THE QUESTION OF MODEL COMPANIONABILITY: POSITIVE AND NEGATIVE ANSWERS

by

Feyza Nur Berksoy

B.S., Mathematics, Boğaziçi University, 2019

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2022

ACKNOWLEDGEMENTS

First of all, I would like to thank to Özlem Beyarslan for all of her exertions, patience and support at every stage of my master study and in writing process of this thesis. She was always interested, enabled me to continue this work with enthusiasm and had a great impact on improving myself in mathematics. Thank you for all the things, it has been a great pleasure working with you.

I also would like to thank to Ayhan Günaydın and Piotr Kowalski for being a member of my thesis committee and their valuable comments on my thesis.

I would like to my family for all their supports in my education life.

As a member, I also would like to thank to Istanbul Technical University for supporting me in my academic research.

Lastly, I would like to thank to TUBITAK for graduate scholarship.

ABSTRACT

THE QUESTION OF MODEL COMPANIONABILITY: POSITIVE AND NEGATIVE ANSWERS

Model companion of a universal theory T is the axiomatization of the existentially closed models of T. This thesis studies the concept of model companionability of theories. We present examples of model companions of certain well known theories. We then give examples of theories without model companions. The main focus of this thesis is to elaborate a technique, which we call "the Compactness Argument". Compactness Argument is used to prove that the model companion of a theory does not exist. We apply Compactness Argument to prove that the following theories do not have model companions: the theory of groups, the theory of rings, two examples of the theory of graphs, the theory of fields with two commuting automorphisms, and the theory of dense linear orders with an automorphism. Several proofs are illustrated by original diagrams to provide a better understanding to the reader.

ÖZET

MODEL EŞİ BULMA SORUNU: POZİTİF VE NEGATİF YANITLAR

Bir evrensel T teorisinin model eşi, T'nin varlıksal kapalı modellerinin teorisinin aksiyomlarından oluşur. Bu tezde teorilerin model eşlenebilirliği kavramı incelenmiştir. Bazı iyi bilinen teorilerin model eşi örnekleri sunulup ardından model eşi olmayan teorilerden örnekler verilmiştir. Bu tezin ana odak noktası "Kompaktlık Argümanı" olarak adlandırdığımız bir tekniğin üzerinde durmaktır. Kompaktlık Argümanı, bir teorinin model eşi olmadığını kanıtlamak için kullanılır. Kompaktlık Argümanını kullanarak model eşi olmadığı kanıtladığımız teoriler şunlardır: gruplar teorisi, halkalar teorisi, çizge teorisinden iki örnek, iki değişmeli otomorfizmalı cisimler teorisi ve otomorfizmalı yoğun lineer sıralama teorisi. Kanıtların daha anlaşılır olmasını sağlamak için kanıtları destekleyen orjinal diyagramlar inşa edilmiştir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

$\bar{v}$	n-tuple $(v_1, v_2, ..., v_n)$
$\mathcal{A}, \mathcal{B}$	Structures
$Diag(\mathcal{A})$	Diagram of $\mathcal{A}$
$\mathcal{L}$	Language
$\mathbb{R}$	Real Numbers
$\mathbb{N}$	Natural Numbers
$\mathbb{N}^*$	Natural Numbers except zero
$\mathbb{Q}$	Rational Numbers

LIST OF ACRONYMS/ABBREVIATIONS

ACF	The theory of algebraically closed fields
DLO	The theory of dense linear orders without endpoints
e.c.	existentially closed
RG	The theory of random graph

1. INTRODUCTION

Model theory is a flourishing area of mathematics which studies interactions between theories: *axioms describing certain mathematical structures*; and their models: *the structures themselves*; like groups, rings, fields or graphs. The techniques and methods that are invented for studying the interactions between theories and models may also become useful in answering questions arising from algebra. A famous example of this is Ehud Hrushovski's proof of Mordell-Lang conjecture using model theory. Hrushovski has given the first proof valid in all characteristics of the "Mordell-Lang conjecture for function fields" using model theoretic techniques.

One important theme which is studied in model theory is the attempt of listing a set of axioms for the structures with desired properties. This kind of research opens up a very fruitful path in model theory letting us understand the mathematical objects using the tools invented. An existentially closed model is one where "anything that can happen happens". More precisely if there is an extension containing a certain element with the desired properties, the element already exists in our model. For example, algebraically closed fields are existentially closed. If a polynomial has a root in an extension of an algebraically closed field, it already is in there. A reoccurring question in model theory is to investigate, if a set of axioms can be listed describing existentially closed models of certain theories.

In this thesis, we show existence and nonexistence of model companions of certain theories. We give several examples of theories where the model companion exists and where they do not. The main emphasis of this thesis is on elaborating a technique for showing nonexistence of model companions. We call this technique as "the Compactness Argument". We explicitly demonstrate the application of this technique in several examples. Another novelty of this thesis can be seen as the use of visual material for making the proofs more comprehensible. Several proofs that are presented in this thesis are accompanied by original diagrams which provide a better understanding to the reader.

In Chapter 2, we first introduce the basic terms and study some key concepts of model theory. Then, we define model completeness and model companions: we reveal equivalent conditions of their definitions and we exhibit some properties in detail. Lastly, we give examples of some well known theories.

The main part of the thesis is Chapter 3, where we give positive and negative examples of model companions. We start by showing some positive examples. Then, we introduce Compactness Argument and prove it to be able to apply it in negative examples. Lastly, we give examples where the model companion does not exist by using the Compactness Argument.

We list several open problems in conclusion.

2. MODEL THEORY

In this chapter, the aim is to introduce main concepts of model theory and to give essential background information for further study. Main references that are used to give basic well-known definitions and theorems are [1–6].

2.1. Basic Concepts

Model theory is the study of mathematical structures, their properties and their relationships with each other in a formal manner. A mathematical structure is basically a set endowed with additional operations and elements. For example, a group structure is a set equipped with a binary function for the group operation and a graph is a set equipped with a binary relation for edge relation. Also, there may exist some significant elements that we want to talk about in these structures. For example in groups, identity element of the operation is an important element for the structure, and in rings identity elements of the both operations are important. Hence, by looking at these examples we can generalize this concept as; a structure is a set endowed with functions, relations and important elements which we call as constants. Before defining a structure explicitly, we need to first define what a language is, this will help us to pursue this study formally.

Basically, a language (or a first order language) is a collection of relations, functions and constant symbols. For example, to talk about a ring structure, we need a language which contains two binary function symbols for operations and two constant symbols representing identity elements of operations. Also, we add one more binary function symbol to the language for practicality in ring theory and denote the language of rings as $\mathcal{L}_R = \{+, \cdot, -, 0, 1\}$. Furthermore, if we want to talk about ordered rings, we need a bigger language which contains again three binary function symbols, two constant symbols and additionally it must contain a binary relation symbol for the order relation. Here, one of the most important things is when we talk about languages, it is basically just a set of symbols which only have syntactic meaning. These set of symbols get their meaning when we associate a set with this language and interpret these symbols according to it. More precisely, if f is a function symbol in a language $\mathcal{L}$, the only important thing about f is its arity; that is, the number of input the function take. Indeed, if f is an n-ary function symbol, then in an $\mathcal{L}$-structure, f can be interpreted as any n-ary function over the underlying set of the structure. Similarly, the only important thing about relation symbol in a language is its arity and the constant symbol is nothing more than a symbol. Therefore, is important to understand that the language is about syntax, and symbols of language get meaning when they are interpreted in a structure.

Now, we can give the formal definitions of language and structure.

Definition 2.1. A language $\mathcal{L}$ is a set of

- function symbols where each function symbol f has arity n_f,
- relation symbols where each relation symbol R has arity n_R,
- consant symbols.

Here, n_f and n_R are positive integers. We denote a language $\mathcal{L}$ as

$$(f_1, f_2, \dots, R_1, R_2, \dots, c_1, c_2, \dots)$$

where each f_i are function symbols, each R_j are relation symbols and each c_k are constant symbols.

Example 2.1. (i.) A very first example is the most basic one, the empty language consisting of no symbols; $\mathcal{L} = \emptyset$.

(ii.) The language of rings $\mathcal{L}_R = \{+, \cdot, -, 0, 1\}$ consists of three binary function symbols $+, \cdot, -$ and two constant symbols $0, 1$. Here, actually two function symbols would be enough, for two operations of ring structure but we also add the symbol $-$ for practicality, since when we check a subset of a ring is a subring, we use "substraction".

(iii.) The language of graphs $\mathcal{L}_G = \{R\}$ consists of one binary relation symbol R.

(iv.) The language of ordered abelian groups is $\mathcal{L} = \{+, 0, <\}$. Indeed, the language of abelian groups consists of a binary function symbol $+$ and a constant symbol 0. To obtain language of ordered abelian groups, we expand the language of abelian groups by adding a binary relation symbol $<$ for the order relation.

When we want to study a mathematical structure or a theory, it is important to choose an appropriate language. After choosing a suitable language and a set that we wish to study, we interpret each symbol of the language $\mathcal{L}$ in terms of the chosen set and we get an $\mathcal{L}$-structure. So now for a chosen language $\mathcal{L}$, we can define what an $\mathcal{L}$-structure is.

Definition 2.2. An $\mathcal{L}$-structure $\mathcal{A}$ consists of the following data:

- A nonempty set A, which is called as underlying set, or the universe of $\mathcal{A}$.
- A function $f^{\mathcal{A}} : A^{n_f} \to A$ for each function symbol $f \in \mathcal{L}$ with arity n_f.
- A relation $R^{\mathcal{A}} \subseteq A^{n_R}$ for each relation symbol $R \in \mathcal{L}$ with arity n_R.
- An element $c^{\mathcal{A}} \in A$ for each constant symbol $c \in \mathcal{L}$.

We denote an $\mathcal{L}$-structure $\mathcal{A}$ by writing the underlying set A of $\mathcal{A}$ and interpretations of each symbol of the language $\mathcal{L}$,

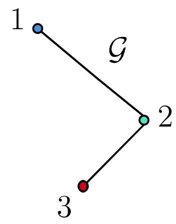
$$\left(A, f_1^{\mathcal{A}}, f_2^{\mathcal{A}}, \ldots, R_1^{\mathcal{A}}, R_2^{\mathcal{A}}, \ldots, c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, \ldots\right).$$

Remark 2.1. Through the text, we denote the structures by curly letters such as $\mathcal{A}$, $\mathcal{B}$, ... and their underlying sets by normal letters A, B, ...; respectively.

Example 2.2. (i.) Let $\mathcal{L} = \{*, e\}$ be the language of groups. The additive group of integers $\mathcal{G} = (\mathbb{Z}, +, 0)$ is an $\mathcal{L}$-structure where interpretation of function symbol $*^{\mathcal{G}}$ is the usual addition $+$ and interpretation of constant symbol $e^{\mathcal{G}}$ is 0, the identity element of $+$. The multiplicative group of rational numbers $\mathcal{G}' = (\mathbb{Q}^*, \cdot, 1)$ is also an $\mathcal{L}$-structure where $*^{\mathcal{G}'}$ is the usual multiplication $\cdot$ and $e^{\mathcal{G}}$ is 1, the identity of multiplication. $GL_n(\mathbb{R})$, the general linear group of order n over $\mathbb{R}$, consisting of $n \times n$ invertible matrices is an $\mathcal{L}$-structure where the function symbol is interpreted as the ordinary matrix multiplication and the constant symbol is interpreted as the identity matrix I_n.

(ii.) Let $\mathcal{L}_G = \{R\}$ be a language consisting of one binary relation symbol. The set of real numbers ordered with usual order relation is an $\mathcal{L}_G$-structure $(\mathbb{R}, \leq)$, where

R is interpreted as $\leq$. A graph $\mathcal{G}$ over the set $\{1,2,3\}$ consisting of two edges between $1-2$ and $2-3$ is an $\mathcal{L}_G$-structure $(\{1,2,3\},\ R^{\mathcal{G}})$. Indeed, the underlying set is $\{1,2,3\}$ and the relation R is interpreted as $R^{\mathcal{G}} = \{(1,2),(2,1),(2,3),(3,2)\}$.



To investigate certain properties of structures, we make use of *first order sentences* and a more general form, *formulae* in model theory. For example, consider the structure $(\mathbb{N},<)$ of natural numbers with usual order relation and the property "There exists a least element". By using symbols of the language $\mathcal{L} = \{<\}$ and some other logical symbols we can express this property formally as a string of symbols

$$\exists x \forall y \, \big(x < y \vee x = y \big)$$

and examine the validity of this sentence in the structure $(\mathbb{N},<)$ or in the other structures. The construction of such sentences in formal languages is similar to the notions of usual languages. More precisely, the language $\mathcal{L}$ can be thought as an alphabet that contains letters to form sentences. To make a meaningful string of letters, we first construct words from letters and then we construct sentences from words. Analogously, in formal languages, we construct terms from symbols of the language and we construct formulae from terms by regarding certain rules.

Basically, a formula of an $\mathcal{L}$-structure is a string of symbols consisting of logical symbols ($\neg$, $\vee$, $\wedge$, $\exists$, $\forall$, $=$), variables (v_i, $i = 0,1,2...$), symbols of the language $\mathcal{L}$ (function, relation and constant symbols) and punctuation marks; comma and parentheses. However, formulae are not arbitrary combination of these kind of symbols; of course, there are rules which makes them understandable. Now, we start defining the basic building blocks, that are terms and then we inductively define what an atomic formula and formula is.

Definition 2.3. *Terms* of the language $\mathcal{L}$ are defined recursively as follows:

i) Constant symbols c of the language $\mathcal{L}$ are terms.
ii) Variable symbols v_i for $i = 1,2,\ldots$ are terms.

iii) If $t_1, t_2, \dots, t_{n_f}$ are terms and f is an n_f-ary function symbol, then $f(t_1, t_2, \dots, t_{n_f})$ is also a term.

and no other strings of symbols are terms.

Example 2.3. Let $\mathcal{L}_R = \{+, \cdot, -, 0, 1\}$ be the language of rings where $+, \cdot, -$ are binary function symbols and $0, 1$ are constant symbols. $+(1, v_1)$, $\cdot(-(v_1, 1), +(v_2, v_3))$ and $+(1, +(1, 1))$ are $\mathcal{L}_R$ terms.

Remark 2.2. In fact, we are familiar with the notation $v_3 \cdot (v_1 + v_2)$ rather than $\cdot(v_3, +(v_1, v_2))$ for binary functions such as $+, -, \cdot$; so whenever it is clear from the context, we will use this notation for simplicity. According to this remark, the terms in Example 2.3 can be written as $(1 + v_1)$, $(1 - v_1) \cdot (v_2 + v_3)$ and $1 + (1 + 1)$, respectively.

Interpretation of terms: Let $\mathcal{A}$ be an $\mathcal{L}$-structure, $t = t(\bar{v})$ be a term containing variables from $\bar{v} = (v_1, v_2, \dots, v_m)$ and let $\bar{a} = (a_1, a_2, \dots, a_m) \in A^m$ be an m-tuple. We interpret t as a function $t^{\mathcal{A}} : A^m \to A$ and define it recursively as follows:

- If t is a constant symbol c, then $t^{\mathcal{A}}(\bar{a}) = c^{\mathcal{A}}$.
- If t is a variable symbol v_i, then $t^{\mathcal{A}}(\bar{a}) = a_i$.
- If t is the term $f(t_1, t_2, \dots, t_{n_f})$ where $t_1, t_2, \dots, t_{n_f}$ are terms and f is an n_f-ary function symbol, then $t^{\mathcal{A}}(\bar{a}) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(\bar{a}), t_2^{\mathcal{A}}(\bar{a}), \dots, t_{n_f}^{\mathcal{A}}(\bar{a}))$.

Example 2.4. Let $\mathcal{L} = \{f_1, f_2, g_1, g_2, c_1, c_2\}$ be a language consisting of two unary function symbols f_1 and f_2, two binary function symbol g_1 and g_2, and two constant symbols c_1 and c_2. Some examples of $\mathcal{L}$-terms are

$$t_1 = g_2(g_1(f_1(v_1), f_2(v_1)), v_2),$$

$$t_2 = f_1(g_2(g_2(g_1(c_2, c_2), c_1), v_1)),$$

$$t_3 = f_2(g_2(g_1(c_1, c_2), v_1)).$$

Now, let $\mathcal{A} = (\mathbb{R}, \sin, \exp, +, \cdot, \pi, 1)$ be an $\mathcal{L}$-structure such that interpretations of symbols are $f_1^{\mathcal{A}} = \sin$, $f_2^{\mathcal{A}} = \exp$, $g_1^{\mathcal{A}} = +$, $g_2^{\mathcal{A}} = \cdot$, $c_1^{\mathcal{A}} = \pi$ and $c_2^{\mathcal{A}} = 1$. Interpretations of t_1, t_2 and t_3 in the structure $\mathcal{A}$ are as

$$t_1^{\mathcal{A}}(a_1, a_2) = (\sin(a_1) + e^{a_1}) \cdot a_2,$$
$$t_2^{\mathcal{A}}(a_1) = \sin(((1+1) \cdot \pi) \cdot a_1) = \sin(2\pi a_1),$$
$$t_3^{\mathcal{A}}(a_1) = e^{(\pi+1)\cdot a_1}.$$

By using $\mathcal{L}$-terms as building blocks, we will define $\mathcal{L}$-formulae.

Definition 2.4. *Formulae* of the language $\mathcal{L}$ are constructed recursively as follows:

i) If t_1 and t_2 are $\mathcal{L}$-terms, then $t_1 = t_2$ is a formula.
ii) If R is relation symbol and t_i are terms of the language $\mathcal{L}$, then $R(t_1, t_2, ..., t_{n_R})$ is a formula.
iii) If ϕ is a formula, then $\neg\phi$ is a formula.
iv) If ϕ and ψ are formulae, then $\phi \wedge \psi$ and $\phi \vee \psi$ are formulae.
v) If ϕ is a formula, then $\forall v_i\, \phi$ and $\exists v_i\, \phi$ are formulae (for any variable v_i).

and no other strings of symbols are formulae. Formulae of the form i) and ii) are called as *atomic formulas* which are basic forms of a formula and like building blocks of longer formulae.

Let $\forall v\, \phi$ (resp. $\exists v\, \phi$) be a formula, the subformula ϕ is called as the *scope* of the quantifier $\forall v$ (resp. $\exists v$). If a variable v_i lies within the scope of a quantifier $\forall v_i$ (resp. $\exists v_i$) in a formula, it is called as a *bound* variable; otherwise it is called as *free* variable. An $\mathcal{L}$-formula with no free variables is called a *sentence*.

Remark 2.3. If a formula ϕ contains free variables from $\bar{v} = (v_1, v_2, ..., v_n)$ we denote the formula as $\phi(\bar{v})$ to indicate the free variables of it.

Example 2.5. Let $\mathcal{L} = \{+, \cdot, -, 0, 1, <\}$ be the language of ordered rings. The following are $\mathcal{L}$-formulas:

- $\phi_1(v_1) : 0 < v_1$ is an atomic formula where v_1 is a free variable.
- $\phi_2 : \exists v_1 \forall v_2\ (v_1 < v_2 \vee v_1 = v_2)$ is a sentence since all variables are bound.

- $\phi_3(v_2) : \exists v_1 \ (v_1 + v_2) = 0$ is a formula where v_1 is bound and v_2 is free.

Now, we will define what it means for a structure to satisfy an $\mathcal{L}$-formula.

Definition 2.5. Let $\phi(\bar{v}) = \phi(v_1, v_2, ..., v_n)$ be a formula with free variables $v_1, v_2, ..., v_n$, let $\mathcal{A}$ be an $\mathcal{L}$-structure and let $\bar{a} = (a_1, a_2, ..., a_n) \in A^n$ be an n-tuple. We define $\mathcal{A} \models \phi(\bar{a})$ recursively as follows:

(i) If ϕ is $t_1 = t_2$, then $\mathcal{A} \models \phi(\bar{a})$ if $t_1^{\mathcal{A}}(\bar{a}) = t_2^{\mathcal{A}}(\bar{a})$.

(ii) If ϕ is $R(t_1, t_2, ..., t_{n_R})$, then $\mathcal{A} \models \phi(\bar{a})$ if $(t_1^{\mathcal{A}}(\bar{a}), t_2^{\mathcal{A}}(\bar{a}), .., .t_{n_R}^{\mathcal{A}}(\bar{a})) \in R^{\mathcal{A}}$.

(iii) If ϕ is $\neg\psi$, then $\mathcal{A} \models \phi(\bar{a})$ if $\mathcal{A} \not\models \psi(\bar{a})$.

(iv) If ϕ is $\psi \wedge \gamma$, then $\mathcal{A} \models \phi(\bar{a})$ if $\mathcal{A} \models \psi(\bar{a})$ and $\mathcal{A} \models \gamma(\bar{a})$.

(v) If ϕ is $\psi \vee \gamma$, then $\mathcal{A} \models \phi(\bar{a})$ if $\mathcal{A} \models \psi(\bar{a})$ or $\mathcal{A} \models \gamma(\bar{a})$.

(vi) If ϕ is $\exists v\, \psi$, then $\mathcal{A} \models \phi(\bar{a}, v)$ if there is $b \in A$ such that $\mathcal{A} \models \psi(\bar{a}, b)$.

(vii) If ϕ is $\forall v\, \psi$, then $\mathcal{A} \models \phi(\bar{a}, v)$ if $\mathcal{A} \models \psi(\bar{a}, b)$ for all $b \in A$.

$\mathcal{A} \models \phi(\bar{a})$ can be read as "$\phi(\bar{a})$ is *true* in $\mathcal{A}$" or "$\mathcal{A}$ *satisfies* $\phi(\bar{a})$".

Definition 2.6. Two formulas $\phi(\bar{v})$ and $\psi(\bar{v})$ are said to be *equivalent* if

$$\emptyset \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v})).$$

Remark 2.4. Note that $\forall v\, \psi$ and $\psi \vee \gamma$ can be omitted from the definition above since they are equivalent to $\neg\exists\neg\psi$ and $\neg(\neg\psi \wedge \neg\gamma)$, respectively. Also, we didn't include the symbols $\rightarrow$ and $\leftrightarrow$ from the beginning of the thesis, but we can also use these logical symbols since $(\psi \rightarrow \gamma)$ is equivalent to $(\neg\psi \wedge \gamma)$, and $(\psi \leftrightarrow \gamma)$ is equivalent to $(\neg\psi \wedge \gamma) \wedge (\neg\gamma \wedge \psi)$.

Example 2.6. Let $\mathcal{L} = \{+, \cdot, -, 0, 1, <\}$ be the language of ordered rings and let $\phi_1(v_1) : 0 < v_1$, $\phi_2 : \exists v_1 \forall v_2 \ (v_1 < v_2 \vee v_1 = v_2)$ and $\phi_3(v_2) : \exists v_1 \ (v_1 + v_2) = 0$ be as in the Example 2.5. Consider an $\mathcal{L}$-structure $\mathcal{A} = (\mathbb{Z}, +, \cdot, -, 0, 1, <)$. We see that $\mathcal{A} \models \phi_1(a)$ if $a > 0$ and $\mathcal{A} \not\models \phi_1(a)$ if $a \leq 0$; $\mathcal{A} \not\models \phi_2$ and $\mathcal{A} \models \phi_3(a)$ for all $a \in \mathbb{Z}$.

Remark 2.5. Sentences have truth values, in the structure they are either true or false. However, formulas with free variables do not have truth values, they may be true for specific elements of the structure and false for others.

Further Definitions: Theories and Models

In model theory, we sometimes take a set of $\mathcal{L}$-sentences, which is called as an $\mathcal{L}$-*theory*, and look at the $\mathcal{L}$-structures, which are called as *models*, that are satisfied by the theory. Conversely, we sometimes look at a collection of $\mathcal{L}$-structures and investigate the properties that they satisfy; that is, look at the $\mathcal{L}$-sentences that are satisfied by all of these structures and try to obtain a *theory* from them. Now, we define related terms.

Definition 2.7. A set of $\mathcal{L}$-sentences T is called as an $\mathcal{L}$-*theory*. Let T be an $\mathcal{L}$-theory and $\mathcal{A}$ be an $\mathcal{L}$-structure; $\mathcal{A}$ is said to be a *model* of T and denoted as $\mathcal{A} \models T$, if $\mathcal{A} \models \phi$ for all sentences $\phi \in T$.

Example 2.7. Let $\mathcal{L} = \{R\}$ be a language consisting of one binary relation symbol and consider the following $\mathcal{L}$-sentences.

$$\phi_1 : \forall v\ \neg R(v, v) \qquad \text{(R is irreflexive)},$$

$$\phi_2 : \forall v \forall w\ R(v, w) \to R(w, v) \qquad \text{(R is symmetric)}.$$

$T = \{\phi_1, \phi_2\}$ is an example of an $\mathcal{L}$-theory, which is called as *Theory of Graphs*. Any set together with an irreflexive, symmetric relation is a *model* of T. For example, the set $S = \{0, 1, 2\}$ together with the relation $R_1 = \{(0, 1), (1, 0), (1, 2), (2, 1), (0, 2), (2, 0)\}$ is an $\mathcal{L}$-structure that is a *model* of T. However, S together with the relation $R_2 = \{(0, 1), (1, 0), (1, 2), (2, 1), (2, 2)\}$ is an $\mathcal{L}$-structure which is *not* a model of T.

(S, R_1) *is a model of* T. $\qquad$ (S, R_2) *is a not a model of* T.

Figure 2.1. Illustrations of (S, R_1) and (S, R_2).

Any set of $\mathcal{L}$-sentences is a theory by definition; however, some theories may have no models. To illustrate, consider the theory

$$T = \{\phi_1 : \exists y\, \forall x\, [(x > y) \vee (x = y)], \phi_2 : \forall x\, \exists y\, (x > y)\}.$$

We see that T has no models since there is no $\mathcal{L}$-structure satisfying both ϕ_1 and ϕ_2. We give a special name for theories which have models:

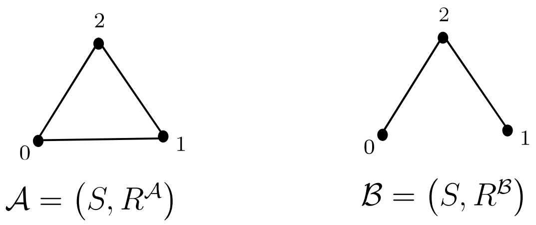
Definition 2.8. An $\mathcal{L}$-theory T is *satisfiable* if it has a model; that is, there is an $\mathcal{L}$-structure $\mathcal{A}$ such that $\mathcal{A} \models T$.

Example 2.8. The theory of graphs which is stated in Example 2.7 is satisfiable. A theory T containing both ϕ and $\neg\phi$ where ϕ is any sentence is not satisfiable.

Definition 2.9. Let T be an $\mathcal{L}$-theory and ϕ be an $\mathcal{L}$-sentence. ϕ is a *logical consequence* of T if whenever $\mathcal{A} \models T$, we have $\mathcal{A} \models \phi$. We write $T \models \phi$.

Definition 2.10. An $\mathcal{L}$-theory T is called *complete* if for any $\mathcal{L}$-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$.

Example 2.9. The theory of graphs which is stated in Example 2.7 is *not* complete. To illustrate, consider two models $\mathcal{A} = (S, R^{\mathcal{A}})$ and $\mathcal{B} = (S, R^{\mathcal{B}})$ of the theory where $S = \{0, 1, 2\}$, $R^{\mathcal{A}} = \{(0,1), (1,0), (1,2), (2,1), (0,2), (2,0)\}$ and $R^{\mathcal{B}} = \{(0,2), (2,0), (1,2), (2,1)\}$. Observe that there is a sentence $\phi : \forall v\, \forall w\ (v \neq w\ \rightarrow R(v,w))$ such that $\mathcal{A} \models \phi$ and $\mathcal{B} \models \neg\phi$. So neither $T \models \phi$ nor $T \models \neg\phi$ is true. Therefore, T is not complete.



$$\mathcal{A} \models \forall v\, \forall w\ (v \neq w\ \rightarrow R(v,w)).$$

$$\mathcal{B} \not\models \forall v\, \forall w\ (v \neq w\ \rightarrow R(v,w)).$$

Figure 2.2. Example 2.9.

Remark 2.6. We gave an example of a theory which is *not* complete. The reason why we don't give an example of a complete theory at this stage is because it is not possible to show that a theory is complete without using any other tools. Later, we will introduce the notion of categoricity and develop a way to show that a theory is complete by this notion and also show that the theory of dense linear orders without endpoints (DLO) is complete.

A way to construct a complete theory is collecting all sentences satisfied by a certain structure. Such theories are called as *theory of a model* and explicitly defined as follows:

Definition 2.11. For an $\mathcal{L}$-structure $\mathcal{A}$, the theory of $\mathcal{A}$ is the set of all sentences satisfied by $\mathcal{A}$ and denoted as

$$Th(\mathcal{A}) = \{\phi : \mathcal{A} \models \phi\}.$$

As it is stated above, we sometimes want to study the theory of a collection of $\mathcal{L}$-structures but it may not exist all the times.

Definition 2.12. A class $\mathcal{C}$ of $\mathcal{L}$-structures is called *axiomatizable* (or *elementary class*) if there is an $\mathcal{L}$-theory T such that $\mathcal{C} = \{\mathcal{A} : \mathcal{A} \models T\}$.

Example 2.10. In Example 2.7, we explicitly write the axioms of the theory of graphs, so the class of graphs is an elementary class. However, if we wish to axiomatize the theory of finite graphs, we can't succeed because the class of finite graphs is *not* elementary. (see Section 2.6 for details)

Until now, we worked with semantic notions such as models of theories, logical consequence and validity of sentences. Now, we also give definitions of syntactic notions such as *proof* and *consistency*. At the end, we see that syntactic and semantic notions are directly related by Completeness Theorem.

Definition 2.13. A *formal proof* of ϕ from a set of formulas T is a *finite* sequence of $\mathcal{L}$-formulas $\phi_1, \phi_2, ..., \phi_n$ such that $\phi_n = \phi$ and each ϕ_i for $i = 1, ..., n$; ϕ_i belongs to T or obtained from previous indexed formulas by applying *logical axioms* or *logical rules* (ex. modus ponens). We write $T \vdash \phi$ if there is a proof of ϕ from T. We will not elaborate on the axioms and logical rules here. A more detailed account of formal proofs can be found in [3, p. 14].

Definition 2.14. A set of $\mathcal{L}$-sentences T is called *inconsistent* if there exists an $\mathcal{L}$-formula ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$. Otherwise, T is called *consistent*.

Completeness and Compactness Theorems

Completeness and Compactness Theorem can be viewed as main theorems of first order logic. Completeness Theorem is first proved by Kurt Gödel in 1930, and states that whenever a sentence is logically followed by a theory, there is also a formal proof of that sentence from the theory. In other words, it states that deductive rules are rich enough to guarantee that every valid argument is deducible by logical rules. The converse of this statement is called as *soundness* of first order logic and it is also true; that is, if some formula is deducible from a set of sentences, then it is valid. Completeness theorem directly links the syntactic notion "*derivability*" and the semantic notion "*satisfiability*".

Theorem 2.1 (Completeness Theorem)**.** *Let T be an $\mathcal{L}$-theory and ϕ be an $\mathcal{L}$-sentence. Then*

$$T \models \phi \ \text{ if and only if } T \vdash \phi.$$

The proof of the Completeness Theorem can be found in [4, p. 61]. We have also the following corollary that relates satisfiability and consistency:

Corollary 2.2. *A theory T is satisfiable if and only if it is consistent.*

Proof. Suppose T is not satisfiable. Then, since there is no model of T every model of T is also a model of $\phi \wedge \neg\phi$; that is, $T \models (\phi \wedge \neg\phi)$. By Completeness Theorem, we have $T \vdash (\phi \wedge \neg\phi)$ which means that T is not consistent. Conversely, assume that T is inconsistent; that is, $T \vdash (\phi \wedge \neg\phi)$. By soundness we have $T \models (\phi \wedge \neg\phi)$ and since a sentence ϕ is either true or false in a structure, $(\phi \wedge \neg\phi)$ cannot be satisfied by a structure. Therefore, T is not satisfiable. □

After we have the Completeness Theorem, Compactness Theorem is a direct consequence of it; because it relates the notion of satisfiability with the notion of formal proofs and as a result the features they have are also shared. To be more precise, because of the fact that the proofs are finite, we also have every finite subset of a satisfiable theory is satisfiable.

Definition 2.15. An $\mathcal{L}$-theory T is called *finitely satisfiable* if every finite subset of T is satisfiable.

Theorem 2.3 (Compactness Theorem)**.** *Let T be an $\mathcal{L}$-theory. T is satisfiable if and only if T is finitely satisfiable.*

Proof. If T is satisfiable, then T is finitely satisfiable since models of T are also models of subsets of T. For the converse, suppose T is not satisfiable. By Corollary 2.2, T is inconsistent. So there is a proof Σ of a contradiction $(\phi \wedge \neg\phi)$ from T. Since proofs are finite, there is a finite subset Δ of T which is used in the proof Σ. Thus, Δ is inconsistent which implies that Δ is not satisfiable by Completeness Theorem. Therefore, not every finite subset of T is satisfiable. □

An application of the Compactness Theorem: If an $\mathcal{L}$-theory T has arbitrarily large finite models, than it has an infinite model.

Proof. Let T be an $\mathcal{L}$-theory and consider the following theory

$$T' = T \cup \{\phi_n : n = 1, 2, 3, ...\}$$

where $\phi_n = \exists x_1 \exists x_2 \ ... \ \exists x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right)$ each stating that there are at least n elements. Clearly, models of T' are infinite models of T. Hence we need to show that T' is satisfiable and actually by compactness, it is enough to show that T' is finitely satisfiable. So let F be a finite subset of T'. Since finitely many ϕ_n's are included in F, take the maximum index m such that $\phi_m \in F$ and $\phi_i \notin F$ for $i \geq m$. We have $F \subseteq T \cup \{\phi_n : i = 1, 2, ..., m\}$ which is satisfiable since T has arbitrarily large finite models. Thus, T' is finitely satisfiable. □

2.2. Relations between $\mathcal{L}$-Structures

In this section, we look at relations between $\mathcal{L}$-structures and define related terms. When we study algebraic structures such as groups, rings etc., to classify such structures we look at the maps between them, that preserve structural properties; that is, we look at homomorphisms and isomorphisms between these algebraic structures. Since we are working with arbitrary structures in model theory, we expand definitions of embeddings, isomorphisms to $\mathcal{L}$-structures and define $\mathcal{L}$-*embeddings* and $\mathcal{L}$-*isomorphisms* as maps preserving structural properties; that is, preserving interpretations of relation, function and constant symbols. But first of all, we define the basic relation between two algebraic structures that is called as *elementary equivalence*.

Definition 2.16. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures. $\mathcal{A}$ and $\mathcal{B}$ are said to be *elementarily equivalent*, denoted as $\mathcal{A} \equiv \mathcal{B}$, if for any $\mathcal{L}$-sentence ϕ we have

$$\mathcal{A} \models \phi \text{ if and only if } \mathcal{B} \models \phi.$$

Remark 2.7. $\mathcal{A} \equiv \mathcal{B}$ if and only if $Th(\mathcal{A}) = Th(\mathcal{B})$. This is just a restatement of the definition because $\mathcal{A} \models \phi$ means that $\phi \in Th(\mathcal{A})$. So $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if for any $\mathcal{L}$-sentences ϕ, we have $\phi \in Th(\mathcal{A})$ if and only if $\phi \in Th(\mathcal{B})$; that is, if $Th(\mathcal{A}) = Th(\mathcal{B})$.

Example 2.11. It is not easy to show that two structures are elementarily equivalent without using other tools since, one needs to check all sentences, so we start by giving some *non-examples*. Later on, we will see in Proposition 2.1 that models of complete theories are elementarily equivalent and in Example 2.9 we see that the theory of dense linear orders without endpoints (DLO) is complete, so by using these two we can say that two models of this theory are elementariliy equivalent. As an example,

$$(\mathbb{Q}, <) \equiv (\mathbb{R}, <).$$

Some non-examples:

i. $\mathbb{Z} \not\equiv \mathbb{Z} \oplus \mathbb{Z}$: Integers and direct sum of two copies of integers are not elementarily equivalent as additive groups in language $\mathcal{L} = \{+, 0\}$. Being an "even" number is definable in language of groups by the formula $\psi(v) : \exists w \ (w + w = v)$. Note that for any two integers x and y either one of them is even or $x + y$ is even. We can express this property in first order as

$$\phi : \forall v_1 \forall v_2 \exists w_1 \exists w_2 \exists w_3 (v_1 = w_1 + w_1 \lor v_2 = w_2 + w_2 \lor v_1 + v_2 = w_3 + w_3).$$

However, ϕ does not hold in $\mathbb{Z} \oplus \mathbb{Z}$. Consider two elements $(0,1)$ and $(1,0)$ of $\mathbb{Z} \oplus \mathbb{Z}$. Clearly, neither of them is even and their sum is not even either. Since $\mathbb{Z} \oplus \mathbb{Z} \not\models \phi$ and $\mathbb{Z} \models \phi$ we get the result.

ii. $(\mathbb{Q}, +, \cdot, 0, 1) \not\equiv (\mathbb{R}, +, \cdot, 0, 1)$: Rationals and reals are not elemetarily equivalent. Indeed, consider the sentence $\exists v\,(v \cdot v = 1 + 1)$ in language of fields. We have $(\mathbb{Q}, +, \cdot, 0, 1) \not\models \exists v\,(v \cdot v = 1 + 1)$ whereas $(\mathbb{R}, +, \cdot, 0, 1) \models \exists v\,(v \cdot v = 1 + 1)$.

Proposition 2.1. *If $\mathcal{A}$ and $\mathcal{B}$ are models of a complete theory T, then $\mathcal{A} \equiv \mathcal{B}$.*

Proof. Let T be a complete theory and let $\mathcal{A} \models T$. First, observe that $T \subseteq Th(\mathcal{A})$ since $\mathcal{A} \models T$. To show that $T = Th(\mathcal{A})$, assume for a contradiction there is a sentence $\phi \in Th(\mathcal{A})$ and $\phi \notin T$. However, since T is complete we would have $\neg\phi \in T$, which is a contradiction since $\mathcal{A} \models T$ and $\mathcal{A} \models \phi$. So for any two models $\mathcal{A}$ and $\mathcal{B}$ of T, we have $T = Th(\mathcal{A}) = Th(\mathcal{B})$. Therefore, we obtain $\mathcal{A} \equiv \mathcal{B}$ by Remark 2.7 since their theories are the same. □

Maps between $\mathcal{L}$-Structures

Definition 2.17. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures. A map $\sigma : A \to B$ from the universes of $\mathcal{A}$ to universe of $\mathcal{B}$ is called as an *$\mathcal{L}$-embedding* if the following three condition are satisfied:

(i.) For all function symbol $f \in \mathcal{L}$ with arity n_f and for all tuple $\bar{a} = (a_1, a_2, ..., a_{n_f}) \in A^{n_f}$, we have $\sigma(f^{\mathcal{A}}(\bar{a})) = f^{\mathcal{B}}(\sigma(a_1), \ldots, \sigma(a_{n_f}))$.

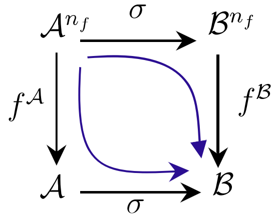


Figure 2.3. Illustration of $\sigma(f^{\mathcal{A}}(\bar{a})) = f^{\mathcal{B}}(\sigma(a_1), \ldots, \sigma(a_{n_f}))$.

(ii.) For all relation symbol $R \in \mathcal{L}$ with arity n_R and for all tuple $\bar{a} = (a_1, a_2, ..., a_{n_R}) \in A^{n_R}$, we have

$$\bar{a} \in R^{\mathcal{A}} \Leftrightarrow (\sigma(a_1), \ldots, \sigma(a_{n_R})) \in R^{\mathcal{B}}.$$

(iii.) For all constant symbol $c \in \mathcal{L}$, we have $\sigma(c^{\mathcal{A}}) = c^{\mathcal{B}}$.

In other words, an $\mathcal{L}$-embedding σ is an injective map that preserves interpretations of all symbols in a language $\mathcal{L}$. If σ is also surjective, then it is called as an $\mathcal{L}$-*isomorphism*. We denote isomorphic structures as $\mathcal{A} \cong \mathcal{B}$. An isomorphism from $\mathcal{A}$ to itself is called as an $\mathcal{L}$-*automorphism*.

Remark 2.8. Elementary equivalence is a notion that generalizes concepts of being isomorphic. Later, we will see that being isomorphic implies being elementarily equivalent. (Theorem 2.4.)

Example 2.12. (i.) Let $\mathcal{L} = \emptyset$ be the empty language. According to this language all maps are $\mathcal{L}$-homomorphisms, all injections are $\mathcal{L}$-embeddings and all bijections are $\mathcal{L}$-isomorphisms.

(ii.) Let $\mathcal{L} = \{f, c\}$ be a language consisting of one binary function symbol and one constant symbol. Consider the structures

$$\mathcal{A} = (\mathbb{N}, +, 0), \qquad \mathcal{B} = (\mathbb{Z}, +, 0), \qquad \mathcal{C} = (\mathbb{R}, \cdot, 1),$$

where f is interpreted as '+' (usual addition) in $\mathcal{A}$ and $\mathcal{B}$ and as '$\cdot$' (usual multiplication) in $\mathcal{C}$; and c is interpreted as identity element, respectively. Let σ_1 be the map defined by $\sigma_1(x) = x$. Clearly, σ_1 is an $\mathcal{L}$-embedding from $\mathcal{A}$ to $\mathcal{B}$, but it is not an $\mathcal{L}$-embedding from $\mathcal{B}$ to $\mathcal{C}$ and $\mathcal{A}$ to $\mathcal{C}$ since interpretation of constant symbol is not preserved under these maps.

Now, consider the map σ_2 defined by $\sigma_2(x) = e^x$. It is an $\mathcal{L}$-embedding from $\mathcal{B}$ to $\mathcal{C}$ and $\mathcal{A}$ to $\mathcal{C}$.

Definition 2.18. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures such that $A \subseteq B$. $\mathcal{A}$ is called as a *substructure* of $\mathcal{B}$ or $\mathcal{B}$ is called as an *extension* of $\mathcal{A}$ if the inclusion map $\iota : A \to B$ defined by $\iota(a) = a$ for all $a \in A$ is an $\mathcal{L}$-embedding. We denote it as $\mathcal{A} \subseteq B$.

Example 2.13. Let $\mathcal{L} = \{*, e\}$ be the language of groups. Consider the $\mathcal{L}$-structures $\mathcal{A} = (\mathbb{R}, +, 0)$, $\mathcal{B} = (\mathbb{Q}, +, 0)$ and $\mathcal{C} = (\mathbb{Q}, \cdot, 1)$ where '+' is usual addition and '$\cdot$' is usual multiplication. $\mathcal{B}$ is a substructure of $\mathcal{A}$ since $\mathbb{Q} \subseteq \mathbb{R}$ and the inclusion map is an embedding, but $\mathcal{C}$ is not a substructure of $\mathcal{A}$, although $\mathbb{Q} \subseteq \mathbb{R}$, the inclusion map does not preserve interpretations of constant and function symbols.

Remark 2.9. If there is an $\mathcal{L}$-embedding from an $\mathcal{L}$-structure $\mathcal{A}$ into an $\mathcal{L}$-structure $\mathcal{B}$, we can view $\mathcal{A}$ as a substructure of $\mathcal{B}$ since $\mathcal{B}$ contains an isomorphic copy of $\mathcal{A}$ as a substructure.

Definition 2.19. $\mathcal{A}$ is called as an *elementary substructure* of $\mathcal{B}$, denoted as $\mathcal{A} \preccurlyeq \mathcal{B}$, if for any $\mathcal{L}$-formula $\phi(\bar{v})$ and for any $\bar{a} \in A^n$, we have

$$\mathcal{A} \models \phi(\bar{a}) \text{ if and only if } \mathcal{B} \models \phi(\bar{a}).$$

Example 2.14. (i.) Let $\mathcal{L} = \{+, \cdot, -, 0, 1\}$ be the language of rings. $(\mathbb{R}, +, \cdot, -, 0, 1)$ is a substructure of $(\mathbb{C}, +, \cdot, -, 0, 1)$, but it is *not* an elementary substructure. More precisely, let $\phi(v)$ be the $\mathcal{L}$-formula $\exists w\ (w \cdot w = v)$.
Then, $(\mathbb{C}, +, \cdot, -, 0, 1) \models \phi(-1)$ but $(\mathbb{R}, +, \cdot, -, 0, 1) \not\models \phi(-1)$.

(ii.) Let $\mathcal{L} = \{<\}$ be the language of orders. $(\mathbb{N}^+, <)$ is a substructure of $(\mathbb{N}, <)$ but not elementary. Consider the formula $\phi(v) : \forall w\ (v < w)$. Clearly $\phi(1)$ is true in $(\mathbb{N}^+, <)$; however, it is not true in $(\mathbb{N}, <)$. Hence, $(\mathbb{N}^+, <)$ is not an elementary substructure of $(\mathbb{N}, <)$. Observe that the structures are even isomorphic by the map sending n to $n + 1$ (it is order preserving one to one and onto function); however, the substructure relation is not elementary.

(iii.) Let $\mathcal{L} = \{*, e\}$ be the language of groups. $(2\mathbb{N}, +, 0)$ is a substructure of $(\mathbb{N}, +, 0)$ but it is not an elementary substructure. Consider the formula
$\phi(2) : \exists v\ (v + v = 2)$ which is true in $(\mathbb{N}, +, 0, <)$ but not true in $(2\mathbb{N}, +, 0, <)$.

Recursive definition of formulas enables us to apply structural induction on the length on formulas, which is a useful tool for proofs in mathematical logic. The following proposition states that substructures are preserved under quantifier-free formulas and the proof uses the method of induction.

Proposition 2.2 ([1, p.11]). *Let $\mathcal{A}$ be a substructure of $\mathcal{B}$, $a \in A^n$ and let $\phi(\bar{v})$ be a quantifier free formula. Then $\mathcal{A} \models \phi(\bar{a})$ if and only if $\mathcal{B} \models \phi(\bar{a})$.*

Proof. The proof is based on induction on the length of formulas. So first we will show that for any term $t(\bar{v})$ and any $\bar{a} \in A^n$, we have $t^{\mathcal{A}}(\bar{a}) = t^{\mathcal{B}}(\bar{a})$ by induction on length of terms.

- If t is a constant symbol, then clearly $t^{\mathcal{A}}(\bar{a}) = c^{\mathcal{A}} = c^{\mathcal{B}} = t^{\mathcal{B}}(\bar{a})$ since $\mathcal{A} \subseteq B$.
- If t is the variable v_i, then $t^{\mathcal{A}}(\bar{a}) = a_i = t^{\mathcal{B}}(\bar{a})$.
- Now assume that t is a function symbol f and $t_1, t_2, ...t_{n_f}$ are terms such that $t_i^{\mathcal{A}}(\bar{a}) = t_i^{\mathcal{B}}(\bar{a})$ for $i = 1, 2, ..., n_f$, then

$$\begin{aligned} t^{\mathcal{A}}(\bar{a}) &= f^{\mathcal{A}}(t_1^{\mathcal{A}}(\bar{a}), t_2^{\mathcal{A}}(\bar{a}), ...t_{n_f}^{\mathcal{A}}(\bar{a})) \\ &= f^{\mathcal{B}}(t_1^{\mathcal{A}}(\bar{a}), t_2^{\mathcal{A}}(\bar{a}), ...t_{n_f}^{\mathcal{A}}(\bar{a})) \quad \textit{since } \mathcal{A} \subseteq B \\ &= f^{\mathcal{B}}(t_1^{\mathcal{B}}(\bar{a}), t_2^{\mathcal{B}}(\bar{a}), ...t_{n_f}^{\mathcal{B}}(\bar{a})) \\ &= t^{\mathcal{B}}(\bar{a}). \end{aligned}$$

Now, we can do induction on formulas to prove the proposition. Let $\phi(\bar{v})$ be a formula,

- If $\phi(\bar{v})$ is $t_1 = t_2$, then

$$\mathcal{A} \models \phi(\bar{v}) \Leftrightarrow t_1^{\mathcal{A}}(\bar{a}) = t_2^{\mathcal{A}}(\bar{a}) \Leftrightarrow t_1^{\mathcal{B}}(\bar{a}) = t_2^{\mathcal{B}}(\bar{a}) \Leftrightarrow \mathcal{B} \models \phi(\bar{v}).$$

- If $\phi(\bar{v})$ is $R(t_1, t_2, ..., t_{n_R})$, then

$$\begin{aligned} \mathcal{A} \models \phi(\bar{v}) &\Leftrightarrow (t_1^{\mathcal{A}}(\bar{a}), t_2^{\mathcal{A}}(\bar{a}), ..., t_{n_R}^{\mathcal{A}}(\bar{a})) \in R^{\mathcal{A}} \\ &\Leftrightarrow (t_1^{\mathcal{A}}(\bar{a}), t_2^{\mathcal{A}}(\bar{a}), ..., t_{n_R}^{\mathcal{A}}(\bar{a})) \in R^{\mathcal{B}} \quad \textit{since } \mathcal{A} \subseteq B \\ &\Leftrightarrow (t_1^{\mathcal{B}}(\bar{a}), t_2^{\mathcal{B}}(\bar{a}), ..., t_{n_R}^{\mathcal{B}}(\bar{a})) \in R^{\mathcal{B}} \\ &\Leftrightarrow \mathcal{B} \models \phi(\bar{v}). \end{aligned}$$

So atomic formulas satisfies the proposition. Further, we check other longer formulas:

- If the proposition is true for $\psi(\bar{v})$ and $\phi(\bar{v}) = \neg\psi(\bar{v})$, then

$$\mathcal{A} \models \phi(\bar{v}) \Leftrightarrow \mathcal{A} \not\models \psi(\bar{v}) \Leftrightarrow \mathcal{B} \not\models \psi(\bar{v}) \Leftrightarrow \mathcal{B} \models \phi(\bar{v}).$$

- If the proposition is true for $\psi(\bar{v})$ and $\theta(\bar{v})$ and if $\phi(\bar{v}) = \psi(\bar{v}) \wedge \theta(\bar{v})$, then

$$\mathcal{A} \models \phi(\bar{v}) \Leftrightarrow \mathcal{A} \models \psi(\bar{v}) \text{ and } \mathcal{A} \models \theta(\bar{v}) \Leftrightarrow \mathcal{B} \models \psi(\bar{v}) \text{ and } \mathcal{B} \models \theta(\bar{v}) \Leftrightarrow \mathcal{B} \models \phi(\bar{v}).$$

We showed that proposition is true for all atomic formulas and if it is true for ψ and θ, then it is true for $\neg\psi$ and $\psi \wedge \theta$. Since the set quantifier free formulas consists of atomic formulas, negation and conjunction quantifier free formulas, the proposition holds for all quantifier free formulas. □

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures and let $\sigma : \mathcal{A} \to \mathcal{B}$ be an $\mathcal{L}$-isomorphism. For any $\mathcal{L}$-formula $\phi(\bar{v})$ and for any tuple $\bar{a} \in A^n$, we have*

$$\mathcal{A} \models \phi(\bar{a}) \quad \text{if and only if} \quad \mathcal{B} \models \phi(\sigma(\bar{a})).$$

In particular, if there is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then we have $\mathcal{A} \equiv \mathcal{B}$.

Proof. The proof of Theorem 2.4 can be found in [1, p. 13]; it depends on induction on the length of formulas similar to the proof of Proposition 2.2. □

Elementary embedding

Definition 2.20. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures. A map $\sigma : \mathcal{A} \to \mathcal{B}$ is called an *elementary embedding* if for all $\mathcal{L}$-formulas $\phi(\bar{v})$ and for all tuple $\bar{a} \in A^n$, we have

$$\mathcal{A} \models \phi(\bar{a}) \quad \text{if and only if} \quad \mathcal{B} \models \phi(\sigma(\bar{a})).$$

Remark 2.10. Isomorphisms are *elementary embeddings* by Theorem 2.4.

Remark 2.11. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures satisfying $\mathcal{A} \subseteq \mathcal{B}$. We have $\mathcal{A} \preccurlyeq \mathcal{B}$ if the inclusion map is an elementary embedding.

An equivalent condition of being an elementary substructure that is stated and proved by Tarski and Vaught is presented as the following theorem. To see that an extension is elementary, it is enough to look at formulas starting with existential quantifiers with parameters from the substructure that is satisfied above and check that if they are also satisfied in the substructure. A similar statement is also presented as Robinson's Test 2.17 in Section 2.4.

Theorem 2.5 (Tarski-Vaught Test)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures such that $\mathcal{A} \subseteq \mathcal{B}$. $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if for any $\mathcal{L}$-formula $\phi(v, \bar{w})$ and for any $\bar{a} \in A^n$; whenever there is $b \in B$ such that $\mathcal{B} \models \phi(b, \bar{a})$, there is $c \in A$ such that $\mathcal{B} \models \phi(c, \bar{a})$.*

Proof. Left to right implication is clear by the definition of elementary substructure. To prove the converse, suppose right handside holds and we will show that $\mathcal{A} \preccurlyeq \mathcal{B}$. We do induction on length of formulas. Previously, we showed in Proposition 2.2 that for any quantifier free formula $\phi(\bar{v})$ and for any $\bar{a} \in A^n$, we have $\mathcal{B} \models \phi(\bar{a}) \leftrightarrow \mathcal{A} \models \phi(\bar{a})$ by using induction on length of formulas. So we just need to prove the existential case to complete the induction since universal formulas are negations of existential formulas. Let $\phi(\bar{a})$ be the existential formula $\exists v\, \psi(v, \bar{a})$. Assume $\mathcal{B} \models \exists v\, \psi(v, \bar{a})$. Then, it means $\mathcal{B} \models \psi(b, \bar{a})$ for some $b \in B$. By assumption of the theorem, we have $\mathcal{A} \models \psi(c, \bar{a})$ for some $c \in A$. Hence, $\mathcal{A} \models \exists v\, \psi(v, \bar{a})$. Conversely, assume $\mathcal{A} \models \exists v\, \psi(v, \bar{a})$, then $\mathcal{A} \models \psi(c, \bar{a})$ for some $c \in A$. By induction, $\mathcal{B} \models \psi(c, \bar{a})$ where $c \in A \subseteq B$ and hence $\mathcal{B} \models \exists v\, \psi(v, \bar{a})$. Therefore, we obtain $\mathcal{B} \models \phi(\bar{a}) \leftrightarrow \mathcal{A} \models \phi(\bar{a})$ for every $\mathcal{L}$-formula $\phi(\bar{v})$ and for all $\bar{a} \in A$ by induction. Hence, $\mathcal{A} \preccurlyeq \mathcal{B}$. □

Definition 2.21. Let $\mathcal{L}$ be a language and let $\mathcal{A}$ be an $\mathcal{L}$-structure. Expand the language $\mathcal{L}$ by adding new constant symbols for each element of $\mathcal{A}$ and call this new language as $\mathcal{L}_A$. We define the *diagram* of the $\mathcal{L}$-structure $\mathcal{A}$ as

$$Diag(\mathcal{A}) = \{\phi(a_1, ..., a_n) : \phi \text{ is atomic or negated atomic } \mathcal{L}\text{-formula} \\ \text{and } \mathcal{A} \models \phi(a_1, ..., a_n)\}.$$

Lemma 2.6 (Diagram Lemma)**.** *Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let $\mathcal{B}$ be $\mathcal{L}_A$-structure (which is naturally an $\mathcal{L}$-structure as well) such that $\mathcal{B} \models Diag(\mathcal{A})$. Then, there exists an $\mathcal{L}$-embedding from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Let $\phi : A \to B$ be a function defined as $\phi(a) = a^{\mathcal{B}}$. We will show that ϕ is an $\mathcal{L}$-embedding of $\mathcal{A}$ into $\mathcal{B}$:

- Let f be a function symbol. We want to show that $\phi(f^A(\bar{a})) = f^B(\phi(\bar{a}))$. If $f^{\mathcal{A}}(\bar{a}) = a_0$, then $f(\bar{a}) = a_0 \in Diag(\mathcal{A})$. Thus, we have $f^{\mathcal{B}}(\bar{a}^{\mathcal{B}}) = a_0^{\mathcal{B}}$ and
$$f^{\mathcal{B}}(\phi(\bar{a})) = f^{\mathcal{B}}(\bar{a}^{\mathcal{B}}) = a_0^{\mathcal{B}} = \phi(a_0) = \phi(f^{\mathcal{A}}(\bar{a})).$$
- Let R be a relation symbol. If $R^{\mathcal{A}}(\bar{a})$, then $R(\bar{a}) \in Diag(\mathcal{A})$ and hence $R^{\mathcal{B}}(\bar{a}^{\mathcal{B}}) = R^{\mathcal{B}}(\phi(\bar{a}))$.
- Let c be a constant symbol. If $c^A = a$ then $c = a \in Diag(\mathcal{A})$. So we have $c^{\mathcal{B}} = a^{\mathcal{B}} = \phi(a) = \phi(c^{\mathcal{A}})$. Hence, $\phi(c^{\mathcal{A}}) = c^{\mathcal{B}}$.
- ϕ is one to one: If a_1 and a_2 are distinct members of A, then $a_1 \neq a_2 \in Diag(\mathcal{A})$ which implies $\phi(a_1) = a_1^{\mathcal{B}} \neq a_2^{\mathcal{B}} = \phi(a_2)$. Hence, ϕ is one to one. □

Categoricity

A theory is named as *categorical* by Oswald Veblen if it has one model up to isomorphism, in 1904 [7]. However, any theory with infinite models is *not* categorical (see Theorem 2.7, since the theory has infinite models it has models of every infinite cardinality $\kappa > |\mathcal{L}|$). Thus, a weaker version of it is defined as follows: a theory T is called κ-categorical for some infinite cardinal κ if it has one model of cardinality κ up to isomorphism. Categoricity is a tool to show that a theory is complete. In 1954, Łos and Vaught independently showed that a satisfiable $\mathcal{L}$-theory with no finite models which is categorical for some infinite cardinal $\kappa > |\mathcal{L}|$ is complete. We present this statement as Vaught's Test in Theorem 2.8.

Definition 2.22. Let κ be an infinite cardinal and let T be an $\mathcal{L}$-theory that has models of size κ. T is called *κ-categorical* if any two models of T of cardinality κ are isomorphic.

Example 2.15. The theory of dense linear orders without endpoints (DLO) in language $\mathcal{L} = \{<\}$ is $\aleph_0$-categorical. Let $(A, <)$ and $(B, <)$ be two countable models of the theory and let $\{a_i : i \in \mathbb{N}\}$ and $\{b_i : i \in \mathbb{N}\}$ be enumerations of elements of these structures, respectively. We construct an isomorphism between them by using an argument called *back and forth*. We will construct partial $\mathcal{L}$-isomorphisms f_i between subsets A_i of A and B_i of B in such a way $\bigcup A_i = A$, $\bigcup B_i = B$ and $f = \bigcup f_i$ wil be the desired $\mathcal{L}$-isomorphism. At odd stages we will guarantee that $a_n \in A_{2n+1}$ and at even steps we will guarantee that $b_n \in B_{2n}$ for all $n \geq 0$ to ensure $\bigcup A_i = A$ and $\bigcup B_i = B$. Since f_i is an $\mathcal{L}$-isomorphism between A_i and B_i, we should have for all $\alpha, \beta \in A_i$, $\alpha < \beta$ if and only if $f_i(\alpha) < f_i(\beta)$. We build these partial bijections by following the steps below.

Step 0. Let $A_0 = B_0 = \emptyset$ and also $f_0 = \emptyset$.

Step n. (n is odd, n=2m+1) In such odd steps, we will guarantee that $a_m \in A_n$. If $a_m \in A_{n-1}$, there is nothing to do just let $A_n = A_{n-1}$, $B_n = B_{n-1}$ and $f_n = f_{n-1}$. If $a_m \notin A_{n-1}$, then we need to find $b \in B \setminus B_{n-1}$ such that for all $a_i \in A_n$ we have $a_i < a_m$ if and only if $f_n(a_i) < b$. Actually, we have three possibilities for the position of a_m relative to elements of A_{n-1}:

i. $a_m < a$ for all $a \in A_{n-1}$ (a_m is less than all elements of A_{n-1}). In this case, choose some $b \in B$ be such that $b_i < b$ for all $b_i \in B_{n-1}$, such element exists since there is no endpoint and B_{n-1} is finite.

ii. There is $a_i, a_j \in A_{n-1}$ such that no element of A_n lies between a_i and a_j, and $a_i < a_m < a_j$ (a_m is in between elements of $\mathcal{A}_n$). So chose some $b \in B$ be satisfying $f_{n-1}(a_i) < b < f_{n-1}(a_j)$. Such element exists since $\mathcal{B}$ is dense.

iii. $a < a_m$ for all $a \in A_{n-1}$ (a_m is greater than all elements of A_{n-1}). In this case, choose some $b \in B$ be such that $b < b_i$ for all $b_i \in B_{n-1}$, such element exists since there is no endpoint and B_{n-1} is finite.

So in each case let $A_n = A_{n-1} \cup \{a_m\}$, $B_n = B_{n-1} \cup \{b\}$ and expand f_{n-1} by letting $f_n(a_m) = b$.

Step n. (n is even, n=2m) In such even steps, we will guarantee that $b_m \in B_n$. Again if $b_m \in B_{n-1}$, nothing to do just let $A_n = A_{n-1}$, $B_n = B_{n-1}$ and $f_n = f_{n-1}$. If $b_m \notin B_{n-1}$, then we need to find $a \in A \setminus A_{n-1}$ such that for all $a_i \in A_{n-1}$ we have

$a < a_i$ if and only if $b_m < f_{n-1}(a_i)$. So we can let $f_n(a) = b_m$. By doing the same argument as in odd stages, we see that such $a \in A$ exists. So let $A_n = A_{n-1} \cup \{a\}$, $B_n = B_{n-1} \cup \{b_m\}$ and expand f_{n-1} by letting $f_n(a) = b_m$.

Hence, we obtained an $\mathcal{L}$-isomorphism $f = \bigcup f_i$ from $A = \bigcup A_i$ to $B = \bigcup B_i$ since all f_i's are partial $\mathcal{L}$-isomorphisms.

Example 2.16. The theory of dense linear orders without endpoints is *not* $\aleph_1$-categorical. To illustrate, let $\mathcal{A} = (\mathbb{R}, <)$ and $\mathcal{B} = ((-\infty, 0] \cup ([0,1] \cap \mathbb{Q}) \cup [1, \infty), <)$ be two models of the theory, both with cardinality $\aleph_1$. However, they are not isomorphic. Suppose, for a contradiction, there is an $\mathcal{L}$-isomorphism f from $\mathcal{A}$ to $\mathcal{B}$. So there is $a_1, a_2 \in \mathcal{A}$, $a_1 < a_2$ such that $f(a_1) = 0$ and $f(a_2) = 1$.

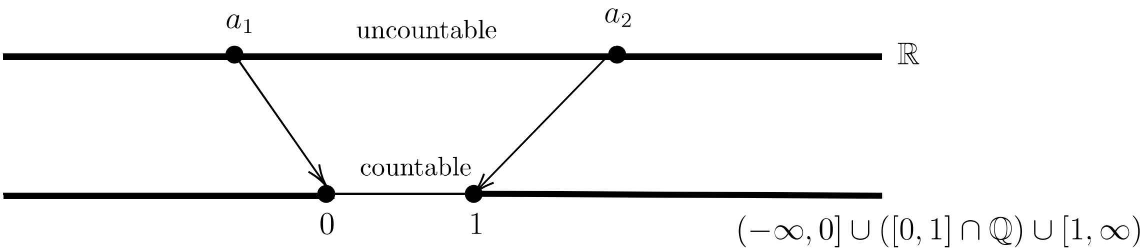


Figure 2.4. Mapping $\mathcal{A}$ and $\mathcal{B}$.

But this leads a contradiction since the interval $[a_1, a_2]$ is uncountable but it has to map to the countable interval $[0,1] \cap \mathbb{Q}$ under the order preserving map, so they cannot be isomorphic.

To prove Vaught's Test, we need to first present the following theorem.

Theorem 2.7. *Let T be an $\mathcal{L}$-theory and let κ be an infinite cardinal such that $\kappa \geq |\mathcal{L}|$. We have the following:*

i. If T is finitely satisfiable, then T has a model of cardinality at most κ.
ii. If T has infinite models, then a model of T with cardinality κ exists.

Proof. i. For the proof, see [1, p. 38, Theorem 2.1.11]. In the proof, a model is constructed by adjoining new constant symbols to the language $\mathcal{L}$ so that we can label all elements of the language by a constant symbol. This method of constructing models is called as *Henkin Construction* which is a useful method that is also used in the proof of Compactness Theorem (without using Completeness Theorem).

ii. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i : i \in I\}$ be a new language obtained by adding κ many new constant symbols to $\mathcal{L}$, where I is an index set of cardinality κ. Also, let $T^* = T \cup \{c_i \neq c_j : i, j \in I;\ i \neq j\}$ be an $\mathcal{L}^*$-theory. Clearly, any model of T^* is a model of T of cardinality at least κ. We need to show that T^* is finitely satisfiable since by showing this we can conclude that it has a model of cardinality κ by part ($i.$). So let J be a finite subset of I and let $\Delta \subseteq T \cup \{c_i \neq c_j : i, j \in J;\ i \neq j\} \subseteq T^*$ be a finite subset of T^*. An infinite model $\mathcal{A}$ of T is clearly a model of Δ since we can interpret each constant symbol c_j for all $j \in J$ with different elements of $\mathcal{A}$. Since T does not contain new constant symbols there is no problem with doing this and since $\mathcal{A}$ is infinite, there are enough elements.

Therefore, T^* is finitely satisfiable and we get the result. □

Theorem 2.8 (Vaught's Test). *Let T be a satisfiable $\mathcal{L}$-theory with no finite models. If T is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$, then T is complete .*

Proof. Assume T is *not* complete. So there is a sentence ϕ such that $T \not\models \phi$ and $T \not\models \neg\phi$. Observe that $T \cup \{\neg\phi\}$ is satisfiable since $T \not\models \phi$ and $T \cup \{\phi\}$ is satisfiable since $T \not\models \neg\phi$. Since T has no finite models, $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ have infinite models and thus by Lemma 2.7, they have models of cardinality κ for any infinite cardinal $\kappa \geq |\mathcal{L}|$. Let $\kappa \geq |\mathcal{L}|$ be an infinite cardinal and let $\mathcal{A} \models T \cup \{\phi\}$ and $\mathcal{B} \models T \cup \{\neg\phi\}$ such that $|\mathcal{A}| = |\mathcal{B}| = \kappa$. We have $\mathcal{A} \not\equiv \mathcal{B}$ since $\mathcal{A} \models \phi$ but $\mathcal{B} \models \neg\phi$. Thus, $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic since they are even not elementariliy equivalent. Therefore, T is not κ-categorical for any infinite cardinal κ. □

Corollary 2.9. *The theory of dense linear orders without endpoints (DLO) is complete by Vaught's Test since it is $\aleph_0$-categorical and has no finite models.*

2.3. Definable Sets and Quantifier Elimination

Given a structure, we study certain subsets of the universe that are called as *definable sets* in order to get information about the structure. More precisely, we study such subsets consisting of elements satisfying a common property; that is, satisfying a common first order formula. Definable sets can also be seen as definable relations of the structure and the study of definable sets are important to understand the structure. The formal definition is as follows.

Definition 2.23. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let X be a subset of A^n. X is called *definable* if there is an $\mathcal{L}$-formula $\phi(v_1, ..., v_n, w_1, ..., w_m)$ and $\bar{b} \in A^m$ such that

$$X = \{\bar{a} \in A^n : \mathcal{A} \models \phi(\bar{a}, \bar{b})\}.$$

We say X is defined by the formula $\phi(\bar{v}, \bar{b})$.

$\bar{b} \in A^m$ is called as parameters of the formula $\phi(\bar{v}, \bar{b})$. If the parameters come from a subset B of A; that is, $\bar{b} \in B^m$ where $B \subseteq A$, then X is called B-definable. If no parameters are used in formula $\phi(\bar{v})$, then it is called $\emptyset$-definable.

Example 2.17.

i. Finite Sets are definable in *any* structure. Indeed, let $\mathcal{A}$ be an $\mathcal{L}$-structure and let $X = \{a_1, a_2, ..., a_n\} \subseteq A$. The formula $\phi(v) : \bigvee_{i=1}^{n}(v = a_n)$ defines X.

ii. Intervals are definable in a linear order. For example, $\mathcal{L} = \{<\}$ and let $(\mathbb{R}, <)$ be an $\mathcal{L}$-structure where $<$ is usual order relation defined on Real Numbers.
For $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha < \beta$,

$$(\alpha, \beta) = \{a \in \mathbb{R} : \alpha < a \wedge a < \beta\} \text{ defined by the formula } \phi(v) : \alpha < v \wedge v < \beta,$$

$$(-\infty, \alpha) = \{a \in \mathbb{R} : a < \alpha\} \text{ defined by the formula } \phi(v) : v < \alpha \ (\{\alpha\}\textit{-definable}),$$

$$(\beta, \infty) = \{a \in \mathbb{R} : \beta < a\} \text{ defined by the formula } \phi(v) : \beta < v \ (\{\beta\}\textit{-definable}).$$

iii. Let $(\mathbb{R}, +, \cdot, -, 0, 1)$ be Real Number Field in the language $\mathcal{L}$ of rings.
   - Algebraic Curves $p(x, y) = 0$ are definable by $\phi(v, w) : p(v, w) = 0$ in $(\mathbb{R}, +, \cdot, -, 0, 1)$.

– The set of nonnegative real numbers $\mathbb{R}^{\geq 0}$ is definable by the formula $\phi(v)$: $\exists w\ (v = w^2)$ in $(\mathbb{R}, +, \cdot, -, 0, 1)$. No parameters are used in formula $\phi(v)$ so it is actually $\emptyset$-definable.

– Order Relation is definable in $(\mathbb{R}, +, \cdot, -, 0, 1)$ by the formula $\phi(v_1, v_2)$: $\exists w\ [(v_2 = v_1 + w^2) \wedge w \neq 0]$. We see that $(\mathbb{R}, +, \cdot, -, 0, 1) \models \phi(a, b)$ if and only if $a < b$. Again no parameters used, so it is $\emptyset$-definable.

iv. Let F be a field and $F[x]$ be a polynomial ring. The Field F is definable in the polynomial ring $(F[x], +, \cdot, -, 0, 1)$. More precisely, $F \setminus \{0\}$ is exactly the set of units of the ring $F[x]$; so F is definable by the formula
$\phi(v) : \exists w\ [(v \cdot w = w \cdot v = 1) \vee v = 0]$.

v. Let $\mathcal{L} = \{*, e\}$ be language of groups and let $\mathcal{G} = (G, *^{\mathcal{G}}, e^{\mathcal{G}})$ be an $\mathcal{L}$-structure. Center of a group G,

$$X = \{g \in G : \forall v\ (g *^{\mathcal{G}} v = v *^{\mathcal{G}} g)\},$$

is definable by the formula $\phi(v) : \forall w\ (w *^{\mathcal{G}} v = v *^{\mathcal{G}} w)$ in the group structure. Also, centralizer of an element $g \in G$ is definable by the formula $\phi(v) : (g *^{\mathcal{G}} v = v *^{\mathcal{G}} g)$.

Quantifier Elimination

Study of definable sets is hard with quantifiers. If we allow more quantifiers, definable sets become more complicated. So we introduce a concept so called quantifier elimination. A theory is said to have *quantifier elimination* if all formulas in the theory are equivalent to quantifier free formulas. If a theory T eliminates quantifiers, then definable sets of models of T are become less complicated. Quantifier elimination is a useful property that lets us to understand the theory better. For example, if T has quantifier elimination, this gives a way to decide whether a sentence belongs to a theory or not (decidability) and gives information about complete extensions of the theory [4].

There are well known examples, which are actually resulting from some algebraic facts, where formulas with quantifiers are shown to be equivalent to quantifier free formulas:

- In $(\mathbb{R},+,\cdot,-,0,1,<)$, an equivalent condition for a quadric polynomial $(av^2+bv+c=0)$ to have a root is having nonnegative discriminant $\delta=b^2-4ac$. So the formula $\phi(a,b,c):\exists v\ (av^2+bv+c=0)$ is equivalent to the quantifier free formula $\psi(a,b,c):\ [a\neq 0\wedge(0\leq b^2-4ac)]\vee(b\neq 0\vee c=0))$. Indeed, we have

$$(\mathbb{R},+,\cdot,-,0,1,<)\models\phi(a,b,c)\text{ if and only if }(\mathbb{R},+,\cdot,-,0,1,<)\models\psi(a,b,c).$$

  In the field of complex numbers $(\mathbb{C},+,\cdot,-,0,1)$, however, a quadric polynomial has a root in any case. So the formula $\phi(a,b,c):\exists v\ (av^2+bv+c=0)$ is equivalent to the quantifier free formula $\gamma(a,b,c):(a\neq 0\vee(b\neq 0\vee c=0))$ in complex numbers. That is,

$$(\mathbb{C},+,\cdot,-,0,1)\models\phi(a,b,c)\text{ if and only if }(\mathbb{C},+,\cdot,-,0,1)\models\gamma(a,b,c).$$

- A second well-known example uses a fact from linear algebra. We know that *a square n by n matrix is invertible if and only if its determinant is* 0. So let $(F,+,\cdot,-,0,1)$ be any field and consider the formula

$$\phi(a,b,c,d):\exists v_1\,\exists v_2\,\exists v_3\,\exists v_4\,[(a\cdot v_1+b\cdot v_3=1)\wedge(a\cdot v_2+b\cdot v_4=0)$$
$$\wedge(c\cdot v_1+d\cdot v_3=0)\wedge(c\cdot v_2+d\cdot v_4=1)],$$

  which indicates that the inverse $\begin{pmatrix}v_1&v_2\\v_3&v_4\end{pmatrix}$ of the matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ exists. So $\phi(a,b,c,d)$ is equivalent to the quantifier free formula $\psi(a,b,c,d):(a\cdot d-c\cdot b)\neq 0$. Hence, we have

$$(F,+,\cdot,-,0,1)\models\phi(a,b,c,d)\textit{ if and only if }(F,+,\cdot,-,0,1)\models\psi(a,b,c,d).$$

The formal definition of Quantifier Elimination is as follows.

Definition 2.24. A theory T has *quantifier elimination* if for any $\mathcal{L}$-formula $\phi(\bar{v})$ there exists a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that

$$T\models\forall\bar{v}\,\big(\phi(\bar{v})\leftrightarrow\psi(\bar{v})\big).$$

In other words, every $\mathcal{L}$-formula is equivalent to a quantifier free $\mathcal{L}$-formula modulo T.

It is not easy to directly show that a theory has quantifier elimination. But some simple theories such as Dense Linear Orders without endpoints (DLO) can be directly shown to have Quantifier Elimination [1, Theorem 3.1.3]. In this part of the text, we want to show that the theory of algebraically closed fields (ACF) eliminates quantifiers. To be able to prove it, we first need to give some tests that enable us to check a theory has quantifier elimination.

Theorem 2.10. *Let T be an $\mathcal{L}$-theory where $\mathcal{L}$ is a language containing at least one constant symbol c and let $\phi(v)$ be an $\mathcal{L}$-formula. The following conditions are equivalent:*

i. $\phi(\bar{v})$ *is equivalent to a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ modulo T; that is,* $T \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

ii. For any $\mathcal{L}$-structure $\mathcal{A}, \mathcal{B}$ that are models of T and for any common substructure $\mathcal{D}$ of $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{A} \models \phi(\bar{d})$ if and only if $\mathcal{B} \models \phi(\bar{d})$ for all $\bar{d} \in D^n$.

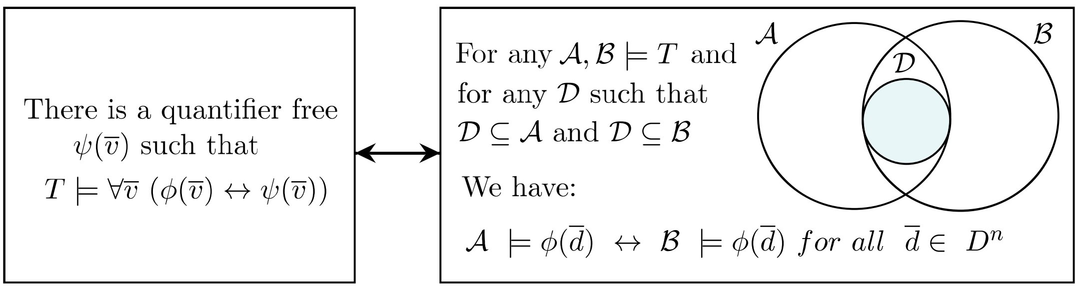


Figure 2.5. Theorem 2.10.

Proof. ($i. \Rightarrow ii.$) Let $\phi(\bar{v})$ be an $\mathcal{L}$- formula and assume there is a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. Let $\mathcal{A}, \mathcal{B} \models T$ and let $\mathcal{D} \subseteq \mathcal{A}, \mathcal{B}$.

We have $\mathcal{A} \models (\phi(\bar{a}) \leftrightarrow \psi(\bar{a}))$ for all $\bar{a} \in A^n$ by assumption. Further, we have:

$$\begin{aligned}
\mathcal{A} \models \phi(\bar{d}) &\Leftrightarrow \mathcal{A} \models \psi(\bar{d})) \textit{ for all } \bar{d} \in D^n \textit{ since } D \subseteq A \\
&\Leftrightarrow \mathcal{D} \models \psi(\bar{d}) \textit{ for all } \bar{d} \in D^n \textit{ by Proposition 2.2 since } \mathcal{D} \subseteq \mathcal{A} \textit{ and} \\
&\qquad \psi(\bar{v}) \textit{ is quantifier free} \\
&\qquad \textit{(quantifier free formulas are preserved under substructures)} \\
&\Leftrightarrow \mathcal{B} \models \psi(\bar{d}) \textit{ for all } \bar{d} \in D^n \textit{ again by Proposition 2.2} \\
&\Leftrightarrow \mathcal{B} \models \phi(\bar{d}) \textit{ for all } \bar{d} \in D^n \textit{ by Assumption (i.).}
\end{aligned}$$

Hence we obtained the result.

(*ii.* $\Rightarrow$ *i.*) Now, let c be a constant symbol in $\mathcal{L}$, let $\phi(\bar{v})$ be an $\mathcal{L}$-formula and assume (*ii.*) is true. We want to show that there is a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. Define the set

$$\Sigma(\bar{v}) = \{\sigma(\bar{v}) : \sigma(\bar{v}) \textit{ is quantifier free and } T \models \forall \bar{v}\ (\phi(\bar{v}) \rightarrow \sigma(\bar{v}))\},$$

which consists of quantifier free consequences of $\phi(\bar{v})$. Now, to get rid of the free variables we introduce new constant symbols $c_1, c_2, ..., c_n$. Replace $\bar{v}$ by $\bar{c} = (c_1, c_2, ...c_n)$ and obtain sentences in extended language.

Claim: $T \cup \Sigma(\bar{c}) \models \phi(\bar{c})$.

Suppose the claim holds. Then, by Compactness Theorem there is a finite subset $\{\sigma_1(\bar{c}), ..., \sigma_m(\bar{c})\}$ of Σ such that $T \cup \{\sigma_1(\bar{c}), ..., \sigma_m(\bar{c})\} \models \phi(\bar{c})$. So we have $T \cup \{\bigwedge_{i=1}^{n} \sigma_i(\bar{c})\} \models \phi(\bar{c})$ and by Deduction Theorem [3, Theorem 1.3.2] we get $T \models \big(\bigwedge_{i=1}^{n} \sigma_i(\bar{c})\big) \rightarrow \phi(\bar{c})$. Equivalently, we obtain

$$T \models \forall \bar{v}\ \Big(\big(\bigwedge_{i=1}^{m} \sigma_i(\bar{v})\big) \rightarrow \phi(\bar{v})\Big) \text{ and hence } T \models \forall \bar{v}\ \Big(\big(\bigwedge_{i=1}^{m} \sigma_i(\bar{v})\big) \leftrightarrow \phi(\bar{v})\Big)$$

by definition of Σ. Let $\psi(\bar{v}) = \bigwedge_{i=1}^{m} \sigma_i(\bar{v})$, we see that $\psi(\bar{v})$ is quantifier free since it is conjunction of m-many quantifier free formulas. Therefore, we have shown that $\phi(\bar{v})$ is equivalent to a quantifier free formula $\psi(\bar{v})$ modulo T.

Now, it only remains to prove the *claim* to finish the proof. But before proving it we first observe that if $T \models \forall \bar{v}\ \phi(\bar{v})$ then we have $T \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow c = c)$ and if $T \models \forall \bar{v}\ \neg\phi(\bar{v})$ then we have $T \models \forall \bar{v}\ (\neg\phi(\bar{v}) \leftrightarrow c = c)$, or equivalently we have $T \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow c \neq c)$. So in both cases $\phi(\bar{v})$ is equivalent to a quantifier free formula modulo T. Since $T \models \forall \bar{v}\ \phi(\bar{v})$ means that $T \cup \{\neg\phi(\bar{v})\}$ is *not* satisfiable and $T \models \forall \bar{v}\ \neg\phi(\bar{v})$ means that $T \cup \{\phi(\bar{v})\}$ is *not* satisfiable, we can assume that $T \cup \{\phi(\bar{v})\}$ and $T \cup \{\neg\phi(\bar{v})\}$ are satisfiable since we considered the cases where they are not satisfiable above.

Proof of Claim. We want to show that $T \cup \Sigma(\bar{c}) \models \phi(\bar{c})$. Suppose, for a contradiction, $T \cup \Sigma(\bar{c}) \not\models \phi(\bar{c})$. It means that $T \cup \Sigma(\bar{c}) \cup \neg\phi(\bar{c})$ is satisfiable, so let $\mathcal{A} \models T \cup \Sigma(\bar{c}) \cup \neg\phi(\bar{c})$ and consider a substructure $\mathcal{D}$ of $\mathcal{A}$ generated by the interpretations of constant symbols $c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, ..., c_n^{\mathcal{A}}$. To use the assumption and to get a contradiction, we want to find an extension of $\mathcal{D}$ which is a model of T satisfying $\phi(\bar{c})$. Such extension exists if $T \cup Diag(\mathcal{D}) \cup \phi(\bar{c})$ is satisfiable and if there is a model of $\mathcal{B}$ of $T \cup Diag(\mathcal{D}) \cup \phi(\bar{c})$, then we obtain $\mathcal{A} \models \phi(\bar{c})$ by assumption (*ii.*) since $\mathcal{A}, \mathcal{B} \models T$, $\mathcal{D} \subseteq \mathcal{A}, \mathcal{B}$ and $\mathcal{B} \models \phi(\bar{c})$, which is a contradiction since $\mathcal{A} \models \neg\phi(\bar{c})$.

It remains to show that $T \cup Diag(\mathcal{D}) \cup \phi(\bar{c})$ is satisfiable. Assume for a contradiction, $T \cup Diag(\mathcal{D}) \cup \phi(\bar{c})$ is not satisfiable. Then $T \cup Diag(\mathcal{D}) \models \neg\phi(\bar{c})$ and by Compactness Theorem, there is a finite subset $\{\gamma_1, ..., \gamma_l\} \in Diag(\mathcal{D})$ such that $T \cup \{\bigwedge_{i=1}^{l} \gamma_i(\bar{c})\} \models \neg\phi(\bar{c})$. Further, we obtain $T \models \big(\bigwedge_{i=1}^{l} \gamma_i(\bar{c}) \rightarrow \neg\phi(\bar{c})\big)$ by Deduction Theorem and also we obtain $T \models \big(\phi(\bar{c}) \rightarrow \bigvee_{i=1}^{l} \neg\gamma_i(\bar{c})\big)$ by taking contrapositive of the statement. However, since each γ_i is quantifier free $\bigvee_{i=1}^{l} \neg\gamma_i(\bar{c}) \in \Sigma$; so $\mathcal{A} \models \bigvee_{i=1}^{l} \neg\gamma_i(\bar{c})$ and this implies $\mathcal{D} \models \bigvee_{i=1}^{l} \neg\gamma_i(\bar{c})$ by Proposition 2.2 since $\mathcal{D} \subseteq \mathcal{A}$. This is a contradiction since each $\gamma_i \in Diag(\mathcal{D})$, we have $\mathcal{D} \models \bigwedge_{i=1}^{l} \gamma_i(\bar{c})$. □

Theorem 2.11. *Let T be an $\mathcal{L}$-theory. If for any quantifier free $\mathcal{L}$-formula $\gamma(w, \bar{v})$, there exists a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}\ [\exists w\ \gamma(w, \bar{v}) \leftrightarrow \psi(\bar{v})]$, then T has quantifier elimination.*

Proof. Let $\phi(\bar{v})$ be any $\mathcal{L}$-formula. We want to show that there is a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that $T \models \forall\bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. We will do induction on the length of formulas.

1. Let $\phi(\bar{v})$ be an atomic formula. Then, it is already quantifier free; so the statement holds for atomic formulas.
2. Let $\phi(\bar{v})$ be $\neg\theta(\bar{v})$, we have $T \models \forall\bar{v}\ (\theta(\bar{v}) \leftrightarrow \psi(\bar{v}))$ for some quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ by induction and equivalently, we have $T \models \forall\bar{v}\ (\neg\theta(\bar{v}) \leftrightarrow \neg\psi(\bar{v}))$. Thus, $T \models \forall\bar{v}\ (\phi(\bar{v}) \leftrightarrow \neg\psi(\bar{v}))$ where $\neg\psi(\bar{v})$ is quantifier free.
3. Let $\phi(\bar{v})$ be $\theta_1(\bar{v}) \wedge \theta_2(\bar{v})$. We have $T \models \forall\bar{v}\ (\theta_1(\bar{v}) \leftrightarrow \psi_1(\bar{v}))$ and $T \models \forall\bar{v}\ (\theta_2(\bar{v}) \leftrightarrow \psi_2(\bar{v}))$ by induction where $\psi_1(\bar{v})$ and $\psi_2(\bar{v})$ are quantifier free formulas. By combining these, we obtain $T \models \forall\bar{v}\ (\theta_1(\bar{v}) \wedge \theta_2(\bar{v}) \leftrightarrow \psi_1(\bar{v}) \wedge \psi_2(\bar{v}))$ and equivalently $T \models \forall\bar{v}\ (\phi(\bar{v}) \leftrightarrow (\psi_1(\bar{v}) \wedge \psi_2(\bar{v})))$ where $\psi_1(\bar{v}) \wedge \psi_2(\bar{v}))$ is conjunction of quantifier frees so it i quantifier free.
4. Let $\phi(\bar{v})$ be $\exists w\ \gamma(w, \bar{v})$. By induction, we have $T \models \forall\bar{v}\forall w\ (\gamma(w, \bar{v}) \leftrightarrow \psi_0(w, \bar{v}))$ where $\psi_0(w, \bar{v})$ is quantifier free. We also have $T \models \forall\bar{v}\ (\exists w\ \gamma(w, \bar{v}) \leftrightarrow \exists w\ \psi_0(w, \bar{v}))$. Now, by using the assumption of the theorem since $\psi_0(w, \bar{v})$ is quantifier free there is a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that $T \models \forall\bar{v}\ (\exists w\ \psi_0(w, \bar{v}) \leftrightarrow \psi(\bar{v}))$. Therefore, $T \models \forall\bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$ where $\psi(\bar{v})$ is quantifier free.

By induction, we have shown that T has quantifier elimination. □

We combine Theorem 2.10 and Theorem 2.11 and obtain the following Corrolary.

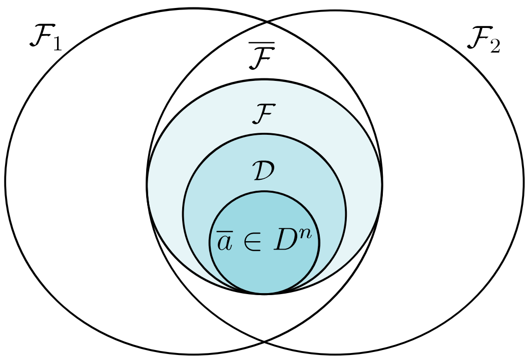
Corollary 2.12. *Let T be an $\mathcal{L}$-theory and let $\mathcal{A}$ and $\mathcal{B}$ be models of T. If for any quantifier free $\mathcal{L}$-formula $\gamma(w, \bar{v})$, for any common substructure $\mathcal{D}$ of $\mathcal{A}$ and B and for any $\bar{d} \in D^n$; whenever there is an $a \in A$ such that $\mathcal{A} \models \gamma(a, \bar{d})$, there is $b \in B$ such that $\mathcal{B} \models \gamma(b, \bar{d})$; then T has quantifier elimination.*

Proof. T eliminates quantifiers if for any quantifier free $\mathcal{L}$-formula $\gamma(w, \bar{v})$, there is a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that $T \models \forall\bar{v}\ [\exists w\ \gamma(w, \bar{v}) \leftrightarrow \psi(\bar{v})]$ by Theorem 2.11. Also, for $\exists w\ \gamma(w, \bar{v})$ such quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ exists if for any $\mathcal{A}, \mathcal{B} \models T$ and for any $\mathcal{D} \subseteq \mathcal{A}, \mathcal{B}$, we have $\mathcal{A} \models \exists w \ \gamma(w, \bar{d})$ *if and only if* $\mathcal{B} \models \exists w \ \gamma(w, \bar{d})$ *for all* $\bar{d} \in D^n$ by Theorem 2.10. So we have the corrolary. □

Theorem 2.13. *The theory of algebraically closed fields (ACF) has quantifier elimination.*

Proof. We will use the previous corrolary; so let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two algebraically closed fields and let $\mathcal{D}$ be a common substructure of them. Also, let $\gamma(w, \bar{v})$ be a quantifier free formula and let $\bar{a} \in D^n$; we will show that if there is $e \in F_1$ such that $\mathcal{F}_1 \models \gamma(e, \bar{a})$, then there is $f \in F_2$ such that $\mathcal{F}_2 \models \gamma(f, \bar{a})$ and by applying Corrolary 2.12 we will obtain *ACF eliminates quantifiers.*

Since the language of fields is $\mathcal{L} = \{+, \cdot, -, 0, 1\}$, a substructure $\mathcal{D}$ of algebraically closed fields is at least an integral domain. Consider the field of fractions $\mathcal{F}$ of $\mathcal{D}$ and also algebraic closure $\overline{\mathcal{F}}$ of field of fractions of $\mathcal{D}$. Clearly, $\overline{\mathcal{F}} \subseteq \mathcal{F}_1$ and $\overline{\mathcal{F}} \subseteq \mathcal{F}_2$ since $\mathcal{F}_1$ and $\mathcal{F}_2$ are algebraically closed fields.



We will show that for any quantifier free formula $\gamma(w, \bar{v})$ and for any $\overline{a} \in (\overline{F})^n$; if there is $e \in F_1$ such that $\mathcal{F}_1 \models \gamma(e, \overline{a})$, then there is $f \in F_2$ such that $\mathcal{F}_2 \models \gamma(f, \overline{a})$.

We proceed by writing the quantifier free formula $\gamma(w, \bar{v})$ in Disjunctive Normal Form (DNF); that is, disjunction of conjunctions of atomic and negated atomic formulas. Every quantifier free formula can be written in this form [2, p. 42]. So we have

$$\gamma(w, \bar{v}) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{l} \phi_{i,j}(w, \bar{v}),$$

where each $\phi_{i,j}(w, \bar{v})$ are atomic or negation of atomic formulas. If $\mathcal{F}_1 \models \gamma(e, \bar{a})$, then we have $\mathcal{F}_1 \models \bigvee_{i=1}^{m} \bigwedge_{j=1}^{l} \phi_{i,j}(e, \bar{a})$ and it means that $\mathcal{F}_1 \models \bigwedge_{j=1}^{l} \phi_{i,j}(e, \bar{a})$ for some $i \in \{1, 2, ..., m\}$. Now, observe that terms in language of fields are consist of addition and multiplication of variables and constant symbols; so atomic and negation of atomic formulas in language of fields are just consist of polynomial equalities $p(\overline{X}) = 0$ and inequalities $p(\overline{X}) \neq 0$ where $p(\overline{X}) \in \mathbb{Z}[\overline{X}]$. By means of integer coefficients, we actually mean the terms $\pm(1+1+...+1)$. If $p(X, Y_1, ..., Y_n) \in \mathbb{Z}[X, Y_1, ..., Y_n]$, then for $\bar{a} \in (\overline{\mathcal{F}})^n$ we have $p(X, \bar{a}) \in \overline{\mathcal{F}}[X]$. Thus, $\bigwedge_{j=1}^{l} \phi_{i,j}(w, \bar{a})$ is equivalent to

$$\bigwedge_{i=1}^{r} p_i(x) = 0 \wedge \bigwedge_{j=1}^{s} q_j(x) \neq 0$$

where each p_i and q_j is in $\overline{\mathcal{F}}[X]$. If any of the p_i's are nonzero, then it means that e is algebraic over $\overline{\mathcal{F}}$ and hence $e \in \overline{\mathcal{F}}$ since it is algebraically closed. So we have $e \in F_2$ since $\overline{F} \subseteq F_2$ and hence $\mathcal{F}_2 \models \bigwedge_{j=1}^{l} \phi_{i,j}(e, \bar{a})$. So assume all p_i's are 0. In this case, we want to find an $f \in F_2$ such that

$$\mathcal{F}_2 \models \bigwedge_{j=1}^{s} q_j(f) \neq 0.$$

Since all q_i's are polynomials, they only have finitely many roots so only finitely many element of $\overline{F}$ does not satisfy $\bigwedge_{j=1}^{s} q_j(x) \neq 0$. Moreover, since algebraically closed fields are infinite, there is at least one element f of $\overline{F}$ satisfying $\bigwedge_{j=1}^{s} q_j(x) \neq 0$. Hence, we again get $\mathcal{F}_2 \models \bigwedge_{j=1}^{l} \phi_{i,j}(f, \bar{a})$. □

2.4. Model Completeness

The term *model completeness* is introduced by *Abraham Robinson* who was influenced by the fact that the maps between algebraic structures are rarely elementary and the cases where they are elementary are important maps such as maps between algebraically closed fields [7]. As Hodges states, Robinson thought that there should be a systematic reason of existence of elementary embeddings and introduced the notions of model completeness and model companions in model theory around 1950s. [2, p. 374] Through this historic motivation, *we can call theories as model complete if all embeddings between its models are elementary.* The formal definition is as follows.

Definition 2.25. Let T be a consistent $\mathcal{L}$-theory. T is called *model complete* if for any two models $\mathcal{A}$ and $\mathcal{B}$ of T such that $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{A} \preccurlyeq \mathcal{B}$. In other words, every extension of a model of T is an elementary extension.

Remark 2.12. We stated that if all embeddings between models of a theory T are elementary, then T is said to be model complete. This statement is equivalent to Definition 2.25. Indeed, assume all embeddings between models of T are elementary, then for any two models of T satisfying $\mathcal{A} \subseteq \mathcal{B}$, we obtain $\mathcal{A} \preccurlyeq \mathcal{B}$ since the inclusion map would be an elementary embedding. Conversely, assume Definition 2.25 and let $\sigma : \mathcal{A} \to \mathcal{B}$ be an embedding between models of T. This means that $\mathcal{B}$ contains an isomorphic copy of $\mathcal{A}$ as an elementary substructure. Hence, σ is an elementary embedding.

Neither model completeness nor completeness implies each other. To illustrate,

- The theory of algebraically closed fields (ACF) is an example of a model complete theory which is it is not complete. (see Example 2.18 and Section 2.6.4)
- The theory of dense linear orders with endpoints is complete theory but it is not model complete. (see Example 2.23)
- The theory of dense linear orders without endpoints (DLO) is both complete and model complete. (see Example 2.22)

An equivalent condition of model completeness is stated in the following theorem.

Theorem 2.14. *A theory T is model complete if and only if for any model $\mathcal{A}$ of T, $T \cup Diag(\mathcal{A})$ is complete.*

Proof. ($\Rightarrow$) First, assume T is model complete. We want to show that $T \cup Diag(\mathcal{A})$ is complete. So let $\mathcal{A}$ and $\mathcal{B}$ be two models of $T \cup Diag(\mathcal{A})$. Since $\mathcal{B}$ is a model of T satisfying $\mathcal{B} \models Diag(\mathcal{A})$, there is an $\mathcal{L}$-embedding σ from $\mathcal{A}$ into $\mathcal{B}$ by Diagram Lemma 2.6. Moreover, since $\mathcal{A}$ and $\mathcal{B}$ are models of a model complete theory T, the $\mathcal{L}$-embedding σ from $\mathcal{A}$ into $\mathcal{B}$ is an elementary embedding. Hence, $\mathcal{A} \equiv \mathcal{B}$. Since arbitrary two models of $T \cup Diag(\mathcal{A})$ are elementarily equivalent, it is complete by the Proposition 2.1.

($\Leftarrow$) Now, suppose T is not model complete. It means that there are models $\mathcal{A}$ and $\mathcal{B}$ of T where $\mathcal{A}$ is a substructure of $\mathcal{B}$; but, $\mathcal{A}$ is not an elementary substructure of $\mathcal{B}$. That is, there is an $\mathcal{L}_A$-sentence $\phi(\bar{a})$ such that $\mathcal{B} \models \phi(\bar{a})$ but $\mathcal{A} \not\models \phi(\bar{a})$. However, this implies that the $\mathcal{L}_A$-theory $T \cup Diag(\mathcal{A})$ is not complete since $\mathcal{A}, \mathcal{B} \models T \cup Diag(\mathcal{A})$, $\mathcal{B} \models \phi(\bar{a})$ but $\mathcal{A} \not\models \phi(\bar{a})$. $\square$

As it is stated previously, quantifier elimination is a nice property that theories may have. In the following theorem, we show that if a theory has quantifier elimination, then it is model complete. So proving quantifier elimination for the theory T is one of the ways to show that a theory is model complete.

Theorem 2.15. *If a theory T eliminates quantifiers, then it is model complete.*

Proof. Assume T has quantifier elimination, and let $\mathcal{A}$ and $\mathcal{B}$ be two models of T such that $\mathcal{A} \subseteq \mathcal{B}$. We want to show that $\mathcal{A} \preccurlyeq \mathcal{B}$. So let $\phi(\bar{v})$ be an $\mathcal{L}$-formula and $\bar{a} \in A^n$. Since T has quantifier elimination, there exists a quantifier free $\mathcal{L}$-formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. So we obtain $\mathcal{A}, \mathcal{B} \models (\phi(\bar{a}) \leftrightarrow \psi(\bar{a}))$ since $\mathcal{A}$ and $\mathcal{B}$ are models of T and $\bar{a} \in A^n \subseteq B^n$. Moreover, since quantifier free formulas are preserved under substructures by *Proposition* 2.2, we get

$$\begin{aligned} A \models \phi(\bar{a}) &\Leftrightarrow \mathcal{A} \models \psi(\bar{a}) \text{ since } \mathcal{A} \models (\phi(\bar{a}) \leftrightarrow \psi(\bar{a})) \\ &\Leftrightarrow \mathcal{B} \models \psi(\bar{a}) \text{ by Proposition 2.2 since } \psi(\bar{v}) \text{ is quantifier free and } \mathcal{A} \subseteq \mathcal{B} \\ &\Leftrightarrow \mathcal{B} \models \phi(\bar{a}) \text{ since } \mathcal{B} \models (\phi(\bar{a}) \leftrightarrow \psi(\bar{a})). \end{aligned}$$

Hence $\mathcal{A} \preccurlyeq \mathcal{B}$. $\square$

Example 2.18. The previous theorem shows that the theory of algebraically closed fields is model complete since it has quantifier elimination (Theorem 2.13).

Remark 2.13. Converse of the Theorem 2.13 is not true. For example, the theory ACFA, the model companion of the theory of fields with an automorphism, is model complete but it does not eliminate quantifiers. (Model companion of a theory will be discussed later in detail.)

Model completeness can be a tool to show that a theory is complete. At the beginning of this section, we saw that model completeness does not imply completeness, but with an additional condition, it actually does.

Definition 2.26. Let T be an $\mathcal{L}$-theory and let $\mathcal{P}$ be a model of T. $\mathcal{P}$ is called a *prime model* of T if it can be embedded into every model of T.

Remark 2.14. A prime model of a theory T is unique up to isomorphism, if it exists.

Example 2.19. Let T be the theory of fields of characteristic 0, the rational number field $\mathbb{Q}$ is a prime model of T. Similarly, if T is the theory of fields of characteristic p, then $\mathbb{F}_p$ is a prime model of T.

Theorem 2.16. *If a theory T is model complete and has a prime model, then it is complete.*

Proof. Let $\mathcal{P}$ be a prime model of T and let $\mathcal{A}$ and $\mathcal{B}$ be two models of T. So there is an embedding from $\mathcal{P}$ to $\mathcal{A}$ and $\mathcal{P}$ to $\mathcal{B}$ by definition of prime model. Since T is model complete, we observe that these embeddings are elementary. Therefore, we obtain $\mathcal{P} \equiv \mathcal{A} \equiv \mathcal{B}$. Since arbitrary two models of T is elementarily equivalent, T is complete by the Proposition 2.1. □

Example 2.20. We have shown that the theory of algebraically closed fields is model complete since it eliminates quantifiers in previous example. Consider the theory of algebraically closed fields with characteristic p, where p is any prime number and denote it as ACF_p. Since ACF is model complete and any model of ACF_p is also a model of ACF, ACF_p is also model complete. Also, since ACF_p has a prime model, we apply the previous theorem and obtain ACF_p is *complete.*

We proceed by defining what does it mean for a model to be *existentially closed.* After that, we will see that it is directly related with *model completeness.* But first of all, we need to give related definitions.

Definition 2.27. i. An $\mathcal{L}$-formula $\phi(\bar{v})$ is called *existential formula* if it is of the form $\exists \bar{w}\, \psi(\bar{v}, \bar{w})$ where $\psi(\bar{v}, \bar{w})$ is quantifier free.

ii. An $\mathcal{L}$-formula $\phi(\bar{v})$ is called *universal formula* $\phi(\bar{v})$ if it is of the form $\forall \bar{w}\, \psi(\bar{v}, \bar{w})$ where $\psi(\bar{v}, \bar{w})$ is quantifier free.

iii. An $\mathcal{L}$-formula $\phi(\bar{v})$ is called $\forall\exists$-*formula* if it is of the form $\forall \bar{v}\, \exists \bar{w}\, \psi(\bar{v}, \bar{w})$ where $\psi(\bar{v}, \bar{w})$ is quantifier free.

An $\mathcal{L}$-theory is called as $\forall\exists$-*theory* if it can be axiomatized by $\forall\exists$-sentences.

Definition 2.28. Let T be an $\mathcal{L}$-theory. A model $\mathcal{A}$ of T is called *existentially closed* model of T if for any extension $\mathcal{B}$ of $\mathcal{A} \subseteq \mathcal{B}$ such that $\mathcal{B} \models T$ and for any quantifier free $\mathcal{L}_A$-formula $\phi(\bar{v})$, we have

$$\mathcal{A} \models \exists \bar{v}\, \phi(\bar{v}) \quad \text{if and only if} \quad \mathcal{B} \models \exists \bar{v}\, \phi(\bar{v}).$$

Example 2.21. Consider two models of the theory of fields $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}]$, and let $\phi : \exists v\ (v \cdot v = 2)$ be an existential $\mathcal{L}_{\mathbb{Q}}$-sentence. Clearly, $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{2}] \models \phi$, but $\mathbb{Q} \not\models \phi$. So $\mathbb{Q}$ is not an existentially closed model of the theory of fields. This also shows that the theory of fields is not model complete due to the fact that there are two models where the substructure relation is not elementary.

In the following theorem, which we named as *Robinson's Test*, we show that if a theory is model complete, then all models of the theory are existentially closed. Also, some other equivalent conditions of model completeness are stated.

Theorem 2.17 (Robinson's Test)**.** *Let T be an $\mathcal{L}$-theory. The following condions are equivalent:*

(i) T is model complete.

(ii) Every model of T is existentially closed.

(iii) For every existential formula $\phi(\bar{v})$, there is a universal formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

(iv) For every formula $\phi(\bar{v})$, there is a universal formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

Proof. $(i) \Rightarrow (ii)$ is clear by definition.
$(ii) \Rightarrow (iii)$ Assume that every model of T is existentially closed. Let $\phi(\bar{v})$ be an existential formula and consider the set

$$\Sigma(\bar{v}) = \{\sigma(\bar{v}) : \sigma(\bar{v}) \textit{ is universal and } T \models \forall \bar{v}(\phi(\bar{v}) \to \sigma(\bar{v}))\}.$$

Expand the language $\mathcal{L}$ by adding new constant symbols $c_1, c_2, ..., c_n$ and replace free variables $\bar{v}$ with $\bar{c} = (c_1, ..., c_n)$ to get rid of free variables and obtain sentences in extended language.

Claim: $T \cup \Sigma(\bar{c}) \models \phi(\bar{c})$.

Assume we have the claim. By Compactness Theorem, there are finitely many sentences $\sigma_1(\bar{c}), \sigma_2(\bar{c}), ..., \sigma_n(\bar{c}) \in \Sigma$ such that $T \cup \{\sigma_1(\bar{c}) \wedge \sigma_2(\bar{c}) \wedge ... \wedge \sigma_n(\bar{c})\} \models \phi(\bar{c})$ and by Deduction Theorem [3, Theorem 1.3.2] we obtain $T \models (\sigma_1(\bar{c}) \wedge \sigma_2(\bar{c}) \wedge ... \wedge \sigma_n(\bar{c})) \to \phi(\bar{c})$. Hence, we get $T \models (\sigma_1(\bar{c}) \wedge \sigma_2(\bar{c}) \wedge ... \wedge \sigma_n(\bar{c})) \leftrightarrow \phi(\bar{c})$ by combining previous result with definition of Σ. Equivalently, we obtain $T \models \forall \bar{v} \; ((\sigma_1(\bar{v}) \wedge \sigma_2(\bar{v}) \wedge ... \wedge \sigma_n(\bar{v})) \leftrightarrow \phi(\bar{v}))$. The desired universal formula $\psi(\bar{v})$ is obtained from $(\sigma_1(\bar{v}) \wedge \sigma_2(\bar{v}) \wedge ... \wedge \sigma_n(\bar{v})$ by moving their quantifiers to the front. We have $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$ where $\psi(\bar{v})$ is universal.

Proof of Claim. We want to show $T \cup \Sigma(\bar{c}) \models \phi(\bar{c})$. We can assume that $T \cup \Sigma(\bar{c})$ is consistent because otherwise the claim is automatically true.
Let $\mathcal{A}$ be a model of $T \cup \Sigma(\bar{c})$ such that $c_i^{\mathcal{A}} = a_i$ for $i = 1, ..., n$. We want to show that $\mathcal{A} \models \phi(\bar{a})$. Since $\phi(\bar{a})$ can be viewed as an existential $\mathcal{L}_A$-formula and $\mathcal{A}$ is existentially closed by assumption, it is enough o show that $\phi(\bar{a})$ is satisfied by some extension of $\mathcal{A}$ that is a model of T. To show that such extension exist, we will show that $T \cup \{\phi(\bar{a})\} \cup Diag(\mathcal{A})$ is satisfiable. By Compactness, it is enough to show that $T \cup \{\phi(\bar{a})\} \cup Diag(\mathcal{A})$ is finitely satisfiable, so let $\{\theta_1(\bar{a}), \theta_2(\bar{a}), ..., \theta_n(\bar{a})\}$ be a finite subset of $Diag(\mathcal{A})$ and let $\theta(\bar{a}) = \theta_1(\bar{a}) \wedge \theta_2(\bar{a}) \wedge ... \wedge \theta_n(\bar{a})$. We can assume that $T \cup \phi(\bar{c})$ is consistent, otherwise we would have $T \models \forall \bar{v} \; \neg\phi(\bar{v})$ which implies that $\Sigma(\bar{c})$ is inconsistent. So it is enough to show that $T \cup \{\phi(\bar{a})\}$ is consistent with $\theta(\bar{a})$; that is, $T \cup \{\phi(\bar{a})\} \not\models \neg\theta(\bar{a})$. Assume, for a contradiction, $T \cup \{\phi(\bar{a})\} \models \neg\theta(\bar{a})$.

Replace each element of A appearing in the formula $\theta(\bar{a})$ by variables v_j except a_i for $i = 1, ..., n$ (Since a_i are interpretations of constant symbols c_i). So the assumption is equivalent to $T \cup \{\phi(\bar{a})\} \models \neg(\exists \bar{v}\ \theta(\bar{v}, \bar{a}))$. By Deduction Theorem [3, Theorem 1.3.2], we obtain $T \models \phi(\bar{a}) \rightarrow \forall \bar{v}\, \neg\theta(\bar{a}, \bar{v})$ which implies that $\forall \bar{v}\, \neg\theta(\bar{a}, \bar{v}) \in \Sigma(\bar{a})$. However, since $\mathcal{A} \models T \cup \Sigma(\bar{a})$, we get $\mathcal{A} \models \forall \bar{v}\, \neg\theta(\bar{a}, \bar{v})$; and also since $\theta_i(\bar{a}) \in Diag(\mathcal{A})$, we have $\mathcal{A} \models \exists \bar{v}\ \theta(\bar{v}, \bar{a})$, which is a contradiction. Hence, $T \cup \{\phi(\bar{a})\} \cup Diag(\mathcal{A})$ is satisfiable.

Since $T \cup \{\phi(\bar{a})\} \cup Diag(\mathcal{A})$ is satisfiable, it has a model $\mathcal{B}$ which is an extension of $\mathcal{A}$ (by Diagram Lemma 2.6) such that $\mathcal{B} \models T$ and $\mathcal{B} \models \phi(\bar{a})$. Hence, $\mathcal{A} \models \phi(\bar{a})$ since $\mathcal{A}$ is existentially closed in $\mathcal{B}$.

$(iii) \Rightarrow (iv)$ Proof depends on induction on length of formulas.

1. If $\phi(\bar{v})$ is an atomic formula, it is clearly equivalent to a universal formula $\psi(\bar{v}) : \forall \bar{w}\ \phi(\bar{v})$ (actually, we can consider quantifier free formulas as universal formulas since we can basically add universal quantifiers to the beginning with new variables, where those variables do not appear in the formula); that is, we have $T \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.
2. Let $\phi(\bar{v})$ be $\neg\theta(\bar{v})$. By induction, we have $T \models \forall \bar{v}\ (\theta(\bar{v}) \leftrightarrow \gamma(\bar{v}))$ for some universal formula $\gamma(\bar{v})$. Equivalently, we have $T \models \forall \bar{v}\ (\neg\theta(\bar{v}) \leftrightarrow \neg\gamma(\bar{v}))$ where $\neg\gamma(\bar{v})$ is an existential formula. By part (iii), we know that every existential formula is equivalent to a universal formula modulo T, so there exists a universal formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}\ (\neg\gamma(\bar{v}) \leftrightarrow \psi(\bar{v}))$. Therefore, we have
$T \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$ where $\psi(\bar{v})$ is universal.
3. If $\phi(\bar{v})$ is $\theta_1(\bar{v}) \wedge \theta_2(\bar{v})$, then by induction we have $T \models \forall \bar{v}\ (\theta_1(\bar{v}) \leftrightarrow \psi_1(\bar{v}))$ and $T \models \forall \bar{v}\ (\theta_2(\bar{v}) \leftrightarrow \psi_2(\bar{v}))$ where $\psi_1(\bar{v})$ and $\psi_2(\bar{v})$ are universal formulas; hence, $T \models \forall \bar{v}\ (\theta_1(\bar{v}) \wedge \theta_2(\bar{v}) \leftrightarrow \psi_1(\bar{v}) \wedge \psi_2(\bar{v}))$. By moving quantifiers of $\psi_1(\bar{v}) \wedge \psi_2(\bar{v})$ to the front, we obtain a universal formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.
4. If $\phi(\bar{v})$ is $\forall \bar{w}\ \theta(\bar{v}, \bar{w})$, then by induction we have $T \models \forall \bar{v} \forall \bar{w}\ (\theta(\bar{v}, \bar{w}) \leftrightarrow \gamma(\bar{v}, \bar{w}))$ where $\gamma(\bar{v}, \bar{w})$ is a universal formula. Hence, $T \models \forall \bar{v}\ (\underbrace{\forall \bar{w} \theta(\bar{v}, \bar{w})}_{\phi(\bar{v})} \leftrightarrow \forall \bar{w} \gamma(\bar{v}, \bar{w}))$ where $\forall \bar{w} \gamma(\bar{v}, \bar{w}))$ is universal. (In this step of induction, we can either check the

formulas with existential quantifiers or universal quantifiers since $\exists\bar{w}\ \gamma(\bar{v},\bar{w}) = \neg\forall\bar{w}(\neg\gamma(\bar{v},\bar{w}))$ and $\forall\bar{w}\ \gamma(\bar{v},\bar{w}) = \neg\exists\bar{w}(\neg\gamma(\bar{v},\bar{w}))$. By showing for one of them, we also have the other since we have already do the induction for negations of formulas.)

$(iv) \Rightarrow (i)$ Let $\mathcal{A},\mathcal{B} \models T$ such that $\mathcal{A} \subseteq \mathcal{B}$. Also, let $\phi(\bar{v})$ be an $\mathcal{L}$-formula and $\bar{a} \in A^n$, we want to show that $\mathcal{A} \models \phi(\bar{a})$ if and only if $\mathcal{B} \models \phi(\bar{a})$. So first assume $\mathcal{B} \models \phi(\bar{a})$. Since $\mathcal{B}$ is a model of T and since $T \models \forall\bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$ for some universal formula $\psi(\bar{v})$ by assumption (statement (iv)), we have $\mathcal{B} \models \psi(\bar{a})$. Because of the fact that $\psi(\bar{a})$ is universal, there is a quantifier free formula $\theta(\bar{a},\bar{v})$ such that $\psi(\bar{a}) = \forall\bar{v}\ \theta(\bar{a},\bar{v})$. So we have $\mathcal{B} \models \theta(\bar{a},\bar{b})$ for all $\bar{b} \in B^m$. Also, since $\mathcal{A} \subseteq \mathcal{B}$ we have $\mathcal{B} \models \theta(\bar{a},\bar{b})$ for all $\bar{b} \in A^m$ and since quantifier free formulas are preserved under substructures by Proposition 2.2, we obtain $\mathcal{A} \models \theta(\bar{a},\bar{b})$ for all $\bar{b} \in A^m$ and this implies that $\mathcal{A} \models \forall\bar{v}\ \theta(\bar{a},\bar{v})$; that is, $\mathcal{A} \models \psi(\bar{a})$. Hence, $\mathcal{A} \models \phi(\bar{a})$ since $\mathcal{A} \models \forall\bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. Therefore, we have $\mathcal{B} \models \phi(\bar{a})$ implies $\mathcal{A} \models \phi(\bar{a})$.

If we assume that $\mathcal{B} \not\models \phi(\bar{a})$ the argument is very similar. It is equivalent to $\mathcal{B} \models \neg\phi(\bar{a})$, so we can easily show that this implies $\mathcal{A} \models \neg\phi(\bar{a})$ same as previous part. So we have $\mathcal{A} \models \phi(\bar{a})$ if and only if $\mathcal{B} \models \phi(\bar{a})$ for all $\bar{a} \in A^n$. Hence, T is model complete. □

Remark 2.15. Statement (iv) is also equivalent to the following statement:

(iv^*) : *For every formula* $\phi(\bar{v})$*, there is an existential formula* $\psi(\bar{v})$ *such that* $T \models \forall\bar{v}\ (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

To observe this, assume a formula $\phi(\bar{v})$ is equivalent to a universal formula. Since every universal formula $\forall\bar{w}\gamma(\bar{w},\bar{v})$ can be written as $\neg\exists\bar{w}\ \neg\gamma(\bar{w},\bar{v})$ and since every (existential) formula is equivalent to a universal formula by statement (iv), by negating this universal formula we obtain an equivalent existential formula. Conversely, assume a formula $\phi(\bar{v})$ is equivalent to a existential formula. Since every existential formula $\exists\bar{w}\gamma(\bar{w},\bar{v})$ can be written as $\neg\forall\bar{w}\ \neg\gamma(\bar{w},\bar{v})$ and since every (universal) formula is equivalent to a existential formula by statement (iv^*), by negating this existential formula we obtain a universal formula. Thus, $(iv) \Leftrightarrow (iv^*)$.

Remark 2.16. In the proof of Theorem 2.17 and Theorem 2.10, we expand the language by adding new constant symbols, then get rid of the free variables and obtain sentences in extended language. It is an important technique since Deduction Theorem [3, Theorem 1.3.2] is only applicable to the cases where there is no free variable.

Example 2.22. The theory of dense linear orders without endpoints (DLO) is model complete. We can show this by using condition (ii) in Theorem 2.17. Let $\mathcal{A} = (A, <)$ and $\mathcal{B} = (B, <)$ be two models of the theory such that $\mathcal{A} \subseteq \mathcal{B}$. We want to show that $\mathcal{A}$ is existentially closed in $\mathcal{B}$. So let $\phi(\bar{a}) : \exists \bar{v}\ \psi(\bar{v}, \bar{a})$ be an existential $\mathcal{L}_A$-sentence. Since the language consists only of an order relation, the existential sentence $\phi(\bar{a})$ can only describe the positions of elements with respect to $a_1, ..., a_n$. So we will show that if such elements exists in B at these positions (or we may say if the positions that are described by the formula is meaningful), we can also find some elements in A lies exactly in the same position with respect to the elements $a_1, ..., a_n$.

Assume, without loss of generality, that a_i satisfies $a_1 < a_2 < ... < a_n$. Consider the position b_1 with respect to a_i's. First case is to consider is $b_1 = a_i$ for some i, but then a_i itself is the element in that position in A; so this is the trivial case. If b_1 is different from all a_i's, then either of the three statement holds: it may be smaller than all a_i, or it may be bigger than all a_i, or it lies in some interval $[a_i, a_{i+1}]$ for some $i = 1, 2, ..., n-1$.

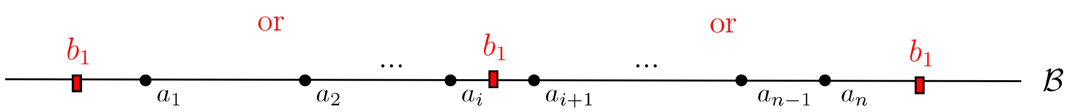


Figure 2.6. All possible places of b_1 relative to a_i's.

In any case, we can find an element c_1 in A having the same position with b_1 with respect to the elements a_i since $\mathcal{A}$ is a dense linear order with no endpoints. Similarly, by doing this for all b_j, $j = 1, 2, ..., m$; we get

$$\mathcal{A} \models \psi(\bar{c}, \bar{a}) \textit{ if and only if } \mathcal{B} \models \psi(\bar{b}, \bar{a}).$$

Hence, $\mathcal{A} \models \phi(\bar{a})$ *if and only if* $\mathcal{B} \models \phi(\bar{a})$ for all $\bar{a} \in A^n$. We showed that an arbitrary model of DLO is existentially closed. Hence DLO is model complete.

Example 2.23. The theory of dense linear orders with endpoints is *not* model complete. Consider two models $\mathcal{A} = ([0,1], <)$ and $\mathcal{B} = ([0,2], <)$ of the theory which satisfies $\mathcal{A} \subseteq \mathcal{B}$. The existential $\mathcal{L}_A$-sentence $\exists v(1 < v)$ is satisfied in $\mathcal{B}$, but it is not satisfied in $\mathcal{A}$. Hence, $\mathcal{A}$ is not existentially closed which shows that the theory is *not* model complete.

2.5. Model Companion

In this section, we will define *model companion of a theory* and discuss properties of theories with model companions, whenever they exist.

Definition 2.29. Let T be an $\mathcal{L}$-theory. An $\mathcal{L}$-theory T^* is called *model companion* of T if the following three conditions are satisfied:

- T^* is model complete.
- Every model of T can be embedded into a model of T^*.
- Every model of T^* can be embedded into a model of T.

A theory is called *companionable* if it has a model companion. Moreover, if a theory is companionable, model companion of the theory is unique up to equivalence of theories; *i.e.*, if T^* and T^{**} are model companions of T, then models of T^* and T^{**} are the same. (Theorem 2.22)

Example 2.24. i. The model companion of the theory of fields is the theory of algebraically closed fields (ACF).

ii. The model companion of the theory of linear orders is the theory of dense linear orders without endpoints (DLO).

We know that DLO and ACF are model complete theories as they are verified before in the text. Also, other conditions in definition can be checked easily. The purpose of this thesis is to study the concept of model companionability in detail. In this section, we will investigate the properties and we will present more examples in Chapter 3.

Definition 2.30. An $\mathcal{L}$-theory T^* is called the *model completion* of an $\mathcal{L}$-theory T if we have the following:

- T^* is the model companion of T.
- For every model $\mathcal{A}$ of T, $T^* \cup Diag(\mathcal{A})$ is complete.

According to definition above, a model companion T^* of T is called model completion if for any two extensions of $\mathcal{A}$ that are models of T^* are elementarily equivalent as $\mathcal{L}_A$-structures.

To give an equivalent condition for being a model completion of a theory, we first define the following property.

Definition 2.31. Let T be an $\mathcal{L}$-theory. T is said to have *amalgamation property* if $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are models of T and $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{A} \to \mathcal{C}$ are $\mathcal{L}$-embeddings, then there exists a model $\mathcal{A}'$ of T with embeddings $f' : \mathcal{B} \to \mathcal{A}'$, $g' : \mathcal{C} \to \mathcal{A}'$ such that the diagram commutes.

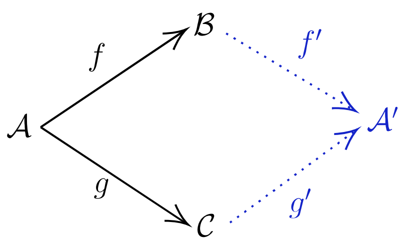


Figure 2.7. Amalgamation property.

Example 2.25. The theory of fields has amalgamation property. More precisely, if two fields F_1 and F_2 have a common subfield F, then we can look at their tensor product over F since they have the same characteristics and the field of fractions of the tensor product $F_1 \otimes_F F_2$ is a field that contains both F_1 and F_2. Therefore, the theory of fields has amalgamation property.

Note that having a common subfield is crucial to extend two fields to a common field. For example, we may have two fields with different characteristics and in this case we cannot extend them to a common field.

Theorem 2.18. *Let T^* be the model companion of a theory T. The following are equivalent.*

- *T^* is the model completion of T.*
- *T has amalgamation property.*

Proof. ($\Rightarrow$) Assume T^* is the model completion of T. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be models of T with embeddings $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \to \mathcal{C}$. Then, embed $\mathcal{B}$ and $\mathcal{C}$ into the models $\mathcal{B}'$ and $\mathcal{C}'$ of T^*, respectively. Observe that $\mathcal{B}'$ and $\mathcal{C}'$ are also models of the *complete* theory $T^* \cup Diag(\mathcal{A})$. We claim that an extension $\mathcal{A}'$ of $\mathcal{B}'$ and $\mathcal{C}'$ satisfying $\mathcal{B}' \subseteq \mathcal{A}'$ and $\mathcal{B}' \subseteq \mathcal{C}'$ exists; that is, we claim that $T^* \cup Diag(\mathcal{A}) \cup Diag(\mathcal{B}) \cup Diag(\mathcal{C})$ is satisfiable. Suppose the claim holds, then we can embed $\mathcal{A}'$ into a model $\mathcal{D}$ of T and this completes the proof since we obtain embeddings $f' : \mathcal{B} \to \mathcal{D}$ and $g' : \mathcal{C} \to \mathcal{D}$ that shows T has amalgamation property.

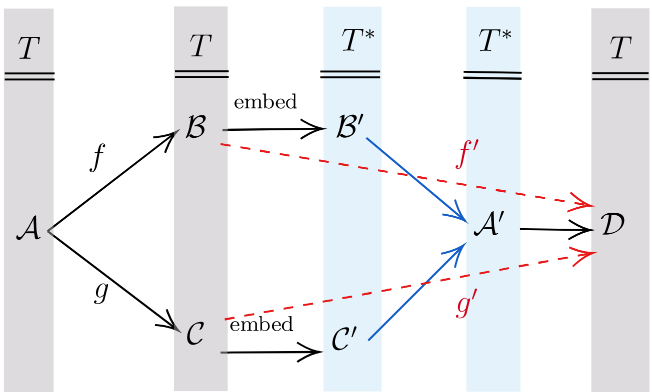


Figure 2.8. Illustration of the first part of the proof of Theorem 2.18.

It remains to show that $T^* \cup Diag(\mathcal{A}) \cup Diag(\mathcal{B}) \cup Diag(\mathcal{C})$ is satisfiable. Assume for a contradiction, it is not satisfiable. Then by Compactness Theorem there are finite subsets $\{\theta_1, \ldots, \theta_n\} \subseteq Diag(\mathcal{A})$ and $\{\gamma_1, \ldots, \gamma_m\} \subseteq Diag(\mathcal{B})$ such that $T^* \cup Diag(\mathcal{A}) \cup \{\bigwedge_{i=1}^{n} \theta_i\} \cup \{\bigwedge_{j=1}^{m} \gamma_j\}$ is not satisfiable. Expand the language by adding new constant symbols for the elements that are used in θ_i and γ_j and denote them as $\theta_i(\bar{c})$ and $\gamma_j(\bar{c})$ in new language. We see that $T^* \cup Diag(\mathcal{A}) \cup \{\bigwedge_{i=1}^{n} \theta_i(\bar{c})\}$ is satisfiable since $\mathcal{B} \models T^* \cup Diag(\mathcal{A})$ and $\theta_i \in Diag(\mathcal{B})$ implies $\mathcal{B} \models \bigwedge_{i=1}^{n} \theta_i(\bar{c})$. Now, observe that if $T^* \cup Diag(\mathcal{A}) \cup \{\exists \bar{v}\ (\bigwedge_{i=1}^{n} \theta_i(\bar{v}))\} \models \exists \bar{v}\ (\bigwedge_{j=1}^{m} \gamma_j(\bar{v}))$, then by interpreting the constant

symbols as witnesses of these existential sentences, we see that
$T^* \cup Diag(\mathcal{A}) \cup \{\bigwedge_{i=1}^n \theta_i\} \cup \{\bigwedge_{j=1}^m \gamma_j\}$ is satisfiable which contradicts with the assumption that it is not satisfiable. So we should have
$T^* \cup Diag(\mathcal{A}) \cup \{\exists \bar{v}\ (\bigwedge_{i=1}^n \theta_i(\bar{v}))\} \not\models \exists \bar{v}\ (\bigwedge_{j=1}^m \gamma_j(\bar{v}))$; that is,
$T^* \cup Diag(\mathcal{A}) \cup \{\exists \bar{v}\ (\bigwedge_{i=1}^n \theta_i(\bar{v}))\} \models \forall \bar{v}\ (\bigvee_{j=1}^m \neg\gamma_j(\bar{v}))$. However, since $\mathcal{B}$ are $\mathcal{C}$ are models of the complete theory $T \cup Diag(\mathcal{A})$, we have $\mathcal{B} \models \exists \bar{v}\ (\bigwedge_{i=1}^n \theta_i(\bar{v}))$ implies $\mathcal{C} \models \exists \bar{v}\ (\bigwedge_{i=1}^n \theta_i(\bar{v}))$ and hence we obtain $\mathcal{C} \models \forall \bar{v}\ \neg\gamma_j(\bar{v})$ for some j, which is a contradiction since $\gamma_j \in Diag(\mathcal{C})$. Hence, $T^* \cup Diag(\mathcal{A}) \cup Diag(\mathcal{B}) \cup Diag(\mathcal{C})$ is satisfiable.

($\Leftarrow$) Assume T has amalgamation property and let T^* be model companion of T. Also, let $\mathcal{A}$ be a model of T and let $\mathcal{B}'$ and $\mathcal{B}''$ be models of the $\mathcal{L}_A$-theory $T^* \cup Diag(\mathcal{A})$. We will prove that $T \cup Diag(\mathcal{A})$ is complete by showing $\mathcal{B}' \equiv \mathcal{B}''$.

1. Since $\mathcal{B}', \mathcal{B}'' \models Diag(\mathcal{A})$, there are embeddings $f : \mathcal{A} \to \mathcal{B}'$ and $g : \mathcal{A} \to \mathcal{B}''$ by Diagram Lemma 2.6.
2. We can embed the models $\mathcal{B}'$ and $\mathcal{B}''$ of T^* into models $\mathcal{A}'$ and A'' of T, respectively since T^* is the model companion of T.
   Thus, there are embeddings from $\mathcal{A} \models T$ into models $\mathcal{A}'$ and $\mathcal{A}''$ of T.
3. By amalgamation property of T, there is a model $\mathcal{A}'''$ of T such that $\mathcal{A}'$ and $\mathcal{A}''$ embeds into $\mathcal{A}'''$; that is, there are embeddings $f' : \mathcal{A}' \to \mathcal{A}'''$ and $g' : \mathcal{A}'' \to \mathcal{A}'''$.
4. As a last step, we embed the model $\mathcal{A}'''$ of T into a model $\mathcal{B}'''$ of the model companion T^* of T.

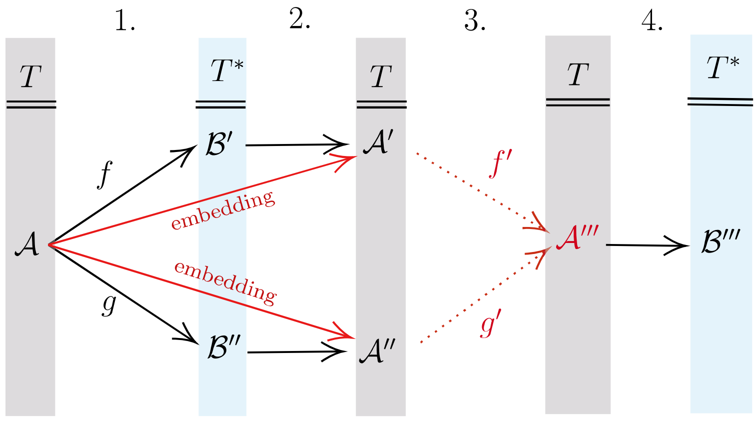


Figure 2.9. Illustration of the second part of the proof of Theorem 2.18.

Since T^* is model complete and $\mathcal{B}' \subseteq \mathcal{B}'''$ and $\mathcal{B}'' \subseteq \mathcal{B}'''$, we have $\mathcal{B}' \preccurlyeq \mathcal{B}'''$ and $\mathcal{B}'' \preccurlyeq \mathcal{B}'''$. Also, since A is contained in both B' and B''; for all $\bar{a} \in A^n$ and for all $\mathcal{L}$-formula $\phi(\bar{v})$ we have

$$\mathcal{B}' \models \phi(\bar{a}) \Leftrightarrow \mathcal{B}''' \models \phi(\bar{a}) \Leftrightarrow \mathcal{B}'' \models \phi(\bar{a}).$$

Hence, $\mathcal{B}' \equiv \mathcal{B}''$ as $\mathcal{L}_A$-structures. Since two arbitrary models of $T^* \cup Diag(\mathcal{A})$ are elementarily equivalent, the theory $T^* \cup Diag(\mathcal{A})$ is complete by Proposition 2.1. Thus, T^* is the model completion of T. □

Corollary 2.19. *ACF is the model completion of the theory of fields.*

Proof. The theory of fields has amalgamation property by Example 2.25, so the model companion ACF of the theory of fields is the model completion of the theory of fields by Theorem 2.18. □

Inductive Theories

Definition 2.32. Let $(I, <)$ be an ordered set. A set of $\mathcal{L}$-structures $(\mathcal{A}_i : i \in I)$ is called:

i. a *chain* if for any $i < j$, we have $\mathcal{A}_i \subseteq \mathcal{A}_j$.
ii. an *elementary chain* if for any $i < j$, we have $\mathcal{A}_i \preccurlyeq \mathcal{A}_j$.

Let $(\mathcal{A}_i : i \in I)$ be a chain. We construct an $\mathcal{L}$-structure $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ satisfying $\mathcal{A}_i \subseteq \bigcup_{i \in I} \mathcal{A}_i$ as follows:

1. The universe of $\mathcal{A}$ is $\bigcup_{i \in I} A_i$.
2. Let c be a constant symbol in $\mathcal{L}$. Observe that $c^{\mathcal{A}_i} = c^{\mathcal{A}_j}$ for all $i < j$ since $\mathcal{A}_i \subseteq A_j$. So we can define $c^{\mathcal{A}}$ as $c^{\mathcal{A}} = c^{\mathcal{A}_i}$ for some $i \in I$.
3. Let f be a function symbol in $\mathcal{L}$, we define $f^{\mathcal{A}}$ as $\bigcup_{i \in I} f_i^{\mathcal{A}_i}$. This definition is well defined because for $\bar{a} \in A^n$, there exists $\mathcal{A}_i$ containing $\bar{a}$ and we have $f^{\mathcal{A}_i}(\bar{a}) = f^{\mathcal{A}_j}(\bar{a})$ for all $j \geq i$ since $\mathcal{A}_i \subseteq \mathcal{A}_j$.

4. Let R be a relation symbol in $\mathcal{L}$ and let $\bar{a} \in A^n$. We define $R^{\mathcal{A}}$ as $\bigcup_{i\in I} R_i^{\mathcal{A}_i}$.

Proposition 2.3. *Let $(\mathcal{A}_i : i \in I)$ be an elementary chain of $\mathcal{L}$-structures, then for all $i \in I$, $\mathcal{A} = \bigcup_{i\in I} \mathcal{A}_i$ is elementary extension of $\mathcal{A}_i$.*

Proof. Let $\phi(\bar{v})$ be an $\mathcal{L}$-formula. We will show that for all $\bar{a} \in A_i^n$,

$$\mathcal{A}_i \models \phi(\bar{a}) \textit{ if and only if } \mathcal{A} \models \phi(\bar{a}). \qquad (*)$$

The proof is by induction on length of formulas. We know by Proposition 2.2 that if $\phi(\bar{v})$ is quantifier free, then the statement $(*)$ holds since $\mathcal{A}_i \subseteq \mathcal{A}$. So if $\phi(\bar{v})$ is an atomic formula we clearly have the statement, and the cases where $\phi(\bar{v})$ is in the form $\neg\psi(\bar{v})$ and $\psi_1(\bar{v}) \wedge \psi_(\bar{v})$ are also shown in Proposition 2.2. So it only remains to show that we have the statement $(*)$ is true for $\phi(\bar{v}) : \exists w\ \psi(w, \bar{v})$. If $\mathcal{A} \models \phi(\bar{a})$, then there exists $b \in A$ such that $\mathcal{A} \models \psi(b, \bar{a})$ and there exist a structure $\mathcal{A}_j$ for some $j \geq i$, such that $b \in A_j$. We have $\mathcal{A}_j \models \psi(b, \bar{a})$ by induction hypothesis and hence $\mathcal{A}_j \models \phi(\bar{a})$. Since the chain is elementary, we have $\mathcal{A}_i \subseteq \mathcal{A}_j$ which implies that $\mathcal{A}_i \models \phi(\bar{a})$. If $\mathcal{A} \not\models \phi(\bar{a})$, then for any $b \in A$, $\mathcal{A} \not\models \psi(b, \bar{a})$. So by induction hypothesis, $\mathcal{A}_j \not\models \psi(b, \bar{a})$ for any $j \geq i$ and for any $b \in A_j$. Hence, $\mathcal{A}_i \not\models \phi(\bar{a})$. □

Definition 2.33. An $\mathcal{L}$-theory T is called *inductive* if it is closed under unions of chains of models. That is, for any chain $(\mathcal{A}_i : i \in I)$ such that $\mathcal{A}_i \models T$ for all $i \in I$, we have $\bigcup_{i\in I} \mathcal{A}_i \models T$.

Example 2.26. (i) The theory of fields is inductive since union of any towers of fields is again a field. Also, the theory of groups, the theory of rings are inductive theories.

(ii) The theory of dense linear orders with endpoints is not inductive. Consider the chain of models $(\mathcal{A}_i : i \geq 1)$ where $\mathcal{A}_i = ([-i, i], <)$. We see that $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ is not a model of the theory of dense linear orders with endpoints since it has no endpoints.

Theorem 2.20. *If T is a model complete theory, then it is inductive.*

Proof. Let T be a model complete theory, let $(I, <)$ be some ordered set and consider a chain $(\mathcal{A}_i : i \in I)$ of models of T. Since T is model complete, $(\mathcal{A}_i : i \in I)$ is an elementary chain, so by Proposition 2.3 we have $\mathcal{A}_i \preccurlyeq \bigcup_{i\in I} \mathcal{A}_i$ for all $i \in I$. Since all $\mathcal{A}_i \models T$, we have $\bigcup_{i\in I} \mathcal{A}_i \models T$. Hence T is inductive. □

Remark 2.17. The converse of the above theorem is *not* true. To illustrate, the theory of fields is inductive but it is not model complete.

The following theorem states that if a theory is inductive, then its axioms consist of $\forall\exists$-sentences; that is; sentences in the form $\forall\bar{v}\exists\bar{w}\ \psi(\bar{v},\bar{w})$ where $\psi(\bar{v},\bar{w})$ is quantifier free. The converse of this statements is also true.

Theorem 2.21. *T is inductive if and only if it is a $\forall\exists$-theory.*

Proof. ($\Leftarrow$) Let T be a $\forall\exists$-theory and let $(\mathcal{A}_i : i \in I)$ be a chain of models of T. We will show that $\bigcup_{i\in I} \mathcal{A}_i \models T$. So take a sentence $\phi \in T$, we know that ϕ is of the form $\forall\bar{v}\,\exists\bar{w}\,\psi(\bar{v},\bar{w})$ for some quantifier free formula $\psi(\bar{v},\bar{w})$. Let $\bar{a} \in \bigcup_{i\in I} A_i$, then there exists $i \in I$ such that $\bar{a} \in A_i^n$. Since $\mathcal{A}_i \models \forall\bar{v}\,\exists\bar{w}\,\psi(\bar{v},\bar{w})$ we have $\mathcal{A}_i \models \exists\bar{y}\,\psi(\bar{a},\bar{y})$.
We proceed as,

$$\begin{aligned}
\mathcal{A}_i \models \exists\bar{w}\,\psi(\bar{a},\bar{w}) &\Rightarrow \mathcal{A}_i \models \psi(\bar{a},\bar{b}) \textit{ for some } \bar{b} \in A_i^m. \\
&\Rightarrow \bigcup_{i\in I} \mathcal{A}_i \models \psi(\bar{a},\bar{b}) \textit{ by Proposition 2.2 since } \mathcal{A}_i \subseteq \bigcup_{i\in I} \mathcal{A}_i \textit{ and } \psi(\bar{v},\bar{w}) \\
&\qquad\qquad \textit{is quantifier free.} \\
&\Rightarrow \bigcup_{i\in I} \mathcal{A}_i \models \exists\bar{w}\,\psi(\bar{a},\bar{w}). \\
&\Rightarrow \bigcup_{i\in I} \mathcal{A}_i \models \forall\bar{v}\exists\bar{w}\,\psi(\bar{v},\bar{w}) \textit{ since } \bar{a} \in \bigcup_{i\in I} \mathcal{A}_i \textit{ was arbitrarily chosen.}
\end{aligned}$$

Hence, $\bigcup_{i\in I} \mathcal{A}_i \models \phi$ for all $\phi \in T$. Therefore, T is inductive.

($\Rightarrow$) Suppose T is an inductive theory and let

$$\Sigma = \{\phi : \phi \textit{ is } \forall\exists\textit{-sentence and } T \models \phi\}.$$

We will show that the models of T and Σ are exactly the same. Clearly, $T \models \Sigma$ by definition. To show that $\Sigma \models T$, let $\mathcal{A} \models \Sigma$. We will build the following chain of $\mathcal{L}$-structures

$$\mathcal{A}=\mathcal{A}_0 \subseteq B_0 \subseteq \mathcal{A}_1 \subseteq B_1 \subseteq \mathcal{A}_2 \subseteq B_2 \subseteq \ldots$$

$$\underbrace{\mathcal{A}_0 \quad \mathcal{A}_1}_{\preccurlyeq} \quad \underbrace{\mathcal{A}_1 \quad \mathcal{A}_2}_{\preccurlyeq} \quad \underbrace{\mathcal{A}_2 \quad \ldots}_{\preccurlyeq}$$

where each $\mathcal{B}_i \models T$ and $(\mathcal{A}_i : i \in \mathbb{N})$ is an elementary chain, and prove that $\mathcal{A} \models T$.

1. First of all, we will show that there exists $\mathcal{B} \models T$ such that for any $\exists\forall$-*sentence* ψ, $\mathcal{A} \models \psi$ implies $\mathcal{B} \models \psi$. So let $\Delta = \{\psi : \psi \textit{ is } \exists\forall\textit{-sentence and } \mathcal{A} \models \psi\}$.

Claim 1: $T \cup \Delta$ is satisfiable.

Assume for a contradiction, $T \cup \Delta$ is not satisfiable. So by Compactness Theorem there is a finite subset $\{\delta_1, ..., \delta_n\} \subseteq \Delta$ such that $T \cup \{\delta_1, ..., \delta_n\}$ is not satisfiable. Add new constant symbols to the language for the elements that are used in each δ_i and denote these sentences as $\delta_i(\bar{c})$ in the new language. Notice that if $T \models \exists\bar{v}\ \bigwedge_{i=1}^{m} \delta_i(\bar{v})$, then by interpreting constant symbols as witnesses to existential sentence we obtain $T \cup \{\delta_1, ..., \delta_n\}$ is satisfiable which contradicts with the assumption that $T \cup \{\delta_1, ..., \delta_n\}$ is not satisfiable. So we have $T \not\models \exists\bar{v}\ \bigwedge_{i=1}^{m} \delta_i(\bar{v})$; that is, we have $T \models \forall\bar{v}\ \bigvee_{i=1}^{m} \neg\delta_i(\bar{v})$ which implies that $\forall\bar{v}\ \bigvee_{i=1}^{m} \neg\delta_i(\bar{v}) \in \Sigma$. Also since $\mathcal{A} \models \Sigma$, we have $\mathcal{A} \models \forall\bar{v}\ \bigvee_{i=1}^{m} \neg\delta_i(\bar{v})$. However, this is a contradiction due to the fact that $\mathcal{A} \models \exists\bar{v}\ \bigwedge_{i=1}^{m} \delta_i(\bar{v})$ by definition of Δ.

2. Now, we will show that there exists $\mathcal{B}_0$ such that $\mathcal{A} \subseteq \mathcal{B}_0$ and $\mathcal{B} \equiv \mathcal{B}_0$.

Claim 2: $Th(\mathcal{B}) \cup Diag(\mathcal{A})$ is satisfiable.

Suppose for a contradiction, $Th(\mathcal{B}) \cup Diag(\mathcal{A})$ is not satisfiable. Then by Compactness Theorem there is a finite subset $\{\theta_1, ..., \theta_m\} \subseteq Diag(\mathcal{A})$ such that $Th(\mathcal{B}) \cup \{\bigwedge_{i=1}^{m} \theta_i\}$ is not satisfiable. Replace the elements that are used in θ_i by new constant symbols and denote sentences in expanded language as $\theta_i(\bar{c})$. Observe that if $Th(\mathcal{B}) \models \exists\bar{v}\ \bigwedge_{i=1}^{m} \theta_i(\bar{v})$, then we obtain $Th(\mathcal{B}) \cup \{\bigwedge_{i=1}^{m} \theta_i\}$ is satisfiable which contradicts with the assumption that $Th(\mathcal{B}) \cup \{\bigwedge_{i=1}^{m} \theta_i\}$ is not satisfiable. So we have $Th(\mathcal{B}) \not\models \exists\bar{v}\ \bigwedge_{i=1}^{m} \theta_i(\bar{v})$; that is, $Th(\mathcal{B}) \models \forall\bar{v}\ \bigvee_{i=1}^{m} \neg\theta_i(\bar{v})$. Note that we have $\mathcal{A} \models \psi$ implies $\mathcal{B} \models \psi$ for all $\exists\forall$-sentences ψ, so $\mathcal{B} \models \neg\psi$ implies $\mathcal{A} \models \neg\psi$ where $\neg\psi$ is a $\forall\exists$-sentence. But then we obtain $\mathcal{A} \models \forall \bar{v} \ \bigvee_{i=1}^{m} \neg\theta_i(\bar{v})$, which is a contradiction since $\theta_i \in Diag(\mathcal{A})$.

3. Lastly, we show that there exist $\mathcal{A}_1$ such that $\mathcal{B}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A} \preccurlyeq \mathcal{A}_1$. Clearly a model of $Diag(\mathcal{B}_0)$ is an extension of $\mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{B}_0$. Due to the fact that the extension should also be an elementary extension of $\mathcal{A}$, it should satisfy all existential $\mathcal{L}_A$-sentences satisfied by $\mathcal{A}$. So let $Th_\exists(\mathcal{A})$ denote the existential $\mathcal{L}_A$-sentences satisfied by $\mathcal{A}$.

Claim 3: $Th_\exists(\mathcal{A}) \cup Diag(\mathcal{B}_0)$ is satisfiable.

Suppose the claim is false. Then by Compactness Theorem there is a finite subset $\{\epsilon_1, ..., \epsilon_k\} \subseteq Th_\exists(\mathcal{A})$ such that $\{\bigwedge_{i=1}^{k} \epsilon_i\} \cup Diag(\mathcal{B}_0)$ is not satisfiable. Like in previous parts, we replace the elements that are used in each ϵ_i by new constant symbols and denote them as $\epsilon_i(\bar{c})$ in new language. We have

$$
\begin{aligned}
Diag(\mathcal{B}_0) \not\models \exists \bar{v} \bigwedge_{i=1}^{k} \epsilon_i(\bar{v}) &\Rightarrow Diag(\mathcal{B}_0) \models \forall \bar{v} \bigvee_{i=1}^{k} \neg\epsilon_i(\bar{v}) \\
&\Rightarrow \mathcal{B}_0 \models \forall \bar{v} \bigvee_{i=1}^{k} \neg\epsilon_i(\bar{v}) \Rightarrow \mathcal{B} \models \forall \bar{v} \bigvee_{i=1}^{k} \neg\epsilon_i(\bar{v}) \text{ since } \mathcal{B} \equiv \mathcal{B}_0 \\
&\Rightarrow \mathcal{A} \models \forall \bar{v} \bigvee_{i=1}^{k} \neg\epsilon_i(\bar{v}) \text{ by definition of } \Delta \\
&\qquad\qquad \text{since } \neg\epsilon_i\text{'s are universal sentences.}
\end{aligned}
$$

It is a contradiction since $\mathcal{A} \models \exists \bar{v} \bigwedge_{i=1}^{k} \epsilon_i(\bar{v})$. Hence there is $\mathcal{A}_1$ such that $\mathcal{A} = \mathcal{A}_0 \subseteq \mathcal{B}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A} = \mathcal{A}_0 \preccurlyeq \mathcal{A}_1$.

By iterating these stages, we obtain the desired chain. Notice that T is inductive and each $\mathcal{B}_i \models T$, so we have $\bigcup_{i=1}^{\infty} \mathcal{B}_i = \bigcup_{i=1}^{\infty} \mathcal{A}_i \models T$ and also since $(\mathcal{A}_i : i \in \mathbb{N})$ is an elementary chain, we have $\mathcal{A} \equiv \bigcup_{i=1}^{\infty} \mathcal{A}_i \models T$; that is, we have $\mathcal{A} \models T$. Therefore, $\Sigma \models T$. Due to the fact that Σ consists of $\forall\exists$-sentences and since models of Σ are exactly the same as models of T, this shows that T is can be axiomatized by $\forall\exists$-sentences. $\square$

Theorem 2.22. *If T is companionable, model companion T^* of T is unique up to equivalance of theories.*

Proof. Let T^* and T^{**} be two model companions of the theory T and let $\mathcal{A}_0$ be a model of T^*. We will show that $\mathcal{A}_0$ is also a model of T^{**}. Since T^* is a model companion, we can embed $\mathcal{A}_0$ into a model $\mathcal{A}_0'$ of T, and since T^{**} is also a model companion, we can embed $\mathcal{A}_0'$ into a model $\mathcal{A}_1$ of T^{**}. Thus, we can embed a model $\mathcal{A}_0$ of T^* into some model $\mathcal{A}_1$ of T^{**}. Similarly, we can embed a model $\mathcal{A}_1$ of T^{**} into a model $\mathcal{A}_2$ of T^* and by proceeding this way, we obtain the chain

$$\frac{T^*}{\mathcal{A}_0} \subseteq \frac{T^{**}}{\mathcal{A}_1} \subseteq \frac{T^*}{\mathcal{A}_2} \subseteq \frac{T^{**}}{\mathcal{A}_3} \subseteq \dots .$$

Moreover, since T^* and T^{**} are model complete theories, we also have two elementary chains,

$$\frac{T^*}{\mathcal{A}_0 \preccurlyeq \mathcal{A}_2 \preccurlyeq \mathcal{A}_4 \preccurlyeq \mathcal{A}_6 \preccurlyeq \dots ,}$$

We have $\mathcal{A}_0 \preccurlyeq \bigcup_{i=0}^{\infty} \mathcal{A}^{2i} = \bigcup_{i=0}^{\infty} \mathcal{A}^i$

by Proposition 2.3.

$$\frac{T^{**}}{\mathcal{A}_1 \preccurlyeq \mathcal{A}_3 \preccurlyeq \mathcal{A}_5 \preccurlyeq \mathcal{A}_7 \preccurlyeq \dots .}$$

We have $\mathcal{A}_1 \preccurlyeq \bigcup_{i=0}^{\infty} \mathcal{A}^{2i+1} = \bigcup_{i=0}^{\infty} \mathcal{A}^i$

by Proposition 2.3.

We obtain $\mathcal{A}_0 \equiv \bigcup_{i=0}^{\infty} \mathcal{A}_i \equiv \mathcal{A}_1$ and hence $\mathcal{A}_0 \models T^{**}$. We showed that a model of T^* is also a model of T^{**}. Therefore, models of T^* and T^{**} are exactly the same; that is, the theories are equivalent $T^* \equiv T^{**}$. □

Theorem 2.23. *Let T be an inductive theory. Every model of T can be extended to an existentially closed model of T.*

Proof. Let $\mathcal{A}$ be a model of T and let $\{\phi_i : i < \kappa\}$ be an enumeration of all existential $\mathcal{L}_A$-formulas. We will construct an extension $\mathcal{A}^1$ of $\mathcal{A}$ such that whenever ϕ_i is satisfied by an extension of $\mathcal{A}^1$ that models T, then it is also satisfied by $\mathcal{A}^1$. We build the chain

$$\mathcal{A} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots \subseteq \mathcal{A}_\lambda \subseteq \dots$$

recursively as follows:

i. Let $\mathcal{A} = \mathcal{A}_0$.
ii. nth step: Let $\mathcal{A}_n$ be constructed. If there is an extension of $\mathcal{A}_n$ that models T and ϕ_n, then let $\mathcal{A}_{n+1}$ be this model. If there is no such model, then let $\mathcal{A}_{n+1} = \mathcal{A}_n$.
iii. At limit ordinals λ, let $\mathcal{A}_\lambda = \bigcup_{i<\lambda} \mathcal{A}_i$. Since T is inductive, we have $\mathcal{A}_\lambda \models T$.
iv. Lastly, let $\mathcal{A}^1 = \mathcal{A}_\kappa$.

Clearly, if there is an existential $\mathcal{L}_A$-sentence satisfied by an extension of $\mathcal{A}^1$ that models T, it is equal to ϕ_i for some $i < \kappa$; so $\mathcal{A}_{i+1} \models \phi_i$. Since ϕ_i is an existential sentence, we can write it as $\exists \bar{v}\ \theta_i(\bar{v})$ where $\theta_i(\bar{v})$ is quantifier free. So $\mathcal{A}_i \models \theta_i(\bar{a})$ for some $\bar{a} \in A_i^n$ and we have $\mathcal{A}^1 \models \theta_i(\bar{a})$ by Proposition 2.2 since $\mathcal{A}_{i+1} \subseteq \mathcal{A}^1$. Hence $\mathcal{A}_1 \models \phi_i$.

Likewise, we can construct an extension $\mathcal{A}^2$ of $\mathcal{A}^1$ by applying the same technique that is presented above and $\mathcal{A}^2$ has the property that whenever an existential $\mathcal{L}_{A^1}$-sentence is satisfied by extension of $\mathcal{A}^2$ that is a model of T, it is also satisfied by $\mathcal{A}^2$. By iterating this process we obtain the chain

$$\mathcal{A} = \mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \mathcal{A}^2 \subseteq \mathcal{A}^3 \subseteq \dots .$$

Take the union of this chain and let $\mathcal{A}_{ec} = \bigcup_{i=0}^{\infty} \mathcal{A}^i$. Let us show that $\mathcal{A}_{ec}$ is existentially closed. Let $\psi(\bar{a})$ existential sentence with parameters from A_{ec}, that satisfied by an extension of $\mathcal{A}_{ec}$ modeling T. Then there exists j such that $\bar{a} \in (\mathcal{A}^j)^n$ and since $\psi(\bar{a})$ is an existential $\mathcal{L}^{\mathcal{A}^j}$- sentence, we have $\mathcal{A}^{j+1} \models \psi(\bar{a})$. Also since $\mathcal{A}^{j+1} \subseteq \mathcal{A}$ and $\psi(\bar{a})$ is an existential sentence, we have $\mathcal{A} \models \psi(\bar{a})$ (this is proved in previous half of the proof). Hence, $\mathcal{A}_{ec}$ is the desired existentially closed extension of $\mathcal{A}$. □

Theorem 2.24. *Let T be an inductive theory.*

i. If the model companion T^* of T exists, then models of T^* are exactly the existentially closed models of T.

ii. T is companionable if and only if the class of existentially closed models is elementary.

Proof. i. Assume that T^* is the model companion of an inductive theory T. First, we assume $\mathcal{A} \models T^*$ and we will to show that $\mathcal{A}$ is an existentially closed model of T. We will first show that $\mathcal{A} \models T$. Start by embedding $\mathcal{A}$ into a model $\mathcal{B}$ of T and then embed $\mathcal{B}$ into a model $\mathcal{A}'$ of T^*. Obviously, we can do this since T^* is model companion of T. Also, observe that $\mathcal{A} \preccurlyeq \mathcal{A}'$ since T^* is model complete.

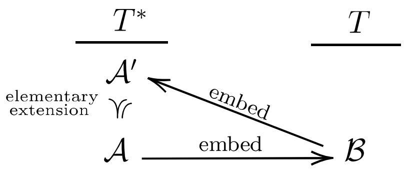


Figure 2.10. First diagram demonstrating the proof of Theorem 2.24.

Now the claim is; every existential sentence with parameters from A satisfied by $\mathcal{B}$ is also satisfied by $\mathcal{A}$. So let $\phi(\bar{v})$ be an existential formula and let $\bar{a} \in A^n$. Assume $\mathcal{B} \models \phi(\bar{a})$. Then since we can view B as a substructure of $\mathcal{A}'$; that is, $\mathcal{B} \subseteq \mathcal{A}'$, we directly have $\mathcal{A}' \models \phi(\bar{a})$. Moreover, since $\mathcal{A} \preccurlyeq \mathcal{A}'$, we have $\mathcal{A} \models \phi(\bar{a})$. Hence, we showed that

$$\mathcal{B} \models \phi(\bar{a}) \textit{ implies } \mathcal{A} \models \phi(\bar{a}).$$

By using this fact, we will show that $\mathcal{A} \models T$. Since T is an inductive theory, T is $\forall\exists$-theory by Theorem 2.21. So let $\psi \in T$ such that $\psi = \forall\bar{v}\exists\bar{w}\,\theta(\bar{v},\bar{w})$. We have $\mathcal{B} \models \forall\bar{v}\exists\bar{w}\,\theta(\bar{v},\bar{w})$ since $\mathcal{B} \models T$ and it means that $\mathcal{B} \models \exists\bar{w}\,\theta(\bar{b},\bar{w})$ for all $\bar{b} \in B^n$. Also, since $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{B} \models \exists\bar{w}\,\theta(\bar{a},\bar{w})$ for all $\bar{a} \in A^n$. Now, we have an existential sentence with parameters from $\mathcal{A}$ that is satisfied in $\mathcal{B}$; by the above fact, we obtain $\mathcal{A} \models \exists\bar{w}\,\theta(\bar{a},\bar{w})$ for all $\bar{a} \in A^n$ which is equivalent to $\mathcal{A} \models \forall\bar{v}\exists\bar{w}\,\theta(\bar{v},\bar{w})$.
Therefore, $\mathcal{A} \models T$.

It remains to show that $\mathcal{A}$ is an existentially closed model of T, so let $\mathcal{B}'$ be an extension of $\mathcal{A}$ such that $\mathcal{B}' \models T$ and let $\exists\bar{v}\,\delta(\bar{v},\bar{a})$ be an existential sentence with parameters from $\mathcal{A}$ such that $\mathcal{B}' \models \exists\bar{v}\,\delta(\bar{v},\bar{a})$. We want to show that $\mathcal{A} \models \exists\bar{v}\,\delta(\bar{v},\bar{a})$. First of all, we can embed $\mathcal{B}'$ to a model $\mathcal{A}''$ of T^*. Observe that $\mathcal{A}''$ is an elementary extension of $\mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{B}' \subseteq \mathcal{A}''$ and $\mathcal{A} \models T^*$. Now, we obtain $\mathcal{A}'' \models \exists \bar{v}\, \delta(\bar{v}, \bar{a})$ since $\mathcal{B}' \models \exists \bar{v}\, \delta(\bar{v}, \bar{a})$ and $\mathcal{A} \subseteq \mathcal{B}'$. Therefore, $\mathcal{A} \models \exists \bar{v}\, \delta(\bar{v}, \bar{a})$ since $\mathcal{A} \preccurlyeq \mathcal{A}''$. We showed that $\mathcal{A}$ is existentially closed model of T.

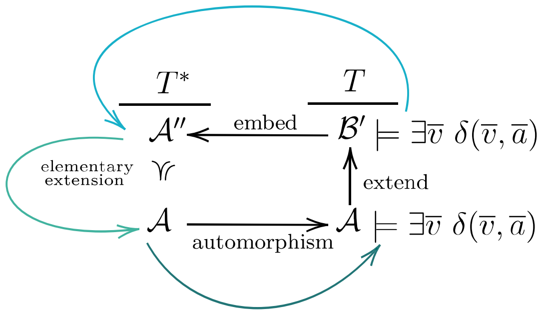


Figure 2.11. Second diagram demonstrating the proof of Theorem 2.24.

For the converse, assume $\mathcal{A}$ is an existentially closed model of T. We want to show that $\mathcal{A} \models T^*$. Firstly, observe that T^* is an inductive theory by Corrolary 2.20 since it is model complete. So it has $\forall\exists$-axiomatization by Theorem 2.21. Now, embed $\mathcal{A}$ into a model $\mathcal{B}$ of T^* and then embed $\mathcal{B}$ into a model $\mathcal{C}$ of T. Let $\forall \bar{v} \exists \bar{w}\, \phi(\bar{v}, \bar{w})$ be an axiom of T^*.

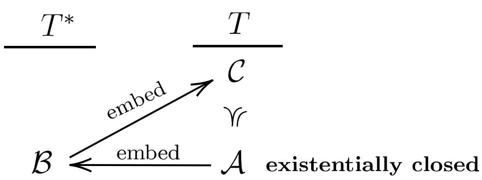


Figure 2.12. Third diagram demonstrating the proof of Theorem 2.24.

Since $\mathcal{B} \models T^*$, we have $\mathcal{B} \models \forall \bar{v} \exists \bar{w}\, \phi(\bar{v}, \bar{w})$ which is equivalent to $\mathcal{B} \models \exists \bar{w}\, \phi(\bar{b}, \bar{w})$ for all $\bar{b} \in B^n$. Also, since $\mathcal{B}$ is embedded into $\mathcal{C}$, the existential sentence $\exists \bar{w}\, \phi(\bar{b}, \bar{w})$ is also satisfied in $\mathcal{C}$; that is, $\mathcal{C} \models \exists \bar{w}\, \phi(\bar{b}, \bar{w})$ for all $\bar{b} \in B^n$. Further, since $\mathcal{A}$ is existentially closed model of $\mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{C}$, we have $\mathcal{A} \models \exists \bar{w}\, \phi(\bar{a}, \bar{w})$ for all $\bar{a} \in A^n$. Thus, $\mathcal{A} \models \forall \bar{v} \exists \bar{w}\, \phi(\bar{v}, \bar{w})$. Therefore, $\mathcal{A} \models T^*$.

ii. Right to left implication is proved in part *i.*. Indeed, if a theory T has model companion T^*, we know that the models of T^* are exactly the existentially closed models of T. Therefore, T^* is an axiomatization for the class of existentially closed models of T.

To prove the converse, assume that the class of existentially closed models of T is elementary and call its theory as T'. Clearly, we can embed a model of T' to a model of T by just sending it to itself since every mod of T' is also a model of T. Conversely, we can extend a model of T to an existentially closed model of T by Theorem 2.23 since T is inductive. So we can embed a model of T into a model of T'. Lastly, we see that T' is model complete since it consists of existentially closed models of T, we clearly see that any such model remains existentially closed in T'. Therefore, T is companionable since T' is the model companion of T. □

Remark 2.18. If there is an inductive theory T, the procedure of finding the model companion of T is trying to find an axiomatization of the class of existentially closed models of T. Likewise, to show that T has no model companion, we may show that the class of existentially closed models of T does not form an elementary class.

2.6. Examples of Theories and Their Properties

2.6.1. Theory of Equivalence Relations

Let $\mathcal{L} = \{E\}$ be a language consisting of a binary relation symbol. An equivalence relation E has three properties: it is reflexive, symmetric and transitive. So the axioms of the theory of equivalence relations are

$$E_1 : \forall v\ E(v, v) \qquad \text{(reflexive)},$$
$$E_2 : \forall v \forall w \big(E(v, w) \to E(w, v)\big) \qquad \text{(symmetric)},$$
$$E_3 : \forall v_1 \forall v_2 \forall v_3 \big((E(v_1, v_2) \wedge E(v_2, v_3)) \to E(v_1, v_3)\big) \qquad \text{(transitive)}.$$

The theory of equivalence relations with infinitely many classes: This theory consists of the axioms of the theory of equivalence relations E_1, E_2 and E_3 that are stated above, and additionally we need to add the set of axioms $\{\psi_n : n \geq 2\}$ where each ψ_n is defined as

$$\psi_n : \exists v_1 \exists v_2 \ldots \exists v_n \left(\bigwedge_{1 \leq i < j \leq n} \neg E(v_i, v_j) \right).$$

Note that each ψ_n says that there are at least n equivalence classes. Hence, E_1, E_2 and E_3 together with the set of axioms $\{\psi_n : n \geq 2\}$ constitutes the axioms of theory of equivalence relations with infinitely many classes.

The theory of equivalence relations with infinitely many infinite classes: The infinite set of axioms $\{\phi_n : n \geq 2\}$ together with the axioms of the theory of equivalence relations with infinitely many classes are the axioms of the theory of equivalence relations with infinitely many infinite classes where each ϕ_n is defined as

$$\phi_n : \forall v \exists v_1 \exists v_2 \ldots \exists v_n \left(\bigwedge_{1 \leq i < j \leq n} v_i \neq v_j \ \wedge \ \bigwedge_{i=1}^{n} E(v, v_i) \right).$$

Notice that each ϕ_n saying that there are at least n elements in an equivalence class. Hence the axioms of this theory are E_1, E_2 and E_3 together with the set of axioms $\{\phi_n : n \geq 2\} \cup \{\psi_n : n \geq 2\}$. Let ER* denote the theory of equivalence relations with infinitely many infinite classes.

Properties:

1. ER* is $\aleph_0$-categorical. (This can be shown by back and forth argument.) Hence, it is *complete* by Vaught's Test 2.8 since it has no finite models.
2. ER* has *quantifier elimination.*
3. ER* is *model complete* since it eliminates quantifiers by Theorem 2.15.
4. ER* is the *model companion* of the theory of equivalence relations. (see Proposition 3.2)

2.6.2. Theory of Dense Linear Orders

Let $\mathcal{L} = \{<\}$ be a language consisting only of a binary relation symbol. Consider the axioms

$$L_1 : \forall v \;\; \neg(v < v) \qquad \text{(irreflexive)},$$
$$L_2 : \forall v_1 \, \forall v_2 \, \forall v_3 \;\; [(v_1 < v_2) \wedge (v_2 < v_3) \to v_1 < v_3] \qquad \text{(transitive)},$$
$$L_3 : \forall v \, \forall w \;\; (v < w \vee v = w \vee w < v) \qquad \text{(trichotomy)},$$
$$D : \forall v_1 \forall v_2 \;\; (v_1 < v_2 \to \exists w (v_1 < w \wedge w < v_2)) \qquad \text{(denseness)}.$$

The theory of *linear orders* consists of the axioms $\{L_1, L_2, L_3\}$ and the theory of *dense linear orders* consists of the axioms $\{L_1, L_2, L_3, D\}$. For example, the structure $(\mathbb{Z}, <)$ of integers with usual order relation is a model of the theory of linear orders, but it is not a model of the theory of dense linear orders since it does not satisfy denseness property; whereas the structure $(\mathbb{Q}, <)$ of rational numbers with usual order relation is model of both of the theories. We can extend the theory of dense linear orders and obtain the theory of dense linear orders with endpoints by adding the axioms

$$\gamma_1 : \exists w_1 \forall v \;\; (w_1 < v \vee v = w_1), \;\; \gamma_2 : \exists w_2 \forall v \;\; (v < w_2 \vee v = w_2). \qquad \text{(endpoints exists)}$$

Instead of the axioms γ_1 and γ_2, we can also add the following axiom to the theory of dense linear orders which states that there is no endpoint, and obtain the theory of dense linear orders without endpoints, which we denote as DLO.

$$\delta : \forall v \, \exists w_1 \, \exists w_2 \;\; (w_1 < v \wedge v < w_2). \qquad \text{(no endpoints)}$$

Let us list some properties of DLO.

Properties:

1. DLO is $\aleph_0$-categorical (see Example 2.15). So it is *complete* by Vaught Test (Theorem 2.8) since it also has no finite models.
2. DLO is *model complete.* (see Example 2.22)
3. The theory of dense linear orders with endpoints is *not* model complete. (see Example 2.23)
4. DLO *eliminates quantifiers.* [1, Theorem 3.1.3]
5. DLO is the *model companion* of the theory of linear orders. (see Proposition 3.4)

2.6.3. Theory of Groups

Let $\mathcal{L}_g = \{\cdot, e\}$ be the language of groups consisting of a binary relation symbol " $\cdot$ " and a constant symbol " e ". A group structure has three properties; associativity of the operation, existence of identity element and inverses. So the axioms of *the theory of groups* are

$$\mathcal{G}_1 : \forall v_1 \forall v_2 \forall v_3 \ [v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3] \qquad \text{(associativity)},$$
$$\mathcal{G}_2 : \forall v \ (v \cdot e = e \cdot v = v) \qquad \text{(identity element)},$$
$$\mathcal{G}_3 : \forall v \, \exists w \ (v \cdot w = w \cdot v = e) \qquad \text{(inverses)}.$$

The theory consisting only of the axiom $\mathcal{G}_1$ is the theory of semigroups, $\{\mathcal{G}_1, \mathcal{G}_2\}$ is the theory of monoids and $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ is the theory of groups. In addition to the axioms $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, if we also add the axiom

$$A : \ \forall v \forall w \ (v \cdot w = w \cdot v),$$

we obtain the theory of *abelian groups*.

Let us also extend the theory of abelian groups.

The theory of torsion free divisible abelian groups: We can extend the theory of abelian groups by the infinite set of axioms $\{\neg\phi_n : n \geq 2\}$ where each ϕ_n defined as

$$\phi_n : \exists v \, (v \neq e \rightarrow (\underbrace{v \cdot v \cdot \ldots \cdot v}_{n\, times}) = e).$$

A group satisfying all $\neg\phi_n$'s is a group which does not contain any element of finite order. So the axioms of abelian groups and $\{\neg\phi_n : n \geq 2\}$ constitute the axioms of the theory of torsion free abelian groups. Moreover, we can extend the theory of torsion free abelian groups by adjoining the infinite set $\{\psi_n : n \geq 2\}$ of axioms, where each ψ_n is defined as

$$\psi_n : \forall v \exists w \ [(\underbrace{w \cdot w \cdot \ldots \cdot w}_{n\, times}) = v],$$

and obtain the theory of torsion free divisible abelian groups.

Let DAG denote the theory of torsion free divisible abelian groups. Let us list some properties of DAG.

Properties:

1. DAG is κ-categorical for any $\kappa > \aleph_0$. (See [1, Proposition 2.2.4]) Hence, it is *complete* by Vaught's Test 2.8.
2. DAG has *quantifier elimination*. (see [1, Theorem 3.1.9].)
3. DAG has is *model complete*. (see [8].)

2.6.4. Theory of Algebraically Closed Fields

Let $\mathcal{L}_r = \{+, \cdot, -, 0, 1\}$ be the language of rings where $+, -$ and $\cdot$ are binary function symbols and 0, 1 are constant symbols. Consider the axioms

R_1: $\forall v_1 \forall v_2 \forall v_3\ [v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3]$ (+ is associative),

R_2: $\forall v\ (v + 0 = 0 + v = v)$ (identity of +),

R_3: $\forall v \exists w\ (v + w = w + v = 0)$ (additive inverses),

R_4: $\forall v \forall w\ (v + w = w + v)$. (commutativity of +),

R_5: $\forall v_1 \forall v_2 \forall v_3\ [v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3]$ (· is associative),

R_6: $\forall v_1 \forall v_2 \forall v_3\ v_1 \cdot (v_2 + v_3) = (v_1 \cdot v_2) + (v_1 \cdot v_3)$ (distributive properties),

R_7: $\forall v_1 \forall v_2 \forall v_3\ (v_1 + v_2) \cdot v_3 = (v_1 \cdot v_3) + (v_2 \cdot v_3)$ (distributive properties),

R_8: $\forall v_1 \forall v_2 \forall v_3\ [(v_1 - v_2) = v_3) \leftrightarrow (v_1 = v_2 + v_3)]$,

which forms the theory of rings. Note that the last axiom is needed just because we add the symbol '$-$' to the language for further use in ring theory. We know that a field is a commutative ring with unity where every element has inverses with respect to the operation " $\cdot$ ", so we add the axioms

$\forall v \forall w\ (v \cdot w = w \cdot v)$ (commutativity of ·),

$\forall v\ [v \neq 0 \rightarrow \exists w\, (v \cdot w = w \cdot v = 1)]$ (multiplicative inverses),

$\forall v\ (v \cdot 1 = 1 \cdot v = v)$. (unity) to the theory of rings consisting of the axioms R_i for $i \in \{1, 2, ..., 8\}$ to obtain the theory of fields. Furthermore, we will extend the theory of fields to the theory of algebraically closed fields, which is denoted as ACF, by adding the set of axioms $\{\phi_n : n \in \mathbb{N}^*\}$ where each ϕ_n is defined as

$$\phi_n : \forall w_0 \ldots \forall w_{n-1} \exists v \ (v^n + w_{n-1}v^{n-1} + \cdots + w_1 v + w_0 = 0).$$

Note that each ϕ_n express existence of roots of all polynomials of degree $n \geq 1$. Let us list some properties of the theory of algebraically closed fields.

Properties:

1. ACF is *not complete*: For a fixed prime p consider the $\mathcal{L}_r$-sentence

   $$\psi_p = \exists v(\underbrace{v + v + ... + v}_{p\,times} = 1),$$

   which is true in algebraically closed fields of characteristic p but not true in other models of the theory of fields which have different characteristics. Therefore, ACF is not complete.
2. Let ACF$_p$ =ACF $\cup\{\psi_p\}$, the theory of algebraically closed fields of characteristic p. ACF$_p$ is *complete.*
3. ACF *eliminates quantifiers.* (see Theorem 2.13)
4. ACF is *model complete.* (see Example 2.18)
5. ACF is the *model companion* of theory of fields. (see Proposition 2.18)

   Moreover, since the theory of fields has amalgamation property, ACF is the *model completion* of the theory of fields.

2.7. Theory of Graphs

Let $\mathcal{L} = \{R\}$ be a language consisting of a binary relation symbol. An $\mathcal{L}$-structure is called as a *directed graph* (or *digraph*) if it is irreflexive and it is called as a *graph* if it is also symmetric. Look at the axioms

$$g_1 : \forall v \, \neg R(v, v) \qquad \text{(irreflexive)},$$

$$g_2 : \forall v \forall w \, \big(R(v, w) \rightarrow R(w, v)\big) \qquad \text{(symmetric)}.$$

The theory of digraphs is $T_{digraphs} = \{g_1\}$ and the theory of graphs is $T_{graphs} = \{g_1, g_2\}$. We can also consider the theory of infinite graphs by adjoining new axioms.

The theory of infinite graphs: The axioms of the class of all infinite graphs consist of axioms of graphs and additionally, we need to say that the structure is infinite. For each $n \geq 1$, consider the sentence

$$\phi_n : \exists v_1 \exists v_2 \ldots \exists v_n \left(\bigwedge_{1 \leq i < j \leq n} v_i \neq v_j \right).$$

Observe that each ϕ_n states that there are at least n elements, so the models satisfying all ϕ_i's are exactly infinite models of the theory of graphs. Therefore, the axioms $\{\phi_n : n \geq 1\}$ together with axioms of all graphs constitutes the axioms of the theory of infinite graphs.

We may also try to find an axiomatization for *the class of finite graphs*. However, unlike the class of infinite graphs, it is not axiomatizable. We show it by using Compactness Theorem. Assume that the theory of finite graphs exists and call it as T_{fin}. Consider $T' = T_{fin} \cup \{\phi_n : n \geq 1\}$ where each ϕ_n states that there are at least n elements, as above. Observe that an arbitrary finite subset

$$\Delta = T_{fin} \cup \{\phi_i : m \geq i \geq 1\}$$

of T' is satisfiable since models of T_{fin} are finite graphs and we can find a graph consisting of m-many elements. T' is finitely satisfiable, so by Compactness Theorem T' is satisfiable. But this shows that T_{fin} has infinite models, contradicting the fact that it is the theory of finite graphs.

The theory of random graph (RG): The theory of random graph consists of the graph axioms $\{g_1, g_2\}$ together with the axiom $\psi_0 : \exists v_1 \exists v_2 (v_1 \neq v_2)$ and also the set of axioms $\{\psi_n : n \geq 1\}$ where ψ_n is defined as

$$\psi_n : \forall v_1, .., v_n \forall w_1, .., w_n \left[\bigwedge_{j=1}^{n} \bigwedge_{i=1}^{n} (v_i \neq w_j) \to \exists z \left(\bigwedge_{i=1}^{n} R(z, v_i) \wedge \bigwedge_{j=1}^{n} \neg R(z, w_j) \right) \right].$$

So RG$= \{g_1, g_2\} \cup \{\psi_n : n \geq 0\}$. A model $\mathcal{G}$ of RG have the property that, for any two disjoint subsets A and B of G, an element $z \in G$ exists where there is an edge between

z and all elements of A and there is no edge between z and any element of B.

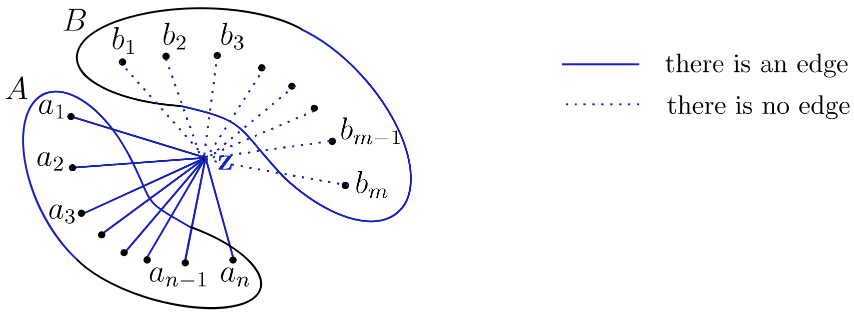


Figure 2.13. Random graph.

Let us list some properties of the theory of random graphs.

Properties:

1. RG is $\aleph_0$-categorical (see [1, Theorem 2.4.2]). Hence, RG is complete by Vaught's Test 2.8 since RG does not have finite models.
2. RG is eliminates quantifiers.
3. RG is is model complete.
4. RG is is the model companion of theory of graphs. (see Proposition 3.3)

3. MODEL COMPANIONABILITY

In this chapter, we study model companions of theories and give examples of theories where the model companions exist and where they do not.

Table 3.1. Theories and model companions.

Theory	**Model Companion**
Theory of sets	Exists [Theory of infinite sets] (Proposition 3.1)
Theory of equivalence relations	Exists [Theory of equivalence relations with infinitely many infinite classes] (Proposition 3.2)
Theory of linear orders	Exists [Theory of dense linear orders without endpoints] (Proposition 3.4)
Theory of graphs	Exists [Theory of random graph] (Proposition 3.3)
Theory of cycle free graphs	NO model companion (Theorem 3.6)
Theory of digraphs with unique successor and predecessor satisfying a certain symmetric relation	NO model companion (Theorem 3.5)
Theory of groups	NO model companion (Theorem 3.3)
Theory of abelian groups	Exists [Theory of divisible Abelian groups having infinitely many elements of order p, for every prime p]
Theory of commutative rings	NO model companion (Theorem 3.4)
Theory of commutative rings without nilpotent elements	Exists [9]
Theory of fields	Exists [Theory of algebraically closed fields] (Proposition 3.5)
Theory of fields with an automorphism	Exists [Theory of algebraically closed fields with a "generic" automorphism (ACFA)] [10]
Theory of fields with two commuting automorphisms	NO model companion (Theorem 3.8)
Theory of dense linear orders without endpoint with an automorphism	NO model companion (Theorem 3.9)

3.1. Model Companions of certain theories

Our aim in this section is to we give examples of theories that have model companions. First example that is given is the most basic one; the theory of sets in empty language $\mathcal{L} = \emptyset$ consisting of no axioms. Let T_{sets} denote the theory of sets. Clearly, any set is a model of T_{sets}. Moreover, T_{sets} is inductive theory since it is closed under unions of chains. So we consider the theory of existentially closed models of T_{sets} to find the model companion by Theorem 2.24 and in the following proposition we show that the theory of existentially closed models of T_{sets} is exactly the theory of infinite sets.

Proposition 3.1. *The model companion of the theory of sets is the theory of infinite sets.*

Proof. We will show that the theory of infinite sets is exactly the theory of existentially closed models of T_{sets}. Let $\mathcal{A}$ be an existentially closed model of T_{sets} and let $\phi(\bar{a})$ be an existential $\mathcal{L}_A$-sentence. First of all, observe that atomic formulas and negations of atomic formulas in the empty language $\mathcal{L}$ can only be in terms of equalities or inequalities of variables. So existential $\mathcal{L}_A$-sentences where existential quantifier only quantify a single variable are basically one of the following two forms.

$$\psi : \exists w \; (w = a).$$

$$\psi_n(\bar{a}) : \exists w \; (\bigwedge_{i=1}^{n} w \neq a_i).$$

Existential formulas can of course be longer and more complex, but it is enough to consider the ones above because they are ultimately conjunctions and disjunctions of these types formulas. Note that ψ states the existence of a single element and $\psi_n(\bar{a})$ indicates the existence of a new element different from the n-tuple $\bar{a} = (a_1, ..., a_n)$. Clearly ψ is satisfied in any nonempty structure, so we focus on the existential $\mathcal{L}_A$-sentences $\psi_n(\bar{a})$. Assume $\mathcal{A}$ is a finite structure with cardinality m. Clearly, we can extend it to a set by adding a single element where $\psi_m(\bar{a})$ is satisfied, but $\mathcal{A} \not\models \psi_m(\bar{a})$; so a finite model is not existentially closed. If $\mathcal{A}$ is infinite, there is no problem since $\psi_n(\bar{a})$ for some fixed n states that there is an element in $\mathcal{A}$ different from $a_1, ..., a_n$ and clearly we have such an element since $\mathcal{A}$ is infinite. Actually, the $\mathcal{L}$-sentences $\psi_n(\bar{v})$ forces a structure to be infinite. Hence, $\mathcal{A}$ must be an infinite set and this shows that the theory of existentially closed models of T_{sets} is exactly the theory of infinite sets. Therefore, by Theorem 2.24 the theory of infinite sets is the model companion of the theory of sets. $\square$

Secondly, we consider the theory of equivalence relations whose axioms are already presented in Section 2.6. Let E be a binary relation symbol and let $\mathcal{L} = \{E\}$ be the language of the theory of equivalence relations. Recall that in a model of the theory of equivalence relations, E is interpreted as a reflexive, symmetric and transitive relation. Let us denote the theory of equivalence relations as T_{eq}. Since T_{eq} consists of $\forall\exists$-sentences, it is an inductive theory by Theorem 2.21. So we are interested in the existentially closed models of T_{eq} to find the model companion. We will show that EQ^*, the theory of equivalence relations with infinitely many infinite classes, is the theory of existentially closed models of T_{eq}.

Proposition 3.2. *The model companion of the theory of equivalence relations is the theory of equivalence relations with infinitely many infinite classes.*

Proof. First of all, note that T_{eq} is an inductive theory, so to find the model companion we will try to find the theory of existentially closed models of T_{eq} and by using Theorem 2.24 we will be able conclude that it is the model companion of T_{eq}. Let $\mathcal{A}$ be an existentially closed model of T_{eq} and let $\phi(\bar{a})$ be an existential $\mathcal{L}_A$-formula. Note that the atomic formulas and negations of atomic formulas in language $\mathcal{L} = \{E\}$ are equalities and inequalities of variables as in empty language, and moreover we have relations of two variables $E(v_1, v_2)$ and their negations $\neg E(v_1, v_2)$. We know that E is a reflexive, symmetric and transitive relation, and it partitions A into equivalence classes.

We analyse the basic forms of existential $\mathcal{L}_A$ sentences. Observe that the basic forms of existential $\mathcal{L}_A$ sentences can be obtained by using atomic formulas of the

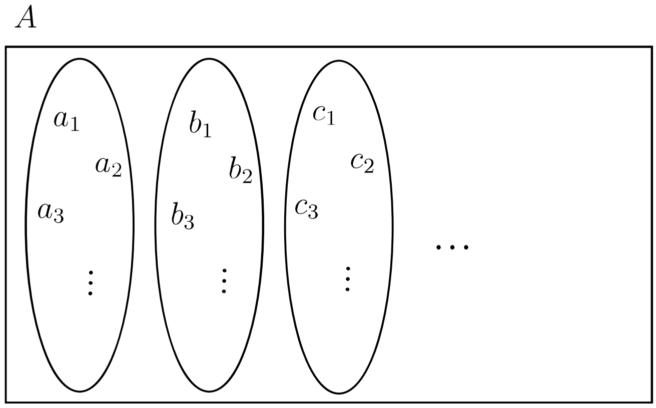


Figure 3.1. Partition of A by the equivalence relation E.

language of equivalence relations so they are one of the following three forms.

$$\psi_n(\bar{a}) : \exists v \ (\bigwedge_{i=1}^{n} v \neq a_i).$$

$$\delta_n(\bar{a}) : \exists v \ \left(\bigwedge_{i=1}^{n} v \neq a_i \ \wedge \ \bigwedge_{i=1}^{n} E(v, a_i) \right).$$

$$\gamma_n(\bar{a}) : \exists v \ \left(\bigwedge_{i=1}^{n} v \neq a_i \ \wedge \bigwedge_{i=1}^{n} \neg E(v, a_i) \right).$$

Note that an existential $\mathcal{L}_A$-sentence can only say something about the number of the elements in an existentially closed model ($\psi_n(a_n)$), number of the elements in an equivalence class ($\delta_n(a_n)$) and number of the classes ($\gamma_n(a_n)$). This shows that if a model A of T_{eq} has finite elements in some of its equivalence classes, let us say it contains m elements in some of its equivalence classes; then take the elements $a_1, a_2, ..., a_m$ in this equivalence class and consider the formula $\delta_m(\bar{a})$. By adding an element to this equivalence class, we can build an extension of $\mathcal{A}$ where $\delta_m(\bar{a})$ is satisfied; but $\mathcal{A} \not\models \delta_m(\bar{a})$. So $\mathcal{A}$ is not existentially closed in this case. Similar argument shows that if there are finite number of equivalence classes in some model $\mathcal{A}$ of T_{eq}, then it cannot be existentially closed. Thus, an existentially closed model of T_{eq} should at least be an equivalence relation with infinitely many infinite classes. Since we indicated all basic forms of existential sentences in language $\mathcal{L} = \{E\}$ above, we see that the equivalence relations with infinitely many infinite classes are clearly existentially closed. Hence, the theory of equivalence relations with infinitely many infinite classes is the theory of existentially closed models of T_{eq}. Therefore, the theory of equivalence relations with infinitely many infinite classes is the model companion of the theory of equivalence relations by Theorem 2.24. □

The third positive example is the model companion of the theory of graphs. Let $\mathcal{L} = \{R\}$ be the language of graphs consisting of a binary relation symbol which is interpreted as an irreflexive, symmetric relation in a model of the theory of graphs. Note that the language is the same as in previous example so the atomic an negations of atomic formulas are the same. Let T_g denote the theory of graphs. As always, we observe that T_g is an inductive theory, since it consists of $\forall\exists$-sentences (the axioms are presented in Section 2.6). So we will look at the theory of existentially closed models of T_g, it will exactly be the model companion of T_g, by Theorem 2.24.

Proposition 3.3. *The model companion of the theory of graphs is the theory of random graph.*

Proof. T_g, the theory of graphs, is an inductive theory, so we will look at the existentially closed models of T_g to find the model companion. So let $\mathcal{A}$ be an existentially closed model of T_g and let $\phi(\bar{a})$ be an existential $\mathcal{L}_A$-sentence. Likewise the Proposition 3.2, the language consists only of a binary relation symbol, so the basic forms of the existential $\mathcal{L}_A$-sentences are similar. But since the relation is irreflexive in this theory, we don't need to indicate that related elements are different. So the existential $\mathcal{L}_A$-sentences can be basically of the following three forms.

$$\psi_n(\bar{a}) : \exists v \; (\bigwedge_{i=1}^{n} v \neq a_i),$$

$$\delta_n(\bar{a}) : \exists v \left(\bigwedge_{i=1}^{n} R(v, a_i) \right), \qquad \gamma(\bar{a}) : \exists w \left(\bigwedge_{i=1}^{n} \neg R(w, a_i) \right).$$

Note that for any chosen n-tuple $\bar{a} \in A^n$, $\delta_n(\bar{a})$ states that there is an element which is connected with all $(a_1, ..., a_n)$ and $\gamma_n(\bar{a})$ states that there is an element which is not connected with any $(a_1, ..., a_n)$.

We clearly see the that the models of the theory of random graph are existentially closed models of the theory of graphs since it satisfies all existential sentences stated

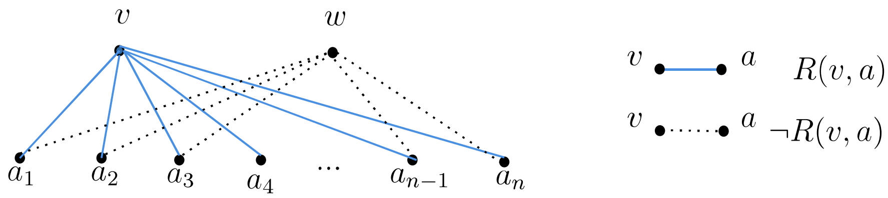


Figure 3.2. Illustrations of $\gamma(\bar{a})$ and $\delta(\bar{a})$.

above. Moreover, take a model containing two sets $B = \{b_1, ..., b_m\}$ and $C = \{c_1, ..., c_l\}$ such that there is no element which is related to all elements of B but not related to any element of C. We can always extend such a model by adding an element and relating it to $b_1, ..., b_m$. So the existential sentence

$$\exists v \left(\bigwedge_{i=1}^{m} R(v, b_i) \wedge \bigwedge_{i=1}^{l} \neg R(v, c_i) \right)$$

is satisfied in some extension. We see that if such an element does not exist in a model, it cannot be existentially closed. Hence, the theory of random graph is exactly the theory of existentially closed models of T_g. Therefore, the theory of random graph is the model companion of the theory of graphs by Theorem 2.24. □

Let $\mathcal{L} = \{<\}$ be the language of orders consisting of a binary relation symbol and let T_{lo} be the theory of linear orders. The axioms of the theory of linear orders and DLO are presented in Section 2.6. We previously showed that DLO, the theory of dense linear orders without endpoints, is model complete in Example 2.22. So actually it is a good candidate for being model companion of the theory of linear orders. In the following proposition, we show that DLO is the model companion of the theory of linear orders.

Proposition 3.4. *The model companion of the theory of linear orders is the theory of dense linear orders without endpoints.*

Proof. T_{lo}, the theory of linear orders, is an inductive theory by Theorem 2.21 since it consists of $\forall\exists$-sentences, so we are interested in the existentially closed models of T_{lo} to find the model companion. We showed in Example 2.22 that DLO is model complete, so the models of DLO are existentially closed models of T_{lo} by Robinson's Test 2.17. If we show that a model which is not dense or has endpoints is not existentially closed, then we can conclude that DLO is exactly the theory of existentially closed models of T_{eq} and this will complete the proof. Let $\mathcal{A}$ be a model of T_{eq} and consider the existential $\mathcal{L}_A$-sentences

$$\phi_1(a_1) : \exists v\ (v < a_1), \qquad \phi_2(a_2) : \exists v\ (a_2 < v),$$
$$\phi_3(a_1, a_2) : \exists v\ (a_1 < v \wedge v < a_2).$$

There is no need to consider the formula $\exists v\,(v = a_i)$ because it is automatically satisfied in any model $\mathcal{A}$ containing a_i, so we omit the equality cases.

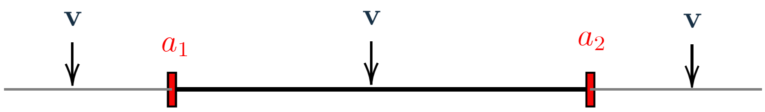


Figure 3.3. Possible positions of v relative to a_1 and a_2.

If there is a_1 which is smaller than all elements or there is a_2 which is bigger than all elements of $\mathcal{A}$, then it cannot be existentially closed since we can add elements and build an extension where $\phi(a_1)$ and $\phi(a_2)$ is satisfied. So an existentially closed model of T_{lo} has no endpoints. Now, assume that it is not dense, so it means that there are two elements a_1 and a_2 such that $a_1 < a_2$ and there is no element between them. But again in this case, we can add an element between them and build some extension where $\phi_3(a_1, a_2)$ is satisfied. So we see that an existentially closed model should be dense. Hence, the theory of dense linear orders without endpoints is exactly the theory of existentially closed models of T_{lo}. Therefore, DLO is the model companion of the theory of linear orders. □

Let T_F be theory of fields in language $\mathcal{L} = \{+, \cdot, -, 0, 1\}$. Axioms of the theory is listed in Section 2.6. We previously showed the the theory of algebraically closed fields is model complete by using the quantifier elimination of ACF and using the Theorems 2.13 and 2.15. So ACF is a candidate of model companion of the theory of fields. In the following proposition, we check this by definition of model companion.

Proposition 3.5. *The model companion of the theory of fields is the theory of algebraically closed fields.*

Proof. First of all, we know that theory of algebraically closed fields is model complete since it eliminates quantifiers by Theorem 2.13 and 2.15. So it can be model companion of theory of fields. We need to check two more things:

i. Since all algebraically closed fields are also models of theory of fields, we can embed a model of the theory of algebraically closed fields into a model of theory of fields by just embedding it into itself.
ii. Any field F has an algebraic closure $\bar{F}$ [11, p. 544]; so we can embed a model F of the theory fields into a model $\bar{F}$ of the theory of algebraically closed fields by just embedding it into its algebraic closure.

Hence, the theory of algebraicaly closed fields is the model companion of the theory of fields. □

Note that the theory of fields is an inductive theory since its models are closed under unions of chains, so ACF is exactly the theory of existentially closed models of the theory of fields by Theorem 2.24. Hence, algebraic closure is equivalent to existential closure for the models of the theory of fields.

3.2. Compactness Argument

Nonexistence of model companion of a theory is not always easy to show. While we were investigating theories without model companions and proofs of non-companionability, we came across [12, p. 239] with a technique which we call as *Compactness Argument*, which is common in all nonexistence proofs. We believe it is important to understand the technique in detail for proving nonexistence of model companions which does not exist in literature.

Definition 3.1. Let T be an $\mathcal{L}$-theory and $\phi(\bar{v})$ be an existential $\mathcal{L}$-formula. An $\mathcal{L}$-formula $\psi(\bar{v})$ is called a *ϕ-obstacle* if $T \cup \{\phi(\bar{v})\} \cup \{\psi(\bar{v})\}$ is "inconsistent"; that is, if there is no model $\mathcal{A}$ of T with a tuple $\bar{a} \in A^n$ which satisfies $\phi(\bar{a})$ and $\psi(\bar{a})$.

Example 3.1. Let $\mathcal{R} = (\mathbb{R}, +, \cdot, -, 0, 1, <)$ be the ordered ring of real numbers and let $T = Th(\mathcal{R})$ be the theory of $\mathbb{R}$ as an ordered ring. Note that $\phi(v) : \exists z\ (v = z^2)$ is an existential formula stating that v is a square. We know that in $\mathbb{R}$, the only squares are non-negative real numbers, so $\psi(v) : v < 0$ is a ϕ-obstacle.

The idea of the Compactness Argument we present below is as follows: We start by assuming that the model companion T^* of an inductive theory T exist. Since T is inductive, models of T^* are existentially closed models of T by Theorem 2.24. We find a set of $\mathcal{L}$-formulas $\Sigma(\bar{v})$ and an existential $\mathcal{L}$-formula $\phi(\bar{v})$ in a way that whenever $\Sigma(\bar{a})$ is satisfied by an existentially closed model $\mathcal{A}$ of T for some $\bar{a} \in A^n$ (or we can say, by a model $\mathcal{A}$ of T^*), $\phi(\bar{a})$ is also satisfied by $\mathcal{A}$. That is, $T^* \cup \Sigma(\bar{v})$ models $\phi(\bar{v})$. So by Compactness Theorem, there exists a finite subset $\Sigma_0(\bar{v})$ of $\Sigma(\bar{v})$ such that $T^* \cup \Sigma_0(\bar{v})$ models $\phi(\bar{v})$. Thus if we show that for any finite subset $\Sigma_0(\bar{v})$ of $\Sigma(\bar{v})$, there is a model of $T^* \cup \Sigma_0(\bar{v})$ which satisfy a ϕ-obstacle $\psi(\bar{v})$, we get a contradiction since $T \cup \phi(\bar{v}) \cup \psi(\bar{v})$ is not satisfiable. Then, we can conclude that the model companion T^* does not exist. Now, we state this argument as the following theorem.

Theorem 3.1 (**Compactness Argument** [12, p. 239])**.** *Let T be an inductive $\mathcal{L}$-theory. If there is an existential $\mathcal{L}$-formula $\phi(\bar{v})$ and a set of $\mathcal{L}$-formulas $\Sigma(\bar{v})$ such that:*

(i) For any existentially closed model $\mathcal{A}$ of T and for all $\bar{a} \in A^n$, we have $\mathcal{A} \models \Sigma(\bar{a})$ implies $\mathcal{A} \models \phi(\bar{a})$.

(ii) For any finite subset $\Sigma_0(\bar{v})$ of $\Sigma(\bar{v})$, there is an existentially closed model $\mathcal{B}$ of T and a ϕ-obstacle $\psi(\bar{v})$ such that $\mathcal{B} \models \Sigma_0(\bar{b})$ and $\mathcal{B} \models \psi(\bar{b})$ for some $\bar{b} \in B^n$.

Then, T has no model companion.

Proof. Assume, for a contradiction, T has a model companion T^*. Since T is inductive, T^* is exactly the theory of existentially closed models of T. Let $\mathcal{A} \models T^*$, we have $\mathcal{A} \models \Sigma(\bar{a})$ implies $\mathcal{A} \models \phi(\bar{a})$ for all $\bar{a} \in A^n$ by part *(i)*. So we have $T^* \cup \Sigma(\bar{v}) \models \phi(\bar{v})$.

By Compactness Theorem, there is a finite subset $\Sigma_0(\bar{v})$ of $\Sigma(\bar{v})$ such that

$$T^* \cup \Sigma_0(\bar{v}) \models \phi(\bar{v}). \quad (*)$$

Now, there is a model $\mathcal{B}$ of T^* and a ϕ-obstacle $\psi(\bar{v})$ such that $\mathcal{B} \models \Sigma_0(\bar{b})$ and $\mathcal{B} \models \psi(\bar{b})$ for some $\bar{b} \in B^n$ by part *(ii)*. Also, since $\mathcal{B} \models \Sigma_0(\bar{b})$ and $\mathcal{B} \models T^*$, we also have $\mathcal{B} \models \phi(\bar{b})$ by $(*)$. However, we obtain $\mathcal{B} \models \phi(\bar{b}) \wedge \psi(\bar{b})$, this contradicts with the fact that $T \cup \{\phi(\bar{v})\} \cup \{\psi(\bar{v})\}$ is not satisfiable. Therefore, T has no model companion. □

3.3. Negative Examples

In this section, we give examples of theories which have no model companion. The theories that are shown to have *no model companion* are as follows:

i. Theory of groups.
ii. Theory of rings.
iii. The theory of digraphs with a unique successor and predecessor, satisfying a certain symmetric relation.
iv. Theory of cycle free graphs.
v. Theory of fields with two commuting automorphisms.
vi. Theory of dense linear orders with an automorphism.

3.3.1. The Theory of Groups

The first negative example that will be presented is *the theory of groups has no model companion.* Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. Recall that the theory of groups consists of the three axioms

$$\forall v_1 \forall v_2 \forall v_3 \ [v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3] \qquad \textit{(associtivity)},$$

$$\forall v (v \cdot e = e \cdot v = e) \qquad \textit{(identity element)},$$

$$\forall v \exists w (v \cdot w = w \cdot v = e) \qquad \textit{(inverses)}.$$

This is obviously an inductive theory since it is axiomatized by $\forall\exists$-sentences. Note that universal sentences can also be counted as $\forall\exists$-sentences, where existential quantifier does not quantify any variable.

The proof uses an intricate group construction known as HNN-extension which was first introduced by G. Higman, B. Neumann, and H. Neumann. They build an extension H of a group G with two isomorphic subgroups A and B where A and B are conjugate in the extended group H.

Theorem 3.2 (HNN-extension)**.** *Let G be a group with subgroups A and B, and let $\sigma : A \to B$ be an isomorphism between the subgroups. There exists an extension H of group G with an element $t \in H$ such that $t^{-1}at = \sigma(a)$ for all $a \in A$.*

The group H introduced in the above theorem is called as *HNN-extension* of G relative to the isomorphism σ. (We refer to [13] for more detail.) If we consider two subgroups generated by a single element, we obtain a corollary of the theorem which we state as Property (A) that will be helpful to prove that theory of groups is not companionable:

Property (A) [13, p.249, Corollary]: Two elements of a group G have the same order if and only if they are conjugate in some group extension H of G.

Theorem 3.3 (Eklof and Sabbagh [14, p. 291])**.** *The theory of groups has no model companion.*

Proof. Let T be the theory of groups and observe that the theory of groups is an inductive theory. To apply Compactness Argument, take the set $\Sigma(v_1, v_2)$ of $\mathcal{L}$-formulas as $\{v_1^n \neq e, v_2^n \neq e : n \in \mathbb{N}^*\}$ consisting of formulas each stating that v_1 and v_2 are not of order n, and the existential $\mathcal{L}$-formula $\phi(v_1, v_2)$ as $\exists w\, (v_1 \cdot w = w \cdot v_2)$ which states that v_1 and v_2 are conjugate. Any elements satisfying $\Sigma(v_1, v_2)$ would be both infinite order.

We first show that $\mathcal{A} \models \Sigma(a_1, a_2)$ implies $\mathcal{A} \models \phi(a_1, a_2)$ for any existentially closed model $\mathcal{A}$ of T and for all $(a_1, a_2) \in A^2$. Let $\mathcal{A}$ be an existentially closed model of T and let $(a_1, a_2) \in A^2$ such that $\mathcal{A} \models \Sigma(a_1, a_2)$. This means that orders of a_1 and a_2 are infinite, so they are conjugate in some extension $\mathcal{A}'$ of $\mathcal{A}$ by Property (A) since their orders are the same. That is, we obtain $\mathcal{A}' \models \phi(a_1, a_2)$. Moreover, since $\mathcal{A}$ is existentially closed, we also have $\mathcal{A} \models \phi(a_1, a_2)$.

Now, take an arbitrary finite subset $\Sigma_0(v_1, v_2) = \{v_1^i \neq e, v_2^i \neq e : 1 \leq i \leq m-1\}$ of $\Sigma(v_1, v_2)$ and an existentially closed model $\mathcal{B}$ of T with elements b_1 and b_2 satisfying $\mathcal{B} \models \Sigma_0(b_1, b_2)$. Actually, we can choose b_1 and b_2 such that the order of b_1 is equal to m and order of b_2 greater than m. Observe that we can always find such elements in an existentially closed group since an existentially closed group $\mathcal{B}$ contains elements of *all orders*. More precisely, the existential formula $\gamma_n = \exists v\, (\bigwedge_{i=1}^{n-1} v^i \neq e \wedge v^n = e)$ is satisfied in some extension of $\mathcal{B}$; for example, in the group $\mathcal{B} \times \mathbb{Z}/n\mathbb{Z}$ and hence γ_n is also satisfied in $\mathcal{B}$.

We have two elements b_1 and b_2 whose orders are different. We will show they cannot be conjugate. Assume for a contradiction, b_1 and b_2 are conjugate and let m and l $(m < l)$ be their orders, respectively. Then, there exists $c \in B$ such that $c^{-1}b_1c = b_2$ and we obtain $(c^{-1}b_1c)^m = c^{-1}b_1^mc = c^{-1}c = e = b_2^m$, which is a contradiction since the order of b_2 is greater than m. So $\psi(v_1, v_2) = (v_1^m = e \wedge v_2^m \neq e)$ is a ϕ-obstacle. We obtained $\mathcal{B} \models \Sigma_0(b_1, b_2)$ and $\mathcal{B} \models \psi(b_1, b_2)$ where $\psi(v_1, v_2)$ is a ϕ-obstacle. Therefore, the theory of groups has no model companion. □

Although the theory of groups has no model companion, the theory of abelian groups has a model companion, namely the theory of divisible abelian groups having infinitely many elements of order p, for all primes p [14, p. 256, Theorem 2.4]. Since the elements of abelian groups are only conjugate to themselves, the above argument does not cause an obstacle.

3.3.2. The Theory of Rings

The second negative example is *the theory of commutative rings has no model companion*. Let $\mathcal{L} = \{+, \cdot, -, 0, 1\}$ be the language of rings. Axioms of the theory of commutative rings is presented in Section 2.6. Note that it is an inductive theory since the axioms are $\forall\exists$-sentences; or we can say it is inductive since models of the theory are closed under unions of chains. We will state and prove the following property which will be useful while proving the theory of commutative rings is not companionable.

Property (B) [15, Lemma 2.1]: Let R be a commutative ring and let $r \in R$. The following statements are equivalent:

i. r is not nilpotent; that is, $r^n \neq 0$ for any $n \in \mathbb{N}^*$.
ii. There is a commutative ring extension R' of R and a nonzero *idempotent element* a (*i.e.*, $a^2 = a$) of R' such that r divides a in R'.

Proof of Property (B). (ii $\Rightarrow$ i) Let $r \in R$ and assume there is an extension R' of R and $a \in R'$ such that $a^2 = a$, $a \neq 0$ and $r.k = a$ for some $k \in R'$. Observe that $a^n = a$ for all $n \in \mathbb{N}^*$ since $a = a^2$. Then, we have $(r.k)^n = r^n.k^n = a^n = a \neq 0$ for any $n \in N^*$. Thus, $r^n \neq 0$ for any $n \in N^*$; that is, r is not nilpotent.

(i $\Rightarrow$ ii) Assume $r \in R$ is an element of R which is not nilpotent and let $R' = R[x]/\langle r^2x^2 - rx\rangle$. Consider the map

$$\begin{aligned} \sigma : R \to R' &= R[x]/\langle r^2x^2 - rx\rangle \\ a &\mapsto a + \langle r^2x^2 - rx\rangle. \end{aligned}$$

Clearly, σ is one to one; so we have $R \subseteq R'$. Moreover, r divides an idempotent in R' which is rx. It remains to show that rx is nonzero; that is, $rx \notin \langle r^2x^2 - rx\rangle$. Assume for a contradiction, $rx = (r^2x^2 - rx)p(x)$ for some $p(x) = \Sigma_{i=0}^n c_i x^i \in R[x]$. So we have $r = -rc_0$, $r^2c_i = rc_{i+1}$ for $i = 0, 1, 2, ..., n-1$ and $r^2c_n = 0$. By using these equations, we obtain $r^2 = -r^2c_0 = -rc_1$, $r^3 = -r^2c_1 = -rc_2$ and by continuing this way we get $r^n = -r^2c_n = 0$ which implies that r is nilpotent, a contradiction. □

Theorem 3.4 (G. Cherlin [15]). *The theory of commutative rings has no model companion.*

Proof. Let T be the theory of commutative rings and observe that T is inductive; that is, models of T are closed under unions of chains. Take $\Sigma(v) = \{v^n \neq 0 : n \in \mathbb{N}^*\}$ consisting of infinitely many $\mathcal{L}$-formulas so that they all together stating that v is not nilpotent and let $\phi(v) = \exists w_1 \exists w_2 [(w_1^2 = w_1) \wedge (w_1 \neq 0) \wedge (v \cdot w_2 = w_1)]$ be an existential $\mathcal{L}$-formula which express that there is a non zero idempotent element that is divisible by v. We will apply the Compactness Argument to show that model companion does not exist.

Let $\mathcal{A}$ be an existentially closed model of T with an element $a \in A$ such that $\mathcal{A} \models \Sigma(a)$. We will show that $\mathcal{A} \models \phi(a)$. We know that $\mathcal{A} \models \Sigma(a)$ means that a is not nilpotent, so by Property (B) there is an extension $\mathcal{A}'$ of $\mathcal{A}$ such that a divides a nonzero idempotent in $\mathcal{A}'$; that is, $\mathcal{A}' \models \phi(a)$. Since $\mathcal{A}$ is existentially closed, we obtain $\mathcal{A} \models \phi(a)$ as well.

Now, take an arbitrary finite subset $\Sigma_0(v) = \{v^i \neq 0 : 1 \leq i < m\}$ of $\Sigma(v)$. Let $\mathcal{B}$ be an existentially closed model of T with an element $b \in B$ such that $b^m = 0$. Note that we can always find such an element since an existentially closed ring $\mathcal{B}$ contains nilpotent elements of all exponents. More precisely, the existential sentence $\gamma_n : \exists v \ (v^n = 0)$ is always satisfied in some extension of $\mathcal{B}$; as an example consider $\mathcal{B} \times \mathbb{Z}/2^n\mathbb{Z}$ satisfying γ_n. Due to the fact that $\mathcal{B}$ is existentially closed, we also obtain $\mathcal{B} \models \gamma_n$.

So far we have $\mathcal{B} \models \Sigma_0(b)$, it only remains to show that there is a ϕ-obstacle $\psi(v)$ such that $\mathcal{B} \models \psi(b)$. We know by Property (B) (i$\rightarrow$ii) that if an element is nilpotent then there is no extension where it divides a nonzero idempotent, so $\psi(v) : v^m = 0$ is the desired ϕ-obstacle. Thus, we have $\mathcal{B} \models (\Sigma_0(b) \wedge \psi(b))$ where $\psi(v)$ is a ϕ-obstacle. Therefore, the theory of commutative rings has no model companion. $\square$

Lipshitz and Saracino figured out that only obstacle to the model companionability of the theory of rings is the existence of nilpotent elements and they showed that the theory of rings without nilpotent elements has a model companion [9].

3.3.3. Two Examples from Graph Theory

We have two negative examples from graph theory whose proofs use Compactness Argument [16]. Let $\mathcal{L} = \{R\}$ be a language consisting of one relation symbol. Recall from Section 2.7 that a digraph is an $\mathcal{L}$-structure where the edge relation R is interpreted as an irreflexive relation and a graph is an $\mathcal{L}$-structure where the edge relation R is interpreted as both an irreflexive and symmetric relation. Also the universes of graph and digraph structures are the set of vertices of the graphs. We first introduce basic terminology from graph theory and after this we will continue with examples of theories extending the theory of graphs and digraphs which have no model companion. [12, p. 240, Example 3.8].

Definition 3.2. i. Let P_n be a graph consisting of the set $\{1, 2, ..., n\}$ of n vertices and edges $\{(i, i+1) : i = 1, 2, ..., n-1\} = R^{P_n}$ where n is a positive integer. A graph is called as an n-path if it is isomorphic to P_n and the length of an n-path is equal to $n - 1$.

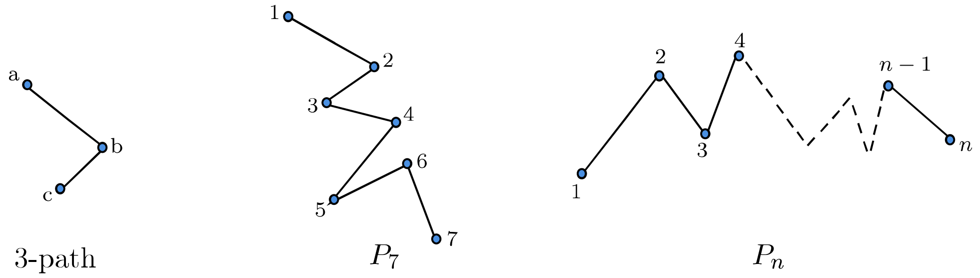
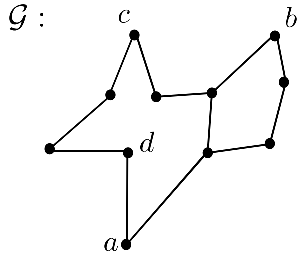


Figure 3.4. Examples of n-paths.

ii. Let $\mathcal{G}$ be a graph and $a, b \in G$. The smallest distance between a and b, denoted as $d_{\mathcal{G}}(a, b)$, is the length of smallest path between a and b.

ii. A graph $\mathcal{G}$ with the set of vertices $G = \{g_1, g_2, ..., g_n\}$ where g_i's are distinct, is called a cycle if $\{(g_i, g_{i+1}) : i = 1, 2, ..., n-1\} \cup \{(g_1, g_n)\} \subseteq R^G$.

A special type of cycle is the graph C_n that consists of the set of n-vertices



$\bullet d_{\mathcal{G}}(a,b) = 3$ $\bullet d_{\mathcal{G}}(a,d) = 1$ $\bullet d_{\mathcal{G}}(b,c) = 3$

$\bullet d_{\mathcal{G}}(a,c) = 4$ $\bullet d_{\mathcal{G}}(b,d) = 4$

Figure 3.5. Illustration of finding $d_{\mathcal{G}}$.

$\{1, 2, ..., n\}$ and edges $\{(i, i+1) : i = 1, 2, ..., n-1\} \cup \{(1, n)\} = R^{C_n}$.

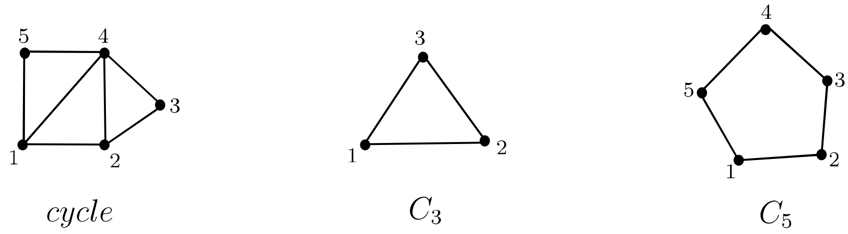


Figure 3.6. Examples of cycles.

iii. A graph is called cycle free if it does not contain any cycle as a subgraph.

The theory of digraphs with a unique successor and predecessor, satisfying a certain symmetric relation: We will write the axioms of a theory extending the theory of digraphs. First of all we introduce the following notations to simplify the axioms:

$D(v,w) = R(v,w) \wedge \neg R(w,v)$,	$S(v,w) = R(v,w) \wedge R(w,v)$.
$v \longrightarrow w$	$v \longleftrightarrow w$

D denotes the directed (antisymmetric) relation and S denotes the symmetric relation between two vertices. Let T_{dg} denote the theory of digraphs with a unique successor and predecessor. The axioms of the theory T_{dg} is as follows:

$d_1 : \forall v \, \neg R(v,v)$. *(irreflexive)*

$d_2 : \forall v \exists w \; D(v,w)$ and $\forall v \forall w_1 \forall w_2 \; [D(v,w_1) \wedge D(v,w_2) \rightarrow w_1 = w_2]$.

(unique successor)

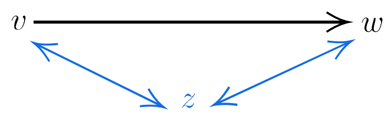
Every element has a *successor* and it is *unique*.

$d_3 : \forall v \exists w\ D(w, v)$ and $\forall v \forall w_1 \forall w_2\, [D(w_1, v) \wedge D(w_2, v) \rightarrow w_1 = w_2]$.

(unique predecessor)

Every element has a *predecessor* and it is *unique*.

$d_4 : \forall v \forall w (D(v, w) \rightarrow \forall z (S(v, z) \leftrightarrow S(z, w)))$.



(symmetric relation)

If two things are related by D, anything which is S related to one is S related to other.

We have the theory $T_{dg} = \{d_1, d_2, d_3, d_4\}$. Observe that the relation D generates orbits in a model of T_{dg}, which we call as D-paths, since it is actually a one to one and onto function by the axioms d_2 and d_3. Notice that D-paths can only be in the form of a cycle or an infinite line due to the fact that every element has a unique successor and predecessor.

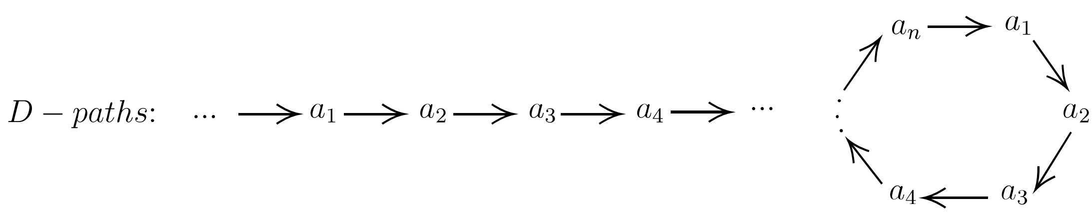


Figure 3.7. D-paths.

Additionally, the axiom d_4 states that if there is an element belonging to some D-path, namely D_1, having a symmetric relation with an element z, then all elements in the path D_1 have a symmetric relation with z.

Moreover, we know that z also belongs to some D-path, namely D_2. Observe that every element in the path D_2 also have a symmetric relation with all elements of

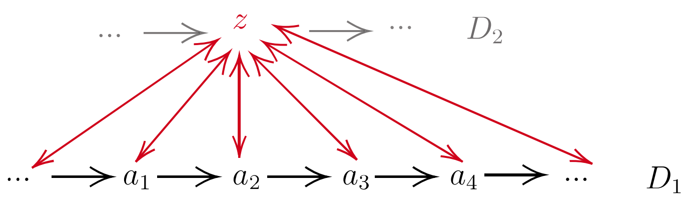


Figure 3.8. Axiom d_4.

the path D_1. The axiom d_4 allows us to say that two elements are not in the same D-path in first order which is a key property that will be used in the proof.

T_{dg} is clearly a satisfiable theory; for example, it is modeled by the $\mathcal{L}$-structure $\mathcal{G} = (\mathbb{Z}, R^{\mathcal{G}})$ where $R^{\mathcal{G}} = \{(c, c+1) : c \in \mathbb{Z}\}$. Also, note that T_{dg} is axiomatized by $\forall\exists$-sentences, so it is an inductive theory. Hence, the models of the model companion T^*_{dg} (if it exists) are exactly existentially closed models of T_{dg} by Theorem 2.24.

Theorem 3.5. *The theory T_{dg} of digraphs with unique successor and predecessor consisting of the axioms $\{d_1, d_2, d_3, d_4\}$ has no model companion.*

Proof. Let $T_{dg} = \{d_1, d_2, d_3, d_4\}$ be the theory of digraphs with unique successor and predecessor. We know that T_{dg} is an inductive theory. We define an $\mathcal{L}$-formula $D^n(v_1, v_2)$ expressing that there exists a directed n-path from v_1 to v_2 for each n as,

$D^n(v_1, v_2) : \exists w_1 \exists w_2 ... \exists w_{n-2}(D(v_1, w_1) \wedge (\bigwedge_{i=1}^{n-3} D(w_i, w_{i+1})) \wedge D(w_n, v_2))$ for $n > 3$,
$D^3(v_1, v_2) : \exists w_1(D(v_1, w_1) \wedge D(w_1, v_2))$,
$D^2(v_1, v_2) : D(v_1, v_2)$.

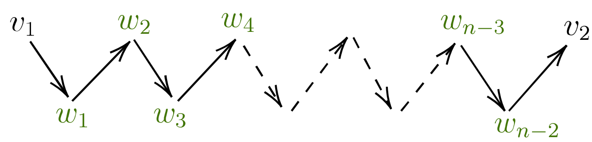


Figure 3.9. $D^n(v_1, v_2)$.

Let $\Sigma(v_1, v_2) = \{(\neg D^n(v_1, v_2) \wedge \neg D^n(v_1, v_2)) : n > 1\}$ be a set of $\mathcal{L}$-formulas which states that v_1 and v_2 are not connected by a directed n-path. Also let $\phi(v_1, v_2) : \exists z(S(z, v_1) \wedge \neg S(z, v_2))$ be an $\mathcal{L}$-formula stating that there is an element having symmetric relation with v_1 and not having the symmetric relation with v_2. Note that in a model, if two elements satisfy $\phi(v_1, v_2)$, then they cannot be in the same D-path by axiom d_4. We will show that for any existentially closed model $\mathcal{A}$ of T_{dg}, $\mathcal{A} \models \Sigma(a_1, a_2)$ implies $\mathcal{A} \models \phi(a_1, a_2)$ for any $a_1, a_2 \in A$.

Let $\mathcal{A}$ be an existentially closed model of T_{dg} and let $\mathcal{A} \models \Sigma(a_1, a_2)$ for some $a_1, a_2 \in A$. Consider an extension $\mathcal{A}'$ of $\mathcal{A}$ whose universe is $A \cup \mathbb{Z}$ (without loss of generality, assume $\mathcal{A} \cap \mathbb{Z} = \emptyset$) and interpretation of the relation R is extended as

$$\begin{aligned} R^{\mathcal{A}'} = R^{\mathcal{A}} &\cup \{(c, c+1) : c \in \mathbb{Z}\} \\ &\cup \{(c, a'), (a', c) : a' \text{ and } a_1 \text{ are connected by a } D\text{-path}; c \in \mathbb{Z}\}. \end{aligned}$$

Note that $\mathcal{A}' \models T_{dg}$ since we just added a D-path and symmetric relations compatible with the axiom d_4. Additionally, we see that $\mathcal{A}' \models \phi(a_1, a_2)$ since any element $c \in \mathbb{Z}$ is a witness of the existential sentence $\phi(a_1, a_2) : \exists z\ (S(z, a_1) \wedge \neg S(z, a_2))$. Due to the fact that $\mathcal{A}$ is existentially closed, we have $\mathcal{A} \models \phi(a_1, a_2)$ as well.

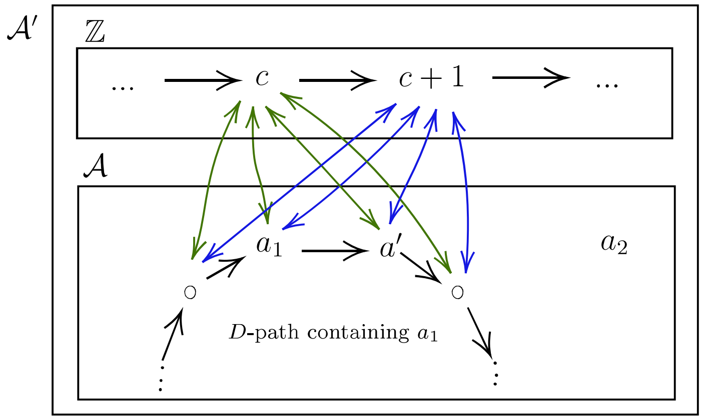


Figure 3.10. Extending $\mathcal{A}$ to $\mathcal{A}'$.

Now, let $\Sigma_0(v_1, v_2) = \{\neg R^i(v_1, v_2) : 1 < i < m\}$ be a finite subset of $\Sigma(v_1, v_2)$. We see that $\psi(v_1, v_2) : D^m(v_1, v_2)$ is a ϕ-obstacle since if two elements are on the same D-path, then they should have the same symmetric relations. So if we find an existentially closed model $\mathcal{B}$ of T with elements b_1 and b_2, satisfying $\mathcal{B} \models \Sigma_0(b_1, b_2) \wedge \psi(b_1, b_2)$, then we are done. Consider a model $\mathcal{G} = (\mathbb{Z}, R^{\mathcal{G}})$ of T_{dg} where $R^{\mathcal{G}} = \{(c, c+1) : c \in \mathbb{Z}\}$. Since T_{dg} is inductive, we can extend $\mathcal{G}$ to an existentially closed model $\mathcal{B}$ of T_{dg} by Theorem 2.23 and we see that the elements 1 and m belonging to $B \supseteq \mathbb{Z}$ cannot be connected by a directed n-path for $n < m$ because if there would be such a path, this contradicts with the fact that there are unique successors and predecessors for each element. So $\mathcal{B} \models \Sigma_0(1, m)$ and we also have $\mathcal{B} \models \psi(1, m)$. Therefore, T_{dg}, the theory of directed graphs with a unique successor and predecessor has no model companion. $\square$

The theory of cycle free graphs: Recall that a graph is called cycle free if it does not contain any cycle as a subgraph. So the theory of cycle free graphs contains:

1) The axioms of the theory of graphs that are

$$g_1 : \forall v \neg R(v, v) \qquad \textit{(irreflexive)},$$

$$g_2 : \forall v \forall w (R(v, w) \rightarrow R(w, v)) \qquad \textit{(symmetric)}.$$

2) An infinite set of axioms $\{\phi_n : n \geq 1\}$ where each states that there is no cycle consisting of n-vertices and defined as

$$\phi_n : \forall v_1 \forall v_2 ... \forall v_n [\bigwedge_{1 \leq i < j \leq n} v_i \neq v_j \rightarrow \neg(\bigwedge_{i=1}^{n-1} R(v_i, v_{i+1}) \wedge R(v_1, v_n)].$$

Let T_{cfg} denote the theory of cycle free graphs. Note that the theory of cycle free graphs consists of universal sentences. Thus, the theory of cycle free graphs is a an *inductive theory.* So the models of T^*_{cfg} (if it exists) are exactly existentially closed models of T_{cfg} by Theorem 2.24.

Theorem 3.6 ([16, p. 86, Proposition 7]). *The theory of cycle free graphs has no model companion.*

Proof. Let T_{cfg} be the theory of cycle free graphs and note that it is an inductive theory. Let $R^n(v_1, v_2) : \exists w_1 \exists w_2 ... \exists w_{n-2} (R(v_1, w_1) \wedge (\bigwedge_{i=1}^{n-3} R(w_i, w_{i+1})) \wedge R(w_{n-2}, v_2))$ denote an $\mathcal{L}$-formula which indicates that there exists a path between v_1 and v_2 consisting of n-vertices for $n > 3$. (n-path)

Let $\Sigma(v_1, v_2) = \{\neg R^n(v_1, v_2) : n > 3\}$ be a set of $\mathcal{L}$-formulas stating that there is no n-path between v_1 and v_2 for any $n > 3$. So if two elements satisfy $\Sigma(v_1, v_2)$, then there could only exist a 2-path or a 3-path between these elements.

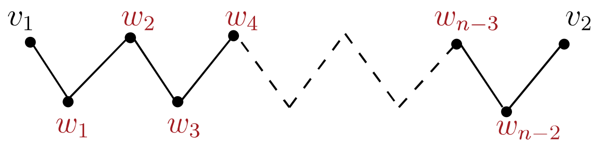


Figure 3.11. $R^n(v_1, v_2)$.

Let $\phi(v_1, v_2) : \exists z(R(v_1, z) \wedge R(z, v_2)) \vee R(v_1, v_2)$ be an $\mathcal{L}$-formula stating that there is either a 3-path or 2-path between v_1 and v_2. We will show that in an existentially closed model $\mathcal{A}$ of T_{cfg} if two vertices a_1 and a_2 are not connected by an n-path for $n > 3$, then they are connected either by a 3-path or by a 2-path. That is, we will show that for any existentially closed model $\mathcal{A}$ and for any $a_1, a_2 \in A$, $\mathcal{A} \models \Sigma(a_1, a_2)$ implies $\mathcal{A} \models \phi(a_1, a_2)$.

Let $\mathcal{A}$ be an existentially closed model of T_{cfg} such that $\mathcal{A} \models \Sigma(a_1, a_2)$ for some $a_1, a_2 \in A$. We will show that $\mathcal{A} \models \phi(a_1, a_2)$. Assume for a contradiction $\mathcal{A} \not\models \phi(a_1, a_2)$; that is, a_1 and a_2 are not connected neither by a 3-path nor by a 2-path. We also know that a_1 and a_2 also not connected by an n-path for $n > 3$ since $\mathcal{A} \models \Sigma(a_1, a_2)$. Thus, a and b are not connected.
Construct an extension $\mathcal{A}'$ of $\mathcal{A}$ by adding a new vertex c and two edges connecting (a_1, c) and (a_2, c). Note that since a_1 and a_2 are not connected, no cycles are formed while extending $\mathcal{A}$ to $\mathcal{A}'$; thus, $\mathcal{A}'$ is a model of T_{cfg}.

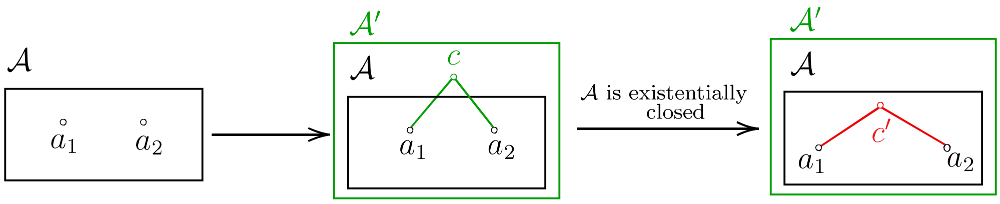


Figure 3.12. Extending $\mathcal{A}$ to $\mathcal{A}'$ by adding a vertex and two edges.

We see that the existential $\mathcal{L}_A$ sentence $\exists z(R(a_1, z) \wedge R(a_2, z))$ is satisfied by an extension of $\mathcal{A}$ that is a model of T_{cfg}, so we have $\mathcal{A} \models \exists z(R(a_1, z) \wedge R(a_2, z))$ since $\mathcal{A}$ is existentially closed. Hence, there is a 3-path between a_1 and a_2 contradicting to the assumption that a_1 and a_2 are not connected. Therefore, we obtain that for any existentially closed model $\mathcal{A}$ of T_{cfg}, $\mathcal{A} \models \Sigma(a_1, a_2)$ implies $\mathcal{A} \models \phi(a_1, a_2)$ for any $a_1, a_2 \in A$.

Now, let $\Sigma_0(v_1, v_2) = \{\neg R^i(v_1, v_2) : 3 < i < m\}$ be a finite subset of $\Sigma(v_1, v_2)$ and let P_m be an m-path with endpoints p_1 and p_m. Clearly, $P_m \models T_{cfg}$ since it does not contain any cycle. Moreover, we can extend P_m to an existentially closed model $\mathcal{P}$ of T_{cfg} by Theorem 2.23 since T_{cfg} is an inductive theory. Observe that we have $\mathcal{P} \models \Sigma_0(p_1, p_m)$; because if not, then there exist a second path between p_1 and p_m of length less than m and a cycle is formed, which contradicts with the fact that $\mathcal{P} \models T_{cfg}$. So far, we found an existentialy closed model $\mathcal{P}$ of T_{cfg} with elements p_1 and p_m satisfying $\mathcal{P} \models \Sigma_0(p_1, p_m)$. Also, we see that $\mathcal{P} \models \psi(p_1, p_m)$ where $\psi(v_1, v_2) : R^m(v_1, v_2)$ is a ϕ-obstacle since there cannot exist two different paths between two elements v_1 and v_2 since we cannot have cycles in the graph. Therefore, the theory of cycle free graphs has no model companion. □

Takeuchi, Tanaka and Tsuboi stated a more general version of the Theorem 3.6 as a corrolary of the Compactness Argument [12]. First, let us introduce and recall some definitions and then we will state the extended version of the previous theorem.

i. A graph is called connected if there is a path from any point to any other point.
ii. A graph is called 2-edge connected if it remains connected even if one edge is removed.
iii. Let K denote a class of finite 2-edge connected graphs and let T_K denote the theory of K-free graphs; that is, the graphs which does not contain any member of K as a subgraph. Note that the class of K-free graphs is elementary that can be axiomatized by the axioms of graphs union the infinite set of axioms $\{\neg\gamma_n : n > 3\}$ where each γ_n expressing the existance of two different n-paths from one point to another. Note that since each γ_n is an existential sentence sentence, negations are universal sentences. Hence T_K consists of universal sentences which implies that T_K is an inductive theory.

As an example, consider $K = \{C_n : n \geq 3\}$ as a class of finite 2-edge connected graphs and observe that T_K denotes the theory of cycle free graphs in this specific case. The following theorem generalizes the result that the theory of cycle free graphs has no model companion to K-free graphs where K is any set of finite 2-edge connected graphs.

Theorem 3.7 ([12, p. 239, Corrolary 3.7]). *Let T_K be the theory of K-free graphs where K denotes a set of finite 2-edge connected graphs. Assume that for any $n \in \mathbb{N}^*$ there is a model G_n of T_K with elements a_n and b_n such that for any $G \supseteq G_n$, we have $d_G(a_n, b_n) \geq n$. Then the model companion of T_K does not exist.*

The proof is exactly similar to the proof of Theorem 3.6. Actually, the properties of cycle free graphs that causes to have no model companion is generalized in this theorem to more various types of graphs.

3.3.4. The Theory of Fields with Two Commuting Automorphisms

In Section 3.1, we showed that the theory of fields has a model companion and the model companion is the theory of algebraically closed fields. Chatzidakis and Hrushovski showed that the theory of fields together with an automorphism also has a model companion which is called as ACFA, the theory of algebraically closed fields with a "generic" automorphism [10]. However, *the theory of fields with two commuting automorphisms has no model companion.* Let $\mathcal{L}$ be language of rings together with two unary function symbols; that is, $\mathcal{L} = \mathcal{L}_{ring} \cup \{\sigma, \tau\} = \{+, \cdot, -, 0, 1, \sigma, \tau\}$. The theory of fields with two commuting automorphisms consists of axioms of the theory of fields, that are already presented in Section 2.6, and additionally the following axioms.

1. Axioms stating σ and τ are one to one :
   $\forall v_1 \forall v_2 \, [(\sigma(v_1) = \sigma(v_2)) \rightarrow v_1 = v_2], \quad \forall v_1 \forall v_2 \, [(\tau(v_1) = \tau(v_2)) \rightarrow v_1 = v_2].$
2. Axioms stating σ and τ are onto :
   $\forall v_1 \exists v_2 \, \sigma(v_2) = v_1, \quad \forall v_1 \exists v_2 \, \tau(v_2) = v_1.$
3. Axioms stating σ and τ are compatible with $+$ and $\cdot$:
   $\forall v_1 \forall v_2 \, \sigma(v_1 \cdot v_2) = \sigma(v_1) \cdot \sigma(v_2), \quad \forall v_1 \forall v_2 \, \sigma(v_1 + v_2) = \sigma(v_1) + \sigma(v_2).$
   $\forall v_1 \forall v_2 \, \tau(v_1 \cdot v_2) = \tau(v_1) \cdot \tau(v_2), \quad \forall v_1 \forall v_2 \, \tau(v_1 + v_2) = \tau(v_1) + \tau(v_2).$

4. Axiom stating σ and τ are commuting :
$\forall v\ [(\tau(\sigma(v)) = \sigma(\tau(v)))$.

The theory of fields with two commuting automorphisms consists of $\forall\exists$-sentences, so it is an inductive theory by Theorem 2.21. We shortly denote a model of the theory as (F, σ, τ) where F denotes a field structure and σ and τ denotes two commuting automorphisms on F.

Property (C) [17, Lemma 3.1]: Let (A, σ, τ) be an existentially closed model of the theory of fields with two commuting automorphisms. For any integer $n \geq 1$, there is $c \in A$ such that the following are satisfied:

1. $\sigma(c) = \tau(c)$,
2. $\sigma^k(c) + \sigma^{k-1}(c) + ... + \sigma(c) + c \neq 0,$ for any $k < n$,
3. $\sigma^n(c) + \sigma^{n-1}(c) + ... + \sigma(c) + c = 0$.

Proof of Property (C). Let T be the theory of fields with two commuting automorphisms and let (A, σ, τ) be an existentially closed model of T . Let $t_0, t_1, ..., t_{n-1}$ be transcendental and algebraically independent elements over A. Let $t_n = -(t_0+t_1+...+t_{n-1})$ and observe that $t_1, t_2, ..., t_n$ are also transcendental and algebraically independent over A. We define an extension of (A, σ, τ) by extending A to $A' = A[t_0, t_1, ..., t_{n-1}]$ and also extending σ and τ to A' by defining $\sigma(t_i) = \tau(t_i) = t_{i+1}$ for $0 \leq i < n-1$. Clearly, σ and τ are two commuting automorphisms on A', so $(A', \sigma, \tau) \models T$. Moreover, $t_0 \in A'$ satisfies:

1. $\sigma(t_0) = \tau(t_0)$,
2. $\sigma^k(t_0) + \sigma^{k-1}(t_0) + ... + \sigma(t_0) + t_0 \neq 0,$ for any $k < n$,
3. $\sigma^n(t_0) + \sigma^{n-1}(t_0) + ... + \sigma(t_0) + t_0 = 0$.

Due to the fact that there is an element t_0 in an extension (A', σ, τ) of (A, σ, τ) satisfying $1, 2$ and 3, we can also find such an element $c \in A$ by existential closedness of (A, σ, τ).

□

Theorem 3.8 (Hrushovski [17, p. 5]). *The theory of fields with two commuting automorphisms has no model companion.*

Proof. Let T be the theory of fields with two commuting automorphisms. Note that T is an inductive theory since it consists of $\forall\exists$-sentences. We will apply the Compactness Argument to show that the model companion T^* of T does not exist. Let

$$\Sigma(v) = \{[(\sigma(v) = \tau(v)) \wedge (\sigma^n(v) + \sigma^{n-1}(v) + ... + \sigma(v) + v \neq 0)] : n \in \mathbb{N}^*\}$$

be an infinite set of $\mathcal{L}$-formulas and let

$$\phi(v) : \exists z\ (z^3 = 1 \wedge \sigma(z) = \tau(z) = z^2) \rightarrow \exists w_1 \exists w_2[(\sigma(w_1) = \tau(w_1) = w_1 + v) \\ \wedge (w_2^3 = w_1) \wedge (\tau(w_2) = z\sigma(w_2))]$$

be an $\mathcal{L}$-formula. Note that $\phi(v)$ states that if the primitive 3^{rd} root of unity ζ belongs to a model and if σ and τ are not acting trivially on ζ; that is, if we have $\sigma(\zeta) = \tau(\zeta) = \zeta^2$; then there exists an element a_1 in the model such that $\sigma(a_1) = \tau(a_1) = a_1 + v$ and image of the third root of a_1 under σ and τ differ by ζ. We will show that for any existentially closed model $\mathcal{A} = (A, \sigma, \tau)$ of T, if there is $a \in A$ such that $\mathcal{A} \models \Sigma(a)$, then we have $\mathcal{A} \models \phi(a)$.

Let $\mathcal{A} = (A, \sigma, \tau)$ be an existentially closed model of T and let a be an element of A such that $\mathcal{A} \models \Sigma(a)$. Note that we can find such an element a in an existentially closed model $\mathcal{A}$ of T by Property (C). We will focus on the existentially closed models where we have $\sigma(\zeta) = \tau(\zeta) = \zeta^2$. First of all, we show that such an existentially closed model exists, otherwise it is meaningless to show that Σ implies ϕ. Let (P, σ_0, τ_0) be a prime field with two commuting automorphisms such that $\zeta \notin P$. (So $p(x) = x^2 + x + 1$ should be irreducible over P. Since the discriminant of $p(x)$ is -3, it is irreducible when -3 is not a square modulo p. Hence, the characteristic of the prime field must be 0 or 2 (mod 3) by quadric reciprocity, see [18, Example 2.4]). We can extend (P, σ_0, τ_0) to (F, σ_0', τ_0') by adjoining ζ to P and extending two automorphisms by sending ζ to ζ^2. Further, since (F, σ_0', τ_0') is a model of the inductive theory T, we can also extend it to an existentially closed model by Theorem 2.23. Hence, we can take an existentially closed model $\mathcal{A} = (A, \sigma, \tau)$ of T with an element $a \in \mathcal{A}$ such that $\mathcal{A} \models \Sigma(a)$ and

$\sigma(\zeta) = \tau(\zeta) = \zeta^2$.

(A, σ, τ) **existentially closed**, $\sigma(\zeta) = \tau(\zeta) = \zeta^2$

(F, σ_0', τ_0') $F = P(\zeta),\ \sigma_0'(\zeta) = \tau_0'(\zeta) = \zeta^2$

(P, σ_0, τ_0) P prime field, $\zeta \notin P \ \rightarrow char(P)$ is 2 $(mod\ 3)$ or 0

Figure 3.13. Building an e.c. model such that $\sigma(\zeta) = \tau(\zeta) = \zeta^2$.

Now, we will construct an extension of $\mathcal{A}$ where there are witnesses of the following existential sentence: (we shortly denote the primitive third root of unity by ζ)

$$\exists w_1 \exists w_2 [(\sigma(w_1) = \tau(w_1) = w_1 + a) \wedge (w_2^3 = w_1) \wedge (\tau(w_2) = \zeta\sigma(w_2)).$$

Let t be a transcendental element over A. We can define an extension $(A(t), \sigma, \tau)$ of $\mathcal{A}$ by extending A to $A(t)$ and also expanding σ and τ on $A(t)$ as $\sigma(t) = \tau(t) = t + a$. Consider $\tau^n(t) = \sigma^n(t) = t + a + \sigma(a) + ... + \sigma^n(a)$. Observe $\sigma^i(t) \neq \sigma^j(t)$ for any $i \neq j$. More precisely, if $\sigma^i(t) = \sigma^j(t)$ for $i > j$, then we have

$$\sigma^i(a) + ... + \sigma(a) + a + t = \sigma^j(a) + ... + \sigma(a) + a + t,$$

$$\sigma^i(a) + ... + \sigma^{j+2}(a) + \sigma^{j+1}(a) = 0,$$

$$\sigma^{j+1}(\sigma^{i-j-1}(a) + ... + \sigma^2(a) + \sigma(a) + a) = 0 \rightarrow \sigma^{i-j-1}(a) + ... + \sigma^2(a) + \sigma(a) + a = 0.$$

We get a contradiction. So $\sigma^i(t) \neq \sigma^j(t)$ for any $i \neq j$. We look at the polynomials $p_i(X) = X^3 - \sigma^i(t)$. Observe that p_i's are irreducible over $A(t)$ since t is transcendental over A and $\zeta \in A(t)$. Let b_i be a root of the polynomial p_i. We extend $A(t)$ by adding b_i's for each i and also extend σ and τ as follows:

$\sigma(b_i) = b_{i+1}$, for any i. $\tau(b_i) = \zeta b_{i+1}$, if i is even,

$\tau(b_i) = \zeta^2 b_{i+1}$, if i is odd.

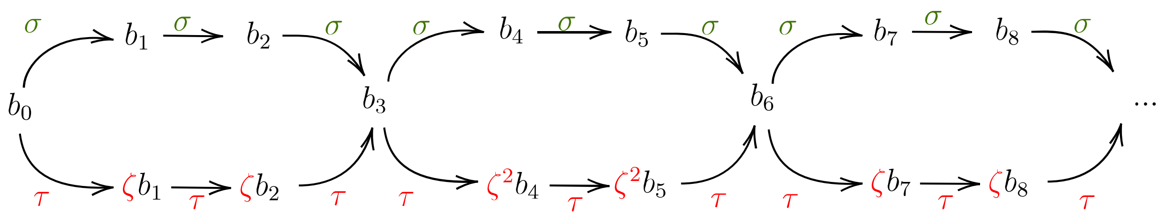
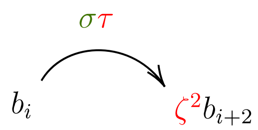


Figure 3.14. Applying σ and τ to b_i's.

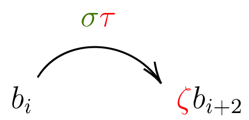
Let $\mathcal{A}'$ be the described extension and let us check that σ and τ commute on $\mathcal{A}'$.

If i is **even,** $$\begin{aligned} \sigma\tau(b_i) &= \sigma(\zeta b_{i+1}) = \boxed{\zeta^2 b_{i+2}} \\ \tau\sigma(b_i) &= \tau(b_{i+1}) = \boxed{\zeta^2 b_{i+2}} \end{aligned}$$



if i is ***even***

If i is **odd,** $$\begin{aligned} \sigma\tau(b_i) &= \sigma\left(\zeta^2 b_{i+1}\right) = \boxed{\zeta b_{i+2}} \\ \tau\sigma(b_i) &= \tau(b_{i+1}) = \boxed{\zeta b_{i+2}} \end{aligned}$$



if i is ***odd***

So $\mathcal{A}'$ is a model of T where the existential sentence $\phi(a)$ is satisfied. Since $\mathcal{A}$ is existentially closed model of T, we also obtain $\mathcal{A} \models \phi(a)$.

Now, let

$$\Sigma_0(v) = \{(\sigma(v) = \tau(v)) \wedge (\sigma^i(v) + \sigma^{i-1}(v) + ... + \sigma(v) + v \neq 0) : i < n\}$$

be a finite subset of $\Sigma(v)$. Let $\mathcal{B}$ be an existentially closed model of T with an element $c \in B$ such that $c + \sigma(c) + \sigma^2(c) + ... + \sigma^k(c) \neq 0$ for any $k < m-1$, and $c + \sigma(c) + \sigma^2(c) + ... + \sigma^{m-1}(c) = 0$ where m is an odd integer greater than n. Notice that we can always find such an element in an existentially closed model of T by Property(C). So we have $\mathcal{B} \models \Sigma_0(c)$. If we also find a ϕ-obstacle $\psi(v)$ such that $\mathcal{B} \models \psi(c)$, this will complete the proof.

We will show that

$$\psi(v) : (v + \sigma(v) + \sigma^2(v) + ... + \sigma^{m-1}(v) = 0) \wedge (\sigma(\zeta) = \tau(\zeta) = \zeta^2)$$

is a ϕ-obstacle if we take m to be an odd integer. Assume that in some model of T we have $\sigma(\zeta) = \tau(\zeta) = \zeta^2$ and there are c, a_1 and a_2 such that $\sigma(a_1) = \tau(a_1) = a_1 + c$, $a_2^3 = a_1$, $\tau(a_2) = \zeta\sigma(a_2)$ and $c + \sigma(c) + \sigma^2(c) + ... + \sigma^{m-1}(c) = 0$. Then, we have $\sigma^m(a_1) = a_1 + c + \sigma(c) + \sigma^2(c) + ... + \sigma^{m-1}(c)$ implying $\sigma^m(a_1) = a_1$. Observe that we also have $\sigma^m(a_2) = \zeta^i a_2$ for some $i = \{0, 1, 2\}$. We will calculate $\sigma^m\tau(a_2)$ in two ways:

$$\begin{aligned}\sigma^m\tau(a_2) &= \sigma^m(\zeta\sigma(a_2)) = \sigma^m(\zeta)\sigma^m(\sigma(a_2)) = \sigma^m(\zeta)\sigma(\sigma^m(a_2)) = \sigma^m(\zeta)\sigma(\zeta^i a_2)\\ &= \boldsymbol{\sigma^m}(\zeta)\sigma(\zeta^{\mathbf{i}})\sigma(\mathbf{a_2}),\\ \sigma^m\tau(a_2) &= \tau\sigma^m(a_2) = \tau(\zeta^i a_2) = \tau(\zeta^i)\tau(a_2) = \sigma(\zeta^{\mathbf{i}})\zeta\sigma(\mathbf{a_2}).\end{aligned}$$

This calculations shows that $\sigma^m(\zeta) = \zeta$, but this is a contradiction since m is odd and $\sigma(\zeta) = \zeta^2$. Hence, $\psi(v)$ is a ϕ-obstacle. Also, we clearly have $\mathcal{B} \models \psi(c)$. Therefore, T has no model companion. □

We showed that the theory of fields with two commuting automorphisms has no model companion; it is also interesting to investigate when an arbitrary theory T with two commuting automorphisms has no model companion [17].

3.3.5. The Theory of Dense Linear Orders without Endpoints with an Automorphisms

Let $<$ be a binary relation symbol and let $\mathcal{L}_{LO} = \{<\}$ be the language of orders. We extend the language of orders by adding a unary function symbol σ and obtain the language $\mathcal{L} = \{<, \sigma\}$. Also, we extend the theory of dense linear orders without endpoints, whose axioms were already listed in Section 2.6, by adding the following axioms stating that σ is an $\mathcal{L}$-automorphism.

1. Axioms stating σ is one to one and onto :
   $\forall v_1 \forall v_2 \, [(\sigma(v_1) = \sigma(v_2)) \rightarrow v_1 = v_2]$ and $\forall v_1 \exists v_2 \, \sigma(v_2) = v_1$.
2. Axiom stating σ preserves the order relation :
   $\forall v_1 \forall v_2 \, (v_1 < v_2 \rightarrow \sigma(v_1) < \sigma(v_2))$.

By expanding DLO by the above axioms, we obtained the theory of dense linear orders without endpoints with an automorphism which we denote as DLO_σ. Like always, we observe that it is an inductive theory by Theorem 2.21 due to the fact that it consists of $\forall\exists$-sentences. In the following theorem, we show that DLO_σ is not companionable.

Theorem 3.9. *The theory of dense linear orders without endpoints (DLO) with an automorphism has no model companion.*

Proof. Let DLO_σ be the theory of dense linear orders without endpoints with an automorphism. We observe that DLO_σ is an inductive theory, so if the model companion exists, then it is exactly the theory of existentially closed models of DLO_σ by Theorem 2.24. We will apply Compactness argument to show that DLO_σ is noncompanionable. Let $\Sigma(v_1, v_2) = \{(v_1 < \sigma(v_1) \wedge \sigma^n(v_1) < v_2) : n \in \mathbb{N}^*\}$ be an infinite set of $\mathcal{L}$-formulas stating that v_1 increases when we recursively apply σ and v_2 is bigger than all of these elements. In other words, we can view $\sigma^n(v_1)$ as an increasing sequence and v_2 as an upper bound for this sequence. Actually, the sequence $\sigma^n(v_1)$ should have a limit point since it is increasing and bounded above by some element and v_2 can be taken as an element beyond the limit point. Also, let $\phi(v_1, v_2) : \exists z\ [(v_1 < \sigma(z)) \wedge (\sigma(z) = z) \wedge (z < v_2)]$ be an $\mathcal{L}$-formula which states that there is an element between v_1 and v_2 whose image remains the same under σ. In short, we may denote ϕ as $\phi(v_1, v_2) : \exists z\ [v_1 < \sigma(z) = z < v_2]$. We will first show that for any existentially closed model $\mathcal{A}$ of DLO_σ, we have

$$\mathcal{A} \models \Sigma(a_1, a_2) \text{ implies } \mathcal{A} \models \phi(a_1, a_2) \text{ for any } a_1, a_2 \in A.$$

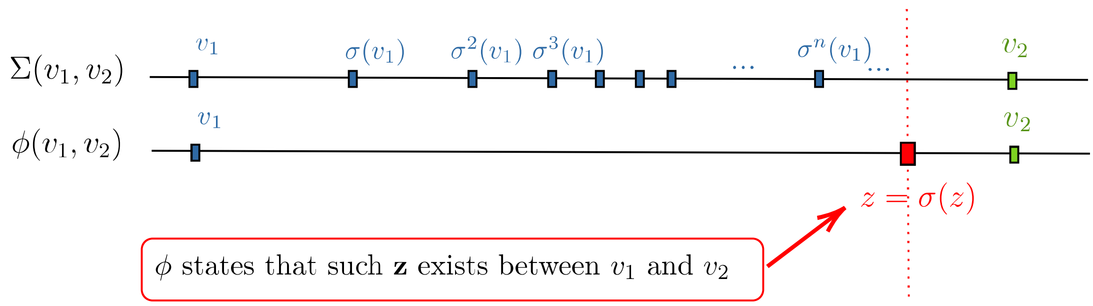


Figure 3.15. $\Sigma(v_1, v_2)$ and $\phi(v_1, v_2)$.

Let $\mathcal{A}$ be an existentially closed model with two elements a_1 and a_2 such that $\mathcal{A} \models \Sigma(a_1, a_2)$ (we can assume that $\mathcal{A}$ is a model containing such elements). Define the sets $X = \{x : x > \sigma^n(a_1) \text{ for all } n \in \mathbb{N}^*\}$ and $Y = A \setminus X$. Clearly, X and Y are nonempty sets, since $a_1 \in X$ and $a_2 \in Y$. Moreover, all elements of X are smaller than all elements of Y and X has no greatest element. So X and Y actually form a Dedekind Cut.

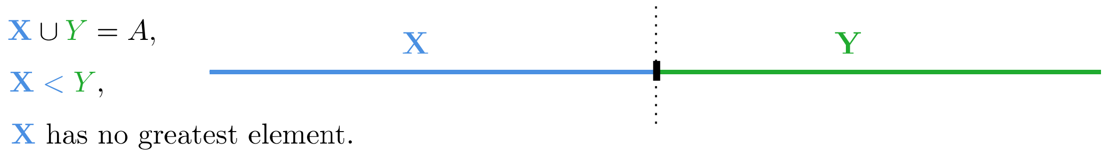


Figure 3.16. X and Y form a Dedekind Cut.

Let us show that X and Y are closed under σ. If $x \in X$, then there is $m \in \mathbb{N}^*$ such that $x < \sigma^m(a_1)$. By applying σ to both sides we get $\sigma(x) < \sigma^{m+1}(a_1)$; hence, $\sigma(x) \in X$. If $y \in Y$, then it means that $\sigma^n(a_1) < y$ for all n and by applying σ to both sides we also obtain $\sigma^{n+1}(a_1) < \sigma(y)$ for all n. So we have $\sigma(y) \in Y$. Hence X and Y are closed under σ. If Y has a least element c corresponding the cut of X and Y, we have $c \leq y$ for all $y \in Y$ and moreover, $\sigma(c) \leq \sigma(y)$ for all $y \in Y$. Since σ is an automorphism and Y is closed under σ, we have $\sigma(c) \leq y$ for all $y \in Y$. Hence, we must have $\sigma(c) = c$. So if Y has a least element, ϕ is automatically satisfied. Assume now Y does not contain a least element. Then, we can add an element c to the cut of X and Y and obtain a model $\mathcal{A}'$ of DLO_σ by extending σ as $\sigma(c) = c$. Since $\mathcal{A}$ is existentially closed there is also an element in $c' \in A$ witnessing the existential sentence $\phi(a_1, a_2)$; that is, there is $c' \in A$ such that $a_1 < \sigma(c') = c' < a_2$. Therefore, if we have $\mathcal{A} \models \Sigma(a_1, a_2)$ for some existentially closed model $\mathcal{A}$ containing a_1 and a_2, then we also have $\mathcal{A} \models \phi(a_1, a_2)$.

Now, let $\Sigma_0(v_1, v_2) = \{(v_1 < \sigma(v_1) \wedge \sigma^n(v_1) < v_2) : 1 \leq n < m\}$ be a finite subset of $\Sigma(v_1, v_2)$. We will find a ϕ-obstacle $\psi(v_1, v_2)$ and an existentially closed model $\mathcal{B}$ with elements b_1 and b_2 such that $\mathcal{B} \models \Sigma(b_1, b_2) \wedge \psi(b_1, b_2)$. First of all, we will show that $\psi(v_1, v_2) : \sigma^m(v_1) = v_2$ is a ϕ-obstacle. Assume for a contradiction, $\phi(c_1, c_2)$ and $\psi(c_1, c_2)$ are satisfied by a model of DLO_σ containing c_1 and c_2. That is, we have $\sigma^m(c_1) = c_2$ and there is an element z such that

$$c_1 < \sigma(z) = z < c_2.$$

But then by applying σ^m to the inequality, we obtain

$$c_2 = \sigma^m(c_1) < \sigma^{m+1}(z) = \sigma^m(z) = \dots = \sigma(z) = z < c_2.$$

This is a contradiction. Hence, $\psi(v_1, v_2)$ is a ϕ-obstacle. Let $\mathcal{B}$ be an existentially closed model of DLO_σ with an element b_1 satisfying $b_1 < \sigma(b_1)$ (note that we can always find such a model of DLO_σ and we can extend it to an existentially closed model). Also, let $b_2 = \sigma^m(b_1)$. We clearly have $\mathcal{B} \models \Sigma_0(b_1, b_2)$ and also $\mathcal{B} \models \psi(b_1, b_2)$ where $\psi(v_1, v_2)$ is a ϕ-obstacle. Therefore, DLO_σ has no model companion. □

Now, let us illustrate what happens in the proof on an explicit model. Consider the model $(\mathbb{Q}, <)$ of DLO and also let σ be an automorphism of $(\mathbb{Q}, <)$ defined as $\sigma(x) = \frac{x}{2}$. Observe that the only element that remains the same under σ is 0. If we take any element $a_1 < 0$, it satisfies $a_1 < \sigma(a_1) < 0$ and for any element a_2 such that $0 < a_2$, we have $0 < \sigma(a_2) < a_2$.

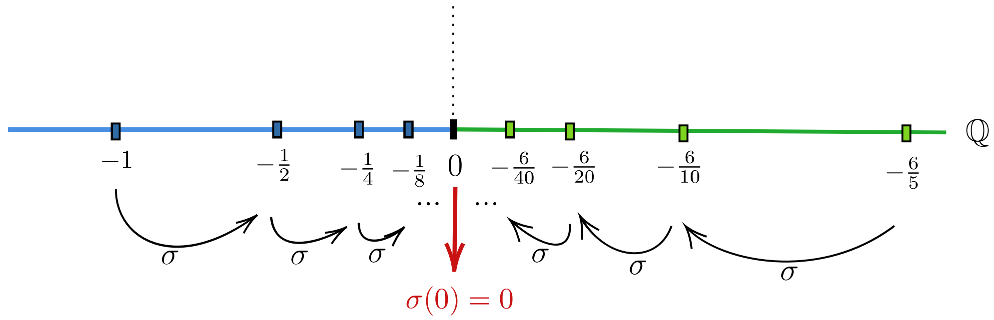


Figure 3.17. Applying $\sigma(x) = \frac{x}{2}$ to -1 and $-\frac{6}{5}$ recursively.

We can extend $(\mathbb{Q}, <, \sigma)$ to an existentially closed model $\mathcal{A}$ of DLO_σ and note that we also have $a < \sigma(a)$ for $a < 0$ and $\sigma(a) < a$ for $0 < a$ in $\mathcal{A}$. Actually, we see that the automorphism σ cuts the linear order into two pieces and 0 is the element corresponding the cut. So if $\mathcal{A} \models \Sigma(a_1, a_2)$, then we have $a_1 < 0 < a_2$ and this implies $\mathcal{A} \models \phi(a_1, a_2)$ since $\sigma(0) = 0$. However, if we take a finite subset Σ_0 of Σ, then it means that we can choose a_2 as $\sigma^m(a_1) < a_2 < 0$. In this situation we cannot find an element between a_1 and a_2 whose image remains fixed under σ.

Kikyo and Shelah generalize the argument that is presented in Theorem 3.9 to prove that whenever there is a theory T with strict order propery, then T together with an automorphism has no model companion [19].

Laskowski and Pal showed that if we restrict our attention to automorphisms σ which has the property $\forall v\ (v < \sigma(v))$, that is called as increasing automorphism, then DLO with an increasing automorphism has model companion. Also, the same is true for decreasing automorphisms. So we see that the obstruction presented in the proof of Theorem 3.9 can be eliminated by putting assumptions on the automorphism. Moreover, they gave a characterisation of all complete and model complete extensions of DLO with an automorphism [20].

SUMMARY:

As a summary, we list below how the formulas $\Sigma(\bar{v})$, $\phi(\bar{v})$ and ϕ-obstacle $\psi(\bar{v})$ mentioned in the Compactness Argument are chosen for the nonexistance proofs.

1. The theory of groups:

$$\Sigma(v_1, v_2) = \{v_1^n \neq e, v_2^n \neq e : n \in \mathbb{N}^*\},$$
$$\phi(v_1, v_2) : \exists w\,(v_1 \cdot w = w \cdot v_2),$$
$$\phi\text{-obstacle: } \psi(v_1, v_2) : (v_1^m = e \wedge v_2^m \neq e).$$

2. The theory of rings:

$$\Sigma(v) = \{v^n \neq 0 : n \in \mathbb{N}^*\},$$
$$\phi(v) : \exists w_1 \exists w_2\,[(w_1^2 = w_1) \wedge (w_1 \neq 0) \wedge (v \cdot w_2 = w_1)],$$
$$\phi\text{-obstacle: } \psi(v) : (v^m = 0).$$

3. The theory of digraphs with a unique successor and predecessor:

$$\Sigma(v_1, v_2) = \{\neg D^n(v_1, v_2) \wedge \neg D^n(v_1, v_2) n \in \mathbb{N}^*\},$$
$$\phi(v_1, v_2) : \exists w(S(w, v_1) \wedge \neg S(w, v_2)),$$
$$\phi\text{-obstacle: } \psi(v_1, v_2) : D^m(v_1, v_2),$$

where $D^n(v_1, v_2) : \exists w_1 \exists w_2 ... \exists w_n (D(v_1, w_1) \wedge (\bigwedge_{i=1}^{n-1} D(w_i, w_{i+1})) \wedge D(w_n, v_2))$.

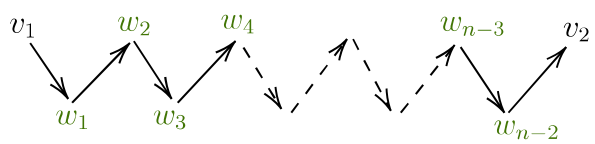


Figure 3.18. $D^n(v_1, v_2)$.

4. The theory of cycle free graphs:

$$\Sigma(v_1, v_2) = \{\neg R^n(v_1, v_2) : n > 3\},$$
$$\phi(v_1, v_2) : \exists z(R(v_1, z) \wedge R(z, v_2)) \vee R(v_1, v_2),$$
$$\phi\text{-obstacle: } R^m(v_1, v_2),$$

where $R^n(v_1, v_2) : \exists w_1 \exists w_2 ... \exists w_{n-2}(R(v_1, w_1) \wedge (\bigwedge_{i=1}^{n-3} R(w_i, w_{i+1})) \wedge R(w_{n-2}, v_2))$ for $n > 3$.

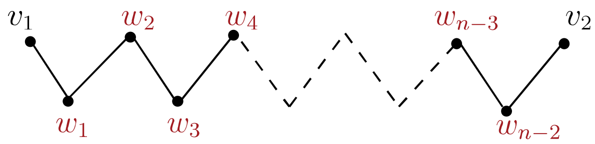


Figure 3.19. $R^n(v_1, v_2)$.

5. The theory of fields with two commuting automorphisms:

$$\Sigma(v) = \{[(\sigma(v) = \tau(v)) \wedge (\sigma^n(v) + \sigma^{n-1}(v) + ... + \sigma(v) + v \neq 0)] : n \in \mathbb{N}^*\},$$

$$\phi(v) : \exists z \ (z^3 = 1 \wedge \sigma(z) = \tau(z) = z^2) \rightarrow \exists w_1 \exists w_2[(\sigma(w_1) = \tau(w_1) = w_1 + v) \wedge (w_2^3 = w_1) \wedge (\tau(w_2) = z\sigma(w_2))],$$

$$\phi\text{-obstacle: } \psi(v) : (v + \sigma(v) + \sigma^2(v) + ... + \sigma^{m-1}(v) = 0) \wedge (\sigma(\zeta) = \tau(\zeta) = \zeta^2).$$

6. The theory of dense linear orders without endpoints with an automorphisms:

$$\Sigma(v_1, v_2) = \{(v_1 < \sigma(v_1) \wedge \sigma^n(v_1) < v_2) : n \in \mathbb{N}^*\},$$
$$\phi(v_1, v_2) : \exists z \ [(v_1 < \sigma(z)) \wedge (\sigma(z) = z) \wedge (z < v_2)],$$
$$\phi\text{-obstacle: } \psi(v_1, v_2) : \sigma^n(v_1) = v_2.$$

4. CONCLUSION

In this thesis, we studied existence and nonexistence of model companions. While studying model companions, we came upon to many areas for future research. We list possible ways of pursuing this research below.

We showed that the theory of fields with two commuting automorphisms has no model companion. It is interesting to generalize this result to arbitrary theories. This question was also asked by Kikyo [17].

Question 1 Let T be an arbitrary theory (not necessarily, the theory of fields). Assume $T_{\sigma\tau}$ is the theory T with two commuting automorphisms. When does $T_{\sigma\tau}$ have a model companion?

We have examples where T has model companion but T_σ has no model companion; for example, if T is the theory of fields with an automorphism, then T has model companion but T_σ has no model companion. We can also investigate the converse of this statement:

Question 2 Is there any example where T has no model companion but T_σ has a model companion?

Question 2 is stated in [12] and there is also one more question in this article that we also want to state. Remember given a theory T, T^* denotes its model companion and T_σ denotes the theory with an automorphism.

Question 3 When do we have $(T_\sigma)^* = ((T^*)_\sigma)^*$?

We also list the other questions now.

Question 4 Can we find a structural property that allow us to prove certain theories do not have model companions? For example, if a theory T has strict order property, then T_σ has no model companion.

Question 5 Can we get information about model companions of theories of fields by interpreting graphs on fields?

REFERENCES

1. Marker, D., *Model Theory: An Introduction*, ISBN 0-387-98760-6, Springer, 2002.

2. Hodges, W., *Model Theory*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1993.

3. Prestel, A. and C. N. Delzell, *Mathematical Logic and Model Theory*, Universitext, Springer, London, 2011.

4. Chang, C. C. and H. J. Keisler, *Model Theory*, Vol. 73 of *Studies in Logic and The Foundations of Mathematics*, Elsevier Science Publisher B.V., 1990.

5. Tent, K. and M. Ziegler, *A Course in Model Theory*, Lecture Notes in Logic, Cambridge University Press, 2012.

6. Poizat, B. and M. Klein, *A Course in Model Theory: An Introduction to Contemporary Mathematical Logic*, Universitext - Springer-Verlag, Springer New York, 2000.

7. Hodges, W. and T. Scanlon, "First-order Model Theory", *The Stanford Encyclopedia of Philosophy*, Winter 2018 edn., 2018, `https://plato.stanford.edu/archives/win2018/entries/modeltheory-fo/`, accessed in February 2022.

8. Zakon, E., "Model-Completeness and Elementary Properties of Torsion Free Abelian Groups", *The Canadian Journal of Mathematics*, Vol. XXVI, No. 4, pp. 829–840, 1974.

9. Lipshitz, L. and D. Saracino, "The Model Companion of the Theory of Commutative Rings without Nilpotent Elements", *American Mathematical Society*, Vol. 38, No. 2, pp. 381–387, 1973.

10. Chatzidakis, Z. and E. Hrushovski, "Model Theory of Difference Fields", *Trans-*

actions of the American Mathematical Society, Vol. 351, No. 8, pp. 2997–3071, 1999.

11. Dummit, D. S. and R. M. Foote, *Abstract Algebra*, Wiley, 2004.

12. Takeuchi, K., Y. Tanaka and A. Tsuboi, “Model Companions of Theories of Graphs”, *Mathematical Logic Quarterly*, Vol. 61, No. 3, pp. 236–246, 2015.

13. Graham Higman, B. N. and H. Neumann, “Embedding Theorems for Groups”, *Journal of the London Mathematical Society*, Vol. s1-24, pp. 247–254, October 1949.

14. Eklof, P. and G. Sabbagh, “Model Completions and Modules”, *Annals of Mathematical Logic*, Vol. 2, pp. 251–295, 1971.

15. Cherlin, G. L., “Algebraically Closed Commutative Rings”, *The Journal of Symbolic Logic*, Vol. 38, No. 3, pp. 493–499, 1973.

16. Naito, T., “Graph Axiom and Model Companion”, *Research Institute for Mathematical Sciences*, Vol. 1988, pp. 83–86, 2014.

17. Kikyo, H., “On Generic Predicates and Automorphisms”, *Research Institute for Mathematical Sciences*, pp. 1–8, 2004.

18. Gorodnik, A., *Lecture 9: Quadratic Residues and the Law of Quadratic Reciprocity*, `https://www.math.uzh.ch/gorodnik/nt/lecture9.pdf`, accessed in November 2021.

19. Kikyo, H. and S. Shelah, “The Strict Order Property and Generic Automorphisms”, *The Journal of Symbolic Logic*, Vol. 67, No. 1, pp. 214–216, 2002.

20. Laskowski, M. C. and K. Pal, “Model Companion of Ordered Theories with an Automorphism”, *Transactions of the American Mathematical Society*, Vol. 367, No. 10, pp. 6877–6902, 2013.