)^2\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}\\
&\lesssim \sup_k \langle k\rangle^{2s_1-6s-2+2\mu(\widetilde{r}_1)+2\epsilon} \sum_{\substack{k_1\simeq\widetilde{r}_1 k\\ n\neq 0}} \frac{1}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}\end{aligned}$$

which is finite provided that $s_1 - s < 2s + 1 - \mu(\widetilde{r}_1)$.

Case A.2. $|n-k| < \delta|k|$

Here $|n-k_1| \geq (\widetilde{r}_1 - 1)|k| - |k-n| - |k_1 - \widetilde{r}_1 k| > (\widetilde{r}_1 - 1 - \delta)|k| - \delta$. In this region for $|\eta_1|, |\eta_2| < \delta$, we may write $n - k = \eta_1 k$ and $k_1 - \widetilde{r}_1 k = \eta_2$. So we have

$$\begin{aligned}|k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3| &= |k^3 - a(\widetilde{r}_1 k + \eta_2)^3 - a\big((1+\eta_1 - \widetilde{r}_1)k - \eta_2\big)^3 + \eta_1^3 k^3| \\ &= |\big(1 - a + 3a\widetilde{r}_1 - 3a\widetilde{r}_1^2 + \mathcal{O}(\delta)\big)k^3 + \mathcal{O}(\delta)k^2 + \mathcal{O}(\delta^2)k + \mathcal{O}(\delta^3)| \gtrsim |k|^3,\end{aligned}$$

the last inequality follows since $\widetilde{r}_1$ is the root of the quadratic $(1-a)x^2 - 3x + 3$. Using these bounds

$$(5.31) \lesssim \sup_k \langle k\rangle^{2s_1 - 4s - 2 + 2\mu(\widetilde{r}_1) + 2\epsilon} \sum_{n \neq 0} \frac{\langle n-k\rangle^{-2s}}{\langle k^3\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1 - 4s - 8 + 6b + 2\mu(\widetilde{r}_1) + 2\epsilon} \lesssim 1$$

for $s_1 - s < s + 4 - 3b - \mu(\widetilde{r}_1)$.

Case A.3. $|n-k_1| < \delta|k|$

In this region, $(\widetilde{r}_1 - 1 - \delta)|k| - \delta < |n-k| < (\widetilde{r}_1 - 1 + \delta)|k| + \delta$. So we may write $n - k = \eta_1 k$ for some η_1 with $|k|^{-1} \leq |\eta_1| \leq \epsilon$ and $k_1 - \widetilde{r}_1 k = \eta_2$ for some η_2 with $|\eta_2| < \delta < \epsilon$. Therefore,

$$\begin{aligned}|k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3| &= |k^3 - a(\widetilde{r}_1 k + \eta_2)^3 - a\big((1+\eta_1 - \widetilde{r}_1)k - \eta_2\big)^3 + \eta_1^3 k^3| \\ &= |\big(1 - a + 3a\widetilde{r}_1 - 3a\widetilde{r}_1^2 + \mathcal{O}(\epsilon)\big)k^3 + \mathcal{O}(\delta)k^2 + \mathcal{O}(\delta^2)k + \mathcal{O}(\delta^3)| \gtrsim |k|^3,\end{aligned}$$

it follows, as in the previous case, that the supremum is bounded for $s_1 - s < s + 4 - 3b - \mu(\widetilde{r}_1)$:

$$(5.31) \lesssim \sup_k \langle k\rangle^{2s_1 - 4s - 2 + 2\mu(\widetilde{r}_1) + 2\epsilon} \sum_{n \neq 0} \frac{\langle n-k_1\rangle^{-2s}}{\langle k^3\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1 - 4s - 8 + 6b + 2\mu(\widetilde{r}_1) + 2\epsilon} \lesssim 1.$$

Case B. $\delta \leq |k_1 - \widetilde{r}_1 k| < \delta|k|$ or $\delta \leq |k_1 - \widetilde{r}_2 k| < \delta|k|$

We assume the first case $\delta \leq |k_1 - \widetilde{r}_1 k| < \delta|k|$; the second one can be treated in a similar fashion. In this region, we have the estimates: $|k_1 - k| < (\widetilde{r}_1 - 1 + \delta)|k|$, $|k_1 - \widetilde{r}_2 k| \geq (\widetilde{r}_2 - \widetilde{r}_1)|k| - |k_1 - \widetilde{r}_1 k| > (\widetilde{r}_2 - \widetilde{r}_1 - \delta)|k|$.

Also $|k_1 - \widetilde{r}_1 k| < \delta |k|$ implies $|k_1| > |k|$. Thus,

$$(5.31) \lesssim \sup_k \langle k\rangle^{2s_1} \sum_{\substack{|k_1|>|k| \\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}|n-k|^2}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}. \tag{5.32}$$

Case B.1. $|n - k_1| \geq \delta|k|$, $|n-k| \geq \delta|k|$

In this case,

$$\begin{aligned}
(5.32) \lesssim& \sup_k \langle k\rangle^{2s_1} \sum_{\substack{|k_1|>|k| \\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s+1}\langle n-k\rangle^{-2s+1}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \\
&+ \sup_k \langle k\rangle^{2s_1} \sum_{\substack{|k_1|>|k| \\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s+1}|k_1-k|}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \\
&\lesssim \sup_k \langle k\rangle^{2s_1-4s} \sum_{\substack{|k_1|>|k| \\ n\neq 0}} \frac{\langle k_1\rangle^{-2s}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \\
&\lesssim \sup_k \langle k\rangle^{2s_1-4s} \sum_{|k_1|>|k|} \langle k_1\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-6s+1} \lesssim 1
\end{aligned}$$

for $s_1 - s \leq 2s - \frac{1}{2}$.

Case B.2. $|n-k| < \delta|k|$

The required estimate specific to this region is

$$|n-k_1| \geq (\widetilde{r}_1 - 1)|k| - |k_1 - \widetilde{r}_1 k| - |n-k| > (\widetilde{r}_1 - 1 - 2\delta)|k|.$$

Also the restriction $|n-k| < \delta|k|$ entailing $|n| \lesssim |k|$ is essential for the summability in the n-variable. Let η_j be some constants satisfying $|\eta_j| < \delta$, $j = 1, 2$, for which $n - k = \eta_1 k$ and $k_1 - \widetilde{r}_1 k = \eta_2 k$. Then

$$\begin{aligned}
|k^3 - ak_1^3 - &a(n-k_1)^3 - (k-n)^3| \\
&= |k^3 - a(\widetilde{r}_1 + \eta_2)^3 k^3 - a(1 + \eta_1 - \widetilde{r}_1 - \eta_2)^3 k^3 - \eta_1^3 k^3| \\
&= |\big(1 - a + 3a\widetilde{r}_1(1-\widetilde{r}_1) + \mathcal{O}(\delta)\big)k^3| \gtrsim |k|^3.
\end{aligned}$$

Using the above estimates

$$\begin{aligned}(5.32) &\lesssim \sup_k \langle k\rangle^{2s_1-2s-6+6b} \sum_{\substack{|k_1|>|k|\\|n|\lesssim|k|}} \langle k_1\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-2s-5+6b} \sum_{|k_1|>|k|} \langle k_1\rangle^{-2s}\\ &\lesssim \sup_k \langle k\rangle^{2s_1-4s-4+6b} \lesssim 1,\end{aligned}$$

as long as $s_1 - s \leq s + 2 - 3b$.

Case B.3. $|n-k_1| < \delta|k|$

Here $|n-k| \approx |k|$, since $(\widetilde{r}_1 - 1 - 2\delta)|k| < |n-k| < (\widetilde{r}_1 - 1 + 2\delta)|k|$. Hence for $s_1 - s \leq s - \frac{1}{2}$, we have

$$\begin{aligned}(5.32) &\lesssim \sup_k \langle k\rangle^{2s_1-2s} \sum_{\substack{|k_1|>|k|\\n\neq 0}} \frac{\langle k_1\rangle^{-2s}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}\\ &\lesssim \sup_k \langle k\rangle^{2s_1-2s} \sum_{|k_1|>|k|} \langle k_1\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-4s+1} \lesssim 1.\end{aligned}$$

Case C. $|k_1 - \widetilde{r}_1 k| \geq \delta|k|$, $|k_1 - \widetilde{r}_2 k| \geq \delta|k|$

We note that $|k_1 - k| \lesssim |k_1 - \widetilde{r}_1 k|$, because $|k_1 - k| \leq |k_1 - \widetilde{r}_1 k| + (\widetilde{r}_1 - 1)|k| \leq \left(\frac{\widetilde{r}_1 - 1 + \delta}{\delta}\right)|k_1 - \widetilde{r}_1 k|$. We need to bound

$$\begin{aligned}(5.31) &\lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{k_1\neq 0\\n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s+1}\langle n-k\rangle^{-2s+1}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}\\ &+ \sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{k_1\neq 0\\n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s+1}|k-k_1|}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}\\ &=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

Case C.1. $|k_1| \geq \delta|k|$

In this case, since $-2s+1<0$,

$$\mathrm{I}_1 \lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{k_1\neq 0\\n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle k_1-k\rangle^{-2s+1}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-4} \sum_{k_1\neq 0} \langle k_1\rangle^{-2s} \langle k_1-k\rangle^{-2s+1} \lesssim \sup_k \langle k\rangle^{2s_1-2s-3} \lesssim 1$$

whenever $s_1 - s \leq \frac{3}{2}$. In the same way, the boundedness of I_2 can be shown provided that $s_1 - s \leq 1$.

Case C.2. $|k_1| < \delta|k|$

In this case, $|k-k_1| \lesssim |k|$ implies the bound

$$\begin{aligned}
\mathrm{I}_1 + \mathrm{I}_2 &\lesssim \sup_k \langle k\rangle^{2s_1-2} \sum_{\substack{k_1\neq 0\\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s+1}\langle n-k\rangle^{-2s+1}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \\
&\quad + \sup_k \langle k\rangle^{2s_1-1} \sum_{\substack{k_1\neq 0\\ n\neq 0}} \frac{\langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s+1}}{\langle k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3\rangle^{2-2b}} \\
&=: \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}$$

Case C.2.1. $|n-k_1| \leq \delta|k|$

Note here that $|n-k| \geq |k| - |k_1| - |n-k_1| \geq (1-2\delta)|k|$. Moreover, since u is mean zero, we have, for some $\eta_j \neq 0$ satisfying $|k|^{-1} \leq |\eta_j| < \delta$, $j = 1, 2$, that $k_1 = \eta_1 k$ and $n - k_1 = \eta_2 k$. Hence the restriction $(\eta_1 + \eta_2)k = n \neq 0$ provides us with a parameter $\eta := \eta_1 + \eta_2$ satisfying $|k|^{-1} \leq |\eta| < 2\delta$, and yielding the following bound

$$\begin{aligned}
|k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3| &= \left|\big(1 - a(\eta_1^3 + \eta_2^3) - (1-\eta)^3\big)k^3\right| \\
&= |k|^3|\eta||3(1-\eta) + \eta^2(1-a) + 3a\eta_1\eta_2| \gtrsim |k|^2.
\end{aligned}$$

Exploiting the above estimates we arrive at

$$\mathrm{J}_1 \lesssim \sup_k \langle k\rangle^{2s_1-2s-5+4b} \sum_{\substack{k_1\neq 0\\ |n|\lesssim|k|}} \langle k_1\rangle^{-2s-2} \lesssim \sup_k \langle k\rangle^{2s_1-2s-4+4b} \lesssim 1$$

and

$$\mathrm{J}_2 \lesssim \sup_k \langle k\rangle^{2s_1-5+4b} \sum_{\substack{k_1\neq 0\\ n\neq 0}} \langle k_1\rangle^{-2s-2}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s+1}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-5+4b} \sum_{k_1\neq 0} \langle k_1\rangle^{-2s-2}\langle k_1-k\rangle^{-2s+1} \lesssim \sup_k \langle k\rangle^{2s_1-2s-4+4b} \lesssim 1$$

provided that $s_1 - s \leq 2 - 2b$.

Case C.2.2. $|n-k| \leq \delta|k|$

In this region, $|n-k_1| \geq |k| - |k_1| - |n-k| > (1-2\delta)|k|$. Then, we write $k_1 = \eta_1 k$ for some η_1 with $|k|^{-1} \leq |\eta_1| < \delta$, and $n-k = \eta_2 k$ for some η_2 with $0 \leq |\eta_2| \leq \delta$. Via these

$$\begin{aligned} |k^3 - ak_1^3 - a(n-k_1)^3 - (k-n)^3| &= |\big(1+\eta_2^3 - a(\eta_1^3 + (1-\eta_1+\eta_2)^3)\big)k^3| \\ &= |k|^3|1-a+\mathcal{O}(\delta)| \gtrsim |k|^3. \end{aligned}$$

As above we get

$$\mathrm{J}_1 + \mathrm{J}_2 \lesssim \sup_k \langle k\rangle^{2s_1-2s-6+6b} \lesssim 1$$

whenever $s_1 - s \leq 3 - 3b$.

5.5. Existence of Global Attractor

This section is devoted to the proof of Theorem 5.2.9. We consider the system

$$\begin{cases} u_t + au_{xxx} + \gamma u + 3a(u^2)_x + \beta(v^2)_x = f \\ v_t + v_{xxx} + \gamma v + 3uv_x = g \\ (u,v)|_{t=0} = (u_0, v_0) \in \dot{H}^1(\mathbb{T}) \times H^1(\mathbb{T}). \end{cases} \tag{5.33}$$

Recall that $\beta < 0$. Firstly we show the existence of an absorbing set corresponding to the system (5.33). To achieve this we use conserved energies (5.2) to obtain:

Lemma 5.5.1. *Let (u,v) be a solution of the system* (5.33) *with data (u_0,v_0), we have the a priori estimate:*

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq C = C(a,\beta,\gamma,\|u_0\|_{H^1},\|v_0\|_{H^1},\|f\|_{H^1},\|g\|_{H^1}),$$

for $t > 0$.

Proof. We start by noting that the constants in the following calculations are denoted by C, C_0, and C_1 whose value may change, their dependence are to be highlighted though. To obtain the L^2 bounds for u and v, we use $E_1(t) := E_1(u,v)(t) = \|u\|_{L^2}^2 - \frac{2\beta}{3}\|v\|_{L^2}^2$. Thus

$$\begin{aligned}\partial_t E_1(t) + 2\gamma E_1(t) &= 2\int uf - \frac{2\beta}{3} vg \, \mathrm{d}x \\ &\leq 2\|u\|_{L^2}\|f\|_{L^2} - \frac{4\beta}{3}\|v\|_{L^2}\|g\|_{L^2} \\ &\leq 2(\sqrt{2}\|f\|_{L^2} + \sqrt{-\beta}\|g\|_{L^2})\sqrt{E_1(t)}.\end{aligned}$$

Setting $E_1(t) = e^{-2\gamma t}F_1(t)$ and using the above inequality we obtain

$$\partial_t \sqrt{F_1(t)} \leq e^{\gamma t}(\sqrt{2}\|f\|_{L^2} + \sqrt{-\beta}\|g\|_{L^2}).$$

Integrating this inequality from 0 to t and then utilizing the resulting inequality in the norms of u and v, we arrive at

$$\begin{aligned}&\|u(t)\|_{L^2} + \sqrt{\frac{-2\beta}{3}}\|v(t)\|_{L^2} \\ &\qquad \leq \sqrt{2}e^{-\gamma t}\sqrt{\|u_0\|_{L^2}^2 - \frac{2\beta}{3}\|v_0\|_{L^2}^2} + \frac{1-e^{-\gamma t}}{\gamma}(2\|f\|_{L^2} + \sqrt{-2\beta}\|g\|_{L^2}).\end{aligned}$$

Regarding the bounds for the spatial derivatives of u and v, we consider $E_2(t) := E_2(u,v)(t) = (1-a)\big(\|u_x\|_{L^2}^2 - 2\int u^3 \mathrm{d}x\big) - 2\beta\big(\|v_x\|_{L^2}^2 - \int uv^2 \mathrm{d}x\big)$. Note that

$$\begin{aligned}(1-a)\|u_x\|_{L^2}^2 - 2\beta\|v_x\|_{L^2}^2 &= E_2(t) + 2(1-a)\int u^3 \, \mathrm{d}x - 2\beta\int uv^2 \, \mathrm{d}x \\ &\leq E_2(t) + C\|u\|_{H^1}\left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2\right) \\ &\leq E_2(t) + C + C\|u_x\|_{L^2}\end{aligned}$$

where the constants depend on the bounds on $\|u\|_{L^2}$, $\|v\|_{L^2}$ in the final inequality. By this inequality, we have

$$\sqrt{1-a}\|u_x\|_{L^2} - \frac{C}{2\sqrt{1-a}} \leq \sqrt{\left(\sqrt{1-a}\|u_x\|_{L^2} - \frac{C}{2\sqrt{1-a}}\right)^2 - 2\beta\|v_x\|_{L^2}^2}$$

$$\leq \sqrt{E_2(t) + C + C^2/4(1-a)} \lesssim \sqrt{|E_2(t)|} + C$$

and

$$\sqrt{-2b}\,\|v_x\|_{L^2} \leq \sqrt{\Big(\sqrt{1-a}\,\|u_x\|_{L^2} - \frac{C}{2\sqrt{1-a}}\Big)^2 - 2\beta\,\|v_x\|_{L^2}^2} \lesssim \sqrt{|E_2(t)|} + C.$$

Thus $\|u_x(t)\|_{L^2} + \|v_x(t)\|_{L^2} \lesssim \sqrt{|E_2(t)|} + C$. To end up the argument it suffices to show that E_2 is bounded. Using this bound and the embedding $H^1 \hookrightarrow L^\infty$ we obtain

$$\begin{aligned}\partial_t E_2(t) + 2\gamma E_2(t) &= 2(1-a)\int f_x u_x - 3fu^2 + \gamma u^3\,\mathrm{d}x - 2\beta \int 2g_x v_x - fv^2 - 2guv + \gamma uv^2\,\mathrm{d}x \\ &\leq C_0(\|u_x\|_{L^2} + \|v_x\|_{L^2}) + C_1 \leq C_0\sqrt{|E_2(t)|} + C_1\end{aligned}$$

where the constants C_0, C_1 depend on the norms $\|f\|_{H^1}$, $\|g\|_{H^1}$, the constants a, β, γ and the bounds on $\|u\|_{L^2}$, $\|v\|_{L^2}$. Setting $E_2(t) = e^{-2\gamma t}F_2(t)$, we get that

$$\partial_t F_2(t) \leq e^{\gamma t}\big(C_0\sqrt{|F_2(t)|} + C_1 e^{\gamma t}\big),$$

from which we have that

$$\begin{aligned}E_2(t) &\leq e^{-2\delta t}E_2(0) + C_1\frac{1-e^{-2\gamma t}}{2\gamma} + C_0\int_0^t e^{-2\gamma(t-t')}\sqrt{|E_2(t')|}\,\mathrm{d}t' \\ &\leq |E_2(0)| + C_1 + C_0\left\|\sqrt{|E_2|}\right\|_{L^\infty([0,t])}\end{aligned}$$

for $t > 0$. This shows that E_2 is bounded from above because if it were the case that t might be the first time at which E_2 assumes its largest value, say C, over $[0,t]$ with $E_2(t) = C \gg |E_2(0)| + C_1 + C_0 =: \widetilde{C}$, then by the above inequality we would have $C \leq \widetilde{C}(1+\sqrt{C})$, but this is impossible for sufficiently large $C \gg 1$. Also the Sobolev embedding and the bounds on $\|u\|_{L^2}$, $\|v\|_{L^2}$ suggest that E_2 is bounded below. □

As a consequence of the Lemma 5.5.1, the existence of an absorbing ball $\mathcal{B}_0 \subset H^1 \times H^1$ follows. As for the verification of the asymptotic compactness of the flow, the second task is to obtain smoothing estimate as done in the non-dissipative case.

Theorem 5.5.2. *Consider the solution of* (5.33) *with initial data* $(u_0, v_0) \in \dot{H}^1 \times H^1$. *Then for any* $\alpha < \min\{\frac{1}{2}, 3 - \mu(\rho_a)\}$, *we have*

$$\left\| u(t) - e^{-(a\partial_x^3+\gamma)t}u_0 - \int_0^t e^{-(a\partial_x^3+\gamma)(t-r)}\rho_2(v,v)(r)\,dr \right\|_{H^{1+\alpha}} + \left\| v(t) - e^{-(\partial_x^3+\gamma)t}v_0 - \int_0^t e^{-(\partial_x^3+\gamma)(t-r)}\rho_3(u,v)(r)\,dr \right\|_{H^{1+\alpha}} \leq C(\alpha, \gamma, \|u_0\|_{H^1}, \|v_0\|_{H^1}, \|f\|_{H^1}, \|g\|_{H^1})$$

where ρ_2 *and* ρ_3 *are as in Proposition 5.3.1.*

Proof. We write the system (5.33) by the Fourier transform as follows

$$\begin{cases} \partial_t u_k - (iak^3 - \gamma)u_k + 3iak \sum\limits_{k_1+k_2=k} u_{k_1}u_{k_2} + i\beta k \sum\limits_{k_1+k_2=k} v_{k_1}v_{k_2} = f_k \\ \partial_t v_k - (ik^3 - \gamma)v_k + 3i \sum\limits_{k_1+k_2=k} k_2 u_{k_1}v_{k_2} = g_k. \end{cases} \tag{5.34}$$

Using the change of variables $y_k = e^{-iak^3t+\gamma t}u_k$, $z_k = e^{-ik^3t+\gamma t}v_k$, and $d_k = e^{-iak^3t+\gamma t}f_k$, $h_k = e^{-ik^3t+\gamma t}g_k$, the above system transforms to

$$\begin{cases} \partial_t y_k = -3iak \sum\limits_{k_1+k_2=k} e^{-iat(k^3-k_1^3-k_2^3)}y_{k_1}y_{k_2} - i\beta k \sum\limits_{k_1+k_2=k} e^{-it(ak^3-k_1^3-k_2^3)}z_{k_1}z_{k_2} + d_k \\ \partial_t z_k = -3i \sum\limits_{k_1+k_2=k} k_2 e^{-it(k^3-ak_1^3-k_2^3)}y_{k_1}z_{k_2} + h_k. \end{cases}$$

After differentiation by parts as in Proposition 5.3.1, the system (5.34) can be written in the form

$$\begin{cases} \partial_t\Big[e^{-iak^3t+\gamma t}u_k\Big] + e^{-\gamma t}\partial_t\Big[e^{-iak^3t+2\gamma t}(B_1(u,u)_k + B_2(v,v)_k)\Big] \\ \qquad = e^{-iak^3t+\gamma t}\big[R_1(u,v,v)_k + R_2(u,u,u)_k + R_3(u,v,v)_k \\ \qquad +2B_1(u,f)_k + 2B_2(g,v)_k + \rho_1(u,u)_k + \rho_2(v,v)_k + f_k\big] \\ \partial_t\Big[e^{-ik^3t+\gamma t}v_k\Big] + e^{-\gamma t}\partial_t\Big[e^{-ik^3t+2\gamma t}B_3(u,v)_k\Big] \\ \qquad = e^{-ik^3t+\gamma t}\big[R_4(u,u,v)_k + \frac{\beta}{3a}R_4(v,v,v)_k + R_5(u,u,v)_k \\ \qquad +B_3(f,v)_k + B_3(u,g)_k + \rho_3(u,v)_k + g_k\big] \end{cases}$$

where B_j, R_j, and ρ_j are as in Proposition 5.3.1.

Integrating these equations from 0 to t leads to the equations

$$
\begin{aligned}
u_k(t) - e^{iak^3t-\gamma t}u_k(0) &= -B_1(u,u)_k - B_2(v,v)_k + e^{iak^3t-\gamma t}\big[B_1(u_0,u_0)_k + B_2(v_0,v_0)_k\big] \\
+\int_0^t e^{(iak^3-\gamma)(t-s)}\big[&-\gamma B_1(u,u)_k - \gamma B_2(v,v)_k + \rho_1(u,u)_k + \rho_2(v,v)_k + f_k + 2B_1(u,f)_k \\
&+2B_2(g,v)_k + R_1(u,v,v)_k + R_2(u,u,u)_k + R_3(u,v,v)_k\big]\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
v_k(t) - e^{ik^3t-\gamma t}v_k(0) &= -B_3(u,v)_k + e^{ik^3t-\gamma t}B_3(u_0,v_0)_k + \int_0^t e^{(ik^3-\gamma)(t-s)}\big[-\gamma B_3(u,v)_k \\
+\rho_3(u,v)_k + g_k &+ R_4(u,u,v)_k + \frac{\beta}{3a}R_4(v,v,v)_k + R_5(u,u,v)_k + B_3(f,v)_k + B_3(u,g)\big]\mathrm{d}s.
\end{aligned}
$$

Note that

$$
\left\|\int_0^t e^{(-a\partial_x^3-\gamma)(t-s)}f(x)\mathrm{d}s\right\|_{H^{1+\alpha}} = \left\|\frac{\langle k\rangle^{1+\alpha}f_k}{iak^3-\gamma}(1-e^{(iak^3-\gamma)t})\right\|_{\ell_k^2} \lesssim \|f\|_{H^{\alpha-2}}\,,
$$

analogous estimate holds for $e^{(-\partial_x^3-\gamma)(t-s)}g$ as well. These bounds, the estimates utilized in obtaining main smoothing result, and the growth bound of Lemma 5.5.1 yield, for $t<\delta$, that

$$
\begin{aligned}
&\left\|u(t) - e^{-(a\partial_x^3+\gamma)t}u_0 - \int_0^t e^{-(a\partial_x^3+\gamma)(t-r)}\rho_2(v,v)(r)\,\mathrm{d}r\right\|_{H^{1+\alpha}} \\
&+\left\|v(t) - e^{-(\partial_x^3+\gamma)t}v_0 - \int_0^t e^{-(\partial_x^3+\gamma)(t-r)}\rho_3(u,v)(r)\,\mathrm{d}r\right\|_{H^{1+\alpha}} \\
&\lesssim \|f\|_{H^{\alpha-2}} + \|g\|_{H^{\alpha-2}} + \big(\|f\|_{H^1} + \|g\|_{H^1} + \|u_0\|_{H^1} + \|v_0\|_{H^1}\big)^2 \\
&+\big(\|u\|_{X_{a,\delta}^{1,1/2}} + \|v\|_{X_{1,\delta}^{1,1/2}}\big)^3 \\
&\le C\big(\alpha,\gamma,\|f\|_{H^1},\|g\|_{H^1},\|u_0\|_{H^1},\|v_0\|_{H^1}\big)
\end{aligned}
$$

where we use the local theory bounds for $X_{a,\delta}^{1,1/2}$, $X_{1,\delta}^{1,1/2}$ norms for the local existence time δ in the final inequality. By virtue of dissipation, this bound also holds for arbitrarily large times making use of the local bound above, for the full discussion, see Section 6 in [17]. □

Proof of Theorem 5.2.9. For the existence of a global attractor, we check the asymptotic compactness of the flow. It suffices to show that for any sequence $(u_{0,r}, v_{0,r})$ in an absorbing set $\mathcal{B}_0$ and for any sequence of times $t_r \to \infty$, the sequence $U_{t_r}(u_{0,r}, v_{0,r})$ possesses a convergent subsequence in $\dot{H}^1 \times H^1$. Next we use Theorem 5.5.2, for almost every $a \in (\frac{1}{4}, 1)$ such that $\alpha < \frac{1}{2}$ and $\rho_2 = \rho_3 = 0$, to write

$$U_{t_r}(u_{0,r}, v_{0,r}) = (e^{-(a\partial_x^3+\gamma)t_r}u_{0,r}, e^{-(\partial_x^3+\gamma)t_r}v_{0,r}) + N_{t_r}(u_{0,r}, v_{0,r})$$

where the nonlinear part $N_{t_r}(u_{0,r}, v_{0,r})$ is contained within a ball in $H^{1+\alpha} \times H^{1+\alpha}$. Therefore by Rellich's theorem the sequence $\{N_{t_r}(u_{0,r}, v_{0,r}) : r \in \mathbb{N}\}$ has a convergent subsequence in $H^1 \times H^1$. This implies the existence of a convergent subsequence of the sequence $\{U_{t_r}(u_{0,r}, v_{0,r}) : r \in \mathbb{N}\}$, since

$$\left\|(e^{-(a\partial_x^3+\gamma)t_r}u_{0,r}, e^{-(\partial_x^3+\gamma)t_r}v_{0,r})\right\|_{H^1\times H^1} \lesssim e^{-\gamma t_r} \left\|(u_{0,r}, v_{0,r})\right\|_{H^1\times H^1} \lesssim e^{-\gamma t_r} \to 0$$

as $t_r \to \infty$ uniformly. Therefore U_t is asymptotically compact. To prove the compactness of the attractor $\mathcal{A}$ in the space $H^{1+\alpha} \times H^{1+\alpha}$ for any $\alpha \in (0, \frac{1}{2})$, we need to show, by using Rellich's theorem, that the attractor is bounded in $H^{1+\alpha+\epsilon} \times H^{1+\alpha+\epsilon}$ for some $\epsilon > 0$ satisfying $\alpha + \epsilon < \frac{1}{2}$. In this regard, it suffices to find some closed ball $\mathcal{B}_{\alpha+\epsilon} \subset H^{1+\alpha+\epsilon} \times H^{1+\alpha+\epsilon}$ such that $\mathcal{A} \subset \mathcal{B}_{\alpha+\epsilon}$ where

$$\mathcal{A} = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} U_t\mathcal{B}_0} =: \bigcap_{\tau\geq 0} V_\tau.$$

As above, using Theorem 5.5.2, we can express each element of V_τ as a sum of linear evolution which decays to zero exponentially and the nonlinear evolution contained by some ball $\mathcal{B}_{\alpha+\epsilon}$ in $H^{1+\alpha+\epsilon} \times H^{1+\alpha+\epsilon}$. This implies that the set V_τ is contained in a δ_τ neighbourhood N_τ of $\mathcal{B}_{\alpha+\epsilon}$ in $H^{1+\alpha+\epsilon} \times H^{1+\alpha+\epsilon}$. Here $\delta_\tau \to 0$ as $\tau \to \infty$ due to the exponential decay of linear evolutions. Therefore,

$$\mathcal{A} = \bigcap_{\tau\geq 0} V_\tau \subset \bigcap_{\tau\geq 0} N_\tau = \mathcal{B}_{\alpha+\epsilon}.$$

$\square$

6. CONCLUSION

In this thesis, we were concerned with the smoothing properties of several dispersive equations on certain domains. In the first part of the thesis, we addressed the Davey–Stewartson system on $\mathbb{R}^2$ and established the smoothing properties, also proved the existence of a global attractor for this system. In the second part, we considered the biharmonic NLS equation on the half line and studied local and global well-posedness and regularity properties of this equation. In the final part, our aim was to establish the smoothing estimates of the Hirota–Satsuma system on the torus. After obtaining the estimates for the nondissipative system, we established the analogous estimates for the dissipative HS system, also with the use of these estimates, we proved the existence of global attractor in the energy space. Our plan for a future project involves proving smoothing effect for the Schrödinger–KdV system with periodic boundary conditions and the Kuramoto–Sivashinsky equation on the half-line.

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APPENDIX A: USEFUL ESTIMATES

In this section, we start by reserving some useful inequalities to be used in the text when necessary. Firstly we start with a lemma which is a consequence of the proof of Theorem 1.3 in [42].

Lemma A.0.1. *When $\mu = 1$ (defocusing nonlinearity), the solutions of the equation* (4.1) *satisfy the following a priori estimate*

$$\|u\|_{H^2(\mathbb{R}^+)} \leq C(\|g\|_{H^2}, \|h_1\|_{H^1}, \|h_2\|_{H^1}).$$

Next lemma is used in the proofs of Proposition 4.4.6 and Proposition 4.4.7.

Lemma A.0.2. *For $m, n, k \in \mathbb{R}$, we have*

$$|m^4 - n^4 + k^4 - (m-n+k)^4| \gtrsim |m-n||n-k|(m^2+n^2+k^2).$$

Proof. Let $g(m,n,k) := m^4 - n^4 + k^4 - (m-n+k)^4$. Then

$$\begin{aligned} g(m,&n,k) \\ &= (m-n)\big[(m^2+n^2)(m+n) - (m-n)^3 - 4(m-n)^2k - 6(m-n)k^2 - 4k^3\big] \\ &= (m-n)(n-k)\big[4m^2 + 2n^2 + 4k^2 - 2mn - 2nk\big] \\ &= (m-n)(n-k)\Big[\frac{5}{2}(m+n)^2 + m^2 + k^2 + 2(n - \frac{1}{2}m - \frac{1}{2}k)^2\Big] \end{aligned}$$

which gives the desired estimate. □

Lemma A.0.3 (See [96])**.** *For $-\frac{1}{2} \leq s \leq \frac{1}{2}$, we have*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^{\frac{1}{2}+}} \|g\|_{H^s}.$$

We state the Gagliardo–Nirenberg inequality [97]:

Theorem A.0.4. *Assume that $g \in L^q(\mathbb{R}^n)$ and $D^m g \in L^r(\mathbb{R}^n)$. Fix $1 \leq q, r \leq \infty$ and $m \in \mathbb{N}$. Then, we have*

$$\left\| D^j g \right\|_{L^p} \lesssim \| D^m g \|_{L^r}^{\alpha} \, \| g \|_{L^q}^{1-\alpha}$$

where $\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n} \right) \alpha + \frac{1-\alpha}{q}$ and $\frac{j}{m} \leq \alpha \leq 1$. When $j = 0$, $rm < n$ and $q = \infty$, assume additional assumption that either $g \in L^s$ for some $s > 0$ or g vanishes at infinity. Also if $m - j - \frac{n}{r}$ is a non-negative integer for $1 < r < \infty$, then we need $\alpha < 1$.

The following lemmas are used frequently in our discussions. For proofs of the first two, see [17] and for the last one we refer [62].

Lemma A.0.5. *If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$ then*

$$\int_{\mathbb{R}} \frac{dx}{\langle x - a_1 \rangle^{\beta} \langle x - a_2 \rangle^{\gamma}} \lesssim \langle a_1 - a_2 \rangle^{-\gamma} \varphi_{\beta}(a_1 - a_2)$$

where

$$\varphi_{\beta}(a) = \sum_{|n| \leq |a|} \frac{1}{\langle n \rangle^{\beta}} \sim \begin{cases} 1 & \beta > 1 \\ \log(1 + \langle a \rangle) & \beta = 1 \\ \langle a \rangle^{1-\beta} & \beta < 1. \end{cases}$$

Lemma A.0.6. *(i) If $\beta \geq \gamma \geq 0$ and $\gamma + \beta > 1$,*

$$\sum_{n} \frac{1}{\langle n - k_1 \rangle^{\beta} \langle n - k_2 \rangle^{\gamma}} \lesssim \langle k_1 - k_2 \rangle^{-\gamma} \varphi_{\beta}(k_1 - k_2)$$

where φ_{β} is defined as in Lemma A.0.5.

(ii) If $\beta > \frac{1}{2}$ and $\gamma > \frac{1}{3}$, then we have

$$\sum_{n} \frac{1}{\langle n^2 + an + b \rangle^{\beta}} \lesssim 1, \text{ and } \quad \sum_{n} \frac{1}{\langle n^3 + an^2 + bn + c \rangle^{\gamma}} \lesssim 1$$

where the implicit constants are independent of a, b and c.

Lemma A.0.7. *For fixed* $\rho \in (\frac{1}{2}, 1)$, *we have*

$$\int \frac{1}{\langle x \rangle^{\rho} \sqrt{|x-a|}} dx \lesssim \frac{1}{\langle a \rangle^{\rho - \frac{1}{2}}}.$$