THEORY OF NONCOMMUTATIVE MOTIVES

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ABSTRACT

THEORY OF NONCOMMUTATIVE MOTIVES

The theory of motives was originally conceived by Alexander Grothendieck as a universal cohomology theory for algebraic varieties. In the decades since it was first introduced, it has become a vast and profoundly sophisticated subject systematically developed in many directions spanning algebraic and arithmetic geometry, homotopy theory and higher category theory. The quest for a fully developed theory of motives as envisioned by Grothendieck drove a great deal of fundamental research in the aforementioned disciplines, while delivering fantastic and long-promised results and settling classical questions as it reached maturity in the past decades. This quest is arguably not complete, since the *abelian* category of mixed motives, originally established by Grothendieck himself as the ultimate *desideratum* of a satisfactory theory of motives, has proven elusive. However, ideas of motivic nature as a programmatic approach to cohomology theories and invariants have proven extremely useful in a variety of other contexts. Noncommutative algebraic geometry is precisely one of these contexts. Following ideas of Maxim Kontsevich, Goncalo Tabuada and Marco Robalo independently developed theories of "noncommutative" motives which fully encompasses the classical theory of motives and helps assemble so-called additive invariants such as Algebraic K-Theory, Hochschild Homology and Topological Cyclic Homology into a motivic formalism in the very precise sense of the word. In this expository work, we will review the fundamental concepts at work, which will inevitably involve a foray into the formalism of enhanced and higher categories. We will then discuss Kontsevich's notion of a noncommutative space, sharpened and made precise over the years by Toën, Tabuada, Robalo and others and introduce noncommutative motives as "universal additive invariants" of noncommutative spaces. We will conclude by offering a brief sketch of Robalo's construction of the noncommutative stable homotopy category.

ÖZET

NONKOMÜTATİF MOTİFLER TEORİSİ

Nonkomütatif Motifler Teorisi cebirsel geometride Alexander Grothendieck tarafından ortaya konulan *motif* fikrinin *değişmesiz* cebirsel geometri alanına taşınmasını konu etmektedir. Maxim Kontsevich tarafından spekülatif bir program olarak 90'larda ortaya atılan bu teori, geçtiğimiz yirmi yıl içinde büyük ölçüde tamamlanmış ve teknik arkaplanı ve uygulamaları bakımından oldukça zengin bir disiplin halini almıştır. Goncalo Tabuada ve Marco Robalo tarafından bağımsız olarak geliştirilen nonkomütatif motifler teorisinin ana objesi zenginleştirilmiş kategori teorisi kullanılarak formalize edilen dg-kategorilerdir. Dg-kategoriler Kontsevich'in fikirlerini takiben en önemli örnekleri arasında zenginleştirilmiş türetilmiş kategoriler bulunan "nonkomütatif uzay" kavramına temel oluşturmuştur. Cebirsel ve aritmetik geometride temel rol oynayan kohomoloji teorilerinin bu objelere genişletilmesi mümkündür. Bu genişletilmelere literatürde "toplamsal sabitler" adı verilmektedir. Toplamsal sabitler dg-kategorilerin yüksek kategorisinden bir stabil simetrik monoidal yüksek kategoriye fonktör olarak formalize edilebilir. nonkomütatif motifler kategorisi bu tür fonktörlerin sahip olması gereken temel özellikleri soyutlayarak elde edilen evrensel özelliklere sahip bir fonktör kategorisidir. Bu kategorinin inşaasında dg-kategoriler teorisi dışında, model kategoriler ve yüksek stabil kategoriler teorisinin de kullanılması gerekmektedir. Bu tez çalışması bunun gibi temel kavramlara bir giriş niteliğindedir. Ana amaç homolojik, homotopik ve kategorisel cebirden bazı kavramların tanıtılması ve sonuç olarak Robalo'nun değişmez stabil homotopi kategorisinin tasviridir.

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LIST OF SYMBOLS

$QCoh(X)$	The abelian category of quasicoherent sheaves on a scheme
$D^b(X)$	The triangulated bounded derived category of a scheme
$Perf_{dg}(X)$	The dg-category of perfect complexes on a scheme
Spc	The ∞-category of spaces
∞-Grpd	The ∞-category of ∞-groupoids
Sp	The ∞-category of spectra
Cat	The category of small categories
Cat_∞	The ∞-category of ∞-categories
$sSet$	The category of simplicial sets
Cat_Δ	The category of simplicial categories
$Fun(C, D)$	The ∞-category of functors between ∞-categories C and D
$K * L$	Join of the simplicial sets K, L
$K^\triangleright$	The cone over a simplicial set K
$C_{/X}, C_{X/}$	Over- and undercategories of $X \in C$
$C_{/p}, C_{p/}$	Over- and under-categories of diagrams $p : K \to C$
$P(C)$	∞-category of presheaves of spaces on C
Pr_L	∞-category of presentable ∞-categories with colimit preserving functors
Cat_∞^{St}	∞-category of stable ∞-categories with exact functors
Pr_{St}^L	∞-category of presentable stable ∞-categories
$DG - Cat_R$	The category of dg-categories over a commutative ring R
$DG - Cat_{idem}$	The category of idempotent complete dg-categories
$(C, \otimes)$	A symmetric monoidal category
Γ	Segal's category of pointed finite sets and its nerve
$Comm^\otimes$	The commutative ∞-operad
$Triv^\otimes$	The trivial ∞-operad
$CAlg(C)$	Commutative algebra objects in C
Op_∞^Δ	The simplicial category of ∞-operads
Op_∞	The ∞-category of ∞-operads

Sym_∞	The ∞-category of symmetric monoidal ∞-categories
$Aff_{ft}^\times$	The nerve of the category of affine schemes of finite type
$SH(R)^\otimes$	The stable homotopy category of schemes over ring R
$ncS(R)^\otimes$	The noncommutative stable homotopy category over R

LIST OF ACRONYMS/ABBREVIATIONS

1. INTRODUCTION

The main objective of this thesis is to offer an expository recapitulation of the theory of noncommutative motives as developed in parallel and independently by Goncalo Tabuada [1] and Marco Robalo [2] and whose roots go back to foundational work by many others over the past decades, such as Kontsevich [3] and Toën [4]. The bulk of the thesis is devoted to an exposition of the fundamental concepts and constructions at work in the context of model categories, dg-categories and higher categories. We have kept the exposition of algebraic geometry minimal and focused on the less accessible categorical aspects of the theory.

As we shall not have the occasion to do so again in the bulk of the text, we take this opportunity to explain concisely what is meant by the mysterious invocation of "noncommutative algebraic geometry". To ward off any fundamental misunderstandings at the outset, this phrase, as used throughout this text, does not refer to the various generalization of SGA-style algebraic geometry to the context of *noncommutative* rings, as has been attempted and carried out by various authors such as [5], [6], among others. Instead, we refer to the *categorical* algebraic geometry which takes as its starting point the study of the derived category of quasicoherent sheaves and its subcategory, the triangulated category of *perfect* complexes of quasicoherent sheaves on a scheme, as carried out by Bondal, Orlov, Kontsevich and others, for instance see [3, 7–9]. Advances in homotopy theory and higher categorical algebra made it possible to fully execute this vision and treat these derived categories as geometric objects in their own right by way of so-called *enhancements* which took care of the foundational difficulties which hobble the formalism of triangulated categories, a topic we explore in great depth in our preliminary chapter. Succinctly put, our model of a *noncommutative* scheme is that of Kontsevich [3]:

Definition 1.0.1. *A noncommutative space is an dg-enhanced triangulated category satisfying some finiteness conditions.*

The main paradigm for this definition is precisely $Perf_{dg}(X)$, the dg-enhanced category of perfect complexes of quasicoherent sheaves on a scheme X. But we have gone too fast and skipped over what we mean by dg-enhanced. As we shall see in our preliminary chapter, dg-categories are *enriched* categories whose Hom-spaces are *chain complexes* of modules over some commutative ring R instead of merely being Hom-sets. We shall see that most relevant constructions familiar from ordinary category theory carry over to this context. But what makes dg-categories so useful is rather the excellent formal properties enjoyed by the category $DG - Cat_R$, the category of dg-categories over some ring R. Further, we shall see that this category admits several model structures which are relevant for algebra and geometry. Dg-enhancement refers to the procedure of replacing triangulated categorical models of derived categories with some associated dg-category by way of an almost canonical procedure which lets us exploit the homotopy theory of dg-categories and carry out categorical operations which are otherwise inaccessible. In later sections, we shall cover the same topics in the context of ∞-categories and compare stable ∞-categories with dg-categories as higher categorical models of abelian/triangulated categories. This brings us to the theory of motives. We refer to [10] as the fundamental reference for the classical theory of motives.

The theory of motives was envisioned by Alexander Grothendieck as the foundation of cohomology theory in algebraic geometry. Motives are formidable objects: their very definition had been the ultimate subject (and in some ways, still is) of several extremely difficult conjectures. By design, motives are cohomological avatars of algebraic varieties over a field k which can be incarnated as specific cohomology theories:

- For char $k = 0$, with an embedding $k \hookrightarrow \mathbb{C}$, we have $H^*(X(\mathbb{C}), \Lambda))$ where $\Lambda \hookrightarrow \mathbb{C}$ and $H^*_{dR}(X(\mathbb{C}))$.
- For char $k > 0$, we have ℓ-adic étale cohomology $H^*_{ét}(X, \mathbb{Q}_\ell)$ and crystalline cohomology $H^*_{cris}(X/W(k))$.

The first glimpse of the motivic picture is to be found in the following *comparison isomorphisms* between cohomology theories, which a priori have no reason to be explicitly comparable.

- There is an isomorphism between Algebraic de Rham and Betti cohomology, given by the period pairing: $H^i_{dR}(X) := \mathbf{H}^i(\Omega^*_X) \cong H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$.
- We have the Artin comparison theorem which gives an isomorphism between étale and Betti cohomologies: $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell) \cong H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$.
- We have the Berthelot-Ogus comparison theorem which gives an isomorphism between de Rham and crystalline cohomology: $H^i_{dR}(X) \cong H^i_{cris}(X_0) \otimes_{W(k_0[1/p])} k$.

These isomorphisms, some of which should also be familiar to algebraic topologists, suggest a deeper picture of cohomology than can be probed by individual theories alone. Somehow they should all be considered together. This is precisely the purpose of the theory of motives. To be more faithful from a historical perspective, it should be mentioned that the theory of motives originated in Grothendieck's programmatic approach to Weil's conjectures on Zeta functions attached to smooth projective varieties over $k = \mathbb{F}_{p^n}$. Having developed étale cohomology theory to provide a geometric interpretation of the zeta function of the variety in terms of the Lefschetz fixed point formula, Grothendieck was perplexed by the picture of the algebraic topology of algebraic varieties that emerged from the Weil conjectures interpreted in this light. For instance, the base change of a smooth variety X over, say, $\mathbb{F}_{p^n}$, to $\mathbb{C}$ appears to govern the arithmetic properties of X. In some sense, there appears to be a single universal cohomology theory. The theory of motives is the materialization of this intuition. From the outset, Grothendieck established as the main objective of the theory of motives the construction of a category whose Hom-spaces could be interpreted as a universal cohomology theory (so-called motivic cohomology) and which possessed realization maps to particular cohomology theories. This turned out to be a formidable challenge.

We now jump ahead in our story. Voevodsky and collaborators achieved the construction of the category of mixed motives, which is a story we mostly avoid in this text. The fundamental sources for the theory of motives and the stable homotopy category of schemes are [11, 12]. What takes the center stage in this thesis is the so-called stable homotopy category of schemes, which is the category of *motivic* spectra, obtained, just like the stable homotopy category of topological spaces, by means of the process of *stabilization*. We will discuss this topic in our last chapter. The stable homotopy category of schemes enjoys a universal property and representability results for algebraic K-theory and other cohomology theories can be formulated within it. Both Tabuada's and Robalo's works reproduce aspects of Voevodsky's construction. Tabuada's approach, which has drawn on a greatly eclectic range of tools and concepts over the past decades, reproduces more classical aspects of the theory of motives, such as a notion of a mixed noncommutative motive, a theory of noncommutative motivic Galois groups, a noncommutative *cycle* theory, that is to say, analogues of great standing questions in algebraic geometry, which are among the most notorious problems in mathematics, such as the Hodge conjecture and the Standard Conjectures. We refer to [13] and later works for an overview. In this text, we shall mostly leave Tabuada's approach untouched and follow Robalo in constructing the stable homotopy category of noncommutative spaces. However, we shall need to build up a tremendous inventory of categorical tools and concepts on our way to this construction.

2. PRELIMINARY OVERVIEW OF HOMOLOGICAL, HOMOTOPICAL AND CATEGORICAL ALGEBRA

2.1. Chain Complexes of Modules

In this section, we give a very cursory overview of some essential concepts from the theory of abelian categories mainly for the purposes of establishing notation. While we do not use specific references, Manin-Gelfand [14], Weibel [15] may be consulted as fundamental references for the concepts in this section.

Throughout this text, R denotes a commutative ring with unity. As usual, denote by Mod_R the category of modules over R and by $Ch(R) := Ch(Mod_R)$ the category of chain complexes of modules over R. Both Mod_R and $Ch(R)$ have a symmetric monoidal structure given by the tensor product of R-modules, where the tensor product of a chain complex of modules is defined componentwise. Recall that a morphism between chain complexes X_i and Y_i consists of maps f_i as shown in the diagram

$$\begin{tikzcd} \ldots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & X_i \arrow[r] \arrow[d, "f_i"] & \ldots \\ \ldots \arrow[r] & Y_{i+1} \arrow[r, "d'_{i+1}"] & Y_i \arrow[r] & \ldots \end{tikzcd} \tag{2.1}$$

such that $f_i d_i^X = d_i^Y f_{i+1}$. In other words, chain morphisms are maps of chain complexes commuting with the differential.

Fix a chain complex $(X_\bullet, d_\bullet)$. Then put $Z_n := ker(d_n)$, $B_n := im(d_{n+1})$, and $H_n(X_\bullet) := Z_n/B_n$. We refer to the latter group as the *homology* group of the chain complex and this assignment gives rise to a functor, the functor of homology, $H(-) : Ch(Mod_R) \to Mod_R$. We say a chain morphism $f : X_\bullet \to Y_\bullet$ is a quasi-isomorphism if it induces an isomorphism $H(X_\bullet) \cong H(Y_\bullet)$. For cochain complexes, all the definitions apply verbatim with arrows reversed.

A chain homotopy between two morphisms of complexes $f_\bullet$ and, $g_\bullet : (X_\bullet, d_\bullet^X) \to (Y_\bullet, d_\bullet^Y)$ is a chain map $h_\bullet : (X_\bullet, d_\bullet^X) \to (Y_\bullet, d_\bullet^Y)$ of degree one satisfying the equation

$$f_n - g_n = h_{n-1} \circ d_n^X + d_{n+1}^Y \circ h_n. \tag{2.2}$$

It is easy to check that this gives rise to an equivalence relation on the morphisms in $Ch(Mod_R)$ which we simply denote by $\sim$. To unburden the notation, we will drop the chain complex indices in what follows, except when additional emphasis is needed. We say two chain complexes X and Y are homotopy equivalent if there exist chain maps $f : X \to Y, g : Y \to X$ such that $f \circ g$ and $g \circ f$ are nullhomotopic. We say a chain complex $X_\bullet$ is contractible if the identity map is nullhomotopic, that is, $X_\bullet$ is homotopy equivalent to the zero complex.

To motivate this definition of chain homotopy, consider topological spaces X,Y and denote by $Sing_\bullet(X)$, $Sing_\bullet(Y)$ the singular chain complexes associated with them. Now assume X and Y are *homotopy equivalent*, that there is a continuous map $f : X \to Y$ with a continuous "homotopy" inverse $g : Y \to X$, that is $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. By the functoriality of the Sing construction, f and g induce chain maps $f : Sing(X) \to Sing(Y)$, $g : Sing(Y) \to Sing(X)$. Then it is a straightforward exercise that $Sing(X)$ and $Sing(Y)$ are (chain) homotopy equivalent when X and Y are homotopy equivalent. We shall later see in the section on homotopical algebra that such analogies are no accident.

A chain complex (and later a dg-algebra) is said to be *connective* if all its negative components vanish and *coconnective* if all its positive components vanish. We denote connective (coconnective) chain complexes by $Ch_{\geq 0}(R)$ (respectively, $Ch_{\leq 0}(R)$). We denote the category of chain complexes which are bounded above by $Ch^+(R)$ and the category of chain complexes bounded below by $Ch^-(R)$. We put $Ch^b(R)$ for the category of chain complexes bounded both below and above.

2.2. Abelian Categories

Mod_R is a particular (in fact, paradigmatic) example of the fundamental notion of an *abelian category.* The formalism of abelian categories emerged from the programmatic reconstruction of algebraic geometry by Grothendieck [16], which crucially required the formalization of the intuitive idea that sheaf categories behave like module categories and the homological calculus of Cartan, Eilenberg etc. makes sense in a far more general setting than that of module theory over some commutative ring. In a nutshell, abelian categories axiomatize this general setting in which one can do homological algebra. We begin by defining additive categories.

Definition 2.2.1 (Additive Category)**.** *A category $\mathscr{A}$ is said to be additive if it has a zero object, admits products and coproducts and the Hom-sets $Hom_{\mathscr{A}}(X, Y)$ are abelian groups with bilinear composition for any $X, Y \in Ob(\mathscr{A})$.*

The fundamental examples of additive categories are categories of vector spaces, modules, (abelian) sheaves on topological spaces.

Definition 2.2.2 (Exact Category)**.** *A category $\mathscr{A}$ is said to be exact if it admits kernels and cokernels and $coim(f) \cong im(f)$ for any morphism f.*

An *abelian* category is a category that is additive and exact in the above sense. There are various, more "functorial" and invariant, definitions of an abelian category, however, this will suffice for our purposes. We collect the following facts without proof.

- When $\mathscr{A}$ is abelian, so is $Ch(\mathscr{A})$, where $Ch(\mathscr{A})$ denotes chain complexes of objects of $\mathscr{A}$.
- Given an abelian category $\mathscr{A}$, the category of $\mathscr{A}$-valued sheaves is abelian.
- A rich harvest of abelian categories: besides paradigmatic examples of module and abelian sheaf categories, we have categories of local systems, perverse sheaves, D-modules, representation categories as examples which are all related precisely *qua* abelian categories.

We say an abelian category is semisimple if all short exact sequences in it split. Recall that a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to split if it is isomorphic to $0 \to A \to A \oplus C \to C \to 0$. The most immediate counterexample is the category $Ab \cong \mathbb{Z}$-Mod; for instance consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, where 2 indicates the multiplication by 2 map. The issue of semisimplicity segues smoothly into that of idempotent completeness.

Recall the useful motto that abelian categories should be seen as abstract forms of $Vect_k$, the category of vector spaces over some field k. The axioms of abelian categories abstractly recapitulate various properties enjoyed by the latter and indeed there are other implicit properties enjoyed by abelian categories among general categories which consolidate the analogy. However, there are some intermediate steps between full *abelianness* (namely, the existence of all (co)kernels) and additivity which are seen in the wild, so to speak. Here we have in mind the crucial theme of *idempotent completeness*, which is also referred to as pseudo-abelian or Karoubian, particularly in the motivic literature, see for instance [16] for a discussion of idempotent completness in the triangulated context. An idempotent endomorphism in an additive category generalizes the notion of an idempotent element, that is, an element e such that $e^2 = e$. Thus, an idempotent in an additive category $\mathscr{A}$ is an endomorphism $f : X \to X$ such that $f^2 = f \circ f = f$, where $X \in \mathscr{A}$. Here it is useful to think of the instructive exercises in elementary linear algebra which exhibits the utility of such reflection maps in probing the structure of vector spaces. We say an additive category is *idempotent complete* if every idempotent admits a kernel or a cokernel (by definition, either one implies the other). Now, the existence of such a kernel is equivalent to whether an idempotent *splits*, that is, if it can be presented as a composition of maps $e = f \circ g$ where g is a section of f, as shown in the diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[rr, bend left, "e"] & Y \arrow[r, "f"'] \arrow[rr, bend left, "id_Y"] & X \arrow[r, "g"'] & Y. \end{tikzcd} \tag{2.3}$$

Every category admits a so-called idempotent completion (also referred to as the Karoubi envelope). The process of idempotent completion is a crucial part of "generalized linear algebra"- that is, the practice of linearizing or abelianizing nonlinear objects. This issue is prominent especially in the theory of motives (e.g. when working with the category of pure Chow motives where idempotents abound) where idempotent completion offers an enlargement of the additive category of correspondences, allowing direct sum decompositions of otherwise simple objects in keeping with geometric intuition regarding them. Note that there is no comparably simple *free* abelianization procedure: this is the whole point of the theory of "additive invariants of categories" and ultimately that of noncommutative motives.

Next we introduce the first example of a fundamental construction that will reappear in many guises throughout this work. We give an extremely informal definition since we will delve into the construction at work more deeply later.

Definition 2.2.3 (Homotopy Category of an abelian category)**.** *The homotopy category of an abelian category is the category whose objects are the objects of $Ch(\mathscr{A})$ and whose morphisms are $Hom(X,Y)/\sim$ for any $X, Y \in Ch(\mathscr{A})$, where, as above, $\sim$ denotes homotopy equivalence. Denote the homotopy category by $Ho(\mathscr{A})$.*

Before we comment on this construction, and on what sort of structure results from it (hint: the abelian structure will not survive), we define what is perhaps the most central object in this text in its various modifications and generalizations.

Definition 2.2.4 (Derived category of an abelian category)**.** *Consider $Ho(\mathscr{A})$ as above. Now define a new category $D(\mathscr{A})$ with the same objects and with morphisms*

$$Hom_{D(\mathscr{A})}(X,Y) = Hom_{Ho(\mathscr{A})}(X,Y)/\sim_{qiso} . \tag{2.4}$$

That is, the derived category consists of chain complexes of objects of $\mathscr{A}$, with morphisms that are considered up to chain homotopy and quasi-isomorphism. We will say much more about this.

It turns out that the abelian category structure on a category $\mathscr{A}$ does not survive the passage to the homotopy and derived categories. However, as first shown by Verdier [17], these categories nonetheless carry a unique and explicit structure which permit the use of homological techniques in dealing with them. This is the topic of Section 2.4.

2.3. Simplicial Objects in Abelian Categories

The topic of simplicial objects in abelian categories offers a perfect segue-way from homological to homotopical algebra. In point of conceptual scope, it is quite clear that the latter subsumes the former completely, but from a practical standpoint, it is useful to build precise dictionaries which mediate between the explicit and computational character of homological algebra and the abstractions of homotopical algebra. The *Dold Kan* correspondence- in its many forms-is the ideal starting point for such a dictionary. We start with the classical correspondence which gives an equivalence between connective chain complexes of abelian groups and simplicial abelian groups. Later we will encounter far more general equivalences of the Dold-Kan type, under which one of the main results underpinning the subject under scrutiny in this text, the equivalence of dg-categories and stable ∞-categories, may also be subsumed. We will not be able to discuss the dg-categorical and ∞-categorical Dold-Kan correspondences here, and will reserve them for the section on the comparison between different models of "stable categories". In what follows, our main references are Section 2.5.6 of [18], and section 8.4 of [15].

Let's introduce simplicial objects in an abelian category. We will have much occasion to study simplicial matters in great detail when we introduce higher categories and hence our discussion here will be quite concise and formal. The reader may consult [19] as a fundamental reference on simplicial sets and simplicial objects. A simplicial set is a functor $X : \Delta^{op} \to Set$, where Δ is the ordinal category with finite ordered strings $[n] := 0 < 1 < .... < n$ as objects and order-preserving maps $[n] \to [m]$ as morphisms. More generally, a simplicial object in any abelian category $\mathscr{A}$ is a functor $\Delta^{op} \to \mathscr{A}$. When $\mathscr{A} = \mathbf{Ab}$, we say $G_\bullet$ is a *simplicial abelian group*. We denote the category of simplicial abelian groups sAb. This is also an abelian category, and this will remain true for all categories of simplicial objects in abelian categories, mirroring the fact that categories of chain complexes of objects of abelian categories are also abelian.

We have the so-called standard n-simplex [1] Δ^n, which is formally the simplicial set represented by $[n]$, that is, $\Delta^n := Hom(-,[n])$. For a general simplicial set X, we call elements of the set $Hom(\Delta^n, X)$ the *n-simplices* of X. A general simplicial X set can be written as a colimit over its sets of simplices: $X = \mathrm{colim}_{\Delta^n \to X} \Delta^n$.

We have a set of special morphisms in Δ, the so-called co-simplicial maps, which generate all the others. The *coface* maps are injective maps $d^i : [n-1] \to [n]$ which, put informally, *omit* i, that is, $i \notin im(d_i)$. The *codegeneracy* maps are surjective maps $s^i : [n] \to [n-1]$ which *repeat* i, that is, such that $s_i(i) = s_i(i+1) = i$. These maps induce the *face* maps $d_i : [n] \to [n-1]$ and *degeneracy* maps $s_i : [n-1] \to [n]$ on simplicial sets. They obey the simplicial identities

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i \quad \text{when} \quad i < j, && (2.5)\\
d_i s_j &= s_{j-1} d_i \quad \text{when} \quad i < j, && (2.6)\\
&= s_j d_{i-1} \quad \text{when} \quad i > j, && (2.7)\\
&= id \quad \text{when} \quad i = j, j+1, && (2.8)\\
s_i s_j &= s_j s_{i-1} \quad \text{when} \quad i > j. && (2.9)
\end{aligned}
$$

A simplex $x_n \in Hom(\Delta^n, X)$ is said to be degenerate if it is in the image of the degeneracy map s_i.

A simplicial set X can be turned into a topological space we denote by $|X|$ by the *geometric realization* operation. Let Δ^n_{Top} be the topological simplex with the set of points $\{x = \{x_i\} \in \mathbb{R}^n_{\geq 0} | x_i \leq 1, \sum x_i = 1\}$. Then we have $|\Delta^n| = \Delta^n_{Top}$. For a general simplicial set X, we have the formula that follows from the fact mentioned above: $|X| := \mathrm{colim}_{\Delta^n \to X} \Delta^n_{Top}$. More abstractly, $|X|$ is the quotient space $\coprod_{n \in \Delta^n} X^n \times \Delta^n_{Top} / \sim$ where $\sim$ is the equivalence relation: $(\delta^*(x), d) \sim (x, \delta_*(d))$ for some map $\delta : [m] \to [n]$

[1] Ordinarily this is denoted by $\Delta[n]$ to avoid confusion with the topological n-simplex, but we will not generally observe this distinction, except in the context of geometric realization where it must be reflected in the notation.

in Δ. Geometric realization and the familiar *singular complex* construction in topology assemble into an adjunction $sSet \to Top$ which is profound significance for the theory of higher categories.

Let us now introduce the Dold-Kan correspondence. We will construct a categorical equivalence $N : sAb \to Ch_{\geq}(\mathbb{Z})$ which will formalize the intuitive equivalence between simplicial objects and chain complexes.

Let $G_\bullet$ be a simplicial abelian group. As mentioned above, we follow 2.5.6 in [18] and section 8 in [15]. We will define two chain complexes (which turn out to be quasi-isomorphic) to adequately describe the Dold-Kan maps. The *Moore complex* of $G_\bullet$ is the chain complex with components $C(G_\bullet)_n := G_n$ and equipped with the boundary map $\partial : C(G_\bullet)_n \to C(G_\bullet)_{n-1}$ where

$$\partial\sigma = \sum(-1)^n d_n(\sigma). \tag{2.10}$$

This defines a complex since the simplicial identities guarantee that

$$\partial^2\sigma = \sum(-1)^{n+m} d_n d_m \sigma = 0. \tag{2.11}$$

We also define the *normalized chain complex* of a simplicial abelian group $G_\bullet$, which is

$$NG_n := \bigcap_{i=0}^{n-1} ker(d_i). \tag{2.12}$$

This construction defines a functor $N : sAb \to Ch(\mathbb{Z})$. We now explore the relationship between the Moore complex and the normalized chain complex. Note that in the definition of the Moore complex we have not yet touched the degeneracy maps of the simplicial abelian group. Removing the degeneracy maps from the definition of a simplicial object leads to the concept of a semisimplicial object, and clearly the Moore complex can be defined for any semisimplicial abelian group. In fact, to get our desired equivalence, we will have to remove the part of our chain complex that come from the degenerate simplices. Denote by $D_n := D(G_\bullet)_n \subset C(G_\bullet)_n$ the subgroup generated by elements in the image of degeneracy maps. First of all, we note that the differential ∂

in fact preserves D_n. Put $\tau = s_i(\sigma)$ for some $\sigma \in G_{n-1}$ and some i. Then we have,

$$\begin{aligned}
\partial\tau &= \sum_{k=0}^{n}(-1)^n d_k\tau \\
&= \sum_{k=0}^{n}(-1)^n d_k s_i\sigma \\
&= \sum_{k=0}^{i-1}(-1)^n d_k s_i\sigma + \sum_{k=i}^{n}(-1)^n d_k s_i\sigma \\
&= \sum_{k=0}^{i-2}(-1)^n d_k s_i\sigma + (-1)^i d_i s_i\sigma + \sum_{k=i}^{n}(-1)^n d_k s_i\sigma.
\end{aligned}$$

Thus the image under the differential of a degenerate element remains degenerate. Then we have a well-defined quotient chain complex $M_* := C_*/D_*$, which we refer to as the *normalized Moore complex* of the simplicial group $G_\bullet$. This construction also defines a functor $N : sAb \to Ch_{\geq}(\mathbb{Z})$. It turns out the normalized chain complex and the normalized Moore complex are quasi-isomorphic and give rise to the same functor. To obtain the desired equivalence, we need to construct an adjoint to this functor, which will be called the Eilenberg-Maclane functor following 2.5.6.3 in [18]. We continue to follow the notation in [15] and Section 2.5 in [18].

Consider the n-simplex Δ^n and denote by $\Delta^n_{\mathbb{Z}}$ its normalized Moore chain complex $N_*(\Delta^n)$.

Definition 2.3.1. *The Eilenberg-Maclane space of a chain complex $M_\bullet$ is the simplicial set $K_*(M_\bullet)$ whose n-th simplices are the Hom-sets $Hom(\Delta^n_{\mathbb{Z}}, M_\bullet)$. This construction gives rise to a functor $K : Ch(\mathbb{Z})_{\geq 0} \to sAb$.*

Proposition 2.3.1. *The Eilenberg-Maclane functor K is right adjoint to N. That is, there is an isomorphism of abelian groups*

$$Hom(N(G_\bullet), M_\bullet) \cong Hom(G_\bullet, K(M_\bullet)_*). \tag{2.13}$$

This adjunction gives rise to an equivalence of categories

$$Ch(\mathbb{Z})_{\geq 0} \underset{N}{\overset{K}{\leftrightarrows}} sAb. \tag{2.14}$$

2.4. Spectra and the Stable Homotopy Category of Topological Spaces

The notion of a simplicial abelian group is only the first step in the rich dictionary between group-like topological objects and more rigid algebraic structures. It may be argued that the full topological analogue of an abelian group is furnished by the notion of a *spectrum*, which we review very superficially below. Spectra are of vital importance in our study, since our end-goal is, essentially, the construction of a *stable* homotopy category, which is an abstract category of "spectra objects". Our higher additive invariants, or cohomology theories, will land not in abelian groups or modules, but in the category of spectra or another symmetric monoidal stable ∞-category. The reader may consult [20] for a modern review of the classical theory of spectra, for which the canonical reference remains [21].

Let us begin by defining a spectrum. By a *space*, we exclusively mean a pointed CW-complex, whose category we denote by CW_*. Denote the homotopy category of spaces by $Ho(CW)$ and put $[X, Y] := Ho(CW)(X, Y)$. We suppress the basepoints in the notation but assume they are fixed and denoted by, for instance, $x_0 \in X$. The smash product $X \wedge Y$, which gives rise to a monoidal structure on CW_* and its homotopy category, is the quotient of the cartesian product $X \times Y$ by the wedge sum of topological spaces $X \vee Y := X \sqcup Y/(x_0 \sim y_0)$.

The n-spheres S^n, with $S^0 = \{0, 1\}$, play a special role in homotopy theory (this is for a variety of reasons which we shall not discuss). Namely, we define the *suspension* operator on some $X \in CW_*$,

$$\Sigma X := S^1 \wedge X. \tag{2.15}$$

This operator admits an adjoint, the so-called loop space functor, Ω, which we shall meet again soon.

Definition 2.4.1. ***A (sequential) spectrum*** *E consists of the data of a sequence $\{E_i\}$ of objects in CW_* and structure maps $\phi : \Sigma E^k \to E^{k+1}$* [2] *. A function $f : E \to F$*

[2] We are giving an extremely naïve discussion of spectra in this section.

between two spectra is a sequence of maps $f^k : E^k \to F^k$ as shown in the commutative diagram

$$\begin{array}{ccc} \Sigma E^k & \longrightarrow & E^{k+1} \\ {\scriptstyle id\wedge f^k}\downarrow & & \downarrow{\scriptstyle f^{k+1}} \\ \Sigma F^k & \longrightarrow & F^{k+1}. \end{array} \tag{2.16}$$

With some minimal modifications, the notion of a function between spectra may be promoted to a proper *morphism*, thereby giving us a category of spectra which we denote by Sp. It is a symmetric monoidal category under the smash product $\wedge$. Using this structure, one can define so-called ring spectra and *module* spectra over ring spectra and the theory that emerges is Waldhausen's *brave new algebra*, one of the main inspirations behind *higher algebra* which is a phantom presence throughout this text. We will only tangentially discuss these, and then, in extreme generality. We denote the homotopy category of spectra by $Ho(Sp)$, where the notion of a homotopy equivalence between spectra is defined in an obvious way. We denote by brackets $[-,-]$ the Hom-sets in $Ho(Sp)$. We note that $Ho(Sp)$ is one of the main examples of a *triangulated* category, a concept we will address at great length in the next chapter. Let us survey a few prominent examples of spectra.

- The sphere spectrum $\mathbb{S}$ is the spectrum $\{S^0, S^1, S^2...\}$ whose structure maps are the identity, since by definition $\Sigma S^n := S^1 \wedge S^n = S^{n+1}$. $\mathbb{S}$ is the unit for the monoidal structure on Sp and in fact the *initial* ring spectrum and, more strikingly put, is to *brave new algebra* what $\mathbb{Z}$ is to commutative algebra.
- More generally, the suspension spectrum $\Sigma^\infty X$ of a topological space X is the spectrum $\{\Sigma^0 X = X, \Sigma^1 X = \Sigma X, \Sigma \circ \Sigma X, ..., \Sigma^n X, ...\}$ with identity as structure maps.
- Variants of K-theory and, more generally, generalized cohomology theories are represented by various spectra. *Brown representability theorem* establishes the conditions under which contravariant functors on topological spaces are representable by spectra, that is, given a cohomology theory $F : Top^{op} \to Ab$, we have

an isomorphism of abelian groups $F(X) \cong [\Sigma^\infty X, K]$ [3] where the spectrum K is said to represent F.

- Recall that for G an abelian group, we have the Eilenberg-Maclane spaces $K(G,n)$ which are CW-complexes characterized up to homotopy equivalence by the fact that $\Pi_k(K(G,n)) \cong G$ when $k = n$ and $\Pi_k(K(G,n)) = 0$ otherwise. Since by the definition of loop spaces we have that $\Pi_{n-1}(\Omega X) \cong \Pi_n(X)$, we have an isomorphism $\Omega K(G,n) \cong K(G,n-1)$. These spaces assemble into a spectrum $\{K(G,n)\}$ with structure morphisms $\Sigma K(G,n-1) \to K(G,n)$ coming from the loop space-suspension adjunction. We denote this spectrum by HG. $H\mathbb{Z}$, the Eilenberg-Maclane spectrum of the integers will play a very important role later on.

2.5. Triangulated Categories

As hinted in the previous section, the homotopy and derived categories of an abelian category are in fact still *additive* categories and they also inherit a sort of weakened exact structure, the structure of a *triangulated* category. A helpful heuristic for triangulated categories is that they are "homotopified" abelian categories. In particular, despite being additive, triangulated categories typically lack the properties that guarantee the existence of kernels and cokernels of maps between objects, which means it is impossible to speak of exact sequences of objects in triangulated categories. It turns out we can still capture (although extrinsically) the exactness behavior of triangulated categories provided we weaken our notion of kernels and cokernels and replace exact sequences with a similarly weakened notion. The exact definition is somewhat complicated and unwieldy, and we provide it below. The canonical references for this section are Section 10.2 in [15] and section 4.1 in [14]. The reader is also advised to consult Neeman's book-length treatment [22] and the excellent and comprehensive treatises by Kashiwara-Shapira [23], [24].

[3] The latter has a canonical group structure because the suspension of a topological space has what's called a homotopy co-group structure, hence homotopy classes of maps from the suspension spectrum inherits an abelian group structure.

Let T be an additive category. A shift functor [4] on T is an endofunctor

$$\Sigma\colon T \to T \tag{2.17}$$

with an "inverse" Σ^{-1} such that $\Sigma \circ \Sigma^{-1} \cong Id$ and $\Sigma^{-1} \circ \Sigma \cong Id$.

Given an additive category T equipped with a shift functor and objects $X, Y, Z \in T$, define a triangle to be a diagram

$$X \to Y \to Z \to \Sigma X. \tag{2.18}$$

A morphism of triangles is given by a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'. \end{array} \tag{2.19}$$

Definition 2.5.1. *A triangulated category is an additive category equipped with a shift functor Σ and a class of triangles called distinguished or exact triangles subject to (and in fact, defined by) the following axioms.*

- **Tri.Ia** Any triangle isomorphic to an exact triangle is exact.
- **Tri.Ib** For all $X \in T$, the "trivial" triangle $X \xrightarrow{id} X \longrightarrow 0 \longrightarrow \Sigma X$ is exact.
- **Tri.Ic** Any morphism $f : X \to Y$ can be completed to an exact triangle $X \to Y \to Z \to \Sigma X$
- **Tri. II** If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is exact, so is the triangle
$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y.$$
- **Tri.III** Given any two exact triangles, $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ and $X' \longrightarrow Y' \longrightarrow Z' \longrightarrow \Sigma X'$, with maps $f : X \to X'$ and $g : Y \to Y'$ such that we have a commutative diagram
$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \longrightarrow & Y' \end{array}$$
there is a map $h : Z \to Z'$ rendering commutative the diagram

[4] also referred to as the suspension functor in the context of stable homotopy theory

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'. \end{tikzcd}$$

- **Tri.IV** (Octahedral Axiom) Consider the exact triangle

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \longrightarrow \Sigma X$$

along with the exact triangle

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \longrightarrow \Sigma Y$$

and the exact triangle

$$X \xrightarrow{gf} Z \xrightarrow{l} Y' \longrightarrow \Sigma X.$$

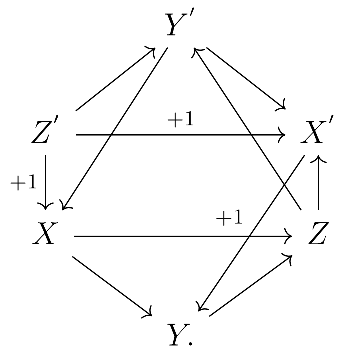
Then there exists a fourth exact triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma Z'$$

such that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y. \arrow[r, "h"] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "k"] \arrow[d] & X' \arrow[r] \arrow[d] & \Sigma Y \arrow[d] \\ X \arrow[r, "gf"] \arrow[d] & Z \arrow[r, "l"] \arrow[d] & Y' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & \Sigma Z'. \end{tikzcd}$$

The diagram given in last axiom can be reorganized in a way that explains the name as the diagram



Definition 2.5.2. *An additive functor F is said to be exact (or triangulated) if it respects the triangulated structure, i.e., if, in addition to being additive, it commutes with the shift functor and preserves exact triangles.*

Definition 2.5.3. *A subcategory $T' \subset T$ is said to be triangulated if the inclusion is a triangulated functor.*

Triangulated categories naturally form a 2-category with exact functors as morphisms and natural transformations as 2-morphisms. However, we will not be working with this 2-category at all since many technical problems that plague the very foundations of the formalism of triangulated categories make it quite ill-behaved. Let us mention one particular issue while deferring a discussion of its consequences to the end of the chapter.

Consider the exact triangle $\Sigma^{-1}X \xrightarrow{0} Y \longrightarrow X \oplus Y \longrightarrow X$. Recall that $Tri.III$ guarantees the existence of a map h giving a morphism of triangles

$$\begin{tikzcd} \Sigma^{-1}X \arrow[r, "0"] \arrow[d] & Y \arrow[r] \arrow[d] & X \oplus Y \arrow[r] \arrow[d, dotted, "h"] & X \arrow[d] \\ \Sigma^{-1}X \arrow[r, "0"] & Y \arrow[r] & X \oplus Y \arrow[r] & X. \end{tikzcd} \tag{2.20}$$

By inspection, however, we can see that for any $f \in Hom_C(X,Y)$, we can put: $h = \left(\begin{smallmatrix} id_X & 0 \\ f & id_Y \end{smallmatrix}\right)$. Therefore we have a lot of freedom in choosing this h and it is far from unique. This renders the mapping cone construction nonfunctorial at the triangulated level. We shall see how to deal with this issue in the later sections.

Definition 2.5.4. *An additive functor $F : T \to \mathscr{A}$ from a triangulated category T to an abelian category $\mathscr{A}$, is said to be cohomological if it sends exact triangles to exact sequences. That is, an exact triangle $A \to B \to C \to \Sigma A$, is mapped by Hom to an exact sequence. $Hom(A,-)$, $Hom(-,A)$ are cohomological functors.*

Let us introduce the two main examples of triangulated categories.

Examples 2.5.1. *(i) Let A be an abelian category and we have as before the homotopy and derived categories of A: $Ho(A) := Ho(Ch(A))$ and $D(A) := D(Ch(A))$ as in the first section. We will discuss the outlines of the triangulated structure on these categories without proof. Put $T = Ho(A)$. The suspension functor is nothing but the shift functor on chain complexes, that is, $(C[1]_n, d_n^{C[1]}) := \Sigma(C_n, d_n^C) = (C_{n-1}, d_{n-1}^C)$. The functor*

$$\Sigma \colon C \mapsto C[1] = (C_{n-1}, -d_{n-1}) \tag{2.21}$$

acts on chain maps in an obvious way.

Given a morphism $f : C \to D$, define the mapping cone M_f

$$M(f)_n := C_{n-1} \oplus D_n \tag{2.22}$$

equipped with the differential

$$d_n^{M(f)} = \begin{bmatrix} -d_{n-1}^C & 0 \\ f_{n-1} & d_n^D \end{bmatrix} \tag{2.23}$$

which can be readily confirmed to square to zero. The distinguished triangles are those isomorphic to the so-called standard triangles

$$X \xrightarrow{f} Y \xrightarrow{(0,id_Y)} M(f) \xrightarrow{(id_X,0)} X[1]. \tag{2.24}$$

(ii) *The stable homotopy category of topological spaces, consisting of so-called spectra, is triangulated. Note, however, that it is not the derived category of an abelian category!*

Before concluding this section, we'd like to advertise the following perspective on triangulated categories which was hinted at in the introduction and which provides an excellent bridge between classical theory and the formalisms discussed in the bulk of this text: a triangulated category is a *homotopified* abelian category. That is, as long as we consider sufficiently weakened or homotopy versions of constructions in abelian categories such as kernels and cokernels, the structure of triangulated categories become intelligible and conceptually more satisfying.

We will not review the concept of homotopy co/limits since we discuss related constructions at length in the context of enriched and higher categories and we have not yet introduced model categories at this stage. The classical reference is [25] and practical introductions can be found in [26] and [27]. The main references in the context of triangulated categories are the books and articles by Neeman and collaborators, for instance, see [22] and [26]. We will focus on the mapping cone M_f introduced above for chain complexes and present it as an example of a homotopy cofiber [5] . We will then construct the functor underlying this construction at the chain complex level and

[5]To be consistent with the notation in the section on abelian categories, one may also refer to a homotopy cofiber as a homotopy cokernel, as we do at the end of the section. But since we wish to describe the construction in a general homotopical context, we'll stick to the topological nomenclature.

discuss why it fails at the derived level. We loosely follow lecture notes by Mazel-Gee [28] for notational consistency but the content is self-evidently universal.

Going back to Mod_R, recall that the *cokernel* of a morphism $f : M \to N$ of R-modules is an R-module $cofib(f)$ equipped with a map $N \to cofib(f)$ which is universal among maps which give zero when composed with f. The situation is exactly replicated for $Ch(Mod_R)$. However, equipped with the notions of chain homotopy and quasiequivalence, we can weaken or *homotopify* this construction as follows. Namely, consider a chain morphism $f : M_\bullet \to N_\bullet$ [6]. The *homotopy* cofiber $hocofib(f)_\bullet$ of this morphism is a chain complex of R-modules equipped with a map $N \to hocofib(f)$ which is universal among maps whose composite with f is *nullhomotopic* to zero.

As expressed by [28] in the spirit of our discussion of colimits in higher categories in later sections, a homotopy cokernel is the homotopy colimit of the diagram of shape $* \longleftarrow \bullet \longrightarrow \bullet$, that is, $hocofib(f)$ fits into the homotopy coherent diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ * \arrow[r] & hocofib(f). \end{tikzcd} \tag{2.25}$$

The reader may consult [29] for the notion of a homotopy coherent diagram.

We have the following fact which illustrates the role of *weakened* or homotopified constructions as akin to resolutions in homological algebra. This is a point that is made very precise by the formalism of model categories.

Lemma 2.5.1. *When f is injective, the canonical map $h : hocofib(f) \to coker(f)$ is a quasi-isomorphism [28].*

This is easy to see by using the fact that a chain map is a quasi-isomorphism if and only if its homotopy cofiber is acyclic.

[6] We will omit bullets from now on.

Now let's return to the derived category of chain complexes of R-modules with its triangulated structure discussed above. For a given morphism $f : X \to Y$, $hocofib(f)$ is nothing but the mapping cone M_f by construction.The standard triangles in the triangulated structure on the derived category exhibit the fact that M_f is a homotopy pushout.

Going back to the abelian setting for a setting and putting $\Delta^1 := [0, 1]$ for the free arrow category, the cokernel construction has an underlying functor $coker : Fun([0 \to 1], Mod_R) \to Ch(Mod_R)$ or, although there is no explicit formula for expressions like $coker(f \circ g)$ where f, g are morphisms in $Ch(Mod_R)$. Moreoever, by simply substituting the definitions, one sees that the $coker(f \circ g)$ is involved in an exact sequence

$$0 \longrightarrow coker(f) \longrightarrow coker(g \circ f) \longrightarrow coker(g) \longrightarrow 0. \tag{2.26}$$

This suggests that there is an exact triangle at the derived level of which this exact sequence is a shadow, and this is in fact the case as shown by the exact sequence

$$M_f \longrightarrow M_{g \circ f} \longrightarrow M_g. \tag{2.27}$$

All our problems originate with the fact that the object $M_{f \circ g}$ is not uniquely defined (it is only unique with respect to non-unique choice), hence we do not have a *strict* functor $hocofib : Fun(\Delta^1, D(R)) \to D(R)$. We would have this problem with any diagram category $Fun(D, T)$ with T triangulated. This is the origin of the theory of derivators [30], [31]. which was incidentally the original framework for the the first theory of noncommutative motives introduced by [1]. This is a point at which all the defects of triangulated categories converge. For instance, even for $D = \Delta^1$, $Fun(D, T)$ is not naturally triangulated. The situation is even more hopeless for more complicated shapes such as $D = \Delta^1 \times \Delta^1$ which describes commutative squares. Hence it is impossible to carry out functorial constructions in the framework of triangulated categories naively. The reader is advised to consult [31] for how the approach via derivators addresses this problem directly.

Let us close this discussion by rephrasing the triangulated category axioms in the light of this new perspective. It is not hard to verify that the following statements are a homotopical reformulation of triangulated category axioms.

(i) The homotopy kernel and cokernel of the identity morphism are zero.
(ii) Every morphism admits a homotopy co/kernel
(iii) Any morphism is the homotopy kernel of its homotopy cokernel and homotopy cokernel of its homotopy kernel.
(iv) Homotopy kernels and cokernels are *weakly* functorial.

We proceed to offer a glimpse of what might be called "triangulated algebra", one prominent branch of categorical algebra which generalizes ring and module-theoretic concepts to the context of triangulated categories. Our aim is basically to define the analogues of ideals (prime, maximal etc.) and quotients for triangulated categories. The reader may consult Balmer's survey of tensor triangulated geometry [32] and Krause's expository text on localization in triangulated categories [33], while the latter is our main reference for this section.

We say a triangulated subcategory $S \subset T$ is *thick* if $X \oplus Y \in S$ implies that X and Y are in S. One motivating example which reinforces the analogy with algebra is the following: let T,$T^{'}$ be triangulated categories and $F : T \to T^{'}$ an exact functor between them. The *kernel* of F is the subcategory $KerF$ with objects $\{X \in T | FX \cong 0\}$. Evidently, an exact functor is in particular additive, so if $X \oplus Y \in KerF$, then $F(X \oplus Y) \cong FX \oplus FY \cong 0$, and hence $X, Y \in KerF$. Now we define a certain localization $T/T^{'}$ of T with respect to a *full* triangulated subcategory $T^{'}$ which is itself triangulated and behaves like a quotient. Denote by M_S the morphisms f in T such that $cone(f) \in T^{'}$. Then we put $T/T^{'} := T[M_S^{-1}]$. By the discussion in Section 4 of [33], localizations of triangulated categories are triangulated and the canonical quotient map $F : T \to T/T^{'}$ is an exact functor. It is easy to check that the objects in $T^{'}$ are annihilated by F and the universal property of the quotient comes from that of localizations. In fact, if $T^{'}$ is a thick subcategory $T^{'} \cong KerF$ and we can establish analogues of isomorphism theorems such as $T/KerF \cong ImT$, where ImT is the essential image of the quotient functor. An immediate example is provided by the derived category of an abelian category, which can be defined as the Verdier quotient of the homotopy category $Ho(A)$ by the subcategory of acyclic complexes. Thereby it is easy to establish the triangulated structure and universal property of the derived category.

We close this section by discussing compactly generated triangulated categories which are a very important stepping stone to regarding categories as "noncommutative spaces" of some kind. The latter perspective revolves around the notion of categorical cohomology theories and which functors qualify as such and whether they are representable. Brown representability theorem we mentioned in the section on spectra comes into play: compactly generated categories afford the correct setting for the study of representability of functors out of triangulated categories and hence that of generalized cohomology theories. The reader may consult [22] for reference on the main definitions, which are standard in category theory.

Let T be an additive category which admits arbitrary coproducts. A compact object in T is an object X such that the functor $Hom(X, -)$ preserves all coproducts, which is to say, for any collection Y_i, we have an isomorphism $Hom(X, \bigoplus Y_i) \cong \bigoplus Hom(X, Y_i)$. This definition in fact works in any category as long as we replace "coproduct" generally with "filtered colimit" and assume the category admits all filtered colimits. However, the reader should beware that "compact object" does not readily translate to more familiar versions of compactness. For instance, a compact object in Top, the category of topological spaces, is *more* than a compact topological space. However, the inspiration for the name comes from the following equivalent characterization of compact objects: they are precisely those objects for which any map $X \to \bigoplus Y_i$ factors through some finite sub-coproduct.

Definition 2.5.5. *A triangulated category T is said to be compactly generated if there is a set of compact objects S such that for any $X \in S$, $Hom(X, Y) = 0$ implies $Y = 0$ for all $Y \in T$.*

Easy examples of compact objects are given by finitely generated modules in module categories. In triangulated categories, we have an exact characterization of compact objects, given by the following result in [34].

Proposition 2.5.1. *(Theorem 3.4 of [34]) Compact objects in a compactly generated triangulated category T with a generating set S are retracts of extensions of shifts of objects in S.*

Going back to the abelian context briefly, somewhat in the spirit of *Morita theory* we will address later, the question of compact generators is best motivated by the following example which is one of the main inspirations behind the notion of "noncommutative algebraic geometry" as expounded, say, in Section 2 of Ginzburg's lectures on noncommutative geometry [35]. It is the most elementary example of a "reconstruction theorem":

Proposition 2.5.2. *Let $\mathscr{A}$ be an abelian category which admits a projective [7] and compact object X. Then $\mathscr{A} \cong (End(X))^{op}$-Mod.*

Proof See Proposition. 2.3 in [35].

The relevance of theorems of this type for "noncommutative algebraic geometry" can be explicated as follows. If for instance, the abelian category in question is the category $QCoh(X)$ of quasicoherent sheaves on a scheme X, which is not necessarily affine, and it admits a projecive generator, then we can think of X as being affine in a noncommutative sense. Noncommutative geometry in this sense was envisioned most explicitly by Kontsevich in the text [3] where he offered the first speculative definition of noncommutative motives associated with noncommutative spaces, as we mentioned in the introduction.

However, such theorems are rarely interesting in the abelian setting. More interesting is the question of *derived* affiness. Then naturally we are led to the question of how to reproduce this theory in the triangulated setting. We will focus on the geometric setting, since this is what is of interest for the theory of noncommutative motives. Then we let $D(X)$ denote the derived category of quasicoherent sheaves on a scheme. We have the following crucial result, 3.1.1 in [36].

[7] A projective object in an abelian category is an object X such that $Hom(X,-)$ is exact. This notion obviously generalizes that of projective modules.

Proposition 2.5.3. *(Proposition 3.1.1 in [36]) Let X be a quasicompact and quasiseparated scheme. Compact objects in $D^b(X)$ are perfect complexes of locally free sheaves and the category admits a compact generator.*

The following related result due to Keller [37] reproduces the abelian reconstruction theorem above in the derived setting.

Proposition 2.5.4. *(Theorem 3.17 and Corollary 3.18 in [36]) Let X be a quasicompact and quasiseparated scheme. Then $D(X)$ is equivalent to the derived category of dg-modules over some dg-algebra A.*

This proposition establishes that quasicompact, quasiseparated schemes are *derived* or noncommutative affine in the sense of Kontsevich and others. This is one of the origins of the philosophy of noncommutative algebraic geometry.

The relevance of compactly generated triangulated categories for the study of cohomology theories comes from *Brown representability theorem*, already discussed briefy when we introduced spectra.

Theorem 2.5.1. *(i) Given an exact functor $F : S \to T$ with T a compactly generated triangulated category and S a triangulated category. For F to admit an right adjoint, it suffices for it to preserve coproducts [22].*

(ii) Let $F : T^{op} \to Ab$ be a cohomological functor from a compactly generated triangulated category to the category of abelian groups. Then F is representable if and only if F preserves coproducts.

The statement of the second item can be found in Section 3 of [34].

Finally, let us define the concept of a *semiorthogonal decomposition* for triangulated categories which generalizes the well-known exceptional collections in algebraic geometry. Our reference for this section is [38].

Definition 2.5.6. *Let T be a triangulated category. A semiorthogonal decomposition of T is a set $\{E_i\}$ of full triangulated subcategories of T such that:*

(i) Any two E_{i_1} and E_{i_2} are Hom-orthogonal when $i_1 \neq i_2$, that is, if $e \in E_{i_1}$ and $f \in E_{i_2}$ with $i_1 \neq i_2$, $Hom(e, f) = 0$.

(ii) $\{E_i\}$ span T in the sense that the smallest triangulated subcategory containing $\{E_i\}$ is T itself. We put $T =< E_i >:=< E_{i_1}, ...E_{i_n} >$ to indicate a semiorthogonal decomposition.

The classical examples of a semiorthogonal decompositions are provided by exceptional collections in derived categories of schemes. Recall that an exceptional object in the derived category of a scheme over some field k is an object $E \in D(X)$ such that $Hom(E, E) = k$ and $Hom(E, E[n]) = 0$ except when $n = 0$. An exceptional collection in a derived category is a collection $\{E_i\}$ of exceptional objects that are Hom-orthogonal, that is, $Hom(E_{i_1}, E_{i_2}[n]) = 0$ for all n. Evidently, putting E_i for the subcategory generated by the exceptional object and $E^{\perp}$ for the subcategory of D which is Hom-orthogonal to the subcategory spanned by $< E_i >$, we have $D(X) =< E_i, E^{\perp} >$, a semiorthogonal decomposition of $D(X)$.

The most famous example of an exceptional collection is provided by Beilinson's exceptional collection, consisting of the twisting sheaves $< O, ..., O(-n) >$ which give a semiorthogonal decomposition of the bounded derived category of the projective space $D^b(\mathbb{P}^n)$. This decomposition turns out to be of vital importance in the foundations of noncommutative motives, as we shall see in the final chapter. Suffice to say for now that these twisting sheaves provide the noncommutative analogues of the Tate twists and are responsible for making the image of the derived category of the projective line automatically invertible in the homotopy category of noncommutative spaces.

2.6. Enriched Categories

We have already encountered a cardinal example of an enriched category in the concept of an additive category, which is a category enriched in vector spaces over some field. Along the same lines, we have the concept of an R-linear category, which is a category enriched in modules over some commutative ring R. Thus, intuitively, a category C enriched over some other category V is nothing but a category whose

Hom-spaces live not only in Set but can be seen as objects of V. Thus ordinary small categories may be described as Set-enriched, since in the absence of further structure, their Hom's form a set. More formally, a V-enriched category consists of the following:

Definition 2.6.1. *Let $(V, \otimes)$ be a monoidal category. C is said to be enriched over $(V, \otimes)$ if given any $x, y \in Ob(C)$, we have an object $Hom(x, y) \in Ob(V)$ equipped with a composition morphism*

$$Hom(x, y) \otimes Hom(y, z) \to Hom(x, z) \tag{2.28}$$

along with a morphism encoding the identity element and commutative diagrams encoding the associativity and unitality of the "composition" product.

We will encounter many enriched categories throughout this work. The ones of immediate interest are additive, topological, differential graded, simplicial and spectral categories. We have already met additive and abelian categories, and the bulk of this thesis is devoted to a detailed study of differential graded (DG) categories. Let us introduce simplicial and spectral categories.

Definition 2.6.2 (Topological categories)**.** *A **topological category** is a category enriched over the monoidal category of topological spaces $Top^{\times}$, with the monoidal structure coming from the Cartesian product.*

Definition 2.6.3 (Simplicial Categories)**.** *A **simplicial category** is a category enriched over $sSet^{\otimes}$. We denote the category of simplicial categories by Cat_{Δ}.*

Definition 2.6.4 (Spectral Categories)**.** *A **spectral category** is a category enriched over (some appropriate version of) the monoidal category of spectra $Sp^{\otimes}$. We denote the category of spectral categories by Cat_{Sp}.*

There are also "linear" version of the enriched categories above, such as the category of simplicial module-enriched categories Cat_{sMod} and the category of spectral module-enriched categories Cat_{Mod_S}.

2.7. DG-categories

In this section, we introduce the elementary theory of dg-categories, which we will gradually refine by way of homotopical algebra and category theory into *the* theory of *noncommutative spaces.* Our references for the fundamental aspects of the theory of dg-categories are the lecture notes by Toën [39] and Keller's sweeping survey [34].

Fix a commutative ring R. Let $V^\bullet, W^\bullet$ be graded R-modules. We fix notation for some fundamental constructions in graded algebra. We say a morphism of graded modules $f : V^\bullet \to W^\bullet$ is of degree n if for all $p \in \mathbb{Z}$, $f(V^p) \subset W^{p+n}$. The tensor product of graded modules $V^\bullet$ and $W^\bullet$ is the graded module

$$(V^\bullet \otimes W^\bullet)^n := \bigoplus_{p+q=n} V^p \otimes W^q. \tag{2.29}$$

Now consider two graded morphisms $f : V_1^\bullet \to V_2^\bullet, g : W_1^\bullet \to W_2^\bullet$ with $deg(f) = p, deg(g) = q$. The tensor product $f \otimes g : (V_1^\bullet \otimes W_1^\bullet)^\bullet \to (V_2^\bullet \otimes W_2^\bullet)^\bullet$ is defined by the Koszul sign rule: $(f \otimes g)(v \otimes w) = (-1)^{pq} f(v) \otimes f(w)$. Conventions for graded algebras (i.e., graded modules equipped with a morphism $A^\bullet \otimes A^\bullet \to A^\bullet$ and a unit $1 \in A^\bullet$) are identical.

A differential graded module $(V^\bullet, d^\bullet)$ is a graded module $V^\bullet$ equipped with a map, called the differential, $d^\bullet : V^\bullet \to V^\bullet$ such that $d^{n+1} \circ d^n = 0$ for all n. The tensor product of graded modules may be extended to that on differential graded modules by putting $(V, d_V) \otimes (W, d_W) := (V \otimes W, d_V \otimes 1_W + 1_V \otimes d_W)$. Note that the data of a differential graded module is in fact equivalent to that of a chain complex of modules. Similarly, a differential graded algebra is a graded algebra where now, in addition to $d^2 = 0$, the differential has to satisfy the Leibniz rule: $d(f \circ g) = df \circ g + (-1)^n f \circ dg$, where $n = deg(f)$. From our point of view on categorical algebra, it is most practical to regard dg-algebras as monoid objects in chain complexes. With these in mind, we arrive at the definition of the most fundamental object in this work.

Definition 2.7.1. *A differential graded category over a ring R, or an R-linear dg-category, is a category enriched in dg-modules. More concretely, we may regard a dg-category as a $Ch(R)$-enriched category, where $Ch(R)$ denotes the category of chain* *complexes of R-modules.*

Let us unpack the concept of enrichment in this concrete context. Let A be a dg-category, denote by $A(X,Y)$ the hom dg-modules for any $X, Y \in Ob(A)$. Then we have a morphism of dg-modules, "the composition law"

$$m : A(X,Y) \otimes A(Y,Z) \to A(X,Z). \tag{2.30}$$

Further, for any object $X \in A$, $A(X,X)$ is a unital dg-algebra, that is we have an element of degree zero $1_X \in A(X,X)_0$, or equivalently, an identity map $1_X : R \to A(X,X)$. Unpacking the aforementioned conditions governing associativity and the identity map gives us the following commutative diagrams.

Associativity of multiplication is encoded by the diagram

$$\begin{tikzcd} A(X,Y) \otimes A(Y,Z) \otimes A(Z,U) \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"] & A(X,Y) \otimes A(Y,U) \arrow[d, "m"] \\ A(X,Z) \otimes A(Z,U) \arrow[r, "m"'] & A(X,U), \end{tikzcd} \tag{2.31}$$

whereas the two-sided identity is encoded by the diagram

$$\begin{tikzcd} R \otimes A(X,Y) \arrow[r, "1_X\otimes id"] \arrow[dr, "\cong"'] & A(X,X) \otimes A(X,Y), \arrow[d, "m"] \\ & A(X,Y), \end{tikzcd} \tag{2.32}$$

and the diagram

$$\begin{tikzcd} A(Y,X) \otimes R \arrow[r, "id\otimes 1_X"] \arrow[dr, "\cong"'] & A(Y,X) \otimes A(X,X) \arrow[d, "m"] \\ & A(Y,X). \end{tikzcd} \tag{2.33}$$

As in ordinary category theory, we denote by A^{op} the opposite dg-category of a dg-category A, defined as the category with hom-complexes $A^{op}(X,Y) := A(Y,X)$.

Examples 2.7.1. • *Any dg-algebra can be considered as a dg-category with one object. Namely, consider a dg-category A with a single object, X. Then, we have* *a dg-algebra structure on A(X,X) induced by the composition law on the hom-complexes of A. On the other hand, given any dg-algebra B, we may define a dg-category with a single object X, with B:= A(X,X), where, in turn, the dg-algebra structure on B gives rise to the composition law on A, see [39]. This construction is actually quite useful and gives us an embedding of dg-algebras (and hence of all associative algebras, considered as dg-algebras with trivial differential) into the category of pointed dg-categories* $DG - Cat_* := Fun(1_{dg}, DG - Cat)$*, that is dg-categories equipped with a zero object. This embedding admits an adjoint which sends a pointed dg-category to the endomorphism dg-algebra of its zero object. For the details about and the importance of this construction, see [40].*

- *The category of chain complexes or R-modules over a ring* $Ch_{dg}(R)$*. This example may be freely generalized to complexes of objects in any abelian category. Also, the category of dg-modules over any dg-algebra also naturally forms a dg-category. We give an explicit description of* $Ch_{dg}(R)$ *as a dg-enrichment of the ordinary abelian category of chain complexes. Given* $A, B \in Ch(R)$*, we have the hom-complex* $Hom(A, B)$ *of morphisms between A and B (I omit the bullets and dots when denoting complexes from now on). Concretely, the R-module of degree n morphisms may be given as*

$$Hom^n(A, B) := \prod_{i \in \mathbf{Z}} Hom(A^i, B^{i+n}). \tag{2.34}$$

The differential of on the Hom-complex can be described explicitly as the map that sends the map $f := \{f^i\}$ *of degree k to the map* $\{d_B \circ f - (-1)^k f \circ d_A\}$*. This defines a differential since*

$$\begin{aligned} d \circ d \circ f &= d_B(d_B \circ f - (-1)^k f \circ d_A) - (-1)^{k+1}(d_B \circ f - (-1)^k f \circ d_A) \circ d_A \\ &= (-1)^k d_B \circ f \circ d_A - (-1)^{k+1} d_B \circ f \circ d_A \\ &= 0. \end{aligned}$$

Setting n=0, we obtain that

$$Hom^0(A, B) = \prod_{i \in \mathbf{Z}} Hom(A^i, B^i) \tag{2.35}$$

which is just the set of chain morphisms between A and B, since a degree zero

morphism is precisely a morphism that "commutes" with or exchanges the differentials.
Finally, graded composition of morphisms gives us the multiplication morphism in $Ch_{dg}(R)$

$$Hom^n(A,B) \times Hom^m(A,B) \to Hom^{n+m}(A,B). \tag{2.36}$$

Thus, we have promoted $Ch(R)$ *to a dg-category* $Ch_{dg}(R)$ *by way of self-enrichment.*

- *Triangulated categories of geometric origin can always be enhanced to a DG-category. In this way, extremely important objects like the derived categories of schemes can be considered as DG-categories. In fact, it is quite essential that we do consider them as such! For instance, the foundations of the theory of spherical functors which arise in the context of homological mirror symmetry and the theory of Fourier-Mukai transforms, rely essentially on the existence of DG-enrichments of derived categories. From this perspective, Fourier-Mukai theory can be considered as a chapter of the Morita theory of dg-categories, as we shall see.*
- *Categories of matrix factorizations associated with a singularity* f. *These appear to be the only examples of dg-categories not of algebraic or geometric origin and are a fertile source of applications for the theory outlined in this text. We will not define or touch on them in any greater detail.*
- *We have the following dg-categories which are crucial for the description of the homotopical cell structure of the category of dg-categories, particularly its structure as a cofibrantly generated category. We faithfully follow Tabuada's dissertation [41] which brought to a rigorous completion the Dwyer-Kan homotopy theory of dg-categories. First of all, we introduce some chain complexes which are analogues of the topological sphere and the disk. Fix an* R*-module* M*, we work inside* $Ch(R)$*. The sphere object in chain complexes* $S^n(M)$ *is a chain complex such that* $S^n(M)_n = M$ *and* $S^n(M)_k = 0$ *otherwise. Further, the disk object* $D^n(M)$ *in chain complexes is the chain complex concentrated in degrees* $n-1$ *and* n *with both components equal to* M*. We put* $S^n := S^n(R)$ *and* $D^n := D^n(R)$*. Clearly we*

have an inclusion $S^{n-1} \to D^n$ which is just the identity on the $n-1$-th component. We let $S(n)$ be the dg-category diagramatically exhibited by the diagram

$$\begin{tikzcd} 1 \arrow[loop above, "R"] \arrow[d, "S^{n-1}"] \\ 2 \arrow[loop below, "R"] \end{tikzcd} \tag{2.37}$$

where the annotations on the arrows indicate that the morphism dg-modules are $Hom_{S(n)}(1,1) = Hom_{S(n)}(2,2) = R$ and $Hom_{S(n)}(1,2) = S^{n-1}$. Similarly, let $D(n)$ be the dg-category presented by the diagram

$$\begin{tikzcd} 3 \arrow[loop above, "R"] \arrow[d, "D^{n-1}"] \\ 4. \arrow[loop below, "R"] \end{tikzcd} \tag{2.38}$$

We have a dg-analogue of sphere inclusion above: $i(n) : S(n-1) \to D(n)$. On objects, we have $i(1) = 3$, $i(2) = 4$. On morphisms $i(n)$ is nothing but the chain complex sphere inclusion $S^{n-1} \to D^{n-1}$ described above. Denote by I the set of dg-functors $i(n)$. We say a dg-category is a dg-cell if the map from the initial dg-category is a transfinite compositions of pushouts of I. These constructions will be relevant later when we discuss the homotopy theory of dg-categories.

- *We have the initial dg-category $\emptyset$ and the terminal dg-category $*$, which is the dg-category with one object 1 and morphism the empty set $Hom(1,1) = *$ We also have the unit dg-category 1_{dg_R}[8] which is nothing but the ring R seen as a dg-category with one object and with trivial differentials. 1_{dg} is a unit for the monoidal structure on dg-categories.*
- *Denote by Δ^1_R the dg-category with objects $\{1,2\}$ and morphisms: $\Delta(1,1) = \Delta(1,2) = R$ and $\Delta(2,2) = \Delta(2,1) = 0$. We call this dg-category the 1-simplex dg-category, in fact, it is the R-linearization of the standard 1-simplex Δ^1 and of the ordinary category $[1] := 1 \to 2$ which represents the latter . It will be important in the description of morphisms in $DG - Cat$, like its simplicial counterpart Δ^1.*

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[8] We will usually omit the base ring R or when it needs to be emphasized write 1_R instead.

Crudely speaking, the assignment $R \to DG - Cat_R$ admits some transparent exceptional functorialities. For any ring morphism $f : R \to S$, we have a forgetful functor acting on the hom-spaces by componentwise restriction of scalars of the S-modules, thus giving us a functor $f^* : DG - Cat_S \to DG - Cat_R$. As usual, this restriction functor admits a left adjoint $- \otimes_R S$ and a right adjoint $Hom(1_S, -)$.

Let us introduce two functors, $Z^0, H^0 : DG - Cat_R \to Cat$ which will serve as a bridge between differential graded and ordinary categories.

We define the *underlying category* $Z^0(A)$ of a dg-category A as the category with the same objects as A and with the morphisms only the degree zero morphisms

$$Z^0(A(X,Y)) := ker(d : A^0(X,Y) \to A^1(X,Y)). \tag{2.39}$$

Evidently,the composition $f \circ g$ of degree 0 morphisms f, g is itself of degree 0, since $d(f \circ g) = df \circ g + f \circ dg = 0$. Hence we have a well-defined composition law on $Z^0(A(X,Y))$ and it is confirmed that this defines a category. It is seen immediately that $Z^0(Ch_{dg}(R)) = Ch(R)$. More generally, $Z^0(Ch_{dg}(A)) = Ch(A)$ where A is a Grothendieck abelian category. In the same vein, the *homotopy category* $H^0(A)$ of a dg-category A is the category with morphisms

$$H^0(A(X,Y)) := A(X,Y)/\sim_{ho} = Hom_{Ho(Ch(R))}(X,Y) \tag{2.40}$$

where $\sim_{ho}$ denotes chain homotopy.

By definition, $H^0(Ch_{dg}(R)) = Ho(Ch(R))$, the usual homotopy category of the abelian category of chain complexes of R-modules.

We say an object X of a dg-category A is *contractible* if the dg-algebras $A(X,X)$ are *acylic*, that is if $H_*A(X,X) = 0$. In particular, if X is contractible, $H_0(A(X,X)) = 0$ and we have an element $h \in A(X,X)_1$ such that $dh = 1_X$. We refer to this h as the nullhomotopy or the contraction of X. In the case of chain complexes, such an h is indeed nothing but the null-homotopy of the identity chain morphism, making the chain complex contractible. Hence the terminology.

Consider dg-categories A, B. A *dg-functor* (or a morphism of dg-categories) $F : A \to B$ consists of

(i) A map $F : Ob(A) \to Ob(B)$,
(ii) Morphisms of dg-modules $F(X,Y) : A(X,Y) \to B(FX, FY)$.

along with diagrams encoding the conditions regarding associativity of multiplication and identity, which can be presented as follows.

For all $X, Y, Z \in A$, the associativity is encoded by a commutative diagram

$$\begin{tikzcd} A(X,Y) \otimes A(Y,Z) \arrow[r, "m_A"] \arrow[d, "F(X,Y)\otimes F(Y,Z)"'] & A(X,Z) \arrow[d, "F(X,Z)"] \\ B(F(X),F(Y)) \otimes B(F(Y),F(Z))) \arrow[r, "m_B"'] & B(F(X),F(Y)). \end{tikzcd} \tag{2.41}$$

For all $X \in A$, identity for the multiplication is encoded by the diagram

$$\begin{tikzcd} R \arrow[r, "id_X"] \arrow[dr, "id_{F(X)}"'] & A(X,X) \arrow[d, "F(X,X)"] \\ & B(F(X),F(X)). \end{tikzcd} \tag{2.42}$$

There is an intuitive notion of a natural transformation of dg-functors, with a slight sublety introduced by the graded nature of the objects at hand. Ultimately, we should seek to obtain a *complex* of natural transformations between dg-functors which can reflect this nature. Hence we first define a natural transformation $\eta : F \to G$ of degree n between dg-functors F, G to be a family of dg-functors $\eta_X \in Hom^n(F(X), G(X))$ parametrized by all $X \in A$ giving a "graded commutative" diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(\phi)"'] & G(X) \arrow[d, "G(\phi)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y). \end{tikzcd} \tag{2.43}$$

That is, we have $F(\phi)\ \eta_Y = (-1)^{|nm|}\ \eta_X\ G(\phi)$, where m is the degree of the morphism $\phi : X \to Y$. These R-modules assemble into a complex which we denote by $Hom(F, G)$, where the confusion with other Hom-spaces should be ruled out by the context. As we shall see, this construction allows us to promote the category of dg-categories to a self-enriched category by way of an internal hom-object which gives rise to a closed monoidal structure.

The appropriate notion of equivalence in the dg-context is *quasiequivalence.*

Definition 2.7.2. *A dg-functor $F : A \to B$ is said to be a quasiequivalence if:*

- *(Quasi-fully faithful) $F(X, Y)$ is a quasi-isomorphism for all $X, Y \in A$*
- *(Quasi-essentially surjective) The induced morphisms between homotopy categories $H^0(F) : H^0(A) \to H^0(B)$ is essentially surjective.*

Dg-categories assemble into a category DG-Cat, which has a natural closed symmetric monoidal structure. Given dg-categories A, B, define $A \otimes B$ as the category with objects those of $A \times B$ and morphisms

$$A \otimes B((X,Y),(X^{'},Y^{'})) := A(X,X^{'}) \otimes B(Y,Y^{'}). \tag{2.44}$$

We denote by $\mathscr{Hom}(-,-)$ the internal Hom of DG-Cat, whose construction we omit. It is easy to verify pointwise that:

$$Hom(A \otimes B, C) \cong Hom(A, \mathscr{Hom}(B,C)). \tag{2.45}$$

which gives the aforementioned *closed* monoidal structure. We denote by $DG - Cat^{\otimes}$ the resulting symmetric monoidal category.

We wil end this introductory section on dg-categories with discussion of module theory over dg-categories, which will be essential in what follows.

Consider a dg-category A. A right (left) $dg - module$ over A is a dg-functor $M : A^{op} \to Ch_{dg}(R)$ (resp.a dg-functor $A \to Ch_{dg}(R)$). To see why we refer to these chain complex valued presheaves on dg-categories as "modules", note that a functor $A^{op} \to Ch_{dg}(R)$ is equivalent to the data of a family of chain complexes M_X, where X varies over A, equipped with an A-action: $M_X \otimes A(X,Y) \to M_Y$ (once again, with the right compatibilities with unit and associativity conditions). A morphism of dg-modules is simply a natural transformation of dg-functors: given dg-modules M, N, $Hom(M,N)$ is the set $Z^0(\mathscr{Hom}(M,N))$. We denote by $A-Mod$ ($A^{op}-Mod$)) the category of left (resp. right) dg-modules over some dg-category A. There is an obvious dg-enrichment of the category $A-Mod$, as suggested by the notation above, which we also denote by $A-Mod$. We have covariant and contravariant dg-Yoneda embeddings: $h_A : A \hookrightarrow A-Mod$ and $h^A : A^{op} \to A-Mod$ which realizes any element $X \in A$ as the dg-modules $A(X,-) : A \to Ch_{dg}(R)$ and $A(-,X)$ it (co)represents. We have the following familiar proposition whose proof mimics that for the ordinary Yoneda lemma and is merely a transcription of the definitions involved:

Proposition 2.7.1. *(dg-Yoneda lemma) The Yoneda embedding is fully faithful and for any dg-module F, we have an isomorphism:*

$$Hom_{A-Mod}(h_X, F) \cong F(X) \tag{2.46}$$

realized by the map $\eta \mapsto \eta_X(1_X)$

Proof See [42] for details.

One may generalize the notion of quasi-isomorphism to dg-modules by putting $H_*(M)(X) := H_*(M(X))$, for $M \in A-dgMod$ and $X \in A$ and defining a quasi-isomorphism to be a dg-module morphism inducing an isomorphism of dg-module homology functors componentwise (that is, such that $H_*(M(X)) \cong H_*(N(X))$ for all $X \in A$. We can generalize other constructions and operations on dg-categories to dg-modules by considering them componentwise: i.e., a map $M \to N$ is said to be an epimorphism if it induces surjections of complexes $M(X) \to N(X)$ for all $X \in A$, likewise for monomorphisms.

Any morphism of dg-categories $f : A \to B$ induces an adjunction of dg-functors between module categories: $f_! : A - Mod \rightleftarrows B - Mod : f^*$.

The homotopy category of dg-modules is defined as the homotopy category of the dg-enhancement of $A-Mod$: $Ho(A-Mod) := H^0(A-Mod)$. Finally, the *derived category* of a dg-category is the localization (see next chapter) of $Ho(A-Mod)$ at quasi-isomorphisms. We denote the derived category of a dg-category by $D(A)$. We should emphasize that the derived category of a dg-category should not be confused- thematically- with that of a ring or a scheme, although this construction does indeed produce the dg-enhancement of the usual derived category when A is an ordinary k-algebra (considered as a dg-algebra with trivial differential). The point of the derived category for us is to provide foundations for *Morita theory* of dg-categories. We regard this theory as foundational in the study of dg-categories *qua* noncommutative spaces for several reasons. First of all, Morita theoretic considerations are essential for fine-tuning the concept of *finiteness* in the non-commutative setting. Secondly, the generalized cohomology theories, or additive invariants, that we shall consider are insensitive to differences between Morita equivalent objects. Thirdly, the generalized representation theory of dg-categories encompasses, at least formally, rich geometric content, such as the theories of Fourier-Mukai transforms and derived equivalences where the fundamental questions can often be reduced to those of representability. These goals may seem contradictory: on one hand, we wish to consider dg-categories *up to* their derived module theory, hence not distinguish those dg-categories with equivalent derived categories. On the other hand, we wish to *disambiguate* derived equivalent dg-categories so as to identify interesting links, find generators and so on. We shall see how this tension plays out when we revisit Morita theory towards the end of this section.

Let us begin by making precise what we mean by *Morita theory* of dg-categories. We say dg-categories T, $T^{'}$ are Morita equivalent if there is a quasi-equivalence of derived categories $D(T) \cong D(T^{'})$. The reason for the designation is as follows. Two rings R and S are said to be *Morita equivalent* if there is an equivalence of categories $Mod_R \cong Mod_S$, where Mod_{-} denotes the category of *right* modules. On the other hand, two rings are said to be *derived Morita equivalent* if there is a *triangulated* (i.e., unenhanced) equivalence of their derived categories $D(R) \cong D(S)$. The main theorem of ordinary Morita theory, the *Eilenberg-Watts theorem* concerns the special place played by the functor $-\otimes M$, with M an R-S-bimodule (or equivalently, an $R^{op}\otimes S$-module), among *colimit* preserving (i.e., cocontinuous) functors $Mod_R \to Mod_S$.

Theorem 2.7.1. *([43,44]) Given any cocontinuous functor $F : Mod_R \to Mod_S$, there is an $R^{op} \otimes S$-module M such that $F \cong - \otimes M$.*

It turns out Eilenberg-Watts theorem admits generalizations to many different contexts such as model categories and dg-categories. By work of Hovey, an Eilenberg-Watts type theorem holds for the (model categories of) *convenient* topological spaces, simplicial sets, chain complexes, spectra, module spectra and so on (which property he refers to as *homotopically self-contained*, see [45]). However, the theorem fails completely in the *derived* setting, as illustrated by the work of Neeman, Keller and Christensen [46] on the failure of Brown representability in derived categories, obstructing the development of derived Morita theory along these lines. Equally severe are the problems with functoriality when we work in the category of triangulated categories, whose formal structure leaves much to be desired. It turns out dg-enhancement of derived categories fixes these issues. Hence, the notion of a derived Morita theory of dg-categories suggests itself immediately. We will be able to revisit this issue after developing a better understanding of the homotopy theory of dg-categories and particulary of the *derived* mapping spaces in $DG - Cat$.

It has been remarked many times up to this point that dg-categories should be considered as enrichments or enhancements, at once, of abelian (e.g., categories of chain complexes) and homotopy and derived categories. What happens to the *triangulated* structure on the latter according to this scheme? Naturally, the formalism of dg-categories has a way of not only reproducing this structure, but also fixing some of its fatal flaws. We start with a tautological and contentless "definition", which is actually a desideratum that will be realized.

Definition 2.7.3 (Mock definition: Pretriangulated dg-category)**.** *A dg-category A is said to be pretriangulated if the homotopy category $H^0(A)$ is triangulated.*

Let us begin by observing that this "tautological" criterion is fulfilled by a class of dg-categories we have encountered: dg-categories of dg-modules, owing to the classical fact that derived categories of modules are triangulated. There are various strategies to induce from this enhanced triangulated structure, at least formally, an analogue of a triangulated structure on dg-categories.More precisely, by considering the dg-Yoneda embedding $A \hookrightarrow A - Mod$, we can investigate the conditions under which one finds an induced "pretriangulated" structure on A. Kapranov and Bondal [47] describe a formalism of triangles for dg-categories which accomplish this, using the concept of twisted complexes. We will note be using this concept and we only note in passing that they and their generalizations allow very explicit constructions which are computationally quite useful.

Recall that for a given dg-category A, we have the dg-Yoneda embedding $h^X := Hom(-, X) : A^{op} \to A - Mod$ for any $X \in A$. In what follows, we freely confuse objects with their representable functors, i.e., representable dg-modules. The shift of an object $X \in A$ is the dg-module $Hom(-, X)[n]$. The mapping cone of a morphism $f : X \to Y$ in A is the usual mapping cone of the morphism of chain complex valued functors $Cone(f) := Cone(Hom(-, X) \to -Hom(-, Y))$. Explicitly, this is the object $Hom(-, Y) \oplus Hom(-, X)[1]$ witn the differential

$$\begin{bmatrix} d_{Hom(-,Y)} & \tilde{f} \\ 0 & -d_{Hom(-,X)} \end{bmatrix}. \tag{2.47}$$

Definition 2.7.4 (Pretriangulated dg-category). *A dg-category is said to be pretriangulated or stable* [9] *if its Yoneda image is closed under the suspension operation and mapping cones, see [41].*

The triangulated envelope of a dg-category A is the closure of its Yoneda image under cones and suspensions. We denote it by $Tri(A)$ or A_{tri}. Evidently, a dg-category is stable or pretriangulated if $A \cong Tri(A)$.

After developing the homotopy theory of dg-categories in later sections, we will describe a model structure on dg-categories, the so-called Morita model structure, which picks out pretriangulated or stable dg-categories as fibrant objects. This model

[9]Note that the nomenclature of *stability* is not standard and there are certainly very good reasons for keeping the explicit reference to triangulated structure (the overabundance of things called "stable" prominent among them).

structure will be quite essential in the construction of *the* category of noncommutative spaces.

2.8. Model Categories and Homotopical Algebra

The localization procedure we briefly touched on in our review of homological algebra and the construction of derived categories admits a far more elegant and comprehensive formalism, which encompasses similar procedures in other areas of mathematics, such as homotopy theory. This is Quillen's theory of model categories, which-in essence- extends to different contexts familiar homotopy theoretic operations and objects such as localization, weak equivalences, fibrations and cofibrations, cones, cylinders and so on beyond mere analogies, providing an invaluable computational tool. This theory gives content to the slogan that homotopical algebra subsumes homological algebra, and should be regarded as a "nonabelian" generalization of the latter. Our reference for this entire section is [48], whom we follow very faithfully. We will often not provide specific references, since the concepts discussed by Hovey are standard and his book is the canonical reference.

A model structure on a category C consists of the data of three subclasses of morphisms (*weak equivalences, fibrations, cofibrations*) abstractly determined by the following properties.

(i) Let $f, g \in Mor(C)$ be composable, i.e., we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \tag{2.48}$$

Then if any two of f, g and gf are weak equivalences, so is the third.

(ii) All three classes of morphisms are closed under *retracts*. Recall that we say a morphism f is a retract of a morphism g if we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_1"'] \arrow[rr, bend left, "f_2\circ f_1=id"] \arrow[d, "f"] & Y \arrow[r, "f_2"'] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r, "g_1"] \arrow[rr, bend right, "g_2\circ g_1=id"'] & T \arrow[r, "g_2"] & Z. \end{tikzcd} \tag{2.49}$$

(iii) Trivial cofibrations have left lifting property (l.l.p.) with respect to fibrations. That is, assume that i is a cofibration that is also a weak equivalence and p is a fibration. Then there is a lift h giving a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ Z \arrow[r, "g"'] \arrow[ur, dashed, "h"] & T. \end{tikzcd} \tag{2.50}$$

(iv) Cofibrations have l.l.p. with respect to trivial fibrations, completely analogous to the previous item.

(v) There exists two *functorial* factorizations of all morphisms $f : X \to Y$ in $Mor(C)$:

- $f = p \circ i$, where p is a fibration and i is a trivial cofibration
- $f = q \circ j$ where q is a trivial fibration and j is a cofibration.

Definition 2.8.1. *A model category is a category admitting all small limits and colimits which is equipped with a particular model structure.*

Note that, by definition, all model categories admit terminal and initial objects, which we denote by $*$ and $\emptyset$ respectively. Thus for any object $X \in M$ there are unique maps $\emptyset \to X$ and $X \to *$. An object $X \in M$ is said to be cofibrant if $\emptyset \to X$ is a cofibration. Conversely, we say it is fibrant if the map $X \to *$ is a fibration.

We can apply the fifth model category axiom to the morphisms $X \to *$ and $\emptyset \to X$ to obtain the so-called fibrant and cofibrant replacement functors, which we denote by $L(-)$ and $Q(-)$ respectively. Thus we have factorizations: $X \to L(X) \to *$ and $\emptyset \to Q(X) \to X$. We refer to $Q(X)$ and $L(X)$ as cofibrant and fibrant replacements respectively. The map $Q(X) \to X$ is a trivial fibration and $X \to L(X)$ a trivial as a consequence of the factorization axioms.

Finally, we put $Ho(M) := M[W^{-1}]$: this is the *homotopy* category of the model category constructed as the localization of M at the weak equivalences. We will introduce the concept of localization shortly, but informally this should be regarded as a category obtained from M in a canonical way and determined uniquely by a universal property in which all the weak equivalences become isomorphisms. It turns out there is another category which is in fact equivalent to the homotopy category, which may therefore be regarded as another model for it. We need some preliminary notions first.

The appropriate notion of a functor between two model categories M and $M^{'}$ is provided by that of a Quillen functor, or more precisely, a Quillen adjunction $F\colon M \rightleftarrows M^{'} :U$.

Definition 2.8.2. *We say an adjunction $F \dashv U$ is a Quillen adjunction if:*

(i) F preserves cofibrations and trivial cofibrations.

(ii) U preserves fibrations and trivial fibrations.

We denote the unit map for this adjunction by $\eta : X \to UFX$ and the counit map by $\epsilon : FUX \to X$. Also denote by ϕ the adjunction isomorphism $M^{'}(FX, Y) \cong M(X, UY)$.

A Quillen adjunction $F \dashv U$ is said to be a Quillen *equivalence* if F induces an equivalence of homotopy categories $Ho(M) \cong Ho(M^{'})$ or equivalently if U induces such an equivalence. We have an equivalent, more explicit, if also more complicated, criterion for an adjunction being a Quillen equivalence: $F \dashv U$ is an equivalence iff for all cofibrant objects $X \in M$ and fibrant objects $Y \in M^{'}$, $f : FX \to Y$ is a weak equivalence if and only if $\phi(f) : X \to UY$ is a weak equivalence.

A Quillen bifunctor is intuitively a Quillen functor in two-variables. We follow Hovey in axiomatizing this notion as follows. Given model categories M, N, L, a Quillen functor is a bifunctor $M \times N \to L$ if for any cofibration $f : X \to Y$ in M and a cofibration $g : Z \to U$, the induced map on the coproduct

$$f \sqcup g : (Y \otimes Z) \sqcup (X \otimes U) \to Y \otimes U \tag{2.51}$$

is also a cofibration and a weak equivalence whenever either f or g is a weak equivalence.

There is a very important class of model categories, in which one has a great deal of control over cofibrations. To be able to define it, we need to plod through some fundamental constructions involving slight set-theoretic technicalities. In this subsection, we follow Sections 2.1.1-2.2.1.3 of [48] very faithfully. Note that this does not contradict our stated goal of completely eschewing any set-theoretic diversions throughout this text, hence the omission of expressions such as $\mathbb{V}$-small etc. that customarily decorate accounts of this subject. This does not mean such issues are not at times quite vital.

Now consider an ordinal number λ, which we regard, as we often do natural numbers and posets, as a category. We will use it as an indexing category for transfinite "sequences" in our categories, which may not really be diagrams in the usual sense of the word in category theory. A λ-sequence is then just a functor $\lambda \to M$, and we put $colim_{\kappa<\gamma} X_\kappa$ for the "transfinite composition" of the string of morphisms indexed by λ up to $\gamma < \lambda$. Now assume M is co-closed and Φ a class of morphisms in M. Let λ be a κ-filtered ordinal for some cardinal κ as in Definition 2.1.2 of [48]. Then we say an object $Y \in M$ is small relative to a collection of morphisms D if for all λ-sequences $X_0 \to X_1 ..... \to X_\alpha ...$ and κ-filtered ordinals, we have isomorphisms

$$colim_{\alpha<\lambda}(Hom_C(Y, X_\alpha)) \cong Hom_C(Y, colim X_\alpha). \tag{2.52}$$

When $D = Mor(C)$, we just say Y is small. It is useful to compare smallness to compactness in triangulated categories. In addition, if κ is a finite cardinal, we say Y is *finite* with respect to D if it is small with respect to D in the above sense.

Let's introduce some notation which formalizes the tangle of ubiquitous lifting properties. Given a subset of morphisms $S \subset Mor(C)$, we denote by $rl(S) \subset Mor(C)$ the collection of morphisms with the right lifting property against morphisms in S and by $ll(S)$ those with the left lifting property against morphisms in S. We denote by $cell(S)$ the collection of morphisms obtained as transfinite pushouts of coproducts of morphisms in S. We denote by $rt(S)$ the set of retracts of the elements of $cell(S)$.

The following theorem is of fundamental importance in the theory of model categories and higher categories since it gives a way to present morphisms in a category in terms of small data:

Theorem 2.8.1. *(2.1.14 in [48]) Let C be a category closed under small colimits and $I \subset Mor(C)$ a class of morphisms in C. Assume the domains of morphisms in I are small with respect to $cell(I)$. Then every morphism $f \in Mor(C)$ admits a functorial factorization as $f = \gamma \circ \delta$ with $\gamma(f) \in cell(I)$ and $\delta(f) \in rl(I)$.*

When a class of morphisms I satisfies the conditions of the theorem above, we say it admits the *small object argument.*

Definition 2.8.3. *We say a model category is cofibrantly generated if there are sets $S, T \subset Mor(C)$ such that:*

(i) $rt(S)$ is the set of cofibrations in C.
(ii) $rt(T)$ is the set of fibrations in C that are weak equivalences.
(iii) S, T admit the small object argument.

Let's proceed to develop the rudiments of abstract homotopy theory in the context of model categories. This machinery allows us to carry out homotopical constructions without reference to particular model structures and specific constructions. The reader may consult Section 1.2 of [48] for details concerning the constructions below.

By definition model categories admit all product and coproducts. Hence for any objects X, Y, we have the product $X \times Y$ and coproduct $X \sqcup Y$. Denote by Δ the diagonal map $X \to X \times X$ and by ∇ the co-diagonal map $X \sqcup X \to X$, which is of course nothing but the map (id_X, id_X).

The formalism of model categories allows us to define the notion of an abstract homotopy between maps $f, g : Y \to X$ in M. We fix such maps f, g in what follows.

Definition 2.8.4 (Cylinder Object). *Apply the fifth model category axiom to the co-diagonal map $\nabla : Y \to Y \sqcup Y$ to obtain a factorization as in the diagram*

$$\begin{tikzcd} Y \sqcup Y \arrow[r, hook, "i=i_1+i_2"'] \arrow[rr, bend left, "\nabla"] & Y' \arrow[r, "d"'] & Y \end{tikzcd} \tag{2.53}$$

where i is a cofibration and d is a trivial fibration. We refer to the Y' with the data of this factorization as the functorial cylinder object for Y, and denote it by $Y \times I$.

Evidently, the cylinder object owes its name to the cylinder object over the circle, $S^1 \times I$, in the category of topological spaces. More generally, a cylinder object over a space X is the product space $X \times I$. Its importance for homotopy theory of topological spaces comes, fundamentally, from the fundamental role played by the "interval object" (see next definition) I. More precisely, the cylinder object features in the definition of the mapping cylinder of a map $f : X \to Y$ of topological spaces, as in the pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d] \\ X \times I \arrow[r] & Cyl(f). \end{tikzcd} \tag{2.54}$$

We note that $Cyl(id_X) = X \times I$. The mapping cylinder is of vital importance in the homotopy theory of spaces since it is used to construct the "homotopy cofiber" or mapping cone C_f in Top. At the heart of this is the fact that $X \to Cyl(f)$ is a *cofibration* and $Cyl(f)$ is homotopy equivalent to Y. Thus, the mapping cylinder allows us to replace every map in Top with a cofibration.

The cylinder object over some simplicial set X in the (model) category of simplicial sets is, predictably, the simplicial set $X \times \Delta[1]$.

The role of the unit interval in $Ch(Mod_R)$ is played by the "normalized chain complex of the simplicial interval" $I := R \to R \oplus R$ where the only nonzero differential

d_0 equals $(id, -id)$. Thus the cylinder object over a chain complex X is nothing but $I \otimes X$. In fact, the notion of chain homotopy can be rephrased in terms of the interval object.

Definition 2.8.5 (Path Object). *Apply the fifth model category axiom to the diagonal map $\Delta : X \to X \times X$ to obtain a factorization as shown in the diagram*

$$\begin{tikzcd} X \arrow[r, "k"'] \arrow[rr, bend left, "\Delta"] & X' \arrow[r, two heads, "e=e_1+e_2"'] & X \times X \end{tikzcd} \tag{2.55}$$

where k is a trivial cofibration and e is a fibration. We refer to X' with the data of this factorization as the functorial path object for X and denote it by X^I.

Definition 2.8.6 (Homotopy). *A left homotopy between morphisms $f, g : Y \to X$ is a map $Cyl(Y) \to X$ such that $Hi_1 = f$ and $Hi_2 = g$. A right homotopy between morphisms is a map $H : Y \to X^I$ such that $e_1 H = f$ and $e_2 H = g$. We say f and g are homotopic, or $f \sim g$, if there is both a left and right homotopy between them.*

The reason we discussed this concept at such length is as follows. When Y is a cofibrant object (or X is a fibrant object), left homotopy (resp. right homotopy) gives rise to an equivalence relation on the set of morphisms $Hom_M(Y, X)$ for any X. When X is fibrant and Y is cofibrant at the same time, homotopy as defined above defines an equivalence relation on $Hom_M(Y, X)$. Now denote the subcategory of bifibrant objects by C_{cf}. Homotopy defines a category-wide equivalence relation when restricted to C_{cf} and $C_{cf}/\sim_{ho}$ is well-defined. This leads to the following proposition.

Proposition 2.8.1. *$C_{cf}/\sim_{ho}$ is a model for the homotopy category of the model category M. That is, we have an equivalence $Ho(M) \cong C_{cf}/\sim_{ho}$.*

Let us proceed to discuss some examples of model structures.

Examples 2.8.1.

- *Any category admitting all small limits and colimits admits the model structure where the weak equivalences are the isomorphisms and fibrations/cofibrations all morphisms.*
- *$Ch(A)$ for any Grothendieck abelian category admits several model structures. Putting $A = Ch(R)$ for the moment, we have very explicit characterization of*

these model structures. The injective model structure consists of the following: the weak equivalences are the quasi-isomorphisms, cofibrations are componentwise monomorphisms and fibrations are componentwise epimorphisms whose kernels are injective modules. The projective structure, on the other hand, consists of the following: the weak equivalences are the same as in the injective model structure, the cofibrations are componentwise monomorphisms with projective kernel, and the fibrations are componentwise epimorphisms. We will expand on these model structures since they are the foundation of the homotopy theory of dg-categories as well. Relatedly, we have similar model structures on globalized versions of these categories, such as $Ch(Coh(X))$, $Ch(QCoh(X))$ *and so on.*

- *If* M *is any model category, then* M^{op} *is a model category in a natural way: fibrations in the former become cofibrations in the latter etc. If* M, N *are model categories then* $M \times N$ *is also a model category: fibrations in* $M \times N$ *are pairs* (f, g) *where* f, g *are fibrations in the model structure on* M *and* N *respectively.*
- *Consider a small category* D *and a model category* M *which is also cofibrantly generated (see below). Then there is a "pointwise" model structure on the category of diagrams* M^D*: for instance,* $f \in Mor(M^D)$ *is a fibration if for all* $d \in D$, $f(n)$ *are fibrations in* M.
- *The homotopy theory of topological spaces occupies an especially canonical role in abstract homotopy theory and most of the concepts in model category may be best grasped by way of analogy with the original constructions and objects in topological spaces. The classical or Quillen model structure on the category of topological spaces* Top *consists of homotopy equivalences as weak equivalences, Serre fibrations as fibrations, and the set of cofibrations generated by the set of maps* $S^{n-1} \to D^n$*, that is, the inclusions of the (n-1)-sphere into the n-disk. In fact, we will soon see that the model structure on chain complexes is roughly a "singular" analogue of this.*
- *The category of simplicial sets* $sSet$ *admits the following model structure: the weak equivalences are the simplicial maps inducing a homotopy equivalence on geometric realizations, cofibrations are levelwise monomorphisms and the fibrations are maps* $f : X \to Y$ *obeying the Kan lifting property. That is, given any horn inclusion* $\Lambda^n_i \hookrightarrow X$*, there exists a lift* $h : \Delta^n \to X$ *giving a commutative*

diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "h"] & Y. \end{tikzcd} \tag{2.56}$$

The relationship between this model structure on sSet and Top goes beyond mere analogy. In fact, there is a Quillen equivalence $Top_{Quillen} \cong sSet_{Quillen}$ realized by the the composite of Sing and $|-|$ functors we touched on in the first section. The fibrant objects in this model structure on sSet, that is, simplicial sets for which there is a lift h for all n as in the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "h"] & * \end{tikzcd} \tag{2.57}$$

are called Kan complexes or- as we shall later see- ∞-groupoids.
There is another model structure on sSet, the Joyal model structure, for which the fibrant objects will be weak Kan complexes, our model of choice for ∞-categories. We will discuss this model structure later.

As promised, let us discuss the model structures on $Ch(\mathscr{A})$ in greater detail. Let us fix $\mathscr{A} = Mod_R$ in what follows. We follow Section 2.3 of [48]. In the spirit of abstract homotopy theory, we introduce the *sphere* and *disk* objects for chain complexes. Recall that for some R-module M, the sphere object $S^n(M)$ is a chain complex such that $S^n(M)_n = M$ and $S^n(M)_k = 0$ otherwise. Further, the disk object is a chain complex $D^n(M)$ such that $D^n(M)_k = M$ when $k = n-1, n$. The sphere object evidently injects into the disk $i : S^{n-1}(M) \to D^n(M)$ for all n. Now put $S^n(R) =: S^n$ and $D^n(R) =: D^n$. We denote the set of maps $S^{n-1} \to D^n$ by I and the set of pointing maps $0 \to D^n$ by J. We have the following proposition on the characterization of cofibrations in the projective model structure.

Proposition 2.8.2. *The maps in I are a generating set for cofibrations in the projective model structure. The maps in J are a generating set of cofibrations that are also weak equivalences.*

In this model structure, every chain complex is transparently *fibrant*. Cofibrant chain complexes are those that can be expressed as filtered unions, where the associated graded objects of the filtration are projective modules. Hence cofibrant chain complexes are *flat*, a fact that will be important later.

2.9. Localization and Function Complexes in Model Categories

In algebra, we may define the localization of a ring R at a multiplicative subset S to be a ring $R[S^{-1}]$ equipped with a ring homomorphism $p : R \to R[S^{-1}]$ which sends all elements $s \in S$ to units in $R[S^{-1}]$ and has the universal property with respect to such maps. More precisely, for any map $q : R \to R^{'}$ which also maps all $s \in S$ to units in some ring $R^{'}$, we have a unique ring homomorphism $r : R[S^{-1}] \to R^{'}$ such that $q = r \circ q$. In the same vein, given a category C and a class of morphisms W, we wish to construct a new category $C[W^{-1}]$ naturally equipped with a functor $L_W : C \to C[W^{-1}]$ which sends morphisms in W to isomorphisms in the new category. In the spirit of ordinary localization, we would also want that $C[W^{-1}]$, or rather the functor L, has the universal property in the appropriate sense. More precisely, the *localization* of a category with respect to a subclass of morphisms W consists of the following.

Definition 2.9.1. *The localization of a category C with respect to a subclass of morphisms W consists of:*

(i) A category $C[W^{-1}]$

(ii) A functor $L_W : C \to C[W^{-1}]$ such that $L_W(w)$ is an isomorphism for all $w \in W$, and given any other category D and functor F such that $F : C \to D$ maps morphisms in W to isomorphisms in D, there is a unique functor $G_W : C[W^{-1}] \to D$ with an isomorphism of functors $F \cong G_W L_W$. Further, consider an arbitary category D and denote by $Fun(C, D)$ the functor category. Then the induced functor $L_W^* : Fun(C[W^{-1}], D) \to Fun(C, D)$ is fully faithful.

Of course, this definition does nothing to explicate the actual shape of this new category or offer a tractable description of it. As we already hinted at in our discussion of homotopy and derived categories, there is a concrete way to describe the result of categorical localization which is, however, not all attractive nor offers any particular practical advantage. Hence we omit this description.

There is a more sophisticated version of localization due to Dwyer-Kan whose output is a *simplicial* category and which is therefore better suited for packaging homotopical (read: higher, foreshadowing ∞-categories) information. Certain fundamental results in the homotopy theory of dg-categories rely crucially on the simplicial structure on the mapping spaces of DG-Cat that results from Dwyer-Kan localization. We will pass over this construction since ∞-categorical methods will be introduced later and the end result of simplicial localization is better described in that context. However, certain crucial results in the theory of quasicategories also rely on methods related to simplicial localization. Hence we will introduce the notation for the so-called *hammock localization* procedure whose output is a simplicial category in this section. For the immediate purposes of this section, we will use this tool to give a description of the "function complex" which is *the* canonical model for derived mapping spaces in homotopy theory. Our specific reference for this section is the key paper by Dwyer and Kan [49].

Definition 2.9.2. *Let (M, W) be a category with weak equivalences. We denote by $L^H(M, W)$ the so-called hammock localization of M with respect to the weak equivalences W whose output is a simplicial category. Consult Section 3.1 in [49] for the explicit construction.*

Definition 2.9.3. *Let M be a model category and $X, Y \in M$. A simplicial resolution of Y [Dwyer-Kan2,4.3] is a an object $Y_\bullet : \Delta^{op} \to M$ with $Y[0]$ weakly equivalent to Y such that:*

(i) Y_0 is a fibrant object.
(ii) All the face maps of $Y_\bullet$ are trivial fibrations.

A cosimplicial resolution is exactly the dual of the simplicial resolution. A more compact way of expressing this is that simplicial resolution is nothing but the Q-functor *(fibrant replacement) for the model structure on the diagram category $\Delta^{op} \to M$.*

Definition 2.9.4. *With M and X, Y as before, denote $X_\bullet$ and $Y_\bullet$ fixed cosimplicial and simplicial resolutions for X and Y respectively. Then we have the bisimplicial set of maps (that is,a map $\Delta^{op} \times \Delta^{op} \to M$) $M(X_\bullet, Y_\bullet)$. The homotopy function complex $M(X, Y)$ is the diagonal simplicial set $Diag(M(X_\bullet, Y_\bullet))$.*

The existence of the simplicial mapping spaces $M(X, Y)$ shows that every model category is *tensored* over the category of simplicial sets, that is, monoidally enriched over the category of simplicial sets.

By the following result of Dwyer-Kan, the function complexes can be modeled by the so-called hammock localization (or equivalently, the simplicial localization).

Proposition 2.9.1. *(Proposition 4.4 in [49] The simplicial hom-set $L^H(X, Y)$ has the same homotopy type as $M(X, Y)$.*

Thus from the point of view of homotopy theory, the function complex is *the* model of mapping spaces in model categories. We will see later that this perspective turns out to be extremely rewarding when applied to the model category of dg-categories.

2.10. Monoidal Model Categories, Enriched Model Categories and $Ch(R)$-model Categories

Next, we describe the notion of a *monoidal* model category. While the definition is straightforward, there are some subtleties in the compatibility relations between the monoidal structure and "homotopy theoretic" content of a model category. Explicitly, we say there is a closed monoidal structure on a model category M if the monoidal product bifunctor $- \otimes -$ is a Quillen bifunctor. In the same vein, we say a model category M is enriched over a monoidal model category C if M is a C-enriched category which is bitensored over C such that the co-tensoring functor $M \otimes C \to M$ is a Quillen bifunctor. [10]

Definition 2.10.1. *A $Ch(R)$-model category is a $Ch(R)$-enriched model category.*

Evidently, $Ch(R)$ with its self-enrichment supplies an immediate example of such a model category. Let us add that the homotopy category of a $Ch(R)$-enriched model category is automatically enriched in $D(R)$.

[10] The immediate rationale behind (originally) Hovey's definition of model categorical enrichment was to be able to induce a monoidal structure at the homotopy level, which can be easily seen by employing the C_{cf} model of the homotopy category presented above.

3. HOMOTOPY THEORY OF DG-CATEGORIES AND NONCOMMUTATIVE SPACES

The contents of this chapter will bring us well over half-way to the theory of noncommutative motives and in fact form the foundation of our subject. Indeed, our subject may be concisely described as the *motivic* homotopy theory of noncommutative spaces. However, the latter- unlike their geometric counterparts already exhibit a great deal of "exactness" phenomena intrinsically, while being completely deficient in extremely rudimentary geometric characteristics: a good notion of a point, for instance or a satisfying notion of "Zariski" or "étale" localeness. Robalo's notion of a noncommutative Nisnevich cover of a dg-category is also precisely novel in that it grafts the exact structure of the category of noncommutative spaces onto the commutative theory, but this cannot be considered a genuinely geometric notion at all, being somewhat ad-hoc. Counterintuitively, all these things argue, not against, but in favor of the program of noncommutative algebraic geometry. Our interest in noncommutative spaces, and the origin of this analogy in the first place, is fundamentally cohomological or *motivic*. The main tasks for the construction of *right* category noncommutative spaces as the target of cohomology theories is that they reflect the noncommutative spaces that arise as derived categories and that they are the minimal class of objects which can be distinguished by cohomology theories or "additive invariants". The motivic picture automatically falls out of these choices, with minimal technical complications.

Without further ado, let us introduce the first model structure on $DG-Cat$ that we will be relying on for much of this thesis. Much of what follows stems from Tabuada's work on the homotopy theory of dg-categories and expositions based around it. Our main reference is Tabuada's dissertation [41].

We need to specify three classes of morphisms to define a model category structure. The $Dwyer-Kan$ model structure on $DG-Cat$ consists of the following:

- The weak equivalences are the quasi-equivalences of dg-categories. Thus $Ho(DG - Cat) := DG - Cat/\sim_{qiso}$ or more accurately $Ho(DG - Cat) := DG - Cat[qiso^{-1}]$.
- The fibrations are the morphisms $f : A \to B$ between dg-categories A and B such that:
  - The induced morphism on the hom-complexes $A(X, X') \to B(f(X), f(X'))$ is a fibration in the model structure on $Ch(R)$, which is to say, it is surjective.
  - Any isomorphism $H^0(f)(X) \cong b$ in $H^0(B)$ may be lifted to an isomorphism in $H^0(A)$.
- The cofibrations are the maps with the right lifting property against the fibrations.

This model structure is cofibrantly generated, which is not completely surprising. It can be immediately observed that every dg-category is fibrant in this model structure. To see this, consider the complex $A(X, Y)$ for some dg-category A. Now, the unique map $A \to *$ is just the one sending any object of A to $*$, and the morphisms complexes $A(X, Y)$ for all $X, Y \in A$ to the zero complex. This map is clearly surjective. As for the second condition, the only morphism in $*$ is the identity morphism of $*$, which is naturally an isomorphism. Evidently, the identity isomorphism of the dg-category A is the lift of this zero morphism. Hence, A is fibrant. The monoidal unit dg-category 1_{dg} is also a cofibrant object as can be seen immediately from the definition. Consequently, the simplex dg-category $\varDelta^1_R$ is also cofibrant since cofibrations are stable under pushouts.

For a given $Ch(R)$-model category M, Toën and collaborators introduced the notation $Int(M)$ for the dg-subcategory of bifibrant objects in M, which they dubbed the "internal dg-category" of M, see Section 3.2 in [39]. This construction provides a way to produce dg-enhancements of homotopy categories. Recall that $Ho(M)$ has a model as the quotient of M_{cf} by homotopy equivalence. Hence, given some $Ch(R)$-model category M, $Int(M)$ is in fact nothing but a dg-enhancement of the homotopy category $Ho(M)$ in that we have an equivalence of categories: $H^0(Int(M)) = Ho(M)$.

Given a dg-category A, there is an induced model structure on $A - mod$ from the previous section, which is essentially the pointwise model structure. That is, for instance, a morphism $f : F \to G$ of dg-modules F and G in $A - Mod$ is a fibration if $f(X) : F(X) \to G(X)$ is a fibration for all $X \in A$ and so on. Then the homotopy category $Ho(A - Mod)$ associated with this model structure on $A - Mod$ is the the derived category $D(A)$ we have already met in the previous sections. Therefore we obtain a dg-enhancement of the derived category $D(A)$ which is nothing but $Int(A - Mod)$. Same story holds for left dg-modules, and for reasons of convenience we will switch to working with them. Thus, we put $\hat{A} := Int(A^{op} - Mod)$ for the dg-category of cofibrant dg-modules over A. Note that every object is already fibrant, so we do not need to worry about restricting to another subcategory. For instance, $\hat{1}_{dg}$ is the dg-category of cofibrant chain complexes of R-modules. We also note that $\hat{T}$ can be expressed in terms of the derived inner Hom-object we introduced below and essentially by definition we have that $\hat{A} \cong \mathbb{R}Hom(A^{op}, \hat{1}_{dg})$.

We can port the discussion of module theory into the homotopical context verbatim because the the dg-Yoneda embedding holds at the derived level, in that the dg-Yoneda embedding in fact defines a functor

$$h^A : A \to \hat{A}. \tag{3.1}$$

This is because of the following very useful result.

Proposition 3.0.1. *For any dg-category A and $X \in A$ the Yoneda dg-module h^X is cofibrant in the model structure on $A - Mod$ and hence defines an element of $\hat{A}$.*

Proof Using dg-Yoneda lemma, one may show that any morphism $f : h_X \to G$ has the lifting propery with respect to any trivial fibration $p : F \to G$ in $A^{op} - Mod$, hence establishing cofibrancy (see Lemma 2.16 in [42] for details). $\square$

Definition 3.0.1. *A dg-module F is said to be quasirepresentable (resp. quasicorepresentable) if $F \cong h_A$ (resp. h^A) for some dg-category A. In other words a quasirepresentable dg-module is in the quasi-essential image of the Yoneda embedding.*

Definition 3.0.2. *(Definition 4.1 in [4]) Denote by* $-\otimes^{\mathbb{L}}-$ *the derived tensor product which endows* $Ho(DG-Cat)$ *with a symmetric monoidal structure. This is nothing but the Quillen left derived functor of the tensor product of dg-categories given by cofibrant replacement:* $-\otimes^{\mathbb{L}}- := Q(-)\otimes -$. *Given two dg-categories* A *and* B, *we say a* $(A\otimes^{\mathbb{L}}B^{op})$*-module is right quasirepresentable if* $i^*(F)$ *is a quasirepresentable* $B^{op}-$*module, where* i *is the map* $1_A\otimes id : B^{op}\to A\otimes^{\mathbb{L}}B^{op}$. *For a bimodule dg-category* T, *we denote the subcategory of right quasirepresentable modules by* T^{rqr}.

The relevance of right quasirepresentable dg-bimodules comes from the the following characterization of the internal Hom-object:

Lemma 3.0.1. *(6.1, [4]) Given dg-categories* T *and* T', *we have an isomorphism*

$$\mathbb{R}Hom(T,S)\cong Int(((T\otimes^{\mathbb{L}}S^{op})-Mod)^{rqr}) \tag{3.2}$$

Let M be a $Ch(R)$-model category such that,

(i) M is cofibrantly generated.

(ii) Every cofibrant object X in M is flat in the homotopical sense, that is, $-\otimes X$ respects weak equivalences.

This is evidently true when a dg-category T is *locally cofibrant*, that is, when $T(x,y)$ is cofibrant for all $x,y\in T$ since cofibrant chain complexes are flat. For any dg-category T, denote the "M-valued" dg modules over T by M^T, a version of $T-Mod$ with a more general coefficient object. Then the following Proposition 1 in [39] characterizes isomorphism classes in the homotopy category of a functor category.

Proposition 3.0.2. *(Proposition 1 [39]) There is a bijection*

$$[T,Int(M)]\cong(Ho(M^T))_{\sim} \tag{3.3}$$

where the subscript $\sim$ *on the right denotes isomorphism classes of objects.*

In the vein of the construction of the derived mapping spaces in a general model category, we now wish to obtain a better description of the mapping spaces in $DG-Cat$.

This description will likewise produce a simplicial set, which is precisely the content of the simplicial enrichment of the category $DG-Cat$. In what follows, we follow Section 4 of [4].

Recall the notion of a (co)simplicial resolution in a model category that we discussed in the Section 2.9 on mapping complexes in general model categories. Let $\Gamma : DG-Cat \to DG-Cat^{\Delta^{op}}$ be the functor of cosimplicial resolution, which amounts to the following data.

(i) We have a quasiequivalence $\Gamma^n(T) \to T$ for any n and $T \in DG-Cat$.

(ii) $\Gamma^n(T)$ are cofibrant for all n and T in the model structure on the diagram category $DG-Cat^{\Delta^{op}}$.

(iii) The initial morphism $\Gamma^0(T) \to T$ is just cofibrant replacement of T in the original model structure.

As in the discussion of simplicial enrichment of model categories in Section 5, we have the simplicial mapping space for dg-categories T and $T^{'}$ whose simplices are

$$Map_{\Delta}(T, T^{'})_n := Hom(\Gamma^n(T), T^{'}) \tag{3.4}$$

We do not need to fibrantly replace T since all dg-categories are fibrant in the Dwyer-Kan model structure as we have seen. Toën proves a weak equivalence between the simplicial mapping space and the following simplicial sets which reflects the algebraic and geometric aspects of dg-categories we have discussed above.

Definition 3.0.3. *Consider the simplicial dg-category whose n-simplices are given by dg-categories $(\Gamma^n(T) \otimes S^{op})$-Mod. Let $\mathscr{M}(\Gamma^n(T), S)$ be the following subcategory:*

- *Objects are quasirepresentable dg-modules F such that $F(x, -)$ for any $x \in \Gamma^n(T)$ is a cofibrant S^{op}-module.*
- *Morphisms are equivalences in $(\Gamma^n(T) \otimes S^{op})$-Mod.*

We put $N(\mathscr{M}(\Gamma^n(T), S))$ for the nerve of the category above, which gives rise to a bisimplicial set[11] . We then have the following fundamental theorem due to Toën [4].

Theorem 3.0.1. *(Theorem 4.2 in [4]) In the model structure on sSet, the map*

$$Map_{\Delta}(T, S) \to Diag(N(\mathscr{M}(\Gamma^n(T), S))) \tag{3.5}$$

is a weak equivalence, and so is the map

$$Diag(N(\mathscr{M}(\Gamma^n(T), S)) \to N(\mathscr{M}(\Gamma^0(T), S)). \tag{3.6}$$

The proof of the theorem is straightforward but extremely technical. We will mention the following crucial results discussed in sections 4.1 and 4.2 of [4] which characterize mapping spaces in $DG - Cat$ and which are immediate consequences of this theorem.

Proposition 3.0.3. *Given dg-categories T and S, put $QRep(S, T)$ for the subcategory of quasirepresentable objects in the homotopy category of bimodules $Ho((T \otimes^{\mathbb{L}} S^{op})) - Mod$. Then there is a bijection:*

$$[S, T] \cong Qrep(S, T)_{\sim}, \tag{3.7}$$

and putting $S = 1_{dg}$, we also have that:

$$[1_{dg}, T] \cong Ho(T)_{\sim}. \tag{3.8}$$

The purely homological notion of pretriangulated or stable dg-category which we introduced in the previous section has interacts fruitfully with the homotopy theory of dg-categories. Recall that a dg-category T is said to be *pretriangulated* if its Yoneda image is closed under cones and suspensions or, in other words, it is quasi-equivalent to its pretriangulated envelope T_{tri}. Recall also that a dg-module in the quasi-essential image of the Yoneda embedding is called *quasirepresentable*. We have the closely related notion of a *triangulated dg-category* due to Toën which characterizes the property of being triangulated, a priori an *exactness* or stability property, as being in fact a *finiteness* property. What interpolates between these is the concept of *compactness*.

[11] See next section for the definition of the nerve functor.

Let T be a dg-category and $D(T^{op})$ the derived dg-module category. A compact object of this triangulated category is precisely (as before) an object $X \in T$ such that $Hom(X, -)$ preserves arbitrary direct sums. We denote by T_c the subcategory of compact objects of the dg-category T. For instance $\hat{T}_c = Int(T^{op} - Mod)_c$ is the dg-category of compact dg-modules. The objects in the image of the dg-Yoneda embedding we gave above are evidently compact, hence h_A for some dg-category A can be further seen as a map $A \to \hat{A}_c$. We then have the following definition/theorem which establishes when the converse holds:

Definition 3.0.4. *A dg-category T is said to be triangulated if and only if every compact object is quasirepresentable. In other words, the homotopical Yoneda embedding is a quasi-equivalence of dg-categories. Passing to homotopy categories, we have that $Ho(T) \cong D(T^{op})_c$.*

We may now address why dg-categories are a satisfying candidate for "enriched triangulated categories" from the point of view of addressing the inadequacies of the unenriched theory. We have already seen that the homotopy theory of dg-categories fixes some of the major flaws of triangulated categories, such as the lack of a good structure on $TriCat$ or in the same vein the definition of meaningful mapping spaces (that is, triangulated functor categories) between triangulated categories. The outstanding issue is the nonfunctoriality of the mapping cone construction we mentioned in the first chapter. We follow Toën's account of how dg-categories resolve this issue, as given in section 5.1 [39].

The following discussion mirrors the account we gave at the end of the first chapter. First we formalize some self-evident points. The dg-category of objects in a dg-category T is the functor dg-category $Fun(1_{dg}, T)$. The hom-sets of its homotopy category, the homotopy classes of maps $[1_{dg}, T]$ classifies the isomorphism classes of objects in $Ho(T)$. Further, the dg-category of morphisms $Mor(T)$ of a dg-category T is the functor dg-category $Fun(\Delta^1_R, T)$, and the homotopy classes of maps $[\Delta^1_R, T]$ correspond bijectively to isomorphism classes of maps in $Ho(T)$. By the results discussed above, $[\Delta^1_R, T]$ also corresponds to quasirepresentable dg-modules. Informally, our aim is to construct a functor $Mor(T) \to T$ which assigns a morphism in a dg-category T its "cofiber". Now let T be a (pre)triangulated dg-category as in [4]. Keeping the previous notation, we have then isomorphisms $RHom(\hat{1}_{pe}, T) \cong RHom(1, T) \cong T$.

Consider the map $\Delta^1_R \to \hat{1}_{pe}$ which sends 0 to 0 and 1 to R. There is an induced map $p : RHom(1, T) \cong T \to Mor(T)$ which sends an object $X \in T$ to the morphism $0 \to X$. The dg-mapping cone will be the adjoint to p, constructed in the following proposition, which is Proposition 5.9 in [39].

Proposition 3.0.4. *The map p admits a left adjoint cone* $: Mor(T) \to T$. *That is, we have quasi-equivalence of dg-modules:*

$$Mor(T)(f, p(Y)) \cong T(cone(f), Y). \tag{3.9}$$

Proof Above, we have obtained homotopically correct representatives of T, $Mor(T)$ and so on. Namely, as a model of T we always consider $T^{'} := QRep(\hat{T}) = \hat{T}_c$ and in the same vein, $Mor(T)^{'} := QRep(Mor(T^{op} - Mod))) := Mor(T^{op} - Mod)_c$ as a model of $Mor(T)$. Having thus taken care of issues regarding cofibrancy etc., we may simply define *cone* $: Mor(T) \to T$ to be the map that sends $f : X \to Y$ in $Mor(T)^{'}$ to $cone(f)$ defined by the diagram in T^{op}-Mod

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & cone(f) \end{tikzcd} \tag{3.10}$$

which is a pushout square. What remains to be verified is that $cone(f)$ is an object of $T^{'}$. This follows from the general fact that (homotopy) colimits preserve compactness.

We take this opportunity to review another problem resolved by the introduction of dg-categories, particularly, the use of dg-enhanced derived categories in geometry, the problem of descent. With X denoting a scheme and $D(X)$ its derived category of quasicoherent sheaves, attempting to assemble the "derived category" assignment $X \mapsto D(X)$ into some kind of higher functor from schemes to triangulated categories $Sch \to Tri$ predictably leads to many technical difficulties. However, even if one took care of the functoriality issues by way of some ad-hoc methods, this functor would not behave as one would expect. The problem is that naive gluing does not work in triangulated derived categories, in other words the assignment $X \to D(X)$ does not form a stack over the category of schemes Sch because of failure of *locality*, as discussed in Section 2.2.4 of [39]. Consider the projective line $\mathbb{P}^1$ over some ring R with standard opens $U_i := SpecR[x^{\pm}]$. Of course, $\mathbb{P_R}^1$ can be constructed by way of the diagram (a localization square, if you will)

$$\begin{tikzcd} SpecR[x^{\pm}] \arrow[r] \arrow[d] & SpecR[x] \arrow[d] \\ SpecR[x^{-1}] \arrow[r] & \mathbb{P}^1_R \end{tikzcd} \tag{3.11}$$

which is a pushout square. Then by standard arguments of descent, the module categories also fit in a localization square, that is, a pullback diagram

$$\begin{tikzcd} QCoh(\mathbb{P}^1_R) \arrow[r] \arrow[d] & Mod_{R[x^{-1}]} \arrow[d] \\ Mod_{R[x]} \arrow[r] & Mod_{R[x,x^{-1}]} \end{tikzcd} \tag{3.12}$$

in the appropriate 2-categorical sense. However, no such gluing is possible when we pass to derived categories, as illustrated by the following famous example. Assume $R = k$ is a field. Consider the extension exact sequence in $D(\mathbb{P}^1_R)$ of line bundles: $0 \to O(-2) \to O(-1)^{\oplus 2} \to O \to 0$, which defines an element of $Hom_{D^b(\mathbb{P}^1_k)}(O, O(-2)[1])$. Any such map must be the zero map over any one of U_i since these are affine and there are no higher Exts among line bundles over affine pieces. Hence this element of the derived category cannot be built out of local information although the geometric object underlying it is. Any attempt to fix this defect by forcing such descent properties is doomed at the outset if one sticks to triangulated categories. We will see such a procedure is the key to the definition of motives and noncommutative motives in the context of ∞-categories.

The theory of dg-enhancements fixes this problem. We will revisit this issue again in the stable ∞-setting but will merely state the following for now, as presented in the section 5.3 of [39].

Proposition 3.0.5. *For a scheme X and open subschemes U and V such that $X = U \cup V$, we have a diagram*

$$\begin{array}{ccc} Perf_{dg}(X) & \longrightarrow & Perf_{dg}(U) \\ \downarrow & & \downarrow \\ Perf_{dg}(V) & \longrightarrow & Perf_{dg}(U \times_X V) \end{array} \tag{3.13}$$

which is a homotopy pushout square in $DG - Cat_R$.

The following discussion will serve to clarify and solidify the central themes of *noncommutative* algebraic geometry.

Following Definition 2.4 in [40], a dg-category A is said to be,

(i) *locally proper* if $A(x, y)$ is a perfect complex [12] .
(ii) *compactly generated* if $Ho(\hat{A})$ is compactly generated as a triangulated category.
(iii) *proper* if it is locally proper and compactly generated.
(iv) *smooth* if A considered as a diagonal $A^{op} \otimes A$ module is perfect in the dg-module category $\widehat{A^{op} \otimes^{\mathbb{L}} A}$.
(v) *saturated* if it proper, smooth and pretriangulated.
(vi) *of finite type* if there is a dg-algebra B which is compact such that $\hat{A}$ is quasiequivalent to $Perf(B) := \hat{B^{op}}$.

Since smooth, proper, compactly generated, and finite-type dg-categories will be the main objects of interest for the rest of this text, we will spend some time developing the background for these concepts and untangling the relationships between them. The following lemma due to Toën and Vaquie is the first step.

Proposition 3.0.6. *(Lemma 2.6 in [40]) The dg-category $\hat{A}_c$ is triangulated.*

Proof Consider the dg-category $\widehat{\hat{A}_c}$. Above, we mentioned that we have an equivalence $\hat{A} \cong \mathbb{R}Hom(A^{op}, \hat{1}_{dg})$, hence we have $\widehat{\hat{A}_c} \cong \mathbb{R}Hom(\hat{A}_c^{op}, \hat{1}_{dg})$. By Lemma 7.5 in [4], we have a quasiequivalence $\mathbb{R}Hom(\hat{A}_c^{op}, \hat{1}_{dg}) \cong \mathbb{R}Hom(A^{op}, \hat{1}_{dg})$ but the latter is

[12] i.e., a bounded complex of projective R-modules, also finitely presented if R is not Noetherian.

just $\hat{A}$. Thus we have a quasiequivalence $\widehat{\hat{A}_c} \cong \hat{A}$. Then we have a quasiequivalence of subcategories of compact objects, and $\hat{A}_c$ is triangulated. □

Proposition 3.0.7. *(Lemma 2.6 in [40]) $\hat{A}$ is compactly generated, smooth or proper if and only if $\hat{A}_c$ is.*

Proposition 3.0.8. *(Lemma 2.6 in [40]) A is compactly generated if and only if there is a quasi-equivalence $\hat{A} \cong \hat{B}^{op}$ for B a dg-algebra. Such an A is proper if and only if B is a perfect dg-algebra and smooth if and only if B is a perfect $B \otimes^{\mathbb{L}} B^{op}$-module.*

Proposition 3.0.9. *(Corollary 2.12 in [40]) A compactly generated dg-category T is of finite type if and only if the dg-algebra as in the previous proposition is a perfect dg-algebra.*

Proposition 3.0.10. *(Corollary 2.13 in [40] A smooth and proper dg-category is of finite type. Conversely, any dg-category of finite type is smooth.*

These propositions should be seen as taxonomic underpinnings for the theory of noncommutative algebraic geometry on the basis of dg-categories. The culmination of the classical period- so to speak- in homotopy theory of dg-categories is the following theorem due to Toën, which are a great deal subtler and deeper than they might at first appear.

Theorem 3.0.2. *There is a closed symmetric monoidal structure on $Ho(DG-Cat)$. That is, given dg-categories X,Y,Z, we have a derived internal hom-object $\mathbb{R}Hom(Y,Z)$ participating in the following derived tensor-hom adjunction expressed by the isomorphism*

$$\mathscr{Hom}(X, \mathbb{R}Hom(Y,Z)) \cong \mathscr{Hom}(X \otimes^{\mathbb{L}} Y, Z). \tag{3.14}$$

Furthermore, the following theorem essentially brings the derived Morita theory of dg-categories to a completion.

Theorem 3.0.3. *(Corollary 7.6 in [4]) Put $\mathbb{R}Hom_a(\hat{X},\hat{Y})$ for the subcategory of the derived internal hom dg-category consisting of additive (that is, direct sum preserving)* *dg-functors. Then we have an isomorphism*

$$\mathbb{R}Hom_a(\hat{X}, \hat{Y}) \cong \widehat{X^{op} \otimes^{\mathbb{L}} Y}. \tag{3.15}$$

The following theorem of Toën's makes evident the geometric content of derived Morita theory. Theorems of this form regarding derived categories have been proven in a multitude of contexts. For instance, they form the cornerstone of the *Tannakian* study of stable ∞-categories, see for instance [50, 51].

Theorem 3.0.4. *(Theorem 8.9 in [4]) Let X, Y be smooth, proper R-schemes and put $Perf_{dg}(X)$ and $Perf_{dg}(Y)$ for the associated dg-enhanced categories of perfect complexes of quasicoherent sheaves. Then we have an isomorphism*

$$Perf_{dg}(X \times Y) \cong \mathbb{R}Hom(Perf_{dg}(X), Perf_{dg}(Y)). \tag{3.16}$$

We will later have one last dive into the foundations of the homotopy theory of dg-categories once we have introduced ∞-categories and construct the ∞-category of dg-categories.

4. STABLE ∞-CATEGORIES

4.1. ∞-categories

We lack the space to give even a cursory treatment of the evolution of higher categories or expound upon what motivates their adoption as the fundamental formalism in this thesis and will treat the theory of ∞-categories mostly as a black-box environment which provides a very convenient context for unifying and generalizing concepts from homological algebra, category theory, homotopical algebra and algebraic topology. As hinted at above, most ∞-categories we encounter arise from some model category as a result of localization at weak equivalences. Hence ∞-categories can be intuitively seen as a way to get a better handle on such derived or "homotopified" structures than what is possible through the mere homotopy 1-category.

While there are various models for ∞-categories, throughout this thesis we will rely almost exclusively on the formalism of quasi-categories developed by Joyal, Lurie and others. In this model, ∞-categories are simplicial sets whose sets of higher simplices ("sets of higher morphisms") obey a certain lifting condition which encode the laws governing the composition of arrows and natural compatibilities among them. We review the fundamental aspects of this theory below with the intention of showing that much that is familiar in category theory can be almost verbatim ported into it.

Unless otherwise indicated, our reference for the elementary material in the first section is Lurie's fundamental treatise Higher Topos Theory [52], supplemented by [18, 53]. To avoid burdensome citations, we do not give specific references in this section as all the facts discussed are by now standard.

4.1.1. Simplicial Sets

We have already made the acquaintance of simplicial sets. We keep the notation from previous sections. As before, we denote the category of simplicial sets, that is, the presheaf category $Fun(\Delta^{op}, Set)$ by $sSet$. By virtue of the central role played by simplicial objects in homotopy theory and their combinatorial properties, $sSet$ enjoys many intimate connections with ordinary category theory and categorical algebra. The nerve construction gives a functor $N : Cat \to sSet$ from the category of small categories to simplicial sets, which, composed with the geometric realization functor $|-| : sSet \to Top$ is nothing but the classifying space construction familiar from algebraic topology and K-theory. For some $C \in Cat$, we denote by BC its classifying space. Intuitively, the homotopy type of this space encodes the behavior of composable morphisms and reflects fundamental properties of the category like the existence of terminal or initial objects (either of which makes the classifying space contractible by an easy argument).

Given a simplicial set X, define as before its sets of n-simplices $X_n := Hom_sSet(\Delta[n], X)$, where $\Delta[n]$, the so-called standard n-simplex, is the simplicial set corresponding to $[n]$ under the Yoneda embedding, i.e., $\Delta[n] : [m] \to Hom([m], [n])$. The "cosimplicial" maps mentioned above induce simplicial maps in sSet, which can be explicitly exhibited in the following mock diagram of $X_\bullet$,

$$X_0 \leftrightarrows X_1 \leftrightarrows X_2.... \tag{4.1}$$

As alluded to in the section on model categories, we have the so-called *Kan* model structure on $sSet$ in which the fibrant objects are called *Kan complexes* or ∞-groupoids. This means that Kan complexes are the simplicial sets for which every lifting problem of the following form admits a solution, as described in the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n. \arrow[ur, dotted] & \end{tikzcd} \tag{4.2}$$

The reason for this name is two-fold: first of all, nerves of ordinary groupoids are precisely Kan complexes. Secondly, and relatedly, they offer an ideal ∞-categorification of the notion of a groupoid as a category in which every morphism is invertible, as we shall see.

The most intuitive way to motivate the presentation of ∞-categories by simplicial sets is by way of the *nerve construction*, which we now proceed to describe, following Section 1.1.2 of [52]. Let C be a small category. We form a simplicial set out of C as follows. Define $N(C)$ to be the simplicial set with simplices

$$N(C)_n := Map(\Delta^n, N(C)) \tag{4.3}$$

where $N(C)_n$ is the set of composable n-strings of morphisms in C which can be presented by the diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} ... \xrightarrow{f_n} C_n. \tag{4.4}$$

The face map $d_i : N(C)_n \to N(C)_{n-1}$ defined above sends the n-string above to the string

$$C_0 \xrightarrow{f_1} ... \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} ... \xrightarrow{f_n} C_n. \tag{4.5}$$

On the other hand, the degeneracy map $s_i : N(C)_n \to N(C)_{n+1}$ sends the n-string to the string

$$C_0 \xrightarrow{f_1} ... \xrightarrow{f_i} C_i \xrightarrow{id} C_i \xrightarrow{f_{i+1}} ... \xrightarrow{f_n} C_n. \tag{4.6}$$

Let's spell out how this construction encodes the structure of a category. Clearly, the set of 0-composable morphisms is just the set of objects of C, thus, $N(C)_0 = Ob(C)$. Equally obvious is the fact that $N(C)_1 = Mor(C)$. The face maps d_0, d_1 are nothing but the source and target maps that appear in the formal definition of a category. The degeneracy maps encode identity morphisms on objects. Higher simplices encode composition of morphisms and rules governing it (associativity etc.). Hence, the data of a category can be put together from that of a simplicial set by interpreting certain simplicial operations category theoretically. We shall see below that simplicial sets that arise in this way obey a certain lifting condition and this will form the underpinning the quasi-categorical model of ∞-categories.

Define the i-th horn Λ_i^n of the n-simplex Δ^n to be the simplex obtained from Δ^n by deleting the i-th face. For instance, consider Δ^2, which is the simplex

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[rr] & & 2. \end{tikzcd} \tag{4.7}$$

with its boundary

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2. \end{tikzcd} \tag{4.8}$$

The respective i-horns are as follows. Λ_1^2 is the simplex

$$\begin{tikzcd} & 1 \arrow[dr, "12"] & \\ 0 \arrow[ur, "01"'] & & 2. \end{tikzcd} \tag{4.9}$$

Λ_2^2 is the simplex

$$\begin{tikzcd} & 1 \arrow[dr, "12"] & \\ 0 \arrow[rr, "02"'] & & 2. \end{tikzcd} \tag{4.10}$$

Λ_0^2 is the simplex

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur, "01"] \arrow[rr, "02"'] & & 2. \end{tikzcd} \tag{4.11}$$

The *outer* horns are the horns Λ_0^n and Λ_n^n, referred to as left and right horns respectively. Horns for which $i \neq 0, n$ are referred to as *inner* horns.

We have remarked many times that a Kan complex is defined by a certain lifting condition. We may now we complete the definition of the Kan complex by spelling this condition out.

Definition 4.1.1. *Let X be a simplicial set. X is said to be a Kan complex if for any diagram*

$$\begin{tikzcd} \Lambda^n_i \arrow[r, hook] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{4.12}$$

with $0 \le i \le n$, there exists a dotted arrow making the diagram commute. In other words, every horn is fillable.

The following theorem is the foundation of the theory of ∞-groupoids and justifies their alternative designation as "homotopy types" [13] . It suggests that ∞-groupoids are the same thing as spaces. First of all, we note that the category Kan complexes or ∞-groupoids themselves assemble into an ∞-category (see below) we denote by Kan, ∞-Grpd, or *Spc*. The reason for the latter designation is the following fortified version of the classical adjunction equivalence between simplicial sets and topological spaces :

Theorem 4.1.1. *There is a Quillen equivalence $Top_{Quillen} \to sSet_{Quillen}$, which may be seen as a shadow of an equivalence of ∞-categories $\infty - Grpd \cong Top$. In particular, the singular complex and geometric realization functors are mutually inverse and give an equivalence $Ho(CW) \cong Ho(Kan)$.*

Finally, we define weak Kan complexes or ∞-categories. Let's pause to note that what we refer to as an ∞-category in this text is more properly called an $(\infty, 1)$-category. While all the categories we deal with in this text such as $DG - Cat$ or Cat^{st}_∞ in fact naturally have a $(\infty, 2)$-categorical structure, we have avoided any discussion of this aspect and will not introduce (∞, n)-categories for $n > 1$. In this notation an ∞-groupoid is an $(\infty, 0)$-category. Here n denotes the dimension starting with which all higher morphisms are invertible. In an ∞-groupoid all n-morphisms are invertible. In an ∞-category, all n-morphisms are invertible except for $n = 0$ since not every

[13] Nomenclature that we largely avoid.

"edge" (see below) is necessarily invertible. With higher n, one loses the automatic invertibility of 2-morphisms, which are guaranteed by the weak Kan filling condition for $n = 1$. Let us state this condition explicitly.

Definition 4.1.2. *A simplicial set is said to be (weak) Kan complex or an ∞-category if for every diagram*

$$\begin{tikzcd} \Lambda^n_i \arrow[r, hook] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{4.13}$$

where $0 < i \leq n$, there exists a dotted arrow making the diagram commute. In other words, every inner horn is fillable.

As mentioned before, there is a model structure on $sSet$, the Joyal model structure, in which the fibrant objects are precisely ∞-categories. By way of a detour through the theory of simplicial categories and constructions involving this model structure, one may show that ∞-categories themselves assemble into an ∞-category Cat_∞ where we again ignore the 2-categorical aspect. We omit the account of the theory of ∞-functors, which is conceptually quite natural but extremely inexplicit in contrast to ordinary functors. Actual ∞-functors are best handled via the theory of *fibrations* which we deal with in the next section.

For a given ∞-category C, it is standard practice to refer to elements of $C[0]$, the vertices of the simplicial set as objects and to those of $C[1]$, the edges of the simplicial set C, as morphisms in the sense of ordinary category theory. This is justified by the example provided by the nerve construction. Just as in the latter, given two composable "morphisms" $f : X \to Y, g : Y \to Z \in C[1]$, there should be a way to describe composition as a map $C[1] \times C[1] \to C[1]$. First of all, let's decode what it means for morphisms to be composable, mirroring the nerve construction. Recall that the face maps $d_0, d_1 : C[1] \to C[0]$ function as source and target maps and morphisms f, g are composable if $d_0(f) = d_1(g)$. A priori, two such morphisms describe a horn-shaped diagram in C, that is, a map $\Lambda^2_0 \to C$ as in the diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] & & Z \end{tikzcd} \tag{4.14}$$

Now observe that the Kan lifting condition above guarantees the existence of a map $h : X \to Z$ completing the horn diagram to a 2-simplex $\Delta^2 \to C$ which is precisely *a* composition of f and g:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dotted, "h"'] & & Z \end{tikzcd} \tag{4.15}$$

But if we end stop the story here and claim we have defined a category structure, with composition laws for morphisms, we'd be forgetting all the higher data that go into the definition of an ∞-category. First issue to address is that the horn-filler h is not unique, but only unique up to homotopy. Let's first define homotopies between edges of a simplicial set.

Definition 4.1.3. *Consider parallel morphisms $f, g : X \to Y$ in C with the same source and target. A homotopy between f and g is a map $\Delta^2 \to C$, that is, a diagram*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "id_Y"] & & Y. \end{tikzcd} \tag{4.16}$$

Lurie (see 1.3.3.7 in [18]) proves that this defines an equivalence relation on the set of edges and, further, gives the following explicit form of the definition which makes manifest the symmetry and the "homotopy map" between f and g.

Lemma 4.1.1. *Morphisms $f, g : X \to Y$ with the same source and target are homotopic if and only if there is a map $H : \Delta^1 \times \Delta^1 \to C$ such that we have a "commutative* *diagram"*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] \arrow[dr, "h"] & Y \arrow[d, "id_Y"] \\ X \arrow[r, "g"'] & Y. \end{tikzcd} \tag{4.17}$$

Now we can properly state the manner in which the composition $g \circ f$ can be said to be well-defined *up to homotopy.*

Lemma 4.1.2. *Given morphisms f and g as above, any two candidates for the composition $g \circ f$ (that is, horn fillers for the diagram above) are homotopic. Hence they descend to a well-defined composition at the level of the homotopy category.*

Note that we have mentioned the homotopy category above but will not be defining it. The reader may consult section 1.2.3 in [52].

This fundamental non-uniqueness is of course replicated at all levels. Hence when we speak of *the* commutative diagram witnessing associativity of composition or some such property, we are going to be invoking the Kan lifting condition to establish that such a diagram must exist, since such conditions merely correspond to higher horns, which can all be filled if the simplicial set is an ∞-category. However, as we've seen with the example of composition, such a diagram will never be unique, but only so up to a yet higher homotopy. This is the fundamental characteristic of higher categories.

We now arrive at the task of defining mapping spaces in an ∞-category C. This task is not as straightforward as it was in ordinary category theory (see the preliminary discussion in 1.2.2 in [52]) for the simple reason that we are in fact secretly dealing with what we called "derived mapping spaces" in the previous sections, where now the homotopical and model theoretic baggage has been concealed under the quasicategorical formalism. In fact there are several candidates for this object and it turns out they are all isomorphic at the level of the homotopy category [14] . The first thing that

[14]We have not introduced the homotopy category of an ∞-category, see 1.2.3 in [52]

comes to mind is to take advantage of the flexibility of the ∞-categorical formalism and use the arrow category $Fun(\Delta^1, C)$ to define the mapping spaces as explained in Definition 1.3.47 in [53]. As in ordinary category theory, we have the source and the target maps $s \times t : Fun(\Delta^1, C) \to C \times C$ whose images can be identified with X and Y respectively and we can define $Map_C(X, Y)$ by the *pullback* diagram [15]

$$\begin{tikzcd} Map_C(X,Y) \arrow[r] \arrow[d] & Fun(\Delta^1, C) \arrow[d] \\ \Delta^0 \arrow[r] & C \times C. \end{tikzcd} \tag{4.18}$$

By general arguments, $Map_C(X, Y)$ turns out to be a Kan complex, hence it is clearly *a* correct candidate for the derived mapping space object in an ∞-category. Indeed, we have a well-defined composition operation

$$Map_C(X, Y) \times Map_C(Y, Z) \to Map_C(X, Z) \tag{4.19}$$

coming from the pullback diagram induced by the functor

$$Map_C(X, Y) \times Map_C(X, Y) \to Fun(\Delta^1, C) \times_C Fun(\Delta^1, C) \to Fun(\Delta^1, C). \tag{4.20}$$

Another candidate for the quasicategorical mapping spaces in an ∞-category is provided by the simplicial set of *right* (or left) morphisms $Hom^R_C(X, Y)$ which is constructed as follows.

Definition 4.1.4. *(Proposition 1.2.2.3 in [52]) Given an ∞-category, the simplicial set of right morphisms $Hom^R_C(X, Y)$ is the simplicial set with simplices $(Hom^R_C(X, Y))_n$ consisting of all maps $f : \Delta^{n+1} \to C$ such that $f|\Delta^{n+1} = Y$ and $f|\Delta^{0,\ldots,n} = X$ with the simplicial maps induced by those of C. In fact, this simplicial set is a Kan complex.*

In what follows, we will simply put $Hom_C(X, Y)$ for the Kan complex of right morphisms between X and Y, with the understanding that $Hom^R_C(X, Y)$ and $Hom^L_C(X, Y)$ represent the same object in the homotopy category. In fact, more is true. It is explained in Proposition 1.2 of [54] that what we called $Map_C(X, Y)$ and $Hom^{L,R}_C(X, Y)$ are models of the model-categorical function com-

[15] See the next section for (co)limits in the ∞-categorical context.

plex $hMap_{\partial/\Delta^1 sSet_{Joyal}}(\Delta^1, C)$ [16] of the Joyal model category of simplicial sets over $\partial\Delta^1$. Hence, from the perspective of homotopic fidelity, these mapping spaces describe the same object.

4.1.2. Limits and Colimits in ∞-categories

Let C, D be ordinary categories. We adopt the notation of Section 1.2.8 in [52]. The join $C * D$ is the category with objects $Ob(C*D) = Ob(C) \sqcup Ob(D)$ and morphisms

$$Mor(C * D) = \begin{cases} Hom(X,Y) & X, Y \in C \\ Hom(X,Y) & X, Y \in D \\ * & X \in C, Y \in D \\ \varnothing & X \in D, Y \in C. \end{cases} \tag{4.21}$$

This operation gives rise to a monoidal structure on Cat, which we denote by $Cat^{\sqcup}$. It admits a natural generalization to the higher context as follows.

Definition 4.1.5. *For $S, S^{'} \in sSet$, the join $S * S^{'}$ is the simplicial set whose n-simplices are*

$$(S * S^{'})_n := S_n \cup S^{'}_n \bigcup_{i+j=n-1} S_i \times S^{'}_j. \tag{4.22}$$

Denote by d_n the face maps acting on sets of simplices S_l and $S^{'}_k$. Explicitly, these are

$$d_j(x,y) = \begin{cases} (d_j x, y), j \leq l, l \neq 0 \\ (x, d_{j-l-1} y), i > l, k \neq 0 \\ x, l = 0 \\ y, k = 0. \end{cases} \tag{4.23}$$

[16]See Section 2.9 on function complexes in model categories.

Likewise, we have the degeneracy maps (keeping the same notation)

$$s_j(x,y) = \begin{cases} (s_j x, y), j \leq l \\ (x, s_{j-l-1} y), i > l. \end{cases} \tag{4.24}$$

One can verify that this definition is the appropriate generalization of the ordinary categorical join in that we have an equivalence

$$N(C * C^{'}) \cong N(C) * N(C^{'}). \tag{4.25}$$

It is instructive to play around with algebra of joins and look at examples in lower dimensions for reasons that will become clear. For instance take $C = C^{'} = \Delta^0$. Then evidently $\Delta^0 * \Delta^0 = \Delta^1$. We may generalize this to joins of other simplices with the point. For instance, $\Delta^0 * \Delta^1$ is the simplicial set

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[rr] & {} & 1. \end{tikzcd} \tag{4.26}$$

Just as in the case of ordinary categories, the simplicial join operation in fact restricts to a monoidal structure on ∞-categories.

Now we have the equipment to discuss limits and colimits in ∞-categories.

Definition 4.1.6. *(1.2.8.4 in [52]) The left (resp. right) cone of a simplicial set K is the simplicial set $K^{\triangleleft} := \Delta^0 * K$ (respectively, $K^{\triangleright} := K * \Delta^0$).*

To define the notions of limits and colimits of K-diagrams $K \to S$ with $K \in sSet, S \in Cat_\infty$, we need the notion of an over- and undercategories in the higher context. Recall that for C an ordinary category, the overcategory $C_{/X}$ is the category whose objects are morphisms $X \to Z$ and whose morphisms are commutative triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[dl] \\ Z. & \end{tikzcd} \tag{4.27}$$

A prominent example in algebraic geometry comes from the *relative* scheme categories over some base scheme S: $Sch_S := Sch_{/S}$. The convention for overcategories is the dual for the undercategories.

We will define over- and undercategories in the higher context by universal properties, following section 1.2.9 in [52]. Given $S, K \in sSet$ and $p : K \to S$, define $S_{/p}$ to be the simplicial set whose n-simplices are

$$(S_{/p})_n := Hom_p(\Delta^n * K, S) \tag{4.28}$$

where the subindex p indicates we are only considering maps $f : Y * K \to S$ which equal p when restricted to K. This simplicial set, more or less tautologically, enjoys the universal property that the simplicial set of maps $Hom_{Set_\Delta}(Y, S_{/p})$ is equal to that of maps of simplicial sets $\Delta^n * K \to S$ which equal p when restricted to K.

Definition 4.1.7. *With p, K as above and C an ∞-category, we say $C_{/p}$ is the the (over)category of C over p. When p is a map $\Delta^0 \to C$ with image an object X of C, we denote the category of C over X by $C_{/X}$. Dualizing in an evident fashion, we denote the undercategory over p by $C_{p/}$.*

As an illuminating example, the undercategory for $p = \Delta^2_0 \to C$ is the ∞-category consisting of diagrams

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 2 \arrow[r] & U. \end{tikzcd} \tag{4.29}$$

The notions of initial and final objects in an ordinary category are self-evident. Since it is by construction impossible to impose conditions such as uniqueness at the level of objects in ∞-categories, the homotopically correct approach to the generalization is given by way of Proposition 1.2.12.4 of [52].

Definition 4.1.8. *An object X of an ∞-category C is said to be final (initial) if the Hom ∞-groupoids $Hom(Y, X)$ are contractible for $X \in C$ (resp. Hom(X,Y)).*

By the following proposition, which is Proposition 1.2.12.9 in [52], these objects satisfy the right analogue of uniqueness in the homotopic context.

Proposition 4.1.1. *The subcategory spanned by final (or initial) objects is empty or contractible.*

We may now define the notion of a (co)limit:

Definition 4.1.9. *A limit of a diagram $p : K \to C$ is a final object of $C_{/p}$. A colimit of a diagram is an initial object of $C_{p/}$. By Remark 1.2.13. 5 in [52], one also refers to the map $p : K^{\triangleright} \to C$ corresponding to the object of the overcategory (undercategory) given by the (co)limit.*

To explicate the last sentence, consider the following situation. Let $p : \varDelta^2_0 \to C$ be the diagram whose limit we are considering. Formally, this limit (a pushout) is a "commutative diagram"

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 2 \arrow[r] & U. \end{tikzcd} \tag{4.30}$$

The point of the remark is that we will refer to the "cone point" U (or rather the join diagram $U * \varDelta^2_0$) as the *limit* of the diagram p if this diagram is in fact a limit diagram as an object of the overcategory.

(Co)limits inherit the desired universal property from their definition as initial (final) objects, as in the case of ordinary categories.

The most important class of colimits are the *filtered* ones, which are colimits of diagrams $D \to C$ where D is a filtered ∞-category. We recall the notion of the filtered category and filtered colimit in ordinary category theory, following the notation of section 5.3.1 of [52].

Definition 4.1.10. *With κ a regular cardinal, category is said to be κ-filtered if:*

(i) For every collection $\{X_i\}$ of finite objects, there is an object equipped with maps $\phi_i : X_i \to X$.

(ii) Given objects X, Y, Z and for any morphisms $f, g : X \to Y$, there is a morphism $h : Y \to Z$ such that $h \circ f = h \circ g$.

A special case of a filtered category is a *directed* poset regarded as a category. Recall that we can see a poset as a category with morphisms given by the partial order relation $\leq$ on the elements. A directed poset is a poset in which for any $a, c \in P$ there is an object b such that $a < b < c$. The object X in the case of directed poset is just upper bound of a given collection of elements computed as a directed limit. The second condition can be interpreted as saying that any two parallel morphisms "eventually" agree.

Definition 4.1.11. *A filtered colimit is the colimit of a diagram $F : I \to D$ where I is a filtered category.*

An illuminating fact about filtered colimits is the following slogan which holds for limits and colimits in the category of sets:

Proposition 4.1.2. *Filtered colimits commute with finite limits.*

Now we define the notion of a filtered ∞-category as given in section 5.3.3 of [52]. Recall that a simplicial set is said to be κ-small for κ a regular cardinal, if its number of non-degenerate simplices is smaller than κ and just *small* when it is ω-small.

Definition 4.1.12. *For a regular cardinal κ, an ∞-category C is said to be κ-filtered if every map $K \to C$ with K κ-small simplicial set, extends to a map $\kappa^{\triangleleft} \to C$. C is said to be filtered if it is ω-filtered.*

There is a notion of compact objects and compact generation in the context of ∞-categories which directly parallel the same notion in the triangulated and dg-contexts, see section 5.3.4 in [52] for details.

Definition 4.1.13. *Let C and D be ∞-categories and κ a regular cardinal. A functor $F : C \to D$ which preserves κ-filtered colimits is said to be κ-continuous. An object $X \in C$ is said to be κ-compact if the Yoneda functor j_X is κ-continuous. When $\kappa = \omega$, we just say C is compact. Denote the subcategory of compact objects of C by C^ω.*

We have the following definition of compactly generated ∞-categories, which will make sense once we unwind it after we introduce these concepts in the next section:

Definition 4.1.14. *An ∞-category is said to be compactly generated if it is presentable and κ-accessible.*

4.2. Presentable ∞-categories

The correct context for the study of adjunctions and the adjoint functor theorem in the ∞-categorical setting is the theory of presentable ∞-categories. The formalism of presentability allows us to study large ∞-categories by way of a *small* amount of data, analogous to how finitely generated, but not necessarily finite, groups, rings, modules and so an are presented by a finite amount of data. We shall see that presentable ∞-categories assemble into an ∞-category which we denote by Pr^L, where we consider left-adjoint, or colimit preserving functors between presentable ∞-categories as morphisms.

As alluded to above, the main motivation for presentable ∞-categories is the following theorem of Lurie generalizing the adjoint functor theorem for ordinary categories.

Theorem 4.2.1. *(Theorem 5.5.2.9 in [52]) Let C, D be presentable ∞-categories. Then a functor $F : C \to D$ admits a right adjoint if and only if it preserves small colimits. Further, a functor $F : C \to D$ admits a left adjoint if and only if it is accessible and preserves small limits.*

For a simplicial set S, denote by $P(S) := Fun(S^{op}, Spc)$ the category of presheaves of spaces on S. We have the fully faithful Yoneda embedding: $S \hookrightarrow P(S)$ given by

Proposition 5.1.3.1 of [52], which is (small) limit preserving. An object of $P(S)$ is said to be representable if it is in the essential image of the Yoneda embedding, that is, if F is of the form $Hom(-,X)$ for some $X \in S$. As in ordinary category theory, it is possible to see $P(S)$ as the free co-completion of S and the free ∞-category generated by S under colimits by way of the Yoneda embedding. In this vein, $P(C)$ has the following universal property. Let C be an ∞-category admitting small colimits. Then we have an equivalence of ∞-categories between the category of colimit preserving functors $Fun^L(P(S),C)$ and the functor category $Fun(S,C)$. That is, $P(S)$ may be characterized by the fact that every functor $S \to C$ uniquely (in the apppropriate sense) extends to a colimit preserving functor $P(S) \to C$.

Let us fix an ∞-category C. Now we focus on a particularly important subcategory of $P(C)$, the category of ind-objects of C. It is more convenient in the ∞-categorical context to define the category of ind-objects as a subcategory of $P(C)$ rather than directly in terms of ind-limits. In this section, we follow Section 5.3.5 in [52].

Definition 4.2.1. *Let C be an ∞-category. Denote by $Ind(C)$ the full subcategory of $P(C)$ spanned by presheaves which correspond to fibrations $\tilde{C} \to C$ with $\tilde{C}$ a filtered category. This category enjoys the following useful properties:*

(i) There is a subfunctor of Yoneda embedding $j : C \to Ind(C)$.
(ii) $Ind(C)$ admits all small filtered colimits.

Definition 4.2.2. *A ∞-category C is said to be (κ)-accessible, for κ some small cardinal, if we have an equivalence:*

$$C \cong Ind(C_0) \tag{4.31}$$

where C_0 is a (κ)-small ∞-category. If, further, C admits all colimits, it is said to presentable. We denote by Pr^L the category of presentable ∞-categories with colimit preserving functors.

By a theorem of Simpson, an ordinary category C is presentable iff there exists a small category D such that C is an *accessible localization* of $P(D)$. Thus we have the slogan that presentable categories are localizations of categories of presheaves. We'll introduce localizations in the context of ∞-categories and see that exactly the same thing holds for presentable ∞-categories.

We will now introduce the concept of idempotent completeness in the context of ∞-categories. We have already met idempotents in ordinary categories and the procedure of idempotent (or Cauchy) completion in our introductory section. These concepts readily generalize to higher categories, as explained in Section 4.4.5 in [52]. Let E^+ be the (nerve of the) category consisting of elements $\{X, Y\}$ with morphisms an idempotent map $e : X \to X$, a retraction $r : X \to Y$, and the section $s : Y \to X$ with relations which encode these properties: $e^2 = e$, $e = sr$, $rs = id_Y$. We will confuse E^+ with its nerve. Let E be the (the nerve of the) subcategory spanned by X.

Definition 4.2.3. *An idempotent morphism in an ∞-category C is a map $E \to C$. An ∞-category is said to be idempotent complete if every idempotent map $E \to C$ extends to a map $E^+ \to C$.*

We will meet examples of idempotent complete ∞-categories later on. For now, we note the following facts, as given by Corollary 4.4.5.16 in [52].

Proposition 4.2.1. *A co-complete ∞-category is idempotent complete. More specifically, an ∞-category admitting κ-filtered colimits is idempotent complete.*

More strikingly, we have the following result which demonstrates the relevance of idempotent completeness:

Proposition 4.2.2. *(5.4.3.6 in [52]) A small ∞-category is accessible if and only if it is idempotent complete.*

Just like in the case of ordinary categories, there is a self-evident procedure of idempotent completion which we will introduce here and will revisit in later sections.

Proposition 4.2.3. *(Proposition 5.4.3.5 on [52]) Given a small ∞-category C, the Yoneda embedding $j : C \to Ind(C)$ is the idempotent completion of C.*

It is self-explanatory to transcribe the notion of adjoint functors into the ∞-categorical context, and indeed the most common definition is a verbatim transcription of the definition of an ordinary adjunction. A more satisfying definition may be given in terms of coCartesian fibrations, which will be covered in the next section. Since this definition is used in many of the sources we rely on throughout this work, let's offer it in passing:

Definition 4.2.4. *Let C, D be ∞-categories. Then an adjunction between C and D is a correspondence between C and D, that is, the data of a simplicial set E equipped with a morphism $f : E \to \Delta^1$ such that*

- *f is a bi-Cartesian fibration.*
- *We have equivalences of ∞-categories $K_0 \cong C$ and $K_1 \cong D$.*

We say functor $F : C \to D$ and $L : D \to C$ are adjoint if they are the functors classified by the (op)fibrations given above.

Finally, we discuss *localization* of ∞-categories which is exactly derived from the theory of Bousfield localization of model categories. In Definition 5.2.7.2 of [52], Lurie defines a localization functor in the ∞-categorical context by way of reverse engineering:

Definition 4.2.5. *A functor $F : C \to D$ between ∞-categories C and D is said to be a localization if it admits a fully faithful right adjoint.*

To make a connection with the more familiar definition, consider the right adjoint $G : D \to C$ which by definition identifies the essential image of G with a subcategory of C. Then the composition $L := GF : C \to C$ is the more familiar analogue of the localization functor in ordinary categories which universally inverts some subset of elements.

4.3. Deeper Into ∞-category Theory

4.3.1. Fibrations in ∞-categories

In this section, we introduce special classes of maps between simplicial sets which blend categorical and topological characteristics to give rise to what may be concisely called "relative" ∞-category theory. Fibrations are characterized by various lifting properties (which are relative forms of the fundamental lifting property characterizing Kan complexes among simplicial sets). The theory of fibrations allows us to work with functors between ∞-categories by way of slick universal arguments and bypasses formidable difficulties involving higher coherences which are necessarily the part of the data defining functors in the ∞-context. The reader may consult [53] for details on fibrations.

We denote by $LHorn$, $IHorn$ and $RHorn$ the left, inner and right horn inclusions. As in the section on model categories, $rl(M)$ ($ll(M)$) denotes the class of maps with the right lifting property (left lifting property) against the class of maps M.

Definition 4.3.1. *A map $f : X \to Y$ of simplicial sets is said to be:*

(i) A left fibration if $f \in rl(LHorn)$, that is, if for all $0 \leq i < n$
(ii) A inner fibration if $f \in rl(IHorn)$, that is, if for all $0 < i < n$
(iii) A right fibration if $f \in rl(RHorn)$, that is,if for all $0 < i \leq n$

there exists a lift h giving a commutative diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "h"] & Y. \end{tikzcd}$$

Definition 4.3.2. *A map $f : S \to T$ of simplicial sets is said to be right, left, inner anodyne if f has left lifting property against right, left, inner fibrations.*

The reason for referring to the study of (left) fibrations as "relative" ∞-category theory is the following observations.

Lemma 4.3.1. *The unique map of simplicial sets $X \to * = \Delta^0$ is a left fibration if and only if X is a Kan complex. More generally, the fiber $f^{-1}(y) := X \times_y Y$ of a left fibration $f : X \to Y$ over any vertex y of Y is a Kan complex. Hence left fibrations should be regarded as relative ∞-categories.*

The main example of a left (resp. right) fibrations we have already encountered are the natural maps $C_{x/} \to C$ where C is an ∞-category and C/x is the ∞-category over x which was defined via a universal property in the previous section (resp. the map from $C_{/x}$).

Definition 4.3.3. *A Kan fibration is a map $f : X \to Y$ of simplicial sets that is at once a left and right fibration.*

To proceed further, we need a few intermediate definitions. In what follows, we follow [55]. A morphism $f \in Fun(\Delta^1, C)$ in an ∞-category C (or more generally, an edge in a simplicial set) is said to be an *isomorphism* if the induced map in the homotopy category is an isomorphism. This is equivalent to the following criterion given in 2.1 of [55].

Lemma 4.3.2. *A morphism $f \in Fun(\Delta^1, C)$ is an isomorphism if and only if it extends to a map $Fun(J, C)$ where J is the nerve of the classifying category for isomorphisms, that is, the category $N(0 \xrightarrow{\sim} 1)$.*

Definition 4.3.4. *Categorical fibration are the fibrations for the Joyal model structure on $sSet$. Namely, a map of simplicial sets $f : X \to Y$ is a categorical fibration if it is an inner fibration that has the left lifting property against the map $\Delta^0 \to J$.*

Definition 4.3.5. *A trivial fibration is a map $f : X \to Y$ of simplicial sets which has the right lifting property with respect to all boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$.*

A trivial fibration is precisely that for the aforementioned Joyal model structure: a categorical fibration which is a weak equivalence. The fibers of a trivial fibration are contractible.

4.3.2. ∞-categorical Grothendieck Construction

Our sole reference for this entire section is Section 2.4 in [52]. We do not provide specific references here since we follow Lurie very faithfully and there are no other elementary sources on this material, which was chiefly developed by Lurie.

Let's begin by recalling the Grothendieck construction for 1-categories. The idea is to make rigorous the intuitively obvious correspondence between categories over some fixed category C and "functors" $F : C \to Cat$ given by the "fiber" of the fibration $D \to C$ as D varies over Cat. Before considering "functors" $F : C \to Cat$, let us first make the elementary observation that set-valued functors $C \to Set$ also admit an interpretation as such a "fibration". Of course regarding sets as discrete categories gives us the embedding $Set \subset Cat$, hence this construction is an easy special case of the Grothendieck construction. The desired fibration is constructed as follows.

Definition 4.3.6 (Category of Elements). *Given $F : C \to Set$, we denote by $\int^C F$ the category given as follows:*

(i) $Ob(\int^C F)$ consists of pairs (x, f) where $x \in Ob(C)$ and $f \in Fx$.
(ii) The morphisms $(x, f) \to (y, g)$ are given by morphisms $u \in Mor(x, y)$ such that $(Fu)x = y$.

It is an easy exercise to see that the set-valued functor F can be recovered as the fiber of the projection $\int^C F \to C$. Our task is to give a simiar interpretation Cat-valued functors on C. Then we will proceed to generalize this construction to ∞-categories, that is, we will describe how we can encode the category of functors $C \to Cat_\infty$ as a subcategory of the ∞-category of ∞-categories over C, $Cat_{\infty/C}$.

The Grothendieck Construction is an abstract reformulation of the kind of relationship that we see between categories like $Vect(X)$ or $QCoh(X)$ and the underlying category Top or Var, Sch etc. as a "fibration" at the level of categories,which should not be confused with model categorical notion of abstract fibration. This is precisely the origin of the notion of fibered categories and stacks, of which $QCoh(-)$ is a prominent example. Note that here we are working with the definition of a stack that includes sheaves of categories, not just groupoids.

As promised, let us start with the notion of a cartesian fibration of ordinary categories, which is the content of the "Grothendieck construction". First, we define the concept of a (co)cartesian morphism.

Definition 4.3.7. *Let $F : C \to D$ be a functor between ordinary categories C and D and $f : X \to Y$ a morphism in C. We say f is F-**cartesian** if for every $W \in C$, the diagram*

$$\begin{array}{ccc} Hom_C(W,X) & \longrightarrow & Hom_C(W,Y) \\ \downarrow & & \downarrow \\ Hom_D(F(W),F(X)) & \longrightarrow & Hom_D(F(W),F(Y)) \end{array} \tag{4.32}$$

*is a pullback square. Dually, we say f is an **F-cocartesian** if for every object $W \in C$, the diagram*

$$\begin{array}{ccc} Hom_C(Y,W) & \longrightarrow & Hom_C(X,W) \\ \downarrow & & \downarrow \\ Hom_D(F(Y),F(W)) & \longrightarrow & Hom_D(F(X),F(W)) \end{array} \tag{4.33}$$

is a pullback square.

It is quite evident what these properties entail for the compatibility between a morphism and a functor. An F-cartesian morphism is "functorially" adapted for the functor F.

Definition 4.3.8. *Let $F : C \to D$ be a functor between ordinary categories C,D. F is said to be a Cartesian fibration if for every object $Y' \in C$ and every morphism*

$f: X \to F(Y')$ in D, there exists an object X' of C such that $F(X') = X$ and an F-cartesian morphism f' in C such that $F(f') = f$. Dually, we say that F is a cocartesian fibration if it satisfies the same property with the arrows reversed.

(co)Cartesian fibrations over some fixed category C can be self-evidently organized into a (2-)category $Cart(C)$ with morphisms between Cartesian fibrations given by by functors sending Cartesian morphisms on one side to Cartesian morphisms on the other side. The following formulation foreshadows what we will soon describe in the context of ∞-categories.

Theorem 4.3.1. *There is an equivalence of categories,*

$$Fun(C^{op}, Cat) \cong Cart(C). \tag{4.34}$$

Note that we can dualize to obtain an equivalence between $coCart(C)$ and $Fun(C, Cat)$.

Now we must reformulate these concepts in the context of ∞-categories. What happens if we attempt to transplant, verbatim, the definitions above into the world of ∞-categories? Already we run into a problem in the definition of an F-cartesian morphism for a given ∞-functor $F: C \to D$ between ∞-categories C,D. Let $f: x \to y$ be an edge in C. Above we used "composition with f on the left" to construct a map $Hom_C(z, x) \to Hom_C(z, y)$. For ∞-categories, such a map is only homotopically well-defined. Since we don't wish to revisit the difficulties of working with hom-spaces of ∞-categories, we will instead proceed with a more direct and precise approach as follows.

Definition 4.3.9. *Let $F: X \to Y$ be a map between simplicial sets X,Y and $f: x \to y$ an edge in X. f is said to be F-cartesian if for every $n \geq 2$, there exists a lift as shown in the diagram*

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "\sigma"] \arrow[d] & X \arrow[d, "F"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \tag{4.35}$$

where the edge $\Delta^{n-1,n}$ in the horn Λ^n_n is mapped to the edge f in X under σ.

Dually, an edge is coCartesian if there exists a lift as shown in the diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\sigma"] \arrow[d] & X \arrow[d, "F"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \tag{4.36}$$

where the edge $\Delta^{0,1}$ in the horn Λ^n_0 is mapped to the edge f under σ.

Now we can proceed to define (co)Cartesian fibrations for ∞-categories. Let $F : X \to Y$ be an inner fibration between simplicial sets X,Y. Recall that this means F satisfies right lifting property with respect to all horn inclusions:

$$\Lambda^n_i \hookrightarrow \Delta^n \tag{4.37}$$

with $0 < i < n$.

Definition 4.3.10. *F is said to be a Cartesian fibration if for every edge $f : x \to y$ in Y and every $b \in X$ with $F(b) = y$ there exists an F-cartesian edge $g : a \to b$ in X such that $F(g) = f$.*

Dually, F is said to be a cocartesian fibration if for every edge $f : x \to y$ in Y and every $a \in X$ with $F(a) = x$, there exists an F-cocartesian edge $g : a \to b$ in X such that $F(g) = f$.

As before, the data of all cartesian fibrations can be organized into an ∞-category which we denote by $Cart(C)$.

We have the following theorem which is absolutely crucial for many essential constructions underlying our work in this thesis.

Theorem 4.3.2. *(Straightening) Let Y be a simplicial set. We have an equivalence of ∞-categories*

$$Fun(Y^{op}, Cat_\infty) \cong Cart(Y). \tag{4.38}$$

4.4. Symmetric Monoidal ∞-categories

Foundations of higher algebra contain a certain tautological aspect which may cause some confusion unless care is exercised, which stems from what has been colloquially called the "macrocosm/microcosm" principle. On one hand, higher algebra studies *algebraic* objects and operations in the context of a higher category, which seems intuitive enough. On the other hand, to be able to discuss such things appropriately, we need to define an algebraic structure on the ambient higher category itself, which in turn requires studying it as an algebraic object in yet another ambient higher category. This aspect is already visible in *categorical* algebra, which is at once the study of algebra-like objects in symmetric monoidal categories and symmetric monoidal categories as monoid objects themselves in the category of small categories. In particular, this section aims to complete the analogy between linear algebra and ∞-categorical homotopical/homological algebra. Particularly, we will delve into the delicate question of how to isolate the right class of "finite" (finite dimensional, perfect, compact, idempotent complete etc.) objects.

In the spirit of higher algebra, we would like to see symmetric monoidal ∞-categories as commutative algebra objects in Cat_∞, where the latter is equipped with the Cartesian symmetric monoidal structure. That is, we would like to put $SymMon_\infty \cong CAlg(Cat_\infty)$. Evidently, this is somewhat tautological and we need to make precise what we mean exactly by a symmetric monoidal structure on an ∞-category. To do this rigorously, we would need to embark on a rather long and technical detour through the theory of ∞-operads, of which symmetric monoidal ∞-categories are a special case. For reasons of economy and space, we will not pursue this path and omit the proper definition of symmetric monoidal ∞-categories, hoping that our intuitive sketches will be sufficiently illuminating.

Essentially, this perspective revolves around the "bar construction" of a monoidal category C as a monoid object in Cat, which results in a simplicial object $\Delta^{op} \to Cat$ with the simplicial structure maps encoding the monoidal structure on C. Now let $(C, \otimes)$ be a symmetric monoidal category and M a monoid object in C with the "multiplication law" $\mu : M \otimes M \to M$ and unit map $\eta : * \to M$. For the purposes of this section one may simply take $C = Vect_k$, the category of vector spaces over some field k, in which case M is an ordinary monoid. The bar construction builds a simplicial monoid (or equivalently by Dold-Kan correspondence, a chain complex) out of M using the multiplication law. Namely, we put $Bar(M)_n := M^n$ for the n-simplex of this simplicial object with face maps given by $\mu : M \otimes \underbrace{....}_{\text{n}} \otimes M \to M \otimes \underbrace{...}_{\text{n-1}} M$. More precisely, μ_i is the map that *eliminates* the i-th factor in the product $M \otimes \underbrace{....}_{\text{n}}$ by applying the multiplication map and the degeneracy maps s_i add an i-th factor by applying the unit map at the i-th position. We denote by $Bar(M)$ the resulting simplicial monoid. On the face of it, $Bar(M)$ appears to contain more information than the bare monoid with its binary multiplication operation. For ordinary monoids, the data of a bar construction is an interesting repackaging of the "space" of n-ary operations which all unproblematicaly collapse to the binary one. However, in the homotopic context, this collapsing itself will involve a great deal of coherence data. Note that the monoid itself (as encoded by the binary multiplication operation μ) can be recovered from the 2-simplex $Bar(M)_2$. An operad is a device that axiomatizes the combinatorics of composition of n-ary operations and concatenations of compositions, which naturally have a tree-like structure (in fact operads can be modeled as "tree-shaped" diagrams in certain target categories). Hence operads allow us to describe "algebraic structures" in different contexts. Since we do not wish to define ∞-operads here, we follow a more streamlined approach. Replacing the ordinal category with the Segal category and adding little more data to what was described above, we can obtain a similar characterization of a *symmetric* monoidal category as a functor $\Gamma^{op} \to Cat$. But by the results of the preceding section, such functors can be precisely characterized as Cartesian fibrations $C \to \Gamma^{op}$. In fact, the definition of Cartesian fibrations naturally takes care of how the symmetric monoidal structure can be naturally codified in this formulation. By construction, these concepts generalize immediately to ∞-categories!

In what follows, we do not distinguish between the Segal category of finite sets Γ and its nerve. As in the chapter on Cartesian fibrations, we denote by $C_[n]$ the "fiber" of a (co)Cartesian fibration over $[n] \in \Gamma$ and so on.

Definition 4.4.1. *(Definition 1.1.2 in [56]) A monoidal ∞-category is a coCartesian fibration of simplicial sets $p : C^{\otimes} \to \Gamma$ such that we have an equivalence of ∞-categories*

$$C^{\otimes}_{[n]} \to C^{\otimes}_{0,1} \times ... \times C^{\otimes}_{\{n-1,n\}} \cong (C^{\otimes}_{[1]})^n. \tag{4.39}$$

The fiber over [1], $C_{[1]}$ is referred to as the underlying ∞-category of $C^{\otimes}$ and we denote it by C unless we wish to emphasize the monoidal structure. The induced functor $C_{[0]} \to C$ encodes the unit object we denote by 1_C. Unpacking the equivalence condition for fibers over $\{1,2\}$ and $\{2,3\}$, we obtain the monoidal product $\otimes : C \times C \to C$. Higher coherence conditions can be verified to reproduce analogues of Maclane's axioms for an ordinary monoidal category in the homotopic setting.

A functor between F symmetric monoidal categories $p : C^{\otimes} \to \Gamma$ and $q : D^{\otimes} \to \Gamma$ is said to be monoidal if (Definition 1.1.18 in [56]) the following conditions are satisfied.

(i) There is a commutative diagram

$$\begin{tikzcd} C^{\otimes} \arrow[r] \arrow[d, "p"'] & D^{\otimes} \arrow[dl, "q"] \\ \Gamma. & \end{tikzcd}$$

(ii) F takes p-coCartesian morphisms to q-coCartesian morphisms.

Monoidal ∞-categories with monoidal functor can be assembled into an ∞-category we denote by Mon_{∞}. As hinted at above, one can define the notion of an algebra object in a monoidal ∞-category. The precise definition is the following.

Definition 4.4.2. *(Definition 1.1.18 in [56]) An algebra object in a monoidal ∞-category is a lax monoidal section of $p : C^{\otimes} \to \Gamma$.*

As a very technical but straightforward sanity check, one can prove that monoidal ∞-categories are precisely monoid objects for the Cartesian monoidal structure on Cat_{∞}.

One can further define *symmetric* monoidal ∞-categories, which requires more serious foundational work and a proper foray into theory of ∞-operads. The construction may be found in section 2.0.0.7 of [57]. We denote the ∞-category of symmetric monoidal ∞-categories by $SymMon_\infty$. In fact, we have $SymMon_\infty \cong CAlg(Cat_\infty)$. That is to say, symmetric monoidal ∞-categories are commutative algebra objects with respect to the Cartesian monoidal structure on Cat_∞. The exact definition of commutative algebra objects in monoidal ∞-categories can be found in section 2.1.3 of [57].

The last topic we need to cover in the symmetric monoidal ∞-categories is the calculus of dualizability, which will help us do a sort of categorified linear algebra with symmetric monoidal ∞-categories.

As one might guess, the origin of *dualizability* as a finiteness property goes back to the well-known fact that one has an isomorphism of vector spaces $V \cong (V^{op})^{op}$ and finite dimensional vector spaces admit a trace formalism which exploits the closed symmetric monoidal structure on their category. Explicitly, this structure gives rise to the trace map $tr : End(V) \to k$ as a certain composition of evaluation and coevaluation maps, which we review below.

Let $(C, \otimes, 1_C)$ be an ordinary symmetric monoidal category. We follow section 2.3.1 in [58]. We say an object is dualizable with dual V^{op} if there exists maps $ev_V : V \otimes V^{op} \to 1_C$ and $coev_V : V^{op} \otimes V \to 1_C$ such that the diagram

$$V \xrightarrow{id_V \otimes coev_V} V \otimes V^{op} \otimes V \xrightarrow{ev_V \otimes id_V} V \tag{4.40}$$

composes to id_V and the diagram

$$V^{op} \xrightarrow{coev_V \otimes id_{V^{op}}} V^{op} \otimes V \otimes V^{op} \xrightarrow{id_{V^{op}} \otimes ev_V} V^{op} \tag{4.41}$$

composes to $id_{V^{op}}$.

A symmetric monoidal category in which every object is dualizable is often called *rigid*. The category of finite dimensional vector spaces is a rigid symmetric monoidal category.

We may port this definition verbatim into the higher context. Namely, we say an object X in a symmetric monoidal ∞-category is dualizable as in Definition 2.1.13 [59] if there is an object X^{op} such that we have an evaluation map $ev_X : X^{op} \otimes X \to 1_C$ and a coevaluation map $coev_X : 1_C \to X \otimes X^{op}$ which form a diagram identical to the one above subject to higher compatibility relations, which we omit.

4.5. Stable ∞-categories

In this chapter, we introduce the final fundamental object of our study, stable ∞-categories. Our ultimate goal is to study the ∞-category Cat^{st}_{∞}, the ∞-category of stable ∞-categories (with exact functors) and Pr^L_{st}, the ∞-category of presentable stable ∞-categories. Formal properties of this category can be probed via comparison with the ∞-category $DG - Cat$ and various model structures on the latter can be interpreted in terms of the former. Naturally, we start with the very definition of a stable ∞-category, which amounts to a creative reverse-engineering of the definitions of an abelian category and triangulated category.

In what follows, we follow Lurie [52, 56, 57].

Definition 4.5.1. *A zero object in an ∞-category C is an object that is both initial and final: that is, an object 0 such that mapping ∞-groupoids (from now on, spaces) $Map(0, X)$ and $Map(X, 0)$ are both contractible for all $X \in C$. The zero object is forced to be "unique" in the appropriate homotopical sense: we have a contractible choice of "zero objects". If C has a zero object, we say it is pointed.*

We proceed to set up the concept that is a simultaneous generalization of "exact sequences" and " exact triangles".

Definition 4.5.2. *A triangle in a pointed ∞-category is a diagram $D : \Delta^1 \times \Delta^1 \to C$*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{4.42}$$

where $f, g \in Fun(\Delta^1, C)$, with 0 a zero object of the pointed ∞-category. We say that

D is a (co)fiber sequence if it is a pullback (pushout) square. This is the notion that simultaneously generalizes the notion of an exact sequence and exact triangle under the right circumstances, as we shall see.

Following Lurie, we expand out the definition of the triangle. First note that implicit in the definition is that we have a diagram witnessing the composition of the morphisms f and g,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \tag{4.43}$$

where h is *a* composite of f and g. We will explain the sense in which this composite is unique.

That the composition is homotopically zero is witnessed by the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "h"'] & 0 \arrow[d] \\ & Z \end{tikzcd} \tag{4.44}$$

where the maps are the zero maps which always exist by the definition of the zero object. We follow Lurie in adopting the suggestive triangulated categorical notation and express a triangle as the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{4.45}$$

for triangles in a pointed ∞-category. Note that we have not as yet imposed any stricter exactness properties on these objects. We proceed now to find appropriate notions of (homotopy) "kernel" and "cokernel" in our context.

Consider a map $f : X \to Y$. The *fiber* of f is presented by the fiber sequence

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y. \end{tikzcd} \tag{4.46}$$

Likewise the *cofiber* of a map $f : X \to Y$ is presented by the cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array} \tag{4.47}$$

Appropriate versions of "uniqueness" for (co)fibers of maps may be established for, see for instance Remark 1.1.1.7 in [57]. Namely, denote the subcategory of cofiber sequences in an ∞-category C by $Ex(C) \subset Fun(\Delta^1 \times \Delta^1, C)$. Then the forgetful functor $\theta : Ex(C) \to Fun(\Delta^1, C)$ which maps a cofiber sequence as above to the morphism f is a *Kan* fibration with contractible fibers. Hence a cokernel is ∞-categorically unique if it exists.

Finally, we reach the definition of our main object.

Definition 4.5.3. *We say a pointed ∞-category is stable if every morphism in C admits a fiber and cofiber and they coincide. It is immediately obvious that this is a direct transcription of the axioms of an abelian category.*

Before we delve further into the lore of stable ∞-categories, we go back to to the foundations of ∞-category theory to introduce a crucial technical lemma on the uniqueness of Kan extensions due to Lurie which is used constantly in the formulations of key concepts in stable ∞-categories. Since it is a rather specialized topic which admits an intuitive but technical generalization in the higher setting, we did not cover Kan extensions in the previous section and we take the opportunity to review them now. Assuming knowledge of the classical theory of Kan extensions, we forge ahead to the ∞-categorical one.

Definition 4.5.4. *(Definition 4.3.2.1 in [52]) Let C be an ∞-category and C^0 a full subcategory as in Remark 1.2.11 in [52]. Given a diagram $p : K \to C$ with K a simplicial set, we put $C^0_{/p} := C_{/p} \times_C C^0$. For $X \in C$, the over-category $C^0_{/X}$ is precisely the full subcategory spanned by morphisms $X' \to X$, where $X' \in C^0$.*

To define Kan extensions in the context of ∞-categories, we need a relative notion of a (co)limit:

Definition 4.5.5. *(Definition 4.3.1.1 in [52]) Let $f : C \to D$ be an inner fibration between ∞-categories. For a diagram $\overline{p} : K^{\triangleright} \to C$, put $p := \overline{p}|K$. $\overline{p}$ is said to be an f-colimit if the induced map of overcategories $C_{\overline{p}/} \to C_{p/} \times_{D_{fp/}} D_{f\overline{p}/}$ is a trivial fibration. When $D = *$, we recover the usual notion of a colimit.*

Definition 4.5.6. *(Definition 4.3.2.2 in [52]) Consider the diagram of ∞-categories*

$$\begin{tikzcd} C^0 \arrow[r, "F_0"] \arrow[d, hook, "i"'] & D \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, "F"] & *. \end{tikzcd} \tag{4.48}$$

We say F is a Kan extension of F_0 if the induced diagram

$$\begin{tikzcd} C^0_{/X} \arrow[r, "F_X"] \arrow[d, hook, "i"'] & D \arrow[d, "p"] \\ C^{\triangleright}_{/X} \arrow[r] \arrow[ur] & * \end{tikzcd} \tag{4.49}$$

exhibits $F(X)$ as a colimit of F_X for all $X \in C$.

Lemma 4.5.1. *("Mysterious" Proposition 4.3.2.15 in [52]) Let C and D be ∞-categories, C^0 be a full subcategory C and $K \subset Fun(C, D)$ the functor subcategory spanned by left Kan extensions of functors in $Fun(C^0, D)$. Denote by K' the subcategory of K consisting of functors $F : C^0 \to D$ such that the induced diagram $C_{/C} \times_C C^0 \to D$ admits a colimit. Then $K \to K'$ is a trivial fibration.*

This statement should be considered as establishing the appropriate ∞-categorical analogue of uniqueness of Kan extensions. The appropriate version of uniqueness is expressed here by the trivial fibration which implies that the fibers of the map are contractible. Its content is best illustrated and demystified, by its crucial role in the constructions that follow.

We first met the phenomenon of stability in our brief review of the category of spectra and then again in more abstract forms in triangulated categories and dg-categories. In these contexts, stability is defined by way of the loop space/suspension adjunction but the latter makes sense in any *pointed* category. Likewise, every pointed ∞-category admits *loop space* Ω and *suspension* Σ functors defined as follows.

Definition 4.5.7. *Given an object $X \in C$, the loop space object ΩX is the fiber $fib(0 \to X)$, defined by the fiber sequence*

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array} \tag{4.50}$$

and the suspension object ΣX is the cofiber $cofib(X \to 0)$ defined by the cofiber sequence

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array} \tag{4.51}$$

It is an equivalent characterization of stable ∞-categories that these functors, which always form an adjunct pair, are also mutually inverse.

To make the loop space and suspension constructions functorial, we need to describe the functor $cofib : Fun(\Delta^1, C) \to C$ (or indeed the functor fib) which assigns to a morphism its cofiber (fiber) as an object in C. This is done by a sort of reverse-engineering, relying crucially on some deft arguments in combinatorics of simplices in ∞-categories and the machinery of Kan extensions in this context. As a result of this construction, we will have solved the most prominent issue plaguing the foundations of the theory of triangulated categories, the dreaded non-functoriality of the mapping cone or homotopy cofiber construction that we addressed in the relevant section. The reader is advised to read this section in tandem with the discussion at the end of that section which formally describes the difficulty with functorial mapping cones in a non-enriched context. The reader is advised at this stage to refer back to the discussion in the section on DG-categories as well, where Toën's proof of the functorality of the dg cofiber is given.

Let us construct the cofiber functor $cofib : Fun(\Delta^1, C) \to C$, following Remark 1.1.1.7 in [57]. From the outset we assume C is stable, although it suffices for it to be merely pointed for the purposes of this construction as we have done so far. The construction involves a pushout, hence we will be interested in diagrams with shapes Δ^1, $\Lambda_0^n = \Delta^1 \vee \Delta^1$ and $\Delta^1 \times \Delta^1$. These classify morphisms, pushouts and commutative diagrams respectively.

Following Lurie, we need to apply the Kan extension lemma twice, since we are trying to show that a morphism can be "completed" to a pushout square in C functorially. We now put ourselves in situation of Lemma 14.1 as in Remark 1.1.1.8 in [57]. Let $C := \Delta_0^2 = \Delta^1 \vee \Delta^1$ and $C^0 := \Delta^1$. Fix some ∞-category D which admits colimits. We begin by applying the lemma to the diagram categories $Fun(C^0, D)$ and $Fun(C, D)$. Let K and $K^{'}$ be as in the statement of the lemma. Then $K \to K^{'}$ is a trivial fibration. We have the following general fact about Kan extensions in the pointed context. Right Kan extensions along inclusions that are *sieves*, that is, fully faithful inclusions i such that whenever there is a morphism $X \to i(Y)$, X itself lies in the image i, are *extensions by zero* explained by [31]. Thus without loss of generality, we may deal with inclusions of the form as shown in the diagram

$$\begin{tikzcd} * \arrow[d] \\ * \end{tikzcd} \hookrightarrow \begin{tikzcd} * \arrow[r] \arrow[d] & * \\ 0 & \end{tikzcd} \tag{4.52}$$

where the the first diagram is inserted into the horizontal segment of the second.

Then, once again, the lemma says that this inclusion is a trivial fibration. Now we go one dimension higher, and consider shapes

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \\ * & \end{tikzcd} \tag{4.53}$$

which, as we've seen so many times already, classify pushouts. The category $K^{'}$ in this instance are precisely the horn-shaped diagrams in C which *can* be completed to pushout squares

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & *. \end{tikzcd} \tag{4.54}$$

The latter, of course, gives us our category K and we have once again by the lemma that $K \to K^{'}$ is a trivial fibration. We may restrict this trivial fibration to those pushout squares whose lower left element is 0 as before, that is, an element of $Fun(\Lambda_1^2 * 0, C)$. Composing everything, we obtain a trivial fibration

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \arrow[r, phantom, "\Longrightarrow"] & * \arrow[d] \\ 0 \arrow[r] & * & * \end{tikzcd} \tag{4.55}$$

where of course the representation is supposed to indicate a fibration of the functor categories, not the shapes themselves.

By Corollary 1.4.5.5 of [18], every trivial fibration admits a section. Denote the section of the trivial fibration above by s. Now denote by $p : Fun(\Lambda_1^2 * 0, C) \to C$ the functor which assigns to a pushout square the object in its lower right corner. The cofiber functor $cofib$ is then defined to be the composition $p \circ s : Fun(\Delta^1, C) \to C$. Lurie remarks in Remark 1.1.1.8 of [57] that $cofib$ is in fact the left adjoint to the left Kan extension functor $C \to Fun(\Delta^1, C)$ which assigns X to the morphism $0 \to X$, from which fact it follows that it preserves colimits by the adjoint functor theorem. The reader should be immediately reminded that this is exactly analogous to Toën's functorial dg-cofiber construction we have covered in previous sections. Toën shows in Proposition 9 in [39] that there is a map $[Mor(T)] \to Mor([T])$ in the notation of Chapter 3 for T a triangulated dg-category which is, however, not *fully faithful* in general. Lurie establishes the identical result for the ∞-categorical cofiber construction. Namely, there is a functor $Ho(Fun(\Delta^1, C)) \to Fun(Ho(\Delta^1), Ho(C)))$ which is not *fully faithful*. The passage to the homotopy class precisely destroys the higher coherence data which deals with the issue of weak functoriality, the *weak* dependence of the tringulated mapping cone construction on the composite of morphisms, precisely by keeping the ambiguity as part of the fundamental datum of a homotopy cofiber.

The following proposition brings together almost everything we have studied so far. It will establish the role of stable ∞-categories as higher-categorical models of abelian/triangulated categories.

Proposition 4.5.1. *The homotopy category $Ho(C)$ of a stable ∞-category C is triangulated.*

Proof There is a variety of ways one can prove this statement. For instance, one can exploit the connection between stable model categories and stable ∞-categories and proceed as Hovey does to describe explicitly the triangulated structure on the homotopy category of a stable model category. We have not discussed stable model categories and will not be following this path, which would be rather circuitous anyhow. Another alternative is to employ the formalism of stable derivators and show that stable derivators give rise to stable ∞-categories and use the fact that the homotopy category constructed from a stable derivator is triangulated. Evidently, all these are rather indirect. Hence we will reproduce Lurie's proof which constructs a triangulated structure on the homotopy category directly. This will consist of several main steps.

(i) Showing $Ho(C)$ is additive.
(ii) Showing the shift functors and triangles really correspond to their namesakes at the homotopy level.
(iii) Verifying the triangulated category axioms

Let us show that $Ho(C)$ is additive, that is the Hom-spaces of $Ho(C)$ are abelian groups and $Ho(C)$ admits small sums. Let us consider the loop space and suspension adjunction discussed previously, which holds for C only pointed and not necessarily stable. We have an equivalence of ∞-groupoids for all X and Y in C

$$Hom(\Sigma(X), Y) \cong \Omega Hom(X, Y). \tag{4.56}$$

Passing to connected components, we have a set bijection $\Pi_0 Hom(\Sigma(X), Y) \cong \Pi_0 \Omega Hom(X, Y)$. By definition of the loop space object, this is in fact a bijection $\Pi_0 \mathscr{H}om(\Sigma(X), Y) \cong \Pi_1 Hom(X, Y)$, where the latter makes sense because we can "point" the hom ∞-groupoid by the zero map. Thus, as in our discussion of spectra, $\Sigma(X)$ is a homotopy co-group object and in fact $\Sigma\Sigma(X)$ is an homotopy abelian co-group object. Since every object in a stable ∞-category is infinitely deloopable, in particular for every object X, there exists some Z such that $X = \Omega\Omega Z$. But then $Hom(X,Y) = Hom(\Sigma\Sigma Z, Y)$, which means $Hom(X,Y)$ carries a homotopy abelian group structure. That is, $Ho(A)$ is Ab-enriched. Thus if we show that $Ho(C)$ (or better C) admits finite coproducts, we will have additivity. We will refer to an ∞-category as additive if its homotopy category is enriched over abelian groups and it admits finite coproducts. Recall that since $\Sigma X = cofib(X \to 0)$, we have isomorphism $X \cong cofib(\Sigma^{-1}X \to 0)$. For any Y, we also have tautologically that $Y \cong cofib(0 \to Y)$. Now the canonical maps $X \to 0$ and $0 \to Y$ admit a coproduct as elements of the functor category and it is nothing but the zero map $X[-1] \longrightarrow Y$. Combining this with the fact that the "cofib" functor described above is additive (i.e., preserves coproducts), we conclude that any X and Y admit a coproduct in C, which is to say, C is an additive ∞-category.

Next, for the second part, we construct the diagrams in C of which the exact triangles of the triangulated homotopy category will be shadows. A diagram in $Ho(C)$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \tag{4.57}$$

is said to be an exact triangle if there exists a diagram in C

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\tilde{g}"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "\tilde{h}"] & W \end{tikzcd} \tag{4.58}$$

where both squares are pushout/pullback diagrams, and hence the larger rectangle also determines a pushout/pullback diagram exhibiting W as the suspension ΣX. The maps $\tilde{f}$,$\tilde{g}$ are lifts of f, g and h is the composition in the homotopy category of $\tilde{h}$ and the equivalence $\Sigma X \cong W$. We need to verify that these objects faithfully reproduce the properties framed by the triangulated category axioms. We follow the notation of the first chapter and go through the axioms in that order.

Let's begin by verifying Tri.I. Denote by $E \subset Fun(\Delta^1 \times \Delta^2, C)$ the subcategory consisting of the expanded exact triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & W. \end{tikzcd} \tag{4.59}$$

Following Lurie, we fall back on the trick that we first introduced above in the construction of the cofiber functor which requires the repeated invocation of the mysterious Lemma 4.3.2.15 in [52]. Consider the restriction map $e : E \to Fun(\Delta^1, C)$ which picks out the map f. Then by the lemma and our discussion of it, e is a trivial fibration, and the whole diagram in E giving our expanded exact triangle can be determined uniquely by specifying such a morphism. Hence every morphism can be extended to an exact triangle, which takes care of Tri.I (i). It is clear that any two isomorphic exact triangles give rise to the same exact triangle, which takes care of Tri.I (ii). Now we construct the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & \Sigma X. \end{tikzcd} \tag{4.60}$$

which is easily seen to be a an exact triangle, which takes care of Tri.I (iii).

Now we verify the rotation axiom, Tri. II. Keeping the notation above, given an exact triangle $T \in E$, consider the amalgamated diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "g"] \arrow[d] & W \arrow[d] \\ & 0 \arrow[r] & V \end{tikzcd} \tag{4.61}$$

and consider the horizontal and vertical rectangles as suspension squares with a map between them induced by the maps in the middle as shown in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[r, phantom, "\Longrightarrow"] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & W & 0 \arrow[r] & V. \end{tikzcd} \tag{4.62}$$

But a map between the suspension squares is a map between the suspensions $W \cong \Sigma X$

and $V \cong \Sigma Y$, as in the diagram

$$\begin{tikzcd} W \arrow[r, "\sim"] \arrow[d] & \Sigma X \arrow[d, dashed, "\Sigma f"] \\ V \arrow[r, "\sim"] & \Sigma Y. \end{tikzcd} \tag{4.63}$$

Now consider the further extended diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"] \\ & 0 \arrow[r] & V \arrow[r] & \Sigma Y. \end{tikzcd} \tag{4.64}$$

By Lemma 1.1.2.13 in [57], the rectangle in the middle above defines an exact triangle

$$Y \longrightarrow Z \longrightarrow \Sigma X \longrightarrow \Sigma Y. \tag{4.65}$$

It may be shown with an argument along similar lines that the converse of Tri.II also holds.

We now proceed to Tri.III. Recall that $e : E \to Fun(\Delta^1, C)$ is a trivial fibration. Consider the exact triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X \tag{4.66}$$

and the exact triangle

$$X' \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow \Sigma X' \tag{4.67}$$

in the homotopy category induced by elements of E we denote by e_1 and e_2. Putting ourselves in the situation of Tri.III, we have maps $t : X \to X'$ and $s : Y \to Y'$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] & \Sigma X \arrow[d, "\Sigma t"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'. \end{tikzcd} \tag{4.68}$$

Now consider the commutative diagram given by the first square. We can lift this diagram to an object of $Fun(\Delta^1 \times \Delta^1, C)$. Commutative diagrams can be regarded as morphisms in $Fun(\Delta^1, C)$, and this diagram defines a morphism $d : e(e_1) \to e(e_2)$. Using once again that e is a trivial fibration, we conclude that d can be lifted to a morphism of $d' : e_1 \to e_2$. But a morphism of elements of E is nothing but a morphism of exact triangles in the homotopy category

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] \arrow[d, "q"] & \Sigma X \arrow[d, "\Sigma t"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'. \end{tikzcd} \tag{4.69}$$

But this is precisely the content of Tri.III, hence we are done.

The demonstration of Tri.IV is an amalgam of diagram chasing techniques and repeated application of the Kan extension lemma in the same vein as the previous demonstrations. Considering the sheer size of diagrams involved in the octahedral axiom, the reader will hopefully forgive us for omitting the exposition of this routine procedure. Thus we have established that the homotopy category of a stable ∞-category is triangulated and therefore stable ∞-categories are a direct categorification of abelian/triangulated categories.

Lurie proves the following proposition which establishes an equivalent and extremely useful criterion of stability for an ∞-category C.

Proposition 4.5.2. *(Proposition 1.1.3.4 in [57]) A pointed ∞-category is stable iff the following are satisfied.*

(i) C admits all limits and colimits.

(ii) A square $\Delta^1 \times \Delta^1 \to C$,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \tag{4.70}$$

is a pushout iff it is a pullback.

To define Cat_{∞}^{st}, we need the notion of an exact functor.

Definition 4.5.8. *A functor $F : C \to D$ between stable ∞-categories is said to be exact if it preserves triangles and carries fiber sequences to fiber sequences. By the very definitions of these objects, it can be shown that it suffices for F to be left or right* *exact in the ordinary sense (commuting with finite (co)limits) to satisfy this notion of exactness.*

We put $Cat_{\infty}^{st} \subset Cat_{\infty}$ for the sub- ∞-category of stable ∞-categories with exact functors. The ∞-category of presentable stable ∞-categories is denoted by Pr_{st}^{L}. Presentability in the stable context admits a simpler description, as given by Corollary 1.4.4.2 in [57].

Proposition 4.5.3. *A stable ∞-category is presentable iff it admits small coproducts, is compactly generated* [17] *and its homotopy category $Ho(C)$ has small Hom-sets.*

Further, the following proposition characterizes stable and presentable ∞-categories as left-exact localizations.

Proposition 4.5.4. *An ∞-category is stable and presentable if and only if there exists a presentable and stable ∞-category D with an accessible left-exact localization functor $L : D \to C$.*

Lurie proves many closure properties for stable ∞-categories, see Section 1.1.3 of [57]. We may list some of them as follows.

(i) If $C \subset C'$ is a full subcategory of a stable ∞-category C' which is closed under cokernels and suspensions, it is also stable.
(ii) When C is stable, so is $Ind(C)$.
(iii) Given a collection $\{C_\alpha\}$ of stable ∞-categories, $\prod C_\alpha$ is stable.
(iv) More generally, Cat_{∞}^{st} admits small limits and filtered colimits and the inclusion $Cat_{\infty}^{st} \subset Cat_{\infty}$ is limit/filtered colimit preserving.

4.5.1. Stabilization of an ∞-category

There is a canonical way to construct a stable ∞-category out of a pointed ∞-category due to Lurie which is a far reaching generalization of the *stabilization* proce-

[17] See the section on mapping spaces in stable ∞-categories.

dure in stable homotopy theory which gives rise to the category of spectra. As before, we follow Section 1.4 in [57] in what follows.

The stabilization of a pointed ∞-category C is the ∞-category of *spectrum objects* in C, meaning the homotopy limit in Cat_∞, of the diagram

$$... \xrightarrow{\Omega} C \xrightarrow{\Omega} C \longrightarrow C. \tag{4.71}$$

While we will not discuss *Goodwillie calculus* at all in this text , it is still appropriate to give an account of the theory of *excisive* functors, which are, crudely put, approximate linearizations of an ∞-category which *converge* to the full *stabilization* given formally by the homotopy limit above.

Definition 4.5.9. *Let C, D be a ∞-categories and let C admit pushouts and a final object 1_C and D admit pullbacks and a final object 1_D. An excisive functor $F : C \to D$ is a functor that sends pushout squares in C to pullback squares in D. A reduced functor $F : C \to D$ is a functor that preserves the final object 1_D of D. The category of excisive functors is denoted by $Exc(C, D)$ and that of reduced functors $Fun_*(C, D)$.*

Now, we use excisive functors to re-characterize spectrum objects more concretely:

Definition 4.5.10. *A spectrum object of a finitely complete ∞-category C is an excisive functor from the category of finite pointed spaces Spc_*^{fin} to C. The category of spectrum objects in C is the functor category: $Sp(C) := Exc(Spc_*^{fin}, C)$.*

We have the following proposition concerning $Sp(C)$, for which see 1.4.2.16 and 1.4.2.17 of [57].

Proposition 4.5.5. *If C is finitely complete, $Sp(C)$ is stable.Further $Sp(C)$ is the universal stabilization of a finitely complete ∞-category.*

Theorem 4.5.1. *(Lemma 1.4.2.19 in [57]) Let C be a finitely cocomplete ∞-category, and D be a finitely complete ∞-category. Let $\Omega^\infty : Sp(D) \to D$ be the functor of evaluating on the point $* \in Spc_*^{fin}$: informally, this functor is the "forgetful functor"* *which sends the sequence ... $\to C \to ... \to C$ to its first entry. Then Ω^∞ induces the the equivalence*

$$Exc_*(C, Sp(D)) \cong Exc_*(C, D). \tag{4.72}$$

Thus the category of spectrum objects in an ∞-category is the universal target of excisive functors into it, hence its "free stabilization". The canonical example- as expected- is the ∞-category of spectra $Sp = Sp(Top_*)$.

4.5.2. Mapping Spaces in Cat_∞^{st} and Compact Objects

A very important issue regarding stable ∞-categories is that they are canonically enriched over *spectra*. We will try to make this as precise as possible without delving into the concept of enriched ∞-categories. Intuitively, this statement implies that the Hom-spaces of a stable ∞-category can be regarded as an element of the symmetric monoidal ∞-category of Sp (which we have mentioned above in the context of stabilization but not yet described) in a canonical way and there is a certain compatibility with this monoidal structure. Our first task is to discuss Sp in greater detail, in particular as an algebra object in Pr^L_{st}.

There are several equivalent characterizations of Sp. We have already seen the ∞-categorical version of the classical one: Sp is the category of spectrum objects in the category of topological spaces Top. That is, Sp is the free stabilization of Spc in the sense. Another attractive characterization is that Sp is the stable ∞-category freely generated under colimits by the sphere spectrum $\mathbb{S}$ which is essentially a consequence of the fact Spc is freely generated under colimits by the point $*$ Further, Sp is in fact the unit object of the monoidal product on Cat_∞ which we will introduce later. First we need a description of the mapping spaces in stable ∞-categories.

The enrichment of a stable ∞-category over spectra is easy to describe explicitly. Fix a stable ∞-category C and consider the Yoneda embedding $C \subset P(C) =$

$Fun(C^{op}, S)$. Applying the stabilization construction to the ∞-category of pointed objects in $P(C)$, we find by evaluating the limit objectwise that $Sp(P(C)_*) = Fun(C^{op}, Sp(S_*)) = Fun(C^{op}, Sp)$. Thus we obtain a functor $Sp(C_*) \cong C \to Fun(C^{op}, Sp)$ which, following Definition 2.15 in [60], we dub the spectral Yoneda embedding (note that this is not actually an embedding, since the functor is not fully faithful). This embedding gives rise to the spectral Hom functor (the adjoint to the Yoneda embedding) which can be described objectwise quite explicitly as the mapping space

$$Map_{C,st}(X, Y)_i = Hom_C(X, \Sigma^i Y). \tag{4.73}$$

We may now discuss the question of compact generation of stable ∞-categories which is a direct generalization of the theory in the triangulated setting. In fact, our definitions will be *verbatim* transcriptions of those in the latter. We have already covered the following in our section on ∞-categories:

Definition 4.5.11. *An object X in a (stable) ∞-category C is said to be ($\omega-$)compact if $Hom(X, -)$ commutes with (ω-)filtered colimits.*

As before, we denote by C^ω the subcategory of compact objects in C. Recall that the ind-completion $Ind(C)$ of an ∞-category C is just the "free filtered co-completion" of C and we say C is compactly generated if the natural functor $Ind(C^\omega) \to C$ is an equivalence of ∞-categories. Denoting by Cat_∞^{perf} the ∞-category of idempotent stable ∞-categories, we have the following lemma.

Lemma 4.5.2. *(Lemma 2.20 in [60]) The inclusion $Cat_\infty^{perf} \subset Cat_\infty^{st}$ exhibits the ∞-category of idempotent complete stable ∞-categories as the reflexive localization (see previous section) of stable ∞-categories. The left-adjoint localization functor L is nothing but the idempotent completion functor $Idem$ covered in previous sections. Indeed, explicitly $Idem(C) \cong Ind(C)^\omega$.*

We may now specify the monoidal structure on Pr^L_{st}, Cat^{perf}_∞ and Cat^{st}_∞. The monoidal structure on the first descends from Pr^L. On the first, it may be defined (as in Section 3.1 of [60]) as the tensor product

$$C \otimes D := (Ind(C) \otimes Ind(D))^{\otimes}. \tag{4.74}$$

As for general stable ∞-categories C and D, we can put

$$C \otimes D := Idem(C) \otimes Idem(D). \tag{4.75}$$

where the monoidal structure on the right-hand side is the one we have defined on Cat^{perf}_∞. We then have the following proposition establishing the closed symmetric monoidal structure on Cat^{st}_∞.

Proposition 4.5.6. *(Proposition 3.1 in [60]) Cat^{perf}_∞ is a closed symmetric monoidal ∞-category with the monoidal structure identified above, with internal mapping object given by the category $Fun^L_c(-,-)$ of colimit and compact-object preserving functors and the unit object the ∞-category of compact spectra Sp^ω.*

In analogy with Toën's results on mapping spaces in dg-categories, this latter internal mapping category can be described alternatively in terms of "module" categories. Namely, let C and D be idempotent complete stable ∞-categories. Then we have an equivalence

$$Fun^L_c(C, D) \cong ((C^{op} \otimes D) - Mod)_{qrep}. \tag{4.76}$$

where the subscript "qrep" indicates right quasirepresentable , to reinforce the analogy with Toën's results, bimodules, which are called *right compact* in Corrollary 3.3 in [60]. We do not spell out this condition explicitly except to say it is quite similar to the former in the context of dg-categories.

4.6. DG-Cat as an ∞-category and Noncommutative Spaces

We now complete our construction of $DG-Cat$ as an ∞-category. More precisely, we will construct $DG - Cat^{idem}$ as the dg-analogue of the ∞-category of idempotent complete stable ∞-categories. We follow [2] to bring everything we have covered so far together.

Recall that dg-categories $T, T^{'}$ are said to be Morita equivalent if they are weakly equivalent in the Morita model structure on DG-Cat, i.e., if we have a weak equivalence $\hat{T} \cong \hat{T}'$. Localizing with respect to these equivalences produces the ∞-category $DG - Cat^{idem}$. More precisely, the following ∞-categories are equivalent, as discussed in Theorem 6.1.2 of [2].

(i) $Cat_{Ch(R)}$ localized with respect to Morita equivalences of dg-categories

(ii) $DG - Cat^{idem}$

(iii) The subcategory of $DG - Cat^{l.c.}$, the category of locally cofibrant dg-categories, spanned by module categories of the form $\hat{T}$ over *small* dg-categories T with colimit and compact-object preserving functors between them.

Recall also that a dg-category T is said to be idempotent complete if the functor $T \to (\hat{T})_c$ is a weak equivalence of dg-categories, where $(T)_c$ is the subcategory of compact objects. This condition implies that T is in fact uniquely determined by its associated "Morita theory". Denoting the subcategory of idempotent dg-categories $DG - Cat^{idem}$, we have the inclusion $DG - Cat^{idem} \subset Dg - Cat$. The functor above, $T \to (\hat{T})_c$, can be considered as the "idempotent completion" functor.

Proposition 4.6.1. *(Proposition 6.1.17 in [2]) The functor $T \mapsto (\hat{T})_c$ is the left adjoint to the inclusion $DG - Cat^{idem} \subset Dg - Cat$.*

Let us say some words on the third item in the list above, since it is a useful model for the "Morita category" of dg-categories. Let's denote by $DG - Cat^c$ the category of dg-categories of the form $\hat{T}$ with morphisms which are continuous (i.e., commute with infinite sums) at the level of the homotopy category. Denote by $DG - Cat^{cc}$ the subcategory of $DG - Cat^c$ with the same objects but compact-object preserving functors. One can fine-tune the results in [4] to obtain an equivalence $DG - Cat^{idem} \cong DG - Cat^{cc}$.

Finally, we construct the ∞-category of noncommutative spaces. Recall from the section on dg-categories that we say a dg-category is of finite type if it is equivalent to the dg-category of *perfect* dg-modules over a *compact* dg-algebra. We put $DG-Cat^{f.t.}$ for the category of dg-categories of finite type. By definition, $T \in DG-Cat^{f.t.}$ is of the form $Perf(A)$ with A compact, hence T is compactly generated. By the results in [4], $DG-Cat^{idem}$ is the ind-completion of the $DG-Cat^{f.t.}$, and the latter is nothing but the subcategory of *compact* objects in the former. We put $Nc(R) := DG-Cat^{f.t.^{op}}$. This latter ∞-category will be the main target of additive invariants and the central object around which the theory of noncommutative motives will be built.

4.7. Equivalence of DG-Categories and Stable ∞-categories

One of the most central results invoked both explicitly and implicitly throughout this text concerns the equivalence of dg-categories and stable ∞-categories. There are different accounts of this equivalence in existence, and the most rigorous and complete ones are those that have been given by Cohn [61] and Faonte [62]. Whereas the former depends on the theory of spectral categories and *enriched* Dold-Kan correspondence, the latter requires a foray into the theory of A_∞-categories which would take us too far afield. Therefore we will instead sketch Cohn's proof of the following theorem (keeping his notation almost verbatim), which makes precise the aforementioned equivalence.

Theorem 4.7.1. *(Corollary 5.7 in [61]) There is an equivalence of ∞-categories*

$$DG-Cat_R^{idem} \cong Mod_{HR-Mod}((Pr^L_{st^{\otimes}})). \tag{4.77}$$

As before, $DG-Cat_R^{idem}$ is the ∞-category obtained by way of the Morita model structure on $DG-Cat_R$ and $HR-Mod$ is the ∞-category of HR-modules, where we regard HR as an object of $Pr^L_{st^{\otimes}}$. The notion of R-linearity is encoded by the structure of a module over $HR-Mod$ on the right hand side, where as usual an HR-module is just a spectrum X with an action $X \wedge HR \to X$ with the right compatibilities and unit relations, and $HR-Mod$ is the symmetric monoidal ∞-category of such module spectra. To prove this equivalence, Cohn goes through several intermediate equivalences, which we will state in the following sections. The rest of the section is devoted to unwinding this theorem by introducing the objects at work.

4.7.1. Spectral Categories

Recall that a spectral category is a category enriched over spectra. To make the analogy with dg-categories explicit, we unwind this, following [63] and [60].

Definition 4.7.1. *A spectral category A consists of the following:*

(i) *Given any $X, Y \in A$, a spectrum $A(X, Y)$*
(ii) *Given any $X, Y, Z \in A$, a composition morphism $A(Y, Z) \wedge A(X, Y) \to A(X, Z)$ in the category of spectra*
(iii) *Given any $X \in A$, a unit map $\mathbb{S} \to A(X, X)$ of spectra.*

We denote the (ordinary) category of spectral categories by Cat_S. Given a spectral category S, an S-module is as usual a functor $S^{op} \to Sp$ and these form a spectral category S-Mod. In exact parallel with the model structures on dg-categories we have already discussed, there are closely related but distinct model structures on Cat_S which capture distinct aspects of the homotopy theory of spectral categories . We will not delve into the details, see [63] for a discussion. We adopt the latter's terminology and refer to the relevant model structures at work as *Dwyer-Kan* and *Morita* model structures. We record as a crucial fact that the associated ∞-categories are presentable (since these model categories are *combinatorial*, see the section on model categories).

We regard the topic discussed in this section as a form of stable or enriched *Dold-Kan* correspondence. Indeed, the following precise formulation of the latter by Tabuada in [63] figures prominently in Cohn's proof.

Theorem 4.7.2. *We have a Quillen equivalence (with respect to the Morita model structure)*

$$DG - Cat_R \cong Cat_{HR-Mod}. \tag{4.78}$$

As before, we need to impose some *finiteness* conditions on our objects to extract our desired result. Later in this section, we will offer some comments on how one might about generalizing these ideas to a "higher" context.

Definition 4.7.2. *A spectral R-module is said to be perfect if it belongs to the smallest stable subcategory generated by R under finite homotopy colimits and retracts of homotopy colimits.*

Definition 4.7.3. *A spectral R-module T is said to be cell if it can be presented as the union of a sequence of R-modules T_i with $T_i = hocofib(\bigwedge R[n] \to T_i)$ where n is arbitrary. We denote by $R - Cell$ and $Perf(R)^{cell}$ the categories of cell R-modules and perfect cell R-modules respectively. In fact,by Proposition 3.14 in [61] the latter category can be characterized as those module spectra generated by R under finite colimits and tensors with finite spectra.*

Following Cohn verbatim, we have the following theorem which gives an economic way to package some finiteness data in terms already familiar to us from the discussion in the dg-context.

Theorem 4.7.3. *Denote by $Mod_{Perf(R)^{cell}}(Cat_S)$ the category of module categories over $Perf(R)^{cell}$, which consists of spectral categories S with an action by $Perf(R)^{cell}$: $S \wedge Perf(R)^{cell} \to S$. The reader should take care not to confuse the monoidal product on spectra and on Cat_S which we do not notationally distinguish. Define the subcategory of $Cat^{f.c.}_{R-Mod}$ which consists of R-module spectra closed under finite colimits and tensors with finite spectra. Define the subcategory $Mod_{Perf(R)^{cell}}(Cat^{f.c.}_S) \subset Mod_{Perf(R)^{cell}}(Cat_S)$ which consists of module categories over $Perf(R)^{cell}$ closed under colimits and tensors with finite spectra. Then we have an equivalence*

$$Cat^{f.c.}_{R-Mod} \cong Mod_{Perf(R)^{cell}}(Cat^{f.c.}_S). \tag{4.79}$$

We will not distinguish between the model categories with the induced model structures discussed above and their associated ∞-categories in what follows. All these associated ∞-categories are presentable and therefore afford an adequate setting for the study of adjoint functors (such as those that appear in the Barr-Beck-Lurie theorem). However, we need to discuss a subtle issue that arises from the incompatibility between the symmetric monoidal structure and the model structure which mirrors the identical difficulty in the dg-context: for instance, cofibrant objects need not be homotopically flat (see section on the homotopy theory of DG-Cat). The monoidal and model structures cannot be reconciled without addressing this incompatibility. The solution is completely identical to that of Toën in [4] which we have already discussed at length. We call a spectral category S *locally cofibrant* if $S(X, Y)$ is cofibrant for all $X, Y \in S$. We denote by $Cat_S^{l.c}$ the subcategory of locally cofibrant spectral categories. Then we have:

Proposition 4.7.1. *A locally cofibrant spectral category S is homotopically flat, i.e., $S\wedge$ preserves cofibrant objects. We therefore put $Cat_S^{flat} := Cat_S^{l.c.}$.*

Now we have finally isolated the correct subcategory of spectral categories which allows us to formulate the following crucial intermediate result. The reader may consult [61, 64].

Theorem 4.7.4. *There is an equivalence of symmetric monoidal ∞-categories,*

$$Cat_S^{flat^{\otimes}} \cong Cat_{\infty}^{perf}. \tag{4.80}$$

More generally, we have an equivalence of ∞-categories,

$$Cat_{R-Mod} \cong Mod_{Perf(R)}(Cat_{\infty}^{perf}). \tag{4.81}$$

Now putting $R = HR$ and using the enriched Dold-Kan equivalence alluded to above, we obtain the equivalence

$$Cat_{HR-Mod} \cong DG - Cat_R \cong Mod_{Perf(HR)}(Cat_{\infty}^{perf}). \tag{4.82}$$

To obtain te final form of the equivalence we had stated in the introduction, we use the Ind-functor as discussed in the section on idempotent complete ∞-categories where Cat_{∞}^{perf} was introduced. The functor $Ind : Cat_{\infty}^{perf}$ induces an equivalence

$$Mod_{Perf(HR)}(Cat_{\infty}^{perf}) \cong Mod_{HR-Mod}(Pr_{st,c}^{L}). \tag{4.83}$$

This equivalence allows us to state Cohn's final result which combines everything discussed in this section, which is the isomorphism

$$DG - Cat_R \cong Mod_{HR-Mod}(Pr^L_{st,c}). \tag{4.84}$$

Therefore dg-categories over some fixed ring R are models of R-linear presentable stable ∞-categories with functors preserving compact objects. This result is the foundation of everything that follows.

5. FROM ADDITIVE INVARIANTS TO NONCOMMUTATIVE MOTIVES

This section is arguably the core of this thesis. It implicitly builds on the foundational work treated so far, however, the full relationship will remain somewhat obscure until the next chapter since this relationship is provided precisely by the theory of noncommutative motives.

For reasons of economy, we work only ∞-categorically and confine our exposition to "cohomology theories" or additive invariants of stable ∞-categories or dg-categories. For instance, we will not give an account of Hochschild homology of algebras or algebraic K-theory of rings or even ring spectra, but only discuss constructions of these theories for stable ∞-categories or dg-categories [18]

We start by reviewing the formalism of universal additive and localizing invariants of stable ∞-categories as formulated by Blumberg, Gepner and Tabuada, since we have not done so in the main body of the text although we have introduced all the main ingredients at work. We work within Cat_{∞}^{idem}, the ∞-category of stable idempotent ∞-categories. First, we need to make precise the notion of an exact sequence of dg-categories or stable ∞-categories.

5.1. Exact Sequences in $DG-Cat$ and Cat_{∞}^{st}

To sum up, all the structures introduced up to this point as enriched or higher-categorical generalizations of abelian and homotopified abelian structures (such as pre-triangulated dg-categories and stable ∞-categories) serve a very clear purpose: they encode "exactness properties" on underlying categories. That is, they give us a way to speak of and work with exact sequences of objects at increasing levels of abstract-

[18] This "or" should be understood both as expressing an ambiguity and distinction: while the theories defined here ultimately agree for stable ∞-categories and dg-categories, as can be shown by a few pages worth of routine diagram chasing, it still makes sense to discuss them separately.

ness. We have touched on how these enriched and higher-categorical models themselves assemble into categories with very good properties which allow the nearly care-free application of "universal" constructions (limits, colimits. functor categories....). Such universal constructions are absolutely crucial for the development of the theory of noncommutative motives, hence our adoption of the language of higher categories. However, these categories do not straightforwardly inherit the exactness properties of their objects. For instance, ∞-category Cat_∞^{st} is not itself a stable ∞-category (example!). Hence some care is called for when we speak of exact sequences of "dg-categories" and stable ∞-categories, which is a notion that we must nonetheless invoke in our account since they are crucial for any reasonable theory of noncommutative motives. While we do not provide an account of these, let us also hint at more recent developments around this subject which allow one to conceive of ∞-category Cat_∞^{st} as the fundamental example of a stable $(\infty, 2)$-category, a notion that axiomatizes the exactness properties of ∞-categories like Cat_∞^{st} of which we will catch glimpses below. On this basis, it is also possible to offer a speculative definition of a *dg*-2-category.

There are many equivalent, or at most subtly different, definitions of an exact sequence of dg-categories and stable ∞-categories in the literature. For the moment, we adopt those offered by Robalo in Sections 6.4.1 and 2.1.24 in [2].

Let us start with exact sequences of dg-categories. We work within $DG-Cat^{idem}$ as usual. A sequence of idempotent dg-categories

$$A \to B \to C \tag{5.1}$$

is said to be an *exact sequence* if the map $A \to B$ is fully faithful and we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & C \end{array} \tag{5.2}$$

which is a pushout square.

By the discussion in Section 6.4.1 [2], we can calculate the pushout C within $DG - Cat$ as the idempotent completion of the Verdier-Drinfeld quotient: $\widehat{B/A}_c$. In addition, every exact sequence in $DG - Cat^{idem}$ can be regarded as an exact sequence in $DG-Cat^{cc}$ by the equivalence we discussed in the last chapter, that is the pushout diagram

$$\begin{tikzcd} \hat{A} \arrow[r] \arrow[d] & \hat{B} \arrow[d] \\ * \arrow[r] & \hat{C} \end{tikzcd} \tag{5.3}$$

represents the same exact sequence in $DG-Cat^{cc}$. Thus compact-object preserving functors become an intrinsic part of the definition of an exact sequence of dg-categories.

An exact sequence of dg-categories,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\ * \arrow[r] & C \end{tikzcd} \tag{5.4}$$

is said to split if the maps i and p admit homotopy sections $j : B \to A$ and $r : C \to B$, that is, we have $j \circ i \cong id_A$ and $p \circ r \cong id_C$. If this pushout square were also a pullback (say if $DG-Cat^{idem}$ were itself a stable ∞-category), then this splitting would exhibit B as a direct sum $A \oplus C$, recovering the familiar notion of a split exact sequence in abelian categories that we saw in the first section. Split exact sequences are of central importance for us. By definition, an additive invariant is a functor that preserves *split* exact sequences. Indeed there are additive invariants which do not preserve general exact sequences, see [60] for a brief discussion.

In keeping with our policy of addressing identical constructions separately for dg-categories and stable ∞-categories (despite their by now abundantly advertised equivalence), let us now introduce *exact sequences* or localization sequences of stable ∞-categories which mirror the construction on the dg-side.

We will work in the ∞-category of presentable stable ∞-categories Pr^L_{st}. We have already covered the Verdier quotient of triangulated categories in the preliminary chapter, the notion for ∞-categories is an immediate generalization.

Keller devised a construction in [34], which was later refined by Drinfeld in [65], of the Verdier quotient in the dg-context, whose details we will not delve into. Suffice to say that given a triangulated dg-category A and a triangulated dg-subcategory B A/B is the dg-category such that $H^0(A/B) = H^0(A)/H^0(B)$ where the latter is the Verdier quotient of dg-categories. The dg-quotient construction provides a way to construct a dg-enrichment of derived categories since, for instance, we may define the dg-derived category of an abelian category as the dg-quotient of $Ch_{dg}(A)$ by the dg-subcategory of acylic complexes.

We follow Section 5.4 of [60] in generalizing the notion of a Verdier quotient to stable ∞-categories. The flexibility of the ∞-categorical formalism allows a slicker definition than had been possible in the triangulated and dg-context.

Definition 5.1.1. *Given a fully faithful, colimit preserving functor $F : C \to D$ between presentable stable ∞-categories, the Verdier quotient is the cofiber $cofib(F)$ as an object of Pr^L_{st}.*

Note that this *cofiber* in Pr^L_{st} should not be confused with the cofiber *within* a stable ∞-category which we spent so much time discussing. However, the notion of a cofiber makes sense in any *pointed* ∞-category, which is precisely what we are using here.

Finally, we say a sequence of presentable stable ∞-categories

$$A \to B \to C \tag{5.5}$$

is an exact sequence if the displayed morphisms compose to zero, the morphism $A \to B$ is fully faithful and we have an equivalence between the Verdier quotient of stable ∞-categories B/A and C. To express this condition in a more familiar form, we say a diagram

$$A \to B \to C \tag{5.6}$$

is an exact sequence of stable ∞-categories if we have the following.

- The morphisms compose to zero.
- The map $X \to Y$ is fully faithful and we have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & C \end{tikzcd} \tag{5.7}$$

which is a pushout diagram.

This is of course precisely the diagram expressing C as the cofiber of the map $f : A \to B$, which we merely repackaged above as a "Verdier quotient" of stable ∞-categories. Thus an exact sequence in Pr^L_{st} or any other pointed ∞-category, is just a cofiber sequence.

It will become necessary to speak of exact sequences of *small* [19] stable ∞-categories, and as might be expected, this is done by passing to their ind-completions. A sequence $A \to B \to C$ of small stable ∞-categories which admit κ-colimits with κ-small colimit preserving functors is said to be exact if, as in Definition 5.12 of [60], there is an exact sequence

$$Ind_\kappa(A) \to Ind_\kappa(B) \to Ind_\kappa(C). \tag{5.8}$$

A *split* exact sequence of stable ∞-categories is slightly subtler to define. We again place ourselves in the ∞-category of small stable ∞-categories which admit κ-colimits with κ-small colimit preserving functors. An exact sequence

$$A \xrightarrow{p} B \xrightarrow{r} C \tag{5.9}$$

is said to split if p and r admit adjoints q and s such that $p \circ q \cong s \circ r \cong Id$.

Now we are finally able to define the notions of additive and localizing invariants. We follow sections 6 and 6 of [60].

Definition 5.1.2. *A functor $F : Cat^{st}_\infty \to D$, where $D \in Pr^{L,st}$. is said to be a (D-valued)* ***additive*** *invariant of stable ∞-categories if it inverts Morita equivalences*

[19] Presentable ∞-categories are never small.

(that is to say, descends to a functor on Cat_∞^{idem}), preserves filtered co-limits (which, by a theorem of Simpson and Lurie is essentially equivalent to additivity in the usual sense) and sends split exact sequences of stable ∞-categories to split cofiber sequences in D.

*If, in addition, F sends all exact sequences of stable ∞-categories to cofiber sequences, rather than just the split ones, it is said to be a **localizing** invariant.*

The reader is welcome to substitute Sp, the ∞-category of spectra (the unit object for the monoidal structure on Cat_∞^{st}) for D. When we only consider additive (resp.localizing) invariants valued in spectra, we shall omit "D-valued" and say "additive (resp. localizing) invariant".

Examples of localizing invariants include nonconnective K-theory and Topological Hochschild Homology. Crucially, *connective* K-theory is additive but not localizing, as discussed in [60].

Now we can finally state the central result of Tabuada (with Gepner and Blumberg).

Theorem 5.1.1. *(Theorem 1.1 of [60]) There exist stable ∞-categories M_{add}, M_{loc} with functors*

$$U_{add} : Cat_\infty^{st} \to M_{add}, \tag{5.10}$$

$$U_{loc} : Cat_\infty^{st} \to M_{loc} \tag{5.11}$$

which are universal additive and localizing invariants in the sense that given any $D \in Cat_\infty^{st}$, we have equivalences

$$Fun_{Pr^{L,st}}(M_{add}, D) \cong Fun_{Add}(Cat_\infty^{st}, D)) \tag{5.12}$$

where Fun_{add} is the category of additive invariants.

This theorem is the final version of the approach to the theory of noncommutative motives developed by Tabuada, Cisinski and their collaborators. For a comparison of this theory and Robalo's approach by way of the noncommutative stable homotopy category, the reader may consult Section 8 of [2].

5.2. Algebraic K-Theory

Algebraic K-theory provides the fundamental paradigm of "additivization" and occupies a special place among additive invariants in the sense discussed above in that it is the universal such, i.e., every additive invariant admits a map from it, which can be seen as a factorization of the motivic realization map. In the commutative world, the spectrum representing algebraic K-theory is a unit object in the category of motivic spectra, and it is in this sense the "universal additive invariant". We will see that the "noncommutative" generalization of algebraic K-theory inherits these features. Moreoever, we can exploit this universal property to obtain new information about the commutative world, see the work of [66].

There is fierce competition among candidates for the *true* algebraic K-theory. In fact, all the relevant flavors and models of algebraic K-theory available can be reconciled when interpreted in the right context and appropriately restricted. We distinguish two main branches of algebraic K-theory: the *exact* K-theory, and the *monoidal* one. The former is represented by Waldhausen K-theory and the latter by Segal K-theory. We are interested in the algebraic K-theory of dg-categories and stable ∞-categories, hence the accounts we present belong to the former branch, however it is certainly possible to extend Segal-type K-theory to general algebra objects in symmetric monoidal ∞-categories.

Essentially, K-theory is a machine for extracting some simplified linear or additive information out of an "additive" category which has some kind of *exact structure*, which is our shorthand for the notion of exact sequences that is appropriate in a given context. We have already met abelian categories (bona fide exact sequences), triangulated categories (exact triangles), dg-categories or stable ∞-categories (cofiber sequences) which have various kinds of such exact structures. In the context of ordinary categories, the notion of exactness is axiomatized by the intermediate notion (due to Quillen) of an exact category, which we will not present here since it is beside the point. The notion of a Waldhausen category subsumes exact categories in that every exact category can be given a Waldhausen structure. Waldhausen ∞-categories a la Barwick is an ∞-categorical generalization of this same idea, where cofibrations become *ingressive morphisms* and the subcategory of weak equivalences is abstracted by a relative or pair category structure.

We now proceed to define Waldhausen structures on categories, which may be seen as an analogue of weakened homotopical structures (e.g., model categories, relative categories etc.) that is better adapted for dealing with towers of "quotient objects" in a category, where quotients are defined by reference to a class of maps called cofibrations. We will revisit the ordinary definition of the Grothendieck group K_0 at the end of this exposition to see, at least conceptually, how Waldhausen categories are related to more familiar constructions in K-theory. Our references for this section are [67] and [68].

5.2.1. Waldhausen Categories

We start with two intermediate structures which can be immediately observed to be *components* of the structure of a model category.

A *category with weak equivalences* is a pair (C, W) where C is a category and W is a class of morphisms in C called weak equivalences such that,

(i) Every isomorphism in C is in W.
(ii) Morphisms in W satisfy the two-out-of-three property as in the model category axioms.

A *category with cofibrations* is a pair (C, Cof) where C is a category and Cof is a class of morphisms called cofibrations in C such that,

(i) Every isomorphism in C is in cof.

(ii) C is a pointed category and every object is cofibrant. [20]

(iii) Given any cofibration $X \rightarrowtail Y$, and any morphism $X \to Z$, we have the pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Y \sqcup_X Z \end{tikzcd} \tag{5.13}$$

where now $Z \rightarrowtail Y \sqcup_X Z$ is also in Cof.

These properties combine to equip a "category with cofibrations" with a lot of structure. For instance, given any two objects X, Y, the pushout of the cofibration $0 \rightarrowtail X$ along $0 \to Y$ is just the coproduct of X and Y, hence C has coproducts. We can also define cokernels or quotient objects in C for any cofibration $X \rightarrowtail Y$: this is just the cofiber Y/X given as the pushout of $X \to Y$ along the canonical map $X \to 0$. Thus we refer to pushouts along cofibrations in a Waldhausen category briefly as cokernels or quotients.

Definition 5.2.1. *A Waldhausen category is a category with cofibrations and weak equivalences which obey the so-called gluing axiom. In the construction of Waldhausen categories, we will often use spans of morphisms*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, tail] & \\ Z & & Y \end{tikzcd} \tag{5.14}$$

which play an important role in the generalizations of the Waldhausen S-construction. The gluing axiom essentially says that spans may be glued along weak equivalences. That is, given a diagram

$$\begin{tikzcd} & Y \arrow[r, "\sim"] & Y' & \\ X \arrow[ur, tail] \arrow[dr] & \arrow[r, "\sim"] & & X' \arrow[ul, tail] \arrow[dl] \\ & Z \arrow[r, "\sim"] & Z' & \end{tikzcd} \tag{5.15}$$

[20] This should be understood in the sense of model categories.

we have a weak equivalence of pushouts $Y \sqcup_X Z \to Y' \sqcup_{X'} Z'$.

It is possible to work dually with the notion of a category with *fibrations* and nothing essential will be affected as long as one is consistent. Naturally, equipping a category C with a Waldhausen structure does not induce a Waldhausen structure on C^{op} (thinking of cofibrations as abstract monomorphisms, this is self-evident). Thus we have the notion of a bi-Waldhausen category where C and C^{op} have Waldhausen structures that are compatible in some sense.

We can define the zeroth K-theory (or Grothendieck group) of a Waldhausen category as follows.

Definition 5.2.2. *The Grothendieck group* $K_0(C)$ *of a Waldhausen category* C *is the abelian group generated by symbols* $[X]$ *with* $X \in C$ *subject to the conditions:*

(i) *When* X *and* Y *are weakly equivalent,* $[X] = [Y]$

(ii) *When we have a cofibration sequence* $X \rightarrowtail Y \to Y/X$, $[X] + [Y/X] = [Y]$.

The reader will immediately recognize that the construction involved here is just the familiar Grothendieck construction, except we have a far more abstract notion of an exact sequence than in the context of abelian categories (such as the category of vector bundles) given by cofibration sequences. $K_0(C)$ respects the additive structure of C in that we clearly have $[X \sqcup_0 Y] = [X] + [Y]$ since we have cofibration sequence $X \rightarrowtail X \sqcup_0 Y \to Y$ and more generally $[X \sqcup_Z Y] = [X] + [Y] - [Z]$.

This discrete construction, which treats objects as points and extracts a group structure from the crude datum of cofibration sequences, can be refined to a homotopy theoretic one in which one obtains a K-theory *space* whose homotopy groups recover the linear/discrete K-theory. To avoid repeating ourselves, we will explore this from an intuitive angle in the context of ∞-categories later in this section, and will now only reproduce Waldhausen's original description of the S-construction for ordinary Waldhausen categories without further embelishment.

The S-construction revolves around filtered objects (i.e., sequences of cofibrations) in Waldhausen categories and a lot of extremely slick bookkeeping to account for all the quotient objects in an economic manner. For a category C, denote by Ar(C) its category of arrows [21] , which is the functor category $Fun([0 \to 1], C)$ where the morphisms are commutative squares. The structures defined above are inherited *pointwise* by Ar(C), for instance, a commutative square involving two morphisms is said to be a cofibration if the morphisms are. Waldhausen defines a subcategory F_1C of Ar(C) which consists of those morphisms in C which are cofibrations. Now, F_1 is also a category with cofibrations (this is not immediately obvious). Denote by $F_1^+(C)$ the category of cofibration sequences $X \rightarrowtail Y \to Y/X$. This category is just $F_1(C)$ with the datum of a quotient object Y/X for every cofibration $X \rightarrowtail Y$ and is therefore also a category with cofibrations. Now we generalize these constructions to n-sequences of cofibrations $X_0 \rightarrowtail X_1 \rightarrowtail .... \rightarrowtail X_n$, denoting the respective categories by $F_n(C)$ and $F_n^+(C)$. In the latter category, we denote by $X_{i,j}$ the fixed quotient object X_j/X_i for every $0 \leq i < j \leq n$. These are also categories with cofibrations for every n, therefore the constructions can be iterated to create new categories with cofibrations $F_nF_m(C)$. Now we reformulate these ideas slightly to obtain Waldhausen's S-construction.

Let C be a Waldhausen category, $[n]$ the ordinal category and $Ar[n]$ the arrow category of the ordinal category, consisting of ordered pairs (i, j) with morphisms $(i, j) < (i', j')$ whenever $i < j$ and $i' < j'$. The constructions given in the previous paragraph give rise to functors $X : Ar[n] \to C$ such that $X(i, j) \rightarrowtail X(i, k)$ is a cofibration when $i \leq j \leq k$ and we have a cofibration sequence $X_{i,j} \rightarrowtail X_{i,k} \rightarrowtail X_{j,k}$. We put $S_n(C)$ for the subcategory of $Fun(Ar[n], C)$ consisting of such functors X. By construction $S_n(C)$ assemble into a simplicial category $\Delta^{op} \to Cat$- the simplicial Waldhausen S-construction of C. Every $S_n(C)$ is a Waldhausen category when C is, giving rise to the simplicial Waldhausen category associated with C.

It is instructive to look at a few of these diagram categories in lower degrees. Evidently, $S_0(C) = *$. Next, for $n = 1$, $S_1(C)$ is the category of diagrams in C of shape $0 \rightarrowtail X_{0,1} \rightarrowtail 0$, and it is therefore equivalent to C. Going further, for $n = 2$, $S_2(C)$ is

[21]This notation is used only in this section in the 1-categorical context

the category of diagrams

$$\begin{tikzcd} 0 \arrow[r] & X_{0,1} \arrow[r] \arrow[d] & X_{0,2} \arrow[d] \\ & 0 \arrow[r] & X_{1,2} \arrow[d] \\ & & 0 \end{tikzcd} \tag{5.16}$$

where it should be noted that this diagrammatic presentation is redundant: we only need the first row to obtain the others as cofibers of the diagonal maps. More generally for any n, $S_n(C)$ is the category of diagrams

$$\begin{tikzcd} X_{0,0} \arrow[r] & X_{0,1} \arrow[r] \arrow[d] & X_{0,2} \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & X_{0,n} \arrow[d] \\ & X_{1,1} \arrow[r] & X_{1,2} \arrow[r] \arrow[d] & \dots \arrow[r] & X_{1,n} \arrow[d] \\ & & \dots \arrow[rr] & & X_{n,n} \end{tikzcd} \tag{5.17}$$

where we continue to indicate all the intermediate quotients to emphasize all the data packaged into these categories.

Now we construct the higher algebraic K-theory of a Waldhausen category. Let $S_\bullet(C) : \Delta^{op} \to Cat$ be the simplicial Waldhausen category we already met above. We denote by $wS_\bullet(C)$ the simplicial subcategory of weak equivalences and by $|wS_\bullet(C)|$ its geometric realization. One might be tempted to immediately define K-groups as homotopy groups of this space, however we need to introduce a "shift" to get the right homotopy type. We then define.

Definition 5.2.3. *The Waldhausen K-theory space $K_{Wald}(C)$ of a Waldhausen category (C, cof, w) is the topological space $\Omega|wS_\bullet(C)|$. The higher algebraic K-theory groups are the homotopy groups of this loop space.*

The above construction can be iterated to give rise to a K-theory *spectrum.*

Definition 5.2.4. *The Waldhausen K-theory spectrum $K(C)$ of (C, cof, w) is the Ω-spectrum whose n-th space is $\Omega....\Omega|wS....S(C)|$, where $wS....S(C)$ is the subcategory of* *weak equivalences of the space obtained by the iteration of the $S_\bullet$-construction outlined above.*

We conclude our review of Waldhausen K-theory by stating the *Waldhausen additivity theorem*, which is the first example of an additivity theorem in this general setting. It is the origin of the notion of "additive invariants" and hence should in fact be seen as a vast generalization of the additivity Euler-type invariants, to which it can be easily shown to descend by way of *decategorification*. Let $E(C)$ be the category of cofibration sequences in C, that is objects of the form $X \rightarrowtail Y \to Z$, with the obvious morphisms. Note that $E(C)$ is naturally a Waldhausen category and by Theorems 1.3.2 and 1.4.2 in [68] we have the following result:

Theorem 5.2.1. *Waldhausen K-theory respects cofibration sequences in the sense that we have a homotopy equivalence*

$$wS_\bullet E(C) \cong wS_\bullet(C) \times wS_\bullet(C). \tag{5.18}$$

5.2.2. K-theory of Stable ∞-categories

There are two state-of-the-art models of algebraic K-theory for a stable ∞-category, one due to Blumberg-Gepner-Tabuada [60] and the other due to Clark Barwick [69] . Blumberg-Gepner-Tabuada K-theory (denoted by K_{BGT}) is specifically designed as a functor $Cat_\infty^{st} \to Sp$ whereas Barwick's construction is more general, giving rise to a functor from so-called Waldhausen ∞-categories to Sp. Now every stable ∞-category can be seen as a Waldhausen ∞-category, and for this reason, among others, Barwick's completely independent construction appears to us to be more fundamental. One might go further and say that Barwick's theory has a strong claim to being an independent theory for noncommutative motives as well. We will not be able to delve into his theory at all, since it relies on a finely constructed but extremely formidable machinery that is as inaccessible as it is entirely self-sufficient- therefore it cannot be distilled or summarized without mutilation. Below, we give an account of Blumberg-Gepner-Tabuada's theory and state some of its fundamental properties which generalize those of ordinary Waldhausen K-theory.

The algebraic K-theory of stable ∞-categories due to Blumberg-Gepner-Tabuada (building on the work of Schlichting, Toën and others) is a more faithful generalization of Waldhausen K-theory to the higher setting. An exact comparison may be made between these approaches, for which the discussion in 7.1.4 and 7.1.5 in [2] may be consulted. In this section, we follow [60].

Let C be a pointed ∞-category which admits finite colimits. Recall the category $Ar[n]$ we introduced above. As has been the custom so far, we will confuse it with its nerve in the higher context. Let $Gap([n], C)$ be the subcategory of $Fun(Ar[n], C)$ spanned by functors F such that for all $i \leq j \leq j$, $F(i, i)$ are zero objects and the diagrams

$$\begin{tikzcd} F(i,j) \arrow[r] \arrow[d] & F(i,k) \arrow[d] \\ F(j,j) \arrow[r] & F(j,k) \end{tikzcd} \tag{5.19}$$

are pushout squares in C.

In analogy with the S-construction, when C is a stable ∞-category, so is $Gap([n], C)$. Proceeding with the analogy, we put $S_n(C) := Gap([n], C)$ for the S-theory of the ∞-category C, which assemble into a simplicial ∞-category $S_\bullet : \Delta^{op} \to Cat_\infty$. As we saw in the construction of Waldhausen K-theory and also Barwick's construction, we may pass to the "subcategory of weak equivalences" or the "underlying space" of this simplicial ∞-category at each level, obtaining a simplicial Kan complex or ∞-groupoid $(S_\bullet(C))_\sim$. Then by exact analogy with ordinary Waldhausen K-theory, $\Omega|(S_\bullet(C))_\sim|$ is the K-theory space of the category C and we put $K(C)$ for the Ω-spectrum consisting of the spaces $|(S_\bullet....S_\bullet(C))_\sim|$.

The precise notion of additivity that is relevant for us and which we discussed at the beginning of this chapter can be obtained as a consequence of Waldhausen's additivity theorem and its higher analogue. Namely, the following is proven in 7.10 in [60]:

Proposition 5.2.1. *The K-theory functor as defined above is an additive invariant, i.e., it inverts Morita equivalences, preserves filtered colimits and sends split exact sequences of stable ∞-categories to cofiber sequences of spectra.*

5.3. Hochschild Homology

As the second main example of an additive invariant, we will discuss Hochschild (co)homology of dg-categories and stable ∞-categories.

The operative slogan in this subsection is that Hochschild (co)homology of a stable ∞-category is the "derived trace map in symmetric monoidal ∞-category". We adopt this seemingly outlandish perspective in this thesis, because it makes certain fundamental properties and structures associated with Hochshild (co)homology almost trivial to state. We will start by introducing the dg-analogue of the same construction, which is more accessible. We follow [4, 39]. The following construction is presented in 5.2.2 of [39].

Definition 5.3.1. *Let T be a dg-category and $End(T) := \mathbb{R}Hom_{(T\otimes^{\mathbb{L}}T^{op})-Mod}(T,T)$ the endomorphism dg-category associated with T, where we consider the derived Hom in the category of $T\otimes^{\mathbb{L}}T^{op}$-modules. Then Hochschild cohomology complex of T is the derived endomorphism dg-module of T in $Ho(DG-Cat)$*

$$HH(T) := \mathbb{R}Hom(T,T)(id,id). \tag{5.20}$$

The Hochschild cohomology groups of T are just the cohomology groups of this chain complex

$$HH^*(T) := H^i(\mathbb{R}Hom(T,T)(id,id)). \tag{5.21}$$

Let's offer a few words to illuminate this definition. Recall that Hochschild cohomology for associative algebras can be defined explicitly at chain level by way of the Bar resolution [70] and also by the "Morita" flavored approach which is the conceptual underpinning of the former. For instance, let's assume we have defined the Hochschild cohomology $HH^*(A)$ for an associative algebra A by applying the Dold-Kan correspondence to the simplicial algebra given by the standard resolution $A \leftarrow A \otimes A \leftarrow A \otimes A \otimes A \leftarrow ...$ and thereby obtaining the Hochscild (co)chain complex. Now if we consider A as an $A \otimes A^{op}$-module, the above computation can be seen to amount to the fundamental fact that in the bimodule category $A \otimes A^{op}$-Mod, $HH^*(A) \cong H^* RHom_{A \otimes A^{op}}(A, A)$. But we are just working with modules here, so the latter derived hom is just the Ext functor, and hence we have $HH^*(A) \cong Ext^*(A, A)$. One has a similar characterization of Hochschild homology in terms of the Tor functor. The definition above can be likewise justified in terms of the bar resolution for dg-categories, and the results discussed in the previous sections make sure all the constructions involved are well-behaved.

One advantage of this definition in the context of dg-categories is that the Morita theory of dg-categories as developed by Toën in [4] and which we discussed at some length makes the Morita invariance of Hochschild homology manifest as argued in 5.2.2 of [39].

Lemma 5.3.1. *We have an isomorphism:* $HH(T) \cong HH(\hat{T}_c)$.

As in our treatment of algebraic K-theory, we now introduce Hochschild (co)homology for stable ∞-categories. The formalism we adopt comes from the style of higher category theory and geometry inspired by the works which offer higher categorifications of fundamental concepts in linear algebra, see for instance [71] . We follow [59] in offering a clear cut formulation particularly of Hochschild homology as a *localizing* and hence *motivic* invariant in the sense we discussed at the beginning of this section.

As an aside, let us try to port the definition above for dg-categories and begin with a false start, which should be considered as a stimulating provocation for work in the future. Following Definition 1.4 in [72], we offer the following definition of Hochschild homology for dualizable objects.

Definition 5.3.2. *For A dualizable in Pr^L_{st}, we have the Hochschild homology*

$$HH(A) := A \otimes_{A \otimes A^{op}} A. \tag{5.22}$$

One immediately notes that this functor has the wrong *categorical* dimension, and lands in some stable ∞-category as a coefficient object. We may therefore refer to it as *categorified* Hochschild homology. As such, the proper home for discussing it would be a theory of 2-motives. This will be the topic of future work , however for the moment see [73] for extremely interesting advances in this direction.

Let us now proceed with the proper definition of a ordinary Hochschild homology for stable ∞-categories. As we have hinted at in the section on symmetric monoidal categories, the calculus of dualizability in the context of symmetric monoidal ∞-categories constitutes a comprehensive categorification of fundamental linear algebraic notions such as dimension and trace, as argued in [72]. Recall that for $(X, X^{op}, ev_X, coev_X)$ a dualizable object in some $(C, \otimes, 1_\otimes) \in Sym_\infty \cong CAlg(Cat_\infty)$, we define the trace of an endofunctor as in Definition 2.8.1 of [59]: define $\phi : X \to X$ to be the functor $Tr(\phi) := \epsilon \circ (\phi \otimes id) \circ \eta : 1_\otimes \to X \otimes X^{op} \to X^{op} \otimes X \to 1_\otimes$. Thus trace gives a map $End(X) \to End(1_C)$. Then we define the dimension of such a dualizable object X as the trace $dim(X) := Tr(id_X) = \epsilon \circ \eta$. These evidently generalize the notions of trace and dimension in the ordinary symmetric monoidal category $Vect_k$.

For instance, let $C = Pr^{L,R}_{st}$, the ∞-category of R-linear presentable stable ∞-categories with the closed monoidal structure we've already discussed. As above, the trace is an endomorphism of the unit object, which is nothing but the stable ∞-enrichment of the triangulated derived category $D(R)$, whose dg-incarnation we have already met as $\hat{1}_{dg}$. We denote this ∞-category also by $D(R)$. From now on assume that $R = k$ is a field, and put $D(k)$ for the derived category of k. We then have the following definition, which is 2.8.7 in [59]:

Definition 5.3.3. *The Hochschild homology functor $HH : Cat^{st}_{\infty,R} \to D(k)$ is the chain-complex valued functor which is the composition*

$$HH := dim \circ Ind : Cat^{st}_{\infty,R} \to D(k) \tag{5.23}$$

This definition may be extended to presentable stable ∞-categories as in 2.8.7 in [59].

The motivic nature of Hochschild homology is underscored by the following result, already completely known and indeed obvious to early workers in the field, but which is proven by Chen for this cutting-edge formulation of Hochschild homology in Proposition 2.9.9 and Corollary 2.9.10 in [59]:

Proposition 5.3.1. *Hochschild Homology is a localizing invariant.*

Let us close this section by discussing a theorem due to Tabuada [74] which offers an interesting perspective concerning the categorification of the famous HKR isomorphism, for which see Theorem 9.13 in [35]. In Theorem 1.2 of [74], Tabuada shows that every mixed commutative motivic realization H extends to the noncommutative world, which extension is denoted by H^{nc}. Now assume X smooth scheme of finite type over a field of characteristic 0 which admits a filtration $* \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2....X_n \hookrightarrow X$ such that $X_i - X_{i-1}$ are all smooth affine schemes of finite type. Now denote by D_X the ring of differential operators on X. D_X-Mod is the famous category of D-modules which features prominently in Geometric Langlands correspondence and is one of the central characters in Riemann-Hilbert correspondence. We put $Perf_{dR}(X) := Perf_{dg}(D_X)$ for the subcategory of the dg-enhanced derived category of D-modules on X consisting of perfect complexes, where the notation is supposed to indicate that is the noncommutative de Rham space associated with a scheme. Then the following theorem says that X and the noncommutative de Rham space are not motivically distinguishable.

Theorem 5.3.1. *(Theorem 3.4 in [74] For every mixed realization H_{nc} extending a given mixed realization H, we have an isomorphism $H_{nc}(Perf_{dg}(X)) \cong H(X)$.*

The interest of this theorem from the standpoint of Hoschschild homology as an additive invariant of noncommutative spaces stems from Remark 3.8 in [74] where Tabuada points out that Hochschild homology in fact does not satisfy it. Namely, that it distinguishes between X itself considered as a noncommutative space by way of the $Perf_{dg}$ functor and its de Rham noncommutative space, in that it is not generally true that $HH(Perf_{dR}(X)) \cong HH(X) := HH(Perf_{dg}(X))$. Hence categorification somewhat complicates the HKR story, according to which we should expect a correspondence between the de Rham and classical aspects of a space. Let's emphasize, however, that this is a philosophical point and we are not saying that HKR isomorphism does not naively generalize to this context! By construction, we certainly have an isomorphism $HH(Perf_{dR}(X)) \cong H_{dR}(X)$. This is not a genuine categorification but merely a translation of the definitions into the higher context. To understand what's going on, we should deal with higher invariants, which we will briefly discuss at the end.

It is of great theoretical interest to generalize results of this kind to explore the structure of genuine noncommutative spaces like $Perf_{dR}(X)$. It appears quite likely that such results could play an important role in the categorification of Riemann-Hilbert type correspondences.

5.4. The Stable Homotopy Category of Noncommutative Spaces

In this final section, we will finally construct the stable homotopy category of noncommutative spaces and justify the claim that it is the noncommutative analogue of Voevodsky's stable homotopy category of schemes and therefore is the proper candidate for the category of noncommutative motives. For reasons of space, we have completely omitted an account of Voevodsky's category $SH(S)$ and Robalo's ∞-categorical reconstruction of it which makes manifest its universal property (see Chapter 4 and 5 of [2]). However, Robalo's procedure for obtaining the noncommutative homotopy category exactly mimics steps in the construction of the homotopy category of schemes. Crudely put, the latter arises as the results of a series of localizations of the category of simplicial presheaves on a certain site consisting of smooth schemes equipped with the *Nisnevich* topology [11, 12]. The resulting category consists of $\mathbb{A}^1$-homotopy invariant simplicial presheaves satisfying a certain excision property with respect to so-called elementary Nisnevich squares, which are commonly referred to as *motivic spaces.* Experience from representability results from topology and K-theory leads one to consider, further, the stable version of this category. This stabilization is carried out with respect to the pointed projective line $(\mathbb{P}^1, \infty) \cong S^1 \wedge \mathbb{G}_m$, where the equivalence is the $\mathbb{A}^1$-homotopy equivalence and $\wedge$ product is the monoidal product on motivic spaces. Let us unpack this statement [22] .

First of all, algebraic geometry offers two distinct candidates as an analogue of the topological 1-sphere, that is, the circle S^1 which- as we saw in our preliminary chapter- plays a distinguished role in homotopy theory. By definition, the suspension of a pointed space X is the smash product $S^1 \wedge X$. Returning to the motivic world, that is, the world of simplicial presheaves on the category of schemes, the first analogue of the topological circle is the *simplicial* circle which is the homotopy quotient $S^1 := \Delta^1 // \Delta^0$ obtained by gluing the endpoints of the interval category Δ^1. The second analogue is the multiplicative group scheme $\mathbb{G}_m \cong \mathbb{A}^1 - \{0\}$. It is therefore natural to consider the smash product $S^1 \wedge \mathbb{G}_m$ as the proper motivic analogue of the topological circle. We claim this wedge product is $\mathbb{A}^1$-homotopy equivalent to the motivic space $(\mathbb{P}^1, \infty)$, which is the motivic space represented by the projective line pointed at ∞. To see this, consider the pushout square which presents $\mathbb{G}_m$ as the intersection of the affine patches which cover $\mathbb{P}^1$, expressed in the diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd} \tag{5.24}$$

where we have omitted the base-points and the base-point preserving maps. Since we are in an $\mathbb{A}^1$-invariant context, we can collapse the copies of $\mathbb{A}^1$ to the point, giving us a homotopy pushout square

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbb{P}^1. \end{tikzcd} \tag{5.25}$$

Thus this diagram is the homotopy pushout of the diagram $* \leftarrow \mathbb{G}_m \rightarrow *$, which is nothing but the suspension $\Sigma\mathbb{G}_m := S^1 \wedge \mathbb{G}_m$ of $\mathbb{G}_m$. Hence $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$.

[22] We adopt the convention of not distinguishing between the scheme and the motivic space it represents

Thus to obtain the *stable* homotopy category of Nisnevich sheaves, one must stabilize with respect to this so-called *mixed circle*, along *both* suspension coordinates, as explained in section 2.3 of [11]. The resulting category of *bispectra*, equipped with an induced monoidal structure we keep referring to as the smash product, is called the stable homotopy category of schemes.

Going back to our task of generalizing these constructions to the noncommutative world, it turns out the process of forcing $\mathbb{A}^1$-invariance and stabilization can be achieved almost unproblematically, modulo some subtleties in the behavior of the monoidal structure under localization and stabilization which the bulk of Robalo's work addresses at great length and with great inventiveness, see Chapter 4 of [2]. The generalization of the Nisnevich property to the noncommutative world turns out to be much more counterintuitive, due to the non-geometric nature of dg-categories and lack of a good notion of "cover" or "topology" in $DG-Cat$. We will see below how Robalo tackled this issue by generalizing the notion of elementary Nisnevich squares to the noncommutative setting. Before discussing this, we review Nisnevich topology in the classical setting.

5.4.1. Nisnevich Squares and Brown-Gersten Property

The Nisnevich topology on the category of smooth schemes consists of the covers $p : U_i \to X$ with p an étale morphism such that given any point $x : Spec(k) \to X$, we have a lift $\tilde{x}_i$ for some i with an induced isomorphism of residue fields $k(x) \cong k(\tilde{x}_i)$. The local rings in this topology are henselizations $O^h_{X,x}$ of the Zariski local rings $O_{X,x}$, see 3.1 in [12]. The Nisnevich topology sits between Zariski and étale topologies, being finer than the former and coarser than the latter. Elementary distinguished squares for the Nisnevich topology are commutative diagrams

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{5.26}$$

of smooth schemes such that $p : V \to X$ is an étale morphism, the i is an open embedding and we have an isomorphism of reduced schemes $p^{-1}(X - U)^{red} \cong (X - U)^{red}$.The pair $\{U, V\}$ form a Nisnevich cover of X and they generate the Nisnevich topology on the category of smooth schemes by Proposition 3.1.4 of [12]. In fact this proposition is of vital importance in that it establishes the Brown-Gersten property.

Proposition 5.4.1. *A (simplicial) presheaf satisfies Nisnevich descent (i.e., is a Nisnevich sheaf) iff it "preserves" squares of this form, i.e., sends them to (homotopy) pullback diagrams in the category of spectra.*

To be more precise, the *Brown-Gersten* property is used with reference to the similar characterization of *Zariski* sheaves in terms of Zariski excision squares but has come to refer to Nisnevich squares as well.

We have not covered model structures on simplicial presheaves on Grothendieck sites, or described the procedure of sheaffification as a model categorical localization and cofibrant replacement. The reader may consult [75–77] for a comprehensive treatment of this topic. At this stage of the construction of *commutative* motives, one localizes with respect to Nisnevich local equivalences and finally along maps $F(X \times \mathbb{A}^1) \to F(X)$, which forces $\mathbb{A}^1$-homotopy invariance. Thereby one obtains the category of motivic spaces. To proceed with this construction, we need an analogue of Nisnevich squares for dg-categories.

5.4.2. Noncommutative Analogue of Nisnevich Squares

To define the analogue of Nisnevich excision squares in the noncommutative setting, The first challenge is to develop a notion of open immersion of noncommutative spaces. Our hope is that the functor $Perf_{dg}$ which interpolates between the commutative and noncommutative worlds is well-behaved enough make such a generalization a merely formal task. Localization theorems of Thomason and Keller for derived categories (rather perfect complexes) give us exactly what we want in this case. After finding a satisfactory notion of an "exact sequence" in the category Cat_{∞}^{st}, we can easily define "elementary squares" of dg-categories by directly applying $Perf_{dg}$. But we have already seen that Verdier cofiber sequences and their various equivalent reformulations give a a perfectly adequate notion of exact sequences of stable ∞-categories or dg-categories. In this chapter, such an "exact sequence" is assumed to be split, a distinction to be explicated later.

Recall from our crash review of Nisnevich topology that the latter is generated by Cartesian squares of the form, called elementary distinguished squares, given by diagrams

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{5.27}$$

with i an open immersion, p is an étale morphism and we have $p^{-1}(X-U) \cong (X-U)^{red}$. We will denote a Nisnevich square like the one above by $sq(X, U, V)$.

Consider the "induced" diagram of stable ∞-categories, which is a (homotopy) pushout diagram as discussed in the model categorical setting in the section on the homotopy theory of dg-categories and in the ∞-categorical setting

$$\begin{tikzcd} Perf_{dg}(U \times_X V) \arrow[r] \arrow[d] & Perf_{dg}(V) \arrow[d] \\ Perf_{dg}(U) \arrow[r] & Perf_{dg}(X). \end{tikzcd} \tag{5.28}$$

We may reverse-engineer the noncommutative analogue of the notion of a Nisnevich square from this example: we want a noncommutative Nisnevich square to be an abstraction of the features enjoyed by this diagram.

5.4.3. Noncommutative Nisnevich Topology

In the previous section we saw that Nisnevich-locality admitted a very practical characterization in terms of the elementary squares which generate the Nisnevich topology: namely that a presheaf F satisfies Nisnevich descent iff it preserves elementary Nisnevich squares. Thus we obtain a characterization of the category of Nisnevich-local objects in $Fun(Sch^{op}, Sp)$ as a reflective localization of a presentable category, which makes it accessible to the machinery developed by Lurie and summarized in the previous sections. We would like to reproduce this practical characterization for noncommutative motives, which should be "sheaffifications" of $\mathbf{A}^1$-invariant additive invariants of dg-categories valued in spectra. The motivation for this condition may seem to be lacking in that there is no reason to expect a precise dictionary between geometry of schemes and "noncommutative" geometry of dg-categories. The first observation is that elementary Nisnevich squares induce- by the theorem of Thomason- "exact" squares of derived categories, which suggest that the analogy is worth pursuing, see the discussion in 6.4.4 by [2]. Thus derived categories satisfy "descent" in the ∞-categorical context and it becomes necessary to recover this feature somehow for invariants valued on them.

The first challenge is to come up with an analogue of open immersion in the noncommutative context. Following Robalo as in 6.4.5 [2], we proceed to define the notion of an open immersion of noncommutative spaces as follows.

Definition 5.4.1. *A morphism $f : X \to Y$ of noncommutative spaces is said to be an open immersion if there exists an idempotent complete dg-category K_{Y-X} along with a fully faithful embedding $K_{Y-X} \hookrightarrow T_Y$ such that f^{op} (that is, f as a morphism of dg-categories of finite type) takes part in the following exact sequence of dg-categories exhibited by the diagram*

$$\begin{array}{ccc} K_{Y-X} & \longrightarrow & T_Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & T_X. \end{array} \tag{5.29}$$

We are now ready to define the notion of a noncommutative Nisnevich square, following 6.4.7 in [2].

Definition 5.4.2. *A commutative diagram of noncommutative spaces*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array} \tag{5.30}$$

is said to be a Nisnevich square if the following hold.

(i) $X \to Y$ *and* $U \to V$ *are open immersions of nc-spaces.*

(ii) The induced morphism of dg-categories $T_V \to T_Y$ *preserves the compact generator of* K_{V-U} *and induces an equivalence* $K_{Y-X} \cong K_{V-U}$*.*

(iii) The diagram is a pushout square.

As explained in 6.4.10 of [2] Every open immersion of noncommutative spaces $f : U \to X$ leads to a Nisnevich square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & Z. \end{tikzcd} \tag{5.31}$$

This follows by simply transcribing the definitions. Denoting as before the associated dg-categories of finite type by T_X etc, we have that K_{X-U} is the dg-category of finite type representing the "closed complement" Z and the map $K_{X-U} \to *$ induces an open immersion $* \to Z$ the above is just the recollement square for the inclusion of an open subset/closed complement, in particular, a pushout square.

We have the following sanity checks which ensure that this definition is in fact viable. The first thing to check is that noncommutative Nisnevich squares are actually generalizations of ordinary ones. Robalo proves in Proposition 6.4.16 in [2] that applying the functor $Perf_{dg}$ to a Nisnevich square $sq(X, U, V)$ produces a pushout diagram of noncommutative spaces which is a noncommutative Nisnevich square. This was indeed to be expected since the definition of the latter is a direct reverse-engineering of the former. But even more interestingly, the converse statement also holds, as discussed in Remark 6.4.17 in [2]. That is, if any pullback diagram that is not a priori necessarily a Nisnevich square *induces* a noncommutative Nisnevich square by way of the $Perf_{dg}$ functor, then it must in fact be a Nisnevich square itself. Therefore there is a very tight link between ordinary and noncommutative Nisnevish squares.

Recall that we have already covered the notion of a semiorthogonal decomposition for triangulated categories, and noted how split exact sequences of dg-categories provide a translation of this concept into $DG - Cat$. Any semiorthogonal decomposition of dg-categories provides an example of a noncommutative Nisnevich cover as explained in 6.4.12 in [2]. Thus for instance Beilinson's exceptional collection on $Ho(Perf_{dg}(\mathbb{P}^n_R)) =< O, O(-1)..., O(-n) >$ gives rise to a Nisnevich cover of $Perf_{dg}(\mathbb{P}^n_R)$ as given in the Example 6.4.12 of [2]. This will be quite important in the final construction of the stable homotopy category.

5.5. The Noncommutative Stable Motivic Homotopy Category Over a Base Scheme

In this section, we put everything together and define Robalo's noncommutative stable homotopy category, which will be the universal recipient of additive invariants of noncommutative spaces. We do not have the resources or the space to go through a demonstration of this result but offer some comments on what makes the intermediate steps work. The following are the main stages of the construction as explained in 6.4.2 of [2].

- *Nisnevich-locality* Start with the bare category $Fun(Nc(R)^{op}, Sp)$ and take the subcategory consisting of the functors preserving Nisnevich squares (i.e., sending Nisnevich squares to fiber sequences in Sp). Let us denote this by subcategory by $Psh_{Nis}(Nc(R))$, this is a monoidal accessible reflective localization of the presentable category $Fun(Nc(R)^{op}, Sp)$ with respect to all morphisms: $j(U) \times_{j(W)} j(V) \to j(X)$, where $j(U)$ etc. denotes the Yoneda embedding and the U, V, X run over all noncommutative spaces fitting in an nc-Nisnevich square (generalized semi-orthogonal decomposition) given by a diagram

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X. \end{array} \tag{5.32}$$

- *Homotopy invariance* Force the analogue of $\mathbb{A}^1$-invariance. For this, we consider the "noncommutative affine line" $Perf(\mathbb{A}^1)$. This dg-category is smooth since $\mathbb{A}^1$

is smooth, hence of finite type. We need to localize $Sh_{Nis}(Nc(R))$ further along the map $Perf(\mathbb{A}^1) \to Perf(Spec(R))$ induced by the map $\mathbb{A}^1 \to Spec(R)$. More precisely, for $X \in Nc(R)$ we wil localize along the maps

$$j(X) \times j(Perf(\mathbb{A}^1) \to j(X) \times j(Perf(Spec(R)) \tag{5.33}$$

which gives a monoidal accessible reflexive localization of a presentable ∞-category. Thus the localization itself is presentable.

- *Stability* Stabilize with respect to (the noncommutative space associated with) $\mathbb{P}^1$, that is the dg-category $Perf(\mathbb{P}^1)$. That is, we monoidally invert the functor $- \wedge Perf(\mathbb{P}^1)$.

The *universality* of the resulting category, the stable homotopy category of noncommutative spaces, will be almost tautological because all the constructions detailed above satisfy the relevant universal properties. The embedding of the commutative theory into the noncommutative one is summarized in 6.4.39 and 6.4.45 of [2] by the crucial diagram

$$\begin{tikzcd}
Aff_{ft}^{\times} \arrow[r, "Perf_{dg}^{\otimes}"] \arrow[d, hook] & ncS^{\otimes} \arrow[d, hook] \\
Fun(Aff_{ft}^{op}, Sp)^{\otimes} \arrow[r, dashed, "P_1"] \arrow[d] & Fun(Dg^{ft}, Sp)^{\otimes} \arrow[d, hook] \\
Fun_{Nis}(Aff_{ft}^{op}, Sp)^{\otimes} \arrow[r, dashed, "P_2"] \arrow[d] & Fun_{Nis}(Dg^{ft}, Sp)^{\otimes} \arrow[d] \\
Fun_{Nis,\mathbb{A}^1}(Aff_{ft}^{op}, Sp)^{\otimes} \arrow[r, dashed, "P_3^{\otimes}"] \arrow[d] & Fun_{Nis,Perf_{dg}(\mathbb{A}^1)}(Dg^{ft}, Sp)^{\otimes} \arrow[d] \\
SH^{\otimes} \arrow[r, "P"] & SH_{nc}^{\otimes}.
\end{tikzcd} \tag{5.34}$$

Let us make some peripheral comments before giving more details. On the left-hand side, we have the ∞-categorical version of the construction of Voevodsky's stable homotopy category of schemes, on which we offered a few comments in the preceding sections. The connection between the commutative and noncommutative worlds is provided by the *perfect derived category* functor $Perf_{dg}$ which we have implicitly used a few times already. Its proper construction as an ∞-functor from affine schemes to $DG-Cat^{idem}$ and indeed $DG-Cat^{f.t.}$ or $ncS(R)$ is quite delicate and relies on the cotangent complex formalism developed by Robalo in sections 3.11.2 and 6.3.3 of [2]. This functor is monoidal and induces monoidal functors on the localized functor categories as we approach the stable homotopy category. The final functor P gives an "embedding" of the commutative world into the noncommutative one. Philosophically, this embedding is not surprising considering the complete correspondence, say, between classical Nisnevich squares and the noncommutative ones induced by them and the fact that the noncommutative procedure is the mirror of the commutative one. However, there are various new "derived" phenomena which make themselves felt only at the noncommutative level. The new "categorical" Nisnevich covers provided by orthogonal decompositions are the most prominent example of such new derived phenomena. The following puzzling theorem explicates this issue:

Theorem 5.5.1. *(Proposition 6.4.19 in [2]) The homotopy category $H_{nc}(R)$ of noncommutative spaces over a base ring R, which consists of $\mathbb{A}^1$-invariant noncommutative Nisnevich sheaves of spectra on $DG - Cat$ is already stable and the Yoneda image of $\mathbb{P}^1_R$ in $H_{nc}(R)$ is already invertible in it.*

Let us unwind the content of this theorem. We folllow the proof of Proposition 6.4.19 in [2]. First of all, denote by ψ the composition of the functor P with the pointing map $H(R) \to H_*(R)$ and point $\mathbb{P}^1$ at ∞ as before. The claim is that $\psi(\mathbb{P}^1, \infty)$ is an invertible object and in fact the unit for the monoidal structure on $H_{nc}(R)$. $(\mathbb{P}^1_R, \infty)$ as a motivic space is the homotopy cofiber of the $\mathbb{A}^1$-localized pointing map $Spec(R) \to \mathbb{P}^1_R$. Putting $l^{nc}_{\mathbb{A}^1}$ for the functor of localization with respect to $Perf(\mathbb{A}^1)$ and l^{nc}_{Nis} for the functor of noncommutative Nisnevich sheaffification and applying the ψ functor defined above first of all gives us an isomorphism

$$\psi(\mathbb{P}^1, \infty)) \cong l^{nc}_{\mathbb{A}^1} j(Perf_{dg}((cofib : Spec(R) \to \mathbb{P}^1_R)) \tag{5.35}$$

where the cofiber is now taken in the category $l^{nc}_{Nis} Fun(ncS(R), Sp)$. This collapses further since the spectral Yoneda embedding commutes with colimits and we have an isomorphism

$$\psi(\mathbb{P}^1, \infty)) \cong l^{nc}_{\mathbb{A}^1}(cofib : j((Perf_{dg}(Spec(R) \to \mathbb{P}^1_R))) \tag{5.36}$$

which is to say, an isomorphism

$$\psi(\mathbb{P}^1, \infty)) \cong l^{nc}_{\mathbb{A}^1}(cofib : j((Perf_{dg}(R)) \to Perf_{dg}(\mathbb{P}^1_R))). \tag{5.37}$$

However, the latter cofiber is just $Perf_{dg}(R)$ itself, as we have a noncommutative Nisnevich cover given by the exceptional collection $Ho(Perf_{dg}(\mathbb{P}^1_R)) = D^b(\mathbb{P}^1_R) =< O, O(-1) >$, that is a diagram

$$\begin{tikzcd} Perf_{dg}(R) \arrow[r] \arrow[d] & Perf_{dg}(\mathbb{P}^1_R) \arrow[d] \\ * \arrow[r] & Perf_{dg}(R) \end{tikzcd} \tag{5.38}$$

which is a pushout square after sheaffification. Hence the cofiber above as computed after noncommutative Nisnevich sheaffification is nothing but $Perf_{dg}(R)$. But $Perf_{dg}(R)$ is precisely the unit for $H_{nc}(R)$, hence the statement follows. It also immediately follows that the $\mathbb{A}^1$-localized homotopy category of functors on noncommutative spaces which preserve noncommutative Nisnevich squares, which we have denoted by $H_{nc}(S)$, should in fact be denoted $SH_{nc}(S)$ since it is equivalent to its $\mathbb{P}^1$-stabilization.

As a result, we have achieved the construction of the noncommutative stable homotopy category. In our concluding chapter, we will briefly discuss why this category is a candidate for the category of motives in the noncommutative context.

6. CONCLUSION

The construction of the noncommutative stable homotopy category described in the last chapter brings the foundations of the theory of noncommutative motives to a close. In conclusion, we would like to discuss some further aspects and avenues which we have not touched on so far.

The first topic that merit discussion is the question of the *representability* of Algebraic K-theory for stable ∞-categories and the various applications of the independent representability theorems to Robalo [2], on one hand, and Tabuada, Cisinski [78, 79], Blumberg and Gepner [60, 64], on the other. While we have not given an account of *nonconnective* algebraic K-theory, which goes back to Bass for algebraic K-theory of rings and Thomason-Trobaugh [80] for the Waldhausen K-theory of the category of perfect complexes on a scheme, the representability theorems for K-theory, which establish its nature as a universal additive invariant, concern precisely this variant. The content of these theorems may be summarized as saying that, in all its various formulations, nonconnective K-theory of dg-categories or stable ∞-categories is the unit noncommutative motive with respect to the monoidal structure on the latter and it constitutes the noncommutative analogue of motivic cohomology in the sense of the following theorem due to Robalo. For convenience, we adopt an exotic notation for noncommutative spaces. For instance, if T is a dg-category of finite type, we denote by $Spec(T)$ the associated noncommutative space. We also let K^S denote the nonconnective K-theory of dg-categories as described in Chapter 7 of [2].

Theorem 6.0.1. *(Theorem 7.0.33 in [2]) We have an isomorphism*

$$Map_{SH_{nc}}(Spec(T), Spec(S)) \cong l^{nc}_{\mathbf{A}^1} K^S(T \otimes \ S^{op}). \tag{6.1}$$

In the same vein, it is of great interest to explore higher analogues of K-theoretic and motivic constructions which take advantage of the exact structure on Cat^{st}_{∞} and $DG-Cat$. This exact structure, while not equivalent to *full* stability was certainly instrumental in setting up the theory of noncommutative motives. One motivation for axiomatizing this exact structure is what might be broadly called *categorified cohomology*. Since Grothendieck's vast generalization of the Riemann-Roch theorem by way of the formalism of derived categories, which is the origin of the six-operations formalism as we discuss below, it has been a commonplace notion that the assignment $X \mapsto D(X)$ of a scheme to its derived category can be interpreted as a sort of *categorified* cohomology theory. The latter can be formalized naively as a functor $Sch_{/S} \to Cat^{st}_{infty}$ or $DG-Cat$ which satisfies certain localization and gluing conditions, and Künneth-type theorems which can be demonstrated by way of the higher analogues of the Tannakian formalism. Indeed, we caught a glimpse of something along these lines at the end of the third chapter. A systematic theory of categorified cohomology would have to concern itself with sheaves, not of spectra, but of stable ∞-categories, giving rise to what one might call parametrized noncommutative algebraic geometry. Going along with the analogy, one can construct higher analogues of $QCoh(X)$ and K-theory for these categorical sheaves. Secondary K-theory, $K^{(2)}(X)$ was introduced by Toën in [39] and further developed by [81], however, this construction is somewhat ad-hoc. It turns out enormous amount of classical cohomological machinery can be made to work for categorified sheaves, whose study is a burgeoning field at the moment. For the study of Chern classes and Grothendieck-Riemann-Roch formalism from the perspective of noncommutative motives, see [82, 83]. For a systematic development of the notion of a stable $(\infty, 2)$-category, of whicht the category of sheaves of stable ∞-categories would be the first obvious example besides Cat^{st}_{∞} and higher noncommutative motives, see [73].

Finally, let us discuss a more foundational and technical topic which is nonetheless of great importance for a complete theory of noncommutative motives. This topic is the so-called six functors formalism, which concerns the exceptional functoriality enjoyed by assignments of the form $S \to D(S)$ and $S \mapsto SH(S)$, which can be slickly formalized as functors out of certain correspondence categories associated to a scheme S into the ∞-category of symmetric monoidal stable ∞-categories, see [84]. This question was settled decisively by Ayoub [85] for the stable homotopy category of schemes and described in the ∞-categorical setting in [86] and [84]. Robalo addressed the construction of a six-functors formalism for the assignment $S \mapsto SH_{nc}(S)$ in [2], although the problem is not fully solved. The reader may consult Chapter 3 of [2] for the state of the art results in this direction.

Both flavors of the theory of noncommutative motives provide an embedding of the commutative world into the noncommutative one. In the case of Robalo's formalism, this embedding is simply induced by the $Perf_{dg}$ functor as illustrated by the diagram in the last section. This is a fact with profound ramifications and has already generated some fascinating results in different directions. Tabuada's extensive program of developing noncommutative analogues of great open questions concerning algebraic cycles and motives, such as the Standard Conjectures, the Hodge and Tate conjectures, and the Period Conjecture, should be mentioned in this connection, see [87]. In a different but equally fascinating direction, recent works by Robalo, Blanc, Toën and Vezzosi [66] building on the work of Toën and Vezzosi [88, 89] reformulate and prove cases of the Bloch conductor conjecture by exploiting the commutative motivic realization functor of noncommutative spaces, obtained as a left adjoint to the functor $SH(S) \to SH_{nc}(S)$ induced by $Perf_{dg} : Aff^{f.t.}_{/S} \to DG - Cat^{f.t.}$. This extremely interesting idea promises to be of systematic interest. Also see works by Pippi [90, 91] for investigations along the same lines.

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