GLOBAL WELL-POSEDNESS OF NLS EQUATIONS

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ABSTRACT

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The thesis is a survey of the I-method. After introducing the method, we discuss the implementation of this method to cubic, defocusing nonlinear Schrödinger equation in the spatial dimension $n = 2$ and quintic, defocusing nonlinear Schrödinger equation in the spatial dimension $n = 1$ with detailed calculations. We will find out on which type of equations one can use the I-method. We then mention our joint work with Engin Başakoğlu on the cubic defocusing fourth order nonlinear Schrödinger equation in the spatial dimension $n = 4$. Lastly, we discuss advantage and disadvantage of the method and share our idea of a study plan of the same equation in the spatial dimensions $n = 2, 3$ as future work.

ÖZET

DOĞRUSAL OLMAYAN SCHRÖDİNGER DENKLEMLERİNDE GLOBAL İYİ KONULMUŞLUK

Bu tez I-metodu üzerine bir araştırmadır. Metodu tanıttıktan sonra, bu metodu uzay boyutu 2 durumunda kübic doğrusal olmayan Schrödinge denklemine ve 1 uzay boyutunda beşinci dereceden (quintic) doğrusal olmayan Schrödinger denklemine uygulamasını detaylı hesaplamalarla tartışacağız. Buradan aslında hangi tip denklemlerde I-metodunun kullanılabileceğini tespit edeceğiz. Sonrasında Engin Başakoğlu ile ortak çalışmamız olan, 4 uzay boyutunda I-metodunun kübik dördüncü dereceden doğrusal olmayan Schrödinger denklemine uygulanmasından bahsedeceğiz. Son olarak bu metodun avantajı ve dezavantajını tartışıp metodu uygulayabileceğimiz gelecek çalışma planımızı paylaşacağız.

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LIST OF SYMBOLS

$\mathbb{C}$	The set of complex numbers
$C_c^\infty(\mathbb{R}^n)$	Space of compactly supported, smooth, complex-valued functions on $\mathbb{R}^n$
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\Re(z)$	Real part of a complex number z
$\overline{u}$	The conjugate of a complex-valued function u
$X \lesssim Y$	Given two quantities $X, Y \geq 0$, there is a universal constant $C > 2$, universal in the sense that C does not depend on X and Y, such that $X \leq CY$
$X \sim Y$	When both $X \lesssim Y$ and $Y \lesssim X$
$X \ll Y$	When $CX < Y$ where $C > 2$ is universal
$X \lesssim_{a_1,\ldots,a_n} Y$	Given two non-negative quantities X, Y there is a constant $C(a_1, \ldots, a_n) > 2$ which is controlled by the parameters $a_1, \ldots, a_n$ such that $X \leq C(a_1, \ldots, a_n)Y$
$X = O(Y)$	$\lvert X\rvert \lesssim Y$
$X = O_{a_1,\ldots,a_n}(Y)$	$\lvert X\rvert \lesssim_{a_1\ldots,a_n} Y$
$\langle X\rangle$	$(1 + \lvert X\rvert^2)^{\frac{1}{2}}$
$X\pm$	We write this symbol instead of $X \pm \varepsilon$ where $\varepsilon > 0$ is a universal arbitrarily small constant
$X_{12\ldots n}$	$X_1 + X_2 + \ldots X_n$
δ	Upper bound for time interval of the local existence of the solutions to the equations
ε	Universal small constant
η	Schwartz function in the time variable
λ	Scaling parameter
ξ	Spatial frequency variable
$\prod$	Product symbol
τ	Time frequency variable

χ_A	The characteristic function on the set A
∇	The gradient operator $(\partial_{x_1}, \ldots, \partial_{x_n})$
$\triangle$	The Laplacian operator $\partial_{x_1}^2 + \cdots + \partial_{x_n}^2$

LIST OF ACRONYMS/ABBREVIATIONS

1D	One Dimensional
2D	Two Dimensional
3D	Three Dimensional
IVP	Initial Value Problem
KdV	Korteweg-de Vries
NLS	Nonlinear Schrödinger
PDE	Partial Differential Equation

1. INTRODUCTION

In this thesis, the main purpose is to present the I-method and its use on nonlinear dispersive partial differential equations. A dispersive PDE is a type of differential equation such that its solution has the velocity which depends on its frequency or wavelength. To be more precise, consider the plane wave solution u to a linear dispersive PDE in 1D

$$i\partial_t u + Lu = 0$$

where L is a linear partial differential operator in the spatial variable, $L = L(\partial_x)$. It is of the form

$$u(t,x) = Ae^{i(kx-wt)}$$

where A, k, w are the amplitude, the wavelength or wave number, and the frequency at the space-time point (t,x), respectively. In general, the frequency w is assumed to be a function of k. The dispersion relation is given by

$$w + L(ik) = 0.$$

For example, let us take $L(\partial_x) = (\partial_x)^2$. Then the dispersion relation becomes

$$w = k^2.$$

The planar waves are called dispersive when the second derivative of $w(k)$ with respect to the variable k is not zero. In the geometric point of view, the characteristic hyperspace $w + L(ik) = 0$, where the nontrivial solutions of the equation persist, has nonzero curvature when dispersion exists. For details, see the $X^{s,b}$-space discussion in Appendix. In physical space, dispersive waves having different wavelengths, hence different frequencies, propagate with different velocities. Therefore, a mixed planar waves of different wavelengths eventually scatter in space. KdV and Schrödinger equations are two pivotal examples of dispersive type of partial differential equations.

We will focus on well-posedness of quintic, defocusing NLS in 1D and cubic, defocusing NLS in 2D when we implement the I-method.

Definition 1.1. *An IVP is called well-posed if the data-to-solution map is at least a continuous function. More precisely, an IVP is well-posed in a Banach space, for example, $H^s(\mathbb{R}^n)$ if for any initial data taken from $u_0 \in H^s(\mathbb{R}^n)$, there is a unique time interval I starting from $t = 0$ such that the unique solution u belongs to $C(I; H^s(\mathbb{R}^n))$ for $(t, x) \in I \times \mathbb{R}^n$.*

The interval of existence, say $[0, T]$ for some $T > 0$, determines that the well-posedness of the equation is either local or global. The local well-posedness theory of a large class of NLS equations can be found in [1]. We restrict our attention to the global well-posedness issue. To this end, the typical approach to show the global well-posedness of an IVP is to use the symmetries or the conservation laws which the solutions obey together with the local theory results. For example, consider the nonlinear Schrödinger equations whose solutions obey the energy conservation and the mass conservation laws. Together with the standard iteration argument, one can conclude NLS equations are globally well-posed in the energy space, $H^1(\mathbb{R}^n)$. Also, with a suitable selection of nonlinear term, for example algebraic type of nonlinearity as in cubic NLS, H^1 global well-posedness implies the global well-posedness of NLS equations in H^s for $s \geq 1$, [2].

The main difficulty below the energy level, $s < 1$, regarding the global well-posedness of NLS type of equations is the absence of conservation laws. There is no conserved quantity for $0 < s < 1$. The energy of a rougher initial data, which is taken from $H^s(\mathbb{R}^n)$ for $s < 1$, is infinite in general. So, we cannot exploit the energy conservation directly to establish global well-posedness results below the energy space. However, if one can generate a quantity which is comparable to the H^s norm of the solution for all fixed time and this quantity does not grow substantially in time, then H^s norm is controlled by the same bound. This would imply the global well-posedness of the equation on H^{s_0} where $s_0 \geq s$. The I-method is a tool to generate such an almost conserved quantity. It was introduced in the papers [2, 3].

This method can be considered as a refinement and modification of J. Bourgain's Fourier truncation approach [4]. In the next chapter, we will start with a brief description of the method. Then we will use it on defocusing, cubic NLS on 2D and defocusing, quintic NLS on 1D. Lastly, we briefly discuss the use of the method on fourth order defocusing cubic NLS. So, the main part of the document consists of the implementation of the method because most of the derived results and estimates including the I-method are heavily related to the equation we are studying on. In the conclusion, we discuss the positive and negative sides of the method and our idea of application of the method on a fourth order NLS as future plan.

2. THE I-METHOD AND ITS APPLICATIONS

In this chapter, we first introduce the operator I and see how it can be used on suitable equations to improve the global well-posedness results. The definition of the operator is taken from [2]. Given $s < 1$ and a cut-off parameter $N \gg 1$, the Fourier multiplier operator I_N is defined as

$$\widehat{I_N u}(\xi) = m_N(\xi)\widehat{u}(\xi) \tag{2.1}$$

where the multiplier m_N is smooth, radial, decreasing in $|\xi|$ and given as

$$m_N(\xi) = \begin{cases} 1 & |\xi| \leq N, \\ |\xi|^{s-1}N^{1-s} & |\xi| \geq 2N \end{cases} \tag{2.2}$$

and smoothly interpolates in the region $N < |\xi| < 2N$. For brevity, we shall drop the subscript N from the notation of the multiplier m_N and the operator I_N and write m and I, respectively. The multiplier m satisfies the condition

$$|\nabla_\xi^j m| \lesssim |\xi|^{-j} \text{ for } j \geq 0, \text{ where } \xi \in \mathbb{R}^n \setminus \{0\}$$

implying that m is a Hörmander-Mikhlin multiplier (see [1] Theorem 2.8). Consequently, the operator I is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Roughly speaking, I acts as an identity operator on the functions with low frequency while the functions of high frequency are $1 - s$ degree smoothened by the action of the operator I.

2.1. On the Cubic Defocusing NLS in 2D

In this section, we consider the following IVP

$$\begin{cases} i\partial_t u + \triangle u = |u|^2 u & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0, x) = u_0(x) \in H^s(\mathbb{R}^2) \end{cases} \tag{2.3}$$

where $s < 1$.

The first result improving the global well-posedness result for the solutions to the equation (2.3) below the energy space, $s = 1$, was established by J. Bourgain in [5] which states that the rough solutions to the equation (2.3) are global in time when $s > \frac{2}{3}$. In [4], he improved this result to $s > \frac{3}{5}$. J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao extend the global well-posedness result to even rougher initial data using the I-method. More precisely, in [2], it is stated that solutions to (2.3) are globally well-posed for the data in $H^s(\mathbb{R}^2)$ for $s > \frac{4}{7}$. In [6], the same authors improved their result to $s > \frac{1}{2}$ by adding a correction term to the almost conserved quantity introduced in [2]. More recently, in 2016, B. Dodson showed in [7] that solutions to the equation (2.3) persist globally in time in $H^s(\mathbb{R}^2)$ for $s \geq 0$. Thus, he reached the best possible global result for (2.3) in spatial dimension 2. However, this section concerns giving insight into the I-method. Therefore, we will mainly follow [2] to understand the use of this approach.

The IVP (2.3) has some useful properties. Namely, a solution u obeys the mass conservation

$$M(u(t)) = \Big(\int_{\mathbb{R}^2} |u(t,x)|^2 dx \Big)^{1/2} = M(u_0)$$

and the energy conservation

$$E(u(t)) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{4} |u(t,x)|^4 dx = E(u_0).$$

for all existence time t. We observe from the energy functional that the energy is always nonnegative. In this case, the IVP (2.3) becomes defocusing. The main result in [2] is stated as follows

Theorem 2.1. *The IVP* (2.3) *is globally well-posed for the initial data* $u_0 \in H^s(\mathbb{R}^2)$ *when* $s > \frac{4}{7}$.

By the standard density argument (see [8]), it is enough to take the initial data $u_0 \in C_c^\infty(\mathbb{R}^2)$ while proving Theorem 2.1.

Also, to establish the result, it is enough to show

$$\|u(t)\|_{H^s_x(\mathbb{R}^2)} \lesssim_{\|u_0\|_{H^s_x(\mathbb{R}^2)}} \langle t\rangle^M \tag{2.4}$$

for some $M > 0$ depending on the H^s-norm of the initial data. We have, for solution u and the initial data u_0,

$$E(Iu(t)) \le \left(N^{1-s}\|u(t)\|_{\dot H^s_x(\mathbb{R}^n)}\right)^2 + \|u(t)\|^4_{L^4_x(\mathbb{R}^n)}, \tag{2.5}$$

$$\|u(t)\|^2_{H^s_x(\mathbb{R}^n)} \lesssim E(Iu(t)) + \|u_0\|^2_{L^2_x(\mathbb{R}^n)}. \tag{2.6}$$

To see (2.5), writing $E(Iu(t))$ explicitly and using Plancherel's theorem

$$\begin{aligned} E(Iu(t)) &= \frac{1}{2}\int_{\mathbb{R}^n} |\xi|^2 m^2(\xi)|\widehat{u}(t,\xi)|^2 d\xi + \frac{1}{4}\|Iu(t)\|^4_{L^4_{t,x}(\mathbb{R}^n)} \\ &\le \int_{|\xi|\le N} |\xi|^2|\widehat{u}(t,\xi)|^2 d\xi + \int_{|\xi|>N} |\xi|^{2s}N^{2-2s}|\widehat{u}(t,\xi)|^2 d\xi + \|u(t)\|^4_{L^4_x(\mathbb{R}^n)} \\ &\le \int_{|\xi|\le N} |\xi|^{2s}N^{2-2s}|\widehat{u}(t,\xi)|^2 d\xi + \int_{|\xi|>N} |\xi|^{2s}N^{2-2s}|\widehat{u}(t,\xi)|^2 d\xi + \|u(t)\|^4_{L^4_x(\mathbb{R}^n)} \\ &= \left(N^{1-s}\|u(t)\|_{\dot H^s_x(\mathbb{R}^n)}\right)^2 + \|u(t)\|^4_{L^4_x(\mathbb{R}^n)}. \end{aligned}$$

For (2.6), by Plancherel's theorem

$$\begin{aligned} \|u(t)\|^2_{H^s_x(\mathbb{R}^n)} &= \int_{|\xi|\le N} \langle\xi\rangle^{2s}|\widehat{u}(t,\xi)|^2 d\xi + \int_{|\xi|>N} \langle\xi\rangle^{2s}|\widehat{u}(t,\xi)|^2 d\xi \\ &\lesssim \|\langle\xi\rangle^{2s-2}\|^2_{L^\infty_\xi(|\xi|\le N)} \int_{|\xi|\le N} \langle\xi\rangle^2|\widehat{u}(t,\xi)|^2 d\xi + \int_{|\xi|>N} \langle\xi\rangle^{2s}|\widehat{u}(t,\xi)|^2 d\xi \\ &\lesssim_s \|u(t)\|^2_{L^2_x(\mathbb{R}^n)} + \int_{|\xi|\le N} |\xi|^2|\widehat{u}(t,\xi)|^2 d\xi + N^{2-2s}\int_{|\xi|>N} |\xi|^{2s}|\widehat{u}(t,\xi)|^2 d\xi \\ &\lesssim E(Iu(t)) + \|u_0\|^2_{L^2_x} \end{aligned}$$

where the mass conservation is applied in the last inequality. By (2.5), (2.6), we can control the modified energy with the H^s-norm of the solution, and control the H^s-norm of u with the modified energy together with the initial mass. Therefore, showing

$$E(Iu(t)) \lesssim \langle t\rangle^M \tag{2.7}$$

with (2.6), for some cut-off parameter N dependent on t, suffices to establish (2.4).

Then we somehow want to control the growth of the modified energy $E(Iu(t))$, which is achieved by the following proposition.

Proposition 2.2. *Given $s > \frac{4}{7}$, $N \gg 1$, and initial data $u_0 \in C_c^\infty$, with $E(Iu_0) \leq 1$, there exists $\delta = \delta(\|u_0\|_{L^2(\mathbb{R}^2)}) > 0$ such that the solution*

$$u \in C([0,\delta], H^s(\mathbb{R}^2))$$

of (2.3) *satisfies*

$$E(Iu(t)) - E(Iu_0) \lesssim N^{-\frac{3}{2}+} \tag{2.8}$$

for all $t \in [0,\delta]$.

So, we can control the growth of the modified energy with a negative power of the large cut-off parameter N. This is why $E(Iu(t))$ is called an almost conserved quantity. Note that Iu is not a solution to (2.3). It is a solution to the following IVP

$$\begin{cases} i\partial_t Iu + \triangle Iu = I(|u|^2 u), \\ \qquad Iu(0,x) = Iu_0(x). \end{cases} \tag{2.9}$$

As stated, the main idea in Proposition 2.2 is to control the growth of almost conserved quantity

$$E(Iu(t)) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla Iu(t,x)|^2 + \frac{1}{4}|Iu(t,x)|^4 dx$$

by means of $-\frac{3}{2}+$ power of N where this power depends on the spatial dimension. Time differentiation gives that

$$\begin{aligned} \partial_t E(Iu(t)) &= \Re \int_{\mathbb{R}^2} \triangle Iu \partial_t \overline{Iu} + Iu\overline{Iu}Iu\partial_t\overline{Iu}\, dx \\ &= \Re \int_{\mathbb{R}^2} \partial_t \overline{Iu}(Iu\overline{Iu}Iu - I(u\bar{u}u))dx. \end{aligned}$$

Integrating this from 0 to δ and using the Plancherel formula yields

$$\begin{aligned}
&E(Iu(\delta))-E(Iu(0))\\
&=\Re\int_0^\delta\int_{\sum_{i=1}^4\xi_i=0}\left(1-\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right)\widehat{\partial_t\overline{Iu}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)\,dt\\
&\lesssim\left|\int_0^\delta\int_{\sum_{i=1}^4\xi_i=0}\left(1-\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right)\widehat{\triangle\overline{Iu}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)\,dt\right|\\
&+\left|\int_0^\delta\int_{\sum_{i=1}^4\xi_i=0}\left(1-\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right)\widehat{\overline{I(|u|^2u)}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)\,dt\right|\\
&=:Term_1+Term_2.
\end{aligned}$$

Thus, the aim is to show $Term_1+Term_2\lesssim N^{-\frac{3}{2}+}$. Before starting the proof of Proposition 2.2, we need the local existence result for the modified equation (2.9).

Lemma 2.3 (Modified Local Existence)**.** *Given* $\frac{4}{7}<s<1$ *and the initial data* u_0 *for the equation* (2.3) *with* $E(Iu_0)\le 1$*, there is a constant* $\delta=\delta(\|u_0\|_{L^2(\mathbb{R}^2)})>0$ *such that the solution* Iu *to the modified IVP* (2.9) *has the following property on* $[0,\delta]$,

$$\|Iu\|_{X_\delta^{1,\frac{1}{2}+}}\lesssim 1. \tag{2.10}$$

Proof. The standard iteration argument will be applied, see [8], to show the local existence of (2.9). For this purpose, we use the following typical $X^{s,b}$ estimates.

$$\|e^{it\triangle}u\|_{X_\delta^{1,\frac{1}{2}+}}\lesssim\|u\|_{H_x^1}, \tag{2.11}$$

$$\left\|\int_0^t e^{i(t-t')\triangle}U(x,t')dt'\right\|_{X^{1,\frac{1}{2}+}}\lesssim\|U\|_{X_\delta^{1,-\frac{1}{2}+}}, \tag{2.12}$$

$$\|U\|_{X_\delta^{1,-b}}\lesssim\delta^{b-b'}\|U\|_{X^{1,-b'}} \tag{2.13}$$

where $0<b'<b<\frac{1}{2}$. For the proof of these type of estimates, see [9]. By Duhamel's principle, one can give the following representation for the solutions of (2.9) as

$$Iu(t,x)=e^{it\triangle^2}Iu_0(x)-i\int_0^t e^{i(t-t')\triangle^2}I(|u|^2u)(t',x)dt' \tag{2.14}$$

for $t \in [0,\delta]$. Using (2.11)–(2.14) we have

$$\begin{aligned}\|Iu\|_{X_\delta^{1,\frac{1}{2}+}} &= \left\| e^{it\triangle}Iu_0 + i\int_0^t e^{i(t-t')\triangle} I(|u|^2u)(t',\cdot)dt' \right\|_{X_\delta^{1,\frac{1}{2}+}} \\ &\lesssim \|Iu_0\|_{H_x^1} + \|I(u\overline{u}u)\|_{X_\delta^{1,-\frac{1}{2}+}} \\ &\lesssim \|Iu_0\|_{H_x^1} + \delta^{0+}\|I(u\overline{u}u)\|_{X_\delta^{1,-\frac{1}{2}++}}.\end{aligned}$$

The next task is to show

$$\|I(u\overline{u}u)\|_{X_\delta^{1,-\frac{1}{2}++}} \lesssim \|Iu\|^3_{X_\delta^{1,\frac{1}{2}+}}. \tag{2.15}$$

Also, note that

$$\|Iu_0\|_{H_x^1} \lesssim (E(Iu_0))^{1/2} + \|u_0\|_{L_x^2} \lesssim 1 + \|u_0\|_{L_x^2}. \tag{2.16}$$

Combining (2.15) and (2.16), we have

$$\|Iu\|_{X_\delta^{1,\frac{1}{2}+}} \lesssim 1 + \|u_0\|_{L^2_{t,x}} + \delta^{0+}\|Iu\|^3_{X_\delta^{1,\frac{1}{2}+}}. \tag{2.17}$$

By applying the bootstrap or continuity argument, see [10] ch.1, we obtain the desired result. The $\|u_0\|_{L_x^2}$ dependence of δ can be seen in (2.17). We also observe that δ must be sufficiently small in order to employ the bootstrap argument. Therefore, establishing (2.15) will complete the proof. Using the interpolation lemma in [11], it is enough to show

$$\|u\overline{u}u\|_{X_\delta^{s,-\frac{1}{2}++}} \lesssim \|u\|_{X_\delta^{s,\frac{1}{2}+}}$$

for $\frac{4}{7} < s < 1$. By the fractional Leibniz rule, it suffices to show

$$\|(\langle\nabla\rangle^s u)\overline{u}u\|_{X_\delta^{0,-\frac{1}{2}++}} \lesssim \|u\|^3_{X_\delta^{s,\frac{1}{2}+}}.$$

We consider the following integral to exploit the duality argument

$$\left| \int_0^\delta \int_{\mathbb{R}^4} (\langle\nabla\rangle^s u)\overline{u}uf\,dxdt \right|. \tag{2.18}$$

where f is taken from $\{f : \|f\|_{X^{0,\frac{1}{2}--}} = 1\}$. It is basically the Riesz representation theorem for $X^{s,b}$ spaces. Applying the Hölder's inequality, we have

$$(2.18) \le \|\langle\nabla\rangle^s u\|_{L^4_{t,x}}\|u\|_{L^4_{t,x}}\|u\|_{L^{4+}_{t,x}}\|f\|_{L^{4-}_{t,x}}.$$

Applying $L^4_{t,x}$-Strichartz estimate (A.14) to the first two factor, we get

$$\|\langle\nabla\rangle^s u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{s,\frac{1}{2}+}_\delta}$$

and

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{2}+}_\delta} \lesssim \|u\|_{X^{s,\frac{1}{2}+}_\delta}.$$

For the third factor, we start applying Sobolev embedding (A.11) and then $L^{4+}_t L^{4-}_x$-Strichartz estimate to obtain

$$\|u\|_{L^{4+}_{t,x}} \lesssim \|\langle\nabla\rangle^{0+} u\|_{L^{4+}_t L^{4-}_x} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}_\delta} \lesssim \|u\|_{X^{s,\frac{1}{2}+}_\delta}.$$

For the last factor, observe that

$$\|f\|_{L^2_{t,x}} \lesssim \|f\|_{X^{0,0}_\delta}$$

and

$$\|f\|_{L^4_{t,x}} \lesssim \|f\|_{X^{0,\frac{1}{2}+}_\delta}.$$

Interpolating the upper bounds of f keeping in mind that the universal small constants from which arise $X^{0,\frac{1}{2}+}$ and $L^{4-}_{t,x}$ norms are not the same but one of them can be regarded as a function of the other one, we obtain

$$\|f\|_{L^{4-}_{t,x}} \lesssim \|f\|_{X^{0,\frac{1}{2}--}_\delta}.$$

Combining all the inequalities and supremum on the set where $\|f\|_{X^{0,\frac{1}{2}--}_\delta} = 1$, we get

$$\|(\langle\nabla\rangle^s u)\overline{u}u\|_{X^{s,-\frac{1}{2}++}_\delta} \lesssim \|u\|^3_{X^{s,\frac{1}{2}+}_\delta}.$$

$\square$

Proof of Proposition 2.2. By the above observation, we may restrict our attention to showing that

$$Term_i \lesssim N^{-\frac{3}{2}+}$$

separately, $i = 1, 2$. For $Term_1$, using the Littlewood-Paley decomposition, define

$$\widehat{u_{N_1}} = \widehat{P_{N_1} \triangle I u}, \quad \widehat{u_{N_2}} = \widehat{P_{N_2} I u}, \quad \widehat{u_{N_3}} = \widehat{P_{N_3} I u}, \quad \widehat{u_{N_4}} = \widehat{P_{N_4} I u}$$

where P_N is the Littlewood-Paley projection operator (A.3) and $N_i = 2^{k_i}$, $k_i \in \{0, 1, 2, \dots\}$ for $i = 1, 2, 3, 4$. For simplicity in terms of writing, we shall denote $m_j = m(\xi_j)$ and $m_{ij} = m(\xi_i + \xi_j)$. $Term_1$ can be controlled via the decomposition

$$\begin{aligned} Term_1 &= \left| \sum_{N_1, \dots, N_4} \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m_{234}}{m_2 m_3 m_4}\right) \widehat{u_{N_1}}(\xi_1) \widehat{u_{N_2}}(\xi_2) \widehat{u_{N_3}}(\xi_3) \widehat{u_{N_4}}(\xi_4) dt \right| \\ &\leq \sum_{N_1, \dots, N_4} \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m_{234}}{m_2 m_3 m_4}\right) \widehat{u_{N_1}}(\xi_1) \widehat{u_{N_2}}(\xi_2) \widehat{u_{N_3}}(\xi_3) \widehat{u_{N_4}}(\xi_4) dt \right| \end{aligned}$$

where $\langle \xi_i \rangle \sim N_i$ for $i = 1, 2, 3, 4$. The symmetry of the variables ξ_2, ξ_3, ξ_4 in the multiplier provides us to restrict our focus to the region $N_2 \geq N_3 \geq N_4$. In this region, we also have $N_1 \lesssim N_2$ because $\xi_{1234} = 0$. Without the loss of generality, we may assume that the Fourier transform of all Littlewood-Paley pieces of u are non-negative, see [12]. To see this assumption, for simplicity let us take $f, g \in L^2(\mathbb{R}^n)$ and set

$$f(x) = f_1(x) + i f_2(x)$$

where f_1, f_2 are real-valued. We have

$$\|f_i\|_{L^2} \leq \|f\|_{L^2}$$

for $i = 1, 2$. Thus, we may initially assume that the dyadic pieces are real-valued. Define

$$f^+(x) = \max\{f(x), 0\} \text{ and } f^-(x) = \max\{-f(x), 0\},$$

and define the same decomposition for g. Then

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) g(x) dx &= \int_{\mathbb{R}^n} f^+ g^+ - f^- g^+ - f^+ g^- + f^- g^- dx \\ &\leq \int_{\mathbb{R}^n} |f^+ g^+| + |f^- g^+| + |f^+ g^-| + |f^- g^-| dx \\ &\leq \|f^+\|_{L^2} \|g^+\|_{L^2} + \|f^-\|_{L^2} \|g^+\|_{L^2} + \|f^+\|_{L^2} \|g^-\|_{L^2} + \|f^-\|_{L^2} \|g^-\|_{L^2} \end{aligned}$$

$$\lesssim \|f\|_{L^2}\|g\|_{L^2}.$$

Thus, we see that taking the dyadic pieces non-negative does not cause the loss of the generality. With this assumption, we can take the multiplier out of the integral with a pointwise bound $C(N_1, N_2, N_3, N_4)$ depending on the size of frequencies N_j in different frequency interactions. By reverting back the Plancherel formula and Cauchy-Schwarz inequality, it is sufficient to show the following estimate

$$C(N_1, N_2, N_3, N_4)\left|\int_0^\delta \int_{\sum_{i=1}^4 \xi_i=0} \widehat{\overline{u_{N_1}}}(\xi_1)\widehat{u_{N_2}}(\xi_2)\widehat{\overline{u_{N_3}}}(\xi_3)\widehat{u_{N_4}}(\xi_4)dt\right|$$
$$\lesssim N^{-\frac{3}{2}+}N_2^{0-}\|u_{N_1}\|_{X^{-1,\frac{1}{2}+}}\prod_{j=2}^{4}\|u_{N_j}\|_{X^{1,\frac{1}{2}+}} \quad (2.19)$$

All L^p and $X^{s,b}$ norms in (2.19) are taken on the domain $[0,\delta]\times\mathbb{R}^2$ and we shall keep this notation for the rest of this chapter. As in the previous term, $Term_2$ can be expressed by

$$\left|\int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0} \sum_{N_4\geq N_5\geq N_6}\Big(1-\frac{m_{456}}{m_4m_5m_6}\Big)P_{N_{123}}\widehat{\overline{I(u\overline{u}u)}}(\xi_1+\xi_2+\xi_3)\right.$$
$$\left.\times\widehat{Iu_{N_4}}(\xi_4)\widehat{\overline{Iu_{N_5}}}(\xi_5)\widehat{Iu_{N_6}}(\xi_6)\right| \quad (2.20)$$

where $P_{N_{123}}$ is the Littlewood-Paley projection operator onto the dyadic shell $N_{123}\sim\langle\xi_1+\xi_2+\xi_3\rangle$. So, we have

$$\left|\int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0} \sum_{N_4\geq N_5\geq N_6}\Big(1-\frac{m_{456}}{m_4m_5m_6}\Big)P_{N_{123}}\widehat{\overline{I(u\overline{u}u)}}(\xi_{123})\right.$$
$$\left.\times\widehat{Iu_{N_4}}(\xi_4)\widehat{\overline{Iu_{N_5}}}(\xi_5)\widehat{Iu_{N_6}}(\xi_6)\right|$$
$$\leq \sum_{N_4\geq N_5\geq N_6}\left|\int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0}\Big(1-\frac{m_{456}}{m_4m_5m_6}\Big)P_{N_{123}}\widehat{\overline{I(u\overline{u}u)}}(\xi_{123})\right.$$
$$\left.\times\widehat{Iu_{N_4}}(\xi_4)\widehat{\overline{Iu_{N_5}}}(\xi_5)\widehat{Iu_{N_6}}(\xi_6)\right|.$$

It is then sufficient to show for the dyadic pieces of (2.20) that

$$\left| \int_0^\delta \int_{\sum_{i=1}^6 \xi_i = 0} \Big(1 - \frac{m_{456}}{m_4 m_5 m_6}\Big) P_{N_{123}} \widehat{\overline{I(u\overline{u}u)}}(\xi_{123}) \widehat{Iu_{N_4}}(\xi_4) \widehat{\overline{Iu_{N_5}}}(\xi_5) \widehat{Iu_{N_6}}(\xi_6) \right| \\ \lesssim N^{-\frac{3}{2}+} N_4^{0-} \|Iu\|^3_{X^{1,\frac{1}{2}+}} \prod_{j=4}^{6} \|Iu_{N_j}\|_{X^{1,\frac{1}{2}+}}. \quad (2.21)$$

We may again assume $N_{123} \lesssim N_4$ and $N_4 \gtrsim N$ to omit the case where the multiplier in (2.21) is zero. N_4^{0-} factor on the right side of (2.21) allows us to sum especially in the dyadic number N_{123}. Therefore, it saves us decomposing the terms of $I(u\overline{u}u)$. At this point, we may again assume that the dyadic pieces have all non-negative spatial Fourier transform. Therefore we can put the symbol out with a bound for each frequency interaction case as

$$\left|1 - \frac{m_{456}}{m_4 m_5 m_6}\right| \lesssim \frac{m(N_{123})}{m(N_4)m(N_5)m(N_6)} \quad (2.22)$$

and undo the Plancherel formula.

Now, we start showing (2.19). We need to consider two frequency interaction cases. The case $N \gg N_2$ is excluded since otherwise the multiplier inside becomes zero.

Case 1: $N_2 \gtrsim N \gg N_3 \geq N_4$ and $N_1 \sim N_2$. Take the multiplier out with the bound

$$\left|1 - \frac{m_{234}}{m_2 m_3 m_4}\right| \lesssim \frac{N_3}{N_2}.$$

Using Hölder's inequality on $u_{N_1} u_{N_3}\ u_{N_2} u_{N_4}$, and applying (A.15), we obtain

$$\text{Left side of (2.19)} \lesssim \frac{N_3}{N_2} \|u_{N_1} u_{N_3}\|_{L^2_{t,x}} \|u_{N_2} u_{N_4}\|_{L^2_{t,x}}$$

$$\lesssim \frac{N_3}{N_2}\frac{N_3^{1/2}}{N_1^{1/2}}\frac{N_4^{1/2}}{N_2^{1/2}}\frac{N_1}{N_2N_3N_4}\|u_{N_1}\|_{X^{-1,\frac{1}{2}+}}\prod_{j=2}^{4}\|u_{N_j}\|_{X^{1,\frac{1}{2}+}}.$$

Hence, it is enough to show

$$\frac{N^{\frac{3}{2}-}N_2^{-2+}N_3^{1/2}}{N_4^{1/2}} \lesssim 1.$$

Since $N_3, N_4 \gtrsim 1$ and $N_2 \gtrsim N$ we get the desired result.

Case 2: $N_2 \geq N_3 \gtrsim N$. We use the following bound for the multiplier

$$\left|1 - \frac{m_{234}}{m_2m_3m_4}\right| \lesssim \frac{m_1}{m_2m_3m_4}.$$

There are two sub-cases coming from which one of N_1, N_3 is comparable to N_2:

Case 2(a): $N_1 \sim N_2 \geq N_3 \gtrsim N$. Applying Hölder's inequality to $u_{N_1}u_{N_4}$ and $u_{N_2}u_{N_3}$, and using (A.15), we obtain

$$\begin{aligned}\text{Left side of (2.19)} &\sim \frac{m(N_1)}{m(N_2)m(N_3)m(N_4)}\|u_{N_1}u_{N_4}\|_{L^2_{t,x}}\|u_{N_2}u_{N_3}\|_{L^2_{t,x}}\\ &\lesssim \frac{m(N_1)}{m(N_2)m(N_3)m(N_4)}\frac{N_4^{1/2}}{N_1^{1/2}}\frac{N_3^{1/2}}{N_2^{1/2}}\frac{N_1}{N_2N_3N_4}\|u_{N_1}\|_{X^{-1,\frac{1}{2}+}}\\ &\times\|u_{N_2}\|_{X^{1,\frac{1}{2}+}}\|u_{N_3}\|_{X^{1,\frac{1}{2}+}}\|u_{N_4}\|_{X^{1,\frac{1}{2}+}}.\end{aligned}$$

Therefore, it suffices to show

$$\frac{N^{\frac{3}{2}-}N_2^{-1+}}{m(N_3)N_3^{1/2}m(N_4)N_4^{1/2}} \lesssim 1. \tag{2.23}$$

The function $m(x)x^{1/2}$ is increasing for $s > 1/2$. Thus, $m(N_3)N_3^{1/2} \gtrsim m(N)N = N$ and $m(N_4)N_4^{1/2} \gtrsim 1$, that is,

$$\text{Left side of (2.23)} \lesssim N^{1-}N_2^{-1+} \lesssim 1.$$

Case 2(b): $N_2 \sim N_3 \gtrsim N$ and $N_1 \lesssim N_2$. Use Hölder's inequality to $u_{N_1}u_{N_2}$ and

$u_{N_3}u_{N_4}$, and apply (A.15) to get

$$\text{Left side of (2.19)} \sim \frac{m(N_1)}{m(N_2)m(N_3)m(N_4)}\|u_{N_1}u_{N_2}\|_{L^2_{t,x}}\|u_{N_3}u_{N_4}\|_{L^2_{t,x}}$$

$$\lesssim \frac{m(N_1)}{m(N_2)m(N_3)m(N_4)}\frac{N_1^{1/2}}{N_2^{1/2}}\frac{N_4^{1/2}}{N_3^{1/2}}\frac{N_1}{N_2N_3N_4}\|u_{N_1}\|_{X^{-1,\frac{1}{2}+}}$$
$$\times\|u_{N_2}\|_{X^{1,\frac{1}{2}+}}\|u_{N_3}\|_{X^{1,\frac{1}{2}+}}\|u_{N_4}\|_{X^{1,\frac{1}{2}+}}.$$

Thus, it is enough to establish

$$\frac{m(N_1)N_1^{3/2}N_2^{-3+}N^{\frac{3}{2}-}}{(m(N_2))^2m(N_4)N_4^{1/2}} \lesssim 1. \tag{2.24}$$

Recall that $m(N_1)N_1^{1/2} \lesssim m(N_2)N_2^{1/2}$, $m(N_4)N_4^{1/2} \gtrsim 1$ and $m(N_2)N_2^{1/2} \gtrsim N^{1/2}$. Then it suffices to show

$$N_1N_2^{-2+}N^{1-} \lesssim N_2^{-1+}N^{1-} \lesssim 1$$

by $N_2 \gtrsim N$. Gathering all the estimates and using the Littlewood-Paley inequality (A.9) give the desired result for $Term_1$. Thus, it is left to show that $Term_2 \lesssim N^{-\frac{3}{2}+}$. First, apply Hölder's inequality to obtain

$$\text{Left side of (2.21)} \lesssim \frac{m(N_{123})}{m(N_4)m(N_5)m(N_6)}\|P_{N_{123}}I(u\bar{u}u)\|_{L^2_{t,x}}\|Iu_{N_4}\|_{L^4_{t,x}}$$
$$\times\|Iu_{N_5}\|_{L^4_{t,x}}\|Iu_{N_6}\|_{L^\infty_{t,x}}. \tag{2.25}$$

Also, the following inequalities will be needed to advance the next step

Lemma 2.4. *Suppose the functions $u, u_{N_4}, u_{N_5}, u_{N_6}$ are as above and the spatial dimension $n = 2$. Then*

$$\|P_{N_{123}}I(u\bar{u}u)\|_{L^2_{t,x}} \lesssim \frac{1}{\langle N_{123}\rangle^1}\|Iu\|^3_{X^{1,\frac{1}{2}+}}, \tag{2.26}$$

$$\|Iu_{N_j}\|_{L^4_{t,x}} \lesssim \frac{1}{\langle N_j\rangle^1}\|Iu_{N_j}\|_{X^{1,\frac{1}{2}+}} \quad j = 4, 5, \tag{2.27}$$

$$\|Iu_{N_6}\|_{L^\infty_{t,x}} \lesssim \|Iu_{N_6}\|_{X^{1,\frac{1}{2}+}}. \tag{2.28}$$

Proof of Lemma 2.4. To show (2.26), it is enough to show

$$\|\langle\nabla\rangle P_{N_{123}} I(u\overline{u}u)\|_{L^2_{t,x}} \lesssim \|Iu\|^3_{X^{1,\frac{1}{2}+}}. \tag{2.29}$$

The pseudo-differential operator $\langle\nabla\rangle I$ is of positive order $s > \frac{4}{7}$. So, it obeys the fractional Leibniz rule. Then it suffices to show the inequality on a typical term

$$\|P_{N_{123}}(\langle\nabla\rangle Iu)\overline{u}u)\|_{L^2_{t,x}}.$$

We apply Hölder's inequality keeping in mind the boundedness of $P_{N_{123}} : L^2_{t,x} \to L^2_{t,x}$ to obtain

$$\|P_{N_{123}}(\langle\nabla\rangle Iu)\overline{u}u\|_{L^2_{t,x}} \lesssim \|\langle\nabla\rangle Iu\|_{L^4_{t,x}}\|u\|_{L^8_{t,x}}\|u\|_{L^8_{t,x}}. \tag{2.30}$$

Using the Strichartz estimate (A.14), we have

$$\|\langle\nabla\rangle Iu\|_{L^4_{t,x}} \lesssim \|\langle\nabla\rangle Iu\|_{X^{0,\frac{1}{2}+}} = \|Iu\|_{X^{1,\frac{1}{2}+}}. \tag{2.31}$$

For the second and the third factor, applying Sobolev embedding (A.11) and the Strichartz estimate (A.14), we get

$$\|u\|_{L^8_{t,x}} \lesssim \|\langle\nabla\rangle^{1/2}u\|_{L^8_t L^{\frac{8}{3}}_x} \lesssim \|\langle\nabla\rangle^{1/2}u\|_{X^{0,\frac{1}{2}+}} \lesssim \|\langle\nabla\rangle Iu\|_{X^{0,\frac{1}{2}+}} = \|Iu\|_{X^{1,\frac{1}{2}+}}. \tag{2.32}$$

(2.31) and (2.32) give the desired inequality (2.26). To show (2.27), we use the Bernstein inequality (A.6) and the Strichartz estimate

$$\|Iu_{N_j}\|_{L^4_{t,x}} \sim \frac{1}{\langle N_j\rangle}\|\langle\nabla\rangle Iu_{N_j}\|_{L^4_{t,x}} \lesssim \frac{1}{\langle N_j\rangle}\|Iu\|_{X^{1,\frac{1}{2}+}} \quad j = 4, 5.$$

To show (2.28), we first utilize the Fourier inversion formula

$$|Iu_{N_6}(x)| \le \int_{\langle\xi\rangle\sim N_6} |\widehat{Iu_{N_6}}(\xi)|d\xi.$$

By Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int_{\langle\xi\rangle\sim N_6} |\widehat{Iu_{N_6}}(\xi)|d\xi \le& \Big(\int_{\langle\xi\rangle\sim N_6} \langle\xi\rangle^2|\widehat{Iu_{N_6}}(\xi)|^2 d\xi\Big)^{1/2}\Big(\int_{\langle\xi\rangle\sim N_6}\langle\xi\rangle^{-2}d\xi\Big)^{1/2}\\ \lesssim& \|Iu_{N_6}\|_{H^1_x}.\end{aligned}$$

Taking essential supremum in x, we conclude that

$$\|Iu_{N_6}\|_{L^\infty_x} \lesssim \|Iu_{N_6}\|_{H^1_x}$$

which yields (2.28) when we also apply $L^\infty_t L^2_x$-Strichartz estimate. $\square$

Combining (2.25) and Lemma 2.4, we have

$$\text{Right side of (2.25)} \lesssim \frac{m(N_{123})}{m(N_4)m(N_5)m(N_6)\langle N_{123}\rangle\langle N_4\rangle\langle N_5\rangle}\|Iu\|^3_{X^{1,\frac{1}{2}+}} \times \prod_{j=4}^{6}\|Iu_{N_j}\|_{X^{1,\frac{1}{2}+}}. \quad (2.33)$$

Thus, it suffices to show

$$\frac{m(N_{123})N^{\frac{3}{2}-}N_4^{0+}}{m(N_4)m(N_5)m(N_6)\langle N_{123}\rangle\langle N_4\rangle\langle N_5\rangle} \lesssim 1. \quad (2.34)$$

There exist two frequency interaction cases arising $N_{123} \sim N_5$ or $N_5 \sim N_4$.

Case 1: $N_4 \sim N_5$ and $N_4 \geq N_5 \geq N_6$ and $N_4 \gtrsim N$. In this case, the left side of (2.34) satisfies

$$\text{Left side of (2.34)} \sim \frac{m(N_{123})N^{\frac{3}{2}-}N_4^{0+}}{(m(N_4))^2 m(N_6)\langle N_{123}\rangle\langle N_4\rangle^2}.$$

Note that $m(y)\langle x\rangle^{1/2} \gtrsim 1$ for $0 \leq y \leq x$ and $m(N_{123})\langle N_{123}\rangle^{-1} \lesssim 1$. Thus, it suffices to show

$$\frac{N^{\frac{3}{2}-}N_4^{0+}}{\langle N_4\rangle^{1/2}N} \lesssim N^{\frac{1}{2}-}N_4^{-\frac{1}{2}+} \lesssim 1.$$

This is achieved because $N_4 \gtrsim N$.

Case 2: $N_{123} \sim N_4$ and $N_4 \geq N_5 \geq N_6$ and $N_4 \gtrsim N$. We observe

$$\text{Left side of (2.34)} \sim \frac{N^{\frac{3}{2}-}N_4^{0+}}{m(N_5)m(N_6)\langle N_4\rangle^2\langle N_5\rangle}.$$

Replacing $m(N_5)\langle N_5\rangle$, $m(N_6)\langle N_4\rangle^{1/2}$ in the denominator with 1 since $m(y)\langle x\rangle^{1/2} \gtrsim 1$ for $0 \le y \le x$, it is sufficient to show

$$N^{\frac{3}{2}-} N_4^{-\frac{3}{2}+} \lesssim 1$$

which can be seen via $N_4 \gtrsim N$. So, we conclude that $Term_2 \lesssim N^{-\frac{3}{2}+}$ which completes the proof of Proposition 2.2. $\square$

Now, we prove Theorem 2.1 via Proposition 2.2. We apply the proposition to the scaled solution

$$u_\lambda(t,x) = \lambda^{-1} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}). \tag{2.35}$$

Using (2.5) for (2.35), we have

$$\begin{aligned} E(Iu_{0,\lambda}) &\lesssim \big(\lambda^{-2s} N^{2-2s} + \lambda^{-2}\big)\big(1 + \|u_0\|_{H^s(\mathbb{R}^2)}\big)^4 && (2.36)\\ &\lesssim C_0 \lambda^{-2s} N^{2-2s}\big(1 + \|u_0\|_{H^s(\mathbb{R}^2)}\big)^4. && (2.37) \end{aligned}$$

Choose the scaling parameter $\lambda = \lambda\big(N, \|u_0\|_{H^s(\mathbb{R}^2)}\big)$ as

$$\lambda = \Big(\frac{1}{2C_0}\Big)^{\frac{1}{2s}} N^{\frac{1-s}{s}} \big(1 + \|u_0\|_{H^s(\mathbb{R}^2)}\big)^{\frac{2}{s}} \tag{2.38}$$

so that $E(Iu_{0,\lambda}) \le \frac{1}{2}$ and then apply the Proposition 2.2 to the scaled initial data $u_{0,\lambda}$ iteratively, until the size of $E(Iu_\lambda(t))$ reaches 1. More precisely, we reapply the proposition at least $C_1 N^{\frac{3}{2}-}$ many times to have

$$E(Iu_\lambda(C_1 N^{\frac{3}{2}-}\delta)) \sim 1. \tag{2.39}$$

For any time parameter $T_0 \gg 1$, we choose $N \gg 1$ so that

$$T_0 \sim \frac{N^{\frac{3}{2}-}}{\lambda^2} C_1 \delta \sim N^{\frac{7s-4}{s}-}. \tag{2.40}$$

Combining (2.38), (2.39) and (2.40), we conclude that

$$E(Iu(T_0)) = \lambda^2 E(Iu_\lambda(\lambda^2 T_0)) \lesssim_{\delta, \|u_0\|_{H^s(\mathbb{R}^4)}} \lambda^2 \sim N^{\frac{4-4s}{s}} \sim T_0^{\frac{8-8s}{7s-4}+}. \tag{2.41}$$

Using (2.6) and (2.41), we get

$$\|u(T_0)\|_{H^s(\mathbb{R}^n)} \lesssim T_0^{\frac{4-4s}{7s-4}+}. \tag{2.42}$$

The desired polynomial bound (2.4) is obtained whenever $s > \frac{4}{7}$.

2.2. On the Quintic Defocusing NLS in 1D

The following IVP will be considered in this section

$$\begin{cases} i\partial_t u + \partial_x^2 u = |u|^4 u & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}) \end{cases} \tag{2.43}$$

when $s < 1$. In [10], it is stated that the equation (2.43) is globally well-posed for $u_0 \in H^s(\mathbb{R})$ when $s > \frac{3}{4}$. The approach in the article does not involve $X^{s,b}$ space estimates when using the I-method.

The first global existence result below the energy level, $s < 1$, for (2.43) was established in [3] by using the I-method. Furthermore, [3] posits that (2.43) is globally well-posed in $H^s(\mathbb{R})$ when $s > \frac{2}{3}$. In [13], the same authors improved their result to $s > \frac{1}{2}$. In [14], this was extended to $s > \frac{1}{3}$. B. Dodson improved the global well-posedness result to $s > \frac{1}{4}$ in [15], and achieved the best possible global existence result in [16]. In this section, we will implement the almost conservation law approach together with the $X^{s,b}$ estimates to obtain the global well-posedness result for $s > \frac{1}{2}$.

The IVP (2.43) has some useful properties. Namely, a solution u obeys the mass conservation

$$M(u(t)) = \Big(\int_{\mathbb{R}} |u(t,x)|^2 dx \Big)^{1/2} = M(u_0)$$

and the energy conservation

$$E(u(t)) = \int_{\mathbb{R}} \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{6} |u(t,x)|^6 dx = E(u_0).$$

for all existence time t. We observe from the energy functional that the energy is always nonnegative. In this case, the IVP (2.43) is called defocusing.

Theorem 2.5. *The IVP* (2.43) *is globally well-posed for the initial data* $u_0 \in H^s_x(\mathbb{R})$ *when* $s > \frac{1}{2}$.

As in the previous equation (2.3), we may take the initial data u_0 from $C_c^\infty(\mathbb{R})$. We again wish to establish a polynomial growth bound in time for the H_x^s-norm of the solution

$$\|u(t)\|_{H_x^s(\mathbb{R})} \lesssim_{\|u_0\|_{H_x^s(\mathbb{R})}} \langle t \rangle^M \tag{2.44}$$

for some $M > 0$. Similar to (2.5) and (2.6), we have

$$E(Iu(t)) \leq \left(N^{1-s}\|u(t)\|_{\dot{H}_x^s(\mathbb{R}^n)}\right)^2 + \|u(t)\|_{L_x^6(\mathbb{R}^n)}^6, \tag{2.45}$$

$$\|u(t)\|_{H_x^s(\mathbb{R}^n)}^2 \lesssim E(Iu(t)) + \|u_0\|_{L_x^2(\mathbb{R}^n)}. \tag{2.46}$$

One can easily show the above inequalities by applying the similar arguments implemented to prove (2.5) and (2.6). Thus, controlling the modified energy with a polynomial bound in time will suffice. The following proposition is the main result for this purpose.

Proposition 2.6. *Given $\frac{1}{2} < s < 1$, $N \gg 1$, and initial data $u_0 \in C_c^\infty(\mathbb{R})$ with $E(Iu_0) \leq 1$, there exists $\delta = \delta(\|u_0\|_{L_x^2(\mathbb{R})})$ such that the solution*

$$u(t,x) \in C([0,\delta], H^s(\mathbb{R}))$$

of (2.43) satisfies

$$E(Iu(t)) - E(Iu(0)) \lesssim N^{-1+} \tag{2.47}$$

for all $t \in [0,\delta]$.

Let us prove Theorem 2.5 assuming Proposition 2.6. We apply the proposition to the scaled solution

$$u_\lambda(t,x)) = \lambda^{-1/2}u(\frac{t}{\lambda^2}, \frac{x}{\lambda}). \tag{2.48}$$

From (2.45), we have

$$\begin{aligned} E(Iu_{\lambda,0}) &\le (N^{2-2s}\lambda^{-2} + \lambda^{-2})(1 + \|u_0\|_{H^s_x(\mathbb{R})})^6 \\ &\lesssim N^{2-2s}\lambda^{-2}(1 + \|u_0\|_{H^s_x(\mathbb{R})})^6. \end{aligned}$$

For $N \gg 1$, we choose the scaling parameter $\lambda = \lambda(N, \|u_0\|_{H^s_x(\mathbb{R})})$

$$\lambda = 2^{1/2}N^{1-s}(1 + \|u_0\|_{H^s_x(\mathbb{R})})^3 \tag{2.49}$$

so that $E(Iu_{\lambda,0}) \le \frac{1}{2}$. We then apply Proposition 2.6 repeatedly until the size of the modified energy $E(Iu(t))$ reaches 1, that is, at least $C_1 N^{1-}$ times

$$E(Iu_\lambda(C_1 N^{1-}\delta)) \sim 1. \tag{2.50}$$

For any time parameter $T_0 \gg 1$, we choose $N \gg 1$ so that

$$T_0 \sim C_1 \frac{N^{1-}}{\lambda^2}\delta \sim N^{2s-1-}. \tag{2.51}$$

Combining (2.49)–(2.51), we have

$$E(Iu(T_0)) = \lambda^2 E(Iu_\lambda(\lambda^2 T_0)) \lesssim_{\delta, \|u_0\|_{H^s_x(\mathbb{R})}} \lambda^2 \sim N^{2-2s} \sim T_0^{\frac{2-2s}{2s-1}+}. \tag{2.52}$$

Using (2.46) and (2.52), we conclude

$$\|u(T_0)\|_{H^s(\mathbb{R})} \lesssim \langle T_0 \rangle^{\frac{1-s}{2s-1}+} \tag{2.53}$$

which is the desired polynomial bound (2.44).

Now, it is left to prove Proposition 2.6. The following proposition concerns with the local existence to the modified equation

$$\begin{cases} i\partial_t Iu + \partial_x^2 Iu = I(|u|^4 u) \\ Iu(0,x) = Iu_0(x). \end{cases} \tag{2.54}$$

Proposition 2.7. *For $\frac{1}{2} < s < 1$ and a given initial data u_0 with $E(Iu_0) \le 1$, there exists $\delta = \delta(\|u_0\|_{L^2_x(\mathbb{R})}) > 0$ so that the solutions Iu to (2.54) satisfies*

$$\|Iu\|_{X_\delta^{1,\frac{1}{2}+}} \lesssim 1 \tag{2.55}$$

on $[0, \delta]$.

The proof of the proposition is similar to the proof of Proposition 2.3. See Theorem 5.1 and Lemma 5.2 in [3] for the details.

Proof of Proposition 2.6. As in the proof of Proposition 2.2, we estimate the increment of the modified energy as

$$\begin{aligned}
E(Iu(\delta)) - E(Iu(0)) =& \Re \int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0} \Big(1 - \frac{m_{23456}}{m_2m_3m_4m_5m_6}\Big)\widehat{\overline{\partial_x^2 Iu}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3) \\
&\times \widehat{Iu}(\xi_4)\widehat{\overline{Iu}}(\xi_5)\widehat{Iu}(\xi_6) \\
&- \Re \int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0} \Big(1 - \frac{m_{23456}}{m_2m_3m_4m_5m_6}\Big)\widehat{\overline{I(|u|^4u)}}(\xi_1)\widehat{Iu}(\xi_2) \\
&\times \widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)\widehat{\overline{Iu}}(\xi_5)\widehat{Iu}(\xi_6) \\
\lesssim& \Bigg| \int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0} \Big(1 - \frac{m_{23456}}{m_2m_3m_4m_5m_6}\Big)\widehat{\overline{\partial_x^2 Iu}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3) \\
&\times \widehat{Iu}(\xi_4)\widehat{\overline{Iu}}(\xi_5)\widehat{Iu}(\xi_6)\Bigg| \\
&+ \Bigg| \int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0} \Big(1 - \frac{m_{23456}}{m_2m_3m_4m_5m_6}\Big)\widehat{\overline{I(|u|^4u)}}(\xi_1)\widehat{Iu}(\xi_2) \\
&\times \widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)\widehat{\overline{Iu}}(\xi_5)\widehat{Iu}(\xi_6)\Bigg| =: Term_1 + Term_2.
\end{aligned}$$

It is then sufficient to show that $Term_1 + Term_2 \lesssim N^{-1+}$. We start with $Term_1$. Using Littlewood-Paley decomposition, define

$$\widehat{u_{N_1}} = \widehat{P_{N_1}\partial_x^2 Iu}, \quad \widehat{u_{N_j}} = \widehat{P_{N_j} Iu},\ j = 2,3,4,5,6.$$

where $N_i = 2^{k_i}$ are dyadic numbers, $k_i \in \{0,1,2,\dots\}$ and $|\xi_i| \sim N_i$, $i = 1,2,3,4,5,6$. Then we have

$$\begin{aligned}
Term_1 \le \sum_{N_1,\dots,N_6} \Bigg| \int_0^\delta \int_{\sum_{i=1}^6 \xi_i=0} \Big(1 - \frac{m_{23456}}{m_2m_3m_4m_5m_6}\Big)\widehat{\overline{u_{N_1}}}(\xi_1)\widehat{u_{N_2}}(\xi_2)\widehat{\overline{u_{N_3}}}(\xi_3) \\
\times \widehat{u_{N_4}}(\xi_4)\widehat{\overline{u_{N_5}}}(\xi_5)\widehat{u_{N_6}}(\xi_6)\Bigg| \qquad (2.56)
\end{aligned}$$

Therefore, it is enough to show for an arbitrary dyadic piece of the right side of (2.56)

$$\text{Right side of (2.56)} \lesssim N^{-1+} N_2^{0-} \|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}} \prod_{j=2}^{6} \|u_{N_j}\|_{X_\delta^{1,\frac{1}{2}+}} \tag{2.57}$$

in the different frequency interaction cases. We may again assume that the spatial Fourier transform of the dyadic pieces are non-negative. By symmetry in the variables $\xi_2, \xi_3, \xi_4, \xi_5, \xi_6$, we restrict our attention to $N_2 \geq N_3 \geq N_4 \geq N_5 \geq N_6$. Then we also have $N_1 \lesssim N_2$. By reverting back the Plancherel formula after the multiplier is taken out of the integral with a pointwise bound, we need to take care of the following frequency interaction cases:

Case 1: $N_1 \sim N_2 \gtrsim N \gg N_3$. In this case, we take the multiplier out using the following bound

$$\left|1 - \frac{m_{23456}}{m_2 m_3 m_4 m_5 m_6}\right| \lesssim \frac{N_3}{N_2}.$$

Then using the Hölder's inequality and $L^6_{t,x}$-Strichartz estimate (A.14), it is enough to show

$$\frac{N_3}{N_2} \frac{N_1}{N_2 N_3 N_4 N_5 N_6} \lesssim N^{-1+} N_2^{0-}. \tag{2.58}$$

This is obvious since $N, N_2 \gg 1$ and their exponents are negative. **Case 2:** $N_2 \sim N_3 \gtrsim N \gg N_4$ and $N_1 \lesssim N_2$. In this case, the bound for multiplier is the following

$$\left|1 - \frac{m_{23456}}{m_2 m_3 m_4 m_5 m_6}\right| \sim \left|1 - \frac{m_{23456}}{m_2 m_3}\right| \lesssim \frac{m_1}{m_2 m_3} \lesssim N^{-1+} N_1^{-1} N_2^{2-}.$$

Then using the Hölder's inequality and the Strichartz estimate (A.14), we have

$$\begin{aligned}\text{Right side of (2.56)} &\lesssim N^{-1+} N_1^{-1} N_2^{2-} \|u_{N_1}\|_{L_t^8 L_x^4} \|u_{N_2}\|_{L_t^8 L_x^4} \|u_{N_3}\|_{L_t^4 L_x^\infty} \prod_{j=4}^{6} \|u_{N_j}\|_{L^6_{t,x}} \\ &\lesssim N^{-1+} N_1^{-1} N_2^{2-} \frac{N_1}{N_2 N_3 N_4 N_5 N_6} \|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}} \prod_{j=2}^{6} \|u_{N_j}\|_{X_\delta^{1,\frac{1}{2}+}}.\end{aligned}$$

Thus, it suffices to show

$$\frac{1}{N_4N_5N_6} \lesssim 1.$$

This is true since $N_4, N_5, N_6 \gtrsim 1$.

Case 3: $N_1 \sim N_2 \geq N_3 \gtrsim N \gg N_4$. In this case, we will implement the approach in [12] used to prove Proposition 4.11. We can not establish (2.57). Fortunately, it will suffice to show for the whole sum

$$\text{Right side of (2.56)} \lesssim N^{-1+}\|\partial_x^2 Iu\|_{X_\delta^{-1,\frac{1}{2}+}}\|Iu\|^5_{X_\delta^{1,\frac{1}{2}+}}$$

since we will be able to handle $N_1 \sim N_2$ sum by Schur's test. Take the multiplier out of the integral with the bound

$$\left|1 - \frac{m_{23456}}{m_2m_3m_4m_5m_6}\right| \sim \frac{1}{m_3} \lesssim N_3^{1-}N^{-1+},$$

and using the Hölder's inequality and the Strichartz inequality, we bound the right side of (2.56) by

$$\begin{aligned}
&\sum_{N_1\sim N_2}\sum_{N_3\geq\cdots\geq N_6} N_3^{1-}N^{-1+}\|u_{N_1}\|_{L_t^8L_x^4}\|u_{N_2}\|_{L_t^8L_x^4}\|u_{N_3}\|_{L_t^4L_x^\infty}\prod_{j=4}^{6}\|u_{N_j}\|_{L^6_{t,x}}\\
&\lesssim \sum_{N_1\sim N_2}\sum_{N_3\geq\cdots\geq N_6} N_3^{1-}N^{-1+}\frac{N_1}{N_2N_3N_4N_5N_6}\|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}}\prod_{j=2}^{6}\|u_{N_j}\|_{X_\delta^{1,\frac{1}{2}+}}.
\end{aligned}$$

Using the last expression above, we have

$$N^{-1+}\Big(\sum_{N_1\sim N_2}\|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}}\|u_{N_2}\|_{X_\delta^{1,\frac{1}{2}+}}\Big)\sum_{N_3\geq\cdots\geq N_6}N_3^{0-}(N_4N_5N_6)^{-1}\prod_{j=3}^{6}\|u_{N_j}\|_{X_\delta^{1,\frac{1}{2}+}} \tag{2.59}$$

We apply the Schur's test for the first sum by following Exercise 9 of ch.1 in [17]. Set

$$K(N_1,N_2) = \|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}}\chi_{\{N_1,N_2:N_1\sim N_2\}}$$

where N_1, N_2 are dyadic numbers. We have by (A.9)

$$\sum_{N_1} \|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}} K(N_1, N_2) \lesssim \sum_{N_1} \|u_{N_1}\|^2_{X_\delta^{-1,\frac{1}{2}+}}$$

for any fixed N_2. Also,

$$\sum_{N_2} \|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}} K(N_1, N_2) \lesssim \|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}}$$

for any fixed N_1. Thus, the operator

$$T := \sum_{N_1} K(N_1, N_2)$$

acting on the square-summable sequences on the dyadic numbers is bounded and has the property

$$\|T\| \lesssim \Big(\sum_{N_1} \|u_{N_1}\|^2_{X_\delta^{-1,\frac{1}{2}+}}\Big)^{1/2}.$$

Therefore, if we set $v(N_2) = \|u_{N_2}\|_{X_\delta^{1,\frac{1}{2}+}}$ for N_2 dyadic, we observe

$$\begin{aligned}
\sum_{N_1 \sim N_2} \|u_{N_1}\|_{X_\delta^{-1,\frac{1}{2}+}} \|u_{N_2}\|_{X_\delta^{1,\frac{1}{2}+}} &= \sum_{N_2} T(v)(N_2) \\
&\leq \sum_{N_2} |T(v)(N_2)| \\
&\lesssim \Big(\sum_{N_2} |T(v)(N_2)|^2\Big)^{1/2} \\
&= \|T(v)\|_{l^2} \\
&\lesssim \Big(\sum_{N_1} \|u_{N_1}\|^2_{X_\delta^{-1,\frac{1}{2}+}}\Big)^{1/2} \Big(\sum_{N_2} \|u_{N_2}\|^2_{X_\delta^{1,\frac{1}{2}+}}\Big)^{1/2} \\
&\sim \|\partial_x^2 Iu\|_{X_\delta^{-1,\frac{1}{2}+}} \|Iu\|_{X_\delta^{1,\frac{1}{2}+}}.
\end{aligned}$$

For the second sum, Cauchy-Schwarz inequality and the Littlewood-Paley inequality (A.9) are used to get

$$(2.59) \lesssim N^{-1+} \|\partial_x^2 Iu\|_{X_\delta^{-1,\frac{1}{2}+}} \|Iu\|^5_{X_\delta^{1,\frac{1}{2}+}}$$

which is what we wish to achieve. Hence, $Term_1 \lesssim N^{-1+}$.

It remains to show $Term_2 \lesssim N^{-1+}$. As in the proof of Proposition 2.2, using the Littlewood-Paley decomposition, we can bound $Term_2$ as

$$Term_2 \lesssim \sum_{N_{12345}} \sum_{N_6 \geq \cdots \geq N_{10}} \left| \int_0^\delta \int_{\sum_{i=1}^{10} \xi_i = 0} \Big(1 - \frac{m_{678910}}{m_6 m_7 m_8 m_9 m_{10}}\Big) P_{N_{12345}} (\widehat{I(|u|^4 u)})(\xi_{12345}) \right. \tag{2.60}$$

$$\left. \times \widehat{Iu}(\xi_6) \widehat{\overline{Iu}}(\xi_7) \widehat{Iu}(\xi_8) \widehat{\overline{Iu}}(\xi_9) \widehat{Iu}(\xi_{10}) \right|$$

where $P_{N_{12345}}$ is the Littlewood-Paley projection operator onto the dyadic shell $N_{12345} \sim \langle \xi_{12345} \rangle$ and by symmetry, the latter sum can be taken over $N_6 \geq N_7 \geq N_8 \geq N_9 \geq N_{10}$. We also have $N_{12345} \lesssim N_6$. So, it suffices to show for any dyadic piece

$$\text{Right side of (2.60)} \lesssim N^{-1+} N_6^{0-} \|Iu\|^5_{X_\delta^{1,\frac{1}{2}+}} \prod_{j=6}^{10} \|Iu_{N_j}\|_{X_\delta^{1,\frac{1}{2}+}}. \tag{2.61}$$

Assuming the spatial Fourier transform of dyadic pieces are non-negative, taking the multiplier out of the integral with the bound

$$\frac{m_{12345}}{m_6 m_7 m_8 m_9 m_{10}},$$

undoing Plancherel formula and applying the Hölder's inequality, we get

$$\text{Right side of (2.60)} \lesssim \frac{m_{12345}}{m_6 m_7 m_8 m_9 m_{10}} \|P_{N_{12345}} I(|u|^4 u)\|_{L^2_{t,x}} \|Iu_{N_6}\|_{L^6_{t,x}} \|Iu_{N_7}\|_{L^6_{t,x}}$$
$$\times \|Iu_{N_8}\|_{L^6_{t,x}} \|Iu_{N_9}\|_{L^\infty_{t,x}} \|Iu_{N_{10}}\|_{L^\infty_{t,x}}. \tag{2.62}$$

We need following estimates given in the lemma below

Lemma 2.8. *Let $u, u_{N_6}, \ldots, u_{N_{10}}$ be defined as above. Then we have*

$$\|P_{N_{12345}} I(|u|^4 u)\|_{L^2_{t,x}} \lesssim \frac{1}{\langle N_{12345} \rangle} \|Iu\|^5_{X_\delta^{1,\frac{1}{2}+}} \tag{2.63}$$

$$\|Iu_{N_j}\|_{L^6_{t,x}} \lesssim \frac{1}{\langle N_j \rangle} \|Iu_{N_j}\|_{X_\delta^{1,\frac{1}{2}+}}, \quad j = 6, 7, 8, \tag{2.64}$$

$$\|Iu_{N_k}\|_{L^\infty_{t,x}} \lesssim \frac{1}{\langle N_k \rangle^{1/2}} \|Iu_{N_k}\|_{X_\delta^{1,\frac{1}{2}+}}, \quad k = 9, 10. \tag{2.65}$$

Proof of Lemma 2.8. The proof of this lemma is similar to the analogous result established for the proof of Proposition 2.2. Thus, we shall prove only the first estimate in order to give at least a partial proof. For the positively ordered pseudo-differential operator $\langle\nabla\rangle I$, we apply fractional Leibniz rule. On a typical term, we have by Hölder's inequality.

$$\|(\langle\nabla\rangle Iu)\overline{u}u\overline{u}u\|_{L^2_{t,x}} \leq \|\langle\nabla\rangle Iu\|_{L^4_t L^\infty_x}\|u\|^4_{L^{16}_t L^8_x}.$$

We apply the Strichartz estimate for the first factor to get

$$\|\langle\nabla\rangle Iu\|_{L^4_t L^\infty_x} \lesssim \|\langle\nabla\rangle Iu\|_{X^{0,\frac{1}{2}+}_\delta} = \|Iu\|_{X^{1,\frac{1}{2}+}_\delta}.$$

For the other factors, we first apply the Sobolev embedding (A.11) and the Strichartz estimate to obtain

$$\|u\|^4_{L^{16}_t L^8_x} \lesssim \|\langle\nabla\rangle^{1/4}u\|^4_{L^{16}_t L^{\frac{8}{3}}_x} \lesssim \|\langle\nabla\rangle^{1/4}u\|_{X^{0,\frac{1}{2}+}_\delta} \lesssim \|Iu\|_{X^{1,\frac{1}{2}+}_\delta}$$

which completes the proof of the first estimate. □

By Lemma 2.8

$$\text{Right side of (2.62)} \lesssim \frac{m_{12345}}{m_6m_7m_8m_9m_{10}}\frac{1}{\langle N_{12345}\rangle\langle N_6\rangle\langle N_7\rangle\langle N_8\rangle\langle N_9\rangle^{1/2}\langle N_{10}\rangle^{1/2}} \times \|Iu\|^5_{X^{1,\frac{1}{2}+}_\delta}\prod_{j=6}^{10}\|Iu_{N_j}\|_{X^{1,\frac{1}{2}+}_\delta}.$$

Thus, it suffices to show

$$\frac{m_{12345}}{m_6m_7m_8m_9m_{10}}\frac{N^{1-}N_6^{0+}}{\langle N_{12345}\rangle\langle N_6\rangle\langle N_7\rangle\langle N_8\rangle\langle N_9\rangle^{1/2}\langle N_{10}\rangle^{1/2}} \lesssim 1. \tag{2.66}$$

There are two frequency interaction cases:

Case 1: $N_{12345} \sim N_6$ and $N_6 \geq N_7 \geq N_8 \geq N_9 \geq N_{10}$ and $N_6 \gtrsim N$. In this case

$$\text{Left side of (2.66)} \lesssim N^{1-}N_6^{-2+}$$

since $m(x)x^{1/2}$ is increasing. The above bound can be controlled by 1.

Case 2: $N_{12345} \lesssim N_6$ and $N_6 \sim N_7 \geq N_8 \geq N_9 \geq N_{10}$ and $N_6 \gtrsim N$. In this case

$$\text{Left side of (2.66)} \sim \frac{m_{12345} N^{1-} N_6^{0+}}{m_6^2 m_8 m_9 m_{10} N_6^2 N_{12345} N_8 N_9 N_{10}}.$$

Using the fact that $m(y)x^{1/2}$ is increasing for $0 \leq y \leq x$ and $m_{12345} N_{12345}^{-1} \lesssim 1$, we can replace $m_j N_j$ with 1 for $j = 8, 9, 10$, and $m_6^2 N_6$ with N

$$\text{Left side of (2.66)} \lesssim \frac{N^{1-} N_6^{0+}}{N_6 N} = N^{0-} N_6^{-1+} \lesssim 1.$$

In both cases, summing all the dyadic pieces as in Case 3 for $Term_1$ and the Littlewood-Paley inequality (A.9) complete the proof. □

2.3. On a Fourth Order Schrödinger Equation

In this section, we shall briefly introduce a joint work with my colleague Engin Başakoğlu which concerns the global well-posedness of the cubic defocusing fourth order NLS in spatial dimension $n = 4$

$$\begin{cases} i\partial_t u - \triangle^2 u = |u|^2 u, & (t, x) \in \mathbb{R} \times \mathbb{R}^4, \\ u(0, x) = u_0(x) \in H^s(\mathbb{R}^4). \end{cases} \tag{2.67}$$

Fourth order type of NLS equations are first introduced in [18, 19] .The local well-posedness results of (2.67) can be found in [20]. In [21, 22], the authors established the global well-posedness and scattering results in the energy space $H^2(\mathbb{R}^n)$ when $1 \leq n \leq 8$. There are several works improving the global well-posedness results below the energy space, see [23–25]. The authors used the I-method to obtain global results for rougher initial data. Lastly, in [26], when the spatial dimension $n = 4$, it is established that the IVP (2.67) is globally well-posed in $H(\mathbb{R}^4)$ for $s > \frac{60}{53}$. The approach in [26] was inspired by [2] which is similar to our attempt. We have discovered this article while we were making literature review. In any case, it is worthwhile to briefly share our work in this document.

A solution u to the IVP (2.67) enjoys two useful conservation laws, namely, the mass and the energy. The mass of u is given by

$$M(u(t)) = \int_{\mathbb{R}^4} |u(t, x)|^2 dx$$

and the energy of u is given as

$$E(u(t)) = \int_{\mathbb{R}^4} |\triangle u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4 dx.$$

As we can see from the energy functional, the energy space is $H^2(\mathbb{R}^4)$. In this context, the operator I is then defined as

$$\widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi)$$

where

$$m(\xi) = \begin{cases} 1 & |\xi| \leq N, \\ |\xi|^{s-2}N^{2-s} & |\xi| > 2N \end{cases}$$

and smoothly interpolates on the region $N < |\xi| \leq 2N$. With this definition and the different nature of the equation (2.67), some modifications are needed to implement the I-method. For example, we need to establish the analogous of (2.5) and (2.6) type of inequalities. First, we have established these type of estimates:

$$E(Iu(t)) \leq \Big(N^{2-s}\|u(t)\|_{\dot{H}^s_x(R^n)}\Big)^2 + \|u(t)\|^4_{L^4_x(\mathbb{R}^n)}, \tag{2.68}$$

$$\|u(t)\|^2_{H^s_x(\mathbb{R}^n)} \lesssim E(Iu(t)) + \|u_0\|^2_{L^2_x(\mathbb{R}^n)}. \tag{2.69}$$

After, we use the I-method on the equation (2.67). Our main result is as follows:

Claim. The IVP (2.67) is globally well-posed for $u_0 \in H^s(\mathbb{R}^4)$ when $s > \frac{8}{7}$.

The strategy here is again to generate an almost conserved quantity which is comparable to the H^s norm of the solution, and to control the H^s norm of the solution with a polynomial bound in time. The growth of the almost conserved quantity $E(Iu)$ can be controlled with a nicer bound:

$$E(Iu(t)) - E(Iu(0)) \lesssim N^{-3+} \tag{2.70}$$

for all $t \in [0,\delta]$ where $\delta > 0$ comes from the local analysis of the modified equation. (2.70) can be viewed as the analogue of Proposition 2.2.

We use (2.70) for the scaled solution

$$u_\lambda(t,x) := \lambda^{-2} u(\frac{t}{\lambda^4}, \frac{x}{\lambda}).$$

Initially, we have

$$\begin{aligned} E(Iu_{0,\lambda}) &\lesssim \big(\lambda^{-2s} N^{4-2s} + \lambda^{-4}\big)\big(1 + \|u_0\|_{H^s(\mathbb{R}^4)}\big)^4 && (2.71)\\ &\lesssim C_0 \lambda^{-2s} N^{4-2s}\big(1 + \|u_0\|_{H^s(\mathbb{R}^4)}\big)^4. && (2.72)\end{aligned}$$

By taking λ large enough, more precisely, taking

$$\lambda = \Big(\frac{1}{2C_0}\Big)^{\frac{1}{2s}} N^{\frac{4-2s}{2s}} \big(1 + \|u_0\|_{H^s(\mathbb{R}^4)}\big)^{\frac{4}{2s}}$$

we have $E(Iu_{0,\lambda}) \le \frac{1}{2}$. Then we apply (2.70) to $u_{0,\lambda}$ at least N^{3-} many times to get the size of $E(Iu_\lambda(t))$ reached 1. Then we determine the value of the cut-off parameter N in terms of any large time parameter T_0 as

$$T_0 \sim \frac{N^{3-}}{\lambda^4} C_1 \delta \sim N^{\frac{7s-8}{s}-}.$$

By combining the above estimates, we get

$$E(Iu(T_0)) = \lambda^4 E(Iu_\lambda(\lambda^4 T_0)) \lesssim_{\delta, \|u_0\|_{H^s(\mathbb{R}^4)}} \lambda^4 \sim N^{\frac{8-4s}{s}} \sim T_0^{\frac{8-4s}{7s-8}+}$$

which helps us to achieve the desired polynomial bound

$$\|u(T_0)\|_{H^s(\mathbb{R}^n)} \lesssim T_0^{\frac{8-4s}{7s-8}+}.$$

whenever $s > \frac{8}{7}$.

3. CONCLUSION

In this chapter, we start discussing the advantage and disadvantage of the I-method. First, in [6], it is mentioned that even though one can show L^2 global well-posedness and H^1 global well-posedness, the global result between these spaces cannot be obtained via this information. The I-method however generates an almost conserved quantity and hence can be implemented to establish global results in $H^s(\mathbb{R}^n)$, $0 < s < 1$. On the other hand, the disadvantage of the method is that one can not obtain the optimal result in terms of global well-posedness for equations of type (2.3) and (2.43). We can observe this from looking at the growth of energy increment (2.8) and the IVP (2.3) as an illustration. Roughly, for small time intervals, the quantity $E(Iu)$ grows at most $N^{-\frac{3}{2}+}$. The exponent of polynomial bound for the H^s norm of a solution u is determined by the exponent of the cut-off parameter N via the scaling argument. The smaller the exponent of N as the bound for the growth of $E(Iu)$ the better global result we may obtain. In fact, if the exponent of N goes to $-\infty$, then the quantity $E(Iu)$ would be a conserved quantity which is impossible to reach, see [6].

For a future plan, we are planning to obtain global results for (2.67) in the spatial dimensions $n = 2, 3$. The challenge in these dimensions is that we can not implement bilinear Strichartz estimate (A.15) for comparable frequencies. Therefore, our intention is to combine the I-method with Morawetz type estimates, see [22, 23].

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APPENDIX A: FUNCTION SPACES, NOTATIONS, HARMONIC AND FOURIER ANALYSIS TOOLS

The nonlinear Schrödinger type equations, i.e., dispersive evolution equations can be analyzed via comparison and interaction between the low and high-frequency components of the solutions. There are valuable tools to make this type of analysis rigorous in terms of a mathematical point of view. We need to introduce notations, function spaces, and results as preliminaries for the tools: the Fourier transform and Littlewood-Paley theory. We shall briefly review these topics in the appendix. We will mostly follow [10]. The space of p-Lebesgue integrable functions is defined as

$$L^p(\mathbb{R}^n) = \{f | f : \mathbb{R}^n \to \mathbb{C},\ \|f\|_{L^p(\mathbb{R}^n)} < \infty\}$$

where $0 < p < \infty$ and

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.$$

For $p = \infty$, we have

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| = \inf\{a \in \mathbb{R} : m(\{x : |f(x)| > a\}) = 0\}$$

where m is the Lebesgue measure on $\mathbb{R}^n$. One can replace the range of the L^p-functions with a Banach space X, and these spaces consist of the functions $f : \mathbb{R}^n \to X$ with the property

$$\|f\|_{L^p(\mathbb{R}^n;X)} = \left(\int_{\mathbb{R}^n} \|f(x)\|_X^p dx \right)^{1/p} < \infty.$$

In addition, we can define mixed, mostly spacetime, Lebesgue spaces $L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^n)$ consisting of the functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ together with the mixed norm

$$\|f\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^n)} = \left(\int_{\mathbb{R}} \|f(t,\cdot)\|_{L_x^q(\mathbb{R}^n)}^p dt \right)^{1/p} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |f(t,x)|^q dx \right)^{p/q} dt \right)^{1/p}.$$

Similarly, if $I \subset \mathbb{R}$ is a time interval, $k \geq 0$ and X is a Banach space, the space $C_t^k(I;X)$ is the space of k-times continuously differentiable functions $f : I \to X$ with the norm

$$\|f\|_{C_t^k(I;X)} = \sum_{j=0}^{k} \|\partial_t^j f\|_{L_t^\infty(I;X)}.$$

We shall also use Sobolev spaces $H^s, \dot{H}^s, W^{s,p}, \dot{W}^{s,p}$ which carry the information for the derivatives of L^p functions, and weighted Sobolev spaces $X^{s,b}$ which carry the dispersive characteristic associated to the equation. Before introducing that function spaces, we need the following preliminary notions. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called rapidly decreasing if

$$\|\langle x\rangle^N f(x)\|_{L_x^\infty(\mathbb{R}^n)} < \infty$$

for all $N \geq 0$. f is called Schwartz class if it is also smooth and all of its derivatives $\partial_x^\alpha f$ are rapidly decreasing for all multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \{0, 1, 2, \ldots\}$ for $i = 1, \ldots, n$ and the operator ∂_x^α is given as

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}.$$

One can observe that f is Schwartz if and only if $\partial_x^\alpha f(x) \lesssim_{f,\alpha,N} \langle x\rangle^{-N}$ for all multi-index α and all N. The space of Schwartz class functions is denoted by $\mathcal{S}(\mathbb{R}^n)$. The topological dual space of $\mathcal{S}(\mathbb{R}^n)$ consists of continuous linear functionals on the Schwartz space. Elements of the dual space are said to be tempered distributions. Hence, the dual space is called the space of tempered distributions and denoted by $\mathcal{S}'(\mathbb{R}^n)$. Now, we define the Fourier transform. It is convenient to define it on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform of a Schwartz function f at the point $\xi \in \mathbb{R}^n$ is given as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

It is a well-known fact that the Fourier transform $f \mapsto \hat{f}$ is a linear automorphism on $\mathcal{S}(\mathbb{R}^n)$ in its topology. The inverse transform is given as

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

The Fourier transform can be extended as a linear isometric isomorphism on $L^2(\mathbb{R}^n)$ together with the well-known Plancherel identity

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi$$

as well as the closely related Parseval identity

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)} dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi.$$

The Fourier transform has many fruitful symmetries. Some of which are the followings:

(i) The Fourier transform of $f(x - x_0)$, $x_0 \in \mathbb{R}^n$, at ξ equals $e^{-ix_0 \cdot \xi}\hat{f}(\xi)$.

(ii) The Fourier transform of $e^{ix\cdot\xi_0} f(x)$ at ξ equals $\hat{f}(\xi - \xi_0)$.

(iii) The Fourier transform of $\overline{f(x)}$ at ξ equals $\overline{\hat{f}(-\xi)}$.

(iv) Let $\lambda \in \mathbb{R} \setminus \{0\}$. Then the Fourier transform of $f(x/\lambda)$ at ξ equals $|\lambda|^d \hat{f}(\lambda\xi)$.

(v) The convolution operation $*$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x - y)dy.$$

The Fourier transform of $(f * g)(x)$ at ξ equals $\hat{f}(\xi)\hat{g}(\xi)$.

(vi) The Fourier transform of $f(x)g(x)$ at ξ equals $(\hat{f} * \hat{g})(\xi)$.

(vii) Let $P : \mathbb{R}^n \to \mathbb{C}$ be a polynomial. Then we can identify $P(\nabla)$ as

$$P(\nabla) = \sum_{|\alpha| \le M} a_\alpha \partial_x^\alpha$$

where the sum is taken over multi-index α with $|\alpha| = \alpha_1 + \cdots + \alpha_n \le M$ and $a_\alpha \in \mathbb{C}$ for all α. Then the Fourier transform of

$$P(\nabla)f(x) = \sum_{|\alpha| \le M} a_\alpha \partial_x^\alpha f(x)$$

at ξ is equal to

$$\sum_{|\alpha| \le M} a_\alpha (i\xi)^\alpha \hat{f}(\xi) = P(i\xi)\hat{f}(\xi).$$

The last property is critical when studying on a partial differential equation. More precisely, a linear partial differential operator with constant coefficients is of the form

$$P(\nabla) = \sum_{|\alpha| \le M} a_\alpha \partial_x^\alpha$$

for multi-index α and $M > 0$. Then by property (vii), we have

$$\widehat{P(\nabla)f}(\xi) = \sum_{|\alpha|\leq M} a_\alpha (i\xi)^\alpha \hat{f}(\xi) = P(i\xi)\hat{f}(\xi).$$

One can observe that differential operators enhance amplitude of functions with high frequencies while decreasing amplitude of functions of low frequencies. Also, if the operator $P(\nabla)$ is skew-adjoint, i.e. $iP(\nabla)$ is a self-adjoint operator, then we have

$$P(\nabla) = ih(\nabla/i)$$

for some real valued polynomial $h : \mathbb{R}^n \to \mathbb{R}$. In this case,

$$\widehat{P(\nabla)f}(\xi) = i\widehat{h(\nabla/i)f}(\xi) = i \sum_{|\alpha|\leq M} \xi^\alpha \hat{f}(\xi) = ih(\xi)\hat{f}(\xi).$$

With this in mind, and the fact that the Fourier transform can be extended to tempered distributions, we can define Fourier multipliers. Let $m : \mathbb{R}^n \to \mathbb{C}$ be a locally integrable function of at most polynomial growth, that is,

$$\sup_{x\mathbb{R}^n} |m(x)| \lesssim \langle x \rangle^N$$

for some $N > 0$. Then we can define the associated multiplier $m(\nabla/i) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ via the Fourier transform as the formula

$$\widehat{m(\nabla/i)f}(\xi) = m(\xi)\hat{f}(\xi)$$

or equivalently in the physical space by

$$m(\nabla/i)f(x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi.$$

We then have

- $m(\nabla/i)^* = \overline{m}(\nabla/i)$;
- $m_1(\nabla/i) + m_2(\nabla/i) = (m_1 + m_2)(\nabla/i)$;
- $m_1(\nabla/i)m_2(\nabla/i) = (m_1 m_2)(\nabla/i)$.

In particular, all Fourier multipliers commute with each other. The function $m(\xi)$ is called the symbol of the multiplier operator $m(\nabla/i)$. For example, we can define fractional differentiation and integration operators $|\nabla|^s$, $\langle\nabla\rangle^s$ associated with the symbols $|\xi|^s$ and $\langle\xi\rangle^s$, respectively, for $s \in \mathbb{R}$.

Now we are ready to introduce Sobolev spaces. The Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ and the homogeneous Sobolev spaces $\dot{W}^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$ and $1 < p < \infty$, are defined as the closure of the Schwartz class functions under the topology induced by the norms

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \|\langle\nabla\rangle^s f\|_{L^p(\mathbb{R}^n)}$$

and

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \||\nabla|^s f\|_{L^p(\mathbb{R}^n)}$$

respectively. When $p = 2$ we write H^s and $\dot{H}^s$ for $W^{s,2}$ and $\dot{W}^{s,2}$, respectively. By Plancherel's identity, we have

$$\|f\|_{H^s_x(\mathbb{R}^n)} = \|\langle\xi\rangle^s \hat{f}\|_{L^2_\xi(\mathbb{R}^n)}.$$

There is an important class of Fourier multiplier operators known as the Littlewood-Paley multipliers. Suppose φ is a real-valued, smooth, radially symmetric, compactly supported bump function such that its support is adapted to the ball $\{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $\varphi \equiv 1$ on $\{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$ and smoothly decays to zero in the region $\{\xi \in \mathbb{R}^n : 1 < |\xi| \leq 2\}$. Define a dyadic number $N = 2^j$ for $j \in \mathbb{Z}$. For each dyadic number N, the Littlewood-Paley multipliers are defined as

$$\widehat{P_{\leq N} f}(\xi) = \varphi(\xi/N)\hat{f}(\xi) \tag{A.1}$$

$$\widehat{P_{>N} f}(\xi) = (1 - \varphi(\xi/N))\hat{f}(\xi) \tag{A.2}$$

$$\widehat{P_N f}(\xi) = (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi). \tag{A.3}$$

Thus, the operators $P_{\leq N}, P_{>N}, P_N$ are smoothed versions of the characteristic functions that project functions to the regions $|\xi| \leq N$, $|\xi| > N$ and $|\xi| \sim N$ respectively. One can observe that

$$P_{\leq N} f = \sum_{M \leq N} P_M f; \quad P_{>N} f = \sum_{M > N} P_M f; \quad f = \sum_M P_M f$$

for all Schwartz class function f. The sums above are taken over the dyadic numbers. Such a decomposition is called the dyadic decomposition. Littlewood-Paley projections are immensely helpful in the quantitatively detailed analysis of PDE.

They separate the rough, i.e., high-frequency, oscillating, low regularity components of a solution from the smooth, that is, low-frequency, slowly varying, high regularity components, see [10]. We see that Littlewood-Paley projections behave as an approximation to the identity in the physical domain and a smooth partition of unity in the Fourier domain. By definition, Littlewood-Paley operators are convolution operators in the physical space:

$$P_{\leq N} f(x) = \int_{\mathbb{R}^n} \widehat{\varphi}(y) f(x + y/N) dy.$$

Since $\widehat{\varphi}$ is a Schwartz function, we can consider $P_{\leq N}$ as an averaging operator that spreads a sort of period of f by a spatial scale controlled by $1/N$ and localizes the frequency of f in a ball of radius controlled by N, which is consistent with the uncertainty principle. From the convolution definition, one can verify that Littlewood-Paley operators are continuous, hence bounded, on every Lebesgue space $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and also on Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, $\dot{W}^{s,p}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $1 < p < \infty$. Moreover, we have the following useful estimates known as Bernstein inequalities. Let $s \geq 0$ and $1 \leq p \leq q \leq \infty$:

$$\|P_{\geq N} f\|_{L^p(\mathbb{R}^n)} \lesssim_{p,s,n} N^{-s} \||\nabla|^s P_{\geq N} f\|_{L^p(\mathbb{R}^n)} \tag{A.4}$$

$$\|P_{\leq N} |\nabla|^s f\|_{L^p(\mathbb{R}^n)} \lesssim_{p,s,n} N^{s} \|P_{\leq N} f\|_{L^p(\mathbb{R}^n)} \tag{A.5}$$

$$\|P_N |\nabla|^{\pm s} f\|_{L^p(\mathbb{R}^n)} \sim_{p,s,n} N^{\pm s} \|P_N f\|_{L^p(\mathbb{R}^n)} \tag{A.6}$$

$$\|P_{\leq N} f\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q,n} N^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq N} f\|_{L^p(\mathbb{R}^n)} \tag{A.7}$$

$$\|P_N f\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q,n} N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p(\mathbb{R}^n)}. \tag{A.8}$$

Bernstein inequalities tell us that when the frequency is localized, we can change low integrability to high integrability with a cost of some powers of N. If we closely look at the inequalities, the cost in terms of powers of N might be, in fact, a gain when the frequency N is low. There is another very powerful and extremely useful estimate:

$$\|f\|_{L^p(\mathbb{R}^n)} \sim_{p,n} \|(\sum_N |P_N f|^2)^{1/2}\|_{L^p(\mathbb{R}^n)}. \tag{A.9}$$

The estimate (A.9) is known as the Littlewood-Paley inequality. It essentially tells us that the dyadic components $P_N f$ of f are mutually almost orthogonal.

Turning back to the Sobolev spaces, we have helpful embeddings. We can continuously embed homogeneous Sobolev spaces into L^p spaces:

$$\dot{W}^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$$

whenever the condition

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$$

holds for $1 < p < q < \infty$, and $s > 0$. In this case, we have for $u \in \dot{W}^{s,p}(\mathbb{R}^n)$,

$$\|u\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q,n} \|u\|_{\dot{W}^{s,p}(\mathbb{R}^n)}. \tag{A.10}$$

This result also implies the inhomogeneous Sobolev embedding:

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$$

whenever $1 < p < q < \infty$, $s > 0$, and $\frac{1}{q} \geq \frac{1}{p} - \frac{s}{n}$. In this case, we have

$$\|u\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q,s,n} \|u\|_{W^{s,p}(\mathbb{R}^n)}. \tag{A.11}$$

It is also beneficial to mention nonlinear operations, for example, multiplication operation $(f, g) \mapsto fg$ because they arise frequently in nonlinear PDEs. There is a very handy principle known as the fractional Leibniz rule. We state this principle as in [10]. Let f, g be functions on $\mathbb{R}^n$, and let D^α be some sort of differential of pseudo-differential operator of positive order $\alpha > 0$ (roughly speaking, it acts like a differentiation operator).

- (High-low interactions) If f has significantly higher frequency than g(e.g. if $f = P_N F$ and $g = P_{<N/8} G$ for some F, G), or is rougher than g(e.g. $f = \nabla u$ and $g = u$ for some u) then fg will have comparable frequency to f, and we expect $D^\alpha(fg) \sim (D^\alpha f)g$. In a similar spirit, we expect $P_N(fg) \sim (P_N f)g$.
- (Low-High interactions) If g has significantly higher frequency or is rougher than f, then we expect fg to have comparable frequency to g. We also expect $D^\alpha(fg) \sim f(D^\alpha g)$, and $P_N(fg) \sim f(P_N g)$.
- (Full Leibniz rule) If there are no frequency assumptions on f and g, we expect

$$D^\alpha(fg) \sim (D^\alpha f)g + f(D^\alpha g).$$

Its nature is explained in [10] as "The fractional Leibniz rule states that if we differentiate the product of functions, it is not needed to apply the whole product rule for differentiation. It is sufficient to take the derivative of the function with the highest frequency. The same is true for taking the Littlewood-Paley projection of products on the level of the function with the highest frequency. Moreover, a more general principle states that when distributing the derivatives, the dominant terms are, in general, the terms in which all the derivatives fall on a single factor. If the factors have unequal degrees of smoothness, the dominant term will be the one in which all the derivatives fall on the roughest (or the highest frequency) factor."

In the remaining part of the appendix, the weighted Sobolev spaces will be introduced. Consider a linear dispersive equation with constant coefficients

$$\partial_t u - Lu = 0$$

where $L = ih(\nabla/i)$ for some real-valued polynomial on $\mathbb{R}^n$. To understand some characteristics of this equation, we implement the space-time Fourier transform given as

$$\widetilde{u}(\tau,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} u(t,x)e^{-it\tau}e^{-ix\cdot\xi}d\xi d\tau.$$

If we take the space-time Fourier transform of the both sides of the equation, we see that

$$(-i\tau + ih(\xi))\widetilde{u}(\tau,\xi) = 0. \tag{A.12}$$

Then there are two cases to consider:

(i) $\widetilde{u} \equiv 0$. In this case, if one applies the inverse Fourier transform, the conclusion will be $u \equiv 0$. So, this case is out of our interest.

(ii) $\tau = h(\xi)$. In this case, the implication is that the space-time Fourier transform of the solution is supported in the hyperspace

$$\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \tau = h(\xi)\}.$$

Now, if one localizes the solution u in time by multiplying u with a smooth cut-off function η having support $\{t = O(1)\}$, the space-time Fourier transform $\widetilde{\eta u}$ will be concentrated in the region $\{(\tau, \xi) : \tau = h(\xi) + O(1)\}$. In other words, the localized version of the solution remains close to the aforementioned hyperspace. If we also add a nonlinear perturbation term in the above equation

$$\partial_t u - Lu = N(u),$$

the distortion effect of the perturbation will not be harmful when we localize the solution in time first. So, for suitable nonlinear terms $N(u)$, the localized solution ηu to the perturbed equation will still be close to the characteristic space. The reason for this is a dispersive smoothing effect for the operator $\partial_t - L$ away from the hyperspace $\tau = h(\xi)$ which can be considered analogous to the more well-known elliptic regularity. The weighted Sobolev spaces $X^{s,b}$, also known as Fourier restricted spaces, Bourgain spaces, or dispersive Sobolev spaces, carry the information of L^2-functions additionally in terms of their spatial and dispersive regularity. The following definition of $X^{s,b}$ spaces is taken from [10].

Definition A.1. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, and $s, b \in \mathbb{R}$. The $X^{s,b}_{\tau=h(\xi)}(\mathbb{R} \times \mathbb{R}^n)$, abbreviated $X^{s,b}(\mathbb{R} \times \mathbb{R}^n)$ or simply $X^{s,b}$, is defined to be the closure of the Schwartz functions $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{R}^n)$ under the norm*

$$\|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} = \|\langle\xi\rangle^s \langle\tau - h(\xi)\rangle^b \widetilde{u}(\tau, \xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R}^n)}. \tag{A.13}$$

The $X^{s,b}_{\tau=h(\xi)}$ spaces are well-adapted to the solutions, $u(t) = e^{tL}u(0)$, of the linear dispersive equation $\partial_t = Lu$, where $L = ih(D) = ih(\nabla/i)$ by the following lemma:

Lemma A.2 (Free solutions lie in $X^{s,b}$)**.** *Let $f \in H^s_x(\mathbb{R}^n)$ for some $s \in \mathbb{R}$ and let $L = ih(\nabla/i)$ for some polynomial $h : \mathbb{R}^n \to \mathbb{R}$. Then for any Schwartz time cut-off $\eta \in \mathcal{S}_t(\mathbb{R})$, we have*

$$\|\eta(t)e^{tL}f\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{\eta,b} \|f\|_{H^s_x(\mathbb{R}^n)}.$$

For further properties of $X^{s,b}$ spaces, we recommend [10] to the reader. We shall only state the results that we use in this document. Now, when $b > 1/2$, we observe that $X^{s,b}$ functions are very close to the free solutions to a dispersive equation $\partial_t u = Lu$.

Lemma A.3. *Let $L = iP(\nabla/i)$ for some polynomial $P : \mathbb{R}^n \to \mathbb{R}$, $s \in \mathbb{R}$, $b > 1/2$ and let Y be a space-time Banach space of functions on $\mathbb{R} \times \mathbb{R}^n$ with the property that*

$$\|e^{it\tau_0}e^{tL}f\|_Y \lesssim \|f\|_{H^s_x(\mathbb{R}^n)}$$

for all $f \in H^s_x(\mathbb{R}^n)$ and $\tau_0 \in \mathbb{R}$. Then we have the embedding

$$\|u\|_Y \lesssim_b \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)}.$$

Applying this to $Y = C^0_t H^s_x$, we obtain

Corollary A.4. *Let $b > 1/2$, $s \in \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then for any $u \in X^{s,b}_{\tau=h(\xi)}(\mathbb{R} \times \mathbb{R}^n)$, we have*

$$\|u\|_{C^0_t H^s_x(\mathbb{R}\times\mathbb{R}^n)} = \|u\|_{L^\infty_t H^s_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim_b \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)}.$$

Moreover, for a Schrödinger admissible pair (p,q), i.e., $(p,q) \in [2,\infty] \times [2,\infty]$, and $(p,q) \neq (2,\infty)$ when the spatial dimension is 2 and $\frac{2}{p} = n(\frac{1}{2} - \frac{1}{q})$, we have the inequality

$$\|u\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{p,q,b} \|u\|_{X^{0,b}_{\tau=|\xi|^2}(\mathbb{R}\times\mathbb{R}^n)}. \tag{A.14}$$

$X^{s,b}$ spaces are suitable to analyzing nonlinear dispersive equations when one localises in time.

Lemma A.5. *Let $\eta \in \mathcal{S}_t(\mathbb{R})$ be a Schwartz function in time. Then we have*

$$\|\eta(t)u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{\eta,b} \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)}$$

for any $s,b \in \mathbb{R}$, any $h : \mathbb{R}^n \to \mathbb{R}$, and any $u \in \mathcal{S}_{t,x}(\mathbb{R}\times\mathbb{R}^n)$. Furthermore, if $-1/2 < b' \leq b < 1/2$, then for any $0 < T < 1$, we have

$$\|\eta(t/T)u\|_{X^{s,b'}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{\eta,b,b'} T^{b-b'}\|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)}.$$

Proposition A.6 ($X^{s,b}$ energy estimate)**.** *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, let $L = ih(\nabla/i)$, and let $u \in C^\infty_{t,loc}\mathcal{S}_x(\mathbb{R}\times\mathbb{R}^n)$ be a smooth solution to the equation $\partial_t u = Lu + F$. Then for any $s \in \mathbb{R}$, $b > 1/2$, and any compactly supported smooth cut-off $\eta(t)$, we have*

$$\|\eta(t)u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{\eta,b} \|u(0)\|_{H^s_x(\mathbb{R}^n)} + \|F\|_{X^{s,b-1}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)}.$$

Lastly, we have an estimate for nonlinear terms:

Lemma A.7 (Bilinear Strichartz estimate)**.** *Let $u_1, u_2 \in X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^n)$, be supported on spatial frequencies $|\xi| \sim N_1, N_2$, respectively. If $N_1 \ll N_2$ when $n = 1$ or $N_1 \leq N_2$ when $n \geq 2$, we have*

$$\|u_1 u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \frac{N_1^{(n-1)/2}}{N_2^{1/2}}\|u_1\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^n)}\|u_2\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^n)}. \tag{A.15}$$