ON THE HYPERSURFACES IN TORIC VARIETIES

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ABSTRACT

ON THE HYPERSURFACES IN TORIC VARIETIES

Theory of toric varieties provides fruitful interactions between algebraic geometry and combinatorics. It is remarkably fertile in terms of connections with many areas of mathematics and has plentiful applications to other disciplines as well. We introduce and study toric varieties and their hypersurfaces in the realm of algebraic geometry with a focus on quasismooth hypersurfaces. This is because quasismooth hypersurfaces are general enough to contain many examples of elements in some special families (e.g. regular hypersurfaces and Calabi-Yau hypersurfaces) that have a frequent appearance in mirror symmetry, complex and differential geometry, and physics; interesting enough to have special roles in some areas of research such as toric GIT and moduli problems; and easy to characterize using combinatorial tools agreeably to the spirit of toric geometry.

ÖZET

TORİK VARYETELERDE HİPERYÜZEYLER ÜZERİNE

Torik varyeteler kuramı, cebirsel geometri ve kombinatorik arasında zengin bir etkileşim yaratır. Bu teori, matematiğin birçok alanıyla derinden bağlantılıdır ve başka disiplinler üzerine de bolca uygulaması vardır. Bu çalışmada, *oldukça-düzgün hiperyüzeyler* odaklı olmak üzere torik varyeteleri ve onların hiperyüzeylerini cebirsel geometrik bağlam içinde tanıtıyor ve çalışıyoruz. Bunun temel nedeni oldukça-düzgün hiperyüzeylerin ayna simetrisi, karmaşık ve diferansiyel geometri, fizik alanlarında sıklıkla karşımıza çıkan bazı özel hiperyüzey ailelerini içerecek kadar büyük; geometrik değişmezler kuramı (GIT) ve moduli problemleri gibi bazı konularda özel bir role sahip olacak kadar ilginç olmaları ve torik geometrinin genel ruhuna uygun bir şekilde kombinatorik kullanarak ayırt edilebiliyor olmalarıdır.

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$[n]$	Set of positive integers less than or equal to n
$[D]$	Class of a Weil divisor in the divisor class group
A_σ	Vanishing set of $(x_\rho)_{\rho\in\sigma(1)}$ in $\hat{X}_\Sigma$
$\pi^*(\mathcal{S})$ and $B_\mathcal{S}$	Base loci of linear systems $\pi^*(\mathcal{S})$ and $\mathcal{S}$
$B(\Sigma)$	(Irrelevant ideal $x^{\hat{\sigma}} \colon \sigma \in \Sigma_{\max}$) of X_Σ
$\mathbb{C}^*$	Multiplicative group of nonzero complex numbers $\mathbb{C}\backslash\{0\}$
$(\mathbb{C}^*)^n$	Standard n dimensional torus
$\mathbb{C}[S]$	Semigroup algebra over complex numbers
$\mathrm{Cone}(S)$	Convex cone generated by S
$\mathrm{Conv}(A)$	Polytope defined as the convex hull of A
$\mathrm{CDiv}(X_\Sigma$	Group of Cartier divisors on X_Σ
$\mathrm{CDiv}_{T_N}(X_\Sigma)$	Group of toric Cartier divisors on X_Σ
$Cl(X_\Sigma)$	Divisor class group of X_Σ
D_ρ	Prime divisor on X_Σ given as the orbit closure of a ray $\rho \in \Sigma$
$\mathrm{Div}(X_\Sigma$	Group of Weil divisors on X_Σ
$\mathrm{Div}_{T_N}(X_\Sigma)$	Group of toric Weil divisors on X_Σ
$\mathscr{F}_M$	Quasicoherent sheaf corresponding to a graded module M
f_Y	Defining homogeneous polynomial of a hypersurface Y
$G = G(\Sigma)$	Group $\mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$
M	Character lattice of a torus
$\{m_\sigma\}_{\sigma\in\Sigma}$	Cartier data of a torus-invariant Cartier divisor on X_Σ
N	Lattice of one-parameter subgroups of a torus
$M_\mathbb{R}$	The vector space $M \otimes_\mathbb{Z} \mathbb{R}$
$M_\mathcal{S}$	R_0-module of polynomials corresponding to hypersurfaces in the linear system $\mathcal{S}$
$N_\mathbb{R}$	The vector space $N \otimes_\mathbb{Z} \mathbb{R}$
$\mathscr{O}_{X_\Sigma}(D)$	Sheaf of a divisor D on X_Σ
$\mathscr{O}_{X_\Sigma}(\alpha)$	Sheaf of the shift $R(\alpha)$ of R
$\mathcal{O}(\sigma)$	T_N orbit of a cone $\sigma \in \Sigma$ in X_Σ

$\overline{\mathcal{O}(\tau)}$	Closure of the orbit of $\tau \in \Sigma$
P_D	Polyhedron associated to a toric divisor D
$Pic(X_\Sigma)$	Picard group on X_Σ
r	Number of rays $\lvert\Sigma(1)\rvert$
R	Cox ring $\mathbb{C}\left[x_\rho \colon \rho \in \Sigma(1)\right]$ of X_Σ
R^G	Subring of invariants of R under the G action
$R_{x^{\hat{\sigma}}}$	Localization of R at the monomial $x^{\hat{\sigma}}$
$R_\alpha = (R)_\alpha$	Graded piece of degree α polynomials in R
$\mathrm{Relint}(\sigma)$	Relative interior of σ
$\mathcal{S}$	Linear system
$\mathcal{S}^{\mathrm{qs}}$	Quasismooth elements of $\mathcal{S}$
S_i	Semigroup genereated by $A_i = A - m_i$ for $m_i \in A$
S_σ	Semigroup $\hat{\sigma} \cap M$
$\mathrm{SF}(\Sigma)$	Group of integral support functions on Σ
$\mathrm{Span}(\sigma)$	Subspace spanned by σ
$\mathrm{Specm}(\mathbb{C}[S])$	Collection of maximal ideals in $\mathbb{C}[S]$
$\sup(\Sigma)$	Support (i.e. the union of cones) of a fan Σ
T_N	Torus $N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}\left(M, \mathbb{C}^*\right)$ associated to N and M
T_σ	Torus isomorphic to $(\mathbb{C}^*)^{\lvert\Sigma(1)\setminus\sigma(1)\rvert}$
u_ρ	Minimal generator of $\rho \cap N$
$U\sigma$	Affine toric variety V_σ as an open set in X_Σ
$\{(U_i, f_i)\}$	Local data of a Cartier divisor on X
$\widehat{\mathbf{V}(I)}$	The intersection $\mathbf{V}(I) \cap \hat{X}$
V_A	Affine toric variety given by A
V_I	Subvariety of X_Σ defined by a G homogeneous ideal $I \trianglelefteq R$
V_σ	Affine toric variety $\mathrm{Specm}(\mathbb{C}[S_\sigma])$ associated to σ
$x^{\hat{\sigma}}$	Monomial associated to σ
X^{reg} and X^{sing}	Smooth and singular loci of X
X_A	Projective toric variety given by A
X_P	Projective toric variety associated to polytope P
X_Σ	Toric variety of the fan Σ

$\hat{X}_{\Sigma}$	Quasiaffine variety $\mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma)$
Y	A hypersurface
$\mathbb{Z}A$	Free abelian group generated by A
$\mathbb{Z}'A$	Character lattice of X_A
$Z(\Sigma)$	Exceptional set $\mathbf{V}(B(\Sigma))$ of X_{Σ}
$\|\alpha\|$	Complete linear system of hypersurfaces in degree α
γ_{σ}	Distinguished point of V_{σ}
$\Gamma(X_{\Sigma}, \mathscr{O}_{X_{\Sigma}}(D))$	Global sections of the sheaf $\mathscr{O}_{X_{\Sigma}}(D)$
$\Delta_{\mathcal{S}}$	Newton polytope of the linear system $\mathcal{S}$
$\Delta_{\mathcal{S}}^{\sigma}(\rho)$	Newton polytope of $f_Y\|_{A_{\sigma}}$
λ^u	One-parameter subgroup of the torus given by $u \in N$
Φ_A	Torus embedding giving by a finite set A of lattice points
π	Canonical quotient map $\hat{X}_{\Sigma} \to X_{\Sigma}$
$\pi^*(Y)$	Pull-back of Y to $\hat{X}_{\Sigma}$
$\pi^*(\mathcal{S})$	Pull-back of the linear system $\mathcal{S}$
ρ	A ray in $N_{\mathbb{R}}$
σ	Rational convex polyhedral cone in $N_{\mathbb{R}}$
$\hat{\sigma}$	Dual cone of σ
$\sigma^{\perp}$	Set $\{m \in M_{\mathbb{R}} \| \langle m, \sigma \rangle = 0\}$
Σ	A fan consisting of rational convex polyhedral cones in $N_{\mathbb{R}}$
Σ_P	Normal fan of a polytope P
$\sigma(1)$	Rays of a strongly convex cone $\sigma \subseteq N_{\mathbb{R}}$
$\Sigma(r)$	Set of r dimensional faces of Σ
$\Sigma_{\max}$	Collections of cones of maximal dimension in Σ
$\tau \preceq \sigma$	τ is a face face of σ
φ_D	Support function of the toric Cartier divisor D
φ_P	Support function of the divisor D_P associated to polytope P
ϕ_p	Semigroup morphism $S_{\sigma} \to \mathbb{C}$ corresponding to a point $p \in V_{\sigma}$
χ^m	A character of the torus given by $m \in M$

1. INTRODUCTION

The aim of this dissertation is to introduce and study toric varieties from an algebraic geometric point of view and to provide the necessary background for an extensive reading on relatively more advanced topics and research papers about or related to hypersurfaces in toric varieties and toric hypersurfaces.

Theory of toric varieties provides fruitful interactions between algebraic geometry and combinatorics. It is remarkably fertile in terms of connections with many areas of mathematics such as algebraic geometry, convex geometry, commutative algebra, topology, differential and symplectic geometry. Besides, it has plentiful applications to other diciplines such as physics, chemistry, biology (e.g. philogenetics), industrial engineering (e.g. optimization) and so on. We start the second chapter by giving the construction of toric varieties as torus embeddings, as spectra of semigroup algebras (for affine toric varieties and affine patches), and using combinatorial objects such as fans and polytopes. We show that the combinatorial data characterizes toric varieties and give a dictionary demonstrating their correspondence. Then, we explicitly describe the torus action and study its orbits. The third chapter is about divisors on toric varieties and the groups of their classes. We define toric divisors and explain how to compute the divisor class group and the Picard group with their help. We also introduce the support functions associated to toric divisors. In chapter four, we give the presentation of toric varieties as quotients of affine spaces and a toric ideal-variety correspondence using the Cox ring. In chapter five, we study the line bundles and quasicoherent sheaves on toric varieties. We give combinatorial conditions for a line bundle to be base point free, ample, and very ample and the correspondence of quasicoherent sheaves on a toric variety and graded modules over its Cox ring. Chapters $2-5$ heavily depend on the introductory texts [1], [2], [3], [4] on toric varieties. In particular, notations and terminologies are mostly adopted from [1] as it is more up to date.

In the last two chapters, we especially focus on quasismooth hypersurfaces as they are general enough to contain many examples of elements in some the special families (e.g. regular hypersurfaces and Calabi-Yau hypersurfaces) that have a frequent appearance in algebraic, complex and differential geometry, and physics; interesting enough to have special roles in some areas of research, and easy to characterize using combinatorial tools agreeably to the spirit of toric geometry. Quasismooth hypersurfaces in toric varieties have been widely used in some of the very sophisticated and highly active areas of research in algebraic geometry; such as mirror symmetry and minimal model program. Furthermore, they have a special role in the construction of the moduli space of hypersurfaces in toric varieties. After developing some familiarity with hypersurfaces and linear systems on toric varieties in chapter six, we show that quasismoothness can be checked using the Newton polytope of the defining equation of a given hypersurface. We make some brief comments about the moduli problem of hypersurfaces in toric varieties as the last words.

Throughout this dissertation, we will be working with toric varieties over the field of complex numbers $\mathbb{C}$. Note that we omit the background material on algebraic geometry and assume some familiarity with basic concepts, notions and results covered in the first two chapters of Hartshorne's book *Algebraic Geometry.* The reader may consult more or less any other standard book on algebraic geometry besides [5] when it is needed. On the other hand, we do not expect any prior knowledge on toric varieties or their combinatorial counterparts.

2. TORIC VARIETIES

2.1. Construction of a Toric Variety

We rather give the definition in the full generality as the definition of, so called, abstract toric varieties. In affine or projective cases, one can simply put the appropriate adjective before the word *variety* to obtain the contextual definition.

Definition 1. *A toric variety V is an irreducible variety containing an algebraic torus T as a Zariski open subset such that the action of torus on itself by multiplication extends to an algebraic action of the torus on V.*

Examples include most of the families of varieties that are in everyday use, such as tori $(\mathbb{C}^*)^n$, affine spaces $\mathbb{C}^n$, projective spaces $\mathbb{P}^n$ and their products, weighted projective spaces $\mathbb{P}(q_0, \ldots, q_n)$, Hirzebruch surfaces $\mathscr{H}_r$ and so on.

On the other hand, there are many ways to describe a toric variety, all of which are equivalent to the original definition given above. One can use lattice points, monomial ideals, semigroup algebras, combinatorial objects such as cones, fans, and polytopes, moment maps, and GIT-like quotients. We shall provide brief explanations to some of these perspectives.

2.1.1. Lattice Points and Torus Embeddings

The name *torus embedding* for toric varieties, as it is called today, was popular in some quarters in the 70s. An affine (resp. projective) toric variety is obtained as the closure of the image of an algebraic torus T under an embedding to affine (resp. projective) space. Such embeddings are given by finitely many characters of the torus which are determined by finitely many integer tuples.

Definition 2. *A character of an algebraic torus T is a morphism $\chi\colon T \to \mathbb{C}^*$ that is a group homomorphism.*

Note that for every $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$,

$$\chi^a(t) = \chi^a(t_1, \ldots, t_n) = t_1^{a_1} \ldots t_n^{a_n}$$

is a character where $t \in T$ and all characters of T is of this form. Hence, the characters of a torus constitute a free abelian group $M \cong \mathbb{Z}^n$ where n is the dimension of T.

Definition 3. *A one-parameter subgroup is a morphism $\lambda \colon \mathbb{C}^* \to T$ that is a group homomorphism.*

Similarly, each $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup

$$\lambda^b(t) = \lambda^b(t_1, \ldots, t_n) = (t^{b_1}, \ldots, t^{b_n})$$

for $t \in \mathbb{C}^*$ and the one-parameter subgroups form a group $N \cong \mathbb{Z}^n$.

Notice that the dot product on $\mathbb{Z}^n$ gives a pairing from $M \times N$ to $\mathbb{Z}$. So, M and N are dual lattices that determines the torus T. Let T be a torus with the lattice of one-parameter subgroups N. Then,

$$\begin{aligned} N \otimes_{\mathbb{Z}} \mathbb{C}^* &\cong T \\ b \otimes t &\mapsto \lambda^b(t) \end{aligned}$$

is a natural isomorphism. Accordingly, we will denote the torus as T_N for the rest of this dissertation.

Let T_N be a torus with the character lattice $M = \mathrm{Hom}(N, \mathbb{Z})$. Let $A = \{m_1, \ldots, m_s\}$ be a finite subset of the character lattice and consider the map

$$\begin{aligned} \varPhi_A \colon T_N &\to \mathbb{C}^s \\ t &\mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)). \end{aligned}$$

Clearly, the image $\varPhi_A(T_N)$ is an algebraic torus and the Zariski closure of the image is an affine toric variety which will be denoted by V_A. In other words,

$$V_A = \mathbf{V}(\mathbf{I}(\varPhi_A(T_N))). \tag{2.1}$$

One may compose the map $\varPhi_A$ with a projection $\varPi \colon \mathbb{C}^s \to \mathbb{P}^{s-1}$ to get the projective toric variety, which we will denote by X_A, defined as the Zariski closure of the image

$\mathit{\Pi} \circ \mathit{\Phi}_A(T_N)$ that is a subtorus of $T_{\mathbb{P}^{s-1}}$. So,

$$X_A = \mathbf{V}(\mathbf{I}(\mathit{\Pi} \circ \mathit{\Phi}_A(T_N)) \subseteq \mathbb{P}^{s-1}. \tag{2.2}$$

Conversely, every affine and projective toric variety arises in this way.

Note that $\mathbb{Z}^n$ is the character lattice of the torus $(\mathbb{C}^*)^n$ of the affine space and

$$\mathcal{M}_{s-1} = \Big\{ (a_1, \dots, a_s) \in \mathbb{Z}^s | \sum_{i=1}^{s} a_i = 0 \Big\}$$

is the character lattice of the torus $T_{\mathbb{P}^{s-1}} = \mathbb{P}^{s-1} \backslash \mathbf{V}(x_0 \dots x_{s-1})$ of the projective space. We may compute the character lattices of the tori in V_A and X_A by the following proposition ([1] Propositions 1.1.8 and 2.1.6).

Proposition 2.1. *Let T_N be a torus with character lattice M and let us fix a finite set $A = \{m_1, \dots, m_s\} \subseteq M$ of lattice points. Then,*

(i) The character lattice of the torus $\mathit{\Phi}_A(T_N)$ in V_A is the sublattice $\mathbb{Z}A \subseteq M$ generated by $A \subseteq M$ and dimension of V_A is equal to the rank of $\mathbb{Z}A$.

(ii) The torus $\mathit{\Pi} \circ \mathit{\Phi}_A(T_N)$ of the toric variety X_A has the character lattice

$$\mathbb{Z}'A = \Big\{ \sum_{i=1}^{s} a_i m_i \,|\, \tilde{a} = (a_1, \dots, a_s) \in \mathcal{M}_{s-1} \Big\}$$

and the rank of the sublattice $\mathbb{Z}'A$ gives the dimension of X_A.

To be convinced of the proposition above, consider the following diagram.

$$\begin{tikzcd}
T_N \arrow[d, "\mathit{\Phi}_A"'] \arrow[dr, two heads] & \\
(C^*)^s \arrow[d, two heads, "\mathit{\Pi}"'] & \mathit{\Phi}_A(T_N) \arrow[l, hook'] \arrow[d, two heads] \\
T_{\mathbb{P}^{s-1}} & \mathit{\Pi} \circ \mathit{\Phi}_A(T_N) \arrow[l, hook']
\end{tikzcd}
\quad \xrightarrow{\mathscr{F}} \quad
\begin{tikzcd}
m_i \in M & \\
e_i \in \mathbb{Z}^s \arrow[u, "\hat{\mathit{\Phi}}_A"] \arrow[r, two heads] & \mathbb{Z}A \arrow[ul, hook] \\
\mathcal{M}_{s-1} \arrow[u] \arrow[r, two heads] & \mathbb{Z}'A \arrow[u]
\end{tikzcd}$$

The dimension of a toric variety is the dimension of the maximal torus contained in it. The equality follows from the diagram since $\mathscr{F}$ is a contravariant functor taking a torus to its character lattice.

2.1.2. Semigroup Algebras

Semigroup algebras over $\mathbb{C}$ that has a finite generating set of lattice points appear to be the coordinate rings of affine toric varieties. Hence, those semigroups, which are called *affine semigroups*, give a way of characterizing affine toric varieties. Affine semigroups may also be considered as a useful tool to describe the affine pieces of toric varieties that are typically not affine.

Definition 4. *An affine semigroup S is a finitely generated semigroup that can be embedded in a lattice M.*

Note that

$$\mathbb{C}[S] = \Big\{ \sum_{m \in S} c_m \chi^m \mid c_m \in \mathbb{C},\ c_m = 0 \text{ for all but finitely many } m \Big\}$$

is an integral domain and finitely generated as a $\mathbb{C}$ algebra. Consequently, $\mathrm{Specm}(\mathbb{C}[S])$ is an affine variety containing a torus that has character lattice $\mathbb{Z}S$. Moreover, if S is the sublattice generated by $A = \{m_1, \dots, m_s\} \subseteq M_{\mathbb{R}}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$ and $\mathrm{Specm}(\mathbb{C}[S]) = V_A$.

Now let X_A be the toric variety defined as the Zariski closure of the torus $\mathit{\Pi} \circ \mathit{\Phi}_A(T_N)$. Recalling that $X_A \subseteq \mathbb{P}^{s-1}$, we can find the affine pieces of the projective toric variety X_A as its intersections with the affine pieces of $\mathbb{P}^{s-1}$. Let $\{x_0, \dots, x_{s-1}\}$ be a fixed homogeneous coordinate system for $\mathbb{P}^{s-1}$. Take the affine open subset $U_i = \mathbb{P}^{s-1} \backslash \boldsymbol{V}(x_i) \cong \mathbb{C}^{s-1}$ of $\mathbb{P}^{s-1}$. Then, the affine component $X_A \cap U_i$ is the affine toric variety given in 2.1. Hence, we get the associated affine semigroup S_i generated by the set $A_i = A - m_i = \{m_j - m_i |\, j \neq i\,\}$.

Proposition 2.2 ([1] Proposition 2.1.8)**.** *Let $X_A \subseteq \mathbb{P}^{s-1}$ for $A = \{m_1, \dots, m_s\}$. Then the affine piece $X_A \cap U_i$ is the affine toric variety*

$$X_A \cap U_i = V_{A_i} = \mathit{Specm}\left(\mathbb{C}\left[S_i\right]\right).$$

Note that $\mathbb{Z}'A = \mathbb{Z}A_i$ for every $i \in [s] = \{1, \dots, s\}$ and consequently the torus of the affine piece V_{A_i} is the torus of X_A.

It also explains how the data of the pieces are put together to constitute the whole. We may give an algebro-geometric description of the inclusions, although it can be done through combinatorial tools as well. We have

$$X_A \cap U_i \supseteq X_A \cap U_i \cap U_j \subseteq X_A \cap U_j \tag{2.3}$$

when $i \neq j$. Notice that $U_i \cap U_j$ contains all points of $X_A \cap U_i$ where $x_j/x_i \neq 0$. In the coordinate ring Specm($\mathbb{C}[S_i]$) of $X_A \cap U_i$, it corresponds to the points satisfying $\chi^{m_j - m_i} \neq 0$. Thus,

$$X_A \cap U_i \cap U_j = \mathrm{Specm}(\mathbb{C}[S_i])_{\chi^{m_j-m_i}} = \mathrm{Specm}(\mathbb{C}[S_i]_{\chi^{m_j-m_i}}).$$

Let S_k be the affine semigroup corresponding to $X_A \cap U_i \cap U_j$, the expression (2.3) can be written as

$$\mathrm{Specm}(\mathbb{C}[S_i]) \supseteq \mathrm{Specm}(\mathbb{C}[S_i])_{\chi^{m_j-m_i}} = \mathrm{Specm}(\mathbb{C}[S_i + S_j]) = \mathrm{Specm}(\mathbb{C}[S_k]),$$
$$\mathrm{Specm}(\mathbb{C}[S_i + S_j]) = \mathrm{Specm}(\mathbb{C}[S_j])_{\chi^{m_i-m_j}} \subseteq \mathrm{Specm}(\mathbb{C}[S_j]).$$

Here, we use the fact $S_k = S_i + S_j$ known as the *separation lemma.*

2.1.3. Cones, Fans, Polytopes, and Co.

The most useful and, perhaps, the most fun thing about the theory of toric varieties is its natural association with combinatorics. Here we are to explore it!

Definition 5. *A convex polyhedral cone generated by a subset $S \subseteq N_{\mathbb{R}}$ is the set*

$$\sigma = Cone(S) = \Big\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \geq 0 \Big\} \subseteq N_{\mathbb{R}}.$$

$\sigma = Cone(S)$ is said to be rational if S is a finite set of lattice points.

More generally, we call an intersection of finitely many half-planes in $N_{\mathbb{R}}$ a *polyhedron* and a polyhedron that is closed under addition a *cone.* Explicitly, for finitely many vectors $\{m_1, \ldots, m_s\}$ in the dual space $M_{\mathbb{R}}$ of the vector space $N_{\mathbb{R}}$,

$$\sigma = \bigcap_{i \in [s]} H^+_{m_i} = \bigcap_{i \in [s]} \{ n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geq 0 \}$$

is a convex polyhedral cone in $N_{\mathbb{R}}$. Actually, this point of view leads us to talk about *sub-cones* of a cone which are officially called *faces.*

The dimension of a polyhedron is defined as the dimension of the smallest subspace in $N_{\mathbb{R}}$ containing it. In particular, an n-dimensional cone σ has faces of dimension zero to n. Faces of dimension and codimension one in σ have special roles; we call them *rays* and *facets*, respectively.

Definition 6. *The dual cone $\hat{\sigma}$ of a convex polyhedral cone σ is*

$$\hat{\sigma} = \{m \in M_{\mathbb{R}} \mid \ \langle m, u \rangle \geq 0\}.$$

We may obtain the dual cone $\hat{\sigma}$ as an intersection of half-planes defined by the rays of σ. Hence, taking the dual preserves rationality. Following lemma states an algebraic finiteness property which yields one direction of the connection between rational polyhedral cones and affine toric varieties.

Lemma 2.3 (Gordan's Lemma)**.** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone. Then the semigroup*

$$S_\sigma = \hat{\sigma} \cap M$$

is finitely generated.

Proof. Let $\{u_1, \ldots, u_s\} \subseteq N_{\mathbb{R}}$ be the set of ray generators of $\hat{\sigma}$. Consider the zonotope Z generated by the vectors $\{\vec{u}_1, \ldots, \vec{u}_s\}$. Since Z is bounded, it contains finitely many lattice points. It is easy to check that any point in S_σ can be written as a sum of those lattice points. Thus, we get a finite set of generators for the semigroup S_σ. $\square$

The idea is to change the playground by considering the cones instead of their affine semigroup skeletons so that we gain a new tool-set of combinatorics. Let us introduce some special families of cones that translate to favourable properties in the context of toric varieties.

Definition 7. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone. σ is said to be*

- *strongly convex if the dual cone $\hat{\sigma}$ that we are working with is full dimensional, or equivalently, if $\{0\}$ is a face of σ.*

- *smooth if its minimal generators form part of a $\mathbb{Z}$ -basis of N.*
- *simplicial if its minimal generators are linearly independent over $\mathbb{R}$.*

Theorem 2.4 ([1] Theorem 1.2.18)**.** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone where $\dim(M_{\mathbb{R}}) = n$. Then,*

$$V_\sigma = Specm(\mathbb{C}[S_\sigma])$$

is an affine toric variety. Moreover, V_σ has dimension n if and only if σ is strongly convex.

Proof. We may take a finite generating set A for the semigroup algebra $\mathbb{C}[S_\sigma])$ by Gordan's Lemma. Then, V_σ is the affine toric variety V_A with the character lattice $\mathbb{Z}S_\sigma$.

V_σ has dimension n if and only if its torus is of dimension n which is equivalent to the character lattice $\mathbb{Z}S_\sigma = M$ is of rank n. Namely, $\hat{\sigma}$ has dimension n because $\mathbb{Z}S_\sigma$ is the smallest sublattice containing $\hat{\sigma}$. In other words, σ is strongly convex. $\square$

Moreover, smooth and simplicial cones correspond to smooth and simplicial affine toric varieties, respectively. From now on, we will assume strict convexity for cones.

One can think of projective or abstract varieties as a collection of affine varieties with some extra data of gluing. Accordingly, the main combinatorial structures that we are dealing with are going to be collections of cones, named *fans*.

Definition 8. *A fan $\Sigma \subseteq N_{\mathbb{R}}$ is a finite collection of polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions;*

- *Each face of a cone in Σ is in Σ.*
- *An intersection of two cones in Σ is a face of both of them.*

A fan Σ is said to be smooth or simplicial if each cone contained in Σ is smooth or simplicial, respectively.

An important class of examples is the fans arising as the duals of bounded lattice polyhedra; namely, normal fans of lattice polytopes.

Definition 9. *Let M and N be dual lattices.*

- *A lattice polytope P of a finite set $A \subseteq M$ is the set*
$$P = Conv(A) = \Big\{ \sum_{a \in A} \lambda_a a \mid \lambda_a \geq 0\,, \sum_{a \in A} \lambda_a = 1 \Big\}.$$
- *The fan Σ_P in $N_{\mathbb{R}}$ that is dual to P is called the normal fan of P.*

Note that the normal fan Σ_P consists of the dual cones $\hat{\sigma_v} \subseteq N_{\mathbb{R}}$ and their faces where $\sigma_v = \mathrm{Cone}(P - v)$ for each vertex v of P. It is clear that the faces of Σ_P covers $N_{\mathbb{R}}$. Fans satisfying this condition are called *complete*.

We denote the toric variety defined by the torus embedding given by A in 2.2 by X_P where $A = P \cap M$. Let $\{v_k\}_{k \in I} \subseteq A$ be the set of vertices of P. Since σ_{v_k} are the maximal cones in Σ_P, the affine pieces $X_P \cap U_k = \mathrm{Specm}(\mathbb{C}[S_k])$ given in 2.2 forms an affine covering

$$X_P = \bigcup_{k \in I} V_{P \cap M - v_k} = \bigcup_{k \in I} (X_P \cap U_k) \subseteq \mathbb{P}^{s-1}. \tag{2.4}$$

Example 1. *Let P be the 2-simplex in $\mathbb{R}^2$.*

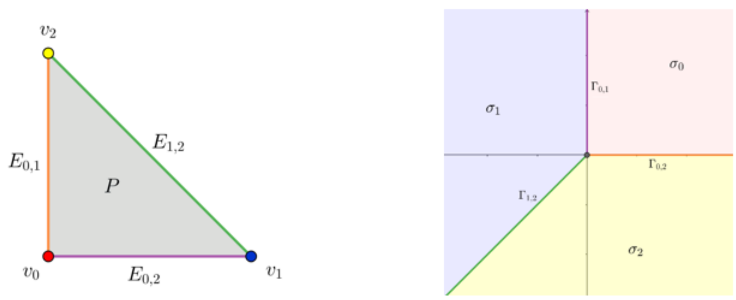

Figure 2.1. regular two-simplex P and its normal fan Σ_P.

The normal fan Σ_P consists of two dimensional cones $\sigma_0 = Cone(e_1, e_2)$, $\sigma_1 = Cone(e_2, -e_1 - e_2)$, and $\sigma_2 = Cone(e_1, -e_1 - e_2)$ corresponding to the vertices

$\{0\}$, $\{e_1\}$, $\{e_2\}$; *one dimensional cones* $\Gamma_{0,1} = Cone(e_2)$, $\Gamma_{1,2} = Cone(-e_1 - e_2)$, *and* $\Gamma_{0,2} = Cone(e_1)$ *corresponding to the edges* $\{E_{0,1}\}$, $\{E_{1,2}\}$, $\{E_{0,2}\}$; *and* $\{0\}$ *corresponding to* P *itself.*

Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan. By the theorem 2.4, each cone $\sigma \in \Sigma$ gives an affine toric variety V_σ. We take this collection of cones and glue them together along their intersections, that is determined by the face relations in the fan Σ, to obtain the toric variety X_Σ. More explicitly, the affine pieces V_{σ_1} and V_{σ_2} corresponding to the cones $\sigma_1, \sigma_2 \in \Sigma$ are glued along their intersection, which is the piece V_τ corresponding to the common face $\tau = \sigma_1 \cap \sigma_2$.

Example 2. *We will give the explicit expressions of the affine pieces and the gluing isomorphisms of the projective toric variety of the fan (of the 2-simplex) that was given in example 1. Observe that*

$$\begin{aligned} V_{\sigma_0} &\simeq Specm(\mathbb{C}[x,y]), \\ V_{\sigma_1} &\simeq Specm(\mathbb{C}[x^{-1}, x^{-1}y]), \\ V_{\sigma_2} &\simeq Specm(\mathbb{C}[xy^{-1}, y^{-1}]) \end{aligned}$$

are the maximal affine components in X_Σ *that corresponds to maximal cones in* Σ. V_{σ_i} *and* V_{σ_j} *patch together along* $V_{\Gamma_{ij}}$ *and the gluing isomorphisms of affine varieties are induced by the following isomorphisms*

$$\begin{aligned} \varphi_{0,1} &: \mathbb{C}[x,y]_x \simeq \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}}, \\ \varphi_{1,2} &: \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}} \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}}, \\ \varphi_{0,2} &: \mathbb{C}[x,y]_y \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}} \end{aligned}$$

of the coordinate rings. We may use homogeneous coordinates for (x_0, x_1, x_2). *In this case, the coordinate change* $x \mapsto x_1/x_0$, $y \mapsto x_2/x_0$ *identifies the standard affine open subsets* $U_i \subseteq \mathbb{P}^2 = X_\Sigma$.

Starting with a lattice polytope P, there are two ways to get a toric variety; via the torus embedding defined by the lattice points $P \cap M$, or via the normal fan Σ_P. Both recipes give the same toric variety when the polytope is ‘crowded enough’. Here is the sufficient conditions for this favourable situation.

Definition 10. *Let P be a lattice polytope of dimension n and let S_v denote the semigroup generated by the set $P \cap M - v$.*

- *P is called normal if for every k, $l \in \mathbb{N}$,*

$$((kP) \cap M) \oplus ((lP) \cap M) = (k+l)P \cap M.$$

- *P is said to be very ample if S_v is saturated for every vertex $v \in P$. That is, for every lattice point $m \in M$ and natural number $n \in \mathbb{N}$, $km \in S_v$ implies $m \in S_v$.*

Note that, any polytope P can be normalized by taking a multiple kP for $k \geq n-1$. Besides, every normal polytope is very ample. So, without loss of generality, we may assume that P has enough lattice points to work with. The toric variety of the lattice points of P, which is isomorphic to X_{Σ_P} (see [1] Proposition 3.1.6), is projective by construction. Conversely, if there is an embedding to the projective space from a toric variety X_Σ, one can always recover the associated polytope.

Theorem 2.5. *Let $\Sigma \subseteq N_\mathbb{R}$ be a fan, the toric variety X_Σ is projective if and only if Σ is the normal fan of a lattice polytope P.*

Proof. Let $\Sigma = \Sigma_P$ be a fan where $P = \mathrm{Conv}(A)$ for $A = \{m_1, \ldots, m_s\}$. Without loss of generality, we may assume that P is very ample. In this case, $X_\Sigma \hookrightarrow\!\!\!\rightarrow X_P \hookrightarrow \mathbb{P}^{s-1}$ by 2.4. Note that this direction also follows from one of the major results (proposition 5.3) of the section 5.1, which provides a more satisfying explanation.

Now suppose that X_Σ is projective. Then the torus T_{X_Σ} is the image of a T_N embedding and has character lattice $\mathbb{Z}'A$ for some finite set $A \subseteq M$ where M is the dual lattice of N. Define $P = \mathrm{Conv}(A)$ and let lP be a very ample multiple with the vertices $\{lv_1, \ldots, lv_t\}$. The affine open $V_{lP\cap M} = V_{\sigma_k}$ where $\hat{\sigma_k} = \mathrm{Cone}(lP \cap M - lv_k)$ for $k \in [t]$. Observing that the set $\{v_1, \ldots, v_t\}$ of vertices of P is a subset of A, $\hat{\sigma_k} = \mathrm{Cone}(P \cap M - v_k)$ is in Σ. It follows that X_P is isomorphic to X_Σ and so Σ is the normal fan of P. $\square$

We may give the major result of this section ([6] Theorem 1.6 and [4] Theorems 1.4 and 1.5) that establishes the relation between fans and toric varieties.

Theorem 2.6. *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan consisting of the strongly convex rational polyhedral cones, the toric variety X_Σ is a normal separated toric variety with the torus T_N and furthermore, all normal separated toric varieties may be seen as the toric variety of a fan of strongly convex rational cones.*

Proof. Let $\{\sigma_1, \ldots, \sigma_t\}$ be maximal cones of the fan Σ. Since each σ_k is strongly convex, the affine open V_{σ_k} is an affine toric variety with the torus T_N. Hence, X_Σ contains the torus T_N as a Zariski open subset. The gluing isomorphisms reduce to identity mapping on the intersections, so the torus actions on affine pieces are compatible and give an algebraic action on X_Σ since V_{σ_k}'s constitute an open affine covering of X_Σ.

Besides, X_Σ is irreducible and normal since the affine component V_{σ_k} is so. Applying separation lemma to the $\mathbb{C}$-algebras, we can observe that the image of an intersection $V_{\sigma_i \cap \sigma_j}$ gives a Zariski closed subset of $V_{\sigma_i} \times V_{\sigma_j}$ under the induced diagonal mapping. It follows that X_Σ is separated.

The converse is an implication of Sumihiro's theorem which states that if T_N acts on a normal separated variety X, that is not necessarily toric, each point $p \in X$ has a T_N invariant affine open neighborhood. Since X is a separated toric variety, we have a finite cover consisting of those neighborhoods that are affine toric varieties. The collection of corresponding cones forms a fan as desired. $\square$

To be able to use the dictionary, we will be working with normal toric varieties and type of fans that we introduced earlier. We can observe these theorems at work in the following basic examples.

Example 3. *Let $\Sigma = \{$faces of $\sigma\}$ for a single polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, then X_Σ is the affine toric variety associated to σ. Conversely, one can consider an affine toric variety V_σ as the variety of the fan consisting of the faces of σ.*

Example 4. *We will classify the one dimensional normal toric varieties.*

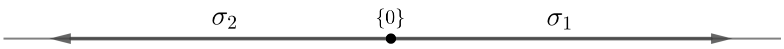

Figure 2.2. All possible cones in dimension one.

There are four possible fans:

$$\begin{aligned} \Sigma_0 &= \{0\}, \text{ then } S_0 = \mathbb{Z} \text{ and } X_{\Sigma_0} = Specm(\mathbb{C}[x, x^{-1}]) \simeq \mathbb{C}^*; \\ \Sigma_1 &= \{0, \sigma_1\}, \text{ so } S_1 = \mathbb{Z}^+ \text{ and } X_{\Sigma_1} = Specm(\mathbb{C}[x]) \simeq \mathbb{C}; \\ \Sigma_2 &= \{0, \sigma_2\}, \text{ so } S_2 = \mathbb{Z}^- \text{ and } X_{\Sigma_2} = Specm(\mathbb{C}[x^{-1}]) \simeq \mathbb{C}. \end{aligned}$$

As one can see, the first three fans give affine varieties since each of them consists of the faces of a single cone. On the other hand, $\Sigma_3 = \{0, \sigma_1, \sigma_2\}$ *is the normal fan of 1-simplex and* $X_{\Sigma_3} \simeq \mathbb{P}^1$ *has the affine open subsets* $U_1 = X_{\Sigma_1}$ *and* $U_2 = X_{\Sigma_2}$.

Example 5. *We may generalize the example 2 to higher dimensional simplexes. Let* $\Delta_n \in \mathbb{R}^n$ *be the regular n-simplex, so its normal fan has the rays generated by*

$$u_i = e_i \text{ for } i \in [n] \text{ and } u_0 = -e_1 - e_2 - \ldots - e_n$$

where any $n-1$ *of these rays generate a maximal cone. Similar to our prior example, each of the maximal cones give the affine space* $\mathbb{C}^n$ *and these pieces intersects outside of a codimension one affine space* $\mathbb{C}^{n-1}$. *It follows that the corresponding toric variety is the projective n space* $\mathbb{P}^n$.

Moreover, products of fans give the products of corresponding varieties. By this way one can get products of projective spaces by considering the products of the fans given above.

Example 6. *Let* Σ *be the fan of the regular n-simplex as in the previous example and let us define* Σ' *by only shrinking the ray generators;*

$$u_i' = u_i / q_i$$

where $q_0, \ldots, q_n$ *are relatively prime positive integers. What we did is equivalent to multiply the lengths of edges of the dual simplex or taking the quotient of the corresponding* *toric variety* $\mathbb{P}^n/\mathbb{Z}_{q_0} \oplus \ldots \oplus \mathbb{Z}_{q_n}$ *which can be identified with the weighted projective space* $\mathbb{P}(q_0, \ldots, q_n)$.

2.2. Properties of Toric Varieties

We present a summary of the bridge between algebraic geometry and combinatorics established via toric varieties in the previous section. Following results first appear in [7] and the first chapters of [6] and [4]. One can also check [1] Sections 1.3, 2.4, and 3.1 for the proofs.

Table 2.1. Toric dictionary.

Geometry	**Combinatorics**
affine toric variety V_σ	cone σ
V_σ: normal	σ: strongly convex
V_σ: smooth	σ: smooth
V_σ: simplicial	σ: simplicial
toric variety X_Σ	fan Σ
X_P: projective	P: very ample
X_P: projectively normal	P: normal
X_Σ: complete	Σ: complete
X_Σ: normal	Σ: fan of strongly convex cones
X_Σ: smooth	Σ: smooth
X_Σ: simplicial	Σ: simplicial

The table shows that many of the properties of toric varieties can be read from the associated combinatorial objects and vice versa. Clearly, smooth (i.e. nonsingular) varieties are normal and projective. On the other side, it means the smooth fans come from polytopes, that are also called smooth in the literature, and are very ample by implication. Conversely, one can generate tons of polytopes that are very ample but not smooth as normal projective toric varieties may be singular.

Example 7. *Take any non-smooth fan Σ_P such that its dual $P \subseteq M_{\mathbb{R}}$ is a polytope. We know that for some $k \in \mathbb{Z}^+$, the multiple kP must be very ample. Observe that $\Sigma_{kP} = \Sigma_P$ implies that kP is not smooth.*

In the same vein, one could ask if every smooth polytope is normal, or if every smooth X_P is projectively normal. This is one of the most important questions concerning toric varieties known as Oda's conjecture, that is still open. Yet, we know that not every projective toric variety is projectively normal. One can check [8] for examples of very ample polytopes that are not normal.

2.3. Action!

Starting with the affine case, we will take a closer look in the fixed points and orbits of the torus action, that is the very reason of the relation between geometry of toric varieties and combinatorics of fans or polytopes.

2.3.1. Fixed Points

Let V_σ be an affine toric variety with the torus T_N as in theorem 2.4. Every point $p \in V_\sigma$ corresponds to a semigroup homomorphism $\phi_p \colon S_\sigma \to \mathbb{C}$. The action

$$\begin{aligned} T_N \times V_\sigma &\to V_\sigma \\ (t,p) &\mapsto \phi_t\phi_p(m) = \chi^m(t)\phi_p(m) \end{aligned}$$

is induced by the $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[S_\sigma] = \mathbb{C}[V_\sigma] \to \mathbb{C}[T_N \times V_\sigma] = \mathbb{C}[T_N] \otimes_{\mathbb{C}} \mathbb{C}[V_\sigma] = \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[V_\sigma]$$

taking $\chi^m \mapsto \chi^m \otimes \chi^m$.

Note that the points of V_σ are in bijection with the maximal ideals of $\mathbb{C}[S_\sigma]$ and the semigroup homomorphisms from S_σ to $\mathbb{C}$. Moreover, for every $t \in T_N$, a fixed point $p \in V_\sigma$ of the action satisfies

$$p = (\varphi_t \cdot \varphi_p)(m) = \chi^m(t)\chi^m(p) = \chi^m(t \cdot p).$$

From here, one can deduce that a fixed point corresponds to the maximal ideal

$$(\chi^m : m \in S_\sigma \backslash \{0\}) \trianglelefteq \mathbb{C}[S_\sigma]$$

and to the semigroup map

$$\begin{aligned} \phi \colon S_\sigma &\to \mathbb{C} \\ m &\mapsto \begin{cases} 1 & \text{for } m \in S_\sigma \cap \sigma^\perp \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.5}$$

that is a morphism when S_σ is pointed (i.e. when $\{0\}$ is the only invertible element in S_σ). Hence, a fixed point should be unique if it exits and it exists precisely when $\sigma \subseteq M_\mathbb{R}$ is full dimensional. We call such a point a *distinguished point* and denote it by γ_σ.

Example 8. *Take* $\sigma = Cone(e_1, e_2)$*. Then,* $S_\sigma = \langle e_1, e_2 \rangle$ *and*

$$V_\sigma = Specm(\mathbb{C}[S_\sigma]) = Specm(\mathbb{C}[x, y]) \cong \mathbb{C}^2.$$

In this case $S_\sigma \cap \sigma^\perp = \{0\}$ *and the point corresponding to* ϕ *in (2.5) is*

$$p = (\phi(1, 0), \phi(0, 1)) = (0, 0),$$

that is the distinguished point. Note that p *is fixed under the torus action since* σ *is of maximal dimension. From the reasoning here, one can see that the origin is only candidate to be the fixed point in the affine case.*

One can compute the distinguished points with a backwards process as the limit point of 1-parameter subgroups.

Proposition 2.7 ([1] Proposition 3.2.2)**.** *Let* $\sigma \in N_\mathbb{R}$ *be a strongly convex rational polyhedral cone and* $u \in Relint(\sigma) = \{u \in N \mid \langle m, u \rangle > 0$ *for all* $m \in \hat{\sigma} \backslash \sigma^\perp\}$*. Then,*

$$\lim_{t \to 0} \lambda^u(t) = \gamma_\sigma \in V_\sigma.$$

Proof. $\lim_{t \to 0} \lambda^u(t)$ is the point given by

$$\begin{aligned} \phi \colon S_\sigma &\to \mathbb{C} \\ m &\mapsto \lim_{t \to 0} \chi^m(\lambda^u(t)) = \lim_{t \to 0} t^{\langle m, u \rangle}. \end{aligned}$$

Clearly the inner product is positive for any $m \in S_\sigma \backslash \sigma^\perp$, so the limit is 0. On the other hand, when $m \in S_\sigma \cap \sigma^\perp$, the inner product is 0 by definition which makes the limit 1. $\square$

2.3.2. Torus Orbits

The key observation is that the orbit of a cone $\sigma \in \Sigma$ can be seen as the orbit of its distinguished point. So,

$$\begin{aligned}\mathcal{O}(\sigma) &= T_N \cdot \gamma_\sigma \subseteq X_\Sigma \\ &= \{\phi\colon S_\sigma \to \mathbb{C} \mid \phi(m) \neq 0 \text{ if and only if } m \in \sigma^\perp \cap M\} \\ &= \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*).\end{aligned}$$

Using the proposition 2.3.1, we will compute the distinguished points for the affine pieces of $\mathbb{P}^2$ and find the orbits that contain them.

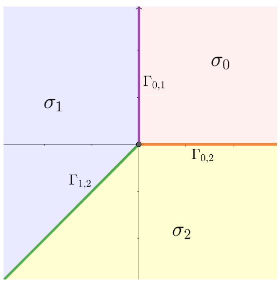

Figure 2.3. The fan of $\mathbb{P}^2$.

Example 9. *Take $X_\Sigma = \mathbb{P}^2$ with $T_N = \{(1, s, t)\colon s, t \in \mathbb{C}^*\}$. Consider $\lambda^u\colon \mathbb{C}^* \to T_N$ taking $t \mapsto (1, t^a, t^b)$. We compute the distinguished point corresponding to each cone in Σ as follows.*

- *Relint* $(\sigma_0) = \{(a, b) : a, b \in \mathbb{Z}^+\} \Rightarrow \lim_{t\to 0} \lambda^u(t) = (1, 0, 0)$.
- *Relint* $(\sigma_1) = \{a < 0,\ a < b\} \Rightarrow \big(1, t^a, t^b\big) = \big(1, t^a, t^{b-a} \cdot t^a\big) \sim (t^{-a}, 1, t^{1,a})$
  $\Rightarrow \lim_{t\to 0} \lambda^u(t) = (0, 1, 0)$.
- *Relint* $(\sigma_3) = \{b < 0, b < a\} \Rightarrow \big(1, t^a, t^b\big) \sim \big(t^{-b}, t^{a-b}, 1\big)$
  $\Rightarrow \lim_{t\to 0} \lambda^u(t) = (0, 0, 1)$.

- *Relint* $(\tau_{0,1}) = \{a = 0, 0 < b\} \Rightarrow \lim_{t\to 0} \lambda^u(t) = (1, 1, 0)$.
- *Relint* $(\tau_{0,2}) = \{b = 0, 0 < a\} \Rightarrow \lim_{t\to 0} \lambda^u(t) = (1, 0, 1)$.
- *Relint* $(\tau_{1,2}) = \{a = b < 0\} \Rightarrow \left(1, t^a, t^b\right) \sim (t^{-a}, 1, 1)$
  $\Rightarrow \lim_{t\to 0} \lambda^u(t) = (0, 1, 1)$.
- *Relint*$(\{0\}) = \{a = b = 0\} \to \lim_{t\to 0} \lambda^u(t) = (1, 1, 1)$.

We may list the orbits as follows.

- $\mathcal{O}_0 = \{(x_0, x_1, x_2) \mid x_1 = x_2 = 0, x_0 \neq 0\}$ *contains* γ_{σ_0}.
- $\mathcal{O}_1 = \{(x_0, x_1, x_2) \mid x_0 = x_2 = 0, x_1 \neq 0\}$ *contains* γ_{σ_1}.
- $\mathcal{O}_2 = \{(x_0, x_1, x_2) \mid x_0 = x_1 = 0, x_2 \neq 0\}$ *contains* γ_{σ_2}.
- $\mathcal{O}_{0,1} = \{(x_0, x_1, x_2) \mid x_2 = 0, x_0, x_1 \neq 0\}$ *contains* $\gamma_{\tau_{0,1}}$.
- $\mathcal{O}_{0,2} = \{(x_0, x_1, x_2) \mid x_1 = 0, x_0, x_2 \neq 0\}$ *contains* $\gamma_{\tau_{0,2}}$.
- $\mathcal{O}_{1,2} = \{(x_0, x_1, x_2) \mid x_0 = 0, x_1, x_2 \neq 0\}$ *contains* $\gamma_{\tau_{1,2}}$.
- $\mathcal{O}_3 = \{(x_0, x_1, x_2) \mid x_0, x_1, x_2 \neq 0\}$ *contains* $\gamma_{\{0\}}$.

For the first three orbits we have $S_\sigma \cap \sigma^\perp = \{0\}$ *and so* $\chi^m(\gamma_{\sigma_i}) = 0$ *for any* $m \neq 0$. *Hence, these orbits only contains the distinguished points. The second three orbits correspond to rays are isomorphic to* $\mathbb{C}^*$ *and finally the last orbit* $\mathcal{O}_3 = T_N \cdot \gamma_{\{0\}} = (\mathbb{C}^*)^2$.

We may now give the main result of this section ([4] Porposition 1.6).

Theorem 2.8 (Orbit- Cone correspondence)**.** *Let* Σ *be a fan and* X_Σ *be the associated toric variety as usual. Then,*

(i) *There is a bijective correspondence between the cones in* Σ *and the orbits of the torus action on* X_Σ.

(ii) $dim(\mathcal{O}(\sigma)) = dim(N_\mathbb{R}) - dim(\sigma)$

(iii) *The affine open subset associated to* σ *can be given as* $V_\sigma = \bigcup_{\tau \preceq \sigma} \mathcal{O}(\tau)$.

iv) *The affine toric variety obtained as the orbit closure of a cone* $\tau \in \Sigma$ *is*

$$\overline{\mathcal{O}(\tau)} = \bigcup_{\tau \preceq \sigma} \mathcal{O}(\sigma).$$

A proof can be found in ([1] theorem 3.2.6). We already confirmed the first two bullets on $\mathbb{P}^2$. Let us go back to our last example to see the last two bullets at work.

Example 10. *For the third bullet of the above theorem, let us look at* σ_0 *again. Its faces are* $\{0\}, \tau_{0,1}, \tau_{0,2}$*, and* σ_0*. Thus,*

$$V_{\sigma_0} = \mathcal{O}_{\{0\}} \cup \mathcal{O}_{\tau_{0,1}} \cup \mathcal{O}_{\tau_{0,2}} \cup \mathcal{O}_{\sigma_0} = \{(x_0, x_1, x_2) : x_0 \neq 0\}\,.$$

Sending $x \mapsto x_1/x_0, y \mapsto x_2/x_0$*, one could see that* $V_{\sigma_0} \simeq \mathbb{C}^2$*.*

For the last bullet, observe that the closure of of $\mathcal{O}_{i,j}$ *is the coordinate axis* $\boldsymbol{V}(x_k)$ *in* $\mathbb{P}^2$*. For instance,*

$$\begin{aligned}
\overline{\mathcal{O}_{0,1}} &= \boldsymbol{V}(x_2) \simeq \mathbb{P}^1 \textit{ contains } (1,0,0) = \gamma_{\sigma_0},\\
\overline{\mathcal{O}_{0,2}} &= \boldsymbol{V}(x_1) \simeq \mathbb{P}^1 \textit{ also contains } (1,0,0),\\
\overline{\mathcal{O}_{\{0\}}} &= \mathbb{P}^2 \textit{ that contains } (1,0,0).
\end{aligned}$$

On the other hand, $\tau_{1,2}$ *is not a face of* σ_0 *and indeed,* $\overline{\mathcal{O}_{1,2}} = \boldsymbol{V}(x_0)$ *does not contain* $(1,0,0)$.

3. DIVISORS AND GROUPS OF DIVISORS

3.1. Weil Divisors and the Divisor Class Group

As a consequence of the orbit- cone correspondence, the orbit closure of a ray $\rho \in \Sigma$ is an irreducible subvariety of codimension one in X_Σ, namely a prime divisor of X_Σ which we denote as D_ρ. Let X_Σ be a toric variety, $\{u_\rho\}_{\rho\in\Sigma(1)}$ be the set of ray generators of the fan and $m \in M$. The divisor of the character χ^m can be written as

$$\operatorname{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho.$$

Moreover, any Weil divisor that remains invariant under the T_N action can be given as a sum of those that are associated to rays of the fan. Hence,

$$\operatorname{Div}_{T_N}(X_\Sigma) = \Big\{ \sum_{\rho\in\Sigma(1)} a_\rho D_\rho \colon a\rho \in \mathbb{Z}\Big\} \simeq \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho \leq \operatorname{Div}(X_\Sigma).$$

Torus invariant divisors are precisely the class of divisors perfectly accordant with the toric data of X_Σ. For this reason, we will call them *toric divisors* to emphasise this harmony. Toric divisors help us compute the divisor class group, which is a very hard task in the general case.

Theorem 3.1 ([1] Theorem 4.1.3). *We have the exact sequence*

$$M \xrightarrow{f} Div_{T_N}(X_\Sigma) \xrightarrow{\pi} Cl(X_\Sigma) \to 0. \tag{3.1}$$

$$m \longmapsto div(\chi^m) \longmapsto [div(\chi^m)]$$

Furthermore, if X_Σ has no torus factor, then we can complete it to the exact sequence

$$0 \to M \hookrightarrow Div_{T_N}(X_\Sigma) \twoheadrightarrow Cl(X_\Sigma) \to 0. \tag{3.2}$$

Proof. It is enough to show that the map π is surjective. Since D_ρ are the irreducible components of $X_\Sigma \backslash T_N$, we have the exact sequence

$$\operatorname{Div}_{T_N}(X_\Sigma) \xrightarrow{p} Cl(X_\Sigma) \xrightarrow{r} Cl(T_N) \to 0.$$

Note that $\mathbb{C}[T_N] = \mathbb{C}[M]$ is a unique factorization domain since M is a free abelian group. In this case, every codimension one prime ideal in $\mathbb{C}[M]$ is principal, which implies that every prime divisor is principal. It follows that $Cl(T_N)$ is trivial and $p = \pi$ is surjective.

For the second part, suppose that $\operatorname{div}(\chi^m)$ is the zero divisor for some $m \in M$. Then, $\langle m, u_\rho \rangle$ for each $\rho \in \Sigma(1)$. In this case $m = 0$ if and only if u_ρ span $N_\mathbb{R}$, that is equivalent to X_Σ having no torus factors. $\square$

Corollary 3.2. *The divisor class group $Cl(X_\Sigma)$ is the cokernel of the map*

$$A\colon \mathbb{Z}^{rank(M)} \to \mathbb{Z}^{|\Sigma(1)|},$$

that is induced by the map f in (3.1). In particular, $Cl(X_\Sigma)$ is a finitely generated abelian group.

Example 11. *Take $X_\Sigma = \mathbb{P}^n$, ray generators of Σ are $u_i = e_i$ for $i \in [n]$ and $u_0 = -e_1 - e_2 - \ldots - e_n$. Accordingly, the exact sequence (3.1) becomes*

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^{n+1} \to \mathbb{Z} \to 0,$$

from which, we conclude that $Cl(\mathbb{P}^n) \cong \mathbb{Z}$.

3.2. Cartier Divisors and the Picard Group

A Cartier divisor is a locally principal Weil divisor on a normal variety. Notice that the divisor of a character is principal and hence is Cartier. Recall that the group of Cartier divisors modulo principal divisors give the Picard group $Pic(X_\Sigma)$ since X_Σ is clearly integral as $\mathbb{C}(S_\sigma)$ are integral domains. So, using the natural inclusions of Cartier divisors into the Weil divisors and the Picard group into the class group, we obtain the Cartier version of the above exact sequence as in [2] Proposition 3.4.1, which is useful for computing the Picard group in the toric case.

Theorem 3.3. *The sequence*

$$M \xrightarrow{f} CDiv_{T_N}(X_\Sigma) \xrightarrow{\pi} Pic(X_\Sigma) \to 0 \tag{3.3}$$

is exact and can be completed to the exact sequence

$$0 \to M \hookrightarrow CDiv_{T_N}(X_\Sigma) \twoheadrightarrow Pic(X_\Sigma) \to 0$$

if and only if X_Σ has no torus factors.

Picard group is a subgroup of the class group, so it is also a finitely generated abelian group. Moreover, it follows from the orbit- cone correspondence that $\mathrm{Pic}(X_\Sigma)$ is torsion free if the fan Σ contains a cone of maximal dimension.

Next, we compare the two exact sequences. For instance, on a smooth variety, the two sequences become the same as two classes of divisors coincide.

Proposition 3.4 ([1] Proposition 4.2.6)**.** *Every Weil divisor is Cartier on X_Σ if and only if X_Σ is smooth.*

Proof. Note that the second direction is true in the general case. Weil and Cartier divisors coincide on any smooth variety. We will explain the first direction, that is special to the toric case.

Assuming $Cl(X_\Sigma) = Pic(X_\Sigma)$, we have $Cl(V_\sigma) = Pic(V_\sigma) = 0$ for any $\sigma \in \Sigma$. Hence, the map

$$M \twoheadrightarrow CDiv_{T_N} \cong \mathbb{Z}^s$$

$$m \mapsto \operatorname{div}(\chi^m) = \sum_{i=1}^{s} \langle m, u_{\rho_i} \rangle D_{\rho_i} \mapsto (\langle m, u_{\rho_1} \rangle, \ldots, \langle m, u_{\rho_s} \rangle)$$

becomes surjective. It follows that $u_{\rho_1}, \ldots u_{\rho_s}$ can be extended to a basis of N, which means that V_σ is smooth. We deduce that X_Σ is smooth since Σ is a smooth fan. □

We can give a combinatorial criteria to determine whether a given toric divisor is Cartier, that is also a solution to [2] Exercise 3.3.5.

Theorem 3.5 ([1] Theorem 4.2.8)**.** *Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a toric divisor on X_Σ. Then, D is Cartier if and only if for each $\sigma \in \Sigma$, there is a point $m_\sigma \in M$ satisfying*

$$\langle m_\sigma, u_\rho \rangle = -a_\rho \tag{3.4}$$

for all rays $\rho \in \sigma(1)$.

Proof. Suppose that D Cartier. Recall that D is principal on the affine open V_σ for all $\sigma \in \Sigma$, by definition. Then, $D\,|_{V_\sigma}$ is a toric Cartier divisor on V_σ. It follows from (3.3) and the orbit- cone correspondence that $D\mid_{V_\sigma}$ is given by a character m_σ. Expanding the sums, $\operatorname{div}(\chi^{m_\sigma})\mid_{V_\sigma}= D\mid_{V_\sigma}$ implies $\langle m_\sigma, u_\rho\rangle = -a_\rho$.

Conversely, let $m_\sigma \in M$ be a point satisfying (3.4). Then,

$$D\mid_{V_\sigma}= \operatorname{div}(\chi^{-m_\sigma})|_{V_\sigma}$$

for every $\sigma \in \Sigma$, which shows that D is principle on affine open sets associated to the cones in the fan. Hence, D is a toric Cartier divisor with the data $\{(V_\sigma, \chi^{-m_\sigma})\}_{\sigma\in\Sigma} \doteq \{m_\sigma\}_{\sigma\in\Sigma}$. □

3.3. Support Functions

There is one other way of describing the toric Cartier divisors which is more favourable in terms of computations.

Definition 11. *Let $\Sigma \in N_{\mathbb{R}}$ be a fan. A function*

$$\varphi\colon \operatorname{sup}(\Sigma) = |\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma \to \mathbb{R}$$

is called a support function if it is linear on each cone $\sigma \in \Sigma$. A support function φ is said to be integral (with respect to the lattice N) if it takes the lattice points to $\mathbb{Z}$, i.e. the lattice points in $\mathbb{R}$. We denote the set of such support functions as $SF(\Sigma)$.

Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a Cartier divisor with data $\{m_\sigma\}_{\sigma\in\Sigma_{\max}}$ such that

$$\langle m_\sigma, u_\rho\rangle = -a_\rho$$

for $\rho \in \sigma(1)$. Then, this data can be encoded into the integral support function

$$\begin{aligned} \varphi_D : |\Sigma| &\longrightarrow \mathbb{R} \\ u &\longmapsto \langle m_\sigma, u\rangle \end{aligned}$$

where σ is the maximal cone containing $u \in N$. Clearly, we have $\varphi_D\,(u_\rho) = -a_\rho$ for all rays $\rho \in \sigma(1)$. So, we can write the divisor D as

$$D = -\sum_\rho \varphi_D\,(u_\rho)\,D_\rho.$$

Theorem 3.6 ([3], II.6.4)**.** *There is a one-to-one correspondence between the toric Cartier divisors and integral support functions on X_Σ.*

Proof. Obviously, $D \mapsto \varphi_D$ gives an injective homomorphism

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \to \mathrm{SF}(\Sigma).$$

We will show that it is also surjective. Let $\varphi \in \mathrm{SF}(\Sigma)$ and $\sigma \in \Sigma$ be arbitrary. By integrality, it gives a linear semigroup map $\varphi|_{\sigma \cap N} : \sigma \cap N \to \mathbb{Z}$, that can trivially be extended to a linear map $\varphi_\sigma : \mathrm{Span}(\sigma) \cap N \to \mathbb{Z}$. One can see that

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Span}(\sigma) \cap N, \mathbb{Z}\right) \simeq M/(\sigma^\perp \cap M).$$

Hence, there exist $m_\sigma \in M$ satisfying $\left.\varphi\right|_\sigma(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$ which suggests the Cartier divisor $D = -\sum_\rho \varphi_D\left(u_\rho\right) D_\rho$ as the preimage of φ. □

Here is an instance to see how the support functions make life easier.

Proposition 3.7. *Let $X_\Sigma = V_\sigma$ be an affine toric variety, then it has a trivial Picard group.*

Proof. Since Σ consists of σ and its faces, every piecewise linear function defined on Σ is in fact linear. Hence, for any $\varphi_D \in SF(\Sigma)$ and $\tau \in \Sigma$, $m_\tau = m \in M$. That is to say $D = \mathrm{div}(\chi^m)$ and so $\mathrm{CDiv}_{T_N} \cong M$. It follows from (3.3) that $Pic(X_\Sigma) = 0$. □

As an immediate corollary of this result and proposition 3.4, affine smooth toric varieties have trivial class group.

4. TORIC VARIETIES AS QUOTIENTS

4.1. Quotient Presentation

One can describe toric varieties as the class of algebraic varieties arising as categorical quotients of smooth quasiaffine varieties. Our main purpose is to present a given toric variety as a quotient

$$X_\Sigma = (\mathbb{C}^r \backslash Z)/G$$

where Z is an exceptional set and G is a group acting on $\mathbb{C}^r \backslash Z$. Note that we will be working with the toric varieties without torus factors since it reproduces the general case.

We give a short recipe to this end. We will choose the smallest affine space that is large enough to guarantee that the quotient contains the given toric variety X_Σ. The group G will be a subgroup of the torus $(\mathbb{C}^*)^r$ that is an invariant of X_Σ. What is remained is to cut out the bad points. To do that, we choose the exceptional set Z as the set of points that are *unstable* under the action. Interested reader may see [9], [10] (chapter 12), [11] and [12] (chapter II.10) for the details of this construction from different perspectives.

Let X_Σ be a toric variety without torus factor. Let us denote T_N by its torus and $r = |\Sigma(1)|$ by the number of rays in the fan Σ. Applying $\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{C}^*)$ to (3.2), we get the exact sequence of algebraic groups

$$1 \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Cl}\left(X_\Sigma\right), \mathbb{C}^*\right) \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1.$$

We set

$$G = G(\Sigma) = \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Cl}\left(X_\Sigma\right), \mathbb{C}^*\right) \leq (\mathbb{C}^*)^{\Sigma(1)}, \tag{4.1}$$

that is going to be the group that we take the quotient by. Notice that G is an invariant of X_Σ as it has $Cl(X_\Sigma)$ as its character group. It follows that G is a product of a torus and a finite abelian group and it is reductive in particular.

Just as the group G, the affine space that we will work on also depends on the rays of Σ. From the exact sequence given above, we observe that the affine space $\mathbb{C}^{\Sigma(1)}$, which is the closure of the torus $(\mathbb{C}^*)^{\Sigma(1)}$, is suitable to work with. We call its coordinate ring

$$R = \mathbb{C}\left[x_\rho \colon \rho \in \Sigma(1)\right] \tag{4.2}$$

the *Cox ring* of X_Σ. We will now describe the exceptional set using the Cox ring.

Let us first define a monomial associated to a cone $\sigma \in \Sigma$ as

$$x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho. \tag{4.3}$$

Then, we take the irrelevant ideal $B(\Sigma) = (x^{\hat{\sigma}} \colon \sigma \in \Sigma_{\max})$ generated by these monomials and the exceptional set as its vanishing locus

$$Z(\Sigma) = \mathbf{V}(B(\Sigma)) \subseteq \mathbb{C}^{\Sigma(1)}, \tag{4.4}$$

that is the union of coordinate subspaces. Note that our quotient may be seen as a GIT quotient when X_Σ is projective. In this case, the set of semistable points under the linearization of the action of G by a character associated to an ample divisor class is exactly what is left after cutting out the irrelevant locus ([13], passim).

Observe that the irrelevant ideal $B(\Sigma)$ controls all the relations of cones in the fan Σ except for the information about the rays. So, $B(\Sigma)$ together with the set of rays $\Sigma(1)$ uniquely determines the fan and consequently associated toric variety. Moreover, $\Sigma(1)$ is controlled by the group G and the affine space $\mathbb{C}^{\Sigma(1)}$. Therefore, the triple $(\mathbb{C}^{\Sigma(1)}, \operatorname{Hom}_{\mathbb{Z}}\left(\operatorname{Cl}\left(X_\Sigma\right), \mathbb{C}^*\right), \mathbf{V}(B(\Sigma))$ encodes the characteristics of the toric variety X_Σ.

There is the algebraic action $(\mathbb{C}^*)^{\Sigma(1)} \curvearrowright \mathbb{C}^{\Sigma(1)}$ of diagonal matrices. Then, the subgroup $G \leq (\mathbb{C}^*)^{\Sigma(1)}$ acts on $\mathbb{C}^{\Sigma(1)}$ by

$$g \cdot t = \left(g\left([D_\rho]\right) t_\rho\right)$$

where $g \colon Cl(X_\Sigma) \to \mathbb{C}^*$ and $t = (t_\rho)_{\rho \in \Sigma(1)} \in \mathbb{C}^{\Sigma(1)}$. Observe that the exceptional set $Z(\Sigma)$ consists of the orbits of the monomials $x^{\hat{\sigma}}$, so the G action restricts to $\hat{X}_\Sigma \doteq \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma)$ since it is clearly G invariant in $\mathbb{C}^{\Sigma(1)}$.

Theorem 4.1 ([9] Theorem 2.1, [1] Theorem 5.1.11)**.** *Let X_Σ be a toric variety without torus factors and $\pi : \hat{X}_\Sigma \to X_\Sigma$. Then,*

(i) π is constant on G orbits.

(ii) π is an almost geometric quotient for the action of G on $\hat{X}_\Sigma$. Thus

$$X_\Sigma \simeq \hat{X}_\Sigma // G.$$

(iii) π is a geometric quotient if and only if X_Σ is simplicial.

Proof. Let us define $\tilde{\sigma} = \text{Cone}(e_\rho \mid \rho \in \sigma(1)) \subseteq \mathbb{R}^{\Sigma(1)}$ for each core $\sigma \in \Sigma$ and $\tilde{\Sigma} = \{\tilde{\sigma} \text{ for } \sigma \in \Sigma \text{ and their faces}\}$. Observe that $\tilde{\Sigma}$ is a subfan of $\tilde{\Sigma}_0 = \{ \text{ faces of Cone}(e_\rho \mid \rho \in \Sigma(1))\}$, that is the fan of $\mathbb{C}^{\Sigma(1)}$. So the toric variety $X_{\tilde{\Sigma}}$ can be obtained by removing the orbits of the cones $\tilde{\Sigma}_0 \backslash \tilde{\Sigma}$ from $\mathbb{C}^{\Sigma(1)}$. We deduce that $X_{\tilde{\Sigma}} = \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma) = \hat{X}_\Sigma$.

Now recall that

$$1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \xrightarrow{\pi} T_N \to 1$$

is an exact sequence where π is induced by the lattice map $\bar{\pi} \colon \mathbb{Z}^{\Sigma(1)} \to N$ taking $e_\rho \mapsto u_\rho$. The map π naturally extends to $\pi : \hat{X}_\Sigma \to X_\Sigma$ which is constant on G orbits since $G = \text{Ker}(\pi)$ in the above exact sequence.

Rest of the proof is rather too long and technical to sketch here. One can consult with [9] and [1] for the details. $\square$

Example 12. *Take $X_\Sigma = \mathbb{P}^n$. We have the ray generators $u_i = e_i$ for $i \in [n]$ and $u_0 = -e_1 - e_2 - \ldots - e_n$. In this case, the number of rays is $|\Sigma(1)| = n + 1$. The maximal cones of Σ are $\sigma_i = Cone(u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$ that are associated to the monomials x_i. Then $B(\Sigma) = (x_0, \ldots, x_n)$ and so*

$$Z(\Sigma) = \boldsymbol{V}(B(\Sigma)) = \{0\}.$$

Moreover, one can compute the class group $Cl(\mathbb{P}^2) \cong \mathbb{Z}$, which implies

$$G = Hom_{\mathbb{Z}}(Cl(\mathbb{P}^2), \mathbb{C}^*) \cong \mathbb{C}^*.$$

Therefore, we may write the geometric quotient

$$\mathbb{P}^2 \cong \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma)/G \cong \mathbb{C}^3 \backslash \{0\}/\mathbb{C}^*.$$

Example 13. *Take $X_\Sigma = \mathbb{P}^n \times \mathbb{P}^m$ and let us write $\Sigma = \Sigma_n \times \Sigma_m$. Let $\rho_1, \ldots, \rho_{n+1}$ and $\rho'_1, \ldots, \rho'_{m+1}$ denote the rays coming from Σ_n and Σ_m, respectively. One can observe that $B(\Sigma) = (x_1, \ldots, x_{n+1}) \cap (y_1, \ldots, y_{m+1})$, hence*

$$Z(\Sigma) = \boldsymbol{V}(x_1, \ldots, x_{n+1}) \cup \boldsymbol{V}(y_1, \ldots, y_{m+1}) = \{0\} \times \mathbb{C}^{n+1} \cup \mathbb{C}^{n+1} \times \{0\}.$$

Besides, $Cl(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z} \oplus \mathbb{Z}$ so $G \cong (\mathbb{C}^*)^2$ and

$$|\Sigma(1)| = |\Sigma_n(1)| + |\Sigma_m(1)| = n + m + 2.$$

Therefore, we get the quotient presentation

$$\begin{aligned} \mathbb{P}^n \times \mathbb{P}^m &\cong \mathbb{C}^{n+m+2} \backslash (\{0\} \times \mathbb{C}^{n+1} \cup \mathbb{C}^{n+1} \times \{0\})/(\mathbb{C}^*)^2 \\ &\cong \mathbb{C}^{n+1} \backslash \{0\}/\mathbb{C}^* \times \mathbb{C}^{m+1} \backslash \{0\}/\mathbb{C}^*. \end{aligned}$$

Example 14. *Take $X_\Sigma = \mathbb{P}(q_0, \ldots, q_n)$ weighted projective space where q_i are positive integers with $gcd(q_0, \ldots, q_n) = 1$. Σ consists of the cones generated by the proper subsets of $\{u_0, \ldots, u_n\}$ where u_i are primitive elements of*

$$N = \mathbb{Z}^{n+1}/(q_0, \ldots, q_n)\mathbb{Z}.$$

Just as in the case of projective spaces, we have $|\Sigma(1)| = n+1$, $Z(\Sigma) = \{0\}$. Consider the exact sequence

$$\begin{gathered} 0 \to M \to \mathbb{Z}^{n+1} \twoheadrightarrow Cl(X_\Sigma) \to 0 \\ m \mapsto (\langle m, u_i \rangle)_i \mapsto \sum_{i=0}^{n} \langle m, u_i \rangle q_i \end{gathered}$$

taking $m \in M$ to the trivial class in $Cl(X_\Sigma)$ since $\left\langle m, \sum_{i=0}^{n} u_i q_i \right\rangle = \langle m, 0 \rangle = 0$. It follows that $Cl(X_\Sigma) = \mathbb{Z}$ and so $G = \mathbb{C}^*$. Note that in this case, G acts on $\mathbb{C}^{n+1}$ by

$$t \cdot (a_0, \ldots, a_n) = (t^{q_0} a_0, \ldots, t^{q_n} a_n).$$

Finally, since $\mathbb{P}(q_0, \ldots, q_n)$ is simplicial, we have the geometric quotient

$$\mathbb{P}(q_0, \ldots, q_n) \cong \mathbb{C}^{n+1} \backslash \{0\}/\mathbb{C}^*.$$

4.2. Cox Rings

The role of the Cox ring R, that was introduced in the previous section, of a toric variety X_Σ may be possibly underestimated at the first glance. Although, it is the key object on the algebraic side for the generalization of the algebra-geometry dictionary of affine and projective varieties to the toric case. Moreover, many aspects of the geometric structure of a toric variety can be given by its Cox ring. In [1], the authors call R *the total coordinate ring* instead, while the name *Cox ring* is rather special to and vastly used in the context of *Mori Dream spaces.* In general, for a normal variety X, Cox ring is defined as the graded ring over $Cl(X)$ where the graded pieces are the global sections $\Gamma(\mathcal{O}_X(D))$. When X is a toric variety, this ring is nothing but $R = \mathbb{C}[x_\rho\colon \rho \in \Sigma(1)]$. Furthermore toric varieties may be characterized as the varieties, or Mori dream spaces, having polynomial rings as Cox rings [14].

As stated by Dolgachev (in [15] section 1.2);

> "It is well known that a $\mathbb{Z}$-grading of a commutative ring is equivalent to an action of a one dimensional algebraic torus on its spectrum."

Since $Cl(X_\Sigma)$ is the character group of G that appears in the quotient construction, it generalizes to the case of toric varieties as the G-action on the $\mathrm{Specm}(R) = \mathbb{C}^{\Sigma(1)}$ being equivalent to a $Cl(X_\Sigma)$-grading on R. Hence,

$$R = \bigoplus_{\alpha \in Cl(X_\Sigma)} R_\alpha$$

such that $R_\alpha \cdot R_\beta \subseteq R_{\alpha+\beta}$ for any $\alpha, \beta \in Cl(X_\Sigma)$.

As in the classical case of projective and weighted projective varieties, the degree of polynomials in R is what remains invariant under the induced action of G on R. Let us take a monomial

$$f = \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho} = x^a \in R.$$

One can see that the orbit of f under the G action gives divisors of the same class.

Accordingly, the degree of f is defined as the class

$$\Big[\sum_{\rho\in\Sigma(1)} a_\rho D_\rho\Big] \in Cl(X_\Sigma)$$

and a polynomial $f \in R_\alpha$ is called G-homogeneous of degree α.

Example 15. *For an affine space $\mathbb{C}^n$, one can notice that the Cox ring has the trivial grading as the class group is trivial. It also means that every polynomial is trivially homogeneous in the generalized sense in this context.*

Example 16. *Take $X_\Sigma = \mathbb{P}^n$, then $R = \mathbb{C}[x_0, x_n]$ and $G = \mathbb{C}^*$. The grading is given by the lattice map $Div_{T_N}(\mathbb{P}^n) \cong \mathbb{Z}^{n+1} \to \mathbb{Z}_n \cong Cl(\mathbb{P}^n)$ that takes $(a_0, \ldots, a_n) \mapsto \sum a_i$.*

Example 17. *For a weighted projective space $\mathbb{P}^n(q_0, \ldots, q_n)$, everything is the same with the previous example except for the lattice map taking $(a_0, \ldots, a_n) \mapsto \sum q_i a_i$ which gives the grading on the Cox ring R.*

Example 18. *Take $\mathbb{P}^n \times \mathbb{P}^m$ and $R = \mathbb{C}[x_0, \ldots x_n, y_0, \ldots y_m]$ as its Cox ring. R is graded over $\mathbb{Z}^2$ where $deg(x_i) = (1, 0)$ and $deg(y_j) = (0, 1)$.*

We may state a weak correspondence for subvarieties of a toric variety X_Σ and ideals of the Cox ring using the quotient map π, as in [1] Section 5.2, without the information of an embedding to an affine, projective or weighted projective space. If π is a geomeric quotient, in other words, if X_Σ is simplicial, we have a toric ideal-variety correspondence ([9] Proposition 2.4) that generalizes the affine and projective cases.

Theorem 4.2. *Let X_Σ be a toric variety, R be its Cox ring and $\pi : \hat{X}_\Sigma \to X_\Sigma$ be the almost geometric quotient as introduced. Then,*

(i) $I \trianglelefteq R$ is a G-homogeneous ideal if and only if

$$V_I = \pi(\widehat{\mathbf{V}(I)}) \doteq \pi(\mathbf{V}(I) \cap \hat{X})$$

is a closed subvariety of X_Σ.

(ii) There is a one-to-one correspondence between the closed subvarieties $V \subseteq X_\Sigma$ and radical G-homogeneous ideals $I \subseteq B(\Sigma) \subseteq R$ if X_Σ is simplicial.

Proof. When $I \trianglelefteq R$ is G-homogeneous, $\mathbf{V}(I) \subseteq \mathbb{C}^{\Sigma(1)}$ is closed and invariant under the G action. Clearly, $\widehat{\mathbf{V}}(I)$ is also closed in $\hat{X}_\Sigma$ and G invariant, since $Z(\Sigma)$ is so. Hence, it maps to a closed algebraic set in X_Σ under a good categorical quotient. Conversely, take a closed subset Y of X_Σ. Then,

$$I_Y = \{f \in R \colon f \text{ is homogeneous and } Y \subseteq V_{(f)}\}$$

can be easily seen to be a G-homogeneous ideal. I_Y can be written as the union of ideals corresponding to the closed sets in X_Σ containing Y. So, $\pi(\mathbf{V}(I)) = Y$.

For the second part, suppose that π is a geometric quotient. Obviously, any ideal $I \trianglelefteq R$ containing $B(\Sigma)$ corresponds to an affine variety $\mathbf{V}(I) \subseteq Z(\Sigma)$ which maps to the empty set in X_Σ. Noting that the irrelevant ideal is radical, there is a one-to-one correspondence between radical ideals I contained in $B(\Sigma)$ and G invariant subvarieties of $\mathbb{C}^{\Sigma(1)}$ containing the irrelevant locus by the classical ideal- variety correspondence. The latter corresponds to G invariant closed subsets of $\hat{X}_\Sigma$ which is exactly the subvarieties of X_Σ since π is submersive. $\square$

One can also describe the homogeneous coordinates using the theorem above. Note that the local coordinates for the affine patches associated to the cones in the fan are compatible with the homogeneous coordinates given by the quotient by the G-action. Let X_Σ be a toric variety of dimension n and $\sigma \in \Sigma$ be smooth. Then, we have the commutative diagram

$$\begin{tikzcd} \mathbb{C}^{\sigma(1)} \arrow[r, hook, "\phi_\sigma"] \arrow[d, two heads, "\pi"] & \hat{X}_\Sigma \arrow[d, two heads, "\pi"] \\ U_\sigma \arrow[r, hook, "i"] & X_\Sigma \end{tikzcd}$$

of toric varieties where i is given by the inclusion and

$$\phi_\sigma \colon (a_\rho)_{\rho \in \sigma(1)} \longmapsto (b_\rho)_{\rho \in \Sigma(1)} = \begin{cases} a_\rho & \text{for } \rho \in \sigma(1) \\ 1 & \text{otherwise} \end{cases}$$

for every smooth $\sigma \in \Sigma_{\max}$. In this case, for a subvariety $V_I = \pi(\mathbf{V}(I))$ for $I \trianglelefteq R$, the intersection $V_I \cap U_\sigma$ is associated to the dehomogenization $\tilde{I} \trianglelefteq \mathbb{C}[x_\rho \colon \rho \in \sigma(1)]$ of I by setting $x_\rho = 1$ for the rays not contained in σ.

5. SHEAVES AND LINE BUNDLES

5.1. Toric Divisors and Line Bundles

The notions of line bundles and Cartier divisors coincide on a normal variety. We stick with the language of divisors to be able to go back and forth between the geometric and combinatorial sides. In [16] (section 4), authors focus on the line bundles linearized with respect to the T_N action as they note that every line bundle has a T_N-linearization. T_N-linearized sheaves (resp. invertible sheaves, i.e. line bundles) correspond to the toric divisors (resp. toric Cartier divisors) on a complete toric variety (see [4] section 2.2). We use support functions and polytopes associated to toric Cartier divisors as two main tools to characterize the line bundles on a toric variety having desirable geometric properties such as ampleness.

Proposition 5.1 ([1] Proposition 4.3.2)**.** *Let D be a toric Weil divisor on X_Σ. Then, the global sections of the sheaf $\mathscr{O}_{X_\Sigma}(D)$ of $\mathscr{O}_{X_\Sigma}$-modules can be given as the direct sum*

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{m \models \star} \mathbb{C} \cdot \chi^m \tag{5.1}$$

over $m \in M$ satisfying the condition $\star \doteq div(\chi^m) + D \geq 0$.

Proof. For any f in the global sections of $\mathscr{O}_{X_\Sigma}(D)$, we know that $\operatorname{div}(f) + D \geq 0$. Restricting it to T_N, we observe that $f \in \mathbb{C}[M]$ since $\mathbb{C}[M]$ is the coordinate ring of the torus. Moreover, $\Gamma\left(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)\right) \subseteq \mathbb{C}[M]$ is invariant under the T_N-action on $\mathbb{C}[M]$ since D is toric. Therefore, the global sections is generated by the characters $\chi^m \in \Gamma\left(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)\right)$ which are precisely the characters of $m \in M$ satisfying $\star$. $\square$

Let us look at the condition $\star$ more closely. Let $D = \sum\limits_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a toric divisor and $m \in M$. If we write it explicitly, the condition $\star$ is equivalent to

$$\langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1).$$

Let P_D be the set of points $p \in M_\mathbb{R}$ satisfying above condition. Since there are finitely many non-zero a_ρ, P_D is obtained as an intersection of finitely many half-spaces and so is a polyhedron. If Σ is complete, P_D is bounded (i.e. a polytope) for any toric divisor D ([2] Proposition 3.4.2). Although, it may not be a polytope in the general case.

Let D, D' be two toric divisors of the same class. So, $D = D' + \operatorname{div}(\chi^m)$ for some $m \in M$. The map $f \mapsto f\chi^m$ induces an isomorphism

$$\Gamma\left(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)\right) \simeq \Gamma\left(X_\Sigma, \mathscr{O}_{X_\Sigma}(D')\right),$$

which implies $P_{D'} = P_D + m$ on the combinatorial side. It follows that P_D is an invariant of the class $[D]$ up to translation. The polyhedron P_D is a useful tool to determine whether a divisor (respectively, a divisor sheaf) has favourable properties.

Proposition 5.2 ([1] Proposition 6.1.1)**.** *Let X_Σ be a toric variety with a full dimensional fan Σ and $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a toric Cartier divisor with the data $\{m_\sigma\}_{\sigma \in \Sigma_{\max}}$. Then, $\mathscr{O}_{X_\Sigma}(D)$ is generated by the global sections (i.e. D is base point free) if and only if $m_\sigma \in P_D$ for all $\sigma \in \Sigma_{\max}$.*

Proof. For any $\sigma \in \Sigma_{\max}$, the character χ^{m_σ} gives a global section s with divisor of zeros $\operatorname{div}_0(s) = D + \operatorname{div}(\chi^{m_\sigma})$ since $m_\sigma \in P_D$. Noting that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for $\rho \in \sigma(1)$, one can deduce s is non-vanishing on the affine piece U_σ as $\operatorname{div}_0(s)$ does not contain it. It proves the second direction since $\{U_\sigma\}_{\sigma \in \Sigma_{\max}}$ is a covering for X_Σ.

Now, suppose that D is base point free and $\sigma \in \Sigma_{\max}$. Recall that

$$\mathcal{O}(\sigma) = \{\gamma_\sigma\} = \bigcap_{\rho \in \sigma(1)} D_\rho$$

by theorem 2.8. In this case, there is a global section s such that the support of $\operatorname{div}_0(s)$ does not contain γ_σ. By the previous proposition, s is given by χ^m for some $m \in P_D \cap M$. Hence,

$$\operatorname{div}_0(s) = D + \operatorname{div}(\chi^m) = \sum_\rho \left(a_\rho + \langle m, u_\rho \rangle\right) D_\rho.$$

Since $\gamma_\sigma \in D_\rho$ for any ray $\rho \in \sigma(1)$, $a_\rho + \langle m, u_\rho \rangle = 0$ for $\rho \in \sigma(1)$. σ being full dimensional implies that $m_\sigma = m \in P_D$. $\square$

Let us give an example to visualize it.

Example 19. *Let X_Σ be the Hirzeburch surface $\mathscr{H}_2$. The corresponding fan Σ consists of the maximal cones $\sigma_1 = Cone(e_1, e_2)$, $\sigma_2 = Cone(e_1, -e_2)$, $\sigma_3 = Cone(-e_2, 2e_2 - e_1)$, and $\sigma_4 = Cone(2e_2 - e_1, e_2)$ and their faces.*

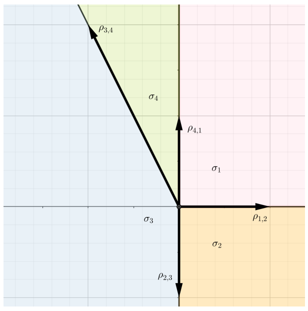

Figure 5.1. Fan of the Hirzeburch surface $\mathscr{H}_2$.

Lets take $D = D_{\rho_{2,3}}$ and $E = D_{\rho_{2,3}} + D_{\rho_{4,1}}$ with the Cartier data $\{m_i\}_{i\in[4]}$ and $\{n_i\}_{i\in[4]}$, respectively. We will compute the numbers m_i and n_i to check whether these toric divisors have base points. By (3.4), we have

$$\langle m_1, u_{\rho_{4,1}}\rangle, \langle m_1, u_{\rho_{1,2}}\rangle = 0,$$
$$\langle m_1, e_2\rangle, \langle m_1, e_1\rangle = 0,$$
$$\therefore\ m_1 = (0,0).$$

Similarly, the equations

$$\langle m_2, e_1\rangle = 0, \qquad \langle m_2, -e_2\rangle = -1,$$
$$\langle m_3, -e_2\rangle = -1, \qquad \langle m_3, 2e_2 - e_1\rangle = 0,$$
$$\langle m_4, 2e_2 - e_1\rangle = 0, \qquad \langle m_4, e_2\rangle = 0$$

show that $m_2 = (0,1)$, $m_3 = (2,1)$, and $m_4 = (0,0)$. If we repeat the process for n_i, we obtain $n_1 = (0,-1)$, $n_2 = (0,1)$, $n_3 = (2,1)$, and $n_4 = (-2,-1)$. Notice that this computation also shows that both P_D and P_E are given by the same supporting hyperplanes with a different ordering. So,

$$P_D = P_E = Conv((0,0),(0,1),(2,1)).$$

We deduce that D is basepoint free while E is not as P_D contains all m_i and on the other hand, $n_1, n_4 \notin P_E$.

Moreover, when P_D is a polytope, the points m_σ lie on the supporting hyperplanes of P_D. So, for base point free divisors, the relation $D \mapsto P_D$ is clearly one-to-one. On the other hand, its inverse can be given using the correspondence of toric Cartier divisors and integral support functions. The toric Cartier divisor associated to a given polytope $P \subseteq M_{\mathbb{R}}$ is defined as the divisor D_{φ_P} of the integral support function

$$\varphi_P(u) = \min(\langle m, u\rangle \mid m \in P).$$

Note that $D_{P_D} = D$ and $P_{D_P} = P$ by construction when P_D is a polytope.

Proposition 5.3 ([1] Proposition 6.1.10)**.** *$P \subseteq M_{\mathbb{R}}$ is a full dimensional polytope (resp. very ample) if and only if D_P is an ample (resp. very ample) base point free divisor on X_{Σ_P}.*

Proof. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional polytope, then D_P is well defined and is base point free by the previous proposition. Lets choose $k \in \mathbb{Z}$ big enough so that $kP = Q$ is very ample. Clearly, D_Q is also base point free, so the global sections $\Gamma(X_{\Sigma_Q}, \mathscr{O}_{X_{\Sigma_Q}}(D_Q)) = W$ that is generated by χ^{m_i} for $m_i \in \{m_1, \ldots, m_s\} = Q \cap M$ give the morphism ϕ_{D_Q} to $\mathbb{P}^{s-1}$. Note that we may write

$$\phi_{D_Q}(p) = (\chi^{m_1})(p), \ldots, \chi^{m_s})(p),$$

from which, we deduce that it factors as

$$X_{\Sigma_Q} \xrightarrow{\alpha} X_{Q\cap M} \hookrightarrow \mathbb{P}^{s-1}.$$

D_Q gives a closed embedding to the projective space $\mathbb{P}^{s-1}$ if and only if the map α is an isomorphism. Since the maximal cones σ_i in Σ_Q are indexed by the vertices m_i of Q, we can state it as for each $i \in I \subseteq [s]$ such that $m_i \in Q$ is a vertex,

$$\alpha|_{\sigma_i} \colon U_{\sigma_i} \longrightarrow X_{Q\cap M} \cap U_i$$

is an isomorphism where U_i are the standard affine patches of $\mathbb{P}^{s-1}$. Equivalently, $S_{m_i} = \langle Q \cap M - m_i\rangle \to \hat{\sigma} \cap M = S_{\sigma_i}$ is a semigroup isomorphism for each $i \in I$, which is the definition of Q being very ample. Recalling that Q is taken arbitrarily, we conclude that Q is very ample if and only if D_Q is very ample. It also implies that P being a polytope is equivalent to D_P being ample since $Q = D_Q = D_{kP} = kD_P$ is shown to be very ample. $\square$

We can give another description for the ample divisors by defining the class of support functions associated to polytopes.

Definition 12. *Let Σ be a fan with a full dimensional convex support and D be a Cartier divisor on X_Σ. The support function φ_D is said to be strictly convex if it is convex and*

$$\varphi_D(u) = \langle m_\sigma, u\rangle \Longleftrightarrow u \in \sigma$$

for every $\sigma \in \Sigma(\max)$.

Note that the condition given for strict convexity of φ_D is to say the points m_σ for the maximal cones $\sigma \in \Sigma$ are distinct. From the reasoning in the proof of the previous proposition, it is clear that such a condition makes it possible to gather the existing data to form a closed projective embedding. Strictly convex support functions characterize the class of ample line bundles on toric varieties.

Theorem 5.4 ([2] Proposition 3.4.3)**.** *$D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a Cartier divisor with the data $\{m_\sigma\}_{\sigma\in\Sigma_{\max}}$ on a complete toric variety on X_Σ and let φ_D be its support function. Then, φ_D is strictly convex if and only if D is ample.*

Proof. Suppose that D is ample. Then, kD is a very ample Cartier divisor for some $k \in \mathbb{Z}$ with Cartier data $\{km_\sigma\}$. Since it is very ample, it is also base point free and so is associated to the very ample polytope $P_{kD} = kP_D$ satisfying $\Sigma = \Sigma_{kP_D}$ as we have shown in proposition 5.3. In this case, the points km_σ coincide with the vertices of kP_D and so are distinct, which implies $\{m_\sigma\}_{\sigma\in\Sigma_{\max}}$ are distinct. Besides, we have $\varphi_D = \varphi_{kD}/k = \varphi_{kP_D}/k$ showing that φ_D is convex by the definition of φ_P.

Now let φ_D be strictly convex. Let us call $P = \mathrm{Conv}(m_\sigma | \sigma \in \Sigma_{\max})$. By definition, $\varphi_P = \varphi_D$ which implies $D_P = D$. It follows from 5.3 that D is ample. □

These two gradual notions that refer to projective morphisms coincide in the good cases ([7] Corollary 1 in page 570), namely when there are no incompatibilities in the local data.

Theorem 5.5. *Every ample divisor is very ample on a smooth complete toric variety X_Σ.*

Proof. Suppose that D is an ample divisor on X_Σ and let k be chosen so that kP_D is very ample. The reasoning in the last proof gives that $\Sigma = \Sigma_{P_D} = \Sigma_{kP_D}$. Since Σ is a smooth fan, kP_D is smooth, and so P_D is also smooth when it is a lattice polytope. Smooth polytopes are very ample, which, by the proposition 5.3, completes the proof. □

5.2. Quasicoherent Sheaves

We will concentrate on the relation of the Cox ring R with the geometric features of X_Σ. In particular, we will show that all quasicoherent sheaves on a toric variety come from graded modules over its Cox ring. Let X_Σ be a toric variety with the full dimensional fan $\Sigma \subseteq N_\mathbb{R}$ and let R be the associated Cox ring. For each $\sigma \in \Sigma$, we have the map

$$\begin{aligned} \mathbb{C}[S_\sigma] &\longrightarrow R_{x^{\hat{\sigma}}} \\ \chi^m &\longmapsto \prod_\rho x_\rho^{\langle m, u_\rho \rangle} \end{aligned}$$

where $R_{x^{\hat{\sigma}}}$ is the localization of R at the monomial of σ. Notice that the monomials $x^{\langle m, \text{-} \rangle}$ in the image have degree

$$[\sum_\rho \langle m, u_\rho \rangle D_\rho] = [\chi^m] = 0$$

and they are precisely the ones that remain invariant under the G-action. Therefore, it induces an isomorphism

$$\phi_\sigma \colon \mathbb{C}\,[\hat{\sigma} \cap M] \longrightarrow (R_{x^{\hat{\sigma}}})^G = (R_{x^{\hat{\sigma}}})_0.$$

So, the right hand side can be considered as the algebraic correspondent to the affine pieces of X_Σ. The intersections of the corresponding copies of affine pieces, i.e. the constant parts of the R-modules $R_{x^{\hat{\sigma}}}$, are compatible since they are given by the gluing data of those pieces.

Lemma 5.6 ([9] Lemma 2.3)**.** *Let τ be a face of σ and $m \in M$ such that $\tau = \sigma \cap m^{\perp}$. The following diagram*

$$\begin{array}{ccc} (R_{x^{\hat{\sigma}}})_0 & \longrightarrow & ((R_{x^{\hat{\tau}}})_0)_{\phi_\sigma(\chi^m)} \\ \uparrow & & \uparrow \\ \mathbb{C}[S_\sigma] & \longrightarrow & \mathbb{C}[S_\tau]_{\chi^m} \end{array}$$

of isomorphisms commutes.

Proof. The condition $\tau = \sigma \cap m^{\perp}$ is to say that the inner product $\langle m, u_\rho \rangle = 0$ for $u_\rho \in \tau(1)$ and is strictly positive for $u_\rho \in \sigma(1) \backslash \tau(1)$. Recalling that taking the constant part commutes with the localization, one can chase the diagram and immediately see that it commutes. □

In the same vein, we can define a sheaf $\mathscr{F}_M$ corresponding to a given graded R-module M.

Proposition 5.7 ([1] Propositions 5.3.3, 5.3.6)**.** *Let M be a graded R-module.*

(i) There exists a quasicoherent sheaf $\mathscr{F}_M$ such that

$$\Gamma(U_\sigma, \mathscr{F}_M) = (M_{x^{\hat{\sigma}}})_0$$

for any $\sigma \in \Sigma$.

(ii) $\mathscr{F}_M$ is coherent if M is finitely generated as a graded R- module.

Proof. Since M is a graded R-module, $M_{x^{\hat{\sigma}}}$ is a graded $R_{x^{\hat{\sigma}}}$ module and $(M_{x^{\hat{\sigma}}})_0$ is an $(R_{x^{\hat{\sigma}}})_0$ module. Hence, it induces a sheaf on

$$\text{Specm}((R_{x^{\hat{\sigma}}})_0) = \text{Specm}(\mathbb{C}[S_\sigma]) = U_\sigma$$

that is compatible on the intersections by the previous lemma. So, the collection $\{(M_{x^{\hat{\sigma}}})_0\}_{\sigma \in \Sigma}$ forms a sheaf that is clearly quasicoherent by construction.

For the second part, we may take G-homogeneous generators of M of degrees $\alpha_1, \ldots, \alpha_r$. For any $\sigma \in \Sigma$ clearly $M_{x^{\hat{\sigma}}}$ is finitely generated over $R_{x^{\hat{\sigma}}}$ with generators

m_i of the same degrees. Now, we multiply each m_i with the generators of $(R_{x^{\hat{\sigma}}})_{-\alpha_i}$ and obtain a generator set of $(M_{x^{\hat{\sigma}}})_0$ over $(R_{x^{\hat{\sigma}}})_0$. Hence, $\mathscr{F}_M$ is coherent. $\square$

Example 20. *For an $\alpha \in Cl(X_\Sigma)$, the shift $R(\alpha)$ gives a coherent sheaf on X_Σ that is denoted by $\mathscr{O}_{X_\Sigma}(\alpha)$.*

In fact, any divisor in the same class gives the same sheaf.

Proposition 5.8 ([1] Proposition 5.3.7)**.** *There is a natural isomorphism $R_\alpha \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha))$ and for any Weil divisor $D = \sum_\rho a_\rho D_\rho$ with the class α,*

$$\mathscr{O}_{X_\Sigma}(D) = \mathscr{O}_{X_\Sigma}(\alpha).$$

Proof. The local sections of $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha))$ are defined as

$$\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(\alpha)) = (R(\alpha)_{x^{\hat{\sigma}}})_0 = (R_{x^{\hat{\sigma}}})_\alpha .$$

Hence, we have the exact sequence

$$0 \longrightarrow \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha)) \longrightarrow \prod_\sigma (R_{x^{\hat{\sigma}}})_\alpha \rightrightarrows \prod_{\sigma,\tau} (S_{x^{\widehat{\sigma\cap\tau}}})_\alpha$$

from which we may conclude that the global sections of $\mathscr{O}_{X_\Sigma}(\alpha)$ has a basis consisting of Laurent monomials $\prod_\rho x_\rho^{a_\rho}$ of degree α such that $a_\rho \geq 0$ for all $\rho \in \Sigma(1)$. Notice that these are exactly the monomials in R of degree α. The first isomorphism follows immediately.

Let us take a divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho \in Cl(X_\Sigma)$ with $[D] = \alpha$. It is enough to show that

$$\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(D)) = \Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(\alpha)),$$

which implies that D gives the same sheaf as α since $(R_{x^{\hat{\sigma}}})_\alpha$ is compatible with the inclusions given by the face poset of Σ. By (5.1), the left hand side is the direct sum of $\mathbb{C} \cdot \chi^m$ over $m \in M$ satisfying $\langle m, u_\rho\rangle \geq -a_\rho$ for each $\rho \in \sigma(1)$. Since the monomial $x^{\langle m,D\rangle}$ has degree $[D] = \alpha$, we may write

$$x^{\langle m,D\rangle} = \prod_\rho x_\rho^{\langle m,u_\rho\rangle + a_\rho} \in (R_{x^{\hat{\sigma}}})_\alpha.$$

So, it gives the map

$$\begin{aligned} \Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(D)) &\longrightarrow (R_{x^{\hat{\sigma}}})_\alpha \\ \chi^m &\longmapsto x^{\langle m, D\rangle}. \end{aligned}$$

Observe that $\langle m, D\rangle = \langle m', D\rangle$ implies $m = m'$ when X_Σ has no torus factors. Thus, above map is injective. Moreover, for any monomial $x^b = \prod_\rho x_\rho^{b_\rho}$ of degree α, we know that

$$\left[\sum_\rho b_\rho D_\rho\right] = \alpha = [D] = [div(\chi^m) + D] = \left[\sum_\rho (\langle m, u_\rho\rangle + a_\rho) D_\rho\right]$$

which guarantees that there exists an $m \in M$ satisfying $b_\rho = \langle m, u_\rho\rangle + a_\rho \geq 0$ for all ρ. Therefore, our map is also surjective and so is an isomorphism. $\square$

Example 21. *Take $X_\Sigma = \mathbb{P}^n$. Recall that the toric prime divisors $D_{\rho_i} = \overline{\mathcal{O}_{\rho_i}}$ are in the class $1 \in \mathbb{Z} = Cl(\mathbb{P}^n)$. So,*

$$\mathscr{O}_{\mathbb{P}^n}(k) \simeq \ldots \simeq \mathscr{O}_{\mathbb{P}^n}(kD_i)$$

with $\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) = R_k$, namely the $\mathbb{Z}$-homogeneous polynomials of degree k.

Therefore, every graded R module gives a quasicoherent sheaf on X_Σ. The converse is also true. For any quasicoherent sheaf $\mathscr{F}$ on X_Σ, there is a graded R-module M such that $\mathscr{F} = \mathscr{F}_M$. Moreover, M can be chosen to be finitely generated over R if $\mathscr{F}$ is coherent. However, this is a many-to-one functor. In other words, many non-isomorphic graded modules may give rise to the same sheaf on X_Σ.

Example 22. *Every finitely generated graded module M over $R = \mathbb{C}[x_0, \ldots, x_n]$ satisfying $\langle x_0, \ldots, x_n\rangle^l M = 0$ for $\ell \gg 0$ gives the trivial sheaf on $\mathbb{P}^n$ ([5], exercise II.5.9).*

6. HYPERSURFACES AND LINEAR SYSTEMS

A *hypersurface* Y in X is an algebraic subset locally of dimension $\dim(X) - 1$. I may be regarded as an effective Weil divisor $\sum_{i\in I} a_i D_i$ where the coefficients a_i are non-negative and D_i are prime divisors of X. If the ground variety X is a projective space, the hypersurface Y is obtained as the zero locus of a single non-constant G-homogeneous polynomial ([5], chapter II.7). It naturally generalizes to the case of toric varieties. In [17] (proposition 2.5.8), it was shown for simplicial toric varieties using the ideal sheaves. We will rather give a more general proof for any (normal) toric variety using the construction in [18] and due to the correspondences of ideals/ subvarieties, and quasicoherent sheaves/ graded modules.

Let X_Σ be a toric variety given by $\pi\colon \hat{X}_\Sigma \to X_\Sigma$ and $Y \subseteq X_\Sigma$ be a hypersurface of degree $\alpha \in Cl(X_\Sigma)$. Let X_Σ^{reg} be the smooth locus of X_Σ. Note that its complement has codimension at least two in X_Σ by normality. The hypersurface Y becomes a Cartier divisor when it is restricted to the smooth locus. Let $\{f_i, U_i\}_i$ be the Cartier data of $Y^{\mathrm{reg}} \doteq Y \cap X_\Sigma^{\mathrm{reg}}$. Note that the pull-back $\pi^*(Y^{\mathrm{reg}})$ is a Cartier divisor on $\hat{X}_\Sigma$ with the data $\{f_i \circ \pi, \pi^{-1}(U_i)\}$.

Definition 13. *The closure of $\pi^*(Y^{reg})$ in $\hat{X}_\Sigma$ is called the pull-back $\pi^*(Y)$ of Y.*

Defining what corresponds to Y in $\hat{X}_\Sigma$ in this way saves us from the problems caused by the singularities of X_Σ that may possibly be worse than quotient singularities.

Proposition 6.1. *Let X_Σ be a toric variety without a torus factor, R be its Cox ring and $\pi\colon \mathbb{C}^r\backslash Z(\Sigma) \twoheadrightarrow X_\Sigma$ be the quotient map given by the G action.*

(i) Any G-homogeneous polynomial $f \in R_\alpha$ inherits a hypersurface in X_Σ of class α.

(ii) For any hypersurface $Y \subseteq X_\sigma$ of class α there is a G-homogeneous polynomial $f_Y \in R_\alpha$ such that $V_{(f_Y)} = Y \subseteq X_\Sigma$.

Proof. For a given G-homogeneous polynomial $f \in R_\alpha$, we know that $V_{(f)} = \pi(\widehat{\mathbf{V}(f)})$ is closed in X_Σ by theorem 4.2. Note that $\mathbf{V}(f)$ is an affine hypersurface and hence, $\widehat{\mathbf{V}(f)} = \mathbf{V}(f) \cap \hat{X}_\Sigma$ must be of dimension $r-1$ since the irrelevant locus $Z(\Sigma)$ has codimension at least two. Since π is an almost geometric quotient, $Y = V_{(f)} = \pi(\widehat{\mathbf{V}(f)})$ is also of codimension one in X_Σ and so is a hypersurface. Besides, the ideal generated by f is a graded R module, so it corresponds to the sheaf $\mathscr{F}_{(f)} = \mathscr{O}_{X_\Sigma}(\alpha)$ since $f \in R_\alpha \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha))$ by proposition 5.8. Therefore, $[Y] = \alpha$.

Now let Y be a hypersurface in X_Σ of class α. In this case, the closure $\overline{\pi^*(Y)}$ of the pull-back in $\mathbb{C}^r$ is a closed G-invariant subset of codimension one, so is given by a single G-homogeneous polynomial $f \in R$. In this case, we have

$$\widehat{\mathbf{V}(f)} = \mathbf{V}(f) \cap \hat{X}_\Sigma = \pi^*(Y).$$

Then, obviously,

$$\pi|_{\pi^{-1}(X_\Sigma^{\text{reg}})}\big(\widehat{\mathbf{V}(f)}\big) = \pi|_{\pi^{-1}(X_\Sigma^{\text{reg}})}(\pi^*(Y)) = Y^{\text{reg}}$$

which implies $\mathbf{V}(f) = \pi(\pi^*(Y)) = Y$. Hence, $f \in R_\alpha$. $\square$

When $f \in R_\alpha$ is a global section of a toric Weil divisor D, the hypersurface $Y = V_{(f)}$ is apparently the zero section of f with respect to the homogeneous coordinates.

6.1. Linear Systems

Similar to the hypersurfaces, the pull-back of a linear system $\mathcal{S}$ is defined as

$$\pi^*(\mathcal{S}) \doteq \{\pi^*(Y) \colon Y \in \mathcal{S}\}$$

and is clearly a linear system on $\hat{X}_\Sigma$ since the preimage of the singular locus of X_Σ has codimension at least two in $\hat{X}_\Sigma$. We denote the collection of polynomials corresponding to the elements of $\pi^*(\mathcal{S})$ by $M_{\mathcal{S}}$ as it is an R_0-module. From the classical point of view (e.g. [5], section II.7), $M_{\mathcal{S}}$ is the subspace is given as

$$\{s \in \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha)) | \text{div}_0(s) \in \mathcal{S}\} \cup \{0\}$$

where $\mathcal{S} \subseteq |\alpha|$.

Definition 14. *A linear system $\mathcal{S} \subseteq |\alpha|$ on a toric variety X_Σ is said to be monomial if $M_\mathcal{S}$ is generated by a set of monomials in R_α.*

Working with the linear systems whose algebraic data can be expressed by monomials brings Newton polyhedra into play. The Newton polytope of $\mathcal{S}$ is defined as

$$\Delta_\mathcal{S} = \mathrm{Conv}\big\{a = (a_\rho)_{\rho\in\Sigma(1)} \colon x^a = \prod_{\rho\in\Sigma(1)} x_\rho^{a_\rho} \in B\big\}$$

where B is a monomial generating set of minimal degree for $M_\mathcal{S}$. Since $M_\mathcal{S} \subseteq R_\alpha$, that is generated by all the monomials of degree α as an R_0 module, we may deduce that $\Delta_\mathcal{S}$ is contained in the polyhedron P_α consisting of the points $a = (a_\rho)_\rho$ where $a_\rho \geq 0$ and $x^a \in R_\alpha$.

Let us denote the base loci of $\mathcal{S}$ and $\pi^*(\mathcal{S})$ by $B_\mathcal{S}$ $B^*_\mathcal{S}$, respectively. When $M_\mathcal{S}$ is generated by monomials $\{x^a \colon a \in B\}$, $B^*_\mathcal{S} = \mathbf{V}(M_\mathcal{S})$ is given as a union of vanishings of x_ρ for $a\rho \neq 0$, which, in particular can be indexed by $\sigma \in \Sigma$. So,

$$B^*_\mathcal{S} = \bigcup_{\sigma\in J} A_\sigma \doteq \bigcup_{\sigma\in S} \left\{x \in \hat{X}_\Sigma : x_\rho = 0, \rho \in \sigma(1)\right\}, \tag{6.1}$$

for some subset $S \subseteq \Sigma$. In fact, the piece $A_\sigma \subseteq B^*_\mathcal{S}$ precisely when the polynomial f_Y corresponding to a general element $Y \in \mathcal{S}$ can be written as

$$f_Y = \sum_{\rho\in\sigma(1)} x_\rho f_\rho. \tag{6.2}$$

The Newton polytope $\Delta_\mathcal{S}$ determines the base locus $B^*_\mathcal{S}$ of a monomial system.

Proposition 6.2 ([18] Lemma 1.1)**.** *Let $\mathcal{S}$ be a monomial linear system on a complete toric variety X_Σ. Then,*

$$B^*_\mathcal{S} = \boldsymbol{V}(x^a) = \boldsymbol{V}\big(\prod_{\rho\in\Sigma(1)} x_\rho^{a_\rho}\big)$$

where a is a vertex of the Newton polytope $\Delta_\mathcal{S} \subset \mathbb{R}^{|\Sigma(1)|}$ of $\mathcal{S}$.

Proof. Let B be a monomial generating set of minimal degree for $M_\mathcal{S}$ and let us denote by I the set of vertices of $\Delta_\mathcal{S}$. It is enough to show that $\mathbf{V}(x^b) \subseteq \bigcup_{a\in I} \mathbf{V}(x^a)$ for any $x^b \in B$. Observing that exponent b of any $x^b \in B$ can be written as $\sum_{i\in[k]} c_i a_i = b$ where $a_i \in I$ are vertices for $i \in [k]$ and $c_i \in (0,1] \cap \mathbb{Q}$ are coefficients satisfying

$\sum_{i\in[k]} c_i = 1$. The desired inclusion follows since the each factor of x^{a_i} for $i \in [k]$ also appear in x^b. $\square$

It follows from above that the monomials having the vertices of the Newton polytope $\Delta_{\mathcal{S}}$ as exponents constitute a basis for $M_{\mathcal{S}}$.

Example 23. *Lets consider the case $X_\Sigma = \mathbb{P}^n$. Recall that $\hat{X}_\Sigma = \mathbb{C}^{n+1}\backslash\{0\}$ and the class group is isomorphic to $\mathbb{Z}$. For a hypersurface $Y \subseteq \mathbb{P}^n$ of class d, f_Y has degree d and $\mathscr{O}_{\mathbb{P}^n}(Y) = \mathscr{O}_{\mathbb{P}^n}(d)$.*

Let $|Y| = \mathcal{S}_d = |d|$ be the complete linear system of hypersurfaces of degree d. Clearly, the homogeneous space $M_{\mathcal{S}_d}$ is generated by the monomials $x^a = \prod_{i\in[n+1]} x_i^{a_i}$ where $a_i \geq 0$ and $\sum_{i\in[n+1]} a_i = d$. So,

$$\Delta_{\mathcal{S}_d} = Conv(\{v_i\}_{i\in[n+1]} | (v_i)_i = d \text{ and } (v_i)_j = 0 \text{ for } i \neq j)$$

and we may compute the base locus as

$$B^*_{\mathcal{S}_d} = \bigcup_{i\in[n+1]} \boldsymbol{V}(x^{v_i}) = \{a = (a_i)_{i\in[n+1]} \in \mathbb{C}^{n+1}\backslash\{0\} | a_i = 0 \text{ for some } i \in [n+1]\}.$$

As one may notice, it becomes messier when the variables x_ρ have different weights. Working on weighted projective spaces, one first needs to solve a partition problem determined by the given information of the dimension of the space, degree of the linear system, and the weights of the variables.

Example 24. *Take $X_\Sigma = \mathbb{P}(1,1,2)$ and consider the complete linear system $\mathcal{S}_3$ of hypersurfaces of degree three.*

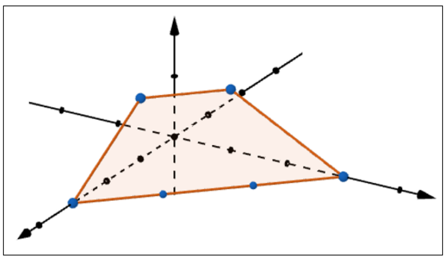

Figure 6.1. The Newton polytope $\Delta_{\mathcal{S}_3}$.

In this case, $M_{\mathcal{S}_3} \subseteq R_3$ has a generating set consisting of all the monomials of degree three, that is $\{x^3, y^3, xy^2, x^2y, xz, yz\}$. The Newton polytope

$$\Delta_{\mathcal{S}_3} = Conv((3,0,0),(0,3,0),(1,0,1),(0,1,1)).$$

Accordingly, one can see that $B^*_{\mathcal{S}_3} = \boldsymbol{V}(x^3, y^3, xz, yz) = \boldsymbol{V}(x) \cup \boldsymbol{V}(y)$ in $\mathbb{C}^3 \backslash \{0\}$.

6.2. Quasismooth Hypersurfaces

Quasismoothness was introduced by Danilov [19] as a regularity notion for hypersurfaces. Quasismooth hypersurfaces in toric varieties have been widely used in some of very sophisticated and highly active areas of research in algebraic geometry; such as mirror symmetry and minimal model program.

Definition 15. *Let X_Σ be a toric variety given by the almost geometric quotient $\pi \colon \hat{X}_\Sigma \to X_\Sigma$. A hypersurface $Y \subset X_\Sigma$ is said to be quasismooth if $\pi^*(Y)$ is smooth in $\hat{X}_\Sigma$.*

If a hypersurface Y is given by the G-homogeneous polynomial $f \in R$, it is equivalent to $\mathbf{V}(f)$ being smooth outside of the exceptional set $Z(\Sigma)$ in $\mathbb{C}^{\Sigma(1)}$. Such a hypersurface may not have singularities except for the ones that are induced by the quotient map. Hence, quasismoothness is equivalent for a hypersurface Y to be simplicial or smooth when the ambient variety X_Σ is simplicial or smooth, respectively. It was shown in [17] (Proposition 2.6.9) that quasismoothness also implies normality. Although the author works with the simplicial toric varieties, it naturally generalizes to the case of any normal toric variety.

Proposition 6.3. *A quasismooth hypersurface $Y \subseteq X_\Sigma$ is normal.*

Proof. Since Y is quasismooth, $\pi^*(Y)$ is smooth and so is normal in $\hat{X}_\Sigma$. Then, $\pi|_{\pi^*(Y)}(\pi^*(Y)) = Y$ is normal by the universal property of good categorical quotients (see [20] section 0.2 or [10] proposition 3.1). □

Following result implies that it is not a very special case for hypersurfaces in toric varieties.

Proposition 6.4 ([18] Proposition 2.2)**.** *Let $\mathcal{S}$ be a linear system of a complete toric variety X_Σ. The subset $\mathcal{S}^{qs}$ of quasismooth hypersurfaces is Zariski open in $\mathcal{S}$.*

Proof. Consider the subset

$$S = \left\{ (x, Y) \in \hat{X} \times \mathcal{S} \colon f_Y(x) = \frac{\partial f_Y}{\partial x_j}(x) = 0, j = [r] \right\}$$

of $\hat{X} \times \mathcal{S}$ where $r = |\Sigma(1)|$ and $f_Y \in R$ is the defining (G-homogeneous) polynomial of $Y \in \mathcal{S}$ and the projection map $p \colon S \to \mathcal{S}$. The fiber $p^{-1}(Y)$ of a hypersurface $Y \in \mathcal{S}$ is empty if Y is quasismooth. Note that p factors through $\bar{p}$ defined on $(\pi \times \mathrm{id})(S)$, that is complete. Hence the image $\bar{\pi}((\pi \times \mathrm{id})(S)) = \mathcal{S} \backslash \mathcal{S}^{\mathrm{qs}}$ is Zariski closed in $\mathcal{S}$ (by [5] exercise II.4.4), which implies $\mathcal{S}^{\mathrm{qs}}$ is open as desired. □

Noting that non-empty open sets are dense in Zariski topology, it immediately follows from the proposition that $\mathcal{S}^{\mathrm{qs}}$ is either empty or is dense in $\mathcal{S}$. In other words, the general element of $\mathcal{S} \subseteq |Y|$ containing Y is quasismooth when Y is quasismooth. In this case, we call $\mathcal{S}$ a *generally quasismooth* linear system. Note that a generally quasismooth linear system $\mathcal{S}$ can also be characterized as a system whose pull back $\pi^*(\mathcal{S})$ is generally smooth.

As we refer in the proof of previous proposition, the condition for quasismoothness may be given as a Jacobian criterion. Accordingly, a hypersurface $Y \subset X_\Sigma$ is quasismooth if and only if the partial derivatives $\{\frac{\partial f_Y}{\partial x_\rho}\}_{\rho \in \Sigma(1)}$ have no common roots. In [16], it was shown that quasismoothness equivalent for a hypersurface to be a suborbifold in the simplicial case ([16] Proposition 3.5). In the light of this result and [15], a characterization of the quasismooth linear systems was given in [21] (Theorem 8.1).

Theorem 6.5. *Let $X_\Sigma = \mathbb{P}(q_0, \ldots, q_n)$ be a weighted projective space, $d \in Cl(X_\Sigma)$ be a class and $I = \{i_0, \ldots, i_{k-1}\} \subseteq [n]$ be an arbitrary non-empty index set. The class d is generally quasismooth precisely in the following cases.*

(i) *There is a variable* x_ρ *of weight* d.

(ii) *There is a monomial* $x_I^{\bar{m}} = x_{i_0}^{m_0} \cdots x_{i_k-1}^{m_k-1} \in R_d$ *of degree* d.

(iii) *There are monomials*

$$x_I^{\bar{m}_t} x_{e_t} = x_{i_0}^{m_{0,t}} \cdots x_{i_k-1}^{m_{k-1,t}} x_{e_t} \in R_d$$

of degree d *for* $t \in [k]$ *and* $\{e_t\}$ *are* k *distinct elements.*

We will present a combinatorial way to determine whether a hypersurface, or its linear system is quasismooth, following the Artebani et. al. [14] that was based on the constructions and results in the classical paper of Khovanskii [22].

Definition 16. *A finite collection* $\{P_i\}_{i\in[k]}$ *of polytopes in* $\mathbb{R}^n$ *is called dependent if there exists a non-emty subcollection* $J \subseteq [k]$ *such that the polytopes in* $\{P_j\}_{j\in J}$ *can be translated to lie on a lower dimensional subspace.*

Using this notion, we can describe the solutions of a given finite system of equations of Laurent polynomials that are general in their support. That is, each member of $\{p_i\}_{i\in[k]} \subseteq \mathbb{C}\left[x_1^{\pm}, \ldots, x_r^{\pm}\right]$ is being general in the collection of Laurent polynomials with the same Newton polytope. The result follows from [22] (sections 2.1 and 2.2).

Lemma 6.6. *Let* $\{p_i\}_{i\in[k]} \subseteq \mathbb{C}\left[x_1^{\pm}, \ldots, x_r^{\pm}\right]$ *be Laurent polynomials that are general in their support. Then the the system* $\{p_i = 0\}_{i\in[k]}$ *of equations have no solutions if and only if the collection of Newton polytopes* Δ_{p_i} *is dependent. The solutions form an analytic manifold of dimension* $(r-k)$ *in* $(\mathbb{C}^*)^r$ *when the system is compatible.*

Now let X_Σ be a complete toric variety, $Y \subseteq X_\Sigma$ be a hypersurface of degree $\alpha \in Cl(X_\Sigma)$ associated to the polynomial $f_Y \in R_\alpha$, and $\mathcal{S}$ be a linear system having Y as a general element. We associate the polytope $\Delta_{\mathcal{S}}^{g}(\rho)$ to the couple $(\mathcal{S}, \sigma)$ for every $\sigma \in \Sigma$, that is the Newton polytope of $f_Y|_{A_\sigma}$.

Although, the expression (6.2) of f_Y may not be unique, it follows from the proposition 6.2 that the Newton polytope has the same vertices. So, $\Delta_{\mathcal{S}}^{g}(\rho)$ are clearly well-defined. Recalling (6.2), when we take the restriction of f_Y to A_σ, the factors of f_ρ indexed by the rays of σ disappears. Therefore,

$$\Delta^\sigma_{\mathcal{S}}(\rho) + e_\rho = \operatorname{Conv}\left(b \in \Delta_{f_Y} \cap \mathbb{Z}^{|\Sigma(1)|} : b_\rho = 1 \text{ and } m_\tau = 0 \text{ for } \tau \in \sigma(1)\backslash\{\rho\}\right).$$

Note that, for each $\sigma \in \Sigma$, the non-empty subcollection of the polytopes $\Delta^\sigma_{\mathcal{S}}(\rho)$ give a collection of faces of the σ-face of Δ_{f_Y} given by $\sum_{\rho\in\sigma(1)} b_\rho = 1$. Hence, the following theorem indicates that the Newton polytope $\Delta_{\mathcal{S}}$ determines whether $\mathcal{S}$ is generally quasismooth.

Theorem 6.7 ([18] Theorem 3.6)**.** *A linear system $\mathcal{S}$ is generally quasismooth if and only if the collection of polytopes $\{\Delta^\sigma_{\mathcal{S}}(\rho)\}_{\rho\in\sigma(1)}$ are dependent for all $\sigma \in \Sigma$ satisfying $A_\sigma \subseteq B^*_{\mathcal{S}}$.*

Proof. Let $\mathcal{S}$ be a linear system on a toric variety X_Σ with a general element $Y \in \mathcal{S}$. Suppose that $\widehat{\mathbf{V}(f_Y)} \subseteq \pi^*(\mathcal{S})$ is singular in $\hat{X}_\Sigma$ and let C be an irreducible component of $\widehat{\mathbf{V}(f_Y)}^{\text{sing}} = \hat{X}_\Sigma \backslash \widehat{\mathbf{V}(f_Y)}^{\text{reg}}$. Let A_σ be the intersection of pieces of $B^*_{\mathcal{S}}$ containing C. Let us write $f_Y = \sum_{\rho\in\sigma(1)} x_\rho f_\rho$. Since A_σ is the smallest piece containing C, f_Y has a singular point on the torus isomorphic to $(\mathbb{C}^*)^{|\Sigma(1)\backslash\sigma(1)|}$ which we will denote by T_σ. Observing that f_ρ are generic in their support as f_Y is general, we get

$$T_\sigma \cap C \subseteq T_\sigma \cap \widehat{\mathbf{V}(f_Y)}^{\text{sing}} \cap A_\sigma = T_\sigma \cap \Big(\bigcap_{\rho\in\sigma(1)} \mathbf{V}(f_\rho) \cap A_\sigma\Big)$$

which implies that $\Delta^\sigma_{\mathcal{S}}(\rho)$ are independent by lemma 6.6.

Similarly, for any piece $A_\sigma \subseteq B^*_{\mathcal{S}}$, the polytopes $\Delta^\sigma_{\mathcal{S}}(\rho)$ are independent, which means that $\{f_\rho\}_{\rho\in\sigma(1)}$ has a common root in the torus T_σ. We may conclude that $\widehat{\mathbf{V}(f_Y)}$ is not smooth as f_Y has singular points on the torus and hence the general element Y of $\mathcal{S}$ is not quasismooth. $\square$

Moreover, we know that $\Delta^\sigma_{\mathcal{S}}(\rho) \subseteq P\alpha$, that spans a linear subspace of dimension $\dim(\pi\,(A_\sigma))$. This fact together with the theorem above has an immediate corollary.

Corollary 6.8. *Let $A_\sigma \subseteq B^*(\mathcal{S})$ for some $\sigma \in \Sigma$. If the number of non-empty polytopes $\Delta^\sigma_{\mathcal{S}}(\rho)$ is greater than dim$(\pi\,(A_\sigma))$, the linear system $\mathcal{S}$ is generally quasismooth.*

7. CONCLUSION

We conclude this dissertation by a discussion on the moduli problem of hypersurfaces in toric varieties and its history as it was the beginning of the history of this dissertation.

7.1. The Moduli Problem

It is known that the study of moduli of hypersurfaces in toric varieties may require non-reductive tools as toric varieties may have non-reductive automorphism groups. Let X_Σ be a complete simplicial toric variety (i.e. toric orbifold), α be an ample divisor class, and let M_α denote the generic coarse moduli space of quasismooth hypersurfaces in X_Σ with the divisor class α. Batyrev and Cox [16] (section 13) have stated that, using the quotient presentation of X_Σ, M_α can be given as a categorical quotient of a *suitable* open subset U, consisting of quasismooth hypersurfaces, of the parameter space $\mathcal{S}_\alpha$ of hypersurfaces of degree α by an algebraic group H when H is reductive. Note that the group H is defined as the centralizer of G (as in 4.1) in $\mathrm{Aut}(\hat{X}_\Sigma)$ and depends on the automorphism group of X_Σ fixing the class α, so is possibly non-reductive. At the time when [16] was published, it was not possible to describe the moduli space more explicitly.

Geometric invariant theory (GIT) is the theory of quotients in the category of varieties that provides very powerful tools for constructing and studying moduli spaces (see [20]). Traditionally, if we have a reductive group G acting on a complex projective variety $X = \mathrm{Proj}(R)$, then the set of closed G-orbits is $X//G = \mathrm{Proj}(R^G)$ and the GIT quotient $\pi\colon X^{ss} \to X//G$ is a good categorical quotient. That is to say, π satisfies what one would expect from such a quotient in the reductive case. On the other hand the famous example of Nagata shows that the ring of invariants may not even be finitely generated if the group G is not reductive. Hence, GIT fails to provide the context to study non-reductive quotients and moduli spaces of unstable objects such as the hypersurfaces of toric varieties.

Moreover, generalizing Mumford's GIT, non-reductive geometric invariant theory (NRGIT) is being developed by Kirwan et. al. for the last two decades. It was shown that many of the results in the classical theory can be generalized to the case of G being a non-reductive complex group with graded unipotent radical [23]. With the new tool-set of NRGIT, the description of the moduli of hypersurfaces in simplicial toric varieties is given by Dominic Bunnett in his PhD thesis [17]. Bunnett shows that quasismooth elements in the complete linear system $\mathcal{S}_\alpha$ in degree α are semistable under the action of $\mathrm{Aut}(X_\Sigma)$ using the A-discriminants defined by Gelfand, Kapranov, Zelevinski.

On the other side of the coin, we have symplectic tools. When G is reductive, the GIT quotient $V//G$ is isomorphic to the symplectic quotient $\mu^{-1}(0)/K$ by the Kempf-Ness theorem where μ is the moment map on V, that is a smooth projective variety, and $K \leq G$ is a maximal compact subgroup. Hence, the notion of quotients in two different worlds is somehow equivalent under favorable circumstances. It is natural to ask whether this equivalence hold in the generalized setting. It would be interesting to investigate the potential applications of the recent works of Kirwan et. al. to moduli problems concerning toric varieties.

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