KADISON-SINGER PROBLEM FROM A BANACH ALGEBRA PERSPECTIVE

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ABSTRACT

KADISON-SINGER PROBLEM FROM A BANACH ALGEBRA PERSPECTIVE

In 1959, Kadison and Singer asked whether every pure state of the diagonal subspace $\mathbb{D}(\ell_2)$ of $B(\ell_2)$ has a unique pure state extension to $B(\ell_2)$. This problem has remained open until 2013; in 2013 it has been solved by a team of computer scientists. In my Master thesis, which is largely based on a paper by Akemann, Tanbay and Ülger, I have tried to learn this problem and the approach considered in this paper.

We identify $\mathbb{D}(\ell_2)$ with $C(\beta\mathbb{N})$. For t in $\beta\mathbb{N}$, δ_t is the Dirac measure at t considered as a functional on $C(\beta\mathbb{N})$. We denote by $[\delta_t]$ the set of the states of $B(\ell_2)$ that extend δ_t. Our main aim is to understand how large the set $[\delta_t]$ is. Using the fact that the von Neumann algebra $B(\ell_2)$ has the Pelczyński's property (V), it is proven that either the set $[\delta_t]$ lies in a finite dimension subspace of $B(\ell_2)^*$ or, in its weak-star topology, it contains a homeomorphic copy of $\beta\mathbb{N}$. We study this result under the so far directly unproven knowledge that $[\delta_t]$ is a singleton.

ÖZET

BANACH CEBİRİ BAKIŞ AÇISIYLA KADISON-SINGER PROBLEMİ

1959 yılında, Kadison ve Singer, $B(\ell_2)$'nin altuzayı olan $\mathbb{D}(\ell_2)$ üzerindeki her pure state'in tek bir şekilde $B(\ell_2)$'ye uzatılıp uzatılamayacağını sordular. 2013 yılına kadar açık olan bu problem, bir grup bilgisayar bilimcisi tarafından çözüldü. Büyük bir kısmı Akemann, Tanbay ve Ülger'e ait olan makaleye dayanan tezimde Kadison-Singer problemini ve bu probleme nasıl yaklaşıldığını öğrenmeye çalıştım.

$\mathbb{D}(\ell_2)$ uzayını $C(\beta\mathbb{N})$ olarak tanımlarsak, $\beta\mathbb{N}$'ye ait her t elemanı için, δ_t'yi $C(\beta\mathbb{N})$ üzerinde bir fonksiyonel olarak görebiliriz. $[\delta_t]$ kümesini, δ_t state'lerin $B(H)$ uzayına uzatılmasıyla elde edilen küme olarak tanımlayalım. Amacımız $[\delta_t]$ kümesinin ne kadar büyük olduğunu anlamaktır. Söz konusu makalede von Neumann cebiri olan $B(\ell_2)$'nin Pelczyński özelliğine sahip olmasını kullanarak, $[\delta_t]$ kümesinin ya $B(\ell_2)^*$ uzayının sonlu boyutlu bir alt uzayı içinde olduğunu ya da zayıf-$*$ topoloji içinde $\beta\mathbb{N}$'nin bir homeomorfik kopyasını içerdiğini ispatlıyoruz. Bu sonucu, direkt bir ispatımız olmadığı halde $[\delta_t]$ kümesinin tek elemanlı olduğunu bilerek inceliyoruz.

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LIST OF SYMBOLS

A'	Set of commutators of a set A
A^*	First dual space of A
A^{**}	Second dual space of A
B(H)	The operator algebra of the bounded linear operators on a Hilbert space H
c_0	The space of sequences converging zero
C(K)	The space of continuous functions on a compact Hausdorff space K
$C_0(\Omega)$	The space of continuous functions vanishing at infinity on a space Ω
$M(K)$	The space of complex signed measures on a compact Hausdorff space K
$P(A)$	The set of pure states on A
$S(A)$	The set of states on A
T^*	The adjoint operator of T

$\beta\mathbb{N}$	The Stone-Čech compactification of $\mathbb{N}$
δ_t	The Dirac delta measure
ℓ^∞	The space of bounded sequences on $\mathbb{N}$
$L^\infty(\Omega,\mu)$	The space of the essentially measurable functions with essential supremum norm for a measurable space (Ω,μ)
$\sigma(a)$	The spectrum of an element a of a complex unital Banach algebra A
$\sigma(X^*,X)$	The weak*-topology on X^* generated by X
$\sigma(X^*,X^{**})$	The weak topology on X^* generated by X^{**}

1. INTRODUCTION

In this dissertation, we are concerned with the uniqueness of extensions of pure states from a maximal abelian self-adjoint algebra(masa) on a Hilbert space H to the algebra of bounded linear operators on that Hilbert space.

Let H be a separable complex Hilbert space and $B(H)$ be the C^*-algebra of bounded linear operators on H. If A is a unital C^*-subalgebra of $B(H)$, then the set of states on A, $S(A)$ is a convex and weak*-compact subset of the dual A^*. Thus, $S(A)$ is the closed convex hull of its extreme points by the Krein-Milman theorem. These extreme points are called pure states of A.

If we consider a measurable space (X, μ), the map $M_f : L^2(X, \mu) \to L^2(X, \mu)$ with $g \mapsto fg$ for $f \in L^\infty(X, \mu)$ generates a masa in $B(L^2(X, \mu))$. If we take $X := [0, 1]$ and define μ as the Lebesque measure, then the masa is called the continuous masa. If we take $X := \mathbb{N}$ and define μ as the counting measure, then the masa is called the discrete masa. Any masa is unitarily equivalent to the continuous masa, the discrete masa, a finite dimensional masa or it can be decomposed as a direct sum of the continuous masa and the discrete masa(or a finite dimensional masa).

Kadison and Singer proved that there are pure states on the continuous masa that do not extend uniquely to the full algebra. They conjectured that pure state extensions of a pure state from the discrete masa to the full algebra are not unique either. The problem has remained open for almost 54 years until it was solved in 2013 by a team of computer scientists. Marcus, Spielman and Srivastava [1] proved that any pure state on $\mathbb{D}(\ell_2)$ has a unique pure state extension on $B(\ell_2)$ by using random polynomials. There is yet no proof given in terms of Banach algebras or Operator algebras.

If K is a compact Hausdorff space, and $C(K)$ the space of continuous functions on K, then a pure state on $C(K)$ is just a complex homomorphism. We know that the spaces $\mathbb{D}(\ell_2), C(\beta\mathbb{N})$ and ℓ^∞ are isomorphic. Since every bounded homomorphism on $C(K)$ for a compact Hausdorff space K can be represented by a Dirac measure δ_t for some t in K, we can reformulate the problem as follows: Given a pure state δ_t on ℓ^∞, can we extend δ_t to a pure state on $B(H)$ in a unique way? In this thesis, which is largely based on a paper by Akemann, Tanbay and Ülger [2], we present a partial solution of this problem.

In the case where the set $[\delta_t]$ is weakly compact, we can deduce that the set $[\delta_t]$ is finite-dimensional. Concerning the weak compactness of $[\delta_t]$, the following result is proved: "the set $[\delta_t]$ is weakly compact if and only if $[\delta_t]$ doesn't include any homeomorphic copy of $\beta\mathbb{N}$". This characterization implies that "either $[\delta_t]$ includes a homeomorphic copy of $\beta\mathbb{N}$ or $[\delta_t]$ is contained in a finite dimensional subspace of $B(H)^*$". It is also shown that "there exists a unique pure state extension of δ_t if and only if the ideal $N_\rho = \{T \in B(H) : \rho(T^*T) = 0\}$ has a positive increasing bounded approximate unit consisting of diagonal operators, where ρ is an extension of δ_t". As a corollary of these main theorems, one obtains that in order to prove the uniqueness of the pure state extension of δ_t, it would be enough to show that $C(\beta\Delta)$ is a Grothendieck space, where Δ denotes the union of compact sets $[\delta_t]$ and that the set $[\delta_t]$ does not contain a homeomorphic copy of $\beta\mathbb{N}$.

2. PRELIMINARIES

In this chapter, we have gathered the basic notions and results that we use in the rest of the thesis. We assume the reader has some knowledge of C^*-algebras. All the results needed can be found in [3], [4] and [5]. Throughout this chapter, we assume that the set X is Hausdorff.

2.1. On weak and weak* topologies

If $(X_i, \tau_i)_{i\in I}$ is a family of topological spaces and $f_i : X \to X_i$ are functions for each $i \in I$, the *weak topology* is the smallest topology on X such that all functions f_i are continuous for $i \in I$.

The following collection, $\left\{ \bigcap_{i\in J} f_i^{-1}(O_i) : O_i \in \tau_i, J \text{ is a finite subset of } I \right\}$ of sets are basis sets for the weak topology on X.

In the case of a non-zero normed linear space X, the weak topology $\sigma(X, X^*)$ on X is defined by the elements of X^*.

For any $\epsilon > 0$ and $f \in X^*$, $V(x_0, f, \epsilon) = \{x \in X : |f(x) - f(x_0)| < \epsilon\}$ is a subbasic open set and for $\{f_i\}_{i=1}^n \subseteq X^*$, a basic open neighborhood of x_0 is in the form of $U(x_0, f_1, .., f_n, \epsilon) = \{x \in X : |f_i(x) - f_i(x_0)| < \epsilon, i \in \{1, ..., n\}\}$. The topological space $(X, \sigma(X, X^*))$ is a Hausdorff topological space.

Theorem 2.1. *Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in a normed linear space X.*

(i) For some $x_0 \in X$, $x_\lambda \to x_0$ in the weak topology iff $f(x_\lambda) \to f(x_0)$ for all $f \in X^*$.

(ii) $x_\lambda \to x_0$ in the norm topology for some $x_0 \in X$, then $x_\lambda \to x_0$ in the weak topology.

Convergence for a Hilbert space H in the weak topology has the following equivalent version; let $\{x_\lambda\}$ be a net in H. For some $x_0 \in H$, $x_\lambda \to x_0$ in the weak topology iff $\langle x_\lambda, x\rangle \to \langle x_0, x\rangle$, $\forall x \in H$ in the norm topology.

As a consequence of the Hahn-Banach theorem, we have the following result.

Proposition 2.2.

(i) Let X be a normed linear space and K be a convex subset of X, then K is closed in the norm topology iff K is closed in the weak topology.

(ii) Let X be a finite dimensional normed linear space, then the norm topology and weak topology coincide on X.

Next we define the weak*-topology on X^*: Let X be a normed linear space. As X^* is also a normed space, we can construct the weak topology on X^*, that is $\sigma(X^*, X^{**})$.

However, there is a weaker one obtained by embedding X into X^{**}. The canonical map $J : X \to X^{**}$ given by $J(x)(f) = f(x)$, $\forall f \in X^*$, which is a linear isometry. Hence, this topology $\sigma(X^*, X)$ is smaller than the weak topology $\sigma(X^*, X^{**})$. This new topology $\sigma(X^*, X)$ is called *weak*-topology* on X^*. For $\epsilon > 0$ and any $f \in X^*$, the set $\{g \in X^* : |f(x_i) - g(x_i)| < \epsilon,$ for $i = \{1, 2, ..., n\}$ and $\{x_i\}_{i=1}^n \subseteq X\}$ is a basic neighborhood of f. This topology is Hausdorff.

Proposition 2.3. *Let $\{f_i\}_{i\in I}$ be a net in the dual of normed linear space X^*.*

(i) $f_i \to f$ in the weak*-topology for some $f \in X^*$ iff $f_i(x) \to f(x)$ in the norm topology for all $x \in X$.

(ii) If $f_i \to f$ in the weak topology for some $f \in X^*$, then $f_i \to f$ in the weak*-topology.

If X is a finite dimensional normed linear space, then $\sigma(X^*, X^{**})$, $\sigma(X^*, X)$ and norm topologies are the same. By the Banach-Alaoglu theorem, we also know that if X is a normed linear space, the closed unit ball of X^* is weak*-compact.

The Goldstine theorem states that if X is a normed linear space, the closed unit ball of X is weak*-dense in the closed unit ball of X^{**}. Thereby, X is weak*-dense in X^{**}.

We can embed X into X^{**} by the canonical map. X is said to be a reflexive space if the canonical map from X to X^{**} is onto.

Theorem 2.4. *Let X be a normed linear space.*

(i) X is reflexive iff the closed unit ball $B = \{x \in X : ||x|| \leq 1\}$ of X is weakly compact.

(ii) Any closed subspace of a reflexive Banach space is reflexive.

(iii) If X is a finite dimensional space, then it is reflexive.

Theorem 2.5. *Let X be a Banach space. Then X is reflexive iff X^* is reflexive.*

Any Hilbert space H is reflexive. The set c_0 is not a reflexive space. So duals ℓ^1 and ℓ^∞ are not reflexive spaces. If $C(K)$ is reflexive for a compact and Hausdorff space K, then K is finite.

Definition 2.6. *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. If T maps bounded subsets of X to relatively (weakly) compact subsets of Y, then T is called a (weakly) compact operator.*

Theorem 2.7. *(Eberlein-Smulian) A subset K of a Banach space is weakly compact iff it is weakly sequentially compact.*

Theorem 2.8. *(Gantmacher) Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Then T is weakly compact iff T^* is weakly compact.*

Proposition 2.9.

(i) Let X, Y and Z be Banach spaces. If $T_1 : X \to Y$ and $T_2 : Y \to Z$ are bounded linear and weakly compact operators, then their composition T_2T_1 is also weakly compact.

(ii) If either X or Y is reflexive Banach space, then every bounded linear operator $T : X \to Y$ is weakly compact.

(iii) *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. If T is a compact operator, then T is a weakly compact operator.*

(iv) *Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. If T is weakly compact, then $T^{**} : X^{**} \to Y$ is a continuous operator when equipped with $\sigma(X^{**}, X^*)$ topology on X^{**} and weak topology on Y.*

2.2. On the Stone-Čech compactification

Let X be a topological space. We say that X has a compactification if there exists a compact Hausdorff space Y and a continuous injective map $J : X \to Y$ such that $\overline{J(X)} = Y$.

Consider the set of continuous functions $C(X, [0, 1])$ for a completely regular topological space (X, τ). Then the space $C(X, [0, 1])$ separates points from a closed subset of X. Define a map $J : X \to \prod\limits_{f \in C(X,[0,1])} [0, 1]$ by $x \mapsto (f(x))_{f \in C(X,[0,1])}$. By the Tychonoff's theorem, $\prod\limits_{f \in C(X,[0,1])} [0, 1]$ is compact and Hausdorff. Then we have a injective map $J : X \to \overline{(J(X))} \subseteq \prod\limits_{f \in C(X,[0,1])} [0, 1]$. In this case, $\overline{(J(X))}$ is called the Stone-Čech Compactification of X and denoted by βX. The Stone-Čech Compactification is the largest compactification.

Theorem 2.10. *Let X be a completely regular space, then every bounded linear operator $T : X \to \mathbb{R}$ has a bounded linear extension $\widetilde{T} : \beta X \to \mathbb{R}$.*

In the case $X = \mathbb{N}$, with the discrete topology, we have $\beta\mathbb{N}$, the Stone-Čech Compactification of $\mathbb{N}$.

The elements of the Stone-Čech Compactification of $\mathbb{N}$ are actually the ultrafilters on $\mathbb{N}$. $\beta\mathbb{N}$ is a compact Hausdorff space and $C(\beta\mathbb{N})$ is isometrically isomorphic to ℓ^∞.

Definition 2.11. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in ℓ^∞ and U be an ultrafilter in $\beta\mathbb{N}$. We say that $U\text{-}\lim a_n = a$ for some $a \in \ell^\infty$ if for any neighborhood N of a, we have $\{n : a_n \in N\} \in U$.*

Lemma 2.12.

(i) *Let* $(a_n)_{n\in\mathbb{N}}$ *be an element in* ℓ^∞ *and* $U_{i_0} \in \beta\mathbb{N}$ *be a principal ultrafilter for some* $i_0 \in \mathbb{N}$*, then we have that* $U_{i_0}\text{-}\lim a_n = a_{i_0}$*.*

(ii) *Let* U *be an ultrafilter in* $\beta\mathbb{N}$ *and* $I \in U$ *be any element. If* $(a_n)_{n\in\mathbb{N}} \in \ell^\infty$*, then* $U\text{-}\lim a_n \in \overline{\{a_n : n \in I\}}$*.*

2.3. On C^* algebras

Definition 2.13. *Let* A *be an algebra provided with an operation* * *satisfying*

(i) $(a+b)^* = a^* + b^*$

(ii) $(ab)^* = b^*a^*$

(iii) $(\lambda a)^* = \overline{\lambda}a^*$

(iv) $(a^*)^* = a$

for all $a, b \in A$ *and* $\lambda \in \mathbb{C}$*. Then* A *is called a* **-algebra.*

If a **-algebra* A *is complete and* $||a^*|| = ||a||$ *for all* $a \in A$*, then* A *is called a Banach* **-algebra.*

If A *is a Banach* **-algebra and* $||a^*a|| = ||a||^2$ *for all* $a \in A$*, then* A *is called a* C^**-algebra.*

In any unital C^**-algebra* A*, the unit of* A *and all projections of* A *have norm* 1*.*

Example 2.14.

(1) *Let* Ω *be a locally compact Hausdorff space, then* $C_0(\Omega)$*,* $C_b(\Omega)$ *and* $\ell^\infty(\Omega)$ *are abelian* C^**-algebras.*

(2) $B(H)$ *is a non-abelian unital* C^**-algebra.*

Definition 2.15. *Let* I *be an ideal of an algebra* A*.* I *is said to be modular if there exists an element* a_0 *of* A *such that* $a - a_0a \in I$ *for all* $a \in A$*.*

Example: Let Ω be a compact Hausdorff space and w_0 be a fixed element in Ω. Define a set $Y := \{f \in C_0(\Omega) : f(w_0) = 0\}$, then Y is modular in $C_0(\Omega)$.

Definition 2.16. *Let A be a unital algebra and a be an element of A. $Inv(A)$ denotes the set of invertible elements of A, then the set $\{\lambda \in \mathbb{C} : \lambda.1 - a \notin Inv(A)\}$ is called the spectrum of a and is denoted by $\sigma(a)$ or $\sigma_A(a)$.*

For a unital Banach algebra A and any element a of A, $\sigma(a)$ is a subset of the closed disc centered at origin with radius $||a||$.

Example 2.17.

(1) Let Ω be a locally compact Hausdorff space. Then for any $f \in C_0(\Omega)$, we have $\sigma(f) = \overline{f(\Omega)}$.

(2) $A = \{(a_{i,j})_{i,j} \in M_{n\times n}(\mathbb{C}) : (a_{i,j})_{i,j}$ is an upper triangle matrix $\}$. Then for any matrix $a = (a_{i,j})_{i,j}$ of A, we have $\sigma(a) = \{a_{11}, a_{22}, ..., a_{nn}\}$.

We can see the spectrum as a generalization of eigenvalues.

Definition 2.18. *Let A be an abelian Banach algebra. Any non-zero continuous homomorphism $\tau : A \to \mathbb{C}$ is called a character.*

The set of all characters on A is denoted by $\Omega(A)$.

Theorem 2.19. *Let A be an abelian Banach algebra and a be an element in A.*

(i) If A is unital, $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}$.

(ii) If A is non-unital, $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\} \cup \{0\}$.

(iii) If A is unital, then for any $\tau \in \Omega(A)$, $||\tau|| = 1$.

A pure state on $C(K)$ for a compact Hausdorff space K is a homomorphism from $C(K)$ into $\mathbb{C}$. The next theorem says that every pure state on $C(K)$ is a Dirac measure at a point $a \in K$.

Theorem 2.20. *Let K be a compact Hausdorff space. Let δ_a be a Dirac delta function(point evaluation function) on $C(K)$ for an element $a \in K$, then $\varphi : K \to \Omega(C(K))$ with $a \mapsto \delta_a$ is a homeomorphism.*

An element a of a C^*-algebra A is said to be positive if it is hermitian and $\sigma(a) \subseteq \mathbb{R}^+$. Any positive element of A is in the form of a^*a for some $a \in A$ and the set of all positive elements of A is denoted by A^+. Since all elements of A can be written as a linear combination of two self-adjoint elements, every element of the closed unit ball of A can be written as a linear combination of four unitary elements.

Definition 2.21. *A net $(u_\lambda)_{\lambda\in\Lambda}$ of a C^*-algebra A is said to be an approximate unit if it consists of increasing positive elements of the closed unit ball of A and for each $a \in A$, $||a - au_\lambda|| \to 0$ and $||a - u_\lambda a|| \to 0$.*

A net $(u_\lambda)_{\lambda\in\Lambda}$ of a C^*-algebra A is said to be a right approximate unit if it consists of increasing positive elements of the closed unit ball of A and for each $a \in A$, $||a - u_\lambda a|| \to 0$.

A net $(u_\lambda)_{\lambda\in\Lambda}$ of a C^*-algebra A is said to be a left approximate unit if it consists of increasing positive elements of the closed unit ball of A and for each $a \in A$, $||a-au_\lambda|| \to 0$.

Theorem 2.22.

(i) If L is a closed left ideal of a C^*-algebra A, then L possesses a right approximate unit $(u_\lambda)_{\lambda\in\Lambda}$.

(ii) Let A and B be two C^*-algebras and $\varphi : A \to B$ be a *-homomorphism. If φ is injective, then it is necessarily isometric.

Definition 2.23. *Let A and B be two C^*-algebras and $\varphi : A \to B$ be a linear map. φ is said to be positive if $\varphi(A^+) \subseteq B^+$.*

A positive linear map between C^*-algebras preserves self-adjoint elements. Every *-algebra homomorphism is a positive linear map.

Example 2.24. *Let A be a C^*-algebra and $\tau : A \to \mathbb{C}$ be a positive linear functional, then $\varphi : A \times A \to \mathbb{C}$ with $(a, b) \mapsto \tau(b^*a)$ is a positive sesquilinear map.*

Theorem 2.25. *Let A be a C^*-algebra and $\tau : A \to \mathbb{C}$ be a positive linear functional.*

(i) *τ is bounded.*

(ii) *$|\tau(b^*a)|^2 \leq \tau(a^*a)\tau(b^*b)$ for all $a, b \in A$.*

(iii) *$\tau(b^*a^*ab) \leq ||a^*a||\tau(b^*b)$ for all $a, b \in A$.*

(iv) *$\tau(a^*) = \overline{\tau(a)}$ and $|\tau(a)|^2 \leq ||\tau||.\tau(a^*a)$ for all $a \in A$.*

Corollary 2.26. *Let A be a unital C^*-algebra and $\tau : A \to \mathbb{C}$ be a bounded linear functional. Then τ is positive iff $\tau(1) = ||\tau||$.*

Definition 2.27. *Let A be a C^*-algebra. If $\tau : A \to \mathbb{C}$ is a positive linear functional and $||\tau|| = 1$, then τ is called a state.*

Theorem 2.28.

(i) *Let A be a C^*-algebra and B be a C^*-subalgebra of A. If $\tau : B \to \mathbb{C}$ is a positive linear functional, then there exists an extension τ' of τ such that $\tau' : A \to \mathbb{C}$ is positive linear functional and $||\tau|| = ||\tau'||$.*

(ii) *(Riesz-Kakutani theorem) Let X be a locally compact Hausdorff space and τ be a positive linear functional on $C_c(X)$, then there exists a unique complex Radon measure μ on X such that for all $f \in C_c(X)$, $\tau(f) = \int\limits_X f(x)d\mu(x)$ and $||\mu|| = ||\tau||$.*

Definition 2.29. *Let A be a C^*-algebra. The pair (H, φ) is called a representation of A if H is a Hilbert space and $\varphi : A \to B(H)$ is a *-homomorphism.*

If φ is injective, then it is called a faithful representation.

Theorem 2.30. *(Gelfand-Naimark-Segal) If A is a C^*-algebra, then there exits a representation (H, φ) of A such that $\varphi : A \to B(H)$ is injective.*

This theorem gives a faithful representation for all C^*-algebras, that is, all C^*-algebras are isometrically isomorphic to a C^*-subalgebra of $B(H)$ for some Hilbert space H.

Definition 2.31. *Let A be a C^*-algebra and x be an element of a convex subset K of A. If for $a, b \in K$ and $t \in (0,1)$, $x = ta + (1-t)b$ implies $x = a = b$, then x is called an extreme point of K.*

Definition 2.32. *Let B be a subset of a C^*-algebra A. The intersection of convex sets of A containing B is called the convex hull of B in A. This set is denoted by $conv(B)$.*

Definition 2.33. *Let A be a C^*-algebra. A state in $S(A)$ is said to be a pure state if it is an extreme point of $S(A)$.*

Definition 2.34. *Let X be a linear space with a Hausdorff topology on it. X is called a locally convex topological vector space if the following holds;*

(i) $X \times X \to X$ *with* $(x, y) \mapsto x + y$ *is continuous.*

(ii) $\mathbb{F} \times X \to X$ *with* $(\lambda, x) \mapsto \lambda x$ *is continuous.*

(iii) There exists a base at the origin consisting of convex sets.

A closed subspace M of a topological vector space X is said to be a complemented subspace if there exists another closed subspace N such that $X = M + N$ and $M \cap N = \{0\}$. The notation of direct sum, $X = M \oplus N$ is sometimes used for the complemented subspaces.

Theorem 2.35. *(Krein-Milman) Let X be a locally convex topological vector space and K be a non-empty compact convex subset of X. Then K is the closed convex hull of its extreme points.*

As the set of states on a C^*-algebra is weak*-compact and convex, it has an extreme point(a pure state).

2.4. On von Neumann algebras

Beside of the norm topology, there are two important topologies on $B(H)$ for some Hilbert space H, which are the strong operator topology(SOT) and the weak operator topology(WOT).

The strong operator topology on $B(H)$ is defined by the following open sets; fix an operator $T \in B(H)$, then for $\epsilon > 0$ and $\xi_1, \xi_2, ..., \xi_n \in H$,

$U(T, \epsilon, \xi_i) := \{S \in B(H) : ||(T - S)\xi_i|| < \epsilon, \text{ for } i = 1, 2, ..., n\}$ is an open set.

The weak operator topology on $B(H)$ is defined by the following open sets; fix an operator $T \in B(H)$, then for $\epsilon > 0$ and $\xi_1, \xi_2, ..., \xi_n, \eta_1, \eta_2, ..., \eta_n \in H$,

$V(T, \epsilon, \xi_i, \eta_i) := \{S \in B(H) : |\langle (T - S)\xi_i, \eta_i \rangle| < \epsilon, \text{ for } , i = 1, 2, ..., n\}$ is an open set.

Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a net in $B(H)$ and T be an element in $B(H)$. Then $T_\lambda \to T$ in the SOT topology iff $||(T_\lambda - T)(\xi)|| \to 0$ for all $\xi \in H$ and $T_\lambda \to T$ in the WOT topology iff $||\langle (T_\lambda - T)(\xi), \eta \rangle|| \to 0$ for all $\xi, \eta \in H$.

The convexity yields the same result here as in the weak topology and the norm topology, that is, for any non-empty convex subset A of $B(H)$, we have $\overline{A}^{SOT} = \overline{A}^{WOT}$. This result gives rise to that the unit ball of $B(H)$ is compact in the WOT topology.

For a subset X of $B(H)$, the set $\{T \in B(H) : TS = ST, \forall S \in X\}$ is the first commutator of X and denoted by X'.

Definition 2.36. *Let $M \subseteq B(H)$ be a self-adjonit unital subalgebra. M is called a von Neumann algebra if it satisfies one of the following conditions;*

(i) $M = M''$.

(ii) *M is closed in the SOT topology.*

(iii) *M is closed in the WOT topology.*

Example 2.37.

(1) *$B(H)$ is a von Neumannn algebra.*

(2) *Let (X, μ) be a measurable space. Then $L^\infty(X, \mu)$ is an abelian von Neumann algebra.*

(3) If M is a *-closed subset of $B(H)$, then M' is a von Neumann algebra.

(4) Let H be a finite dimensional Hilbert space. Then any unital *-subalgebra M of $B(H)$ is a von Neumann algebra.

(5) If M is a finite dimensional unital *-subalgebra of $B(H)$, then it is a von Neumann algebra.

Definition 2.38. *Let $M \subseteq B(H)$ be a von Neumann algebra. A projection p in M is said to be a minimal projection if $p \neq 0$ and $pMp = \mathbb{C}p$.*

2.5. On the Arens multiplication

Let A be a Banach algebra. It is clear that the second dual A^{**} is a Banach space. But, being an algebra is not obvious. To make the second dual space an algebra, we need to clarify the multiplication on A^{**}.

The first Arens product on A^{**} is defined as follows;

(i) For all $a, b \in A$ and $f \in A^*$, we have $fa \in A^*$ and $(fa)(b) = f(ab)$.

(ii) For all $a \in A$, $f \in A^*$ and $F \in A^{**}$, we have $Ff \in A^*$, $Ff(a) = F(fa)$.

(iii) For all $F, G \in A^{**}$ and $f \in A^*$, we have $FG(f) = F(Gf)$.

The second Arens product on A^{**} is defined as follows;

(i) For all $a, b \in A$ and $f \in A^*$, we have $a * f \in A^*$ and $(a * f)(b) = f(ba)$.

(ii) For all $a \in A$, $f \in A^*$ and $F \in A^{**}$, we have $f * F \in A^*$ and $(f * F)(a) = F(a * f)$.

(iii) For all $F, G \in A^{**}$ and $f \in A^*$, we have $(F * G)(f) = F(f * G)$.

Each of two Arens multiplications makes A^{**} a Banach algebra. In general, the first and the second Arens multiplications are not the same. If they are equal, it is called Arens regular. All C^*-algebras are Arens regular.

If the algebra A is Arens regular, then these multiplications on A^{**} are weak* to weak* continuous in each variable when the other is kept fixed. i.e. if in A^{**}, $F_\lambda \to F$ in weak*-topology, then $F_\lambda G \to FG$ and $GF_\lambda \to GF$ for all $G \in A^{**}$ in the weak*-topology. If A is a C^*-algebra, then for $a \in A$(or in A^{**}), $f \in A^*$ and $F \in A^{**}$, we have $\langle aF, f\rangle = \langle a, Ff\rangle$, $\langle aF, f\rangle = \langle F, fa\rangle$ and $\langle aFa, f\rangle = \langle F, afa\rangle$.

2.6. On the Grothendieck and Pelczyński properties

Definition 2.39. *Let A be a Banach space. A series $\sum\limits_{n=1}^{\infty} a_n$ is said to be a weakly unconditionally Cauchy series in A if the series $\sum\limits_{n=1}^{\infty} |f(a_n)|$ converges for all $f \in A^*$.*

Definition 2.40. *Let X and Y be two Banach spaces and $T : X \to Y$ be a bounded linear operator. T is said to be an unconditionally converging operator if T maps any weakly unconditionally Cauchy series in X to an unconditionally series in Y.*

Every weakly compact operator between Banach spaces is an unconditionally converging operator.

Definition 2.41. *Let X be a Banach space. We say that X has the Pelczyński's property(property V) if every bounded linear operator T from X to any Banach space which is unconditionally converging is also weakly compact [6].*

Theorem 2.42. *Let X be a Banach space. X has the property V iff for any $Y \subseteq X^*$ with $\lim\limits_{n} \sup\limits_{f\in Y} |f(a_n)| = 0$ for any weakly unconditionally Cauchy series $\sum\limits_{n=1}^{\infty} a_n$ of X, Y is relatively weakly compact.*

Remark 2.43.

(i) Any quotient space of a Banach space X has the property V if X has the property V.

(ii) Any closed complement subspace of a Banach space X has the property V if X has the property V.

Definition 2.44. *Let X be a Banach space. X is said to have the property V^* if for any $K \subseteq X$ with $\lim_n \sup_{x \in K} |f_n(x)| = 0$ for all weakly unconditionally Cauchy series $\sum_{n \geq 1} f_n$ of X^*, K is relatively weakly compact.*

Theorem 2.45. *Let X be a Banach space.*

(i) If X has the property V^*, then it is weakly sequentially complete.

(ii) If X has the property V, then X^* is weakly sequentially complete.

(iii) X is reflexive iff X has both the property V and the property V^*.

Example 2.46.

(1) If K is a compact Hausdorff space, then $C(K)$ has the property V. So, for any measurable space (X, μ), $L^\infty(X, \mu)$ has the property V.

(2) All C^*-algebras have the property V. So, the von Neumann algebras have the property V [7].

Definition 2.47. *Let X be a Banach space. X is said to be a Grothendieck space if each weak* converging sequence of X^* converges weakly.*

Any quotient space of a Grothendieck space is a Grothendieck space. If a space has the property V and it is a dual of some Banach space, then it has the Grothendieck property [8].

Theorem 2.48. *Let X be a Banach space. Then the following are equivalent;*

(i) X has the Grothendieck property.

(ii) Every bounded linear operator T from X to any separable Banach space is weakly compact.

(iii) Every bounded linear operator T from X to c_0 is weakly compact.

Example 2.49.

(1) If K is a Stonean space(i.e. compact Hausdorff totally disconnected space), then $C(K)$ is a Grothendieck space.

(2) Any reflexive space is a Grothendieck space. So, for a measurable space (X, μ) and $1 < p < \infty$, $L^p(X, \mu)$ is a reflexive Grothendieck space.

(3) ℓ^∞ is a Grothendieck space.

3. THE KADISON-SINGER PROBLEM

Let $B(H)$ be the algebra of bounded linear operators on a separable Hilbert space H. In 1958, Kadison and Singer proved that there are pure states on the continuous masa of $B(H)$ whose extensions to $B(H)$ are not unique. Then they hinted that the same result holds for the discrete masa.

A separable Hilbert space is isomorphic to ℓ_2 and the subalgebra of diagonal operators $\mathbb{D}(\ell_2)$ in $B(\ell_2)$ is isomorphic to ℓ^∞. Any pure state on ℓ^∞, as a linear functional extends to $B(\ell_2)$ with the same norm by the Hahn-Banach theorem. It extends to a state as follows: let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for ℓ_2. Given a pure state f on ℓ^∞, define a map $E : B(\ell_2) \to \ell^\infty$ by $T \mapsto (\langle Te_n, e_n\rangle)_{n\in\mathbb{N}}$. Then $\widetilde{f} := f \circ E : B(\ell_2) \to \mathbb{C}$ is a linear map as the composition of two linear maps is linear. Since f is a state, then $\widetilde{f}(1) = f(E(1)) = f(1) = 1$. If $T \in B(\ell_2)$ is positive, then $\langle Ta, a\rangle \geq 0$ for all $a \in \ell_2$. So, $f(E(T)) = f((\langle Te_n, e_n\rangle)_{n\in\mathbb{N}}) \geq 0$. This implies that $\widetilde{f}$ is a state extension of f. So, the set of state extensions of f is not empty.

It is also known that the set of state extensions of f is weak*-compact and convex. So, it has extreme points by the Krein-Milman theorem. Thus, a pure state has a unique pure state extension to $B(\ell_2)$ iff it has a unique state extension to $B(\ell_2)$.

Kadison and Singer asked whether every pure state extension of f has a unique extension to $B(\ell_2)$? In 2013, the Kadison-Singer problem was solved by Adam Marcus, Daniel Spielman and Nikhil Srivastava. They approached this problem by using random polynomials. There is yet no proof given from a Banach Algebras perspective.

In 1979, Anderson stated the Paving Conjecture and proved that it implies the Kadison-Singer problem. Given a fixed orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$, we give the Paving Conjecture:

Paving Conjecture: For every operator T in $B(\ell_2)$ with zeros on its diagonal, there exists a $k-$partition $\{A_1, ..., A_k\}$ of $\mathbb{N}$(a paving) such that for all $\epsilon > 0$ and $i = 1, 2, ..., k$ $||P_{A_i}TP_{A_i}|| < \epsilon.||T||$, where P_{A_i} are the orthogonal projections in $B(\ell_2(\mathbb{N}))$ onto the closed linear span of $\{e_n : n \in A_i\}$.

The crucial point about this Conjecture is that k doesn't depend on n. Let us give a simple proof that the Paving Conjecture implies the Kadison-Singer problem:

Suppose the Paving Conjecture holds. Let $f : \ell^\infty \to \mathbb{C}$ be a pure state and $E : B(\ell_2) \to \ell^\infty$ be the conditional expectation as defined above. Let f' be any pure state extension of f. Then for any $T \in B(\ell_2)$,

$$f'(T) = f'(T - E(T)) + f'(E(T)) = f'(T - E(T)) + f(E(T)).$$

$T - E(T)$ is zero diagonal and $f(E(T))$ is an extension of f. So, it is enough to show that $E(T) = 0$ implies $f'(T) = 0$. Let $\epsilon > 0$ be given. Suppose $E(T) = 0$. Let $P_{A_1}, P_{A_2}, ..., P_{A_k}$ be a partition such that $\sum_{i=1}^{k} P_{A_i} = 1$ and $||P_{A_i}TP_{A_i}|| \leq \epsilon$.

If we apply f' to the sum $\sum_{i=1}^{k} P_{A_i} = 1$, we obtain $f'(P_{A_{i_0}}) = 1$ for some i_0 in $\{1, 2, ..., k\}$ and $f'(P_{A_i}) = 0$ for $i \neq i_0$. $|f'(TP_{A_i})|^2 \leq f'(T^*T)f'(P^*_{A_i}P_{A_i}) = 0$ for $i \neq i_0$ as f' is positive. Similarly, we have $f'(P_{A_i}T) = 0$ for $i \neq i_0$. So, $f'(P_{A_i}TP_{A_j}) = 0$ for $i \neq i_0$ or $j \neq i_0$.

Hence, $|f'(T)| = |f'(\sum_{i=1}^{k} P_{A_i}T\sum_{i=1}^{k} P_{A_i})| = |f'(P_{A_{i_0}}TP_{A_{i_0}})| \leq ||P_{A_{i_0}}TP_{A_{i_0}}|| \leq \epsilon.$

4. A BANACH ALGEBRA APPROACH TO THE KADISON-SINGER PROBLEM

Let H be a separable Hilbert space with a fixed orthonormal basis $\{e_n\}_{n\geq 1}$ and $B(H)$ be the von Neumann algebra of bounded linear operators on H. Let $C(K)$ be the set of continuous functions on K for some compact Hausdorff space K.

For a non-zero linear functional f on $C(K)$, f is a pure state iff it is a homomorphism from $C(K)$ to $\mathbb{C}$ [5]. Then pure states on $C(\beta\mathbb{N})$ are the non-zero homomorphisms as $\beta\mathbb{N}$ is a compact Hausdorff space. Since $\mathbb{D}(\ell_2)$, $C(\beta\mathbb{N})$ and ℓ^∞ are isomorphic and $\mathbb{D}(\ell_2)$ is a subalgebra of $B(\ell_2)$, we can consider $C(\beta\mathbb{N})$ and ℓ^∞ as subalgebras of $B(\ell_2)$. Define the point evaluation function $\delta_t : C(\beta\mathbb{N}) \to \mathbb{C}$ by $f \mapsto f(t)$ for $t \in \beta\mathbb{N}$. As a non-zero homomorphism on $C(\beta\mathbb{N})$ can be represented by a Dirac measure δ_t(a point evaluation function) for $t \in \beta\mathbb{N}$ [3], we convert the Kadison-Singer problem to the following question: Can we extend a pure state δ_t from $C(\beta\mathbb{N})$ to $B(\ell_2)$ in a unique way?

Throughout this chapter, $[\delta_t] \subseteq B(\ell_2)^*$ denotes the set of pure state extensions of a Dirac measure δ_t to $B(\ell_2)$ for $t \in \beta\mathbb{N}$.

In the article [2], two main results are obtained about the uniqueness problem and also a significant result about the weakly compact subsets of von Neumann algebras is given:

(1) Let A be a von Neumann algebra and K be a weak*-compact subset of A^*. Then the following are equivalent;
   (i) K is a weakly compact subset of A^*.
   (ii) K doesn't contain a homeomorphic copy of $\beta\mathbb{N}$.

As a consequence of this result, we have the following results

(2) $\forall t \in \beta\mathbb{N}$, either $[\delta_t]$ includes a homeomorphic copy of $\beta\mathbb{N}$ or $[\delta_t]$ is contained in a finite dimensional subspace of $B(H)^*$.

(3) Let $\rho = \delta_t + \lambda$ be a pure state extension of δ_t from $C(\beta\mathbb{N})$ to $B(H)$ for a $t \in \beta\mathbb{N}$. Then the following are equivalent;

  (i) The maximal left ideal $N_\rho = \{T \in B(H) : \rho(T^*T) = 0\}$ has a right approximate unit $(U_\lambda)_{\lambda\in\Lambda}$ consisting of positive bounded diagonal operators.

  (ii) ρ is the unique pure state extension of δ_t.

4.1. The continuous extension property

Let X be a Banach space and K be a compact Hausdorff space. Suppose that $C(K)$ is contained in X as a closed subspace. For all $t \in K$, the Dirac measure $\delta_t : C(K) \to \mathbb{C}$ is the point evaluation map at t with $\delta_t(f) = \langle \delta_t, f\rangle = f(t)$ for all $f \in C(K)$. The Hahn-Banach theorem guarantees the existence of extensions of δ_t with the same norm. Suppose δ_t has a unique extension $\overline{\delta_t}$ to X, then

(i) $||\overline{\delta_t}|| = 1$.

(ii) $\rho : K \to X^*$ with $t \mapsto \overline{\delta_t}$ is *weak** continuous.

Regardless of uniqueness, the set $[\delta_t]$ of all extensions of δ_t of norm 1 from $C(K)$ to X is weak*-closed and convex subset of X^*. So, $[\delta_t]$ is weak*-compact by the Banach-Alaoglu theorem.

Also observe that for $t_1 \neq t_2$, we have $[\delta_{t_1}] \cap [\delta_{t_2}] = \emptyset$: otherwise, if $f \in [\delta_{t_1}] \cap [\delta_{t_2}]$, it is both an extension of δ_{t_1} and an extension of δ_{t_2} hence, necessarily f restricted to $C(K)$ is both δ_{t_1} and δ_{t_2}, thus $\delta_{t_1} = \delta_{t_2}$(contradiction).

By the Axiom of Choice, let us select an element $\overline{\delta_t}$ in each set $[\delta_t]$ and define a function $\rho : K \to X^*$ with $t \mapsto \overline{\delta_t}$. The mapping ρ is called a selection mapping if $\rho(t)$ is a norm-preserving extension of the functional δ_t for each $t \in K$.

If there exists a *weak** continuous function ρ as defined above, then we shall say that (K, X) is said to have the continuous extension property.

Remark 4.1.

(i) If the multi-valued function $f : K \to 2^{X^*}$ with $t \mapsto [\delta_t]$ is lower semi-continuous in the weak*-topology on X^*, then the pair (K, X) has the continuous extension property [9].

(ii) If the weak*-topology is the same as the norm topology on the unit ball of X^* (Kadec-Klee property), then the pair (K, X) has the continuous extension property.

4.2. Main Results

Lemma 4.2. *Let X be a Banach space and $C(K)$ be a closed subspace of X for some compact Hausdorff space K. Then the following are equivalent;*

(i) (K, X) has the continuous extension property.

(ii) There exists a contractive projection $P : X \to C(K)$.

Proof. Suppose that (K, X) has the continuous extension property, hence we have a mapping $\rho : K \to X^*$ that is continuous from K into (X^*, w^*) such that for $f \in C(K)$ and $t \in K$, $\langle f, \rho(t)\rangle = \langle f, \delta_t\rangle = f(t)$.

Define a function $\varphi : X \to C(K)$ by $\varphi(x)_{(t)} = \langle x, \rho(t)\rangle$ for $x \in X$ and $t \in K$. Let's first show that $\varphi(x)$ is continuous on K.

$\varphi(x) \in C(K)$ if $\varphi(x) : K \to \mathbb{C}$ with $t \to \langle x, \rho(t)\rangle$ is continuous. So, we must show that for any net $t_\lambda \to t$ in K, we have $\varphi(x)(t_\lambda) \to \varphi(x)(t)$. By weak* continuity of ρ;

$$\begin{aligned}\big|\varphi(x)(t_\lambda) - \varphi(x)(t)\big| = \big|\langle x, \rho(t_\lambda)\rangle - \langle x, \rho(t)\rangle\big| &= \big|\rho(t_\lambda)(x) - \rho(t)(x)\big| \\ &< \epsilon.\end{aligned}$$

Now, we will show that $\varphi : X \to C(K)$ with $\varphi(x)(t) = \langle x, \rho(t)\rangle$ is linear.

(i) Let $x_1, x_2 \in X$ and $t \in K$ be arbitrary elements.

$$\begin{aligned}\varphi(x_1+x_2)(t) = \langle x_1+x_2, \rho(t)\rangle = \langle x_1, \rho(t)\rangle + \langle x_2, \rho(t)\rangle &= \varphi(x_1)(t) + \varphi(x_2)(t)\\ &= \big(\varphi(x_1)+\varphi(x_2)\big)(t).\end{aligned}$$

(ii) For all $x \in X$, any scalar α and $\forall t \in K$

$\varphi(\alpha x)(t) = \langle \alpha x, \rho(t)\rangle = \alpha\langle x, \rho(t)\rangle = \alpha\varphi(x)(t) = (\alpha\varphi(x))(t).$

Secondly, we will show that $\varphi\big|_{C(K)} = id$.

For $f \in C(K) \subseteq X$ and $t \in K$, $\varphi(f)(t) = \langle f, \rho(t)\rangle = \langle f, \delta_t\rangle = f(t)$. Then $\varphi(f) = f$ which implies $\varphi\big|_{C(K)} = id$.

Finally, we will show that $||\varphi|| \leq 1$.

$$\begin{aligned}||\varphi|| = \sup_{||x||=1} ||\varphi(x)|| = \sup_{||x||=1, t\in K} |\varphi(x)(t)| = \sup_{||x||=1, t\in K} |\langle x, \rho(t)\rangle| &\leq \sup_{t\in K, ||x||=1} ||x||.||\rho(t)||\\ &= \sup_{t\in K} ||\overline{\delta_t}|| = 1.\end{aligned}$$

So, $||\varphi|| \leq 1$.

Combining the last two observations, we get $||\varphi|| = 1$. Thus, φ is a contractive projection.

For the reverse direction, suppose that $P : X \to C(K)$ is a contractive projection. Since $M(K) \simeq C(K)^*$(by the Riesz representation theorem) and $P^* : C(K)^* \to X^*$, we have that $P^* : M(K) \to X^*$ is continuous in the weak*-topology. Here, P^* is the adjoint of P.

For $f \in C(K) \subseteq X$ and $t \in K$, $\langle P^*(\delta_t), f\rangle = \langle \delta_t, P(f)\rangle = \langle \delta_t, f\rangle = f(t)$.

$$||P^*(\delta_t)|| = \sup_{||x||=1, x\in X} |P^*(\delta_t)(x)| \geq \sup_{||f||=1, f\in C(K)} |P^*(\delta_t)(f)| = \sup_{||f||=1, f\in C(K)} |f(t)| = 1.$$

Then $1 \leq \big|\big|P^*(\delta_t)\big|\big| \leq ||P^*||.||\delta_t|| = ||P^*|| \leq 1$. So, $\big|\big|P^*(\delta_t)\big|\big| = 1$.

Now observe that

$$\begin{aligned}\delta_{t_\lambda} \to \delta_t \text{ in weak}^*\text{-topology in M(K)} \quad & \text{iff } \delta_{t_\lambda}(f) \to \delta_t(f) \text{ for all f} \in C(K)\\ & \text{iff } f(t_\lambda) \to \text{ f(t) for all f} \in C(K)\\ & \text{iff } t_\lambda \to \text{t in K.}\end{aligned}$$

So, on the Gelfand spectrum $\{\delta_t,\ t \in K\}$ of $C(K)$, the $weak^*$-topology induced by $\sigma(C(K)^*, C(K))$ is the same as the original topology of K . Thus, $\rho : K \to X^*$ with $t \mapsto P^*(\delta_t)$ is weak* continuous.

Furthermore, for $f \in C(K)$, $\langle \rho(t), f\rangle = \langle P^*(\delta_t), f\rangle = \langle \delta_t, P(f)\rangle = \langle \delta_t, f\rangle = f(t)$. So, ρ is a continuous extension mapping. Therefore, (K, X) has the continuous extension property. $\square$

Recall that a bounded projection P is said to be an $\mathcal{M}$-projection if for each $x \in X$, $||x|| = max\big\{||P(x)||, ||x - P(x)||\big\}$.

If $P : X \to X$ is an $\mathcal{M}$-projection, $Q : X \to Z$ is a contractive projection and $Ran(P) = Ran(Q)$, then P and Q are identically equal.

Corollary 4.3. *Let X be a Banach space and $C(K)$ be a closed subspace of X for some compact Hausdorff space K. If $P : X \to C(K)$ is an $\mathcal{M}$-projection, then the only continuous extension map is $\rho : K \to X^*$ with $t \mapsto P^*(\delta_t)$.*

Proof. Let's show the existence of the continuous extension mapping:

Since P is an $\mathcal{M}$-projection, then we have the projection $P : X \to C(K)$ such that $||x|| = max\{||P(x)||, ||x - P(x)||\}$, $\forall x \in X$.

P is a contractive projection as $||P(x)|| \leq ||x||$ for all $x \in X$.

By the previous lemma, there exists a continuous extension property of (K, X) as defined in the lemma, $\rho : K \to X^*$ with $\rho(t) = P^*(\delta_t)$.

Next, we will show the uniqueness: Assume that there is another continuous extension $\rho' : K \to X^*$. By the previous lemma, there exists such a contractive projection $Q : X \to C(K)$ for ρ'. Using the preceding observation about $\mathcal{M}$-projection yields $P = Q$. □

Lemma 4.4. *Let X and Y be Banach spaces. If an isomorphic copy of ℓ^∞ doesn't lie in Y, then every bounded linear operator T from X^* to Y is an unconditionally converging operator. Moreover, if X^* has the Pelczyński's property and an isomorphic copy of ℓ^∞ doesn't lie in Y, then every bounded linear operator T from X^* to Y is a weakly compact operator.*

Proof. Suppose that an isomorphic copy of ℓ^∞ doesn't lie in Y and $T : X^* \to Y$ is a bounded linear operator.

For a contradiction, assume that T is not an unconditionally converging operator. So, there exists a subspace $M \subseteq X^*$ such that $M \simeq c_0$ and $T\big|_M : M \to T(M)$ is an isomorphism (Pelczyński) [6].

Let $i : M \to X^*$ be a natural injection, indeed an isometry. Hence, the map $i^{**} : M^{**} \to X^{***}$ is injective and linear. Let $P : X^{***} \to X^*$ be the restriction projection.

Since $M^{**} \simeq \ell^\infty$ and an isomorphic copy of ℓ^∞ doesn't lie in Y, then the linear operator $T \circ P \circ i^{**} : M^{**} \to X^{***} \to X^* \to Y$ is weakly compact by Rosenthal(Proposition 1.2) [10].

For $m \in M$, $T \circ P \circ i^{**}(m) = T \circ P(m) = T(m)$. The first equality comes from the fact that M can be embedded in M^{**} and $i^{**} : M^{**} \to X^{***}$ is injection. The second equality is because of that $M \subseteq X^*$.

So, $T \circ P \circ i^{**}$ and T are the same on M. Thus, $T\big|_M : M \to T(M)$ is weakly compact. But, $T\big|_M : M \to T(M)$ is an isomorphism and $M \simeq c_0$ (contradiction).

Therefore, if Y doesn't contain any isomorphic copy of ℓ^∞ and $T : X^* \to Y$ is a bounded linear operator, then T is an unconditionally converging operator. Moreover, if addition to the conditions given in the theorem, X^* has property V, T is weakly compact by the definition of the Pelczyński property. $\square$

H. Pfitzner showed that every von Neumann algebra A has the property (V). Consequently, every von Neumann algebra is a Grothendieck space. So, any weak* convergent sequence in A^* is weakly convergent [7].

Theorem 4.5. *Let A be a von Neumann algebra and K be a weak*-compact subset of A^*. Then the following are equivalent;*

(i) K is a weakly compact subset of A^*.

(ii) K doesn't include any homeomorphic copy of $\beta\mathbb{N}$.

Proof. Suppose that K is a weakly compact space. Any weakly compact subset of a Banach space is weakly sequentially compact. $\beta\mathbb{N}$ doesn't have any infinite convergent sequence. So, the homeomorphic copy of $\beta\mathbb{N}$ is not in K.

Conversely, suppose that K doesn't have a homeomorphic copy of $\beta\mathbb{N}$. So, there is no isomorphic copy of ℓ^∞ in $C(K)$ [11].

Define $\psi : A \to C(K)$, $a \mapsto \psi(a)$ with $\psi(a)(t) = \langle a, t \rangle$.

Clearly ψ is a bounded linear operator. Since $C(K)$ doesn't include any isomorphic copy of ℓ^∞, by the previous lemma, ψ is an unconditionally converging operator.

Since A is a von Neumann algebra, it has the property V. Therefore, ψ is a weakly compact operator. $\psi^* : C(K)^* \to A^*$ is also weakly compact.

If we take δ_t Dirac measure(point evaluating map at t) for some $t \in K$, then we have $\langle a, \psi^*(\delta_t) \rangle = \langle \psi(a), \delta_t \rangle = \psi(a)(t) = \langle a, t \rangle$ for all $a \in A$. Hence, $\psi^*(\delta_t) = t$. So, $K \subseteq \psi^*(X)$, where X is a closed unit ball of Banach space $M(K)$(regular complex Borel measures on K).

As ψ^* is weakly compact, $\psi^*(X)$ is relatively weakly compact. Now we have that $K \subseteq \overline{\psi^*(X)} \subseteq A^*$ where K is weak*-compact and $\overline{\psi^*(X)}$ is weakly compact. Since weak*-topology is weaker than weak topology, K is weakly closed in $\overline{\psi^*(X)}$. Therefore, K is weakly compact. $\square$

As a consequence of this theorem, if A is a von Neumann algebra, then any weak*-compact subset K of A^* with $card(K) < 2^c$ is weakly compact(c stands for the cardinality of $\mathbb{R}$) because $card(\beta\mathbb{N}) = 2^c$.

Thus, if A is a von Neumann algebra and K is weak*-compact subset of A^* with $card(K) < 2^c$, then every net of K has a weakly convergent subnet. It is obvious that the conclusion of the previous theorem is not only true for a von Neumann algebra, but also true for any dual Banach space having the property V.

Corollary 4.6. *Let A be a C^*-subalgebra of a von Neumann algebra B. A doesn't contain an isomorphic copy of ℓ^∞, Then the following are equivalent;*

(i) There exists a bounded projection $P : B \to A$.

(ii) The dimension of A is finite.

Proof. Suppose there exists a bounded projection $P : B \to A$. Since A doesn't include any isomorphic copy of ℓ^∞ and B is a von Neumann algebra, P is weakly compact. Let X be a closed unit ball of B. So, $\overline{P(X)}$ is weakly compact. As P is a surjective bounded linear mapping, $\overline{P(X)}^\circ \neq \emptyset$ by the Open mapping theorem. Hence, A is reflexive [4].

Since A is reflexive, then A has both the property V and the property V^*. Hence, A is weakly sequentially complete by Theorem 2.45. Therefore, dim(A)$< \infty$ by [12] (Proposition 2).

Conversely, we know that in any Banach space B, every finite dimensional subspace A of B is complemented in B. Hence, there exists a bounded projection from B onto A. □

Remark 4.7. *We refer the reader to the paper of Archbold mentioned above to see the pairs of C^*-algebras (A, B), where B is a C^*-algebra and A is a C^*-subalgebra of B such that there exists a unique projection $P : B \to A$ with $||P|| = 1$ [13].*

Anderson proved under the continuum hypothesis that there exists an infinite compact subset $K \subseteq \beta\mathbb{N}$ such that δ_t can be extended uniquely to a pure state $\overline{\delta_t}$ on $B(H)$ for all $t \in K$ (Theorem 6) [14].

So, $\rho : K \to B(H)^*$ with $\rho(t) = \overline{\delta_t}$ is the continuous extension mapping. The restriction mapping $T : C(\beta\mathbb{N}) \to C(K)$ with $T(f) = f\big|_K$ is bounded linear and also onto by the Tietze extension theorem. Since T is surjective, it is not weakly compact. So, by Lemma 4.4, $C(K)$ includes an isomorphic copy of ℓ^∞. So, if we consider the result of Anderson and the preceding theorem together, we conclude that even though an isomorphic copy of ℓ^∞ lies in $C(K)$, there is a unique pure state extension of a pure state from $C(K)$ to $B(H)$.

For the reverse direction, Akemann and Weaver showed that there exists a pure state on $B(H)$ such that its restriction to any masa is not pure.

Lemma 4.8. *Let B be a Banach space. If A is a complemented Banach subspace of B^*, then A is complemented in A^{**}.*

Proof. Since $B^{***} = B^* \oplus B^{\perp}$, we have a natural projection $P : B^{***} \to B^*$. Let $Q : B^* \to A$ be a bounded projection, then $Q^{**} : B^{***} \to A^{**}$ is a natural projection. Then the map $Q \circ P \circ Q^{**} : B^{***} \longrightarrow A^{**} \hookrightarrow B^{***} \longrightarrow B^* \longrightarrow A$ is a projection and $Q \circ P \circ Q^{**}(a) = a$ for all $a \in A$.

Therefore, the restriction of $Q \circ P \circ Q^{**}$ on A^{**} is a projection onto A. □

Theorem 4.9. *Let X be a Banach space and K be an infinite compact Hausdorff space. Suppose that $C(K)$ lies in X^* as a closed subspace.*

If $\rho : K \to X^{**}$ is a weak continuous extension mapping, then $C(K)$ is complemented in $C(K)^{**}$.*

Moreover, in this case, $C(K)$ is a Grothendieck space and contains an isomorphic copy of ℓ^∞.

Proof. By Lemma 4.2, if (K, X^*) has the continuous extension property, then there exists a contractive projection $P : X^* \to C(K)$. So, $C(K)$ is complemented in X^*. By the preceding lemma, $C(K)$ is complemented in its second dual. By the Krein-Milman theorem, $C(K)^{**}$ has the Grothendieck property [15]. Thus, its complemented subspace $C(K)$ has the Grothendieck property as well [16].

Finally, we will show that an isomorphic copy of ℓ^∞ lies in $C(K)$. Assume for a contradiction that $C(K)$ doesn't contain an isomorphic copy of ℓ^∞. Then by Lemma 4.4, the bounded linear projection $P : C(K)^{**} \to C(K)$ is weakly compact. By the Open mapping theorem, $C(K)$ is reflexive. This happens only when K is finite. Therefore, $C(K)$ contains an isomorphic copy of ℓ^∞. □

Remark 4.10. *R. Haydon showed that there exists a compact Hausdorff space K such that $C(K)$ is a Grothendieck space and doesn't contain an isomorphic copy of ℓ^∞ [11].*

However, if an isomorphic copy of ℓ^∞ lies in $C(K)$, then a homeomorphic copy of $\beta\mathbb{N}$ lies in the compact Hausdorff space K. The converse is not true in general [11].

Therefore, the main issue for a unique extension of pure states from $C(K) \subseteq X$ to X is not about K containing a homeomorphic copy $\beta\mathbb{N}$ but about an isomorphic copy of ℓ^∞ lying in $C(K)$ or not.

The Kadison-Singer problem:

Let H be a separable Hilbert space and $\{e_n\}$ be an orthonormal basis in H. Take a bounded sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}}$. Define a map $T_\lambda : H \to H$ by $\sum\limits_{i\geq 0} x_i e_i \mapsto \sum\limits_{i\geq 0} \lambda_i x_i e_i$. Let $x = \sum\limits_{i\geq 0} x_i e_i$ and $y = \sum\limits_{i\geq 0} y_i e_i$ be any two elements in H and c be a constant, then

$$\begin{aligned} T_\lambda(cx+y) = T_\lambda(\sum cx_ie_i + \sum y_ie_i) = T_\lambda(\sum(cx_i+y_i)e_i) &= \sum \lambda_i(cx_i+y_i)e_i \\ &= c\sum \lambda_i x_i e_i + \sum \lambda_i y_i e_i = cT_\lambda(x) + T_\lambda(y). \end{aligned}$$

$$\begin{aligned} ||T_\lambda(x)||^2 = ||\sum \lambda_i x_i e_i||^2 = \langle \sum \lambda_i x_i e_i, \sum \lambda_i x_i e_i \rangle &= \sum |\lambda_i x_i|^2 \leq \sup|\lambda_i|^2 . \sum |x_i|^2 \\ &= \big(\sup|\lambda_i|\big)^2 . ||x||^2 = ||\lambda||_\infty^2 . ||x||^2. \end{aligned}$$

So, $||T_\lambda|| \leq ||\lambda||$.

Thus, T_λ is a bounded linear map on H. The corresponding map $\varphi : \ell^\infty \to B(H)$ with $\lambda \to T_\lambda$ is a *isometry because if λ and ν are arbitrary two elements in ℓ^∞ and c is a scalar, then for any $x = \sum\limits_{i\geq 0} x_i e_i \in H$, we have the following

$T_{\lambda+\nu}(x) = \sum\limits_{i\geq 0}(\lambda_i + \nu_i)x_i e_i = \sum\limits_{i\geq 0} \lambda_i x_i e_i + \sum\limits_{i\geq 0} \nu_i x_i e_i = T_\lambda(x) + T_\nu(x)$.

$T_{c\lambda}(x) = \sum\limits_{i\geq 0} c\lambda_i x_i e_i = c\sum\limits_{i\geq 0} \lambda_i x_i e_i = c.T_\lambda(x) = (c.T_\lambda)(x)$.
$T_{\lambda\nu}(x) = \sum\limits_{i\geq 0} \lambda_i \nu_i x_i e_i = T_\lambda T_\nu(x)$. So, φ is a homomorphism.

Let T_λ^* be the Hilbert adjoint operator of T_λ, then for any $x = \sum\limits_{i\geq 0} x_i e_i$ and $y = \sum\limits_{i\geq 0} y_i e_i$ in H,

$$\begin{aligned}\langle T_\lambda^* x, y\rangle = \langle x, T_\lambda y\rangle = \langle \sum_{i\geq 0} x_i e_i, \sum_{i\geq 0} \lambda_j y_j e_j\rangle = \sum_{i\geq 0} \langle x_i e_i, \lambda_i y_i e_i\rangle &= \sum_{i\geq 0} \langle \lambda_i^* x_i e_i, y_i e_i\rangle \\ &= \langle \sum_{i\geq 0} \lambda_i^* x_i e_i, \sum_{i\geq 0} y_j e_j\rangle = \langle T_{\lambda^*} x, y\rangle.\end{aligned}$$

So, $T_\lambda^* = T_{\lambda^*}$ which implies that φ is *- homomorphism. $\varphi(\lambda) = \varphi(\nu)$ implies that $T_\lambda(x) = T_\nu(x)$ for all $x \in H$. If we choose $x = 1.e_1 + 0.e_2 + ...$, we get $\lambda_1 = \nu_1$. Applying this method respectively to each term, we will obtain $\lambda = \nu$. Thus, we have an injective *-homomorphism. Hence, φ is necessarily a *-isometry (see theorem 3.1.5 in Murphy [3]). We can consider ℓ^∞ as a von Neumann subalgebra of $B(H)$. Then $D : B(H) \longrightarrow \ell^\infty$ with $T \longmapsto (\langle T(e_n), e_n\rangle)_{n\geq 1}$ is a contractive positive projection.

$$\begin{aligned}D(T^*) = (\langle T^*(e_n), e_n\rangle)_{n\geq 1} = (\langle e_n, T(e_n)\rangle)_{n\geq 1} = (\overline{\langle T(e_n), e_n\rangle})_{n\geq 1} &= \overline{(\langle T(e_n), e_n\rangle)}_{n\geq 1} \\ &= \overline{D(T)}.\end{aligned}$$

$D(T^*T) = (\langle T^*T(e_n), e_n\rangle)_{n\geq 1} = (\langle T(e_n), T(e_n)\rangle)_{n\geq 1} = (||Te_n||^2)_{n\geq 1}$, which is a positive sequence.

$$\begin{aligned}||D(T)|| = ||(\langle T(e_n), e_n\rangle)_{n\geq 1}|| = \sup_n |\langle Te_n, e_n\rangle| \leq \sup_n ||Te_n||.||e_n|| &\leq \sup_n ||T||.||e_n||.||e_n|| \\ &= ||T||.\end{aligned}$$

So, D is contractive.

Since φ is an isometric *isomorphism between ℓ^∞ and its image in $B(H)$, we can consider λ as an operator T_λ.

If we evaluate T_λ at each point e_i, then $T_\lambda(e_i) = \lambda_i e_i$.

$D(T_\lambda) = (\langle T_\lambda e_n, e_n\rangle)_{n\geq 1} = (\langle \lambda_n e_n, e_n\rangle)_{n\geq 1} = (\lambda_n \langle e_n, e_n\rangle)_{n\geq 1} = (\lambda_n)_{n\geq 1} = \lambda$. Thus, D is a projection.

We identify ℓ^∞ with abelian C^*-algebra $C(\beta\mathbb{N})$, then for any $t \in \beta\mathbb{N}$, $D^*(\delta_t)$ is given by $D^*(\delta_t)(T) = \lim\limits_t \langle Te_n, e_n\rangle$. So, we have

$$\begin{array}{lll} D^* : \ell^{\infty *} \to B(H)^* & D^*(\delta_t) : B(H) \to \mathbb{C} & \delta_t : \ell^\infty \to \mathbb{C}. \\ \quad \delta_t \mapsto D^*(\delta_t) & \qquad T \mapsto \lim\limits_t \langle Te_n, e_n\rangle & \end{array}$$

The limit is taken over ultrafilter t for the bounded sequence $(\langle Te_n, e_n\rangle)_{n\in\mathbb{N}}$. Since D is positive and contractive for each $t \in \beta\mathbb{N}$, $D^*(\delta_t)$ is a state (indeed, a pure state by J. Anderson [14]), extension of δ_t to $B(H)$.

So, $\rho : \beta\mathbb{N} \to B(H)^*$ with $t \mapsto D^*(\delta_t)$ is a weak* continuous extension mapping. As D is a projection, we have the decomposition $B(H) = \ell^\infty \oplus B_\circ(H)$, where $B_\circ(H)$ is the kernel of D. Hence, $B(H)^* = \ell^{\infty *} \oplus \ell^{\infty\perp}$, where $\ell^{\infty\perp}$ is the annihilator of ℓ^∞ in $B(H)^*$. Now fix a $t \in \beta\mathbb{N}$. Since $B(H)^* = \ell^{\infty *} \oplus \ell^{\infty\perp}$, every extension of δ_t to $B(H)$ is of the form $\rho = \delta_t + \lambda$ with $\lambda \in \ell^{\infty\perp}$ and $\delta_t \in \ell^{\infty *}$ so that λ vanishes on the diagonal operators. For each t, $[\delta_t]$ is the set of all state extensions of δ_t to $B(H)$, then $[\delta_t]$ is weak*-compact convex subset of $B(H)^*$.

To continue, we need some preliminary results.

Let A be a C^*-algebra and $P(A)$ be the set of all pure states of A. Let's show that for $\tau \in P(A)$, $N_\tau = \{a \in A : \tau(a^*a) = 0\}$ is a maximal modular left ideal.

If a and b are arbitrary elements in N_τ, then we have $\tau(a^*a) = 0$ and $\tau(b^*b) = 0$.

$$\tau\Big((a+b)^*(a+b)\Big) = \tau(a^*a + a^*b + b^*a + b^*b) = \tau(a^*a) + \tau(a^*b) + \tau(b^*a) + \tau(b^*b)$$
$$= \tau(a^*b) + \tau(b^*a).$$

Since τ is positive, $|\tau(a^*b)|^2 \leq \tau(b^*b).\tau(a^*a) = 0$. So, $\tau(a^*b) = 0$. Similarly, we have $\tau(b^*a) = 0$. Hence, $a + b \in N_\tau$.

For $a \in N_\tau$ and $b \in A$, $\tau\big((ba)^*(ba)\big) = \tau(a^*b^*ba) \leq ||b^*b||.\tau(a^*a) = 0$, So $ba \in N_\tau$. Thus, N_τ is a left ideal.

A is a C^*-algebra, then it admits an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ consisting of increasing positive elements in the closed unit ball of A.

$||\tau\big((a - u_\lambda a)^*(a - u_\lambda a)\big)|| \leq ||\tau||.||(a - u_\lambda a)^*(a - u_\lambda a)|| \to 0$ for an appropriate λ. Thus, N_τ is modular.

Now suppose that N_τ is not a maximal left ideal. So, there exists a proper left ideal L containing N_τ properly. As the left ideal L is modular and proper, $\overline{L}$ is also a proper left ideal. By Theorem 5.3.3 in Murphy [3], there exists a pure state τ' of A such that $\overline{L} \subseteq N_{\tau'}$. So, we have $N_\tau \subseteq N_{\tau'}$, hence $\tau = \tau'$. Thus, $N_\tau = L$ which implies that N_τ is a maximal left ideal. As $\overline{N_\tau}$ is also a left ideal and N_τ is maximal, $N_\tau = \overline{N_\tau}$. Hence, N_τ is closed.

We have $N_\tau + N_\tau^* = Ker\tau$, where $N_\tau^* = \{a^* : a \in N_\tau\}$(see Theorem 5.3.4 in Murphy [3]). Since N_τ is closed left ideal, it admits a right approximate identity $(v_\lambda)_{\lambda\in\Lambda}$ consisting of increasing positive elements in the closed unit ball of N_τ. So, the left ideal N_τ^{**}(the second dual of N_τ, an ideal in von Neumann algebra A^{**}) has a right unit e_τ. Since e_τ is positive and idempotent, it is a projection in A^{**}.

Observe that $(Ker\tau)^{**} = \{m \in A^{**} : \langle m, \tau\rangle = 0\}$. As $(N_\tau)^{**}$ is a subspace of $(Ker\tau)^{**}$, then $\langle e_\tau, \tau\rangle = 0$. Also, as e_τ is self-adjoint and a right unit of N_τ^{**}, we have $ne_\tau = n$ and $e_\tau n^* = n^*$ for all $n \in N_\tau^{**}$.

Lemma 4.11. *Let τ be a pure state on a C^*-algebra A and 1 be a unit of the von Neumann algebra A^{**}. Then there exists a minimal projection $e \in A^{**}$ with $\langle \tau, e\rangle = 1$. Moreover, if there exists another pure state τ' with $\langle \tau', e\rangle = 1$, then $\tau = \tau'$.*

Proof. Let τ be a fixed pure state on A, then we have $\langle \tau, 1 - e_\tau\rangle = 1$ by our previous discussion. We will show that $e = 1 - e_\tau$ is a minimal projection in A^{**}.
As N_τ is an embedded subspace of $(N_\tau)^{**}$, we have

$$eae = ea(1 - e_\tau) = ea - eae_\tau = ea - ea = 0.$$

$$ea^*e = (1 - e_\tau)a^*e = a^*e - e_\tau a^*e = a^*e - a^*e = 0 \text{ for all } a \in N_\tau.$$

By using these two facts with $N_\tau + N_\tau^* = Ker\tau$, we obtain $eae = 0$, $\forall a \in Ker\tau$. As the multiplication in A^{**} is separately weak* continuous and $Ker\tau$ is weak*-dense in $(ker\tau)^{**}$, we have $eae = 0$ for all $a \in (Ker\tau)^{**}$. Since $A^{**} = (Ker\tau)^{**} \oplus \mathbb{C}.e$, then e is a minimal projection in A^{**}.

Now, assume τ' is another pure state with $\langle \tau', e\rangle = 1$. Consider τ as an element of A^{***} and ae_τ as an element of A^{**} for all $a \in A$.
$|\langle \tau, ae_\tau\rangle|^2 \leq \tau(e_\tau^* e_\tau).\tau(a^*a) = \tau(e_\tau).\tau(a^*a) = \langle \tau, e_\tau\rangle.\tau(a^*a) = 0$. So, $\langle \tau, ae_\tau\rangle = 0$. Hence, we have $\langle \tau, ae\rangle = \langle \tau, a(1 - e_\tau)\rangle = \langle \tau, a\rangle$. After a similar calculation, we obtain $\langle \tau, ea\rangle = \langle \tau, a\rangle$.

Thus, $\langle \tau, eae\rangle = \langle \tau, ea\rangle = \langle \tau, a\rangle$ for all $a \in A$. By hypothesis, we have $\langle \tau', e\rangle = 1$. So, $\langle \tau', eae\rangle = \langle \tau', a\rangle$. Since e is a minimal projection, for each $a \in A$, we can find $\exists c \in \mathbb{C}$ such that $eae = c.e$.
$\langle \tau, a\rangle = \langle \tau, eae\rangle = \langle \tau, ce\rangle = c.\langle \tau, e\rangle = c = c\langle \tau', e\rangle = \langle \tau', ce\rangle = \langle \tau', eae\rangle = \langle \tau', a\rangle$.

Therefore, $\tau = \tau'$. $\square$

Lemma 4.12. *Let B be a unital C^*-algebra and A be a C^*-subalgebra of B sharing the same unit. Let τ be a pure state on A and e_τ be the right unit in A^{**} as defined in the preliminary result. Consider A^{**} as a von Neumann subalgebra of B^{**}, then*

(i) $\{f \in S(B) : f|_A = \tau\} = \{f \in S(B) : f(1 - e_\tau) = 1\}$.

(ii) *$span\{f \in S(B) : f|_A = \tau\}$ is a finite dimensional subspace of B^* if and only if $(1 - e_\tau)B^{**}(1 - e_\tau)$ is a finite dimensional subspace of B^{**}.*

(iii) *The pure state τ has a unique extension to B if and only if $(1 - e_\tau)B^{**}(1 - e_\tau)$ is a one-dimensional space(so that $(1 - e_\tau)B^{**}(1 - e_\tau)$ includes a unique state).*

Proof. Suppose that $f(1 - e_\tau) = \langle f, e\rangle = 1$ for $f \in S(B)$, then $f|_A = \tau$ by Lemma 4.11.

For the converse, suppose that $f\big|_A = \tau$ for $f \in S(B)$, then $1 = \langle \tau, e\rangle = \langle f\big|_A, e\rangle$. Hence, $fe = f(1 - e_\tau) = 1$.

For the rest of lemma, first observe that $e : B^* \to B^*$ is a projection. Then we have $B^* = Ker(e) \oplus Ran(e) = e_\tau B^* e_\tau \oplus eB^*e$. Also, $B^{**} = eB^{**}e \oplus e_\tau B^{**} e_\tau$. Since $\langle eb^{**}e, e_\tau b^* e_\tau\rangle = \langle b^{**}, ee_\tau b^* e_\tau e\rangle = \langle b^{**}, 0\rangle = 0$ for any element $eb^{**}e \in eB^{**}e$ and $e_\tau b^* e_\tau \in e_\tau B^* e_\tau$, then the dual space of eB^*e is $eB^{**}e$.

Let ege be an element of eB^*e for $g \in B^*$. Since $eae = 0$ for all $a \in Ker\tau$, we have $\langle ege, a\rangle = \langle g, eae\rangle = 0$. So, $Ker\tau \subseteq Ker(ege|_A)$. Then we have that $ege|_A = \lambda\tau$ for some $\lambda \in \mathbb{C}$. So, $eB^*e \subseteq \{f \in B^* : f|_A = \lambda\tau \text{ for } \lambda \in \mathbb{C}\} = span\{f \in S(B) : f|_A = \tau\}$. Define a map $\psi : span\{f \in S(B) : f|_A = \tau\} \to eB^*e$ by $f \mapsto efe$. So, the map ψ is linear. Since $||efe|| \leq ||f||$, ψ is bounded. Let f be a nonzero element in $Ker\psi$. Then $efe = 0$ implies that $\langle efe, b\rangle = 0$ for all $b \in B$. As B is weak*-dense in B^{**}, it also holds for all $b^{**} \in B^{**}$. In particular, $\langle efe, e\rangle = 0$. Hence, we have $\langle f, e\rangle = 0$(contradiction by i). So, f must be a zero functional. Hence, the map ψ is linear bijective. If one of these spaces is finite dimensional, then the other has the same dimension. $\square$

Theorem 4.13. *$\forall t \in \beta\mathbb{N}$, either $[\delta_t]$ is contained in a finite dimensional subspace of $B(H)^*$ or a homeomorphic copy of $\beta\mathbb{N}$ lies in $[\delta_t]$.*

Proof. Suppose that $[\delta_t]$ doesn't lie in a finite dimensional subspace of $B(H)^*$. We know that $\tau = \delta_t$ is a pure state on $C(\beta\mathbb{N})$. So, by Lemma 4.11, there exists a minimal projection $e \in C(\beta\mathbb{N})^{**}$ with $e = 1 - e_\tau$ such that $\langle \tau, e_\tau \rangle = 0$. Since $B(H)$ is a unital C^*-algebra and $C(\beta\mathbb{N})$ is a closed C^*-subalgebra of $B(H)$ sharing the same unit, we can consider $C(\beta\mathbb{N})^{**}$ as a von Neumann subalgebra of $B(H)^{**}$. As δ_t is a pure state, $\{f \in S(B(H)) : f\big|_{C(\beta\mathbb{N})} = \delta_t\} = \{f \in S(B(H)) : f(1 - e_\tau) = 1\}$ by Lemma 4.12. So, $[\delta_t] \subseteq \{f \in S(B(H)) : f\big|_{C(\beta\mathbb{N})} = \delta_t\}$.

$[\delta_t] \subseteq span([\delta_t])$ and $span([\delta_t])$ is an infinite dimensional subspace of $B(H)^*$, then $eB(H)^*e$ is infinite dimensional. Let's say $S := \{f \in eB(H)^*e : \text{f is a state}\}$ and $P := \{f \in eB(H)^*e : \text{f is a pure state}\}$. Then $span(S)$ is an infinite dimensional subspace of $eB(H)^*e$.

Since $span(S)$ is infinite dimensional (Proposition 3.11.9 [17]), $span(P)$ is an infinite dimensional subspace of $eB(H)^*e$ by the Krein-Milman theorem. Since each pure state in $span(P)$ is supported by a minimal projection of $eB(H)^{**}e$, the set of minimal projections spans an infinite dimensional subspace. For two distinct minimal projections e_{τ_i} and e_{τ_j}, we have a projection $e_{\tau_i}e_{\tau_j}$ such that $e_{\tau_i}e_{\tau_j} \leq e_{\tau_i}$. This happens only when $e_{\tau_i}e_{\tau_j} = 0$ as e_{τ_i} is a minimal projection. So there exists a sequence of orthogonal minimal projections $\{e_{\tau_n}\}_{n\in\mathbb{N}}$ and each e_{τ_n} supports a pure state τ_n of $eB(H)^*e$. So, we have a sequence of orthogonal pure states $\{\tau_n\}_{n\in\mathbb{N}}$ in $eB(H)^*e$ in terms of orthogonality of minimal projections. The orthogonal pure states $\{\tau_n\}$ in $eB(H)^*e$ is homeomorphic to $\mathbb{N}$ so that $[\delta_t]$ contains a homeomorphic copy of $\beta\mathbb{N}$. $\square$

Remark 4.14. *If $[\delta_t]$ is contained in a finite dimensional subspace of $B(H)^*$, it is a compact set in the norm topology, hence it is norm separable. This theorem gives rise to an important question as stated below.*

Question: How to prove that $[\delta_t]$ doesn't contain a homeomorphic copy of $\beta\mathbb{N}$?

Let t be an ultrafilter of $\beta\mathbb{N}$ and $\rho = \delta_t + \lambda$ be a pure state extension of δ_t. We have already proved that $N_\rho = \{T \in B(H) : \rho(T^*T) = 0\}$ is a closed maximal left ideal. This ideal is not always a weak*-closed subspace of $B(H)$. It may not even have a right unit. Since it is a closed left ideal in $B(H)$, it admits a right approximate unit consisting of positive bounded linear operators in the closed unit ball. Moreover, this approximate unit consists of diagonal operators if $t \in \mathbb{N}$. Thus, it is an upcoming question to ask that when the approximate unit of N_ρ consists of diagonal operators?

Theorem 4.15. *Let $\rho = \delta_t + \lambda$ be a pure state extension of δ_t from $C(\beta\mathbb{N})$ to $B(H)$ for any $t \in \beta\mathbb{N}$, where λ vanishes on the diagonal operators. Then the following are equivalent;*

(i) The maximal left ideal $N_\rho = \{T \in B(H) : \rho(T^*T) = 0\}$ has a right approximate unit $(U_i)_{i\in I}$ consisting of positive bounded diagonal operators.

(ii) ρ is a unique pure state extension of δ_t.

Proof. Suppose that δ_t has a unique pure state extension ρ from ℓ^∞ to $B(H)$. So, by Lemma 4.12, $eB(H)^{**}e$ is one dimensional for a minimal projection e of $\ell^{\infty**}$ with $\langle \delta_t, e\rangle = 1$. Since N_ρ is a closed left ideal, it admits a right approximate unit $(U_i)_{i\in I}$ consisting of increasing positive elements in the closed unit ball of N_ρ.

Since $B(H)^{**} = \ell^{\infty**} \oplus \ell^{\infty*\perp}$, where $\ell^{\infty*\perp}$ is the set of annihilator of $\ell^{\infty*}$ in $B(H)^{**}$ and $e\ell^{\infty**}e = \mathbb{C}e$, then $eB(H)^{**}e = \mathbb{C}e$. As U_i is a right approximate identity for N_ρ, there exists a right unit e_ρ for $(N_\rho)^{**}$(second dual of N_ρ). Hence, we have that $ee_\rho e = c.e$ for some $c \in \mathbb{C}$. So $e_\rho \in \ell^{\infty**}$. Thus, the corresponding approximate unit $(U_i)_{i\in I}$ consists of diagonal operators.

For the reverse direction, suppose that the right approximate unit $(U_i)_{i\in I}$ of N_ρ is diagonal. As $N_\rho + N_\rho^* = Ker\rho(N_\rho^*$ is the set of involution elements of $N_\rho)$, $N_\rho \subseteq Ker(\rho)$.

Since U_i are diagonal in $Ker\rho$, we have $\rho(U_i) = 0$ and $\lambda(U_i) = 0$ for each $i \in I$. Hence, $\delta_t(U_i) = 0$ for all $i \in I$.

Let T be any element in $Ker\rho$. Then $T = T_1 + T_2$ for some $T_1 \in N_\rho$ and $T_2 \in N_\rho^*$. As $T = T_1 + T_2 = \lim_i(T_1U_i) + \lim_i(U_iT_2)$, we have

$$\delta_t(T) = \delta_t(\lim_i(T_1U_i) + \lim_i(U_iT_2)) = \lim \delta_t(T_1)\delta_t(U_i) + \lim \delta_t(U_i)\delta_t(T_2) = 0.$$

Thus, $Ker\rho = Ker\delta_t$. Let ρ_1 and ρ_2 be two pure state extensions of δ_t from ℓ^∞ to $B(H)$. Since $Ker\rho_1 = Ker\delta_t$ and $Ker\rho_2 = Ker\delta_t$, then $\rho_1 = \lambda\rho_2$ for some $\lambda \in \mathbb{C}$. As ρ_1 and ρ_2 are pure states, $\rho_1(1) = \rho_2(1) = 1$. So $\lambda = 1$ which shows that the extension of δ_t is unique. $\square$

Let us consider the set $E := \{t : t \in \beta\mathbb{N},\ [\delta_t] \text{ is weakly compact}\}$. Given the results of MSS [1], we know that this set is all of $\beta\mathbb{N}$. But, we don't have a direct proof. We want to look at the topological properties of E in $\beta\mathbb{N}$. By using Theorem 4.5 and Theorem 4.13, we have that the weak compactness of $[\delta_t]$ implies the norm compactness. E is non-empty and it is bigger than the set of integers under the Continuum hypothesis since integers are in E. Let's define $\Delta := \bigcup_{t\in E}[\delta_t]$ and put a metric d on this set inherited from the norm topology on $B(H)^*$.

Lemma 4.16. *The metric space (Δ, d) defined as above is a complete locally compact space. Furthermore, the set $\Delta' = \bigcup_{t\in D}[\delta_t]$ is clopen in Δ for any subset $D \subseteq E$.*

Proof. Let t and s be two different elements in E. Then we can find an operator $T \in \ell^\infty$ with $||T|| = 1$ such that $\langle\delta_t, T\rangle = 1$ and $\langle\delta_s, T\rangle = -1$ by the Urysohn lemma. So, we have $||\rho - \rho'|| \geq |\langle\delta_t + \lambda - \delta_s - \lambda', T\rangle| = 2$ for any extensions ρ and ρ' of δ_t and δ_s respectively. Thus, $d([\delta_t], [\delta_s]) = 2$ for the closed sets $[\delta_t]$ and $[\delta_s]$. Let x be any element in $\overline{\bigcup[\delta_t]}$. Then there exists a net $\{(x_i)\}_{i\in I} \subseteq \bigcup[\delta_t]$ such that for any $\epsilon > 0$, $||x_i - x|| < \epsilon$ for large i. Choose $\epsilon = 1/2$, then for large i, all x_i are necessarily in the same set $[\delta_t]$ for some $t \in E$. After reindexing the net, we will have a net(say the same net x_i) in $[\delta_t]$. As $x_i \to x$ and $[\delta_t]$ is a closed set, $x \in [\delta_t]$. Therefore, the set $\varDelta$ is a norm closed subspace of $B(H)^*$. Therefore, $(\varDelta, d)$ is complete. By the same argument, we can obtain that $\varDelta' = \bigcup_{t \in D} [\delta_t]$ is closed in $B(H)^*$ for any nonempty subset D of E. As $\varDelta \setminus \varDelta'$ is also closed in $\varDelta$ for a subset $D' \subseteq E$, $\varDelta'$ is clopen in $\varDelta$. Thus, for $t \in E$, the set $[\delta_t]$ is both closed and open in $\varDelta$. As a consequence, every pure state extension of δ_t in $\varDelta$ has a compact neighborhood. Therefore $(\varDelta, d)$ is a locally compact space. $\square$

Since $\varDelta$ is locally compact, we can take its Stone-Čech compactification $\beta(\varDelta)$. Suppose $[\delta_t]$ doesn't contain any homeomorphic copy of $\beta\mathbb{N}$ for some $t \in \beta\mathbb{N}$, then the set $[\delta_t]$ is a compact open subset of $\varDelta$ by Lemma 4.13 and Lemma 4.16. Hence, it is also a compact subset of $\beta(\varDelta)$. Now consider the function $\chi_{[\delta_t]} : \beta\varDelta \subseteq B(H)^* \to \mathbb{R}$. For any converging net $f_\lambda \to f$ in $\beta\varDelta$, we have $|\chi_{[\delta_t]}(f_\lambda) - \chi_{[\delta_t]}(f)| \to 0$ as the set $[\delta_t]$ is both open and closed. So, $\chi_{[\delta_t]} \in C(\beta\varDelta)$. Thus, we can define a bounded projection $P : C(\beta\varDelta) \to C([\delta_t])$ by $g \mapsto g.\chi_{[\delta_t]}$. Then $C([\delta_t])$ is a complemented ideal of the C^*-algebra $C(\beta\varDelta)$. Since $[\delta_t]$ is compact and metric, $C([\delta_t])$ is separable(see Riesz's theorem section 12.3, page 251 [4]). If $C(\beta\varDelta)$ has the Grothendieck property, then P is weakly compact(Grothendieck, 1953, Theorem 2.48). Hence, the space $C([\delta_t])$ is finite dimensional. So, $[\delta_t]$ is finite. Since $[\delta_t]$ is convex, this set must be a set of a single point.

Hence, if $C(\beta\varDelta)$ has the Grothendieck property and $[\delta_t]$ doesn't contain any homeomorphic copy of $\beta\mathbb{N}$, then $[\delta_t]$ is a set of one-single point. Therefore, we still have the following questions.

(1) How can we show that $C(\beta\varDelta)$ is a Grothendieck space?

(2) How can we show that E is a closed subset of $\beta\mathbb{N}$?

5. CONCLUSION

Let H be a separable infinite dimensional Hilbert space. The C^*-algebra $C(\beta\mathbb{N})$ sits in $B(H)$ as a complemented weak-star subalgebra. For each t in $\beta\mathbb{N}$, the Dirac measure δ_t is a pure state of the algebra $C(\beta\mathbb{N})$ and a priory has many state extensions to $B(H)$. We denote by $[\delta_t]$ the set of these state extensions. The set $[\delta_t]$ is a weak-star compact convex subset of $B(H)^*$. The authors of the paper [1], by a method completely different from the one used here, solved the famous Kadision-Singer problem by showing that the set $[\delta_t]$ contains only one element so that δ_t has a unique pure state extension to $B(H)$. Our aim in this thesis was to analyze this problem and to obtain this same result by functional analytic methods. In this endeavor, we were only partially successful. Our main result says that either the set $[\delta_t]$ lies in a finite dimensional subspace of $B(H)^*$ or it is very large and contains a homeomorphic copy of $\beta\mathbb{N}$. Since $B(H)^* = \ell^{\infty *} \oplus \ell^{\infty\perp}$, every element of the set $[\delta_t]$ is of the form $\rho = \delta_t + \lambda$ for some $\lambda \in \ell^{\infty\perp}$. Taking this into account, we showed that the set $[\delta_t]$ contains only one point iff the left ideal $N_\rho = \{T \in B(H) : \rho(T^*T) = 0\}$ has a bounded approximate identity consisting of positive diagonal operators.

Although our investigation of the Kadison-Singer problem was not quite successful in solving the Kadison-Singer problem, our approach has raised several interesting questions for the further investigations and showed that the geometric properties of Banach spaces, such as Grothendieck property and (V) property, can be effectively used in the investigations of the problems raised.

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