3+1 FORMULATION IN NEWTON-CARTAN GRAVITY

by

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ABSTRACT

3+1 FORMULATION IN NEWTON-CARTAN GRAVITY

In this thesis, we will give a review of torsion-free Newton-Cartan theory. We will start by doing a large c expansion of general relativity and we will restrict ourselves to a few leading orders. After obtaining Newton-Cartan theory in this way, we will explore its symmetries. Through a special set of combined symmetries, we will construct a split space and time structure. Lastly, we will work out the field equations and equations of motion for a particle in the 3+1 formulation we have obtained.

ÖZET

NEWTON-CARTAN KÜTLEÇEKİMİNDE 3+1 FORMÜLASYONU

Bu tezde torsiyonsuz Newton-Cartan teorisini inceleyeceğiz. Genel göreliliğin büyük c açılımı ile başlayıp kendimizi ilk bir kaç mertebeye kısıtlayacağız. Newton-Cartan teorisine bu yol ile vardıktan sonra, teorinin simetrilerini inceleyeceğiz. Bir takım özel simetrilerin birleşimiyle uzay ve zamanın ayrıştığı bir yapı kuracağız. Son olarak, elde ettiğimiz 3+1 formülasyonunda alan ve parçacık hareket denklemlerimizi çıkaracağız.

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LIST OF SYMBOLS

$i, j, k, l, m, \ldots$	3 dimensional spatial indices (1,2,3)
$\mu, \nu, \lambda, \rho, \sigma, \ldots$	4 dimensional space-time indices (0,1,2,3)
C_μ	Newton-Cartan gauge field
$g_{\mu\nu}$	Metric tensor
$g^{\mu\nu}$	Inverse metric tensor
$G_{\mu\nu}$	Einstein tensor
h_{ij}	Newton-Cartan metric in 3+1 formulation
h^{ij}	Newton-Cartan inverse metric in 3+1 formulation
$K_{\mu\nu}$	Newton-Cartan field strength tensor
$R_{\mu\nu\rho\lambda}$	Riemann curvature tensor
$R_{\mu\nu}$	Ricci tensor
$\hat{R}_{\mu\nu}$	Newton-Cartan Ricci tensor
R_{ij}	Spatial Newton-Cartan Ricci tensor of metric h
R	Ricci scalar
$T_{\mu\nu}$	Stress-energy tensor
$\mathcal{T}_{\mu\nu}$	Trace-reversed stress-energy tensor
$T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n}$	An arbitrary (m, n) tensor
$T'^{\mu'_1 \cdots \mu'_m}{}_{\nu'_1 \cdots \nu'_n}$	An arbitrary (m, n) tensor in a different coordinate basis
$\overset{(i)}{T}{}^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n}$	An arbitrary (m, n) tensor of order i
$\varGamma^\lambda_{\mu\nu}$	Christoffel symbols
$\hat{\varGamma}^\lambda_{\mu\nu}$	Newton-Cartan Christoffel symbols
$\varGamma^k_{ij}$	Spatial Newton-Cartan Christoffel symbols of metric h
$\varDelta$	Laplacian
$\delta^{ij},\ \delta_{ij}$	Kronecker delta symbol
$\delta^\mu_\nu,\ \delta^i_j$	Kronecker delta $(1, 1)$-tensor or identity $(1, 1)$-tensor
ξ^μ	Infinitesimal diffeomorphism generating vector field

$\tau_\mu,\ h^{\mu\nu}$	Newton-Cartan fundamental fields
$\tau^\mu,\ h_{\mu\nu}$	Newton-Cartan derived fields
$\varPhi$	Boost invariant Newtonian potential
ϕ	Newtonian potential
χ_μ	Milne boost one-form
∇_μ	Covariant derivative of metric g
$\hat{\nabla}_\mu$	Newton-Cartan covariant derivative
∇_i	Spatial Newton-Cartan covariant derivative of metric h

LIST OF ACRONYMS/ABBREVIATIONS

GR	General Relativity
LO	Leading Order
NC	Newton-Cartan
NLO	Next to Leading Order
U(n)	Unitary Group of Degree n
WEP	Weak Equivalence Principle

1. INTRODUCTION

Even though Einstein's general theory of relativity is a more detailed description of gravity, Newtonian gravity is still valid and useful for our daily non-relativistic lives. We all know the phrase *"all objects freely fall at the same rate"*. It explains the mass independence of Newton's equations of motion, which means that formulating Newtonian gravity in a geometrical framework might be possible. This was first done by Ellie Joseph Cartan in 1923 [1, 2].

In Chapter 2, we will briefly discuss the history of theories of gravity, Newtonian gravity and Einstein's general theory of relativity. The weak equivalence principle (WEP) implies that Newtonian gravity can be interpreted as a geometric theory.

In Chapter 3, we will introduce the fundamentals of our framework, differential geometry by following the works of [3–5]. The definiton of tensors will be our starting point and we will proceed to build our coordinate independent framework. Later, a discussion about *quantifying* the curvature of our space will take place and the introduction of Riemann and Ricci tensors will prove useful.

In Chapter 4, we will briefly introduce Einstein's general theory of relativity and the trace-reversed forms of the Einstein field equations.

In Chapter 5, we will start expanding our metric and metric-representable objects such as the Levi-Civita symbol and the Ricci tensor by following [6]. Next, we will show a simple expansion of trace-reversed Einstein Field Equations. The expansion of a perfect fluid will be an interesting way to get the Poisson equation of Newtonian gravity.

In Chapter 6, we will start with a truncated expansion by restricting ourselves to a few highest-ordered fields. As a result, Newton-Cartan theory will arise. We will take a look at the particle equations based on the expanded particle action.

Chapter 7 will be the study of the symmetries of Newton-Cartan theory. The main three symmetries we will be exploring are U(1), diffeomorphisms and Milne boosts.

In Chapter 8, we will fix the coordinates and the gauge which will provide us a time-dependent spatial diffeomorphism (TDSD) structure. This will give us the opportunity to explore how the field equations and the particle equations of motion behave in the 3+1 formalism. Next, starting with infinitesimal coordinate transformations, we will see how our fields transform under infinitesimal TDSDs and later try to apply the same principles to the finite TDSDs. Finally, we will introduce a rather peculiar notation for our computations.

Chapter 9 will be the apllications of our 3+1 formalism in a few basic situations. First, we will try to understand how we can describe an expanding space. Then we will look at a rotating frame of reference.

2. GRAVITY

Gravity has always been one of the most important phenomena in nature. Newton's law of universal gravitation was the most successful description at its time. It worked on the surface of the Earth and it worked with celestial objects, well, mostly. It couldn't explain all of the phenomena revolving around gravity such as the perihelion precession of Mercury's orbit or the gravitational deflection of light.

In his 1915 paper [7], Albert Einstein had worked out the field equations of his general theory of relativity and presented the world a new description of gravity in the language of differential geometry. Einstein's theory was successful in explaining the aforementioned phenomena and currently is the most successful theory of gravity we have.

2.1. Mass Equivalence

Fundamentally there can be three different mass properties of an object. First, we have the mass which determines how strong the gravitational field around the object is. It is the source of gravity, we will call this the active mass, m_a. The second mass determines how an object is affected by an external gravitational field. We will call it the passive mass, m_p. And finally, the third mass is the inertial mass, m_i which is the resistance to the acceleration caused by a force acting on the body.

All of our understanding of gravitational physics relied on the *assumption* that these three masses are equivalent until we had some experimental results of the equivalence. The MICROSCOPE experiment results [8] show that there aren't any statistically significant differences between the masses up to 10^{-15} order, therefore

$$m_a = m_p = m_i \tag{2.1}$$

holds for our work.

2.2. Newton's Law of Universal Gravitation

Without the equivalence principle, Newton's Law of Universal Gravitation reads as

$$m_i \ddot{\vec{\mathrm{x}}} = -m_p \vec{\nabla}\phi, \tag{2.2}$$

$$\Delta\phi = 4\pi G_N \rho_a, \tag{2.3}$$

where $\vec{\mathrm{x}}$ is the position of the particle, ϕ is an external gravitational potential that the particle is exposed to, G_N is the gravitational constant and ρ_a is the active mass density. When we consider the equivalence principle, the picture changes drastically. We are left with the equation

$$\ddot{\vec{\mathrm{x}}} = -\vec{\nabla}\phi, \tag{2.4}$$

which is *independent of the properties of the object.* In this equation, the motion of the object is only determined by the gradient of the gravitational potential. There are no references to any properties of the object itself. With the help of differential geometry, we can build a geometrical theory for Newtonian gravity.

Another great interpretation, perhaps the greatest interpretation, of Equation (2.4) is that the effects of gravity can be thought of as an acceleration. This notion is the basis of Einstein's theory of general relativity. The *equivalence* of an accelerating frame of reference and the presence of gravity in an inertial frame of reference.

3. DIFFERENTIAL GEOMETRY

Differential geometry is the heart of Einstein's Theory of General Relativity. It is a great framework which can be used when working on curved spaces. We have gathered a short introductory summary based on various sources [3–5]. This is not an in depth discussion about differential geometry by any means, but a practical one which will suit our needs.

3.1. Tensors

A (m, n)-tensor T defined on a vector space V and its dual space V^* is a multilinear map, which can be expressed as

$$T : \underbrace{V^* \times \cdots \times V^*}_{m} \times \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}. \tag{3.1}$$

We can express a tensor by its *components*. Given the basis $\{e_\mu\}$ for V and $\{f^\mu\}$ for V^*, the components of a tensor can be expressed as

$$T = \sum_{\mu_1,\cdots,\nu_n} T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} e_{\mu_1} \otimes \cdots \otimes f^{\nu_n}. \tag{3.2}$$

If we are working with tensor equations, we can do our calculations on the component basis which will be useful in some areas but more painful than abstract notation in some cases. Well, nothing is perfect.

Hence, from here on, when we are talking about tensors, we will strictly use the components of the tensors when we are referring to them. There are a few useful notational conventions which we will adapt in this thesis:

- Repeated indices are summed over.
- Greek letters take values $0, 1, \ldots, d$ Latin letters take values $1, \ldots, d$. In this work, $d = 3$ unless it is stated otherwise.
- Tensor product symbol $\otimes$ is dropped for brevity.

An example for the notational conventions can be

$$T = T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} e_{\mu_1} \cdots e_{\mu_m} f^{\nu_1} \cdots f^{\nu_n}, \tag{3.3}$$

where we applied all the rules at once.

Now, when we apply this tensor notation to an arbitrary manifold $\mathcal{M}$ we can do some more. First of all, we can use a coordinate basis. At an arbitrary point p, the tangent space V_p will have the basis $\{\partial/\partial x^\mu\}$ and its dual, the cotangent space V_p^* will have the basis $\{\mathrm{d}x^\mu\}$. If we apply a coordinate transformation, the components of an (m, n)-tensor T will transform as

$$T'^{\mu'_1 \cdots \mu'_m}{}_{\nu'_1 \cdots \nu'_n} = T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} \frac{\partial x'^{\mu_1 \prime}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\nu_n}}{\partial x'^{\nu_n \prime}}. \tag{3.4}$$

3.2. Metric Tensor and Connection

The *metric tensor* $g_{\mu\nu}$ is a $(0, 2)$ symmetric tensor. Generally, it is non-degenerate, which ensures that its inverse $g^{\mu\nu}$ exists. The metric identity equation for a non-degenerate metric can be expressed as

$$g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda. \tag{3.5}$$

By using the metric, an inner product can be defined as

$$\langle A, B \rangle = A^\mu B^\nu g_{\mu\nu}. \tag{3.6}$$

In order to raise and lower indices on other tensors, we can use the metric and its inverse as follows

$$A_\nu = A^\mu g_{\mu\nu}, \tag{3.7}$$

$$B^\nu = B_\mu g^{\mu\nu}. \tag{3.8}$$

By definition, the partial derivative operator is coordinate dependent. Therefore, we should be careful when we are working with the derivatives of tensorial objects especially when we are trying to construct a coordinate independent formulation. We can easily show that the partial derivative operator does not transform like a tensor. A simple calculation yields

$$\frac{\partial A'^{\nu'}}{\partial x'^{\mu'}} = \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \left(\frac{\partial^2 x'^{\nu'}}{\partial x^n \partial x^{\mu}} A^{\nu} + \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial A^{\nu}}{\partial x^{\mu}} \right) \neq \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial A^{\nu}}{\partial x^{\mu}}. \tag{3.9}$$

Hence, we need a derivative operator which transforms accordingly which we can use in all coordinate systems. We can construct such an operator by adding a *correction term* to the partial derivative. Of course, this is not a rigorous way to construct a mathematical structure, but it is a practical way nonetheless. A more rigorous discussion of this topic can be found in [3, 4].

When acted on an (m, n) tensor T, the covariant derivative can be expressed as follows

$$\begin{aligned} \nabla_{\lambda} T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} = \partial_{\lambda} T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} + \Gamma^{\mu_1}_{\lambda\rho} T^{\rho \cdots \mu_m}{}_{\nu_1 \cdots \nu_m} + \cdots + \Gamma^{\mu_m}_{\lambda\rho} T^{\mu_1 \cdots \rho}{}_{\nu_1 \cdots \nu_n} \\ - \Gamma^{\rho}_{\lambda\nu_1} T^{\mu_1 \cdots \mu_m}{}_{\rho \cdots \nu_n} - \cdots - \Gamma^{\rho}_{\lambda\nu_1} T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \rho}, \end{aligned} \tag{3.10}$$

which is a metric compatible operator. The compatibiliy condition is

$$\nabla_{\lambda} g_{\mu\nu} = 0. \tag{3.11}$$

The *symbol* Γ is not a tensor. It is called a *Christoffel symbol* or a *Levi-Civita connection*. We can express its components in terms of the metric as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\mu} g_{\rho\nu} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} \right), \tag{3.12}$$

where $\Gamma^{\lambda}_{\mu\nu}$ are called *connection coefficients*.

3.3. Geodesic Equation

Geodesics define the straight lines in a given space. One can derive the geodesic equation by minimizing the distance between two points a and b, which follows as

$$L = \int_a^b ds = \int_a^b \sqrt{-g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}} = \int_a^b \sqrt{-g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}} \mathrm{d}\tau. \tag{3.13}$$

The minimization of L by the principle of least action will yield us

$$\ddot{x}^{\lambda} + \Gamma^{\lambda}_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = 0. \tag{3.14}$$

As one expects, in a flat space with Cartesian coordinates, the geodesic equation will reduce to

$$\ddot{x}^{\lambda} = 0, \tag{3.15}$$

which defines a straight line.

3.4. Curvature

One can ask, does it matter in which order we take the covariant derivative? Do they commute? The commutation can be expressed as

$$[\nabla_{\mu}, \nabla_{\nu}]A^{\rho} = R^{\rho}{}_{\lambda\mu\nu}A^{\lambda}, \tag{3.16}$$

where $R^{\rho}{}_{\lambda\mu\nu}$ or $R_{\rho\lambda\mu\nu}$ is called the Riemann curvature tensor, which tells us how curved our manifold is. We can say that our space is flat if and only if all components of the Riemann tensor are zero, which also means that covariant derivatives commute.

Some properties of the Riemann tensor are:

- Skew symmetry on first two and last two indices: $R_{\mu\nu\lambda\rho} = -R_{\mu\nu\rho\lambda} = -R_{\nu\mu\lambda\rho}$,
- Pairwise interchange symmetry: $R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$.

We can contract two indices of the Riemann curvature tensor as follows

$$g^{\lambda\rho}R_{\lambda\mu\rho\nu} = R^{\rho}{}_{\mu\rho\nu} = R_{\mu\nu}, \tag{3.17}$$

where $R_{\mu\nu}$ is called the Ricci tensor. If we take one step further and contract the Ricci tensor by the metric tensor as follows

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{3.18}$$

we get the Ricci scalar.

4. GENERAL RELATIVITY

As we have discussed in Chapter 2, general relativity is a description of gravity. The main notion of the theory is that the matter changes the structure of spacetime around it and free objects follow the geodesics in the spacetime. Perhaps the most intriguing part of the theory is that time is not an absolute parameter, since it is another dimension in the manifold it can also *bend* for different observers.

4.1. Einstein Field Equations

The main idea of general relativity is to relate the curvature of spacetime to the matter density. But we know that the Ricci tensor is not conserved even though the stress-energy tensor is. The conservation relation can be expressed as

$$\nabla^\mu R_{\mu\nu} \neq 0, \tag{4.1}$$

$$\nabla^\mu T_{\mu\nu} = 0. \tag{4.2}$$

So, there has to be a better candidate than the Ricci tensor. We have an object that is conserved which is closely related to the curvature, it can be expressed as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{4.3}$$

and the conservation equation is

$$\nabla^\mu G_{\mu\nu} = 0, \tag{4.4}$$

where $G_{\mu\nu}$ is called the Einstein tensor. The Einstein field equations without the cosmological constant can be expressed as

$$G_{\mu\nu} = \frac{8\pi G_N}{c^{-4}} T_{\mu\nu}. \tag{4.5}$$

which can be derived by varying the Einstein-Hilbert action with respect to the metric. The action can be written as

$$S = \int \left[\frac{1}{16\pi G_N} R + \mathcal{L}_M \right] \sqrt{-g}\, \mathrm{d}^n x \,, \tag{4.6}$$

where $\mathcal{L}_M$ is the matter Lagrangian. The variation of the matter Lagrangian can be defined as the stress-energy tensor, which follows as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M . \tag{4.7}$$

It will be useful to derive the trace-reversed version of the Einstein field equations in our context, which can be written as

$$R_{\mu\nu} = \frac{8\pi G_N}{c^{-4}} \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) . \tag{4.8}$$

5. LARGE c EXPANSION

There have been various attempts of approximating general relativity. The post-Newtonian expansion and the post-Minkowskian expansion are two examples of this effort. In this section of the thesis, we will treat the speed of light c as a parameter and we will expand our fields by the powers of c. At the end, we will study the leading order (LO) and next-to-leading order (NLO) terms, since they will be the most dominant terms in the context of *large* c. [9, 10]

5.1. Metric Expansion

We can start by expanding our metric. In this thesis, we will restrict ourselves only to the even powers of c, although there is no a priori reason not to include the odd powers, as we can see in [11]. The general structure of the expansion will be as follows

$$g_{\mu\nu} = \sum_{i=-1}^{\infty} \overset{(2i)}{g}_{\mu\nu} c^{-2i}, \qquad g^{\mu\nu} = \sum_{i=0}^{\infty} \overset{(2i)}{g}{}^{\mu\nu} c^{-2i}. \tag{5.1}$$

We can gain some insight about how the ordered terms are constrained by using the metric identity. For instance, the LO term will be in the order of c^{-2}, which implies

$$\overset{(0)}{g}{}^{\mu\lambda} \overset{(-2)}{g}_{\mu\nu} = 0. \tag{5.2}$$

Now we can start *naming* our fields. The LO terms can be decomposed into a one-form and a $(2, 0)$ symmetric tensor as follows

$$\overset{(-2)}{g}_{\mu\nu} = -\tau_\mu \tau_\nu, \tag{5.3}$$

$$\overset{(0)}{g}{}^{\mu\nu} = h^{\mu\nu}. \tag{5.4}$$

If we continue in the same fashion, the metric expansion in terms of the first few fields becomes

$$g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + h_{\mu\nu} + 2\tau_{(\mu}C_{\nu)} + c^{-2}(2B_{(\mu}\tau_{\nu)} - C_\mu C_\nu - h_{\mu\rho}h_{\nu\sigma}\beta^{\rho\sigma}) + \mathcal{O}(c^{-4}), \quad (5.5)$$

$$g^{\mu\nu} = h^{\mu\nu} + c^{-2}(-\tau^\mu\tau^\nu + 2\tau^{(\mu}h^{\nu)\lambda}C_\lambda + \beta^{\rho\sigma}) + \mathcal{O}(c^{-4}). \quad (5.6)$$

Of course, using Equation (3.5), we get

$$h^{\mu\nu}\tau_\mu = 0, \quad (5.7)$$

$$\tau_\mu\tau^\nu + h_{\mu\rho}h^{\rho\nu} = \delta^\nu_\mu, \quad (5.8)$$

$$\tau^\mu\tau^\nu h_{\mu\nu} = 0, \quad (5.9)$$

$$\tau_\mu\beta^{\mu\nu} = 0. \quad (5.10)$$

In our discussion, we won't be seeing much of B_μ and $\beta^{\mu\nu}$ terms since they appear at higher orders only.

We now have a powerful tool in our hands, we can expand all of our metric-representable objects by our fields such as the covariant derivative, Riemann and Ricci tensors. We will make an assumption here, and take τ_μ as closed, which can be expressed as

$$\mathrm{d}\tau = 0. \quad (5.11)$$

This will yield us a *torsionless* theory. This assumption will set most of our terms to zero. The LO term of the Levi-Civita connection will be

$$\overset{(-2)}{\Gamma}{}^\lambda_{\mu\nu} = \frac{1}{4}h^{\lambda\rho}\big[\tau_\mu\partial_{[\nu}\tau_{\rho]} + \tau_\nu\partial_{[\mu}\tau_{\rho]}\big] = 0. \quad (5.12)$$

The highest order non-zero term will be

$$\overset{(0)}{\Gamma}{}^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} = \tau^\lambda\partial_\mu\tau_\nu + \frac{1}{2}h^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) - h^{\lambda\rho}K_{\rho(\mu}\tau_{\nu)}, \quad (5.13)$$

where we defined $K_{\mu\nu}$ as

$$K_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu. \tag{5.14}$$

$\hat{\Gamma}^\lambda_{\mu\nu}$ will be our Newton-Cartan connection. We can define a covariant derivative and a curvature tensor with this connection and build our theory from here. Which we will do in Chapter 6. Newton-Cartan covariant derivative can be expressed as

$$\begin{aligned}\hat{\nabla}_\lambda T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} = \partial_\lambda T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} + \hat{\Gamma}^{\mu_1}_{\lambda\rho} T^{\rho\cdots\mu_m}{}_{\nu_1\cdots\nu_m} + \cdots + \hat{\Gamma}^{\mu_m}_{\lambda\rho} T^{\mu_1\cdots\rho}{}_{\nu_1\cdots\nu_n} \\ - \hat{\Gamma}^{\rho}_{\lambda\nu_1} T^{\mu_1\cdots\mu_m}{}_{\rho\cdots\nu_n} - \cdots - \hat{\Gamma}^{\rho}_{\lambda\nu_1} T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\rho}.\end{aligned} \tag{5.15}$$

One can also observe that due to $\overset{(-2)}{\Gamma}{}^\lambda_{\mu\nu} = 0$, LO and NLO terms of the Ricci tensor will vanish. To see that, we can use the definition of Ricci tensor in terms of the connection coefficients which can be written as

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}. \tag{5.16}$$

Then, the expansion will follow as

$$\overset{(-4)}{R}{}_{\mu\nu} = \overset{(-2)}{\Gamma}{}^\rho_{\rho\lambda}\overset{(-2)}{\Gamma}{}^\lambda_{\nu\mu} - \overset{(-2)}{\Gamma}{}^\rho_{\nu\lambda}\overset{(-2)}{\Gamma}{}^\lambda_{\rho\mu} = 0, \tag{5.17}$$

$$\overset{(-2)}{R}{}_{\mu\nu} = \partial_\rho \overset{(-2)}{\Gamma}{}^\rho_{\nu\mu} - \partial_\nu \overset{(-2)}{\Gamma}{}^\rho_{\rho\mu} + \overset{(-2)}{\Gamma}{}^\rho_{\rho\lambda}\overset{(0)}{\Gamma}{}^\lambda_{\nu\mu} - \overset{(-2)}{\Gamma}{}^\rho_{\nu\lambda}\overset{(0)}{\Gamma}{}^\lambda_{\rho\mu} + \overset{(0)}{\Gamma}{}^\rho_{\rho\lambda}\overset{(-2)}{\Gamma}{}^\lambda_{\nu\mu} - \overset{(0)}{\Gamma}{}^\rho_{\nu\lambda}\overset{(-2)}{\Gamma}{}^\lambda_{\rho\mu} = 0. \tag{5.18}$$

Hence, the highest order non-zero term of the Ricci tensor will be

$$\overset{(0)}{R}{}_{\mu\nu} = \partial_\rho \overset{(0)}{\Gamma}{}^\rho_{\nu\mu} - \partial_\nu \overset{(0)}{\Gamma}{}^\rho_{\rho\mu} + \overset{(0)}{\Gamma}{}^\rho_{\rho\lambda}\overset{(0)}{\Gamma}{}^\lambda_{\nu\mu} - \overset{(0)}{\Gamma}{}^\rho_{\nu\lambda}\overset{(0)}{\Gamma}{}^\lambda_{\rho\mu}, \tag{5.19}$$

which can be written as

$$\hat{R}_{\mu\nu} = \partial_\rho \hat{\Gamma}^\rho_{\nu\mu} - \partial_\nu \hat{\Gamma}^\rho_{\rho\mu} + \hat{\Gamma}^\rho_{\rho\lambda}\hat{\Gamma}^\lambda_{\nu\mu} - \hat{\Gamma}^\rho_{\nu\lambda}\hat{\Gamma}^\lambda_{\rho\mu}. \tag{5.20}$$

$\hat{R}_{\mu\nu}$ will be our Newton-Cartan Ricci tensor.

5.2. Equations of Motion

We can use the trace reversed form of the Einstein field equations as follows

$$R_{\mu\nu} = \frac{8\pi G_N}{c^{-4}} \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \tag{5.21}$$

and define the trace-reversed stress-energy tensor as

$$\mathcal{T}_{\mu\nu} = c^{-4} \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \tag{5.22}$$

Finally, we can expand our Ricci tensor and trace-reversed stress-energy tensor in order to obtain our equations of motion. It follows as

$$\overset{(n)}{R}_{\mu\nu} = 8\pi G_N \overset{(n)}{\mathcal{T}}_{\mu\nu}, \tag{5.23}$$

where $n = -4, -2, 0, 2, \ldots$. As it is discussed in [9], LO term of the stress-energy tensor $\overset{(-4)}{\mathcal{T}}$ assumed to be zero. And when NLO term $\overset{(-2)}{\mathcal{T}}$ is zero, we will have a torsionless theory. As we will see in Section 5.3, the large c expansion of a perfect fluid can give us a good description of a torsion free theory. Eventually, we will recover the Newton-Cartan theory in full detail when we restrict ourselves only to the highest non-zero order.

5.3. Perfect Fluid Expansion

In [12], it is proposed that the Poisson equation for Newtonian gravity can be achieved by the large c expansion of perfect fluid. The stress-energy tensor of a perfect fluid can be written as

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) U^\mu U^\nu + p g^{\mu\nu}, \tag{5.24}$$

where ρ is the rest frame mass density, p is pressure and U^μ is the 4-velocity field of the fluid. Then, the contracted stress-energy tensor becomes

$$T = \left[\left(\rho + \frac{p}{c^2} \right) U^\lambda U^\kappa + p g^{\lambda\kappa} \right] g_{\lambda\kappa}, \tag{5.25}$$

$$= \left(\rho + \frac{p}{c^2} \right) (-c^2) + 4p, \tag{5.26}$$

$$= 3p - c^2 \rho. \tag{5.27}$$

Finally, the trace reversed stress-energy tensor for the perfect fluid can be expressed as

$$T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu + p g_{\mu\nu} - \frac{1}{2}(3p - c^2\rho) g_{\mu\nu}, \tag{5.28}$$

$$= \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu - \frac{1}{2}(p - c^2\rho) g_{\mu\nu}. \tag{5.29}$$

We can also expand the 4-velocity field as follows

$$U^\mu = \overset{(0)}{u}{}^\mu + \frac{1}{c^2}\overset{(2)}{u}{}^\mu + \mathcal{O}(c^{-4}). \tag{5.30}$$

The normalization condition for our velocity field will yield us some useful information regarding the ordered terms. It follows as

$$U^\mu U^\nu g_{\mu\nu} = -c^2 \tag{5.31}$$

$$= \left(\overset{(0)}{u}{}^\mu + \frac{1}{c^2}\overset{(2)}{u}{}^\mu\right)\left(\overset{(0)}{u}{}^\nu + \frac{1}{c^2}\overset{(2)}{u}{}^\nu\right)\left(-c^2\tau_\mu\tau_\nu + \hat{h}_{\mu\nu} + c^{-2}\hat{b}_{\mu\nu}\right), \tag{5.32}$$

where the fields are truncated after $\mathcal{O}(c^{-4})$. LO and NLO equations will yield us

$$\overset{(0)}{u}{}^\mu \tau_\mu = 1, \tag{5.33}$$

$$\frac{1}{2}\hat{h}_{\mu\nu}\overset{(0)}{u}{}^\mu\overset{(0)}{u}{}^\nu = \tau_\mu \overset{(2)}{u}{}^\mu, \tag{5.34}$$

and the highest nonzero term of $\mathcal{T}$ will be

$$\overset{(0)}{\mathcal{T}}_{\mu\nu} = \frac{1}{2}\rho\tau_\mu\tau_\nu. \tag{5.35}$$

As we have discussed in Section 5.2 this distribution satisfies the torsionlessness conditions, which are

$$\overset{(-4)}{\mathcal{T}}_{\mu\nu} = \overset{(-2)}{\mathcal{T}}_{\mu\nu} = 0. \tag{5.36}$$

When we restrict ourselves only to the highest non-zero equations as follows

$$\hat{R}_{\mu\nu} = 4\pi G_N \rho\tau_\mu\tau_\nu, \tag{5.37}$$

what we have is the Poisson equation for Newton-Cartan gravity.

6. NEWTON-CARTAN THEORY

Now, it is time to put the pieces in place and construct the Newtonian gravity in a geometrical perspective. This was first done by Élie Joseph Cartan in 1923. [1, 2]

6.1. Fundamental Fields

The fundamental fields of NC theory are the symmetric 2-tensor $h^{\mu\nu}$, the one-form τ_μ and the gauge field C_μ. τ_μ is the only zero-eigenvector of the $h^{\mu\nu}$ tensor, which can be expressed as

$$h^{\mu\nu}\tau_\mu = 0. \tag{6.1}$$

τ_μ also defines a direction of time. As we have mentioned previously, in our discussion we will consider τ to be closed, which will yield a torsion-free theory as opposed to torsional Newton-Cartan theories [9, 10]. It can be expressed as

$$d\tau = 0, \tag{6.2}$$

alternatively, in index notation it becomes

$$\partial_{[\mu}\tau_{\nu]} = 0. \tag{6.3}$$

Defining a field strength of the gauge field will also be useful, as we have seen in Equation (5.13). It can be expressed as

$$K = dC, \tag{6.4}$$

again, in index notation it becomes

$$K_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu. \tag{6.5}$$

6.2. Derived Fields

Instead of using the connection in Equation (5.13) directly to define a covariant derivative in our manifold, we can try another perspective and construct it with a few conditions. The covariant derivative will be compatible in the NC structure, which can be expressed as

$$\hat{\nabla}_\mu \tau_\nu = 0, \qquad \hat{\nabla}_\mu h^{\nu\rho} = 0. \tag{6.6}$$

Using Equations (6.2) and (6.6), we can see that the connection coefficients will be symmetric. It follows as

$$\partial_\mu \tau_\nu - \hat{\Gamma}^\rho_{\mu\nu} \tau_\rho = \partial_\nu \tau_\mu - \hat{\Gamma}^\rho_{\nu\mu} \tau_\rho, \tag{6.7}$$

$$\hat{\Gamma}^\rho_{\mu\nu} = \hat{\Gamma}^\rho_{\nu\mu}. \tag{6.8}$$

Note that the connection coefficients are not uniquely determined by the compatibility conditions. We can introduce an arbitrary two-form $K_{\mu\nu}$, and under a shift, the new connection coefficients will also satisfy Equation (6.6). [13] It can be expressed as

$$\hat{\Gamma}^\lambda_{\mu\nu} \to \hat{\Gamma}^\lambda_{\mu\nu} - h^{\lambda\rho} K_{\rho(\mu}\tau_{\nu)}. \tag{6.9}$$

Now, in order to write our connection coefficients in terms of our fields, we can introduce two new fields τ^μ and $h_{\mu\nu}$ which will be the solutions to

$$\tau^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu} = \delta^\mu_\nu, \tag{6.10}$$

$$\tau^\nu \tau^\mu h_{\mu\nu} = 0, \tag{6.11}$$

and after using our identity, Equation (6.1), we get

$$\tau^\mu \tau_\mu = 1, \tag{6.12}$$

$$\tau^\mu h_{\mu\nu} = 0. \tag{6.13}$$

We can see that these new derived fields are not uniquely determined. With the introduction of an arbitrary one-form χ_ν, we can define

$$\tilde{\tau}^{\mu} = \tau^{\mu} - h^{\mu\nu}\chi_{\nu}, \tag{6.14}$$

$$\tilde{h}_{\mu\nu} = h^{\mu\nu} + 2\tau_{(\mu}\chi_{\nu)}, \tag{6.15}$$

which will also satisfy Equation (6.10) given that

$$\tau^{\mu}\chi_{\mu} = 0. \tag{6.16}$$

χ_{μ} is considered infinitesimally small. Now, we can define our connection coefficients by utilizing these new derived fields, it follows as

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \tau^{\lambda}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\lambda\rho}(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\mu\rho} - \partial_{\rho}h_{\mu\nu}), \tag{6.17}$$

which satisfies Equation (6.6). Given the non-uniqueness of the derived fields, we can explore how it affects our connection. After a χ_{μ} boost, we get

$$\begin{aligned}\tilde{\hat{\Gamma}}^{\lambda}_{\mu\nu} &= \tilde{\tau}^{\lambda}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\lambda\rho}(\partial_{\mu}\tilde{h}_{\nu\rho} + \partial_{\nu}\tilde{h}_{\mu\rho} - \partial_{\rho}\tilde{h}_{\mu\nu}), \\ &= \hat{\Gamma}^{\lambda}_{\mu\nu} + \frac{1}{2}h^{\lambda\rho}\left[(-\partial_{\rho}\chi_{\mu} + \partial_{\mu}\chi_{\rho})\tau_{\nu} + (-\partial_{\rho}\chi_{\nu} + \partial_{\nu}\chi_{\rho})\tau_{\mu}\right].\end{aligned} \tag{6.18}$$

One can observe that if we define the shift in our gauge field C_{μ} as follows

$$\tilde{C}_{\mu} = C_{\mu} - \chi_{\mu}, \tag{6.19}$$

then the Equation (6.18) will read as

$$\tilde{\hat{\Gamma}}^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} + h^{\lambda\rho}\delta K_{\rho(\mu}\tau_{\nu)}, \tag{6.20}$$

where $K_{\mu\nu}$ is the field strength of C_{μ}.

As we can see, we get the same transformation rule as we predicted in Equation (6.9). Hence, we can define a new connection as follows

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \tau^{\lambda}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\lambda\rho}(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\mu\rho} - \partial_{\rho}h_{\mu\nu}) - h^{\lambda\rho}K_{\rho(\mu}\tau_{\nu)}, \tag{6.21}$$

which will be invariant under the arbitrary χ_{μ} transformations and will be consistent with Equation (5.13) which we have derived from the large c expansion.

Lastly, we can define the Newtonian potential as follows

$$\phi = -\tau^{\mu}C_{\mu}. \tag{6.22}$$

6.3. Boost Invariant Objects

By using our gaugefield, we can define some objects which are invariant under Milne boost as follows

$$\hat{\tau}^{\mu} = \tau^{\mu} - h^{\mu\nu}C_{\nu}, \tag{6.23}$$

$$\hat{h}_{\mu\nu} = h_{\mu\nu} + 2C_{(\mu}\tau_{\nu)}. \tag{6.24}$$

We can also express our NC connection in terms of these boost invariant fields as follows

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \hat{\tau}^{\lambda}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\lambda\rho}(\partial_{\mu}\hat{h}_{\nu\rho} + \partial_{\nu}\hat{h}_{\mu\rho} - \partial_{\rho}\hat{h}_{\mu\nu}). \tag{6.25}$$

The boost invariant Newtonian potential can be defined as

$$\Phi = \phi + \frac{1}{2}h^{\mu\nu}C_{\mu}C_{\nu}. \tag{6.26}$$

Finally, we can combine our newly defined boost invariant objects to get

$$\hat{\tau}^{\mu}\hat{h}_{\mu\nu} = -2\Phi\tau_{\nu}, \tag{6.27}$$

which is a boost invariant relation.

6.4. Particle Motion

The action principle for a point particle can be expressed as

$$S = \frac{m}{2}\int \frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{\tau_{\mu}\dot{x}^{\mu}}\mathrm{d}s, \qquad \dot{x}^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}, \tag{6.28}$$

which is a parametrization invariant action. A simple calculation about this can be found in Appendix A.2. By varying this action, we get

$$\ddot{x}^{\gamma} + \hat{\Gamma}^{\gamma}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \dot{x}^{\gamma}\frac{\dot{N}}{N}, \tag{6.29}$$

where

$$N = \dot{x}^{\mu}\tau_{\mu}. \tag{6.30}$$

The detailed computation of the variation can be found in Appendix A.3.

6.5. Gravity

We can use our connection to define the NC Ricci tensor as follows

$$\hat{R}_{\mu\nu} = \partial_\rho \hat{\Gamma}^\rho_{\nu\mu} - \partial_\nu \hat{\Gamma}^\rho_{\rho\mu} + \hat{\Gamma}^\rho_{\rho\lambda}\hat{\Gamma}^\lambda_{\nu\mu} - \hat{\Gamma}^\rho_{\nu\lambda}\hat{\Gamma}^\lambda_{\rho\mu}. \tag{6.31}$$

As we have discussed in Section 5.3, the Poisson equation in NC theory can be written as

$$\hat{R}_{\mu\nu} = 4\pi G_N \rho \tau_\mu \tau_\nu. \tag{6.32}$$

Finally with the Poisson equation, we have a fully covariant description of the Newtonian gravity.

6.6. Newtonian Conditions

Since we have a gauge freedom in our theory, we can always go back to our Newtonian theory simply by choosing a gauge and a coordinate system. This is a good exercise to test if our theory is consistent with the Newtonian theory. Our gauge and coordinate choices are

$$\tau_\mu = \delta^0_\mu, \tag{6.33}$$

$$\hat{\tau}^\mu = \delta^\mu_0, \tag{6.34}$$

$$h^{ij} = \delta^{ij}, \tag{6.35}$$

which will yield us

$$h^{\mu 0} = 0, \tag{6.36}$$

$$\hat{h}_{00} = -2\Phi, \tag{6.37}$$

$$\hat{h}_{i0} = 0. \tag{6.38}$$

Hence, our h fields in their spatial components will be inverses of each other, which can be expressed as

$$h^{ij}\hat{h}_{jk} = \delta^i_k. \tag{6.39}$$

With the information on hand, we can easily see that the non-zero terms of our connection coefficients (6.25) can come only from $\mu = \nu = 0$, since the only non-zero derivative of $\hat{h}_{\mu\nu}$ is from $\hat{h}_{00}$. It follows as

$$\begin{aligned}\hat{\Gamma}^\rho_{00} &= \hat{\tau}^\rho \partial_0 \tau_0 + \frac{1}{2} h^{\rho\sigma}(\partial_0 \hat{h}_{0\sigma} + \partial_0 \hat{h}_{0\sigma} - \partial_\sigma \hat{h}_{00}),\\ &= h^{\rho\sigma}\partial_0 \hat{h}_{0\sigma} - \frac{1}{2} h^{\rho\sigma}\partial_\sigma \hat{h}_{00},\\ &= h^{\rho 0}\partial_0 \hat{h}_{00} - \frac{1}{2} h^{\rho\sigma}\partial_\sigma \hat{h}_{00},\\ &= -2h^{\rho 0}\partial_0 \Phi + h^{\rho\sigma}\partial_\sigma \Phi.\end{aligned} \tag{6.40}$$

The only non-zero terms for Newtonian conditions are

$$\hat{\Gamma}^\rho_{00} = \begin{cases} 0 & \rho = 0, \\ \delta^{ij}\partial_j \Phi & \rho = i. \end{cases} \tag{6.41}$$

Finally, under these conditions the particle equations of motion, Equation (6.29), will simplify to

$$\ddot{x}^i + \hat{\Gamma}^i_{00} = 0, \tag{6.42}$$

$$\ddot{x}^i = -\delta^{ij}\partial_j \Phi, \tag{6.43}$$

which is Newton's second law for a particle under gravitational force. The Poisson equation (6.32) will simplify to

$$\hat{R}_{00} = 4\pi G_N \rho, \tag{6.44}$$

which yields the standard Newtonian gravity,

$$\Delta\Phi = 4\pi G_N \rho. \tag{6.45}$$

So far, we have not put any restrictions on our C field. If we adopt *strong Newtonian conditions*, $C_i = 0$, the Poisson equation equation becomes

$$\Delta\phi = 4\pi G_N \rho. \tag{6.46}$$

7. SYMMETRIES

We can study the symmetries of the Newton-Cartan theory in three different parts. First, since our theory is a covariant theory, we have a d-dimensional diffeomorphism invariance in our hands. Second, we have the Milne boost invariance. Lastly, we have U(1) symmetry. We can combine these two different approaches to have a unique set of transformations which will leave our theory invariant.

7.1. Diffeomorphisms

Under an arbitrary coordinate transformation, all of our fields will transform as tensors and our equations will stay invariant. We have shown the general tensor transformation in Equation (3.4). We can also explore the infinitesimal diffeomorphisms in our theory. Let us take an arbitrary coordinate transformation in the form of

$$\tilde{x}^\mu(x) = x^\mu - \xi^\mu(x) + \mathcal{O}(\xi^2). \tag{7.1}$$

Under this, our fundamental fields will transform as follows

$$\delta_\xi \tau_\mu = \mathcal{L}_\xi \tau_\mu, \quad \delta_\xi h^{\mu\nu} = \mathcal{L}_\xi h^{\mu\nu}, \quad \delta_\xi C_\mu = \mathcal{L}_\xi C_\mu, \tag{7.2}$$

where $\mathcal{L}_\xi$ is Lie derivative along ξ^μ. Of course, this is equivalent to being carried along ξ^μ field. If ξ^μ leaves our fundamental fields invariant, it is a Newton-Cartan Killing vector.

7.2. Milne Boosts

As we have discussed in Section 6.2, our theory is invariant under a boost generated by χ_μ. These infinitesimal boosts have a great importance in our theory and they can be expressed as

$$\tilde{\tau}^{\mu} = \tau^{\mu} - h^{\mu\nu}\chi_{\nu}, \tag{7.3}$$

$$\tilde{h}_{\mu\nu} = h^{\mu\nu} + 2\tau_{(\mu}\chi_{\nu)}, \tag{7.4}$$

$$\tilde{C}_{\mu} = C_{\mu} - \chi_{\mu}, \tag{7.5}$$

which are also called *Milne boost.* The finite form of the boosts can be expressed as

$$\tilde{\tau}^{\mu} = \tau^{\mu} - h^{\mu\nu}\chi_{\nu}, \tag{7.6}$$

$$\tilde{h}_{\mu\nu} = h^{\mu\nu} + 2\tau_{(\mu}\chi_{\nu)} + h^{\lambda\sigma}\chi_{\lambda}\chi_{\sigma}\tau_{\mu}\tau_{\nu}, \tag{7.7}$$

$$\tilde{C}_{\mu} = C_{\mu} - \chi_{\mu} - \frac{1}{2}h^{\lambda\sigma}\chi_{\lambda}\chi_{\sigma}\tau_{\mu}. \tag{7.8}$$

A detailed calculation about them can be found in A.1.

7.3. U(1)

One can easily show that, $K_{\mu\nu}$ has a local U(1) symmetry parametrized by $\lambda(x^{\mu})$ as follows

$$C'_{\mu} = C_{\mu} + \partial_{\mu}\lambda, \tag{7.9}$$

$$\begin{aligned} K_{\mu\nu} &= \partial_{\mu}C_{\nu}{}' - \partial_{\nu}C_{\mu}{}', \\ &= \partial_{\mu}C_{\nu} - \partial_{\mu}C_{\nu}. \end{aligned} \tag{7.10}$$

Since we have only $K_{\mu\nu}$ appearing in our equations of motion, we can say that our theory has U(1) symmetry. Hence, we can call C_{μ} our gauge field.

8. 3+1 FORMULATION

The absence of torsion enables us to choose a frame where time is absolute. [14] We can use this frame to fix our gauge. If we combine coordinate transformations and Milne boosts, our diffeomorphism invariance breaks down into *time dependent spatial diffeomorphism* invariance. Following the work done in [6] we will build the TDSD structure. Let us fix our theory to the the gauge and coordinate choice of

$$\tau^{\mu} = \delta^{\mu}_{0}, \tag{8.1}$$

$$\tau_{\mu} = \delta^{0}_{\mu}. \tag{8.2}$$

The identities in Equation (6.10) will yield

$$h^{0\mu} = 0, \tag{8.3}$$

$$h_{0\mu} = 0, \tag{8.4}$$

$$h^{ij} h_{jk} = \delta^{i}_{k}. \tag{8.5}$$

Here, we can easily see that h_{ij} and h^{ij} are reduced to 3 dimensional metrics which we can use to define a connection. Also, since we have an absolute direction of time due to τ and our transformations leave time invariant up to a constant, we can try to separate the temporal component of all the fields from our notation. This will prove useful when we combine them with our *new* spatial metric. As for the temporal parts of the fields, we will just rename them.

This reasoning allows us to separate the time derivative too. We will use the conventional dot notation for the time derivatives. It can be expressed as

$$\partial_0 A_{ij} \equiv \dot{A}_{ij}. \tag{8.6}$$

One should be careful when it comes to the raising and lowering indices of the dotted terms because of the ambiguity of

$$\dot{A}^{ij} \stackrel{?}{=} h^{ik}h^{jl}\dot{A}_{kl}, \tag{8.7}$$

$$\dot{A}^{ij} \stackrel{?}{=} \partial_0(h^{ik}h^{jl}A_{kl}). \tag{8.8}$$

We will follow the first of these notations,

$$\dot{A}^{ij} = h^{ik}h^{jl}\dot{A}_{kl}. \tag{8.9}$$

Although, there is an exception to this notation. One can easily observe

$$\partial_0\left(h^{ij}h_{jk}\right) = \dot{h}^{ij}h_{jk} + h^{ij}\dot{h}_{jk} = 0, \qquad \dot{h}^{ij} = -h^{ik}h^{jl}\dot{h}_{kl}, \tag{8.10}$$

which is a physical equation rather than a notational convention.

Let us take a look at the field equations (6.32) in this gauge and coordinate choice. We can separate them into three parts as follows

$$\hat{R}_{00} = 4\pi G_N\rho, \tag{8.11}$$

$$\hat{R}_{i0} = 0, \tag{8.12}$$

$$\hat{R}_{ij} = 0. \tag{8.13}$$

The explicit calculations for the connection coefficients and components for the Ricci tensor can be found in Appendix A.4. The field equations can be expressed as follows

$$-\nabla_i G^i = \frac{1}{2}h^{ik}\ddot{h}_{ik} + \frac{1}{4}\dot{h}^{ik}\dot{h}_{ik} - \frac{1}{4}K^{ik}K_{ik} + 4\pi G_N\rho, \tag{8.14a}$$

$$-\nabla^j K_{ji} = 2h^{kl}\nabla_{[i}\dot{h}_{k]l}, \tag{8.14b}$$

$$R_{ij} = 0, \tag{8.14c}$$

where we defined

$$G_i \equiv K_{i0}, \quad G^i = h^{ij}K_{i0}. \tag{8.15}$$

Next, we can explore the particle equations of motion we derived in Equation (6.29).

The right hand side of the equation had an interesting term,

$$N = \tau_\mu \dot{x}^\mu = 1 \Longrightarrow \dot{x}^0 = 1, \qquad \dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}, \tag{8.16}$$

which means that we can use the reparametrization invariance and choose our parameter as the time coordinate. It follows as

$$s = x^0 = t, \qquad \dot{x}^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}, \tag{8.17}$$

which will yield us

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = -\Gamma^i_{00} - 2\Gamma^i_{0j}\dot{x}^j, \tag{8.18}$$

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = h^{ij}K_{j0} - h^{ik}(\dot{h}_{kj} - K_{kj})\dot{x}^j. \tag{8.19}$$

We can use the definiton of G_i here. It follows as

$$K_{j0} = -\partial_j\phi - \dot{C}_j = G_j. \tag{8.20}$$

Hence, the particle equations of motion becomes

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = -\partial^i\phi - h^{ij}\dot{C}_j - h^{ik}\dot{h}_{kj}\dot{x}^j + h^{ik}K_{kj}\dot{x}^j, \tag{8.21}$$

$$= G^i - h^{ik}\dot{h}_{kj}\dot{x}^j + h^{ik}K_{kj}\dot{x}^j. \tag{8.22}$$

In the following sections, we will take a look at how the fields change under certain transformations. First, we will investigate the infinitesimal coordinate transformation case, then we will work out the finite coordinate transformation case.

8.1. Infinitesimal Coordinate Transformations

Let us do an infinitesimal coordinate transformation in the form of Equation (7.1). The course we will follow is simple. In order for τ fields to be invariant under combined transformations, we will try to find the Milne boost which will cancel out the coordinate transformation.

We can gather some information about the temporal component of the coordinate transformations when we examine τ^{μ},

$$\tilde{\tau}_{\mu} = \tau_{\mu} + \delta_{\xi}\tau_{\mu} = \delta^{0}_{\mu}, \tag{8.23}$$

$$\delta_{\xi}\tau_{\mu} = \mathcal{L}_{\xi}\tau_{\mu} = 0, \tag{8.24}$$

$$\xi^{\rho}\partial_{\rho}\tau_{\mu} + \tau_{\rho}\partial_{\mu}\xi^{\rho} = \partial_{\mu}\xi^{0} = 0. \tag{8.25}$$

Which means that the coordinate transformations can only translate the time coordinate by a constant, hence, we have *spatial diffeomorphism* structure. So far we don't have any restrictions on the spatial components of the transformations.

Next, we can examine $h_{\mu\nu}$ even though we do not require it to be invariant. The crucial point here is that when we fix τ, Equations (8.3) are satisfied. Hence, all temporal components of h fields will vanish, no matter the transformations we have. So, we can take a look at the transformation of h_{0i} which follows as

$$\tilde{h}_{0i} = h_{0i} + \delta_{\xi}h_{0i} + \delta_{\chi}h_{0i} = h_{0i} = 0. \tag{8.26}$$

Infinitesimal coordinate transformations will yield us

$$\mathcal{L}_{\xi}h_{\mu\nu} = \delta_{\xi}h_{\mu\nu} = \xi^{\lambda}\partial_{\lambda}h_{\mu\nu} + h_{\lambda\nu}\partial_{\mu}\xi^{\lambda} + h_{\mu\lambda}\partial_{\nu}\xi^{\lambda}, \tag{8.27}$$

$$\delta_{\xi}h_{0i} = \xi^{\lambda}\partial_{\lambda}h_{0i} + h_{\lambda i}\partial_{0}\xi^{\lambda} + h_{0\lambda}\partial_{i}\xi^{\lambda}, \tag{8.28}$$

$$= h_{ji}\dot{\xi}^{j}, \tag{8.29}$$

and the Milne boost will yield us

$$\delta_{\chi}h_{\mu\nu} = \tau_{\mu}\chi_{\nu} + \tau_{\nu}\chi_{\mu}, \tag{8.30}$$

$$\delta_{\chi}h_{0\nu} = \chi_{\nu} + \delta^{0}_{\nu}\chi_{0}, \tag{8.31}$$

$$\delta_{\chi}h_{0i} = \chi_{i}. \tag{8.32}$$

Finally, the combined transformations become

$$\tilde{h}_{0i} = h_{0i} + \delta_{\chi}h_{0i} + \delta_{\xi}h_{0i} = \chi_{i} + h_{ij}\dot{\xi}^{j} = 0, \tag{8.33}$$

which gives us

$$\chi_i = -h_{ij}\dot{\xi}^j. \tag{8.34}$$

This is a great result. For every infinitesimal coordinate transformation, we can find the proper Milne boosts that will leave our theory invariant. The last piece of information we need is the temporal part of the Milne boost, which we can obtain from

$$\tau^\mu \chi_\mu = 0, \qquad \chi_0 = 0. \tag{8.35}$$

If we want to draw a general picture including all the fields, it follows as

$$\tau^\mu = \delta^\mu_0, \quad \tau_\mu = \delta^0_\mu, \quad h^{\mu 0} = 0, \quad h_{\mu 0} = 0, \tag{8.36}$$

and the transformations, not including the U(1) symmetry, can be written as

$$\delta h_{ij} = \mathcal{L}_\xi h_{ij}, \tag{8.37}$$

$$\delta h^{ij} = \mathcal{L}_\xi h^{ij}, \tag{8.38}$$

$$\delta C_i = \mathcal{L}_\xi C_i + h_{ij}\dot{\xi}^j, \tag{8.39}$$

$$\delta C_0 = \mathcal{L}_\xi C_0 - C_i\dot{\xi}^i. \tag{8.40}$$

Since we have

$$\phi = -\tau^\mu C_\mu = -C_0, \tag{8.41}$$

Equation (8.40) can be written as

$$\delta\phi = \mathcal{L}_\xi \phi + C_i\dot{\xi}^i. \tag{8.42}$$

8.2. Finite Coordinate Transformations

Infinitesimal coordinate transformations are useful, but they are restrictive in the sense that, well, they are infinitesimal. For a broader picture, we can try to apply our gauge and coordinate choice to the finite case. Tensor components transform under an arbitrary transformation as follows

$$\tilde{\tau}^\mu(\tilde{x}) = \Lambda^\mu{}_\nu \tau^\nu(x) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\tau^\nu = \partial_\nu \tilde{x}^\mu \tau^\nu. \tag{8.43}$$

For brevity we will be using the following notation for the inverse transformations,

$$\Lambda^\mu{}_\nu \check{\Lambda}^\nu{}_\lambda = \delta^\mu_\lambda, \qquad \partial_\nu \tilde{x}^\mu \tilde{\partial}_\lambda x^\nu = \delta^\mu_\lambda. \tag{8.44}$$

Again, the procedure will be similar; first, we will do an infinitesimal boost, then a finite coordinate transformation in such a way that $\tilde{\tau}^\mu = \tau^\mu = \delta^\mu_0$ stays invariant. The boost can be expressed as

$$\tau^\mu_1 = \tau^\mu - h^{\mu\nu}\chi_\nu, \tag{8.45}$$

and the coordinate transformation will yield us

$$\tilde{\tau}^\mu = \Lambda^\mu{}_\nu \tau^\nu_1 = \Lambda^\mu{}_\nu(\tau^\nu - h^{\nu\lambda}\chi_\lambda) = \delta^\mu_0, \tag{8.46}$$

$$\delta^\mu_0 = \Lambda^\mu{}_\nu \tau^\nu_0 - \Lambda^\mu{}_\nu h^{\nu\lambda}\chi_\lambda, \tag{8.47}$$

$$\check{\Lambda}^\alpha{}_\mu \delta^\mu_0 = \delta^\alpha_\nu \delta^\nu_0 - \delta^\alpha_\nu h^{\nu\lambda}\chi_\lambda, \tag{8.48}$$

$$\check{\Lambda}^\alpha{}_0 = \delta^\alpha_0 - h^{\alpha\lambda}\chi_\lambda. \tag{8.49}$$

What if we do this in reverse order? Let's look at the coordinate transformation first, it follows as

$$\tau^\mu_1 = \Lambda^\mu{}_\nu \tau^\nu, \qquad h^{\mu\nu}_1 = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho h^{\sigma\rho}, \qquad \chi^1_\mu = \check{\Lambda}^\nu{}_\mu \chi_\nu, \tag{8.50}$$

and now when we apply the boost, we get

$$\tau^\mu_2 = \tau^\mu_1 - h^{\mu\nu}_1 \chi^1_\nu = \Lambda^\mu{}_\nu \tau^\nu_0 - \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho h^{\sigma\rho} \check{\Lambda}^\lambda{}_\nu \chi_\lambda, \tag{8.51}$$

$$\delta^\mu_0 = \Lambda^\mu{}_\nu \tau^\nu_0 - \Lambda^\mu{}_\sigma h^{\sigma\lambda}\chi_\lambda, \tag{8.52}$$

$$\check{\Lambda}^\alpha{}_\mu \delta^\mu_0 = \delta^\alpha_\nu \delta^\nu_0 - \delta^\alpha_\sigma h^{\sigma\lambda}\chi_\lambda, \tag{8.53}$$

$$\check{\Lambda}^\alpha{}_0 = \delta^\alpha_0 - h^{\alpha\lambda}\chi_\lambda. \tag{8.54}$$

We get the same equation as expected, so the order in which we apply the transformations does not matter.

Now we can investigate how fundamental fields transform to get more information. We want τ_μ to stay invariant under coordinate transformation since it is already boost invariant. It follows as

$$\tilde{\tau}_\mu = \partial_\mu \tilde{x}^\nu \tau_\nu = \partial_\mu \tilde{t} = \delta^0_\mu, \tag{8.55}$$

which means

$$\frac{\partial \tilde{t}}{\partial t} = 1, \qquad \frac{\partial \tilde{t}}{\partial x^i} = 0. \tag{8.56}$$

Hence, time stays the same up to a constant under these transformations. This is the same result as the infinitesimal case, which is of course expected. We should be able to go to the infinitesimal case from the finite case. Now we can take a look at τ^μ again. We start with

$$\delta^\mu_0 = \Lambda^\mu{}_0 - \Lambda^\mu{}_j h^{jk} \chi_k, \tag{8.57}$$

for $\mu = 0$ we get

$$\Lambda^0{}_0 = 1 + \Lambda^0{}_j h^{jk} \chi_k = 1 + \frac{\partial \tilde{t}}{\partial x^j} h^{jk} \chi_k = 1. \tag{8.58}$$

This doesn't give us any new information, we already knew that new time coordinate does not depend on the spatial coordinates. For $\mu \neq 0$ we get

$$\Lambda^i{}_0 = \Lambda^i{}_j h^{jk} \chi_k, \tag{8.59}$$

using the explicit form of Λ we get

$$\partial_0 \tilde{x}^i = \partial_j \tilde{x}^i h^{jk} \chi_k. \tag{8.60}$$

Here, in order to solve for χ_k we can use the inverse of the spatial transformations. To obtain this, we can explicitly solve

$$\Lambda\check{\Lambda} = \begin{pmatrix} 1 & 0 \\ \vec{v} & M \end{pmatrix} \begin{pmatrix} 1 & \vec{w}^T \\ \vec{u} & N \end{pmatrix} = \begin{pmatrix} 1 & \vec{w}^T \\ \vec{v} + M\vec{u} & \vec{v}\vec{w}^T + MN \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{I}_3 \end{pmatrix}, \tag{8.61}$$

which yields

$$\vec{w} = 0, \tag{8.62}$$

$$N = M^{-1}, \tag{8.63}$$

$$\vec{u} = -M^{-1}\vec{v}, \tag{8.64}$$

we can express these in their component forms as

$$\check{\Lambda}^0{}_i = 0, \tag{8.65}$$

$$\check{\Lambda}^i{}_j = \Lambda^{-1i}{}_j, \tag{8.66}$$

$$\check{\Lambda}^i{}_0 = -\check{\Lambda}^i{}_j\Lambda^j{}_0. \tag{8.67}$$

The middle equation tells us that the spatial coordinate transformations are the inverses of each other, as we have expected. Finally, for the combined boost relation, we have

$$\chi_i = h_{ij}\check{\Lambda}^j{}_k\partial_t\tilde{x}^k. \tag{8.68}$$

Again, this is an amazing result. If we are given an arbitrary time-dependent spatial transformation, we can find the boosts that will leave our theory invariant.

Now, we can try to see how the fields transform under this combined transformations. h fields are pretty straightforward,

$$\tilde{h}^{ij} = \Lambda^i{}_k\Lambda^j{}_l h^{kl}, \tag{8.69}$$

$$\begin{aligned} \tilde{h}_{ij} &= \check{\Lambda}^\mu{}_i\check{\Lambda}^\nu{}_j(h_{\mu\nu} + \tau_\mu\chi_\nu + \tau_\nu\chi_\mu), \\ &= \check{\Lambda}^k{}_i\check{\Lambda}^l{}_j h_{kl}. \end{aligned} \tag{8.70}$$

For C_i, we have

$$\begin{aligned}\tilde{C}_i &= \check{\Lambda}^\mu{}_i(C_\mu - \chi_\mu) = \check{\Lambda}^j{}_i(C_j - \chi_j),\\ &= \check{\Lambda}^j{}_i C_j - \check{\Lambda}^j{}_i h_{jk}\check{\Lambda}^k{}_l \partial_t \tilde{x}^l,\\ &= \check{\Lambda}^j{}_i C_j - \tilde{h}_{ij}\partial_t \tilde{x}^j. \end{aligned} \tag{8.71}$$

For C_0, we have

$$\begin{aligned}\tilde{C}_0 &= \check{\Lambda}^\mu{}_0(C_\mu - \chi_\mu - \frac{1}{2}h^{ij}\chi_i\chi_j\tau_\mu),\\ &= C_0 - \frac{1}{2}h^{ij}\chi_i\chi_j + \check{\Lambda}^i{}_0(C_i - \chi_i),\\ &= C_0 - \frac{1}{2}h^{ij}\chi_i\chi_j - \check{\Lambda}^i{}_k\Lambda^k{}_0(C_i - \chi_i),\\ &= C_0 - \frac{1}{2}\tilde{h}_{ij}\partial_t\tilde{x}^i\partial_t\tilde{x}^j - \check{\Lambda}^i{}_k\Lambda^k{}_0(C_i - h_{ij}\check{\Lambda}^j{}_l\partial_t\tilde{x}^l),\\ &= C_0 - \frac{1}{2}\tilde{h}_{ij}\partial_t\tilde{x}^i\partial_t\tilde{x}^j - \tilde{C}_k\partial_t\tilde{x}^k + \tilde{h}_{kl}\partial_t\tilde{x}^k\partial_t\tilde{x}^l,\\ &= C_0 - \tilde{C}_k\partial_t\tilde{x}^k + \frac{1}{2}\tilde{h}_{ij}\partial_t\tilde{x}^i\partial_t\tilde{x}^j. \end{aligned} \tag{8.72}$$

Again, we know $C_0 = -\phi$, therefore we can write

$$\tilde{\phi} = \phi + \tilde{C}_k\partial_t\tilde{x}^k - \frac{1}{2}\tilde{h}_{ij}\partial_t\tilde{x}^i\partial_t\tilde{x}^j. \tag{8.73}$$

8.3. Bar Notation

For the finite transformation case, we can try to simplify our calculations by expressing our fields as tensor transformations such as

$$\tilde{h}_{ij} = \tilde{\partial}_i x^k \tilde{\partial}_j x^l \bar{h}_{kl}, \tag{8.74}$$

$$\tilde{h}^{ij} = \partial_k \tilde{x}^i \partial_l \tilde{x}^j \bar{h}^{kl}, \tag{8.75}$$

$$\tilde{C}_i = \tilde{\partial}_i x^j \bar{C}_j. \tag{8.76}$$

If we can find the *bar form* of all the fields, the transformations will simply be applying coordinate transformation to a few fields. Luckily for us, they are not so hard to work out. If we define a *frame shift vector* as follows

$$v^i = -\chi^i = -\tilde{\partial}_j x^i \partial_t \tilde{x}^j, \tag{8.77}$$

$$\tilde{v}^i = -\partial_t \tilde{x}^i, \tag{8.78}$$

then the bar forms of our fields become

$$\bar{h}_{ij} = h_{ij}, \tag{8.79}$$

$$\bar{h}^{ij} = h^{ij}, \tag{8.80}$$

$$\bar{C}_i = C_i + v_i, \tag{8.81}$$

$$\bar{\phi} = \phi - C_i v^i - \frac{1}{2} v_i v^i. \tag{8.82}$$

We can also obtain the bar form of K_{ij} and G_i by using the bar forms of other fields. They can be expressed as

$$\bar{K}_{ij} = K_{ij} + \partial_i v_j - \partial_j v_i, \tag{8.83}$$

$$\bar{G}_i = G_i + K_{ij} v^j - \partial_t v_i - v^j \nabla_j v_i. \tag{8.84}$$

One can expand upon this notion to explore the theory through bar forms. For instance, one can check whether or not the field equations and the particle equations of motion stay same in the bar form. But in this thesis we will solely focus on the computational advantage of the bar forms while doing coordinate transformations in the 3+1 formulation.

9. DISCUSSION

Now that we have a tangible framework in our hands, we can apply it to a few cases where we observe how the particle motion.

9.1. Expansion of Space

We can start with an interesting application; an expanding space. We can choose Cartesian coordinates with an arbitrary gravitational potential, which can be expressed as

$$h_{ij} = \delta_{ij}, \quad C_i = 0, \quad K_{ij} = 0, \quad G_i \neq 0, \tag{9.1}$$

and expand our coordinates by an arbitrary factor as follows

$$\tilde{x}^i = \alpha(t) x^i. \tag{9.2}$$

The frame shift vector becomes

$$v^i = -\frac{\dot{\alpha}}{\alpha} x^i, \tag{9.3}$$

and the bar forms of our fields become

$$\bar{C}_i = -\frac{\dot{\alpha}}{\alpha}\delta_{ij}x^j, \tag{9.4}$$

$$\bar{K}_{ij} = 0, \tag{9.5}$$

$$\bar{G}^i = h^{ij}\bar{G}_j = G^i + \left[\frac{\ddot{\alpha}}{\alpha} - 2\left(\frac{\dot{\alpha}}{\alpha}\right)^2\right] x^i. \tag{9.6}$$

In the new frame, we can express the metric as follows

$$\tilde{h}_{ij} = \alpha^{-2}\delta_{ij}, \quad \tilde{h}^{ij} = \alpha^2\delta^{ij}. \tag{9.7}$$

Particle equation in the new frame can be expressed as

$$\ddot{\tilde{x}}^i = \tilde{G}^i - \tilde{h}^{ik}\dot{\tilde{h}}_{kj}\dot{\tilde{x}}^j = \alpha G^i + \left[\frac{\ddot{\alpha}}{\alpha} - 2\left(\frac{\dot{\alpha}}{\alpha}\right)^2\right]\tilde{x}^i + 2\frac{\dot{\alpha}}{\alpha}\dot{\tilde{x}}^i. \tag{9.8}$$

For $\alpha = e^{\lambda t}$, this reduces to

$$\ddot{\tilde{x}}^i = e^{\lambda t}G^i - \lambda^2\tilde{x}^i + 2\lambda\dot{\tilde{x}}^i. \tag{9.9}$$

9.2. Rotating Frame

In [15] there is a simple yet effective description of the vectorial description of a rotating frame and the Coriolis field. We can easily demonstrate the same effect in our framework.

We will start with strong Newtonian conditions and apply a coordinate transformation. Our x coordinates will represent stationary frame S, and r coordinates will represent the rotating frame S'.

Particle equation of motion in the stationary frame is

$$\ddot{\mathrm{x}}^i = g^i, \qquad G^i = g^i, \tag{9.10}$$

where g^i is an arbitrary gravitational field. And the coordinate transformation can be expressed as

$$x^i = R^i{}_j(t)r^j + x_0^i(t), \tag{9.11}$$

$$r^i = R^{Ti}{}_j(x^j - x_0^j). \tag{9.12}$$

where x_0^i is the origin of the rotating frame in the stationary frame. Now, we can start exploring the gravity in the rotating frame. Let us start with the computation of $\tilde{v}^i$. It follows as

$$\tilde{v}^i = -\partial_t r^i = -\partial_t R^{Ti}{}_j(x^j - x_0^j) + R^{Ti}{}_j\partial_t x_0^j. \tag{9.13}$$

In the stationary frame, v^i will be

$$v^i = \tilde{\partial}_j x^i \tilde{v}^j = R^i{}_j(-\partial_t R^{Tj}{}_k(x^k - x_0^k) + R^{Tj}{}_k \partial_t x_0^k). \tag{9.14}$$

Here, we can define a *rotation vector* as follows

$$\partial_t R^i{}_j R^{Tj}{}_k = \epsilon^i{}_{jk} \Omega^j, \tag{9.15}$$

and our frame shift vector becomes

$$v^i = \epsilon^i{}_{jk} \Omega^j (x^k - x_0^k) + \partial_t x_0^i. \tag{9.16}$$

Now we can find how G and K fields transform. It follows as

$$\bar{G}^i = G^i - \partial_t v^i - v^j \partial_j v^i, \tag{9.17}$$

$$= g^i - \epsilon^i{}_{jk} \partial_t \Omega^j (x^k - x_0^k) + \epsilon^i{}_{jk} \Omega^j \partial_t x_0^k - \partial_t^2 x_0^i$$
$$- \left[\epsilon^j{}_{mn} \Omega^m (x^n - x_0^n) + \partial_t x_0^j\right] \partial_j (\epsilon^i{}_{kl} \Omega^k x^l), \tag{9.18}$$

$$= g^i - \partial_t^2 x_0^i - \epsilon^i{}_{jk} \partial_t \Omega^j (x^k - x_0^k) - \epsilon^i{}_{kj} \Omega^k \epsilon^j{}_{mn} \Omega^m (x^n - x_0^n). \tag{9.19}$$

Similarly, for K we have

$$\bar{K}_{ij} = \partial_i v_j - \partial_j v_i = \epsilon_{jkl} \Omega^k \delta_i^l - \epsilon_{ikl} \Omega^k \delta_j^l = 2\epsilon_{ijk} \Omega^k. \tag{9.20}$$

From now on, we should express everything in terms of r coordinates by using

$$R^i{}_j r^j = x^i - x_0^i. \tag{9.21}$$

We can observe that our definition of Ω is not unique, therefore at first it seems inconsistent with the source material. In their work, they define their Ω as,

$$R^T \dot{R} = \tilde{A}, \qquad \tilde{A}_{ij} = \epsilon_{ikj} \tilde{\Omega}^k, \tag{9.22}$$

whereas we have defined ours as

$$\dot{R} R^T = A, \qquad A_{ij} = \epsilon_{ikj} \Omega^k. \tag{9.23}$$

By a simple calculation we can get

$$\vec{\Omega} = R \vec{\tilde{\Omega}}. \tag{9.24}$$

Finally, our particle equation reads as

$$\ddot{r}^i = \tilde{G}^i + \tilde{K}^i{}_j \dot{r}^j, \tag{9.25}$$

in vector form it becomes

$$\ddot{\vec{\mathrm{r}}} = R^T(\vec{\mathrm{g}} - \ddot{\vec{\mathrm{x}}}_0) - R^T(\vec{\Omega} \times (R\vec{\mathrm{r}})) - R^T(\vec{\Omega} \times (\vec{\Omega} \times (R\vec{\mathrm{r}}))) + 2R^T[(\dot{R\vec{\mathrm{r}}}) \times \vec{\Omega}]. \tag{9.26}$$

Using the rotational invariance of cross product, we will have

$$\ddot{\vec{\mathrm{r}}} = R^T(\vec{\mathrm{g}} - \ddot{\vec{\mathrm{x}}}_0) - (\dot{\tilde{\vec{\Omega}}} \times \vec{\mathrm{r}}) - \tilde{\vec{\Omega}} \times (\tilde{\vec{\Omega}} \times \vec{\mathrm{r}}) + 2\dot{\vec{\mathrm{r}}} \times \tilde{\vec{\Omega}}, \tag{9.27}$$

which is consistent with the paper.

This shows the flexibility of our framework. We can have an arbitrary coordinate transformation and explore gravity in the new frame simply by calculating the frame shift vector and the fields.

10. CONCLUSION

We have started with the weak equivalence principle to show that the Newtonian gravity can be geometrically formulated. We have developed the necessary tools and started with an expansion of GR. By restricting ourselves to the leading order fields, we have ended up with a covariant theory, Newton-Cartan gravity.

After exploring the symmetries of NC gravity, we have adopted a set of transformations that combines diffeomorphisms with Milne boosts. The combined transformations enabled us to separate time coordinate from the space coordinates and provided us a time-dependent spatial diffeomorphism invariant theory. 3+1 formulation is a powerful tool in the sense that we can explore the gravity in all frames of reference, inertial or non-inertial. Of course, we have to stay in the non-relativistic regimes for NC theory to hold.

In our work, we have emphasized that we are adopting a torsion free version of the Newton-Cartan theory. And our main assumption was $d\tau = 0$. But this does not have to be the case for a more general picture, work done in [9, 10] does include the torsional approach.

I can think of a few further research topics regarding the 3+1 formulation, or more generally a d+1 formulation. For the torsional Newton-Cartan theories, it would be interesting to apply a foliation approach, similar to ADM formalism [16] in the GR, and expand the foliated space-time structure. This might yield a fully covariant, torsional, non-relativistic and strong gravity solutions in the first few leading orders. Another topic might be trying to find solutions to the field equations (8.14). At first glance, it seems like an intimidating task for sure. But one can make some simplifying assumptions using U(1) symmetry and TDSD structure.

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APPENDIX A: EXTRA CALCULATIONS

Here we have included some extra explicit calculations that we have used throughout the thesis.

A.1. Finite Milne Boosts

Let us start with a finite boost χ_μ. It follows as

$$\tilde{\tau}^\mu = \tau^\mu - h^{\mu\nu}\chi_\nu, \tag{A.1}$$

$$\tilde{h}_{\mu\nu} = h^{\mu\nu} + 2\tau_{(\mu}\chi_{\nu)} + h^{\lambda\sigma}\chi_\lambda\chi_\sigma\tau_\mu\tau_\nu, \tag{A.2}$$

where χ_μ is an arbitrary one form satisfying $\tau^\mu\chi_\mu = 1$. We can define a scalar that can help us with the computations as follows

$$X = h^{\lambda\sigma}\chi_\lambda\chi_\sigma. \tag{A.3}$$

We will see if the identity equation holds, which can be expressed as

$$\tau_\mu\tilde{\tau}^\nu + \tilde{h}_{\mu\rho}h^{\rho\nu} \stackrel{?}{=} \delta^\nu_\mu. \tag{A.4}$$

We can plug our fields in. It follows as

$$\begin{aligned}\tau_\mu\tilde{\tau}^\nu + \tilde{h}_{\mu\rho}h^{\rho\nu} &= \tau_\mu(\tau^\nu - h^{\nu\rho}\chi_\rho) + (h_{\mu\rho} + \chi_\mu\tau_\rho + \chi_\rho\tau_\mu + X\tau_\mu\tau_\rho)h^{\rho\nu}, \\ &= \tau_\mu\tau^\nu + h_{\mu\rho}h^{\rho\nu} - h^{\nu\rho}\tau_\mu\chi_\rho + h^{\nu\rho}\tau_\mu\chi_\rho + h^{\rho\nu}\tau_\rho\chi_\mu \\ &\quad + h^{\rho\nu}\tau_\rho\tau_\mu X.\end{aligned} \tag{A.5}$$

Using $h^{\mu\nu}\tau_\mu = 0$ we can see that the last two terms vanish. Finally we get

$$\tau_\mu\tilde{\tau}^\nu + \tilde{h}_{\mu\rho}h^{\rho\nu} = \tau_\mu\tau^\nu + h_{\mu\rho}h^{\rho\nu} = \delta^\nu_\mu, \tag{A.6}$$

which shows that the identitiy equation holds. Next, we can check whether or not

$$\tilde{\tau}^\mu\tilde{\tau}^\nu\tilde{h}_{\mu\nu} \stackrel{?}{=} 0 \tag{A.7}$$

is satisfied. It follows as

$$\tilde{\tau}^\mu\tilde{\tau}^\nu\tilde{h}_{\mu\nu} = (\tau^\mu - h^{\mu\rho}\chi_\rho)(\tau^\nu - h^{\nu\sigma}\chi_\sigma)(h_{\mu\nu} + \chi_\mu\tau_\nu + \chi_\nu\tau_\mu + X\tau_\mu\tau_\nu), \tag{A.8}$$

$$\begin{aligned} &= \tau^\mu\tau^\nu h_{\mu\nu} + h_{\mu\nu}(h^{\mu\rho}h^{\nu\sigma}\chi_\rho\chi_\sigma - \tau^\mu h^{\nu\sigma}\chi_\sigma - \tau^\nu h^{\mu\rho}\chi_\rho) \\ &\quad + (\tau^\mu\tau^\nu + h^{\mu\rho}h^{\nu\sigma}\chi_\rho\chi_\sigma - \tau^\mu h^{\nu\sigma}\chi_\sigma - \tau^\nu h^{\mu\rho}\chi_\rho)(\chi_\mu\tau_\nu \\ &\quad + \chi_\nu\tau_\mu + X\tau_\mu\tau_\nu). \end{aligned} \tag{A.9}$$

Using the non-boosted versions of the identity equations we get

$$\tilde{\tau}^\mu\tilde{\tau}^\nu\tilde{h}_{\mu\nu} = h_{\mu\nu}h^{\mu\rho}h^{\nu\sigma}\chi_\rho\chi_\sigma + \tau^\mu\tau^\nu X\tau_\mu\tau_\nu - \tau^\mu X\tau_\mu - \tau^\nu X\tau_\nu. \tag{A.10}$$

We can also use $\tau_\mu\tau^\mu = 1$ to simplify the equation further as follows

$$\begin{aligned} \tilde{\tau}^\mu\tilde{\tau}^\nu\tilde{h}_{\mu\nu} &= h_{\mu\nu}h^{\mu\rho}h^{\nu\sigma}\chi_\rho\chi_\sigma - X, \\ &= (\delta^\rho_\nu - \tau_\nu\tau^\rho)h^{\nu\sigma}\chi_\rho\chi_\sigma - X, \\ &= h^{\nu\sigma}\chi_\nu\chi_\sigma - X = 0. \end{aligned} \tag{A.11}$$

Now we can explore how C_μ must transform for $\hat{\Gamma}^\lambda_{\mu\nu}$ to be invariant. It can be expressed as

$$\tilde{\hat{\Gamma}}^\lambda_{\mu\nu} = \tau^\lambda\partial_\mu\tau_\nu + \frac{1}{2}h^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) - \frac{1}{2}h^{\lambda\rho}(K_{\rho\mu}\tau_\nu + K_{\rho\nu}\tau_\mu), \tag{A.12}$$

which explicitly looks like

$$\begin{aligned} \tilde{\hat{\Gamma}}^\lambda_{\mu\nu} &= (\tau^\lambda - h^{\lambda\sigma}\chi_\sigma)\partial_\mu\tau_\nu + \frac{1}{2}h^{\lambda\rho}\Big[\partial_\mu(h_{\nu\rho} + \chi_\nu\tau_\rho + \chi_\rho\tau_\nu + X\tau_\nu\tau_\rho) \\ &\quad + \partial_\nu(h_{\mu\rho} + \chi_\mu\tau_\rho + \chi_\rho\tau_\mu + X\tau_\mu\tau_\rho) - \partial_\rho(h_{\mu\nu} + \chi_\mu\tau_\nu + \chi_\nu\tau_\mu + X\tau_\mu\tau_\nu)\Big] \\ &\quad - \frac{1}{2}h^{\lambda\rho}\Big[(K_{\rho\mu} + \delta K_{\rho\mu})\tau_\nu + (K^0_{\rho\nu} + \delta K_{\rho\nu})\tau_\mu\Big]. \end{aligned} \tag{A.13}$$

We can simplify this by using the identities. We get

$$\begin{aligned} \tilde{\hat{\Gamma}}^\lambda_{\mu\nu} &= \hat{\Gamma}^\lambda_{\mu\nu} - h^{\lambda\sigma}\chi_\sigma\partial_\mu\tau_\nu + \frac{1}{2}h^{\lambda\rho}\Big[\chi_\nu\partial_\mu\tau_\rho + \partial_\mu(\chi_\rho\tau_\nu) + X\tau_\nu\partial_\mu\tau_\rho \\ &\quad + \chi_\mu\partial_\nu\tau_\rho + \partial_\nu(\chi_\rho\tau_\mu) + X\tau_\mu\partial_\nu\tau_\rho - \partial_\rho(\chi_\mu\tau_\nu + \chi_\nu\tau_\mu + X\tau_\mu\tau_\nu)\Big] \\ &\quad - \frac{1}{2}h^{\lambda\rho}\Big[\delta K_{\rho\mu}\tau_\nu + \delta K_{\rho\nu}\tau_\mu\Big]. \end{aligned} \tag{A.14}$$

We can use $\partial_{[\mu}\tau_{\nu]} = 0$ to see that $\partial_\mu \tau_\nu = \partial_{(\mu}\tau_{\nu)}$ and simplify the equation further. We get

$$\begin{aligned}\tilde{\hat{\Gamma}}^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{2}h^{\lambda\rho}\Big[&\chi_\nu\partial_\mu\tau_\rho + \partial_\mu\chi_\rho\tau_\nu + X\tau_\nu\partial_\mu\tau_\rho + \chi_\mu\partial_\nu\tau_\rho + \partial_\nu\chi_\rho\tau_\mu + X\tau_\mu\partial_\nu\tau_\rho \\ &- \partial_\rho\chi_\mu\tau_\nu - \chi_\mu\partial_\rho\tau_\nu - \partial_\rho\chi_\nu\tau_\mu - \chi_\nu\partial_\rho\tau_\mu - \partial_\rho(X\tau_\mu\tau_\nu)\Big] \\ &- \frac{1}{2}h^{\lambda\rho}\Big[\delta K_{\rho\mu}\tau_\nu + \delta K_{\rho\nu}\tau_\mu\Big].\end{aligned} \tag{A.15}$$

Again, by using $\partial_{[\mu}\tau_{\nu]} = 0$ we can see that most of these terms vanish. We are left with

$$\begin{aligned}\tilde{\hat{\Gamma}}^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{2}h^{\lambda\rho}\Big[&\partial_\mu\chi_\rho\tau_\nu + \partial_\nu\chi_\rho\tau_\mu - \partial_\rho\chi_\mu\tau_\nu - \partial_\rho\chi_\nu\tau_\mu - \partial_\rho X\tau_\mu\tau_\nu\Big] \\ &- \frac{1}{2}h^{\lambda\rho}\Big[\delta K_{\rho\mu}\tau_\nu + \delta K_{\rho\nu}\tau_\mu\Big].\end{aligned} \tag{A.16}$$

Using the definition of K we get

$$\delta K_{\rho\mu} = \partial_\rho\delta C_\mu - \partial_\mu\delta C_\rho. \tag{A.17}$$

Let us use this in

$$\begin{aligned}\tilde{\hat{\Gamma}}^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{2}h^{\lambda\rho}\Big[&\partial_\mu\chi_\rho\tau_\nu + \partial_\nu\chi_\rho\tau_\mu - \partial_\rho\chi_\mu\tau_\nu - \partial_\rho\chi_\nu\tau_\mu - \partial_\rho X\tau_\mu\tau_\nu \\ &-(\partial_\rho\delta C_\mu - \partial_\mu\delta C_\rho)\tau_\nu - (\partial_\rho\delta C_\nu - \partial_\nu\delta C_\rho)\tau_\mu\Big],\end{aligned} \tag{A.18}$$

and try to gather everything under the derivatives as follows

$$\begin{aligned}\tilde{\hat{\Gamma}}^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{2}h^{\lambda\rho}\Big[&\partial_\mu(\chi_\rho + \delta C_\rho)\tau_\nu + \partial_\nu(\chi_\rho + \delta C_\rho)\tau_\mu \\ &-\partial_\rho(\chi_\mu + \delta C_\mu)\tau_\nu - \partial_\rho(\chi_\nu + \delta C_\nu)\tau_\mu - \partial_\rho X\tau_\mu\tau_\nu\Big].\end{aligned} \tag{A.19}$$

One possible candidate is

$$\delta C_\mu = -\chi_\mu - mX\tau_\mu, \qquad m \in \mathbb{R}. \tag{A.20}$$

Let us see if it satisfies the invariance. We get

$$\tilde{\hat{\Gamma}}^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} + \frac{1}{2}h^{\lambda\rho}\Big[\partial_\mu(-mX\tau_\rho)\tau_\nu + \partial_\nu(-mX\tau_\rho)\tau_\mu$$
$$-\partial_\rho(-mX\tau_\mu)\tau_\nu - \partial_\rho(-mX\tau_\nu)\tau_\mu - \partial_\rho X\tau_\mu\tau_\nu\Big]. \quad (A.21)$$

Using $h^{\lambda\rho}\tau_\rho = 0$ we can see that two terms vanish and we are left with

$$\tilde{\hat{\Gamma}}^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} + \frac{m}{2}h^{\lambda\rho}\Big[-X\partial_\mu\tau_\rho\tau_\nu - X\partial_\nu\tau_\rho\tau_\mu + X\partial_\rho\tau_\mu\tau_\nu + X\partial_\rho\tau_\nu\tau_\mu$$
$$+ 2\partial_\rho X\tau_\mu\tau_\nu - \frac{1}{m}\partial_\rho X\tau_\mu\tau_\nu\Big]. \quad (A.22)$$

Using closedness, we get

$$\tilde{\hat{\Gamma}}^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} + \frac{m}{2}h^{\lambda\rho}\Big[2\partial_\rho X\tau_\mu\tau_\nu - \frac{1}{m}\partial_\rho X\tau_\mu\tau_\nu\Big]. \quad (A.23)$$

We can easily see that

$$\tilde{\hat{\Gamma}}^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} \quad \text{for} \quad m = \frac{1}{2}, \quad (A.24)$$

satisfies the invariance. Hence, the combined finite boosts that leave $\hat{\Gamma}$ invariant are

$$\tilde{C}_\mu = C_\mu - \chi_\mu - \frac{1}{2}h^{\rho\sigma}\chi_\rho\chi_\sigma\tau_\mu, \quad (A.25)$$

$$\tilde{\tau}^\mu = \tau^\mu - h^{\mu\nu}\chi_\nu, \quad (A.26)$$

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \chi_\mu\tau_\nu + \chi_\nu\tau_\mu + h^{\rho\sigma}\chi_\rho\chi_\sigma\tau_\mu\tau_\nu. \quad (A.27)$$

We can see that when χ is an infinitesimal quantity, quadratic terms will vanish and we will get our infinitesimal Milne boosts.

A.2. Parametrization Invariance of the Point Particle Action

The action for the point particle can be expressed as

$$S = \frac{m}{2}\int ds \frac{\hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{\tau_\mu\dot{x}^\mu}. \quad (A.28)$$

We will apply a parameter change. The chain rule follows as

$$ds = \frac{\partial s}{\partial\tilde{s}}d\tilde{s}, \quad (A.29)$$

when we appy this to the action, we get

$$S = \frac{m}{2}\int d\tilde{s}\frac{\partial s}{\partial \tilde{s}}\frac{\partial \tilde{s}}{\partial s}\ \frac{\hat{h}_{\mu\nu}\mathring{x}^{\mu}\mathring{x}^{\nu}}{\tau_{\mu}\mathring{x}^{\mu}} = \frac{m}{2}\int d\tilde{s}\frac{\hat{h}_{\mu\nu}\mathring{x}^{\mu}\mathring{x}^{\nu}}{\tau_{\mu}\mathring{x}^{\mu}}, \tag{A.30}$$

where

$$\mathring{x}^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tilde{s}}. \tag{A.31}$$

A.3. Variation of the Point Particle Action

The Lagrangian for the point particle can be expressed as

$$L = \frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{\tau_{\mu}\dot{x}^{\mu}} = \frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{N}. \tag{A.32}$$

We will derive the equations of motion by varying this Lagrangian. We can start with

$$\partial_{\alpha}L = \frac{\partial_{\alpha}\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{N} - \frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\partial_{\alpha}N}{N^2}. \tag{A.33}$$

We will use the following shorthand notation

$$\frac{\partial}{\partial\dot{x}^{\alpha}} \equiv \dot{\partial}_{\alpha} \tag{A.34}$$

in order to avoid some fractions. We have

$$\dot{\partial}_{\alpha}N = \dot{\partial}_{\alpha}(\tau_{\mu}\dot{x}^{\mu}) = \tau_{\alpha}. \tag{A.35}$$

Next, the computation follows as

$$\begin{aligned}\dot{\partial}_{\alpha}L &= \frac{\hat{h}_{\mu\nu}\dot{\partial}_{\alpha}(\dot{x}^{\mu}\dot{x}^{\nu})}{N} - \frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\dot{\partial}_{\alpha}N}{N^2} = \frac{\hat{h}_{\mu\nu}\dot{\partial}_{\alpha}(\dot{x}^{\mu}\dot{x}^{\nu})}{N} - \frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\tau_{\alpha}}{N^2}, \\ &= \frac{2\hat{h}_{\mu\alpha}\dot{x}^{\mu}}{N} - \frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\tau_{\alpha}}{N^2}.\end{aligned} \tag{A.36}$$

The derivative of this will yield

$$\frac{d}{ds}\dot{\partial}_{\alpha}L = \frac{2\frac{d}{ds}(\hat{h}_{\mu\alpha}\dot{x}^{\mu})}{N} - \frac{2\hat{h}_{\mu\alpha}\dot{x}^{\mu}\dot{N}}{N^2} - \frac{\frac{d}{ds}(\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\tau_{\alpha})}{N^2} + 3\frac{\hat{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\tau_{\alpha}\dot{N}}{N^3}. \tag{A.37}$$

We can plug in both terms into Euler-Lagrange equation to get the equations of motion. Euler-Lagrange equation can be expressed as

$$\partial_{\alpha}L = \frac{d}{ds}\dot{\partial}_{\alpha}L. \tag{A.38}$$

After plugging in the terms, we get

$$\frac{\partial_\alpha \hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{N} - \frac{\hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\partial_\alpha N}{N^2} = \frac{2\frac{d}{ds}(\hat{h}_{\mu\alpha}\dot{x}^\mu)}{N} - \frac{2\hat{h}_{\mu\alpha}\dot{x}^\mu\dot{N}}{N^2} - \frac{\frac{d}{ds}(\hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\tau_\alpha)}{N^2} + 3\frac{\hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\tau_\alpha\dot{N}}{N^3}. \tag{A.39}$$

We can contract both sides with $h^{\gamma\alpha}$ and use $h^{\gamma\alpha}\tau_\alpha = 0$ to simplify the equation. Also, let's multiply both sides by N to simplify it further. It follows as

$$h^{\gamma\alpha}\partial_\alpha \hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{h^{\gamma\alpha}\hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\partial_\alpha N}{N} = 2h^{\gamma\alpha}\partial_\nu\hat{h}_{\mu\alpha}\dot{x}^\mu\dot{x}^\nu + 2h^{\gamma\alpha}\hat{h}_{\mu\alpha}\ddot{x}^\mu - \frac{2h^{\gamma\alpha}\hat{h}_{\mu\alpha}\dot{x}^\mu\dot{N}}{N} - \frac{h^{\gamma\alpha}\hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\dot{\tau}_\alpha}{N}. \tag{A.40}$$

We can use the symmetry of h and $\hat{h}$ to split the first term on the right hand side with some re-indexing and we can gather some terms to the same side. After applying these, we get

$$2h^{\gamma\alpha}\hat{h}_{\alpha\mu}\ddot{x}^\mu + h^{\gamma\alpha}(\partial_\nu\hat{h}_{\mu\alpha} + \partial_\mu\hat{h}_{\nu\alpha} - \partial_\alpha\hat{h}_{\mu\nu})\dot{x}^\mu\dot{x}^\nu + \frac{h^{\gamma\alpha}\hat{h}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu(\partial_\alpha N - \frac{d\tau_\alpha}{ds})}{N} = \frac{2h^{\gamma\alpha}\hat{h}_{\alpha\mu}\dot{x}^\mu\dot{N}}{N}. \tag{A.41}$$

We can easily show

$$\partial_\alpha N - \frac{d\tau_\alpha}{ds} = \dot{x}^\beta\partial_\alpha\tau_\beta - \dot{x}^\beta\partial_\beta\tau_\alpha = 2\dot{x}^\beta\partial_{[\alpha}\tau_{\beta]} = 0, \tag{A.42}$$

holds. Therefore, we are left with

$$2h^{\gamma\alpha}\hat{h}_{\alpha\mu}\ddot{x}^\mu + h^{\gamma\alpha}(\partial_\nu\hat{h}_{\mu\alpha} + \partial_\mu\hat{h}_{\nu\alpha} - \partial_\alpha\hat{h}_{\mu\nu})\dot{x}^\mu\dot{x}^\nu = \frac{2h^{\gamma\alpha}\hat{h}_{\alpha\mu}\dot{x}^\mu\dot{N}}{N}. \tag{A.43}$$

Using the identity, $\hat{\tau}^\gamma\tau_\mu + h^{\gamma\alpha}\hat{h}_{\alpha\mu} = \delta^\gamma_\mu$, we get

$$2\ddot{x}^\gamma + h^{\gamma\alpha}(\partial_\nu\hat{h}_{\mu\alpha} + \partial_\mu\hat{h}_{\nu\alpha} - \partial_\alpha\hat{h}_{\mu\nu})\dot{x}^\mu\dot{x}^\nu - 2\hat{\tau}^\gamma\tau_\mu\ddot{x}^\mu = \frac{2\dot{x}^\gamma\dot{N}}{N} - \frac{2\hat{\tau}^\gamma\tau_\mu\dot{x}^\mu\dot{N}}{N}. \tag{A.44}$$

By using the following relation

$$\tau_\mu\ddot{x}^\mu = \frac{d}{ds}(\tau_\mu\dot{x}^\mu) - \dot{\tau}_\mu\dot{x}^\mu, \tag{A.45}$$

we can focus on the rightmost term on both sides. It follows as

$$\cdots - 2\hat{\tau}^{\gamma}(\dot{N} - \dot{\tau}_{\mu}\dot{x}^{\mu}) = \cdots - \frac{2\hat{\tau}^{\gamma}\tau_{\mu}\dot{x}^{\mu}\dot{N}}{N}, \tag{A.46}$$

$$\cdots - 2\hat{\tau}^{\gamma}(\dot{N} - \dot{\tau}_{\mu}\dot{x}^{\mu}) = \cdots - 2\hat{\tau}^{\gamma}\dot{N}. \tag{A.47}$$

Then, we get

$$2\ddot{x}^{\gamma} + h^{\gamma\alpha}(\partial_{\nu}\hat{h}_{\mu\alpha} + \partial_{\mu}\hat{h}_{\nu\alpha} - \partial_{\alpha}\hat{h}_{\mu\nu})\dot{x}^{\mu}\dot{x}^{\nu} + 2\hat{\tau}^{\gamma}\dot{\tau}_{\mu}\dot{x}^{\mu} = \frac{2\dot{x}^{\gamma}\dot{N}}{N}. \tag{A.48}$$

Rearranging the terms yields

$$\ddot{x}^{\gamma} + \hat{\tau}^{\gamma}\partial_{\nu}\tau_{\mu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2}h^{\gamma\alpha}(\partial_{\nu}\hat{h}_{\mu\alpha} + \partial_{\mu}\hat{h}_{\nu\alpha} - \partial_{\alpha}\hat{h}_{\mu\nu})\dot{x}^{\mu}\dot{x}^{\nu} = \frac{\dot{x}^{\gamma}\dot{N}}{N}. \tag{A.49}$$

Finally, after using the definition of $\hat{\Gamma}$ we get

$$\ddot{x}^{\gamma} + \hat{\Gamma}^{\gamma}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \dot{x}^{\gamma}\frac{\dot{N}}{N}. \tag{A.50}$$

A.4. Field Equations in 3+1 Formulation

In this section, we will explicitly calculate the field equations for the 3+1 formulation. We will start by separating (6.32) into three parts.

$$\hat{R}_{00} = 4\pi G_N \rho, \tag{A.51}$$

$$\hat{R}_{i0} = 0, \tag{A.52}$$

$$\hat{R}_{ij} = 0, \tag{A.53}$$

Let us start with Equation (A.51). It can explicitly be expressed as

$$\hat{R}_{00} = \partial_{\rho}\hat{\Gamma}^{\rho}_{00} - \partial_{0}\hat{\Gamma}^{\rho}_{\rho 0} + \hat{\Gamma}^{\rho}_{\rho\lambda}\hat{\Gamma}^{\lambda}_{00} - \hat{\Gamma}^{\rho}_{0\lambda}\hat{\Gamma}^{\lambda}_{\rho 0} = 4\pi G_N \rho. \tag{A.54}$$

First, we should calculate the components of $\hat{\Gamma}$ in this gauge. It follows as

$$\hat{\Gamma}^{\rho}_{00} = \tau^{\rho}\partial_{0}\tau_{0} + \frac{1}{2}h^{\rho\lambda}(\partial_{0}h_{0\lambda} + \partial_{0}h_{0\lambda} - \partial_{\lambda}h_{00}) - h^{\rho\lambda}K_{\lambda(0}\tau_{0)}. \tag{A.55}$$

Using Equations (8.1) and (8.3) we get

$$\hat{\Gamma}^{\rho}_{00} = -h^{\rho\lambda}K_{\lambda 0}. \tag{A.56}$$

We can further decompose this into

$$\hat{\Gamma}^i_{00} = -h^{ij}K_{j0}, \tag{A.57}$$

$$\hat{\Gamma}^0_{00} = 0. \tag{A.58}$$

Now let's look at other components, we have

$$\hat{\Gamma}^0_{\mu\nu} = \tau^0\partial_\mu\tau_\nu + \frac{1}{2}h^{0\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}) - h^{0\lambda}K_{\lambda(\mu}\tau_{\nu)}. \tag{A.59}$$

Again, using Equations (8.1) and (8.3) we get,

$$\hat{\Gamma}^0_{\mu\nu} = 0. \tag{A.60}$$

Next, we have

$$\hat{\Gamma}^\mu_{0\nu} = \tau^\mu\partial_0\tau_\nu + \frac{1}{2}h^{\mu\lambda}(\partial_0 h_{\nu\lambda} + \partial_\nu h_{0\lambda} - \partial_\lambda h_{0\mu}) - h^{\mu\lambda}K_{\lambda(0}\tau_{\nu)}. \tag{A.61}$$

By substituting our fields in, we get

$$\hat{\Gamma}^\mu_{0\nu} = \frac{1}{2}h^{\mu\lambda}\dot{h}_{\nu\lambda} - h^{\mu\lambda}K_{\lambda(0}\tau_{\nu)}, \tag{A.62}$$

$$\hat{\Gamma}^\mu_{0\nu} = \frac{1}{2}h^{\mu\lambda}(\dot{h}_{\nu\lambda} - K_{\lambda 0}\tau_\nu - K_{\lambda\nu}). \tag{A.63}$$

From (A.60), we know that for $\mu = 0$, this becomes zero, and for $\nu = 0$ it reduces to (A.57). Therefore the only new information we get from this is

$$\hat{\Gamma}^i_{0j} = \frac{1}{2}h^{ik}(\dot{h}_{kj} - K_{kj}). \tag{A.64}$$

The remaining components are

$$\hat{\Gamma}^k_{ij} = \tau^k\partial_i\tau_j + \frac{1}{2}h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}) - h^{kl}K_{l(i}\tau_{j)}, \tag{A.65}$$

$$\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} = \frac{1}{2}h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}), \tag{A.66}$$

which are just a 3 dimensional Christoffel symbols. After using these in Equation (A.51), we get

$$\hat{R}_{00} = \partial_i(-h^{ij}K_{j0}) - \partial_0(\frac{1}{2}h^{ik}(\dot{h}_{ki} - K_{ki})) - \Gamma^i_{ij}h^{jk}K_{k0} \\ - \frac{1}{4}h^{ik}(\dot{h}_{kj} - K_{kj})h^{jl}(\dot{h}_{li} - K_{li}). \quad (A.67)$$

As we have discussed in Chapter 8, we can use h to raise or lower indices, which follows as

$$\hat{R}_{00} = -\nabla_i K^i{}_0 - \frac{1}{2}\dot{h}^{ik}(\dot{h}_{ki} - K_{ki}) - \frac{1}{2}h^{ik}(\ddot{h}_{ki} - \dot{K}_{ki}) \\ - \frac{1}{4}(-\dot{h}^{li}\dot{h}_{li} - 2\dot{h}^{li}K_{li} + K^{il}K_{li}). \quad (A.68)$$

Finally, plugging $\hat{R}_{00}$ back into Equation (A.51) will yield us

$$-\nabla_i K^i{}_0 = \frac{1}{2}h^{ik}\ddot{h}_{ik} + \frac{1}{4}\dot{h}^{ik}\dot{h}_{ik} - \frac{1}{4}K^{ik}K_{ik} + 4\pi G_N\rho. \quad (A.69)$$

Now let us look at $\hat{R}_{i0} = 0$, it can explicitly be expressed as

$$\hat{R}_{i0} = \partial_\rho \hat{\Gamma}^\rho_{i0} - \partial_0 \hat{\Gamma}^\rho_{\rho i} + \hat{\Gamma}^\rho_{\rho\lambda}\hat{\Gamma}^\lambda_{i0} - \hat{\Gamma}^\rho_{0\lambda}\hat{\Gamma}^\lambda_{\rho i}. \quad (A.70)$$

Plugging in the connection coefficients, we get

$$\hat{R}_{i0} = \partial_k \left[\frac{1}{2}h^{kj}(\dot{h}_{ji} - K_{ji})\right] - \partial_0 \Gamma^k_{ki} + \Gamma^k_{kl}\left[\frac{1}{2}h^{lj}(\dot{h}_{ji} - K_{ji})\right] \\ - \Gamma^l_{ki}\left[\frac{1}{2}h^{kj}(\dot{h}_{jl} - K_{jl})\right]. \quad (A.71)$$

By combining all the connection coefficients we can get a covariant derivative as follows

$$\hat{R}_{i0} = \frac{1}{2}h^{kj}\nabla_k\left(\dot{h}_{ji} - K_{ji}\right) - \partial_0 \hat{\Gamma}^k_{ki} = 0. \quad (A.72)$$

Further calculation will yield us

$$\nabla^j(\dot{h}_{ji} - K_{ji}) = 2\partial_0 \Gamma^k_{ki}, \tag{A.73}$$

$$= \partial_0(h^{kl}\partial_i h_{kl}), \tag{A.74}$$

$$= \dot{h}^{kl}\partial_i h_{kl} + h^{kl}\partial_i \dot{h}_{kl}, \tag{A.75}$$

$$= \dot{h}^{kl}\nabla_i h_{kl} + \dot{h}^{kl}(\Gamma^j_{ik}h_{jl} + \Gamma^j_{il}h_{kj}) + h^{kl}\nabla_i \dot{h}_{kl} + h^{kl}(\Gamma^j_{ik}\dot{h}_{jl} + \Gamma^j_{il}\dot{h}_{kj}), \tag{A.76}$$

$$= \dot{h}^{kl}(\Gamma^j_{ik}h_{jl} + \Gamma^j_{il}h_{kj}) + h^{kl}\nabla_i \dot{h}_{kl} + h^{kl}(\Gamma^j_{ik}\dot{h}_{jl} + \Gamma^j_{il}\dot{h}_{kj}). \tag{A.77}$$

We can combine the connection coefficients to simplify the equation as follows

$$\nabla^j(\dot{h}_{ji} - K_{ji}) = h^{kl}\nabla_i \dot{h}_{kl} + \Gamma^j_{ik}\partial_0(h^{kl}h_{jl}) + \Gamma^j_{il}\partial_0(h^{kl}h_{kj}), \tag{A.78}$$

which can be expressed as

$$-\nabla^j K_{ji} = h^{kl}\nabla_i \dot{h}_{kl} - h^{jk}\nabla_k \dot{h}_{ji}. \tag{A.79}$$

Relabeling some dummy indices will yield us

$$-\nabla^j K_{ji} = h^{kl}\nabla_i \dot{h}_{kl} - h^{kl}\nabla_k \dot{h}_{il}. \tag{A.80}$$

Finally gathering the terms by using the anti-symmetry, we get

$$-\nabla^j K_{ji} = 2h^{kl}\nabla_{[i}\dot{h}_{k]l}, \tag{A.81}$$

which is our second equation.

Lastly, we can easily show that the NC Ricci tensor will just be a 3 dimensional Ricci tensor of the h metric as follows

$$\hat{R}_{ij} = \partial_\rho \hat{\Gamma}^\rho_{ji} - \partial_j \hat{\Gamma}^\rho_{\rho i} + \hat{\Gamma}^\rho_{\rho\lambda}\hat{\Gamma}^\lambda_{ji} - \hat{\Gamma}^\rho_{j\lambda}\hat{\Gamma}^\lambda_{\rho i}, \tag{A.82}$$

$$= \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}, \tag{A.83}$$

$$= R_{ij}. \tag{A.84}$$

For notational convenience, we can define

$$K^i{}_0 = G^i \tag{A.85}$$

and our field equations can be expressed as

$$R_{ij} = 0, \tag{A.86}$$

$$-\nabla_i G^i = \frac{1}{2} h^{ik} \ddot{h}_{ik} + \frac{1}{4} \dot{h}^{ik} \dot{h}_{ik} - \frac{1}{4} K^{ik} K_{ik} + 4\pi G_N \rho, \tag{A.87}$$

$$-\nabla^j K_{ji} = 2h^{kl} \nabla_{[i} \dot{h}_{k]l}. \tag{A.88}$$

APPENDIX B: LARGE c EXPANSION WITH VIELBEIN APPROACH

There is a great approach to large c expansion that is adopted in [10]. Instead of expanding the metric directly, they have used the vielbein formalism and expanded the vielbein to get a detailed and a systematic expansion of GR.

They start their work with an orthonormal vielbein choice as follows

$$E_A^\mu E_\nu^A = \delta_\nu^\mu, \tag{B.1}$$
$$E_A^\mu E_\mu^B = \delta_B^A, \tag{B.2}$$

and define 8 fields, T_μ, T^μ, $\mathcal{E}_\mu^a$, and $\mathcal{E}_a^\mu$, as

$$E_\mu^A = cT_\mu \delta_0^A + \mathcal{E}_\mu^a \delta_a^A, \tag{B.3}$$
$$E_A^\mu = -\frac{1}{c} T^\mu \delta_A^0 + \mathcal{E}_a^\mu \delta_A^a. \tag{B.4}$$

Using the orthonormality, we can get some identities related to the fields as follows

$$T_\mu \mathcal{E}_a^\mu = 0, \tag{B.5}$$
$$T^\mu \mathcal{E}_\mu^a = 0, \tag{B.6}$$
$$T_\mu T^\mu = -1, \tag{B.7}$$
$$\mathcal{E}_a^\mu \mathcal{E}_\mu^b = \delta_b^a, \tag{B.8}$$
$$\mathcal{E}_a^\mu \mathcal{E}_\nu^a = \delta_\nu^\mu + T^\mu T_\nu. \tag{B.9}$$

Then, we can express the metric in terms of the fields as

$$g_{\mu\nu} = \eta_{AB} E_\mu^A E_\nu^B = -c^2 T_\mu T_\nu + \delta_{ab} \mathcal{E}_\mu^a \mathcal{E}_\nu^b, \tag{B.10}$$
$$g^{\mu\nu} = \eta^{AB} E_A^\mu E_B^\nu = -\frac{1}{c^2} T^\mu T^\nu + \delta^{ab} \mathcal{E}_a^\mu \mathcal{E}_b^\nu, \tag{B.11}$$

and we can define new fields to simplify some calculations as

$$\mathit{\Pi}_{\mu\nu} = \delta_{ab}\mathcal{E}^a_\mu\mathcal{E}^b_\nu, \qquad \mathit{\Pi}^{\mu\nu} = \delta^{ab}\mathcal{E}^\mu_a\mathcal{E}^\nu_b. \tag{B.12}$$

Finally, the metric decomposition becomes

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \mathit{\Pi}_{\mu\nu}, \tag{B.13}$$

$$g^{\mu\nu} = -\frac{1}{c^2}T^\mu T^\nu + \mathit{\Pi}^{\mu\nu}. \tag{B.14}$$

So far, we haven't done anything but reformulating the metric in terms of new arbitrary (but not independent) fields. At this point, we can define some connections and express everything in terms of the fundamental fields and reformulate GR. We don't need those for our work but the original paper is very extensive and detailed in every step. A connection to define the Ricci tensor should be enough for now. We can take the c-independent part of the Levi-Civita connection. Of course, our fields can depend on c themselves, but this connection does not have *explicit* c dependence. It can be expressed as

$$C^\rho_{\mu\nu} = -T^\rho\partial_\mu T_\nu + \frac{1}{2}\mathit{\Pi}^{\rho\sigma}(\partial_\mu\mathit{\Pi}_{\nu\sigma} + \partial_\nu\mathit{\Pi}_{\mu\sigma} - \partial_\sigma\mathit{\Pi}_{\mu\nu}). \tag{B.15}$$

One must be careful here. This connection is not torsion free due to the first term. We do not require T_μ to be closed like we did in our expansion.

We can define a covariant derivative with the connection, $\overset{(C)}{\nabla}_\mu$, and the associated Riemann tensor as

$$[\overset{(C)}{\nabla}_\mu, \overset{(C)}{\nabla}_\nu]X^\lambda = \overset{(C)}{R}_{\mu\nu\sigma}{}^\lambda X^\sigma. \tag{B.16}$$

Lastly, Ricci scalar can be expressed in terms of these new objects as

$$R = \frac{c^2}{4}\mathit{\Pi}^{\mu\nu}\mathit{\Pi}^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma} + \mathit{\Pi}^{\mu\nu}\overset{(C)}{R}_{\mu\nu} - \frac{1}{c^2}T^\mu T^\nu\overset{(C)}{R}_{\mu\nu}, \tag{B.17}$$

where we used

$$T_{\mu\rho} = \partial_\mu T_\rho - \partial_\rho T_\mu. \tag{B.18}$$

From now on, we will focus on the expansion of the vielbein. Of course, our vielbein are T and $\mathcal{E}$. The expansion of Π is derived from the expansion of $\mathcal{E}$. It follows as

$$T_\mu = \tau_\mu + c^{-2} m_\mu + c^{-4} B_\mu + \mathcal{O}(c^{-6}), \tag{B.19}$$

$$\Pi_{\mu\nu} = h_{\mu\nu} + c^{-2}\Phi_{\mu\nu} + \mathcal{O}(c^{-4}), \tag{B.20}$$

and the inverses are

$$T^\mu = v^\mu + c^{-2}(v^\mu v^\rho m_\rho - e^\mu_b v^\rho \pi^b_\rho) + \mathcal{O}(c^{-4}), \tag{B.21}$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^{-2}(2h^{\rho(\mu} v^{\nu)} m_\rho - h^{\mu\rho} h^{\nu\sigma} \Phi_{\rho\sigma}) + \mathcal{O}(c^{-4}). \tag{B.22}$$

Applying the vielbein expansion to the metric, we get

$$g_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + h_{\mu\nu} - 2\tau_{(\mu} m_{\nu)} + c^{-2}(\Phi_{\mu\nu} - m_\mu m_\nu - 2B_{(\mu}\tau_{\nu)}) + \mathcal{O}(c^{-4}), \tag{B.23}$$

$$g^{\mu\nu} = h^{\mu\nu} + c^{-2}(v^\mu v^\nu + 2h^{\rho(\mu} v^{\nu)} m_\rho - h^{\mu\rho} h^{\nu\sigma} \Phi_{\rho\sigma}) + \mathcal{O}(c^{-4}). \tag{B.24}$$

From this point on, the work is almost identical to Chapter 5, except for a few differences in field definitions.