AN EXAMINATION OF ELEMENTARY AND LOWER SECONDARY LEVEL JAPANESE AND TURKISH MATHEMATICS CURRICULA THROUGH QUANTITATIVE AND (CO)VARIATIONAL REASONING IN TERMS OF THE TREATMENT OF FUNCTIONAL RELATIONSHIPS

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Nurdan Turan

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ABSTRACT

AN EXAMINATION OF ELEMENTARY AND LOWER SECONDARY LEVEL JAPANESE AND TURKISH MATHEMATICS CURRICULA THROUGH QUANTITATIVE AND (CO)VARIATIONAL REASONING IN TERMS OF THE TREATMENT OF FUNCTIONAL RELATIONSHIPS

This study investigated the similarities and differences between Japanese and Turkish curriculum materials, i.e. the course of study, teachers' guides, and textbooks, in terms of quantitative and covariational reasoning depicted in the concepts of functional relationships, i.e. linear functions, proportions, rate, and ratio. The analysis was focused on the overall structures of textbooks and the learning opportunities of functional relationships in regard to quantitative and covariational reasoning in the curriculum materials. The tasks, problem situations, questions asked of students, and the use of representations were examined as potential learning opportunities. The finding showed that Japanese textbooks allocated more pages to introduce functional relationships than Turkish textbooks regardless of grade level. Findings also displayed that Japanese curriculum has a spiral nature in terms of covariational reasoning, ratio conception, and task variables. In particular, it has a clear focus on supporting learners' quantitative reasoning and covariational reasoning which seems to gradually raise up to *continuous covariation*. Whereas Turkish curriculum seems to support students to reach up to *coordination of values* level of covariational reasoning. Moreover, in terms of ratio and proportion, the Japanese curriculum starts with supporting students' additive strategies on tasks and gradually introduces multiplicative strategies; whereas Turkish curriculum materials do not cover all multiplicative strategies and do not seem to have spiral nature in terms of ratio conception.

ÖZET

FONKSİYONEL İLİŞKİLERİN JAPONYA VE TÜRKİYE İLKÖĞRETİM MATEMATİK MÜFREDATLARININ NİCEL VE KOVARYASYONEL MUHAKEME AÇISINDAN KARŞILAŞTIRMALI İNCELENMESİ

Bu çalışmada, fonksiyonel ilişkiler Japonya ve Türkiye müfredat materyalleri, yani öğretim programları, öğretmen kılavuzları ve ders kitapları, nicel muhakeme ve kovaryasyonel muhakeme açısından benzerlikler ve farklılıklar ortaya konulacak şekilde incelenmiştir. Fonksiyonel ilişkiler kapsamında doğrusal fonksiyonlar, oran ve orantı konularına bakılmıştır. Analiz yapılırken ders kitaplarının genel yapıları ve müfredat materyallerinde nicel ve kovaryasyonel muhakemeyi destekleyebilecek öğrenme fırsatlarına odaklanılmıştır. Etkinlikler, problem durumları, öğrencilere sorulan sorular ve gösterim biçimlerinin kullanımı potansiyel öğrenme fırsatları olarak incelenmiştir. Bulgular, Japonya ders kitaplarının sınıf düzeyi ne olursa olsun Türkiye ders kitaplarına göre fonksiyonel ilişkileri tanıtmak için daha fazla sayfa ayırdığını göstermiştir. Ayrıca Japonya müfredatının kovaryasyonel muhakeme, oran kavramı ve etkinlik değişkenleri açısından sarmal bir yapıya sahip olduğu bulunmuştur. Özellikle, öğrencilerin nicel muhakemesini desteklemeye net bir şekilde odaklanan Japonya müfredatı, kovaryasyonel muhakemede *sürekli kovaryasyon* seviyesine kadar kademeli bir yükselişi destekler görünmektedir. Oysa Türkiye müfredatının, öğrencileri kovaryasyonel muhakemenin *değerlerin koordinasyonu* seviyesine kadar ulaşmasını destekleyebilir nitelikte olduğu bulunmuştur. Dahası, oran ve orantı açısından, Japonya müfredatı öğrencilerin toplamsal stratejilerini desteklemekle başlayıp aşamalı olarak çarpımsal stratejiler sunar; ancak Türkiye müfredatı tüm çarpımsal stratejileri kapsamaz ve oran kavramı açısından sarmal bir yapıya sahip görünmemektedir.

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LIST OF ACRONYMS/ABBREVIATIONS

APEC	Asia–Pacific Economic Cooperation (APEC)
APOS	Action Process Object Schema
CCK	Common Content Knowledge
COS	Course of Study
HCK	Horizon Content Knowledge
IBDP	International Baccalaureate Diploma Programme
ICME	International Congress on Mathematical Education
IEA	International Association for the Evaluation of Educational Achievement
KCC	Knowledge of Content and Curriculum
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
MoNE	Ministry of National Education
MI	Mathematics International
OECD	Organization of Economic Cooperation and Development
PCK	Pedagogical Content Knowledge
PISA	Programme for International Student Assessment
SCK	Specialized Content Knowledge
SMK	Subject Matter Knowledge
TIMMS	Third International Mathematics and Science Study

1. INTRODUCTION

School curriculum has been designed to improve students' achievement and thereof undergone some reforms (Cai, 2017). With the research study of the Third International Mathematics and Science Study (TIMMS), mathematics textbooks had been analyzed in order to make sense of the learning opportunities presented in the textbooks by particular countries. Particularly, researchers analyzed different aspects of the textbooks conducting cross-national textbook analysis research. The aim of the cross-national textbook analysis studies was to gain some possible insights about opportunities for students to learn mathematics (e.g., Son and Senk, 2010; Fan and Zhu, 2007; Hadar, 2017; Gracin, 2018; Remillard *et al.*, 2014; Kul *et al.*, 2018), teacher's teaching and learning (e.g., Son and Kim, 2015; Ma, 1999; Ball, 1996; Ball and Cohen 1996; Remillard, 2005; Özgeldi, 2011), and educational preferences of a particular country (e.g. Usiskin, 2013; Howson, 2013; İlhan and Aslaner, 2019) (Valverde *et al.*, 2002; Cai, 2017).

In general, researchers examine textbooks regarding either the treatment of a specific mathematics topic (e.g. Charalambous *et al.*, 2010; Son and Senk, 2010; Watanabe *et al.*, 2017; Cai *et al.*, 2002; Lo *et al.*, 2001; Li, 2000; Jones and Fujita, 2013; Miyakawa, 2017; Takeuchi and Shinno, 2019; Sağlam and Alacacı, 2012; Yavuz and Baştürk, 2011; Wang *et al.*, 2017; Ding and Li, 2010) or the overall structure of textbooks without considering specifically on any topic (e.g. Son and Diletti, 2017). The former analysis done on specific mathematics topics is called vertical analysis (i.e. microanalysis), while the latter is named as horizontal analysis (i.e. macroanalysis). Researchers (Charalambous *et al.*, 2010; Watanabe *et al.*, 2017; Li, 2000) recommended to use both the horizontal and the vertical analysis to investigate a treatment of a mathematics topic so that the topic could be examined in terms of the place it has in specific educational system, related topics to it, and learning opportunities provided in it.

Generally, textbooks are examined with frameworks to determine the mathematical, contextual, and response type features of problems as well as the cognitive features. Cognitive features include cognitive demands and cognitive expectations, which include conceptual and procedural knowledge, problem solving, reasoning, and representation. Generally, researchers (e.g. Sağlam and Alacacı, 2012; Yavuz and Baştürk, 2011; Lo *et al.*, 2001) examine the topics in terms of how they are presented in the textbooks. For example, Yavuz and Baştürk (2011) started their analysis of Turkish and French textbooks by considering the introduction problems in the topic of functions, then they investigated the definitions, the types of the problems and the representations placed in the textbooks. Similarly, Lo and her colleagues (2001) examined the ratio and proportion topics in Chinese, Japanese, Taiwanese, and American textbooks regarding the definitions and the type of problems. More specifically, researchers (e.g. Son and Senk, 2010; Sağlam and Alacacı, 2012; Cai *et al.*, 2002) examined the problems given in the textbooks in order to obtain information about whether they are prepared for conceptual or procedural understanding. Regarding cognitive features, Bloom's taxonomy, cognitively guided instruction, Stein and her colleagues' (1996) classification of cognitive demands are some examples of frameworks used by researchers.

Regarding reasoning, some mathematics topics requires to think quantitatively or covariationally for a successful acquisition and conceptualization of the topic. Although textbooks do not provide data to examine students' conceptions, they might trigger students and teachers to focus on quantitative and covariational situations inherent to the topics. Therefore, researchers (Thompson and Carlson, 2017; Taşova *et al.*, 2018) suggested examining curriculum materials in terms of quantitative and covariational reasoning to reveal the extend they might trigger students' conceptions. Especially, functions (Thompson, 1994b; Carlson *et al.*, 2002; Thompson and Carlson, 2017; Taşova *et al.*, 2018) were suggested to be examined due to their link with elementary and collegiate mathematics. Thompson (1994b) argued:

> When analyzing students' concepts of function, we need to keep in mind that the imagery and understandings evoked in students by our probing is going to be textured by their pre-understandings of such things as expressions, variables, arithmetic operations, and quantity. We also need to keep in mind that their mathematical learning has, for the most part, happened in schools, which means that

> our interpretations of students' performance must be conditioned by our knowledge that they are taught by teachers with their own images of what constitutes mathematics, and that both the learning and teaching of mathematics are conditioned by the cultures (school, ethnic, and national) in which they occur. (p. 1-2)

This suggests that it might be worth examining the textbooks of different cultures in terms of how functional relations are treated focusing also on the related topics. Particularly, there are topics in elementary and lower secondary school mathematics such as ratio, rate and proportion to support the understanding of functions (Thompson, 1994a); likewise, functions form a basis for many collegiate mathematics topics (Thompson, 1994b; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008; Thompson and Carlson, 2017). Since functions are introduced to the elementary and lower secondary students with only focusing on linear functions and also the conceptions of ratio, rate, and proportion are closely linked with linear functions (Lamon, 2012; Thompson 1994a), in this study, I examine ratio, rate, proportion, and linear functions given in elementary and lower secondary mathematics textbooks of Japan and Turkey through quantitative and covariational reasoning frameworks. Textbooks are used by students as a guide to learn a topic and by teachers as both a source to study topics and as a guide to design their instruction (Cai, 2017). Therefore, the presentation of the functional relationships in textbooks might contribute to the literature in terms of displaying to what extent the textbooks fulfill their role to contribute to effective understanding of functional relationships. In this study, I selected Japanese textbooks because of both Japanese students' higher achievements on international examinations like PISA and TIMMS, and, Japanese curriculum's focus on quantitative and covariational situations in elementary and lower secondary years (Thomson and Carlson, 2017; Thompson, 1994b). I also selected to examine Turkish textbooks published in 2005 not only because 2005 is the year the reform movement in education had a start in Turkey but also unfortunately, Turkish students' average points in mathematics in PISA and TIMMS have always been below the average of Organization of Economic Cooperation and Development (OECD) countries. Especially, the Turkish students' average point about mathematical literacy in PISA 2015 was lower than the average scores of PISA 2012 and PISA 2009. In addition, there has not been any systematic research on neither Japanese nor Turkish textbooks regarding quantitative and covariational reasoning. In this regard, the purpose of this study is to conduct a cross-national textbook analysis to examine the similarities and differences of the treatment of functional relationship topics in terms of quantitative and covariational reasoning.

2. LITERATURE REVIEW

There has been a growing body of research about school textbooks in the last three decades, just after the Third International Mathematics and Science Study (TIMSS), in which many textbooks were examined from various countries in order to grasp what is provided as the learning opportunities in different educational systems all over the world. In order to figure out the role of the textbooks in educational settings, in the following sections, I briefly explain what curriculum and textbook means. Since in this study, mathematics textbooks are used, from this point on, I use the term *textbook* for referring to *mathematics textbook*. Then, I present some of the research studies in terms of the framework used in the analysis, their results, and their contributions to the field of textbook research. Next, I explicate the reasons behind the choice of Japanese and Turkish textbooks for this study. Following, I explain the topics of functional relationships, including linear functions, rate, ratio and proportion. Lastly, I describe the theoretical framework used in this study with a focus on the rationale of the framework with its relation to the topic and the textbooks selected for this study.

2.1. Curriculum and Textbook

Curriculum is a broad term including intended curriculum, implemented curriculum, attained curriculum and potentially implemented curriculum. In order to comprehend what exactly curriculum means, it is imperative to define these terms in relation to each other. Particularly, a common definition of *intended curriculum* is emphasized as a planned set of actions for students to achieve a specific goal (Schmidt *et al.*, 1999; Valverde *et al.*, 2002; Foxman, 1999; Ornstein and Hunkins, 2004). Then, *implemented curriculum* refers to how the goals of intended curriculum are implemented within classrooms (Schmidt *et al.*, 1999; Ornstein and Hunkins, 2004; Valverde *et al.*, 2002; Foxman, 1999). Finally, *attained curriculum* is about what is learned by students (Valverde *et al.*, 2002; Foxman, 1999). In other words, system level of curriculum corresponds to intended curriculum, while classroom level refers to implemented (i.e. enacted) curriculum and student level corresponds to attained curriculum (Cai, 2017).

Within the span of curriculum, textbooks are designed to cover intended curriculum goals, i.e. curriculum standards, by including introductions of topics, activities, and practices, which are agents of classroom teaching. In other words, textbooks have potential to serve implemented curriculum and they are prepared accordingly to the intended curriculum. Valverde and his colleagues stated that "textbooks are intended as mediators between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms. This suggests that textbooks have a strong impact on what occurs in classrooms." (Valverde *et al.*, 2002, p.2). Therefore, textbooks are classified as *potentially implemented curriculum* (Valverde *et al.*, 2002). Son and Diletti (2017) further argued that since teachers' guidebooks and other supplementary materials such as assessments, review materials, etc. have also potential impact on classroom teaching, they might be classified as potentially implemented curriculum, too. Figure 2.1 given below shows the International Association for the Evaluation of Educational Achievement (IEA)'s model of curriculum, (the tripartite model), including the textbooks to display mediating role of them, which forms the base for TIMMS textbook research.

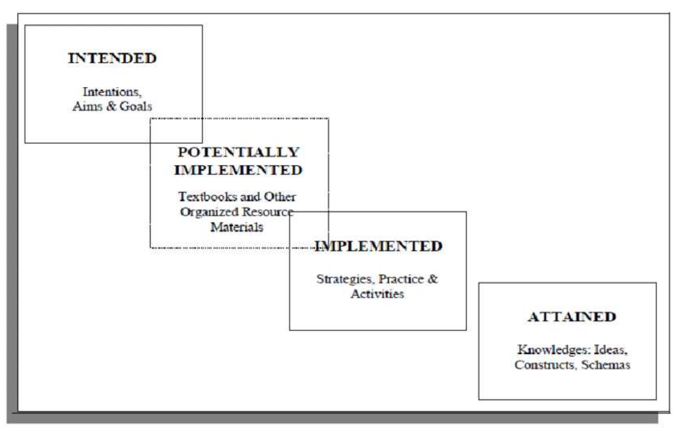


Figure 2.1. The Revised Tripartite Model showing mediator role of textbooks (Valverde et al., 2002, p.13).

In addition to its role as a mediator between intended and implemented curriculum, textbooks provide information about pedagogical predispositions, nature of subject matter, arrangements of topics, physical characteristics about books, and level of complexity. Therefore, textbooks have been accepted as tools to be investigated in order to gain some possible insights about opportunities for students to learn mathematics, teacher's teaching, and educational preferences of a particular country (Valverde *et al.*, 2002; Cai, 2017). In other words, textbooks have been examined as a dependent variable, an independent variable, or a subject of research. In the following sub-section, I explain previous research done on textbooks.

2.1.1. Previous Research on Textbook Analysis

Due to their informative and contributable nature, there have been many research studies on textbook analysis. In fact, examining research questions on textbook research, Fan (2013) conducted a survey study and clustered research on textbook analysis discussed in ICME-10, ICME-11 and the 5th APEC-Tsukuba International Conference. He asserted that existing textbook research studies discussed in the conferences with a special topic of discussion as "mathematics textbooks" could be classified into three broad categories: (1) textbooks as subject of study, depicting what textbooks look like or what the features of textbooks are, (2)textbooks as a dependent variable, revealing how textbooks are affected by different factors such as, the roles of socio-cultural and political factors on the development of textbooks, the roles of government, teachers, curriculum specialists, mathematics educators, and mathematicians on the development of textbooks and (3) textbooks as an independent variable, revealing how textbooks affect different factors, such as instruction, students' and teachers' use of the books, and students' achievements. In addition to Fan's categorization, there are other researchers (Charalambous *et al.*, 2010; Li *et al.*, 2009; Watanabe *et al.*, 2017) who classified textbook analysis research considering the cross-national studies. For example, Charalambous *et al.* (2010) categorized research on cross-national textbook analysis into three main sub-categories: horizontal, vertical, and contextual. On the other hand, Li *et al.* (2009) classified textbook analysis research based on contents of textbooks such as micro level analysis and macro level analysis. Further, Watanabe and his colleagues' (2017) classification of cross-national textbook analysis comprised of overall structures of textbooks, treatments for a particular mathematical topic, and mathematical processes. In the following paragraphs, I discuss the results of the aforementioned studies on textbook analysis in detail.

Regarding the categorization of Fan (2013), studies considering textbooks as a subject of research usually focused on the overall structure of textbooks, such as, size and length of the textbook, use of pictures, sequence of topics, and repetition of topics (Cai *et al.*, 2002; Choi and Park, 2013; Hong and Choi, 2014; Kang, 2014; Li *et al.*, 2009; Son, 2012; Son and Senk, 2010). Though, such studies generally were conducted as cross-national studies in order to examine similarities and differences of different textbooks. Regarding research on the possible factors influencing textbook development, researchers (Fan *et al.*, 2013) emphasized that pedagogical dispositions of an educational system, socio-cultural and political values of a nation, intended curriculum, challenges about technological development and integration of those development into textbooks, as well as textbook authors' perspectives might be accepted as possible factors affecting textbook development. For instance, Fan and Zhu (2007) compared textbooks from China, the USA, and Singapore with respect to problem-solving procedures. They found out that, in contrast to China and the USA, Singaporean textbooks included a separate topic as problem-solving. They interpreted Singaporean textbook authors' approach to problem-solving as a content in mathematics courses to represent the authors' perspectives on mathematics. In other words, Singaporean textbook authors' pedagogical thoughts about "problem solving" affect the way of presentation of problem solving in the textbooks.

Regarding studies in which textbooks were considered as an independent variable affecting other factors include two common assertations: That there may be some influences of textbooks on students' achievement (e.g., Son and Senk, 2010; Fan and Zhu, 2007; Hadar, 2017; Gracin, 2018; Remillard *et al.*, 2014) and that examining textbooks could be considered as learning sources for teachers rather than students since teachers use textbooks as a source such as for planning their lessons, studying topics, selecting activities, and assigning homework (e.g., Son and Kim, 2015; Ma, 1999; Ball and Cohen 1996; Remillard, 2005; Özgeldi, 2011). For example, Hadar (2017)'s study examined two 8th grade mathematics textbooks from Israel to investigate possible effects of cognitive demands of textbooks on students' achievement. For that, after two groups of students got instruction, one group having the instruction depending on the lower cognitive demand textbook and the other having the instruction depending on the higher cognitive demand textbook, Hadar compared the national examination scores of students. She found a correlation between students' scores on standardized national examination and learning opportunities presented in the textbooks; hence, she concluded that one of the textbooks had higher cognitive demands than the other and students using the textbook including higher cognitive demand tasks got higher scores. Hence, the study showed that textbooks might have an impact on students' achievement.

Regarding Charalambous and his colleagues' (2010) classification of cross-national textbook analysis, *horizontal analysis* studies dealt with the whole textbook, i.e. the overall characteristics of textbooks. Investigation of textbooks with its structural components, for instance, lists of topics, topic placement, textbook size/ length, and use of technology in the book was determined as horizontal analysis. Such analysis also corresponds with Li and his colleague's (2009) definition of "macroanalysis" which provides an overview about the topic placements in terms of the grade levels across different educational systems. However, such analysis is acknowledged as being weak in terms of deducing the learning opportunities in the textbooks. With the needs of expending knowledge on the learning opportunities presented in textbooks, researchers decided to select a specific topic to elaborate on the analysis, which was also called *vertical analysis*. In other words, studies aiming to figure out how a particular mathematical topic was treated and what/how was the mathematical processes planned for the topic were assorted as vertical analysis (Charalambous *et al.*, 2010). Such analysis also corresponds with "microanalysis" (Li *et al.*, 2009). Although vertical analysis is more representative in terms of learning opportunities in the textbooks than horizontal analysis, researchers (Watanabe *et al.*, 2017; Charalambous *et al.*, 2010; Li *et al.*, 2009) argued that it has still some deficiencies about representativeness, such that focusing on only a topic for the analysis contradicts with the integral nature of mathematics (Watanabe *et al.*, 2017). For instance, a study of learning opportunities on functional relationships might not fully grasp all the opportunities given in functions unit because there are also opportunities in earlier grades about multiplicative relationships, rate, ratio and proportions. Therefore, it is not enough to examine functions for functional relationships in textbooks rather it is more valuable to examine the related topics as well (Thompson and Carlson, 2017; Carlson *et al*., 2002; Moore *et al.*, 2009; Thompson, 1994b). Therefore, researchers suggested using both the horizontal and vertical analysis with the inclusion of the analysis of the related topics (Watanabe *et al.*, 2017).

Similar to Charalambous *et al.* (2010) and Li *et al.* (2009), Watanabe and his colleagues (2017) classified cross-national mathematics textbook analysis under three categories: overall structures of textbooks (Son and Diletti, 2017; Fan, 2013; Fan *et al.*, 2013), treatments for a particular mathematical topic (Watanabe *et al.*, 2017; Cai *et al.*, 2001; Son and Senk, 2010; Li, 2000; Charalambous *et al*., 2010), and mathematical processes (Fan and Zhu, 2007; Mayer *et al.*, 1995). What is different in their categorization is that they divided vertical analyses, i.e. microanalyses, further into two categories dealing with the treatments of a specific topic and mathematical processes.

For instance, Fan and Zhu (2007) stated that textbook analysis studies about problem solving generally are about representation of problem types (e.g. Zhu and Fan, 2006); hence textbooks' presentation of problem-solving procedures has not been often examined. Therefore, Fan and Zhu (2007) examined Chinese, Singaporean, and American textbooks in terms of problem solving procedures, which could be classified as textbook analysis regarding mathematical processes.

Although, there are different reasons to categorize textbook analysis research differently, the categories have overlaps in significantly similar ways: The vertical analysis aligns with the microanalysis; the horizontal analysis aligns with the macro analysis. Though, as stated earlier, both horizontal and vertical analyses have some strengths and weaknesses. Hence, researchers (e.g., Charalambous *et al.*, 2010; Li *et al.*, 2009; Watanabe *et al.*,2017) tend to combine them by conducting horizontal analysis for particular textbooks such that the textbooks first are investigated with their structural components to get an insight about the educational system in which they were developed; then, analyzing treatments of a particular topic in the textbooks is put under investigation.

In this study, the main purpose is to examine the mathematical processes, namely quantitative and covariational reasoning, and the treatment of a particular topic, specifically functional relationships. Therefore, the study might be classified as vertical or micro level analysis. In addition, taking into consideration the researchers' suggestions, horizontal analysis, i.e., macroanalysis, is also conducted in this study. In the following heading, I discuss the results of previous research that was conducted using both types of analysis simultaneously.

2.1.2. Research on Textbooks Conducted with Horizontal and Vertical Analysis

There has been a vast variety of research combining the horizontal, i.e., macro level, and the vertical, i.e., micro level, analyses. Particularly, researchers examined primary and lower secondary level of mathematics regarding treatments of fractions and fraction operations (Charalambous *et al.*, 2010; Son and Senk, 2010; Watanabe *et al.*, 2017); treatments of arithmetic average (Cai *et al.*, 2002), treatments of ratio and proportion (Lo *et al.*, 2001), treatments of addition and subtraction of integers (Li, 2000), treatments of geometry topics (Jones and Fujita, 2013; Miyakawa, 2017; Takeuchi and Shinno, 2019), and treatments of algebraic thinking, such as distributive property (Ding and Li, 2010). In addition, although high school mathematics topics are minority in textbook research field (Son and Diletti, 2017), some studies were conducted focusing on secondary level mathematics topics such as, quadratics (Sağlam and Alacacı, 2012), functions (Yavuz and Baştürk, 2011) and linear functions (Wang *et al.*, 2017). Though in general, studies were conducted to compare high achieving Asian counties' textbooks with others (Lo *et al.*, 2001; Cai *et al.*, 2002; Watanabe *et al.*, 2017; Son and Senk, 2010; Sağlam and Alacacı, 2012). There are also studies considering a specific country's textbooks to investigate learning opportunities based on some criteria or frameworks (Kul *et al.*, 2018; Gracin, 2018) as well as summarization studies having subject of a particular country like the USA (Usiskin, 2013), the UK (Howson, 2013), and Turkey (İlhan and Aslaner, 2019). In order to render cross-national textbook analysis research, I now briefly explain some of the studies and their conclusions.

Particularly, Charalambous and his colleagues (2010) inquired treatments of addition and subtraction of fractions in primary school mathematics textbooks from Cyprus, Ireland and Taiwan. They developed and used their framework which included both horizontal analysis by examining background information about the textbook development process and overall structure of the textbooks; and, vertical analysis by investigating mathematical content, practices and attitudes, potential cognitive demands, types of responses, as well as connections between textbooks and mathematics topics and other classroom works. They found that although textbooks of Taiwan and Ireland had more common characteristics with each other compared with the textbooks of Cyprus, the topic of addition and subtraction of fractions was developed based on *numbers* in Taiwanese and Cypriot textbooks, but on numerical *operations* in Irish textbooks. Specifically, presenting fraction operations of addition and subtraction with similar and dissimilar denominators, they have argued that the learning perspectives of Irish textbooks are mathematical rather than cognitive (Charalambous *et al.*, 2010). Also, they revealed that Taiwanese textbooks had higher cognitive demands than the others and, differently from Cypriot and Irish textbooks, they systematically include “how much” and “how many” questions for a given situation in the problem.

Similarly, Son and Senk (2010) conducted a research about presentation of multiplication and division of fractions in a USA and a Korean textbook. They used an analytical framework to examine both the content (i.e. when/how the topic was introduced, what were the learning goals, the ways conceptual and procedural understanding were developed) and the nature of problems (i.e. how much opportunities were provided for the advancement of conceptual and procedural understanding, what/how the problems were introduced, what were the expectations). As a result of their study, they hypothesized five features of Korean textbooks:

> (1) more time allocated to key topics, (2) opportunity to learn both concepts and procedures simultaneously, rather than sequentially, (3) opportunity to learn additional computational strategies, (4) opportunities to engage in more challenging problems, and (5) opportunity to engage in the use of various models in problems, as well as in lessons. (Son and Senk, 2010, p. 136)

They suggested to test these hypotheses and further proposed that these characteristics might have contributed to the explanation of advancement of Asian students' performances on international examinations.

In addition, Watanabe and his colleagues (2017) conducted a research study to examine textbooks of Japan, Korea and Taiwan in terms of the treatment of fractions and fraction operations. They analyzed the textbooks by considering the sequence and grade placement of fraction concept, the treatments of addition and subtraction in terms of the problem types, the use of diagrams, and the target algorithms. Then they used Cognitively Guided Instruction framework to analyze the word problems. They found that although there were some grade-based differences, topics were introduced in a similar order and all of the books indicated the idea that non-unit fractions were formed by unit fractions. In addition, although addition and subtraction operations had similarities, multiplication and division operations included some differences among the textbooks under investigation. For instance, the Japanese textbook series differentiated from both the Korean and Taiwan textbooks with the aspect of developing the algorithm of $\frac{a}{b} \div n = \frac{a}{b \times n}$, while the others supported the invert and multiply algorithm. Moreover, the word problems of Japanese textbooks about division of fractions were stated as partitive division problems whereas Korean and Taiwanese textbooks use quotative division problems. Japanese textbook series also used the same problem context for both the introduction of multiplication and division of fractions. Similarly, Japanese textbooks did not wait to finish multiplication of fraction concept to give a place to the division of fraction concept. However, the other two series introduced the division of fractions with a supply of different word problems after the multiplication of fraction concept was over. Researchers suggested further textbook analyses to study other related topics in those textbooks such as, multiplication and division of decimal numbers, ratios, and proportions.

Furthermore, Cai and his colleagues (2002) studied the treatments of arithmetic averages in K-12 education with respect to the US, Chinese, Japanese, and Taiwanese series. They conducted the analysis by considering procedural and conceptual understanding of the arithmetic average concept as an algorithm and as statistics in order to describe, interpret, and compare data sets, respectively. They found that the Asian series centered upon procedural and conceptual understanding of arithmetic average concept as an algorithm, whereas the US series they used for the analysis focused more on the statistical aspects of the concept. Nonetheless, they claimed that the interpretation about the effectiveness of the textbooks might not be done due to the lack of examination of both the Asian and the US students' understandings of the arithmetic average concept, except Cai's (2000) findings revealing that Chinese students are better than American students in computation of arithmetic averages. Cai *et al.* (2002) claimed that the difference about the introduction of average might result from the language difference because of the Japanese and Chinese word corresponding "average" and "mean" are not defined separately.

Finally, Fan and Zhu (2007) investigated mathematical problem solving procedures of Chinese, Singaporean, and the USA lower secondary mathematics textbooks to present the strengths and weaknesses of these three textbook series to inform teachers, policy makers, and curriculum developers, and textbook authors. They used Pólya's four stage model of problem solving (1971) and 17 specific problem-solving heuristics, such as restating the problem, making a table, looking for a pattern, etc., suggested in national syllabuses and standards as their framework to conduct the analysis. They found that Chinese series placed all stages of the Pólya's model in any problem solution, but the USA textbook gave place on only "solution" without any specification about the stages. Also, there was not any specification about stages in Singaporean textbooks. In addition to the analysis with respect to Pólya's model, they found that Chinese textbooks included eleven problem-solving heuristics, while Singaporean textbooks contained sixteen and the USA textbooks had fourteen out of the seventeen heuristics listed. In total, nine of the heuristics were found common in all the textbooks. Therefore, they concluded that Chinese textbooks had a strength on introducing problem solving with the consideration of Pólya's model, despite not stating heuristics specifically on their syllabuses. Also, the USA textbooks had some differences from others as they gave more importance on the visualization and restatement of the problems.

Similarly, Sağlam and Alacacı (2012) examined Singaporean, Turkish and the International Baccalaureate Diploma Programme (IBDP) mathematics textbooks in terms of quadratics units, including quadratic equations, inequalities, and functions. Their content analysis included both qualitative and quantitative research characteristics and they found that each textbook had different weights, time allocations, and position of units of quadratics. They claimed that the position of the topics in the textbooks suggested connections of the topics with other related topics. For instance, quadratics unit was presented just after polynomials in Turkish textbooks which indicates that quadratics are approached as a kind of polynomials. Similar to Turkish textbooks, polynomials also formed a basis for quadratics in Singaporean textbooks. IBDP textbooks differed from the others by connecting the quadratics with binomial theorem. Their results also suggested that mathematical issues were presented separately within different chapters in Turkish textbooks, rather than providing space for students to work on and investigate the mathematical topics by themselves.

Moreover, Kul and his colleagues (2018) examined Turkish and Canadian middle school textbooks in terms of the question types used in the textbooks, the cognitive and the knowledge dimension of the questions. They conduct content analysis using the Synthesized Bloom Taxonomy which enabled to investigate the textbooks by both cognitive and knowledge dimensions. They found out that both textbooks were similar in terms of the cognitive and the knowledge dimensions of the questions; however, Canadian textbooks placed more constructed answer type questions requiring higher cognitive skills.

In addition, Yavuz and Baştürk (2011) investigated the organization of the concept of functions by analyzing two Turkish and two French mathematics textbooks. They found that functions were presented after the sets unit in Turkish curriculum, which indicated the intended connection of the set theory with the definition of functions. Whereas, in the French curriculum the topic of functions was presented after the number unit and the curriculum did not have a unit for the concept of sets. They further found that Turkish textbooks introduced the topic starting with activities, but French textbooks introduced the topic by a section called recalling prior knowledge and provided activities of grade-8 topic on of linear functions to prepare students for the functions topic in the 9^{th} grade. Moreover, French curriculum and textbooks emphasized the use of tables to examine the changes and values of functions, graphs and algebraic representation by explaining the use of these representations detailly. However, there was not found any such detailed explanation about the use of tables in Turkish textbooks. Also, French textbooks gave place to increasing and decreasing functions at the beginning of high school. Nevertheless, Turkish textbooks placed the increasing and decreasing functions within the topic of differentiation which was discussed during the last year of high school.

In sum, as the aforementioned research had pointed as well as Son and Diletti's (2017) study on 31 articles about cross-national textbook analysis between the USA and five high achieving Asian counties, namely, Korea, Japan, China, Singapore, and Taiwan, the macroanalyses displayed that elementary school topics were subject to most of the studies. In particular, fractions were the most frequently studied topic; and following it, whole number operations and algebraic thinking were also common topics among researchers. Further, researchers suggested textbook analyses to study topics such as ratios and proportions (Watanabe *et al.*, 2017).

Studying ratios and proportions would inform researchers examining fractions, functional relationships, numbers, division and multiplication, and multiplicative relationships (Watanabe *et al.*, 2017; Thompson, 1994a; Lo *et al.*, 2001). Moreover, ratio, rate, proportion and linear functions, which are classified under functional relationships, are important for students' mathematical conceptions not only for elementary and middle school but also high school and college (Thompson, 1994b; Carlson *et al.*,2002; Moore *et al.*, 2009; Thompson and Carlson, 2017). Thus, examining the curriculum materials, i.e. national standards, textbooks, and teacher manuals, might be beneficial to inform about how these topics presented in textbooks and in what ways the presentation of the topics might intendedly trigger students' reasoning/ understanding. Thompson and Carlson (2017) suggested examining curricula materials in terms of quantitative, variational and covariational reasoning contributing to students' functional understanding. They claimed that students who do not have variational and covariational reasoning abilities experience difficulties about functional relationships (Thomson and Carlson, 2017; Carlson *et al.*, 2002). Similarly, variational and covariation reasoning are essential for the mathematical development of students (Thomson and Carlson, 2017). Also, these reasoning abilities are accessible for not only high school and college students dealing with higher level mathematics but also for elementary and middle school students (Thompson, 1994b). Similarly, Thompson (1994b) claimed that Japanese elementary mathematics curricula promotes students' conceptualizations of dynamic situations involving continuous variation; however, neither covariational view of functions is emphasized in K-14 mathematics education nor the variation is provided in the textbooks of K-9 education in the US. Thus, in this study, I examine functional relationships presented in elementary and middle school curricular materials in terms of quantitative, variational and covariational reasoning frameworks. In the following sections, I explain functional relationships and the importance of having advanced conceptions of it.

2.2. Functional Relationships

Functional relationships are defined by correspondence relationships between elements of two sets as well as representing covariation in quantities (Blanton *et al.*, 2011). In this study, definition of functional relationships through covariation is considered since covarying quantities allow students to work with variables as varying quantities (Blanton *et al.*, 2011) as well as supporting the definition of functional relationships as correspondence. Functional relationships exist in not only mathematics but also physics, chemistry, biology, engineering and even in daily life. Specifically, understanding of functional relationships is essential for collegiate mathematics (Thompson and Carlson, 2017; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008). For example, calculus topics like limit, continuity, derivative, and differential equations require the understanding of functional relationships (Cooney *et al.*, 2010; Thompson and Carlson, 2017; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008; Moore *et al.*, 2009; Leinhardt *et al.*, 1990). However, research have revealed that undergraduate students start their collegiate study with weak understanding of functions which later also leads to deficiency about making sense of the models of dynamic events (Thompson, 1994; Carlson *et al.*, 2002). Therefore, researchers (Thompson and Carlson, 2017; Carlson *et al.*, 2002; Moore *et al.*, 2009; Thompson and Smith, 2007; Leinhardt *et al.*, 1990; Oehrtman *et al.*, 2008) suggested designing instruction and curriculum in ways to improve the understanding of functional relationships.

Researchers also suggested that studying ratio, rate, proportion, linear functions, and quantitative relationships could contribute to students' understanding of functional relationships (Thompson and Carlson, 2017; Thompson, 1994; Thompson and Smith, 2007; Lobato *et al.*, 2010). This is because not only these topics closely link with functions, but also these topics could be discussed in elementary and middle school years (Thompson and Carlson, 2017; Thompson, 1994; Thompson and Smith, 2007; Lobato *et al.*, 2010). For instance, Thompson (1994) proposed to focus on rate to develop students' understanding of functional relationships and dynamic events starting with earlier grades rather than waiting until undergraduate study since interpretation of dynamic events especially has a crucial role in understanding calculus (Carlson *et al.*, 2002). Similarly, Leinhardt *et al.* (1990) emphasized that understanding the logical relationships between quantities in a proportion like a/b= c/d is necessary for functional understanding. Furthermore, Thompson (1994) stated that it is not possible to study the development of concept of function without considering the other related topics since human thinking is not isolated concept by concept.

Considering the aforementioned reasons, in this study I specifically focus on the topics of ratio, rate, proportion, and linear functions while I examine textbook series of Japan and Turkey in terms of functional relationships. Thus, in the following headings, first I discuss linear functions and then focus on ratio, rate, and proportion. Introducing these topics by giving definitions, I point to previous research on students' conceptions of the topics. I also share different types of representations under the heading of linear functions since representations have an important role on the students' understanding of linear functions (Oehrtman *et al.*, 2008; Leinhardt *et al.*, 1990). Regarding ratio, rate, and proportion, I further express the necessary constructs in order to make the definitions more clearly. Again, fed by previous research, I present the strategies used by students to approach problems related to ratio and rate, and task variables within ratio, rate, and proportion problems because of their link with students' understanding. Doing so might contribute to examining the problem situations and tasks provided in the textbook series in order to reveal how the textbooks approach these topics and what is intended by presenting the topics in particular ways.

2.2.1. Linear Functions

In this section, I first define function and linear function, then discuss research on students' conception of functions, and different types of representations.

2.2.1.1. Definition and conceptions of functions. There are two ways of defining functions: one refers to covariation between two variables and the other refers to correspondence between two sets (Leinhardt et al., 1990; Cooney et al., 2010; Thompson, 1994b). Although mathematicians accept the definition of function as a correspondence between two sets, namely domain and range, and textbooks widely present function by referring to the correspondence definition (the modern definition), researchers (Thompson, 1994b; Carlson et al., 2002; Oehrtman et al., 2008; Smith, 2003; Cooney et al., 2010) suggest introducing covariation definition of functions to students. Specifically, Oehrtman and his colleagues (2008) asserted:

> to use the modern definition of function in an introduction to the function concept is to present students with a solution to problems of which they cannot conceive. We recommend that school curricula and instruction include a greater focus on understanding ideas of covariation and multiple representations of covariation (e.g., using different coordinate systems), and that more opportunities be provided for students to experience diverse function types emphasizing multiple representations of the same functions. College curricula could then build on this foundation. This would promote a more flexible and robust view of functions – one that does not lead to inadvertently equating functions and formulas. (p. 4)

In particular, correspondence definition of function refers that for every element in one set (domain) corresponds to the elements in the other set (range) and it applies in many situations in mathematics (Thompson, 1994b). Whereas covariation definition involves the relationship between the two varying quantities relating one another (Cooney *et al.*, 2010). Researchers argued that students easily make sense of covariational situations since they could recognize them in real life situations and that covariation view of function can contribute to the development of correspondence relationship (Thompson, 1994b; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008; Smith, 2003; Cooney *et al.*, 2010). Similarly, some researchers found that covariational reasoning contributes to students' conception of function which involves dynamic problem situations albeit that covariation definition falls short for some mathematical situations (Saldanha and Thompson, 1998; Carlson *et al.*,2002; Oehrtman *et al.*, 2008). Yet, such situations are not considered in primary and middle grades, rather they are in more advanced level mathematics (Oehrtman *et al.*, 2008). Moreover, covariation meaning of functions is considered to be appropriate for modeling situations (Weber, 2012). Regarding the covariation perspective of function, researchers also emphasized the need to focus students' attention on rate of change to represent real world phenomena, since it indicates the covariation between two variables (Cooney *et al.*, 2010). Specific to linear functions defined in Real Numbers written as f(x)= mx+b, or y=mx+b symbolically, the rate of change refers to the invariant, i.e. constant, "m" called also as slope or steepness. In direct proportions as a subset of linear functions with the expression y=mx, "m" also represents the invariant relationship between the two quantities.

Taking attention to the covariation definition of functions, previous research revealed students' conception of functions. Particularly, using APOS theory (Dubinsky and McDonald, 2001) research has pointed that undergraduate students' conceptions of functions could be classified as action conception, process conception and object conception (Weber, 2012). Specifically, Dubinsky and Harel (1992) stated that action conception of functions refers to using algebraic formula of functions to calculate the result of any number exposed to the functional relations. Therefore, action conception is considered as static in nature. This suggested that a student having action conception of functions would not think many steps at a time (Dubinsky and Harel, 1992). In other words, although the action conception's corresponding meaning for quantitative reasoning refers to "… thinking of two quantities varying in tandem, which does not require a symbolic representation." (Weber, 2012, p.26) and that students having action conception and using quantitative reasoning could *recognize* the invariant relationship between two quantities, action conception still does not contribute to the thinking about invariant relationships (Weber, 2012).

Process conception is more advanced than action conception such that “a process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity.” (Dubinsky and Harel, 1992, p.85). Therefore, students having process conception recognize a function as a dynamic input-output process (Thompson, 1994b; Weber, 2012; Kabael, 2010; Moore, 2010). The differences between action and process conceptions were further scrutinized (Oehrtman *et al.*, 2008) (See Table 2.1).

Table 2.1. Conception of functions (Oehrtman, Carlson and Thompson, 2008, p.10).

Action view	Process view
"A function is tied to a specific rule, formula, or computation and requires the completion of specific computations and/or steps." (p.10)	"A function is a generalized input-output process that defines a mapping of a set of input values to a set of output values." (p.10)
"A student must perform or imagine *each action*." (p.10)	"A student can imagine the *entire process* without having to perform each action." (p.10)
"The "answer" depends on the formula." (p.10)	"The process is independent of the formula." (p.10)
"A student can only imagine a single value at a time as input or output (e.g. x stands for a specific number)." (p.10)	"A student can imagine all input at once or "run through" a continuum of inputs. A function is a transformation of entire spaces." (p.10)
"Composition is substituting a formula or expression for x." (p.10)	"Composition is *coordination* of two input-output processes; input is processed by a second function." (p.10)
"Inverse is about algebra (switch y and x then solve) or geometry (reflect across y=x)." (p.10)	"Inverse is the *reversal of a process* that defines a mapping from a set of output values to a set of input values." (p.10)

Table 2.1. Conception of functions (Oehrtman, Carlson and Thompson, 2008, p.10) (cont.).

"Finding domain and range is conceived at most as an algebra problem (e.g. the denominator cannot be zero, and the radicand cannot be negative)." (p.10)	"Domain and range are produced by operating and reflecting on the set of all possible inputs and outputs." (p.10)
"Functions are conceived as static." (p.10)	"Functions are conceived as dynamic." (p.10)
"A function's graph is a geometric figure." (p.10)	"A function's graph defines a specific mapping of a set of input values to a set of output values." (p.10)

In addition to action and process conception, there is also object conception. An object conception of function requires to think about the correspondence of the input to the process and the outputs from the process so that one can start to reason formally about functions (Dubinsky and Harel, 1992). That is, a process is an interiorized action, and an action is "any repeatable physical or mental manipulation that transforms objects (e.g., numbers, geometric figures, sets) to obtain objects" (Breidenbach *et al.*, 1992, p. 249). Therefore, if a process is transformed by an action, the process is encapsulated to become an object (Breidenbach *et al.*, 1992). Thus, the learner can reason about functions as objects (Thompson, 1994b).

While I examine functional relationships in this study, I consider action and process conception of functions since referring to higher levels of abstraction, object conception of functional relationships might not be expressed in elementary and middle school level. Particularly, considering the Table 2.1. about action and process conceptions of functions, it suggests that the process conception of functions allows students to recognize the dynamic input-output processes so that they do not need to perform actions to understand the whole process. Such conception provides students with thinking about mapping between input and output elements, which contributes to the correspondence view of function. In addition, since students do not need to perform all actions to recognize the process of the function, process conception enables to conceive functions as dynamic in nature, which promotes covariation view of functions (Weber, 2012). Specifically, if students recognize the dynamic nature of a linear function symbolized as $y = mx$, s/he does not need to perform all the actions to determine the domain and range, because they realize that the mapping between input and output constitutes an invariant relationship.

2.2.1.2. Functional representations. There are three main representations of functions generally used in school mathematics: algebraic, tabular, and graphical. Graphs are the most common representation type used in schools, because they illustrate the behavior of functions and ease explanations of increasing, decreasing, maxima, minima, concavity, and inflection points (Selden and Selden, 1992). However, students generally consider graphs as representing images of physical situations, rather than focusing on the mapping of between inputs and outputs, which contributes to their success in advanced mathematics (Oehrtman et al., 2008). The objects for functions and graphs are variables (Leinhardt et al., 1990). There are two ways to define variables: First, variables are thought to express patterns, and second, variables are approached by regarding the simultaneous change in one corresponding to the other (Leinhardt et al., 1990). The former one is static definition, while the latter one is dynamic.

Researchers emphasized the need to represent functions through different representations (tabular, graphical, and algebraic) (Oehrtman *et al.*, 2008; Leinhardt *et al.*, 1990; Cooney *et al.*, 2010). Especially the graphical and algebraic representations of functions are intertwined with each other; so, researchers claimed that it is important to present functions considering both representations (Leinhardt *et al.*, 1990; Cooney *et al.*, 2010). In fact, using covariational reasoning while introducing the function concept provides students with making sense of different representations, such as tables, graphs, and algebraic formula since relationships between varying quantities could be displayed with these representations (Saldanha and Thompson, 1998; Carlson *et al.*,2002; Oehrtman *et al.*, 2008). For instance, using tables might trigger students to recognize the covariation between variables, since the values of variables are listed correspondingly. Therefore, considering the emphasis on the contribution of different representations on students' conceptions of functions (Oehrtman *et al.*, 2008; Leinhardt *et al.*, 1990; Cooney *et al.*, 2010), while I examine how covariational reasoning is depicted in the Japanese and Turkish textbooks, I focus also on the ways the curricular materials use different representations, the explanations about representation types, and connections between different types of representations.

Aforementioned research suggests the inclusion of the definition of functions in school curriculum as a covariation between variables since it also allows to grasp functions as a correspondence between the elements of two sets. Similarly, the covariation definition of functions also supports the understanding of the invariant structure in the mathematical situation such as the rate of change of the variables in linear functions. Furthermore, covariational definition of functions possible to be triggered through the process conception supports students' making sense of the different representations of functions. Therefore, as I have mentioned earlier, in this study, I will be focusing on the covariational definition of functions, specifically the linear functions, while examining functional relationships in Japanese and Turkish curricular materials.

Due to its importance in the construction of functional relationships, in the following paragraphs, I explain ratio, rate and proportion.

2.2.2. Ratio, Rate, and Proportion

Ratio and proportion are the core concepts in which students' multiplicative reasoning is both required and developed (Lobato *et al.*, 2010) in the elementary and middle school years (Heinz,2000; Lobato *et al.*, 2010; Simon and Placa, 2012). Multiplicative reasoning further is essential for students' mathematical development and supports understanding of functions and other collegiate level mathematics (Blanton *et al.*, 2011; Thompson, 1994; Smith and Confrey, 1994). Moreover, focusing on multiplicative structures constitutes proportional reasoning which is essential for the conception of secondary level mathematics (Izsak and Jacobson, 2017). The National Council of Teachers of Mathematics (NCTM, 1989) asserted:

> The ability to reason proportionally develops in students throughout grades 5-8. It is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning. (p.82)

Proportional reasoning is defined as the reasoning in situations involving invariant relationships between two covarying quantities (Lamon, 2012). However, there is no consensus among researchers about the definition of rate and ratio (Heinz, 2000; Karagöz Akar, 2007). In this study, pointing to and acknowledging different researchers' definitions of ratio and rate, I will be specifically adopting Thompson's (1994a) definition of ratio and rate. Also, being considered one of the most complex, difficult to learn and hard to teach concepts (Lamon, 2007), it becomes imperative to define some other constructs inherent in ratio and rate. Therefore, in the following sub sections, I first define the necessary constructs about ratio, rate, and proportion. Then, I share definitions of the concepts. Following, fed by previous research results, I discuss students' conceptions and strategies while approaching problem situations regarding these concepts. Lastly, I talk about the task variables and explicate the relation of these concepts with functional relationships.

<u>2.2.2.1. Necessary constructs for rate, ratio and proportions</u>. Due to their importance for the understanding of the definitions regarding ratio and rate conceptions and strategies, I first explain some constructs such as quantity, quantitative operations and numerical operations, covariance and invariance, within ratios and between ratios and repeated addition and multiplication.

<u>*2.2.2.1.1. Quantity, quantitative operations and numerical operations*</u>. In mathematics education, there are two different approaches to define what quantity is (Heinz, 2000). One

is focusing on formal analysis of a quality of an object and the measure of the quality of the object from the point of view of such as an advanced student or a mathematician (Schwartz, 1988). The other one is focusing on a learner's conception of a situation rather than a formal analysis (Thompson, 1994a). Since in this study I use quantitative reasoning framework (Thompson, 1994a, 2017) as my theoretical framework, I will be discussing the related terminology about quantity and mental operations in the theoretical framework part. The reason I further share in this section Schwartz's (1988) contribution and Thompson's (1994a) contribution about quantity is to examine the way the textbooks present topics and their intents to trigger students' mental operations. I now explain both approaches briefly.

According to Schwartz (1988), quantities can be derived by counting or measuring, as well as, conducting some operations with already derived quantities. Every quantity has a referent, and a new quantity could have the same or different referents compared with the quantities constituting it. If the operation used to get a new quantity allows all the quantities to share the same referent, the operations are called "referent preserving composition". Addition and subtraction are examples of these operations. However, if the operation used to compose a new quantity bring along a new quantity having a different referent from the quantities constituting it, the operation is defined as "referent transforming composition". Multiplication and division operations are examples of referent transforming composition. For instance, if we add the number of 3 pencils to the number of 5 pencils, we get the number of 8 pencils and it has the same referent with the earlier quantities. If we divide the number of 18 pencils to the number of 6 students to find the number of pencils per student, we get 3 pencils per student and this new quantity has a different referent compared to the others. The former is an example of a referent preserving composition, while the latter example is called referent transforming composition. Schwartz (1988) argued that these two types of compositions are important tools for modeling mathematical situations, thus they contribute to the mathematical learning and teaching.

Schwartz (1988) also defined two different type of quantities called "extensive" and "intensive" quantities because of the referent transforming compositions. "An extensive quantity is one that indicates how much, or the extent, of a given quality of an object" (Heinz, 2000, p. 19). An extensive quantity can be gained by direct measurement and its referent is a single entity, such as, 3 minutes, 2 kilograms, 5 nuts. As it is seen by these examples, an extensive quantity indicates amount of a quality of an object. On the other hand, an intensive quantity is not obtained by directly counting or measuring and its referent includes two entities. An intensive quantity describes a quality of an object and does not express the amount of the quantities (Schwartz, 1988). Schwartz (1988) argues that the use of "per" in the expression of a quantity is an indication of an intensive quantity, but there is no requirement such that every intensive quantity includes the expression of "per". For instance, speed, acceleration, power etc. express the relationships between two quantities, so all of them are intensive quantities. Specifically, speed indicates the relationship between distance and time (two extensive quantities). Acceleration expresses the relationship between speed (an intensive quantity) and time (an extensive quantity). Power indicates change in mechanical energy (an intensive quantity) per time (an extensive quantity). A person' use of extensive and intensive quantities indicates how s/he makes sense out of quantities and operations that s/he performed with quantities (Simon and Blume, 1994).

Differently from Schwartz's (1988) classification of quantities, Thompson (1994a) emphasized that "Quantities are conceptual entities. They exist in people's conceptions of situations" (p.184). By this definition, he points out an individual student's understanding could be different from others and a quantity depends on how an individual makes sense out of the quality of an object. Further, he asserted that to define a quantity, one does not need to measure the quality of the object. For instance, one can compare the heights of two persons and determine that one is higher than the other without actually measuring the exact amounts of the heights of two persons. According to Thompson (1994a), there are two different mental operations: quantitative and numerical operations a person acts upon during the formation of quantities. Quantitative operations are used to create a new quantity; whereas numerical operations are used only to make an evaluation (i.e. measure) about quantities. Some examples of quantitative operations are unitizing, segmenting, measuring, grouping, equal partitioning, counting, and matching (Karagöz Akar, 2016; Thompson 1994a). Also, covariance and invariance of quantities play an important role in the formation of intensive quantities such as ratio and rate. In the following sub-section, I briefly explain these terminologies and express the relationship of these terms with quantity and the concepts of ratio, rate, and proportion. Then under the theoretical part, I explain them thoroughly.

2.2.2.1.2. Covariance and Invariance. Understanding quantities and the relationships between quantities are mental activities bonded with both variational and covariational reasoning (Moore, 2014). A person's conception of the varying value of a quantity within a context is called variable (Thompson and Carlson, 2017). According to Castillo-Garsow (2010), students can reason variation either discretely or continuously, which further could be categorized as chunky continuous or smooth continuous.

Covariational reasoning is defined "to be the cognitive activities involved in coordination two varying quantities while attending to the ways in which they change in relation to each other." (Carlson *et al.*, 2002, p. 354). For instance: A person uses 2 apples and 5 oranges to get a special taste of a juice. If s/he wants to get more juice of the same taste, s/he might use 12 apples and 30 oranges. Therefore, the change in the number of apples used also requires a change in the numbers of oranges needed to be used in order to get exactly the same taste. This simultaneous change is called *covariation* which Thompson and Thompson (1996) also expressed as "bidirectional relationship" to emphasize the two-way variation (the covariance of quantities). When we consider the taste itself the multiplicative relationship between 2 apples and 5 oranges remains constant in the case of 12 apples and 30 oranges also. This constant relationship is then called *invariance*. Further, the invariance among quantities in ratio and rate bring about the necessity to discuss within and between ratios.

2.2.2.1.3. Within ratio and Between ratio. There is no consistency among mathematics education researchers in terms of the definition of the terms of within ratio and between ratio (Heinz, 2000; Karagöz Akar, 2007). Some researchers (Lamon, 1994) suggest focusing on measure space to determine the ratio either within or between states. For example, let us consider the problem situation: 3 pencils have cost of 5 dollars, and what is the cost of 18 pencils? From these researchers' point of view, if a person considers the situation by focusing on 3 pencils: 18 pencils and 5 dollars: x dollars, the person is focusing on within state ratios since s/he focuses on the same measure space quantities. If the person considers the ratios of 3 pencils: 5 dollars and 18 pencils: x dollars, it is said that the person is considering between state ratios.

However, not every researcher agrees with this kind of classification. Some researchers (Noelting, 1980; Karagöz Akar; 2007) defined within and between state ratios by considering the system. For instance, if a person focuses on the initial situation (3 pencils: 5 dollars) and compared the ratio of initial situation to the ratio of other system (18 pencils: x dollars) to find the missing value, the person is focusing on within state ratio. In other words, the focus is on the system rather than measure space for within state ratio (Noelting, 1980; Karagöz Akar, 2007). Similarly, if a person constitutes ratios by using quantities coming from different systems having the same measuring space, such as 3 pencils: 18 pencils: and 5 dollars: x dollars, the person is focusing on between ratios. In this study, I consider the classification of latter type for within and between ratios.

2.2.2.1.4. Repeated Addition and Multiplication. Students' conception of ratio requires multiplicative reasoning; however, students' understanding of multiplication in ratio concept could be classified multiplication as *repeated addition* and multiplication as *times as large* (Heinz, 2000; Karagöz Akar, 2007). These terms are important to better explain students' conceptions and strategies while approaching ratio, rate, and proportion problems.

Students could perform repeated addition rather than multiplication in multiplicative situations. When examining the problem situations involving 3 pencils' cost as 5 dollars and asking for the cost of 18 pencils, students might approach this problem by repeating the initial system until they reach the intended number of pencils. This suggests that they use repeated addition to find the missing value. In particular, students might think that 3 pencils have the cost value of 5 dollars; 6 pencils have the cost value of 10 dollars; 9 pencils have the cost value of 15 dollars; 12 pencils have the cost value of 20 dollars; 15 pencils have the cost value of 25 dollars; and finally, 18 pencils have the cost value of 30 dollars. Students stop adding the initial value when they reach the intended situation which is 18 pencils such that the missing value is found as 30 dollars. In fact, such reasoning is called identical groups conception (Heinz, 2000). Students generally prefer to use their additive strategies rather than multiplicative ones (Thomson and Saldanha, 2003), for this reason they sometimes experience difficulty to approach problems requires multiplicative strategies. I explain such situations further under the heading of strategies students use to approach different problem situations.

Though, Thompson and Saldanha (2003) argue that multiplication requires more than repeated addition. In order to comprehend multiplication multiplicatively students need to consider the reciprocal relationship between the multiplier, multiplicand, and the product (Thomson and Saldanha, 2003). For instance, a student reasoning about multiplication of a and b, a×b, multiplicatively reasons that the product a×b is a times as large as b, or b times as large as a. Moreover, a is 1/b times as large as a×b, and b is 1/a times as large as a×b (Thompson and Saldanha, 2003; Karagöz Akar, 2007).

<u>2.2.2.2. Definitions of ratio, rate, and proportion.</u> As mentioned earlier, there is no consensus among mathematics education researchers about definition of rate and ratio (Heinz, 2000). Some researchers defined ratio as a binary relation including two ordered pairs of quantities and rate as an intensive quantity (Lesh *et al.*, 1988); while some defined ratio as a relationship between quantities representing the same quality of an object with the same units of measure and rate as a relationship between two different quantities representing different quality (Vergnaud, 1988). Differently from these definitions, Ohlsson (1988) defined ratio as numerical representation of the amount of one quantity by relating it with another quantity; and, he defined rate as a specific ratio which indicates the relationship between a quantity and a time period. However, Thompson (1994a) suggested focusing on learners' mind actions and argued that if the focus was on the mind activities of a person, we would not need a categorization for rate and ratio. He defined ratio as "the result of comparing two quantities multiplicatively" (Thompson, 1994a, p.190). Since the mental action used in this concept is multiplicative comparison, the dimensions of quantities do not play a role to determine if the comparison is ratio or rate. Therefore, the result of a multiplicative comparison having the same or different unit dimension is ratio. "A rate is a reflectively abstracted constant ratio." (Thompson, 1994a, p.18). He particularly stated:

> When one conceives of two quantities in multiplicative comparison, and conceives of the compared quantities as being compared in their, *independent, static* states, one has made a ratio. As soon as one re-conceives the situation as being that the ratio applies generally outside of the phenomenal bounds in which it was originally conceived, then one has generalized that ratio is a rate (i.e., reflected it to the level of mental operations). (Thompson, 1994a, p.19)

This definition indicates that a rate is a linear function in the form of $f(x) = mx$, i.e., linear equation of $y = mx$ (Thompson, 1994a). In other words, rate refers to the set of equal ratios (Lobato *et al.*, 2010). Thompson's definition of rate also resembles the claims of Kaput and West (1994) about rate-ratio. Kaput and West (1994) presented two terms called particular ratio and rate-ratio. They claimed that particular ratio refers to a particular intensive quantity, while rate-ratio corresponds to rate intensive quantity. If there is a situation and the ratio (representing an intensive quantity itself) is used to define this particular situation, ratio is called particular ratio. Rate-ratio differs in a way that the ratio represents the relationship between two measures without specifically indicating a particular situation or the exact value of the quantities under comparison. Since the rate-ratio describes the relationship, Kaput and West (1994) claimed that "m", coming from the form of the linear equation $y = mx$, represents a rate-ratio. The idea behind their definition of m as rate-ratio stems from the work of Karplus and his colleagues (1983) on proportional reasoning, emphasizing that linear functional relationships represent situations by a proportion, i.e. a constant ratio which corresponds to a rate.

In addition, the multiplicative comparison, which is underlying the representation of rate as linear function in the form of $f(x) = mx$, is important for proportional reasoning, because proportionality could be represented by any linear function, which has the expression of $y = mx + b$ (Lobato *et al.*, 2010). A person cannot reason proportionally without conceiving the multiplicative relationship between quantities (Lesh *et al.*, 1988). Ratio, rate, and proportion are necessary constructs for proportional reasoning (Ben-Chaim *et al.*, 2012) such that proportional reasoning is defined as "reasoning in a system of two variables between which there exists a linear functional relationship" (Karplus *et al.*, 1983). The definition of proportional reasoning made by Lamon (2012) earlier in this chapter also aligns with Karplus's definition in such a way that a linear functional relationship is a mathematical model for a direct proportional situation (Lamon, 2012). Hence, proportional reasoning requires to work with constant ratio. Proportion is defined as a second-order relationship between two ratios in that there is a constant ratio between the values of one pair to the other pair (Ekawati *et al.*, 2015; Lo and Watanabe, 1997; Ben-Chaim *et al.*, 2012). In other words, a proportion is symbolized by the equality of two ratios as "a/b=c/d" (Lamon, 2012; Son, 2013). Thus, proportion refers to the constant ratio, i.e. invariant, stemming from comparison of two particular equivalent ratios, whereas rate corresponds to the invariant of which represented by sets of all equivalent ratios (Lobato *et al.*, 2010).

In sum, aforementioned discussion points that ratio, rate, and proportion are part of a big idea of multiplicative comparisons such that they are intertwined with each other and concept development of one has an impact on the other (Lobato *et al.*, 2010). Furthermore, multiplicative reasoning is essential for the conception of ratio and proportion (Simon and Placa, 2012). It also compounds quantities and the relationships between quantities, so it is part of quantitative reasoning (Simon and Blume, 1994). Although it is necessary for concepts of rate, ratio, and proportion, the development of multiplicative reasoning is not an easy process (Schwartz, 1988). Therefore, curriculum and instruction have to be designed in a way to support students' multiplicative reasoning, which constitutes partly of quantitative and covariational reasoning, starting with primary and middle schools, so that students might conceive more complex and collegiate level mathematics better (Thompson and Carlson, 2017).

2.2.2.3. Strategies students use to approach different problem situations. Previous research showed students' use of different strategies while approaching problems in the context of ratio and proportion. They mainly could be categorized as informal and formal strategies (i.e., the multiplicative strategies). In this section, I briefly explain them.

Building up. A person uses build up strategy by coordinately increasing (or decreasing) the two quantities in ratio by preserving the quality of the situation (Heinz, 2000, p. 32). As mentioned earlier, multiplication as repeated addition is used by students using building up strategy, but students' use of multiplication does not indicate that they are thinking multiplicatively. Rather, students use additional reasoning while they are using build up strategy to approach a problem situation. They cannot conceive the invariant multiplicative relationship between quantities while they are using multiplication in building up strategy. In other words, understanding the invariant relationship of within state ratio or between state ratio is not required to use building up strategy (Heinz, 2000).

Similarly, Lamon (1993) claimed that students who successfully perform ratio problems by using building up strategy, could not solve problems requiring understanding of proportionality. Also, Lesh *et al.* (1988) claimed that this strategy is weak in terms of indicating the proportionality; thus, building up strategy is related with pre-proportionality, a term used by Piaget. Still, although building up strategy is not very useful for problem situations involving proportional reasoning, Thompson (1994a) claimed that use of this strategy might contribute to students' interiorization of the ratio concept, i.e. the conception of rate.

Abbreviated building up. Similar to the building up strategy, the underlying reasoning for the abbreviated building up strategy is also additive reasoning. In addition, students using this strategy are not required to recognize the invariant relationships between quantities either. What is different between abbreviated building up strategy and building up strategy is that a person uses a factor to multiply the quantities of the ratio given to reach what is intended. For example, think of the missing value problem situation shared above while discussing repeated addition – 3 pencils: 5 dollars; 18 pencils: x dollars –. The illustration of a student thinking as 3 pencils have the cost value of 5 dollars; 6 pencils have the cost value of 10 dollars… 18 pencils have the cost value of 30 dollars, referred to building up strategy. However, once the students divide the number of 18 pencils to the number of 3 pencils to get the factor of 6; and then, multiply the factor 6 with 5 dollars to get the intended value as 30 dollars, the students use abbreviated building up strategy. That is, students use division and then multiplication to find the repeated amount of the ratio unit in order to get the intended value (Karagöz Akar, 2007).

<u>*Unit factor (per one) approach.*</u> The main aim to use this strategy is to find the unit factor using division and then multiplying the unit factor by the quantity given to find the intended one. Again, thinking of the same missing value problem situation, $5 for 3 pencils, once students use unit factor strategy, they firstly find the cost of 1 pencil dividing 5 dollars by 3. Whenever they get the cost of 1 pencil as $5/3, they then multiply this unit factor by 18 (pencils) and get 30 dollars. According to Heinz (2000), the unit factor strategy could be used to deal with divisibility failure. For instance, if the cost of 17 pencils is asked in the problem situation $5 for 3 pencils, students may not simply find the answer by using building up strategy, or students trying for the abbreviated building up might not easily make sense of the factor 17/3 as representing how many times the unit is repeated, since it represents $5\frac{2}{3}$.

In addition to per one strategy, Lo and Watanabe (1997) presented ratio-unit / build-up method. A student using this method first finding the ratio-unit by using partitive division, and second repeatedly adding the ratio-unit to find intended value (Lo and Watanabe, 1997). By doing so, the student does not need to deal with fractions and decimal numbers. For example, if the cost of 18 pencils is $30 and the cost of 24 pencils is asked, a student using ratio-unit/ build-up strategy starts with finding ratio-unit which is $5 for 3 pencils. Then add these ratio-unit twice to initial situation and get 24 pencils have cost value of $40. The student used extensive quantities to find the answer rather than forming an intensive quantity by using this strategy.

<u>*Incorrect addition.*</u> Students who do not have the conception of ratio, i.e. do not consider the multiplicative relationships between quantities, tend to focus on the difference between the quantities. Regarding the same example, $5 for 3 pencils, if a student finds the difference between 3 pencils and 18 pencils as 15 and use the difference value 15 to find the intended quantity by adding 15 to $5 concluding 20 dollars, s/he uses incorrect addition strategy. Kaput and West (1994) explained the rationale of students using this strategy as follows:

- Since students are encountered mostly with problem situations requiring additive comparison in their early grades, they may use the constant difference to solve problems requiring multiplicative comparison.

- Since students first met with multiplication as repeated addition, they tend to develop additive reasoning which may lead them to use addition or subtraction to find the intended value in problem situations.

Multiplicative strategies. Heinz (2000) defined multiplicative strategies as

> those in which the relationship between two quantities are appropriately represented with a ratio, and then the ratio is applied as needed to the reminder of the applicable data in the problem to either find the missing value or to make comparison across cases in a multiplicative situation. (Hart,1982,1984; Karplus *et al.*, 1974; Karplus and Karplus, 1972; Karplus and Peterson, 1970; Noelting, 1980a; Tourniaire and Pulos, 1985, as cited in Heinz, 2000, p. 44)

There are three multiplicative strategies to use and form ratios to solve problems: within strategy, between strategy, and proportion equation strategy (Heinz, 2000). Within and between strategies require to recognize the invariant relationship between quantities, so students' conceptions are more advanced compared to other strategies shared above. As mentioned before, I use the within and between ratio terms in this study regarding to definition made by Noelting (1980). Differently from Noelting, Heinz (2000) stated the within ratio strategy as scalar strategy compared the quantities having same measuring space, so the ratio created by this comparison is nothing but a scalar. Between strategy is also called as functional strategy since the comparison of different measuring spaces which represents a functional relationship is considered. Proportion equation strategy involves cross multiplication of the quantities involved in a proportion. However, Kaput and West (1994) claimed that the use of this strategy might not indicate that the person reasons proportionally. In other words, a student might be using such strategy without really understanding the multiplicative relationship between the quantities involved in the problem situation.

As mentioned earlier, strategies students use might imply their reasoning and conceptions; therefore, it is necessary for this study to examine the documents in terms of strategies. Particularly, I examine the documents to find out whether the textbooks, teachers' manuals, and curriculum recommend, trigger or placed the use of some strategies tacitly or explicitly. I also examine which strategies are covered in the textbooks, teachers' manuals, and curriculum.

2.2.2.4. Students' conceptions. In the literature, students' conceptions of ratio are categorized as identical group conception and ratio-as-quantity (Heinz, 2000), ratio-as-measure conception (Simon and Blume, 1994), between state ratio conception and within state ratio conception (Noelting, 1980; Karagoz Akar, 2007).

As already briefly stated earlier, the identical groups conception involves students' mental actions of the collection of sets of the extensive quantities or breaking down the ratio into equal parts. Students having identical group conception use addition; and, the repeated addition version of multiplication to find the collection of sets of the quantities or partitive division to form the equal parts from the ratio. In the former, building up or abbreviated building up strategy is conducted to approach the problem presented, while in the latter, unit factor strategy is used to solve problems. Students having identical groups conception understand that there is a quality of interest and they need to sustain this quality such as taste of a mixture or the cost value in a purchase when they try to reach the intended value. Thus, students realize that the relationship between quantities does not stem from the difference, rather the quantities form a quality. However, since students having this conception does not realize the multiplicative relationship between quantities (Heinz, 2000; Karagöz Akar, 2007), they might not solve all ratio problems, especially ones including divisibility failure where quantities in the problem situation are non-integer multiples of each other (Heinz, 2000).

Contrary to the identical groups conception, which is additive in nature, one of the students' conceptions requiring the understanding of the multiplicative relationship is ratio-as-measure (Simon and Blume, 1994). Simon and Blume (1994) defined ratio-as-measure as quantification of a given attribute such as steepness, taste of a mixture, or density. Ratio-as-measure conception requires the understanding of ratio as an intensive quantity such that the ratio itself indicates a measure that cannot be directly measured, such as speed of a car, saltiness of a water, squareness of a region. Since the ratio is recognized as an intensive quantity, the referent transforming composition is used to form it.

Ratio- as- quantity conception also requires multiplicative reasoning, but this conception requires understanding the invariant relationship between quantities indicating amount of change rather than defining a measure (Heinz, 2000). For example, 6 grams of an object with 10 dm^3 volume, has 0.6 g/ dm^3 density. Density is an intensive quantity that cannot be measured directly as mass or volume of the object. If a student conceives the multiplicative relationship between mass and volume and if s/he conceives the co-variational relationship between these two extensive quantities, then s/he constitutes density having the conception of ratio-as-measure. If it is asked to find the mass of the object having 50 dm^3 volume, the student uses the density to find the mass. However, if s/he considers 50 dm^3 is 5 times bigger than 10 dm^3 and has the understanding that 5 indicates amount of change, s/he can multiply 5 by 6 to get the intended mass of the object. This then could be thought as an example of ratio-as-quantity conception.

Researchers also reported on two more conceptions of ratio held by undergraduate students (Karagoz Akar, 2007, 2017). Based on Noelting's (1980) and Lamon's (1993) definitions of between state ratio and within state ratio, Karagöz Akar (2007) presented the differences of between state ratio and within state ratio conceptions from both identical groups and ratio-as- measure conceptions. It is important to discuss about between and within state ratio conceptions, because there is a link between conceptions of between and within state ratios, as well as proportions (Noelting, 1980; Karagöz Akar 2017), and proportions are included in functional relationships (Thompson and Carlson, 2017).

Consider again the missing value problem asking the cost of 18 pencils if 3 pencils have the cost value of $5. Since between state ratios are formed by regarding the quantities having the same measure space from two different situations, 18 pencils:3 pencils ratio is a between state ratio. If a student uses the ratio18/3, or the results of such ratio i.e. 6, and multiply this with $5 to find the intended cost; then she thinks of 6 as a ratio indicating the relationship between two extensive quantities. Such understanding is different from the abbreviated building up strategy in the sense that the student understands that by the ratio of 18:3, the number of pencils (an extensive quantity) is increased by 600%, so well the cost has to be increased by the same proportional amount. Thus, a student having between state ratio conception might understand ratio as both the amount of increment or enlargement/

amount of change in a quantity. Though, students holding between ratio conception might not necessarily understand unit factor approaches (Karagöz Akar, 2007). Therefore, the conception of between state ratio differs from the identical groups conception in which students might use unit factor approaches. Similarly, students having between state ratio conception might handle the divisibility failure the students with identical groups conception cannot deal with (Karagoz Akar, 2017). Regarding within state ratio conception, the student compares the quantities existing in a situation, such as 3 pencils and 5 dollars, to form a ratio. The ratio composed of the two extensive quantity 5/3 might be thought by the students as the cost of one pencil. However, more advanced level of understanding of within ratio constitutes $y = \frac{5}{3}x$, where x denotes the number of pencils and y corresponds to the cost for that amount of pencil. Thus, within state ratio conception both includes and surpass the identical groups conception. Yet, at the early stages of within state ratios, students might not necessarily understand between state ratios as amount of change or enlargement. Therefore, ratio as measure conception surpass and include within state ratio conception (Karagoz Akar, 2007).

Regarding the strategies student use, students having between state ratio conception could use multiplicative strategies, such as within strategy, and handle the divisibility failure (Karagöz Akar, 2007); however, as already mentioned students having identical groups conceptions are not able to deal with divisibility failure (Heinz, 2000). For instance, think of the example asking the cost of 19 pencils when 3 pencils have the cost value of $5. Since the question has divisibility failure, students having identical groups conception are not able to solve it, but students having between state ratio conception might use the within ratio, 5/3, to multiply with 19 to find out the answer. Although they might use the ratio, 5/3, they do not recognize the multiplicative relationship between the quantities, rather they might think the ratio 5/3 as an expression of the association between the quantities. The understanding of 5/3 representing the multiplicative relationships between the quantities necessitates ratio as measure conception or later stages of within state ratio conception. Moreover, students having identical groups conception might use per one approach (i.e., the unit factor) in problem situations and find the corresponding value of a quantity considering one unit of the other quantity. Nevertheless, students having between state ratio conception are not required of understanding per one strategy, such that even they use 5/3, they might not recognize that the ratio represents the cost of one pencil. Karagöz Akar (2007) argued that per-one strategy (i.e., unit factor) may contribute to the development of the conception of within state ratios, whereas building up strategies may contribute to the construction of the conception of between state ratios. Similarly, as already mentioned, the ratio-as measure conception requires students to recognize that ratio is a single intensive quantity representing the quality of interest, so it pervades within state ratio conception (Karagöz Akar, 2007; Karagöz Akar, 2017). Karagoz Akar (2007) further claimed that within state ratio or between state ratio conceptions *solely* might be sufficient for students to solve missing value problems. Therefore, curriculum developers and teachers need to be aware of students' conceptions and are suggested to develop and use tasks, problems, and representations to further enhance these conceptions in order to reach higher stages of knowing such as ratio as measure or ratio as a quantity conceptions (Karagöz Akar, 2007).

Although there is no data involving students' reflections about tasks, problems, introductory texts, and representations coming from the textbooks, still expressing conceptions is necessary for this particular study, since the tasks, problems, introductory texts, and representations of the textbooks might trigger students' quantitative and covariational reasoning, as well as, some specific underlying conceptions. That is, since quantity, covariation, invariance etc. are the necessary constructs for the conceptions of ratio, rate, and proportion, analyzing the documents with respect to quantitative and covariational reasoning frameworks might indicate the intended conceptions: which of the aforementioned conceptions might be possibly triggered in the textbooks.

<u>2.2.2.5. Task Variables.</u> Proportional reasoning involves both quantitative and qualitative processes to make sense of multiplicative relationships between quantities (Boyacı, 2019; Lamon, 2012). Therefore, the way of presenting a problem situation in both instruction and textbooks is crucial. In the literature, there are three types of task variables classified: numerical features, semantic features, and context of the task/ problem situation (Kaput and West, 1994; Karagöz Akar, 2007). The numerical, semantic and contextual features are

interrelated with each other, so the studies investigating these relationships are also needed (Heinz, 2000).

Regarding the numerical features of the tasks, Kaput and West (1994) stated that students tend to use within state and between state ratios if the problem is composed of familiar multiples to them. That is, when they can easily realize the multiplicative relationship between quantities without conducting an operation. In particular, if the quantities are integer multiples of each other as in the example of 5 dollars for 3 pencils, x dollars for 18 pencils, students can easily recognize the multiplicative relationship. Similarly, students might also use building up strategy or abbreviated building up strategy more effectively if the repetition of the ratio unit easily give the value of the given quantity (Kaput and West, 1994). Heinz (2000) further stated that students find non-integer ratios harder than integer ratios. Moreover, she further claimed that students possibly might use additive thinking if the given ratio pairs are constituted by close numbers, such as 3:4 and 2:3.

In addition to the numerical features, semantic features of tasks also might affect students' approaching the tasks. According to Kaput and West (1994), the expressions of "for every" and "for each" support students' use of building up strategy because of the indication to form unit and iterate it. Moreover, students find problems including ratios requiring the comparison of two discrete and representable quantities easier than those that cannot be representable and continuous quantities (Heinz, 2000). This is because students might solve problems by matching and counting if the quantities are representable (e.g. concrete objects, such as number of pencils, number of pizza slice, number of people, etc.) and discrete (Heinz, 2000).

Finally, the problem context might have an impact on both students' choice of strategy and the level of difficulty experienced (Heinz, 2000). For instance, students generally conduct building up strategy to solve problems including part-part-whole comparisons (Lamon, 1993). If the problem is designed to form a mixture, students have difficulty to identify the quantities contributing to the mixture since the quantities lose their own nature (Heinz, 2000). Kaput and West (1994) also pointed that mixture problems are the most difficult problems for students.

In this study, tasks presented in the textbooks are analyzed in terms of their numerical, semantic, and contextual features, because the selection of these features within the textbooks might indicate the pedagogical preferences underlying the textbooks. Particularly, how task variables are presented in the problem situations might indicate intended predispositions to trigger students' quantitative and covariational reasoning.

2.3. Theoretical Framework

In this section, I introduce quantitative reasoning and variational and covariational reasoning as the theoretical frameworks. I used these frameworks to analyze Japanese and Turkish curricular materials in terms of the questions, tasks, representations, and introductory texts presented with respect to the topics of functional relationships, rate, ratio, proportion.

2.3.1. Quantitative Reasoning

Quantitative reasoning framework was developed by Patrick W. Thompson in 1989. Thompson (1990) defined quantitative reasoning as " the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (p. 12). In order to understand what quantitative reasoning means, it is imperative to define the terms such as quantity and quantitative relationships first.

A quantity is a measurable quality of an object such that it is a conceptual entity – i.e. cognitive object- coming into being with a person's conception of a situation regarding the measurable quality of an object (Thompson, 1994a; Moore *et al.*, 2009). Thompson (1994a) characterized quantity as schematic which means that a quantity is composed of four components: an object, a quality of the object, a proper unit or dimension for the measure of the quality of the object, and a process of assigning a numerical value to the measure of the quality. The last component*, process of assigning numerical values to the measure of quantities,* is named quantification. That is to say, quantification is the process of direct or indirect measurement (Thompson, 1990). The numerical result of a quantification is called a quantity's value. There are two quantification types: Gross quantification and extensive quantification. Roots of a gross quantification lie on experiential criteria of an object. For example, if a person wants to determine which tree is taller, then whenever s/he examines the height of two trees by only referring the appearance of them, s/he is doing gross quantification. However, extensive quantification requires numerical elements of qualities, i.e., it comes up by operations of unitizing and segmenting.

The other term used to define quantitative reasoning is quantitative relationships. "A quantitative relationship is the conception of three quantities, two of which determine the third by a quantitative operation." (Thompson, 1990, p.11). The term entails to know quantitative operations, so a brief explanation on what refers to quantitative operations is needed. "A quantitative operation is the conception of two quantities being taken to produce a new quantity." (Thompson, 1990, p. 9). For example, eight quantitative operations for the comprehension of quantitative situations existing in algebra and arithmetic are: "(1) combine two quantities additively, (2) compare two quantities additively, (3) combine two quantities multiplicatively, (4) compare two quantities multiplicatively, (5) instantiate a rate, (6) generalize a ratio, (7) combine two rates additively, and (8) compose two rates or two ratios" (Thompson 1990, p.10; Thompson, 1994a, p. 9-10). Therefore, basically, quantitative operations are mental operations arising from actions (Thompson, 1994a). For example, if we think the example of comparing heights of two trees again, if a person wants to determine how much a tree is taller than the other one, s/he needs to match the height of two trees to decide the excess. In this situation, quantity is the height of the trees and matching two quantities with the goal of determining the excess is the mental action. The quantitative operation of this example is therefore *comparing two quantities additively*. However, if the person just subtracts the heights to determine which one is taller, s/he does not use a quantitative operation, but a numerical operation. Numerical operations, also called arithmetic operations, are used to determine a quantity's value, whereas quantitative operations are used to produce a new quantity. Thompson (1994a) pointed that quantitative and numerical operations are simultaneously carried out in the formation of quantities and quantitative relationships. Though, researchers have emphasized that it is also important to differentiate quantitative operations from numerical operations since quantitative operations lead a person to comprehend a situation and the meaning of the numerical values she gets by examining a situation (Moore *et al.*, 2009; Thompson and Smith, 2007). Therefore, once someone analysis a situation into a quantitative structure, a network of quantities and quantitative relationships, through quantitative and numerical operations then we might conclude that the person is engaged in quantitative reasoning. Figure 2.2. further summarizes the quantitative reasoning framework.

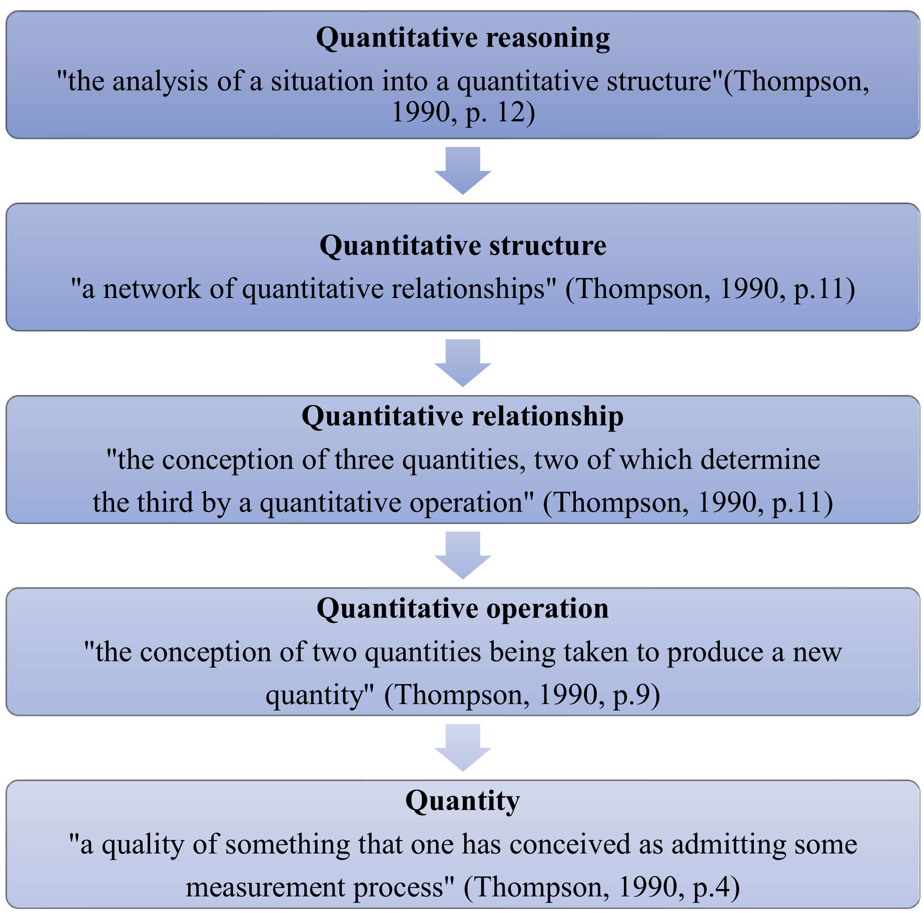


Figure 2.2. The quantitative reasoning framework (Thompson, 1990, p. 4-12).

2.3.2. Variational and Co-variational Reasoning

Co-variational reasoning theory was developed by the works of Thompson, Confrey, and Carlson in the late 1980s and early 1990s. Each researcher defined covariation in a slightly different way. The widely accepted definition of covariational reasoning among mathematics educators was made by Carlson her colleagues (2002):

> We define covariational reasoning to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other… images of covariation are developmental… to mean that the images of covariation can be defined by level and that the levels emerge in an ordered succession. (p. 354)

Carlson and her colleagues (2002) conducted a research study to present a framework for covariational reasoning and used the framework to examine mental actions existing in covariational reasoning "when interpreting and representing dynamic functional events" (p.352). In order to make the definition of covariational reasoning clearer, it is necessary to define what these researchers mean by the terms "image", "developmental", and "level".

The "image" stated in the definition refers to "dynamic, originating in bodily actions and movements of attention, and as the source and carrier of mental operations" (Thompson, 1994a, p.231). The images of covariational reasoning are found developmental parallel to the Piaget's (1970) notion of developmental such that it means "the images of covariation can be defined by level and that the levels emerge in an ordered succession" (Carlson *et al.*, 2002, p. 354). That is to say that, when a person reveals a mental action, the person is thought to be able to perform all the mental actions at the corresponding level as well as the levels lower than the revealed one. In their covariational reasoning framework, researchers presented five mental actions originating in five covariational reasoning levels. The mental actions and corresponding levels of covariational reasoning are shown in Table A.1 in Appendix A.

As a foundational construct for covariation, Castillo-Garsow (2010) further investigated the idea of "variation". Variation refers to the varying value of a quantity (Thompson and Carlson, 2017). Results of his study showed that students might think about varying values of a quantity discretely or continuously. Castillo-Garsow further sub-categorized continuous variation as chunky and smooth. Based on the results of Castillo-Garsow (2010) study explicating the reasoning on how a quantity's value might vary, Thompson and Carlson (2017) suggested to revise the prior covariational reasoning framework (See Table A.3 in Appendix A): 1) considering students' variational reasoning processes separately from their covariational reasoning; 2) examining students' use of multiplicative objects of quantities' values and variational reasoning to coordinate the images of quantities' values varying. They further presented a variational reasoning framework also developmental considering the work of Castillo-Garsow (2010, 2012) and Thompson (2008, 2011) (See Table A.2 in Appendix A). Though, since the framework needs to be verified empirically, it does not have a 'definitive description' yet.

Since the aim of this study is to examine textbooks with regards to the topics functional relationships, ratio, rate, proportion, and linear functions both variational (see Table 2.3.) and covariational reasoning (See Table 2.4.) frameworks will be used in this study in addition to quantitative reasoning framework. Continuous variational and covariational reasoning are fundamental for both teachers' and students' conception of function effectively, eventually for their mathematical development as well (Thomson and Carlson, 2017).

In the following section, I explicate how I embrace the textbook analysis and the quantitative and covariational reasoning frameworks as related to each other.

2.3.3. Relationships of Quantitative Reasoning, (Co)variational Reasoning, and Textbook Research

Both quantitative and covariational reasoning is about conceiving a situation. Quantitative reasoning requires a person to conceive a situation in terms of a quantitative structure, which includes quantities and the quantitative relationships (Thompson and Carlson, 2017; Moore *et al.*, 2009). However, some situations are needed to be conceived dynamically which means thinking about the quantities' values varying. That's is to say, a person should think the changes of values with respect to each other in some situations. In a situation that a person conceives quantitatively, meanwhile thinking the dynamic nature of the situation too, the person might reason (co)variationally. In other words,

> Variation and covariation became necessary in Thompson's theory of quantitative reasoning to explain the reasoning of students who conceptualized a situation quantitatively and at the same time took it as dynamic-they envisioned quantities in their conceptualized situations as having values that varied. (Thompson and Carlson, 2017, p. 424)

Researchers emphasized that it is important for students to reason variationally and covariationally in order to model dynamic situations (Moore and Carlson, 2012; Carlson *et al.*, 2002). For instance, the topics of functional relationships which will be analyzed in this study, requires students of covariational reasoning skills. Particularly, researchers (Moore *et al.*,2009; Carlson *et al.*, 2002) found the ability of covariational reasoning as 'critical' and 'essential' for the conception of functions and central concepts of calculus. According to Moore (2011):

> …allowing students the opportunity to immerse in a situation in which they can conceptualize quantities and the relationships enables formulas and functions to emerge in a way such that they describe a conceived situation. Note that a process conception of function can somewhat "come free" through this perspective. (p.6)

Similarly, Thomson and Carlson (2017) discussed the necessity of continuous variation and covariation reasoning skills for both teachers and students for effective conception of functions. In addition, Thompson (1994a) suggested rate as a foundational idea for functional relationships. Therefore, they suggested focusing on quantitative and covariational reasoning skills on curriculum to improve students' understanding of and ability to use functions. Moreover, there is a necessary role for curriculum and pedagogy to support students' quantitative reasoning (Thompson and Smith, 2007). Both quantitative reasoning and (co)variational reasoning happens in the mind of students. However, textbooks are acknowledged and treated as a bridge between intended and implemented curriculum. It emphasized that curriculum has impact on instructional practices, thus affects students leaning (Cai, 2017). Therefore, it is plausible to analyze textbooks using quantitative and covariational reasoning as a framework (Taşova *et al.*, 2018) to identify the kind of reasoning students are likely to be required of.

It is in this regard that in this study, I will examine the tasks and problems of Japanese and Turkish textbooks about functional relationships including the topics of ratio, rate, proportion, and linear functions using the quantitative and (co)variational reasoning frameworks. The underlying reason of the selection of Japanese textbook series stems from both their success in international examinations (PISA and TIMMS) and their focus on "quantitative relations" throughout their course of study for both elementary and lower secondary levels. Particularly, quantities, quantitative relations, and quantities' values varying are emphasized both in course of study and teachers' guide. Further, Thompson and Carlson (2017) stated that Japanese elementary school mathematics textbooks give a clear point to quantitative and covariational reasoning skills. Thus, analysis of those textbook series would help to examine their way of presenting mathematical topics and the correspondence of their textbooks with their aims and objectives. Compared with Japan, Turkish national textbooks and curriculum might not have a clear focus on quantitative and covariational reasoning. Still, with the purpose to show how the Turkish and Japanese textbook series place quantitative and covariational reasoning in the topics of functional relationships in elementary and lower secondary levels, I will conduct a textbook analysis study. In the next chapter, I explain the methodology of the study.

3. SIGNIFICANCE OF THE STUDY

Studies have indicated that function is a complex topic such that students have difficulty to conceive it (Thompson, 1994b; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008; Cooney *et al.*,2010; Leinhardt *et al.*, 1990). In order to improve students' conception of functions, Thompson (1994) suggests focusing on rate and relationships between covarying quantities. Ratio, rate, and proportion concepts are linked with the concept of linear function which is generally introduced in lower middle school years (Lamon, 2012). Functional relationships, which are composed of the concepts of ratio, rate, proportion and linear functions, are required of to emphasize the quantitative and covariational relationships between quantities (Thompson and Carlson, 2017; Thompson, 1994a; Cooney *et al.*,2010). Therefore, the extend of which these relationships are triggered in curriculum materials might be revealed by textbooks analysis (Thompson and Carlson, 2017; Taşova *et al.*, 2018).

Similarly, the topics of functional relationship and the use of quantitative and covariational situations provide students to model dynamic events (Carlson *et al.*, 2002; Oehrtman *et al.*,2008). Dynamic events can be seen in many science topics of chemistry, biology, physics, engineering, mathematics and even in daily life (Carlson *et al.*, 2002; Oehrtman *et al.*,2008). Therefore, to contribute to students' conception of functional relationships, their modeling ability, and their mathematical development, quantitative and covariational reasoning might be used in primary, secondary, and college level mathematics courses (Thompson, 1994b; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008).

Furthermore, cross-national textbook analysis research gives insights about the possible learning opportunities of students provided in different textbooks. By doing so, the current pedagogical and mathematical perspectives of textbooks and thereof a country might be interpreted, and this interpretation might enable to make sense the achievement differences on international examinations (Valverde *et al.*, 2002; Cai, 2017).

4. STATEMENT OF THE PROBLEM

The purpose of this research study is to examine Japanese and Turkish elementary and lower secondary level mathematics curriculum materials, which includes national standards, textbooks, and teachers' manual, through quantitative and (co) variational reasoning frameworks. I examine the curricular materials in terms of functional relationships presented in the curriculum in elementary and lower secondary level mathematics.

4.1. Research Questions

1. What are the similarities and differences in terms of the overall structures of Japanese and Turkish textbooks regarding ratio, rate, proportion, and linear functions?
2. What are the similarities and differences in the Japanese and Turkish curricular materials in terms of quantitative reasoning and (co)variational reasoning depicted in the concepts of functional relationships focusing on linear functions and proportions?
    i. In particular, what are the similarities and differences with respect to the tasks and problem situations triggering quantitative and covariational reasoning?
    ii. In particular, what are the similarities and differences with respect to the questions required of students to answer?
    iii. In particular, what are the similarities and differences with respect to the use of representations including algebraic, graphical, and tabular?
    iv. In particular, what are the similarities and differences with respect to the students' strategies, students' conceptions, and task variables depicted in the concepts of ratio, rate, and proportion?

5. METHODOLOGY

The purpose of this study is to examine Japanese and Turkish textbooks with specifically focusing on the topics of functional relationships including ratio, rate, proportion and linear functions based on quantitative and (co)variational reasoning. In this chapter, first, I introduce Japanese and Turkish textbook series. Then, I explain the design of the study, data collection and analysis procedures.

5.1. Textbooks Series

The education systems of both Japan and Turkey are centralized with national curriculum standards published by Ministry of National Education. In Japan, elementary and middle secondary school textbook series are prepared with respect to the 2008 standards. The 2008 standards are determined with respect to the reform movement in school education in 2007. With the reform, the goal of education is changed as:

> Particular attention must be paid to achieve basic knowledge and skills, to cultivate thinking, decision making and expressing ability to solve problems by using those knowledge and skills, and to nurture the attitude to willingly pursue leaning in order to lay foundation for lifelong learning. (Isoda, 2010, p.i)

The 2008 standards took place starting with 2009 and were completely implemented in 2011. The textbook series published by Tokyo Shoseki are used in this study. The series of Tokyo Shoseki is one of the six series which have been approved by Ministry of National Education. Tokyo Shoseki is a leading textbook publisher in Japan and the textbook series of it are most widely used in elementary mathematics (Peterson, 2008; Watanabe, Lo, and Son, 2017). With the collaboration with Global Educational Resources in 2011, the English version of the textbook series are named “Mathematics International” (MI) and covered grades 1 to 9 which includes both elementary (grades 1 to 6) and lower secondary (grades 7 to 9) school mathematics. In this study the Mathematics International textbook series of

Tokyo Shoseki for elementary mathematics (Fujii and Iitaka, 2012) and lower secondary mathematics (Fujii and Matano, 2012) are used to examine the functional relationships of the tasks and problems given in the textbooks in terms of the theoretical frameworks of the study. In addition to Japanese textbook series, the course of study and teachers' guides which is supported by Ministry of National Education of Japan are also investigated.

Regarding Turkish textbook series, in this study they are also examined in the same way as Japanese series. In Turkey, the national textbooks are published by Ministry of National Education and given to students free. Elementary and lower secondary school mathematics education had a reform by placing the students-centered learning and constructivism into the national mathematics education system in 2005. After 2005 standards, there have been changes in the standards in 2005, 2009, 2013, 2017, and 2018. Ilhan and Aslaner (2019) conducted a document analysis to examine all the standards from 2005 to 2018. They found that in terms of the objectives of the course of study, the 2005 and 2009 standards are similar while the 2013, 2017, and 2018 standards are similar. Particular to the objectives for the topics under investigation of this study, they are not changed in the 2009,2013,2017, and 2018 standards, with the exception that the only teacher guides published was for the 2005 standards. Therefore, in this study, the textbook series prepared for the 2005 standards are used. Although the number of goals and objectives are pretty reduced through 2005 to 2018, there are some new objectives added to the standards (see Table B.1 in Appendix B).

Moreover, course of studies, teachers' guides, and textbooks are analyzed as curriculum materials in this study. I use the phrase *curriculum materials* and *curriculum* to refer course of studies, teachers' guides, and textbooks in this study. In the following parts, I introduce Japanese and Turkish curriculum materials.

5.1.1. Japanese curricula materials

The National curriculum standards is named and published as “course of study” (COS) in Japan. In this study, for the National curriculum standards, I use the *Japanese course of study for mathematics* published in 2008. There are two sections; one is for elementary grades and the other is for lower secondary grades. The mathematics for elementary school section (grade-1 to grade-6) starts with stating the objectives, then sharing the explanations for the objectives grade by grade and remarks for concerning content.

In their explanations of objectives regarding elementary school section, Japanese curriculum authors categorized the contents into four domains: A. Number and Calculations, B. Quantities and Measurements, C. Geometric Figures, and D. Quantitative Relations. Differently from elementary school mathematics, the classification of contents for the lower secondary school section (grade-7 to grade-9) is categorized as A. Numbers and Mathematical Expressions, B. Geometric Figures, C. Functions, and D. Data Handling. The main structure of the lower secondary grade sections is similar with the elementary grade section.

In addition to the course of study, teachers’ guide is defined as “a guidebook on Japanese curriculum standards” (Isoda, 2010, p. i). In the teachers’ guide document, each domain, which are listed in the former paragraph, is explained clearly by sharing the main ideas needed to be emphasized by teachers. The main focus in the teachers’ guide is on quantitative relationships. In other words, tasks and questions placed in textbooks should be considered by teachers with the lens of quantities and relationships between quantities. In the document, the goals of mathematical activities are also given a place. Mathematical activities take up a great place in Japanese mathematics education; therefore, I clarify the meaning of mathematical activities in Japanese mathematics education.

It is emphasized in the teachers’ guide that the mathematics instruction is aimed to be done through *mathematical activities* to cultivate students’ ability to think and express, as well as their attitude to use mathematics intentionally in their daily lives. The mathematical activities, which students works on willingly with an aim, placed in mathematics instruction could involve various hands-on activities and experiments where concrete materials usually are needed and students' thinking about problems, explaining, and representing are considered. Complete practice problems and teachers' explanations where students are passive and just listening to are not considered as mathematical activities (Isoda, 2010a). In other words, the Japanese curriculum is mostly designed as students centered activities such that in the summary of mathematical activities for elementary grades, there is an emphasis on quantitative operations, use of different representations, and proportions as shown in the table below.

Table 5.1. Content related mathematical activities for elementary school mathematics (Isoda, 2010, p. 10).

Grade level	Mathematical Activity
4	Activities to investigate relationships among quantities in their surroundings.
5	Activities to choose and to use appropriate graphs and tables for different purposes.
6	Activities to learn relations between units. Activities to solve problems by using proportions.

Moreover, the explanation made in *Quantitative Relations* domain in the elementary school section of the course of study clarifies the Japanese mathematics education's emphasis on quantitative relationships and covariation. The statement "*In the "Quantitative Relations" domain, expressing quantitative relationships with words, numbers, algebraic expressions, tables and graphs, and rising the idea of a function as the change and correlation between two quantities as considered to be important*" (Takahashi *et al.*, 2008, p. 5-6) puts an emphasis on quantitative relationships, correlation, and the use of different representations. Such emphasis might be beneficial for the development of the conception of ratio, rate, proportion, and linear functions on the part of students.

In particular, the concept of ratio is mentioned in the 5th grade the first time; however, in the 4th grade there is an objective requiring the investigation of two simultaneously varying quantities. Since I begin explaining my findings about the concepts under investigation -which includes ratio, rate, proportion, and linear functions- in the findings part stating with the 4th grade, I briefly summarize the aim of mathematics education for the first 3 years regarding the concepts under investigation in the below paragraph.

In the *first grade*, addition and subtraction are taught such that development of students' number sense is emphasized. Then, in the *second grade*, multiplication is introduced. In the content section the focus is "to consider a number in relation to other numbers in ways such as regarding it as product of other number". In the *third grade*, "10 and 100 times as much/many as or 1/10 of another number" and the use of soroban (kind of an abacus) is given importance. In addition to the effort of developing the conception of quantities, measurement of quantities, relationships between quantities and the representation of these relationships, in the third grade it is expected for students to use mathematical expressions as a representation. In the quantitative relations section in the *third grade*, it is stated that "Students will understand mathematical expressions that represent relationships of quantities and will be able to use them" (Takahashi *et al.*, 2008, p.8) such that students also need to represent quantitative relationships both through mathematical expressions and related diagrams. The usage of "□" in mathematical expressions is further suggested to represent the unknown. These are the signs of early algebra and the algebraic expressions built on in this early action. These further suggest that algebra in Japanese mathematics education is introduced by highlighting the quantitative relationships. I examine the other grades under the findings chapter. In below, I introduce Turkish curriculum materials and summarize the first four grades before I present findings.

5.1.2. Turkish curricula materials

In 2005, the primary grades (1-5) and the lower secondary grades (6-8) had separated curriculum documents. Since 2005 curriculum is a kind of reform movement, the curriculum materials are heavy and includes detailed explanations and examples. For instance, when the problem solving strategies are listed, there is a classroom teaching case involving teacher student interaction about finding interior angles of any convex polygon. For the successful implementation of the curricula, there are teaching strategies suggested to teachers. These strategies involves (i) starting teaching with concrete experiences, (ii) aiming meaningful learning, (iii) assisting students' interactions with mathematics, (iv) noticing the connections in the mathematics, real life situation and with other subjects, (v) paying regard to students' motivations, (vi) using technology effectively, (vii) paying attention on cooperative learning, and (viii) planning instruction meticulously. The learning areas of the curriculum of grade-1 to 5 are decided as "Numbers, Geometry, Measurement, and Data"; however, the learning areas are differently categorized for lower secondary curriculum (grade6 to 8) as "Numbers, Geometry, Measurement, Probability and Statistics, and Algebra". Similar to Japanese course of study, each learning area is explained, then objectives are shared and clarified through examples.

Differently from Japanese teachers' guide which is aimed to explain the standards, Turkish teachers' guide is published as separated books for each grade level, and it is aimed to explain the textbook and direct teachers. Therefore, I present my findings for the Japanese curriculum materials starting with the content related objectives, teachers' guide parts and textbooks. However, I share my findings for the Turkish curriculum materials beginning with the content related objectives, textbooks and teachers' guide parts. All the textbook parts, i.e. tasks, questions, images, etc., that were shared in this study was recreated by reying the original textbooks and translated from Turkish to English without changing the structure of the tasks and questions.

The definition of the Turkish teachers' guidebooks is expressed that teachers' guidebook involves the directions and explanations of textbooks and students' workbooks. After the book is introduced, the sub-learning areas and the content related objectives are shared in the beginning of each unit. Then, a small scale textbook page is given with the explanations and directions for teachers next to this textbook page. For instance, ratio is given in the 5th grade teachers guide including the small scale textbook page related to the ratio unit with the side explanations for teachers, which involves preliminary preparation ideas, statement of the objectives, mini test ideas, activities, and connections to other subjects.

Similar to my summary of Japanese mathematics education, I make a summary for Turkish mathematics education herein. In the *first grade*, addition and subtraction with natural numbers are presented in the numbers learning area. In the *second grade,* multiplication and division with natural numbers are introduced in addition to arithmetic operations of addition and subtraction with natural numbers. In the *third and fourth grade*, students' understanding of the operations with natural numbers is aimed to be developed. Fractions are introduced first time in grade-1, but the operations with fractions are presented in grade-4. In Turkish curriculum, ratio and proportion are first seen in grade-5 as a sub-learning area under the learning area called "numbers". In this grade level, students are expected to form a ratio by comparing part to whole or whole to part. Students are expected to show part to whole relationship in fractions unit prior to 5th grade, but the fractions formed to represent part to whole relationship are not connected to ratio topic until 5th grade. In the following chapter, I express other grades by going on details because there will be contents directly related with functional relationships.

Since the textbooks, the course of study and the teachers' guides are the documents under analysis, qualitative content analysis (Weber, 1990; Krippendorff, 1989; Yıldırım and Şimşek, 2005; Bowen, 2009) is used in this study. In the following paragraphs, I explain the analysis procedures and processes in detail.

5.2. Research Design and Data Collection

In this study, qualitative research design is used to examine Japanese and Turkish elementary and lower secondary level textbook series. Qualitative research is originated in the idea that "knowledge is constructed by people in an ongoing fashion as they engage in an make meaning of an activity, experience, or phenomenon" (Merriam, 2002, p. 23) The aim for a qualitative study is to reveal and interpret the constructed meanings. In this study, the aim is to analyze and interpret how Japanese and Turkish mathematics textbooks present functional relationships, ratio, rate, proportionality, and linear functions in elementary and lower secondary levels in terms of quantitative and covariational reasoning. Data sources include Japanese and Turkish curriculum materials: The Japanese textbooks published by

Tokyo Shoseki publishing company, the teachers' guide, and the course of study; and, Turkish textbooks, the teachers' guide and the course of study. The translated version Japanese curriculum materials to English are used in this study.

5.3. Data Analysis

Content analysis method is mostly used in health studies (Hsieh and Shannon, 2005). Although content analysis method could be conducted by both qualitatively and quantitatively, I will use only qualitative content analysis method (Weber, 1990) in this study. Holsti (1969) offers a broad definition of content analysis as, "any technique for making inferences by objectively and systematically identifying specified characteristics of messages" (p. 14). Similarly, Krippendorff (1989) defined content analysis as "a research technique for making replicable and valid inferences from data to their context" (p. 403). Moreover, data of a content analysis could be a written document, visual and verbal representation (Krippendorff, 1989; Yıldırım and Şimşek, 2005).

In the educational research, content analysis is used mostly for analyzing textbooks. The procedure of a content analysis study includes six criteria listed: design, unitizing, sampling, coding, drawing inferences, and validation (Krippendorff, 1989). In the design phase, regarding the Krippendorff's (1989) explanation, the researcher has to define the context of the study, e.g. textbook analysis, examine the data sources, e.g. Turkish and Japanese textbook series, and conduct an analytical construct, e.g. quantitative and covariational reasoning frameworks. An analytical construct is used to display the relationship between the data and context. In this study, literature of textbook analysis research, selection of Japanese and Turkish textbook series, and quantitative and covariational reasoning frameworks are explained in detail to form a rationale for the design of the study.

The second phase is unitizing that means determining the chunks of data for analysis. In this study, functional relationships, including ratio, rate, proportion, and linear function topics are selected to be investigated in the textbook series. Therefore, the chunks of data include Japanese and Turkish course of study, teachers' guides and the problems and tasks with respect to the topics selected within the textbooks.

The sampling is important phase of content analysis studies because of the need of unbiased selection of representative data sources. Since the textbooks series are kind of participants in this study, selection of textbook series is considered in terms of its representativeness. Although there are six textbook series published and used in elementary and lower secondary school level of mathematics education in Japan, Tokyo Shoseki's mathematics textbooks are widely used in those levels. Regarding the Turkish context, textbooks are published by Ministry of National Education and given to *every* student as free in Turkey.

The next phase is coding. The coding could be done by emergent or priori coding (Stemler, 2000). The codes are formed by analysts in an emergent coding process, while in a priori coding process, the codes are coming from the analytical constructs of the study. Priori coding process is relevant to what Clement (2000) defined as "coded analysis". According to Clement (2000), "convergent purposes usually lead to a coded analysis …attempts to provide reliable, comparable, empirical findings…" (p. 558) Researcher conducting a convergent study usually starts with predefined categories. The data analysis process is started defining criteria to find out the examining theory or phenomenon. Then, the analyst codes the relevant part of the transcript or a document and list all the places coded. Since the analysis will be conducted in terms of quantitative and covariational reasoning frameworks, the possible codes will be pre-determined in a way. Therefore, the analysis is conducted according to the coded analysis, i.e., priori coding.

Moreover, there are two ways for coding in a content analysis: coding done by trained human coders and computer coding. Although computer coding is fast and reliability is sustained by the software used in coding, it is not effective for analysis requiring the extraction of meaning of the data. However, human coders could analyze the data to get the meaning of the data, but it is kind of time-consuming process and reliability issues should be considered. Generally, interrater agreement is used to check the reliability in content analysis studies having human coders. Thus, interrater reliability is checked for this study which I explain in the following paragraphs.

The most important phase is drawing inferences in which the researcher examines relationships of the codes and phenomena under investigation. Therefore, the last phase includes validation. The most common method to sustain validity for a qualitative study is triangulation of data, which means collecting and analyzing various sources of data to make an inference about the topic of investigation. In order to validate the findings of the study, courses of studies and teachers' guides are also examined in addition to the textbooks series of both countries.

In this study, first, I and my advisor came together and examined the Japanese COS, i.e. standards, and the teachers' guide for the contents of functional relationships in regard to quantitative and covariational reasoning lenses. As it is said in the literature (e.g. Thompson and Carlson, 2017), we were also agreed on that there are explicit referring to quantitative and covariational reasoning in these curriculum materials. Then, I started analysis of the Japanese and Turkish course of studies. First, I examined the COS both for Japan and Turkey to determine the grade levels in which the functional relationships are covered. The topics of functional relationships, i.e. namely, rate, ratio, proportion, and linear function, are covered in Grade 4 to Grade 8 in Japanese curriculum and in Grade 5 to Grade 8 in Turkish curriculum. Starting from Grade 4, I analyzed the COS, teachers' guide, and the units of textbooks for the Japanese curriculum for each grade level, and then I executed the same processes for the Turkish textbooks starting by Grade 5. Moreover, I tracked down the overall structures of Japanese and Turkish textbooks to answer the first research question of the study while I examined each grade level.

Since the quantitative and covariational reasoning frameworks are designed to make interpretations about learners' thinking, in order to use them to examine curriculum materials and search answers for the second research question, I formed tables (see Tables C.1 and Table C.2 in Appendix C) by benefiting from existed literature of covariational and quantitative reasoning regarding to functional relationships. For the quantitative reasoning framework, I included the definitions of kinds of quantities, quantitative operations, and numerical operations. Second, I linked the concepts, i.e. ratio, rate, and proportions, students' conceptions of ratio, and students' strategies of ratio with the quantitative reasoning framework regarding the definitions of kinds of quantities, quantitative operations, numerical operations. For instance, the term ratio is defined as "the result of comparing two quantities multiplicatively" by Thompson (1994a) which indicates that Thompson defined ratio as an intensive quantity. For this definition of ratio, possible students' conceptions might be ratio as quantity, ratio as measure, within-state ratio, and between-state ratio. Thus, for a task defining or triggering students' thinking to ratio as a result of multiplicative comparison, I examined the questions, task variables, examples of students' ideas shared in the task. While I examined a task I searched for the questions, such as: What is expected for students to answer the questions? How the questions phrased? Are there questions to expect students to determine and identify quantities in the situations? Do the questions trigger any students' strategies? What are the values of quantities and/or variables? What guidance given in the textbooks for students to let them think about quantitative relationships/ covariational relationships? I compared the questions on the task to the questions and findings shared in the literature (e.g. using the results of the studies of Thompson 1990, 1994; Simon and Blume, 1994, Heinz, 2000; Karagöz Akar 2007, 2010, 2017; Noelting, 1980, etc.).

Moreover, I formed a table for covariational reasoning to analyze especially proportions and linear functions by involving the categories of covariational reasoning presented by Thompson and Carlson (2017), mental actions corresponding to the covariational reasoning levels tabled by Carlson *et al.*, 2002, Taşova and his colleagues' framework to analyze written curriculum in regard to covariation, and students' conceptions of linear functions. I examined how the relationships between covarying quantities are questioned and which mental actions might be triggered. Depicting the mental actions possible to be triggered through the tasks and questions, I explained the covariational reasoning level corresponds to the mental actions. I also paid attention to the use of representations due to its importance for students' understanding of functional relationships.

6. FINDINGS

In this section, I present the findings from the curricula materials. Starting with the answering first research question, I share a table to display and compare the overall structures of Japanese and Turkish textbook series in regard to topic placement, grade placement of topics, number of pages of the textbook parts comprises functional relationships, and the total number of pages of the textbooks. To express the findings regarding second research question, I share the grade-based goals relating to the functional relationships in the related parts on the teachers' guide and textbooks. At first, I present the findings regarding to the research question 2-iv which express the learning opportunities presented in the concepts of ratio, rate, and proportions, because concepts of ratio are serving a base for proportional relationships and linear functions. Then, I share the findings from the concepts of linear functions and proportions in order to express the possible learning opportunities provided in the curriculum materials regarding to covariational reasoning. After I report findings from Japanese curricula materials, I share the findings stemming from Turkish curriculum materials by following the similar procedure as I have used for Japanese curricula materials. In contrast to teachers' guide in Japan in which the course of study is further explained for teachers, in Turkey, the teachers' guide is designed to explain the tasks and questions of the textbook to teachers. Therefore, I mention the teachers' guide when I share my finding of textbooks for Turkish curricula materials.

Particularly, for the Japanese curricula, first I express the content related objectives and the teachers' guide explanations for the specific objectives. Then, I share the findings about units of textbook, namely *Mathematics International* (MI), designed for the targeted objectives. Throughout the presentation of the findings regarding textbooks, I also share content related units, extending mathematical thinking parts, and additional parts, respectively. The additional parts involve Mastery Problems, Power Builders, Wonderful Problems and Additional Problems given at the end of the textbooks. I include additional parts only if there are important data. For the lower secondary level mathematics, the structure of the Japanese textbooks is slightly different from the elementary ones. Thus, the names of additional parts for lower secondary level includes Window to Mathematics, Daily Living and Mathematics, etc.

6.1. Findings regarding the overall structures of textbooks

In order to present similarities and differences between Japanese and Turkish textbooks as one of the curriculum materials, I examined the topic placements regarding to grade levels, total number of units, number of units involving functional relationships, number of pages of the units involving functional relationships, and the total number of pages of the textbooks. The Table 6.1 shared below involves the topics of functional relationships regarding their grade placements, the number of pages allocated for those topics and total numbers of pages of textbooks. The table involving the overall topic placements of textbooks is shared in the Appendix B (see Table B.2).

Table 6.1. The topics of functional relationships regarding their grade placements and number of pages of textbooks.

	Japanese textbooks				**Turkish textbooks**			
Grade level	Number of units of the textbook	Topics/units depicting functional relationships	Number of pages of the units	Total pages of the textbook	Number of units of the textbook	Topics/units depicting functional relationships	Number of pages of the units	Total pages of the textbook
4th grade	16	Broken line graphs Properties of operations	19	266	-	-	-	-
5th grade	15	Per unit quantity Percentages and graphs	39	256	6	Fractions (involving *percentages* sub-unit) The world of decimal fractions (involving *ratio* sub-unit)	7	224

Table 6.1. The topics of functional relationships regarding their grade placements and number of pages of textbooks (cont.).

	Japanese textbooks				Turkish textbooks			
Grade level	Number of units of the textbook	Topics/units depicting functional relationships	Number of pages of the units	Total pages of the textbook	Number of units of the textbook	Topics/units depicting functional relationships	Number of pages of the units	Total pages of the textbook
6th grade	14	Letters and math sentences Ratios and values of ratios Speed Direct and Inverse proportional relationships	58	226	6	Mathematics and art (involving *everyone should learn algebra* sub-unit) Reflection from numbers to probability (involving *ratio and proportion* sub-unit)	12	255
7th grade	7	Letters in algebraic expressions Equations Direct and inverse proportions	40	267	6	We started with proportions Algebra and probability (involving *linear equations and their graphs* sub-unit)	19	238
8th grade	6	Linear functions	36	216	6	Triangles and algebra (involving *number patterns and identities* sub-unit) Starting point and end point: Geometry (involving *Examination of a Line* sub-unit)	12	224

As it is seen from the Table 6.1, functional relationships started to be introduced in 4th grade in Japanese textbooks but in 5th grade in Turkish textbooks. In the Turkish textbooks, the names of units are broader than Japanese textbooks. That is, the naming of the units seems abstract and compact in a way that involving the goals of different learning domains, such as geometry, algebra, and numbers are covered in the unit *Triangles and Algebra* in 8th grade. Thus, the number of units in Turkish textbooks are less than Japanese textbooks' number of units in general. In terms of the page numbers allocated to functional relationships in textbooks, Japanese textbooks have more pages to introduce and develop functional relationships than Turkish textbooks regardless of grade level. Moreover, there is no mention on functions in Turkish textbooks while Japanese textbooks have a unit for linear functions in 8th grade.

6.2. Findings from Japanese Curriculum Materials

In Japanese curriculum, there are contents directly related with functional relationships starting with the *fourth grade*. Therefore, in the following subsections, I expand on the grades starting with the 4th grade by examining the *content related objectives*, the *explanations given in the teachers' guide* for the contents, and *the units of textbooks* involving functional relationships. In particular, after I present the content related objectives and the explanations given in the teachers' guide for the contents, I first share the findings coming from the unit of textbooks regarding the second research question (specifically, 2-iv) which involves the concepts of ratio, rate, and proportions in order to portray the supported students' strategies, students' conceptions, and task variables. Second, I present the findings coming from the unit of textbooks regarding the topics of linear functions and proportions in order to express the emphasis on the quantitative and covariational reasoning. However, some tasks seem to be fruitful for both triggering students' covariational reasoning and conceptions of ratio. For those tasks, I present the findings of all sub-questions of the second research question when I introduced the task first time in order to avoid unnecessary repetitions. Moreover, it is worth mentioning that concepts of functional relationships are interwoven. In this study, I share the findings chapter regarding *covariation* and *concepts of ratio* separately under two headings in order to emphasize the main conclusions of this study.

6.2.1. Grade-4 Japanese Curriculum Materials

6.2.1.1. Content related objectives and the explanations given in the teachers' guide. In the *fourth grade*, the content related objective with the functional relationships is that:

> "*Students will be able to represent and investigate numbers, quantities, and their relationships by using words, numbers, and mathematical expression as well as diagrams, tables, and graphs.*" (Takahashi *et al*., 2008, p.9).

Following that, in the quantitative relations domain, functional relationships and covariation are given point with the statement of:

> "*(1) Students will be able to represent and investigate the relationship between two quantities as they vary simultaneously. (1.a.) To represent how the quantities vary on a broke-line graph and to interpret the features of their variation.*" (Takahashi *et al*., 2008, p. 11).

These objectives suggest that the *Japanese course of study* explicitly states and expects students to examine relationships between two quantities varying simultaneously. These also suggest that the *course of study* gives importance to represent quantitative relationships in different forms such as algebraically, in graphs and in tables. Similarly, mathematical activities suggested to be used in mathematics classrooms to examine this objective are shared in *teachers' guide* with the expression of "*Activities to find two quantities in everyday life that vary in proportion to each other, and to represent and investigate the relationships of numbers/ quantities in tables and graphs.*" (Isoda, 2010a, p. 116). This further suggests that teachers are expected to provide students with opportunities to examine especially the linear relationship between quantities.

Moreover, in the *course of study* in the fourth grade, the objective "*Students will understand mathematical expressions that represent quantitative relationship and be able* *to use them*" (Takahashi *et al.*, 2008, p.11) suggested the representation of quantitative relationships by using Δ and □. Students are expected to substitute numbers for them such that the use and formation of a formula to represent quantitative relationships are targeted. In the *teachers' guide*, the objective is further explained that "*The objectives of grade 4 are to represent quantitative relationships in algebraic expressions, to improve in their ability to interpret algebraic expressions… and to utilize algebraic expressions appropriately*" (Isoda, 2010a, p. 118). This explanation seems to suggest that Japanese curriculum gives a place for understanding, explicating, and representing quantitative relationships. Also, it is stated that the parentheses and expressions involving multiplication or division sometimes indicate a quantity (Isoda, 2010a). On the top of the understanding of quantitative relationships and expressing them by algebraic expressions, the idea of *formula* is introduced in the fourth grade. The statements of "*It is important to represent quantitative relations using formulas and to interpret formulas in concrete situations. Consideration should be taken so that students understand the merit of formulas that generalize quantitative relationships through these activities*." (Isoda, 2010a, p. 118) given in *teachers' guide* indicates the importance given on the examination and generalization of quantitative relationships.

In sum, all the explanations in both *the course of study* and the *teachers' guide* suggest that in Japan, starting from the fourth grade quantity and quantitative relationships between quantities including simultaneous variation is given importance. Similarly, the use of algebraic expressions and formulas and the need for students to conceive those algebraic notions in concrete examples from real life situations is emphasized. In addition, the use of "number and quantity" together in the same objective seems to indicate that Japanese curriculum differentiate them while giving importance to both. This is interesting in the sense that Thompson (1994a, 2011) points that quantity surpass number albeit having common cognitive actions (Steffe, 1991) on the part of the learner.

6.2.1.2. The units of textbooks regarding the concepts of ratio. Regarding these content related objectives and the related explanations in the teachers' guide, in the 4^{th} grade mathematics textbook, the quantitative relations are held under two topics. The first one gives place to broken-line graphs in the fifth unit with the title *Let's represent how the quantities change in graphs*. Students are introduced to covarying quantities and expected to examine their relationships through this unit. The second topic gives place to properties of arithmetic operations and algebraic expressions in the tenth unit with the title of *Let's investigate the properties of operations*. Since students are expected to examine multiplicative relationships, it might be important for both their quantitative and covariational reasoning as well as their conception of ratio. I share the finding coming from the *Let's investigate the properties of operations* unit in this section. Findings of the other unit is presented in The units of textbooks regarding covariation section.

Let's investigate the properties of operations. The unit of *Let's investigate the properties of operations* is presented under two sub-unit: First, Order of Operation and second, Properties of Operations.

Task1. The introductory task of shopping food, which is presented at beginning of *Order of Operation* sub-unit, involves that a kid named Kenji has 500 yen coin in his hand to buy some food. The prices of specific foods are presented as in the below Figure 6.1.

Figure 6.1. The task for introducing order of operation (Fujii and Iitaka, 2012, Grade 4, p. B8).

Students are asked to represent this situation using mathematical expressions, such as total price that Kenji paid and amount of change that Kenji took. Students are triggered to use parentheses and apply the order of operations to determine the amount of change. Then this task is used to provide students to practice operations. There are different problem situations provided to students to interpret the situation, practice writing mathematical expressions with using parentheses, and perform calculation to find out what is asked. What is important about this unit in terms of the topics under investigation that parentheses are sometimes used to express a quantity and students are supposed to realize and name the new quantity. In the example shared in the Figure 6.2, the two quantities inside of the parentheses are together expressing a new quantity representing the price of a set of a pencil and a pencil cap.

3 A pencil that costs 50 yen and a pencil cap that costs 25 yen will be bought as a set.
How many sets can you buy with 600 yen?
Using the math sentence with words shown below, write a math sentence, and then find the answer.

Amount available ÷ Price of a set = Number of sets

$600 \div (\square + \square) = 600 \div \square$

$= \square$

Figure 6.2. Example displaying the use of parentheses to represent a quantity (Fujii and Iitaka, 2012, Grade 4, p. B10).

In addition, in the *Properties of Operations* sub-unit, there are arithmetic examples to determine the equivalence of two algebraic/ arithmetic expressions. This section does not involve a task, rather it involves arithmetic examples. By doing so the usage of equal sign and equivalence are explained in 4th grade. I think, they might develop students' understanding of algebra gradually in each grade starting with the third grade in which the symbols are used in place of quantities and numbers.

At the end of the sub-unit of properties of operations, multiplication is given place. I want to share the example below given in Figure 6.3 to demonstrate the use of arrow mark which might be used to consider *times as large* relationship.

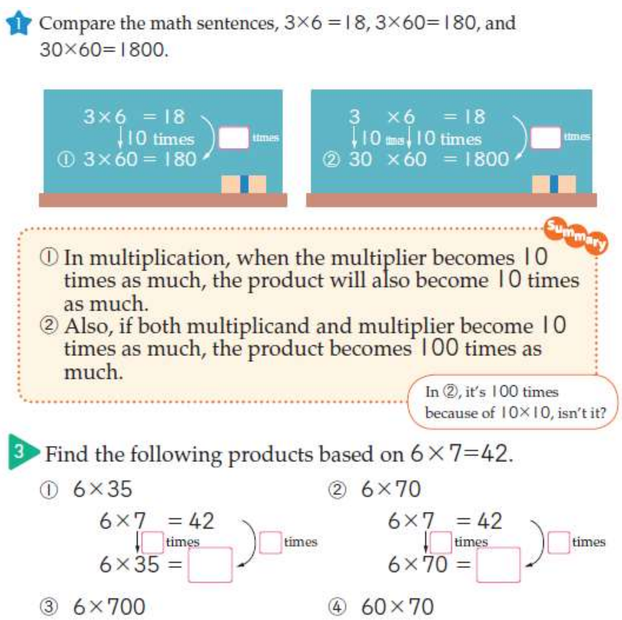


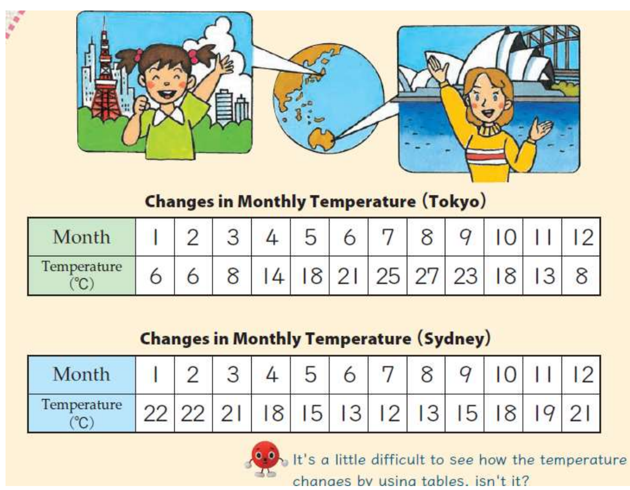
Figure 6.3. Multiplication example (Fujii and Iitaka, 2012, Grade 4, p. B14).

Although, this example is not directly related with the functional relationships, I share it to express both the equivalence relations and representation of multiplication. That is, this example displays the emphasis on the relationship of the result and the amount of increase for each number in the multiplication, which might contribute to students' proportional understanding.

<u>6.2.1.3. The units of textbooks regarding covariation.</u> As I mentioned before in this section, I share the findings coming from the unit with the title of *Let's represent how the quantities change in graphs* given which seems to trigger students' quantitative and covariational reasoning.

<u>*Let's represent how the quantities change in graphs.*</u> The unit of *Let's represent how the quantities change in graphs* starts with the introductory story including the temperatures of two cities Tokyo (from Japan), located in the north hemisphere of the World, and Sydney (from Australia), located in the south hemisphere of the World. There are four pictures with four thermometer pictures, each representing the temperature of the month April, July, October, and January. The pictures are used to show seasonal and temperature related differences between these two cities.

<u>*Task2*</u>. The unit starts with *a table* including monthly temperatures for both Tokyo and Sydney given in below Figure 6.4. Then, it is asked to represent temperature change with *a different graph*. *Bar graph* is given in the textbook to display the values of temperature for each month. In order to examine and represent change in quantities *the broken-line graph* (Figure 6.5) is introduced in the textbook.



Changes in Monthly Temperature (Tokyo)

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temperature (℃)	6	6	8	14	18	21	25	27	23	18	13	8

Changes in Monthly Temperature (Sydney)

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temperature (℃)	22	22	21	18	15	13	12	13	15	18	19	21

Figure 6.4. Tables for monthly temperatures of Tokyo and Sydney (Fujii and Iitaka, 2012, Grade 4, p. A77).

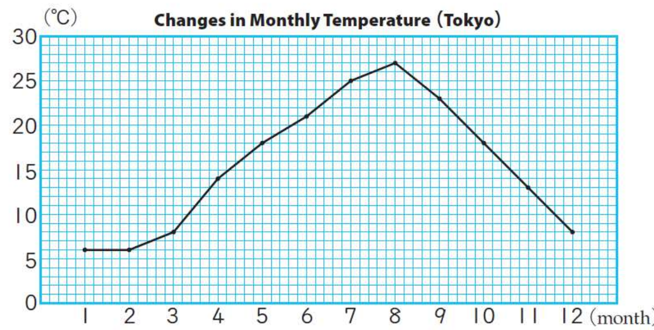


Figure 6.5. Broken-line graph representing monthly change in temperature in Tokyo (Fujii and Iitaka, 2012, Grade 4, p.A78).

With this *broken-line graph* for temperature changes in Tokyo given above, students are asked to express which quantities are placed on the *x* and *y* axes, the intervals on the *y*-axis, the highest temperature, the temperature of March, and the month having the temperature of 18°C. These questions are important to trigger the understanding of covariation because verbal indication of axes and the coordination of values, such as, highest temperature corresponding to the specific month, triggers the *mental action 1*, which is defined by Carlson and her colleagues. As mentioned in the theoretical framework, according to Carlson and her colleagues (2002), mental actions defined for covariation framework categorize students' behaviors when they engage in a task involving covariation. The behavior for mental action 1 is given as "labeling the axes with verbal indications of coordinating the two variables" (Carlson *et al.* 2002, p. 357).

The following questions asked of students to examine the temperature change using the broken-line graph, given below in the Figure 6.6, more explicitly indicate the intend for triggering the mental action 1.

1. Between which months is the temperature going up?
   Between which months is the temperature going down?

2. Between which months is the temperature not changing?

3. Between which two months does the temperature go up the most from January to August?
   Between which two months does the temperature go down the least from August to December?

Figure 6.6. Questions to examine change in quantities. (Fujii and Iitaka, 2012, Grade 4, p. A79).

In addition to the *mental action 1* (*MA1*), data above further suggests that the *mental action 2* (*MA2*) which requires to recognize direction of change is intended to be examined through these question (given in Figure 6.6). I infer the intention of mental action 2 because the steepness of the slant of the broken-lines formed by joining the two successive data are examined as increase, decrease, or no change. These seems to indicate the intention to guide student to realize the direction of change. I deepen my claim further while presenting more of the unit of the textbook.

In the task, the next question for students is to form *a broken-line graph* for Sydney, just like the previous one given for Tokyo. Then, students are asked to draw the broken-line graph for Tokyo on the same graph paper that they draw the graph for Sydney so that they could examine the similarities and differences between the two data sets.

Furthermore, there are two more questions to examine broken-line graphs and change in quantities. Yet, they are not using the quantities given in Task2. These questions are similar to the questions asked in Task2, but they have their own context. The first one asks to draw *a- broken line graph* to represent hourly temperature change on a specific day (June25). And the other one is presented with a figure including *a bar graph* to display amount of tomatoes sold in a city market and *a broken-line graph* to indicate price of 1kg tomatoes, shared below in the Figure 6.7. The figure involves two different kinds of data:

one is amount and the other is price. Students are supposed to answer questions, such as when the greatest amount of tomatoes sold and when the price is highest.

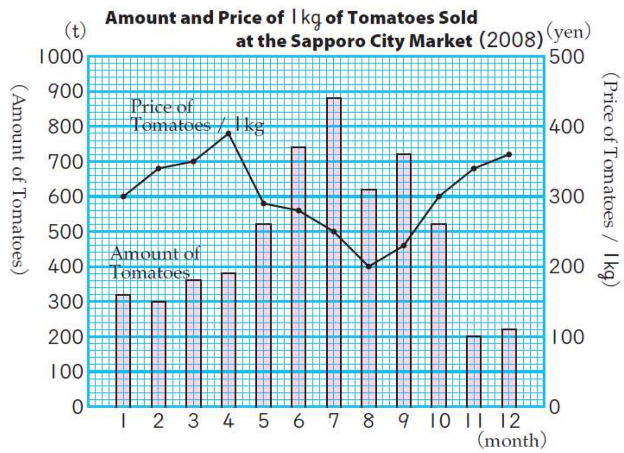


Figure 6.7. Amount and price of tomatoes (Fujii and Iitaka, 2012, Grade 4, p. A83).

As the data above show, in the unit Task2 presented above includes quantities chosen from *real life situations*. The situation in tables and in graphs (both *bar graph* and *broken-line graph*) are given and students are asked to represent the quantities through graphs and are supposed to explain the relationships between two quantities that vary simultaneously by taking advantage of these tables and graphs. For instance, regarding the task, the questions given in Figure 6.6 corresponding to the tables and graphs shown in Figure 6.4; and Figure 6.5 are designed to unveil the relationship between two covarying quantities one of which is temperature and the other of which is time (month). Temperature could be considered as a *continuous variable* and the name of the month could be considered as a *categorical variable* such that students are asked to think about the increase, decrease and constancy of temperature in different months. Similarly, they are asked to compare the temperature between different months. These suggest that although students are asked to think about both the name of the month and the temperature simultaneously, given the months they are asked to think about the change of temperature in chunks. That is, students are expected to focus on the direction of change (increase, decrease, constancy) of temperature given the name of the months such that in Figure 6.2 the horizontal line with numbers 1,2,3,..12 corresponds to the names of the months. This suggests that from the *covariational perspective*, the task seems to contribute to the images of covariation that "…can support the mental action of coordinating the change of one variable with change in the other variable." (Carlson *et al.*, 2002, p.358). In other words, students' being asked to match 1,2,3,…,12 with the names of the months and being asked to think about the increase, decrease of temperature within these months seems to suggest that students are asked to think about 1,2,3 ….,12 as corresponding to one-month periods. In *the teachers' guide* it is also stated that

> To interpret characteristics of change on a broken-line graph means to visually perceive the change of one of the two quantities which vary simultaneously while the other increases, and to make the relationship between two varying quantities clear. To do this, it is necessary to see if quantities increase or decrease from the gradient of line. (Isoda, 2010a, p.117)

Therefore, I construe the data given above as appropriate to the *gross coordination of values* which is a sub-category of covariational reasoning framework shared by Thompson and Carlson (2017). Taşova and his colleagues (2018) also explained this category in their framework, which they developed to analyze written curriculum by considering (co)variation reasoning. They stated, *"Gross coordination of values* involves representing two variables or quantities whose values increase or decrease together without mentioning the individual values of variables as varying together in the narratives." (Taşova *et al.*, 2018, p. 1530). Since both coordination of values and direction is aimed in the course of study and supported through tasks and questions in the textbook, it might be said that in the 4th grade, mental actions in both *MA1* and *MA2* are intended to be triggered, which were involved in the *gross coordination of values* level.

To sum, the relationship between two quantities that vary simultaneously is actually a functional relationship. Choosing the covarying quantities, i.e. quantities having proportional relationship, and representing their relationship by tables and broken-line graphs form a base for linear functions, because covariation is a necessary concept for functional relationships. However, in this grade level, students are not supposed to consider covarying quantities which are directly chosen as continuous variables. Thus, the idea of linear function and developed sense of covariation are not aimed in this level. My argument follows: First, the mental actions involved in *MA1* and *MA2* seemed to be triggered on the part of learners. Similarly, the idea of linear function and a developed sense of covariation requires someone to engage in higher mental actions. Furthermore, according to Noelting (1980), covariation is multiplicative in nature. Thus, the placement of this objective in the course of study might indicate to the focus on developing students' multiplicative schemes, which might enhance students' understanding of ratio, rate, and proportion in the future. Hence, the idea of linear function and a developed sense of covariation has not been fully aimed at 4th grade. On the other hand, although there is no specific objective about ratio, the intention of using proportional quantities might indicate the aim of preparing students by focusing on quantitative relations and covarying quantities during the process of developing students' conceptions of ratio, algebra, arithmetic, and functions. Moreover, students are provided with tasks to emphasize times as measure issue might contribute their understanding of the multiplication as a numerical operation and their multiplicative reasoning.

6.2.2. Grade-5 Japanese Curriculum Materials

<u>6.2.2.1. Content related objectives and the explanations given in the teachers' guide.</u> In the *fifth grade*, ratio is mentioned the first time under the quantities and measurements domain. The content related objective is stated as

> "…*Students will understand the average of measured quantities and the ratio of two unlike quantities*." (Takahashi *et al.*, 2008, p. 13).

The ratio of two unlike quantities part of the objective is further explained in the content section as "*Students will understand how to represent and compare quantities that are obtained as a ratio of two unlike quantities. (a) To become aware of per-unit quantities.*" (Takahashi *et al.*, 2008, p. 14).

In the teachers' guide, this objective is clarified that there are some quantities that could be represented through multiplicative comparison of two quantities, i.e. the ratio of two quantities, such as population density. The (intensive) quantity of *population density* is formed by comparing two unlike quantities: the population and the area. It is suggested that teachers create situations for students to compare quantities such as length with the quantities forming the ratio of two different quantities. That is, students are introduced to *intensive quantities* like population density and are supported to compare and contrast intensive quantities (e.g., ratio) with extensive quantities (e.g., length). I infer that working with intensive quantities which actually expresses a measure, such as population density, and examining situations involving intensive quantities might contribute students' conception of *ratio as measure*. Similarly, students' comparison of extensive quantities and intensive quantities might contribute to their focusing on and differentiating different types of quantities which in turn might enable them to differentiate proportional situations from non-proportional situations.

Moreover, linking the objective in the fifth grade with the objective in the fourth grade involving the examination of the relationships between two quantities varying simultaneously, in the fifth grade, students are expected to deepen their understanding about covarying quantities. In the content section of the course of study, it is stated that "*Students will use tables to consider the relationships between two quantities as the quantities vary simultaneously. (a) To recognize proportional relationships in simple cases.*" (Takahashi *et al.*, 2008, p.14). Through this objective, students' understanding of functional relationships is targeted to be developed. Tables are suggested to be used to examine the relationships between covarying quantities. It is further stated in the teacher's guide that "*It is important to deepen the way of viewing quantitative relations through interpreting the characteristics of how quantities correspond and how they vary by utilizing tables.*" (Isoda, 2010a, p. 140). This goal is consistent with the *dynamic definition of variable* which supports the idea of considering the simultaneous change in one quantity corresponding to the other quantity (Leinhardt *et al.*, 1990). As I explained in the literature review, variables are objects of functions and developing dynamic meaning of variable contributes to the conception of functions by realizing dynamic relationships between input and output values. Moreover, the expression of *simple cases* in the above quote indicates proportional relationships between two quantities varying simultaneously, particularly, which one quantity becomes two, three, four times bigger, while the other one also becomes two, three, four times bigger (Isoda, 2010a).

Again, in the quantitative relations domain in the fifth grade's course of study, algebraic expression is given place as

> *Students will deepen their understanding of mathematical expressions that represent quantitative relationships. They will be able to recognize the correspondence between two quantities and aspects of variation in a quantitative relationship that can be represented by a simple mathematical expression*. (Takahashi *et al.*, 2008, p. 14)

In the teachers' guide, this goal is explained for teachers as follows:

> To be able to interpret the meaning of algebraic expressions, teachers need to provide ample opportunities to examine the characteristics of how two quantities that vary simultaneously correspond or change by examining tables, and ample opportunities to represent the relationship between two quantities with algebraic expression using words. Also, to deepen the understanding of the meaning of algebraic expressions, it's important to broaden students' functional thinking. (Isoda, 2010a, p. 141)

These objectives and teachers' guide explicitly display that students are expected to consider and express quantitative relationships through using different representations, i.e. tabular, graphical, and algebraic. Similarly, it is emphasized that students focus on quantities varying simultaneously.

<u>6.2.2.2. The units of textbooks regarding the concepts of ratio.</u> In the 5th grade textbook, the content related objective is introduced in the unit *Let's think about how to compare*. In the textbook there are two units with this title. The first one is focused on averages and per-unit quantities; and the other is concentrated on percentages and graphs. The objective about covarying quantities are held under the *Extending Mathematical Thinking*. In the fifth grade, there are two extending mathematical thinking parts constituting tasks to highlight quantitative relationships of covarying quantities. In the following sections, first, I share findings about the units of the textbook. Then, I present my findings about extending mathematical thinking parts.

<u>*Let's think about how to compare (1).*</u> The unit which is presented as 7th unit in the textbook includes two sub-titles: 1. Average and 2. Per unit quantity. The first part of the unit involving averages starts with three cases including the action of evening out. The meaning of the expression "evening something out" is explained prior to the beginning task in the unit. The beginning task includes different amount of juice when 6 different oranges are squeezed. The aim is to introduce *average* concept and provide students to interpret situations by using and working with averages.

Later on, the *per unit quantity* part of the unit begins with an introductory problem expressed with the title "Which is more crowded?". The introductory part involves three cases to compare and make interpretation about crowdedness of two pools including the same number of people but different sizes, two classrooms having the same size but different number of students, and two playgrounds having the same size and the same number of students, but the placement of students are different as shown in Figure 6.8 below. After this introduction, to emphasize the quantities needed to be considered for an interpretation about crowdedness, the beginning task is shared.

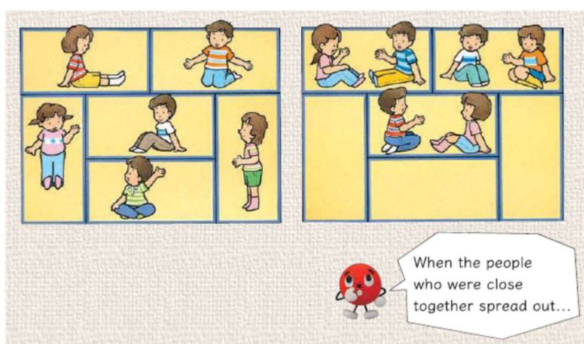


Figure 6.8. Example from the introductory problem (Fujii and Iitaka, 2012, Grade 5, p. A92).

<u>*Task1A.*</u> A task about rabbit cages (the introduction of the task is given below in Figure 6.9) are presented to students in order to investigate population density.

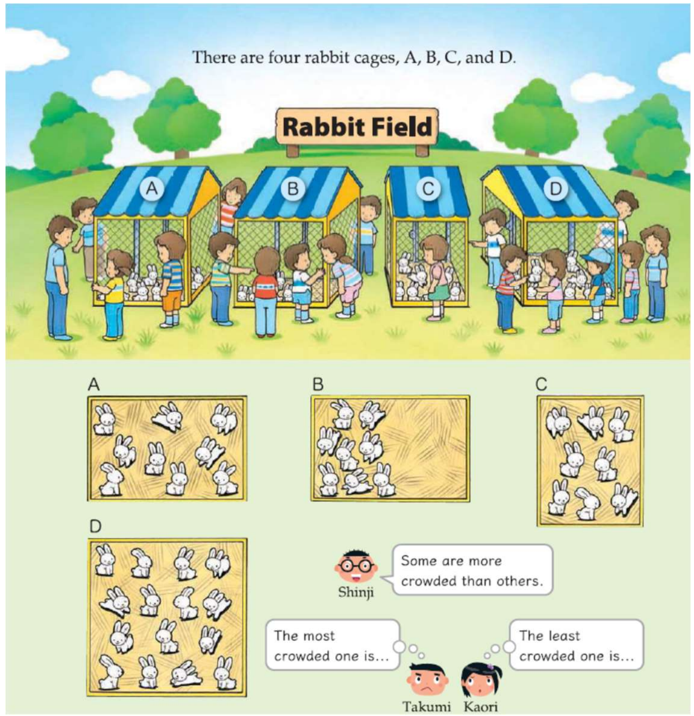


Figure 6.9. Rabbit cage problem (Fujii and Iitaka, 2012, Grade 5, p. A93).

As a first question, quantities which are needed to compare crowdedness of cages are asked. Then, a table involving area of the cages and number of rabbits in each cage are shared in response to the question (shown in Figure 6.10).

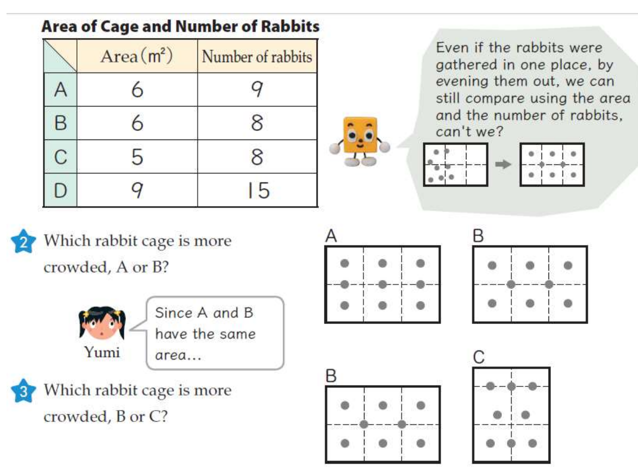


Figure 6.10. Table of quantities to determine crowdedness (Fujii and Iitaka, 2012, Grade 5, p. A94).

As a second question, the cages A and B are asked students to compare in terms of the crowdedness. In the third question, it is asked for the comparison of the cages B and C. These two questions include cases which have one of the two quantities is common either area of cages or number of rabbits. Students could answer these questions (i.e., compare crowdedness) with using additive reasoning such that fixing the common quantity such as the area of cages or the number of rabbits, they can focus on only the other extensive quantity. Thereafter, in the fourth question, the cages A, C, and D are asked to compare regarding crowdedness, but they do not have a common quantity. To compare these cages, three different student strategies are presented as shown in Figure 6.11.

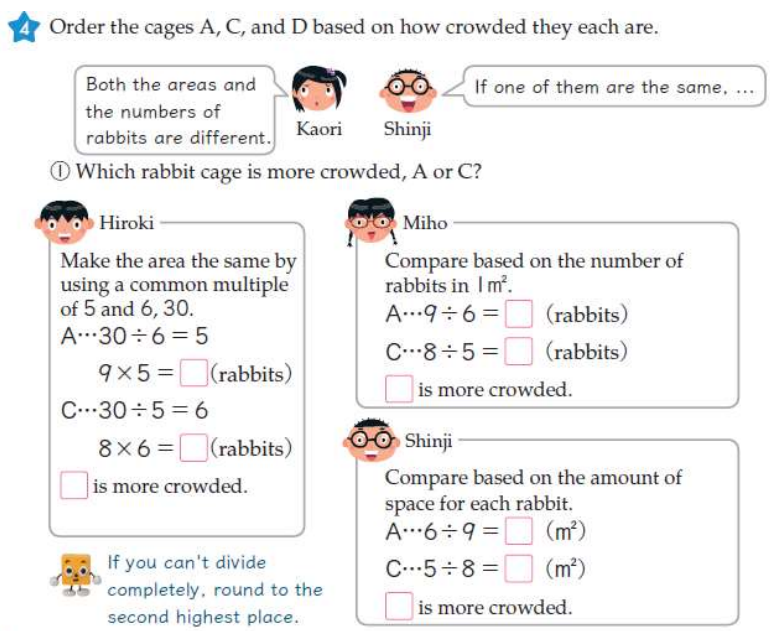


Figure 6.11. Strategies of students (Fujii and Iitaka, 2012, Grade 5, p. A94).

The strategy used by Hiroki is compatible with *abbreviated building up* strategy and the others used by Miho and Shinji are compatible with *per-one strategy*. Kaput and West (1994) expressed that abbreviated building up strategy is placed in the quotative division scheme, while per-one strategy is positioned in the partitive division scheme. That is, for abbreviated building up, thinking of 6 m^2 with 9 rabbits and 5 m^2 with 8 rabbits as associated extensive quantities in groups, one might think of how many 6 there are in 30 and how many

5 there are in 30. Finding out the results as 5 and 6 respectively and thinking that the associated groups need to be kept the same for the crowdedness to stay the same, one can determine that she needs to multiply 5 with 9 and 6 with 8, resulting in 45 rabbits and 48 rabbits. That is, as it is seen from the Figure 6.11, with the Hiroki's method to approach the problem, students' quotative division scheme might be activated. Again, it is important to re-state that Hiroki's strategy is additive in nature: the focus of the student is on fixing one extensive quantity (such as making the area 30 m^2) and comparing the other extensive quantity (e.g., the number of rabbits). Similarly, the result is constructing identical groups of 6 m^2 with 9 rabbits and 5 m^2 with 8 rabbits. That is, five groups of 6 m^2 with 9 rabbits and 6 groups of 5 m^2 with 8 rabbits are made. In addition, the partitive division scheme might be activated with the other students' methods displayed in the figure. For instance, Miho seems to think that for the associated group of 6 m^2 with 9 rabbits, for 1 m^2, there are 1,5 rabbits and for the associated group of 5 m^2 with 8 rabbits, for 1 m^2 there are 1,6 rabbits. Again the important point is that the per-one strategy is additive in nature. These two strategies are also emphasized in the teachers' guide that teachers should use them in their teaching.

In the teachers' guide, it is stated that teachers are expected to "help students think about how they can compare and quantify the situations" (Isoda, 2010a, p.134) with the aim for students to compare situations by using proportional relationships between quantities. It is exemplified in the teachers' guide as follows:

> In general, since there are two kinds of quantities involved, we can usually use the method of fixing one of the quantities and compared other quantity. When we use this way of thinking, we assume that two quantities are in proportional relationships… For example, … when we want to compare the degree of crowdedness between 7 people in a 10 m^2 room and 10 people in a 15 m^2 room, if we make the area of the rooms 30 m^2, the number of people in each room becomes 21 and 20 people respectively, so that we can come up with the idea of comparing by equalizing the size of the areas. (Isoda, 2010a, p.134)

The excerpt above seems to aim at supporting students to use *abbreviated building up strategy* when they involve situations to compare two ratios, each is formed by two different quantities. This excerpt also fits with what is shown in Figure 6.11 in the textbook. In addition, *per one strategy* is also given place in teachers' guide to present students. In the document it is stated

> …it is more efficient to use per-unit quantities when comparing quantities that involve a ratio of two different kinds of quantities, particularly when we compare three or more such quantities or wish to be able to make comparison at any time. (Isoda, 2010a, p.134)

The above excerpt from the teachers' guide is also fostered in Figure6.12 below. It is explicitly stated in both the teachers' guide and the textbook that per- unit quantities should be used to compare quantities.

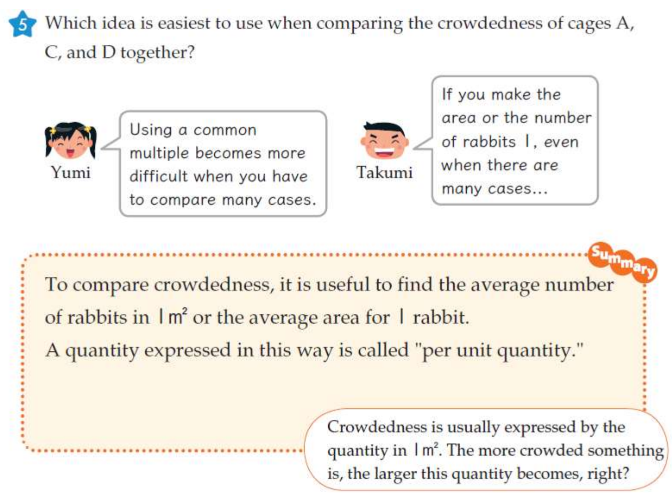


Figure 6.12. Per-unit quantity definition in the textbook (Fujii and Iitaka, 2012, Grade 5, p. A95).

There are some important issues Figure 6.12 points to: First, giving voice to two students, Yumi and Takumi, both the advantage of one strategy (*per one)* over the other strategy (*abbreviated building up)* and also in which situations this strategy is more effective is provided. Second, crowdedness is explicitly stated as a quantity (a per one quantity). This is very important because as Thompson (1990, 1994a, 2011) emphasized quantity is a measurable attribute of an object or a situation. The situation here is "a room with students inside". And the students are expected to think of a measurable attribute of such situation, the crowdedness. Third, although it is implicit, the measure of such quantity, i.e. crowdedness, is given as expressed by how many people in 1 m^2. More importantly, students are expected to think reversibly such that measuring crowdedness is possible by either finding the "average number of rabbits in 1 m^2" or the "average area for 1 rabbit".

In the unit, following the Task1A, there are two exercises presented to students to practice crowdedness of two situations (e.g., cabins and pool) with two unlike quantities which does not have any common value. In addition to crowdedness examples given through Task1A and the following two exercises, there are 3 different tasks to examine and use per-unit quantities.

Task1B. The task is to examine crowdedness of two cities involving quantities of populations and areas (given with the unit of km^2). Then, the term *population density* is defined as "the population per 1 km^2" (Fujii and Iitaka, 2012, Grade 5, p. A96). This definition might cultivate the understanding of intensive quantities, although the per-one strategy does not require the understanding of intensive quantities (Heinz, 2000).

Task1C. The task involves harvest of rice fields given and areas of the fields given below in Figure 6.13.

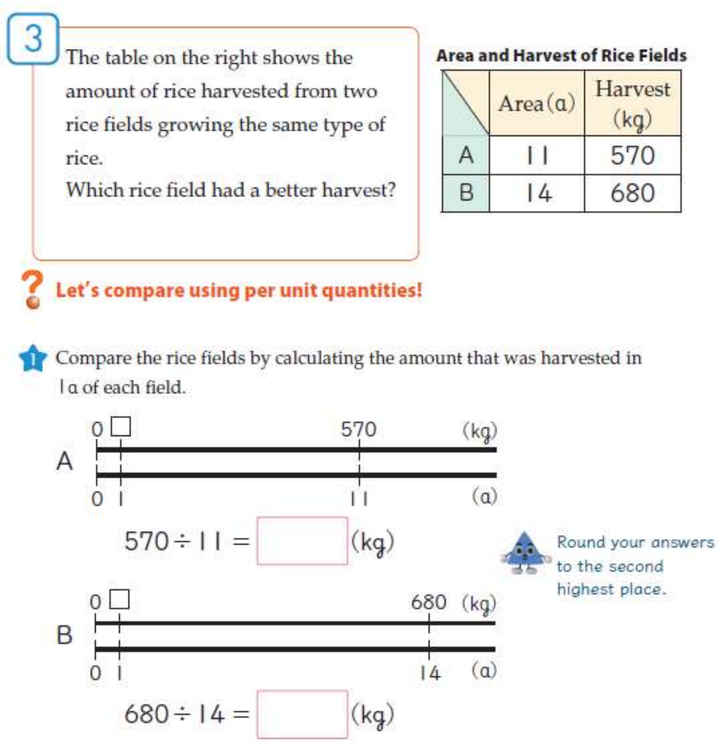


Figure 6.13. Harvest task (Fujii and Iitaka, 2012, Grade 5, p. A97).

As Figure 6.13 shows, students are asked to compare two fields regarding their harvests with the use of per-unit quantities. The representation used here in this task support the idea of matching the two unlike quantities to form a quality of the situation, better harvest, and then trying to keep the quality the same to find out per unit quantity (the measure of better harvest). One important distinction of Task1B from Task 1A, is that students are provided the number line representation. This is important for the following reasons: First, the match of 570 kg of rice on one of the number lines with the 11 m^2 on the other number line as well as the match of 1 m^2 with □ kg of rice suggests the simultaneous use of two quantities. That is, it seems that without explicitly stating, the idea of quantities varying simultaneously is emphasized. Secondly, *per one strategy* is used by dividing the two unlike quantities to find out how much of rice harvested for 1 m^2. Together these suggest that students are expected to go through the mental action of matching with the goal of sharing. Yet, the idea seems to be triggering the *identical groups conception* in the sense that the quality (attribute) in the situation, the better harvest, is kept the same in corresponded values of 570 kg and 11 m^2; and, □ kg and 1 m^2. That is, the values in between 1 and 11 or □ and 570 have not been emphasized. So, from the perspective of covariation framework, these data suggest triggering students' mental actions in *MA1*, because of the indication of coordination of dependence of one variable with respect to the other variable. Finally, although the per unit quantities are within state ratios in this task, there is no emphasis on it.

In addition to the task, there are two exercises to compare situations with using per unit quantities. One is about expensiveness of pencils and the other is about performance of two cars considering amount of gasoline and travelled distance. These also suggest that different situations are provided, and students are expected to think about the attribute in the situations such as expensiveness and performance. Similarly, they are expected to reason about ratio such that it is the measure of such attribute. Therefore, it is plausible to claim that, in the textbooks, students are provided with plenty of different real life situations for making sense of *ratio as a measure* of an attribute.

Task1D. For the task, students are supposed to express per unit quantities and compare situations using them. There are two questions shared in this task. It is given in the task that a wire is used in an artwork and it is 7g per 1m. The total weight of artwork is given as 52.5 g and students are asked to find out how long wire is used in this artwork as a first question (Fujii and Iitaka, 2012, Grade 5). This question requires to use given per-unit quantities to find out missing information. As it is seen at the Figure 6.14 below, students are supported to consider between state ratios. That is, it seems to explicitly points to the between state ratios such that 52.5 g is 7.5 times as large as 7 g and similarly 7.5 m is 7.5 times as large as 1 m. I argue that it is explicit because in the textbook the multiplication operation is provided (Figure 6.14) On the other hand, I claim that there might be an implicit emphasis on the within state ratios by the use of two line segments. In other words, the use of line segments seems to suggest that in the world of length, measured in meter, one seventh times of the quantity in the world of weight, measured in gram.

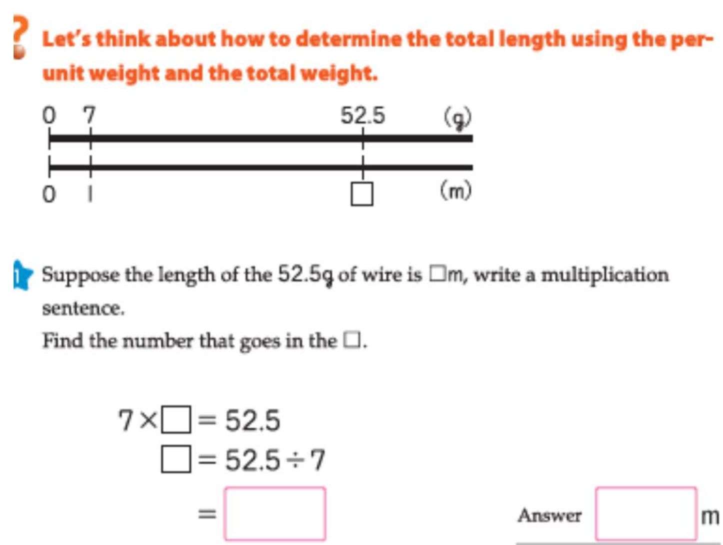


Figure 6.14. Art-work task (Fujii and Iitaka, 2012, Grade 5, p. A98).

Moreover, in the second question the weight of an artwork is asked if 6.4m wire is used. Students are supposed to express the differences between these two questions. Similar to the Figure 6.14 shared above, there are two line segments shared to display the situation: The top line segment represents weight while the bottom line segments shows length. In order to find the missing value, students might either use within state ratios (arithmetically speaking, 7 times larger than 6.4) or between state ratios (arithmetically speaking, 6.4 times larger than 7g) This task is important for the following reasons: First, it seems that between state ratios are explicitly used, while within state ratios are implicitly given in the textbook. Second, proportional relationships are emphasized with the focus on quantities. Third, in terms of task variables, both the numerical values of quantities are non-integer, and the multiples of quantities are selected from non-integer numbers. It is important to have non integer values of quantities and their multiples to challenge students in terms of divisibility failure.

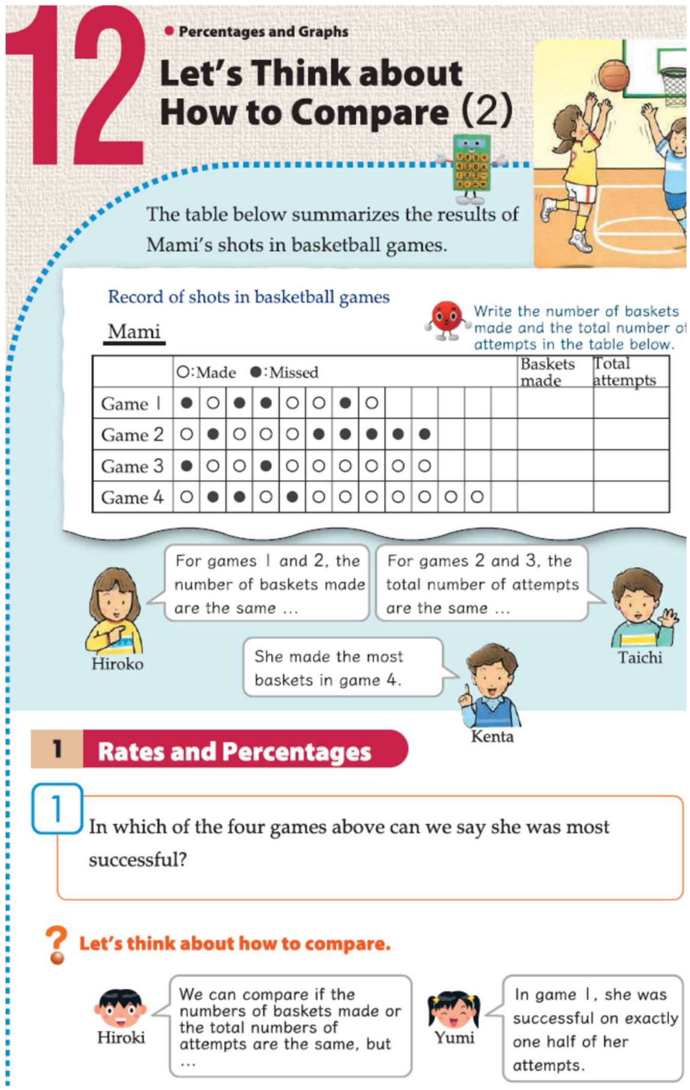
Additional parts. At the end of the unit, there are three parts named *Let's try*, *Power Builder*, and *Mastery Problems* respectively. In the *Let's try* section there are different real life situations where per-unit quantities are used and there are exercises to practice population densities of different prefectures of Japan. In the *Power Builder* section, there are problems using similar contexts with the problems given in the unit. Differently from the other problems in the unit, the first question in the Power Builder section requires students to think about relationships between number of students and the playground area. There is an inverse relationship between the playground area for each student and the total amount of playground area: When the populations of students decreased by 20 percent, the playground area per student increased by 1.25. In the last section called *Mastery Problems*, the rabbit cage context is used again to review how to compare with the use of per unit quantities. All these examples suggest that students are expected to make sense of and use *per one strategy* and *building up strategy*.

Moreover, there is an *Extension* section for students to practice the unit of per-unit quantities. The main theme for this section is to determine the cheapest beef among three different beefs given with one quantity as the weight and the other as the price for given weight of beef. Both amount of beef for 1 yen and price of 1gram beef for each case are asked. Then, it is asked to determine which value of price makes 100g beef cheaper than the given three and which value of weight makes the beef having 1000yen cost cheaper than other. I conclude that price per item and population density are used frequently as contexts throughout the unit.

Let's think about how to compare (2). The unit *Let's think about how to compare (2)* is introduced in the textbook with the relation of percentages and graphs. The unit is sub-categorized under three headings: (1) Rates and Percentages, (2) Percentage Problems, and (3) Graphs Representing Rates, respectively.

In the beginning of the unit, there are three cases to make a comparison. First case involves two identical go-game boards with different number of white and black stones and students are asked to compare which one has more white stones on it. The other one is to make a comparison for two classrooms having different number of students but the same number of students raising their hands in each. Students are asked to interpret in which class more people raising their hands. The last one comprises a table of records of shots in three basketball games which involves number of attempts (having the values of 8,8,10 respectively) and successful shots (having the values of 5,6,6 respectively). The unit starts with the task similar to the last case given in the introduction part, shared in Figure 6.15 below.

Task2A. The task given below involves successful shots and the total number of attempts for four basketball games. The scores and attempts are designed in a way that there are two situation having common value of one quantity. For the game 1 and 2, number of baskets are identical while for game 2 and 3, number of attempts are the same.



12 ● Percentages and Graphs

Let's Think about How to Compare (2)

The table below summarizes the results of Mami's shots in basketball games.

Record of shots in basketball games

Mami

Write the number of baskets made and the total number of attempts in the table below.

	○:Made ●:Missed												Baskets made	Total attempts
Game 1	●	○	●	●	○	○	●	○						
Game 2	○	●	○	○	○	●	●	●	●	●				
Game 3	●	○	○	●	○	○	○	○	○	○				
Game 4	○	●	●	○	●	○	○	○	○	○	○	○		

Hiroko: For games 1 and 2, the number of baskets made are the same …

Taichi: For games 2 and 3, the total number of attempts are the same …

Kenta: She made the most baskets in game 4.

1 Rates and Percentages

1 In which of the four games above can we say she was most successful?

Let's think about how to compare.

Hiroki: We can compare if the numbers of baskets made or the total numbers of attempts are the same, but …

Yumi: In game 1, she was successful on exactly one half of her attempts.

Figure 6.15. Basketball game task (Fujii and Iitaka, 2012, Grade 5, p. B51).

After the task is presented, students are asked to compare the success of Mami for the four games. Then, as it is displayed in the Figure 6.15, two students' ideas are shared: One of them, i.e. Hiroki's idea, triggers the *abbreviated building up strategy* to make a comparison after making one of the quantities identical. The other, i.e. Yumi's idea, triggers proportional relationship between shots and number of attempts which is actually success rate. The first question asked in the task is "In which game, was she successful on more than half of her attempts?" (Fujii and Iitaka, 2012, Grade 5, p. B51). The next question is about comparing game 3 and 4 and students are provided a table including the values of the quantities. To answer the second question a student idea (idea of Miho given in Figure 5.16) is shared, and the third question asked to determine a ratio.

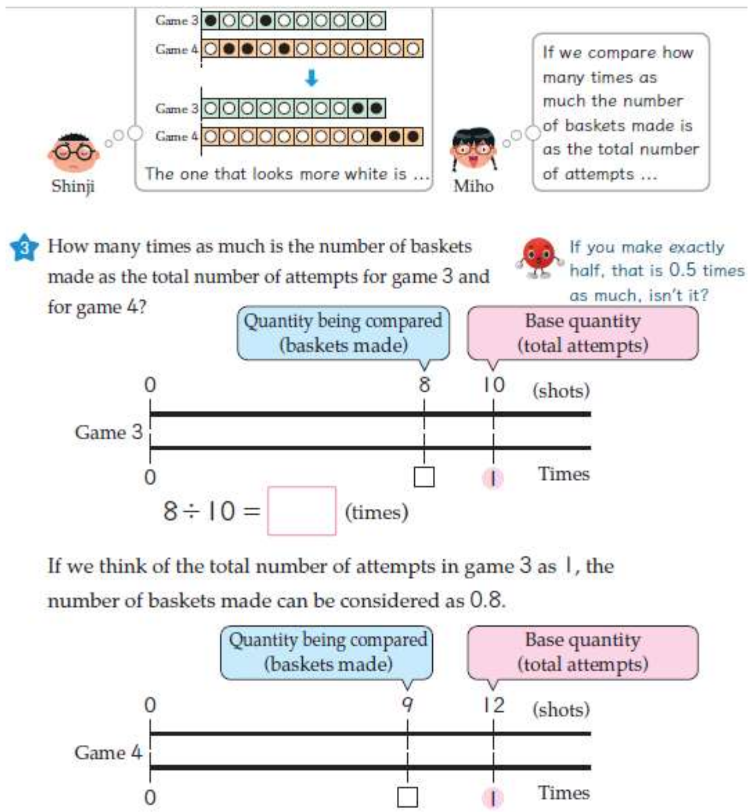


Figure 6.16. Rate example (Fujii and Iitaka, 2012, Grade 5, p. B52).

As it is seen in Figure 6.16 above, students are supposed to use a similar representation, number line, that they used to calculate *per unit quantities* in a previous unit. Differently from the previous unit, in this unit, two quantities are classified as *base quantity* and *quantity being compared.* The main aim is to provide students to express the situation by using *rate*. Rate is defined in the textbook as "*The number that expresses how many times as much a quantity is compared to the base quantity is called the rate*." (Fujii and Iitaka, 2012, Grade 5, p. B53). After the definition is shared, students are provided to a formula for rate which is formed by dividing the quantity being compared with the base quantity in a situation. Then, it is asked to compare game 1 and 2 by using rates. What is interesting about Figure 6.16 follows: First, the number line representation both involved part-whole situation where total number of attempts refers to the whole (the base quantity) and the baskets made refers to the part (the quantity being compared). More interestingly, this part whole relationship is further emphasized as a measurement process: With the inclusion of the second number line where the whole (the base quantity) is made into 1 required the part (the quantity being compared) to be times as much of the multiplicative relationship between the whole and 1. That is, students are expected to make sense of the fact that the multiplicative relationship between such as 10 and 1 is such that 1 is 1/10 times as much of 10, which also holds for the parts such that 0,8 is 1/10 times as much of 8. This is very important for three reasons: First, once students understand that given any whole (the base quantity), they can think of 1 in terms of that quantity (the whole) such that 1 is always (1/the value of the base quantity) times as much of the whole. That is, they measure 1 in terms of 10 and 1 is 1/10 times as much of 10. This might allow them to also take such multiplicative relationship as an operator such that they might think of the new part in terms of the original part (the quantity being compared). In the example above, the new part is always 1/10 times of the original part 8, (the quantity being compared). This in turn might allow them to think of the new part (0,8) in terms of the new base quantity (1). That is, the new part (0,8) is 8/10 times as much of 1. Secondly, this understanding might also allow students to make sense of percentages as the multiplicative relationship between the part and the whole. Third, this understanding might further allow students to think of *per one* as such that it is the result of measuring the first quantity in terms of the second quantity. The last claim actually is supported by the definition of rate given in the book "*The number that expresses how many times as much a quantity is compared to the base quantity...*" (Fujii and Iitaka, 2012, Grade 5, p. B53). What is striking is that "*the base quantity*" could represent the whole in a situation about a part-whole relationship or could represent the second extensive quantity in a situation as shown in Figure 6.13 (The Harvest task). In addition, this understanding seems to be very strong that students might make sense of proportionality.

In this task, the invariant relationship between quantities are not generalized and the sophisticated rate definition is not aimed to be introduced. Also, the quantities are discrete in nature; therefore, they are not much supportive to express the linear relationship that the concept of rate embodies.

Thereupon, a problem given a survey results of students' interest on committees are shared as given below (Figure 6.17). Students are expected to use the formula of rate to examine the questions which requires to compare situations. This problem is important to show an example to students that the rate could be bigger than 1, because the quantity being compared is outnumbering the base quantity. By doing so, students might be prevented to have a misconception about rate that the rate should be smaller than 1.

1 Tsuyoshi's school took a survey to see how interested students are in serving on various committees. The table on the right shows part of the results.

Survey on Student Interest in Committees

Committee	Maximum number of members	Number of students interested
Library Committee	30	24
Communication Committee	20	35

① For the Library Committee, find the rate of the number of students interested compared to the maximum number of members.

Figure 6.17. Students' interest on committees problem (Fujii and Iitaka, 2012, Grade 5, p. B53).

After the Task2A and the problem above, that is illustrated at Figure 6.17, Task2B is introduced.

Task2B. The task presented at below Figure 6.18 is aimed to provide students exercise the rate formula and introduce the percentage with the use of rate.

2 At Midori's school, there are 80 fifth graders, and 12 of them belong to the soccer club.
Find the rate of fifth graders who belong to the soccer club compared to the total number of fifth graders.

Figure 6.18. Introduction task to percentage (Fujii and Iitaka, 2012, Grade 5, p. B54).

The first question is to express the situation with a mathematical sentence. The second, it is asked to find rate of soccer players to all fifth graders at the school. Then, the rate value of 0.01 is defined 1 percent (1%) and representation of rate by using percent is defined *percentage* in the textbook. At the third and last question, students are expected to express the rate that found as a percentage.

Similar to actions aimed at the Task2A, students are expected to consider 1 is 1/80 times as much of 80 and the unknown would be 1/80 times much as 12. The task is chosen to use part to whole relationship between quantities to form rate and then to express rate as percentage. It is explained in the teachers' guide that "*When we consider that relationship between the whole and part of the data, or among parts of the data, we often use ratio.*" (Isoda, 2010a, p.141). The percentage is also defined as a ratio since it stems from the multiplicative comparison of two quantities (Parker and Leinhardt, 1995). As it is seen so far, in fifth grade Japanese textbook, rate and percentage are introduced to students at about the same time.

Moreover, Japanese terminology of proportion, which is named as *buai*, is presented in the textbook with a real life case. In regard to buai, rate of 0.1 is called 1 *wari*, rate of 0.01 is 1 *bu*, and rate of 0.001 is 1 *rin*. For instance, rate value of 0.315 could be expressed as 3 wari 1 bu 5 rin. After introducing rate, providing rate problems for students to practice, defining percentage and buai, *percentage problems* are given in the textbook.

The main aim in the percentage problems is to use given rate and base quantity to find *quantity being compared* or to use given rate and quantity being compared to find *base quantity*. In addition, there are problems which do not give the rate directly and requiring to find out *quantity being compared* or *base quantity*. The book is classified these three scenarios and provide students tasks for each, then give exercise problems for students to practice what they learn from each task.

In tasks 2C, 2D and 2E, first, students are expected to determine both the *base quantity* and *the quantity being compared* in the problem situations. Second, they are expected to reason about multiplicative relationship given in the problem situations.

<u>*Task2C.*</u> The task involves the consideration of fruit and water composition to examine a juice as given below Figure 6.19. The main goal in this task to find out *quantity being compared*.

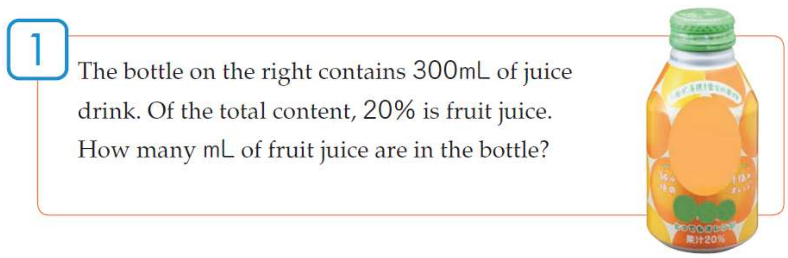


Figure 6.19. Juice task (Fujii and Iitaka, 2012, Grade 5, p. B56).

It is asked to students to express base quantity and quantity being compared. Then, 20% is asked to be expressed with decimal numbers. Finally, students are supposed to form a mathematical sentence and to perform calculation for quantity being compared. The task basically provides to use *arithmetic operations* to calculate the quantity being compared; however, it also requires to students to think about quantities and to express the multiplicative relationship between quantities. For instance, it is given in the book that "Since 20% of 300mL is 0.2 times as much of 300mL, 300×0.2= □" (Fujii and Iitaka, 2012, Grade 5, p. B57) and the answer is expressed with the unit of mL. Supporting the mathematical sentences to express multiplicative relationships between quantities might trigger the *quantitative operation* of "comparing quantities multiplicatively" (Thompson, 1994a).

In addition, the exercise problems involve the context of comparison of fruit and water composition for a juice, the capacity of a bus in terms of its passengers, and the capacity of a car in terms of the distance traveled and the fuel consumption. The percentage in some of them is excessing the base quantity, such as 120%.

<u>*Task2D.*</u> The task comprises the evaluation of the birth weight of a cat, which is a *base quantity* in this situation, when the percentage of its birth weight and current weight is given. As it is displayed in the Figure 6.20, the situation is illustrated with a graph, and the questions asked in the task is similar to the questions of Task2C presented above.

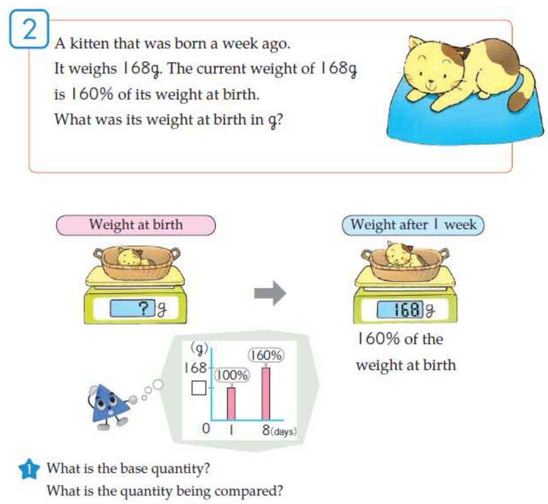


Figure 6.20. A cat's weight task (Fujii and Iitaka, 2012, Grade 5, p. B58).

Both for the Task2C and Task2D, percentages or rates are representing the *between state ratios*. However, the base quantity is <u>multiplied</u> by the rate to find quantity being compared in the former, the quantity being compared is <u>divided</u> by the rate to find base quantity in the latter. For both of them, rate is given as *times as much* of the base quantity. In the cat's weight task, the rate is 1.6 and it is expressed that *1.6 times as much of base quantity is 168g*. To find the base quantity students are supported to divide 168 with 1.6. Again, I could say that both arithmetic and quantitative operations are tried to be offered in the textbook.

In addition to the task, there are two examples for students the practice their learning. The examples of percentage problems requiring to find out the value of the base quantity includes calculations of price of a milk when the price is reduced and the protein consumption of a child's diet.

Task2E. The off price of a marker and the percentage of the sale is given in the task which is shown as Figure 6.21 below.

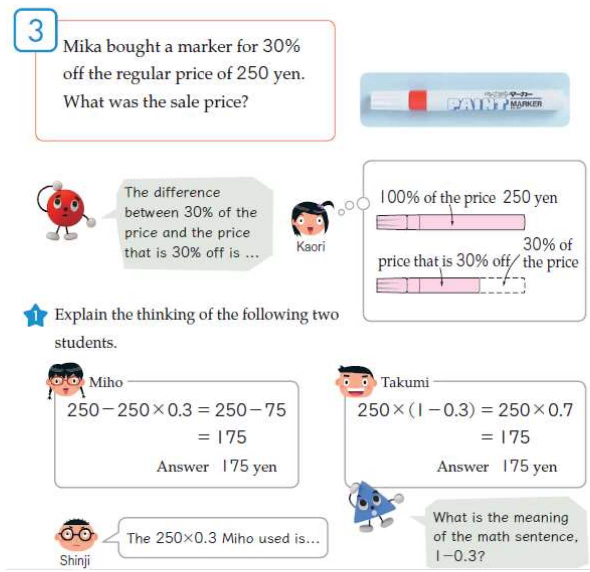


Figure 6.21. Marker task (Fujii and Iitaka, 2012, Grade 5, p. B60).

To find the sale price there are two students ideas shared. One could find either the off price and subtract it from the regular price like Miho did, or the rate of sale price which is 0.7 and multiply it with the regular price like Takumi did. Students are asked to examine both ideas. Students could express the difference by executing the arithmetic operation which is reducing the parenthesis. However, students are supported with the figures, tables, shared students' ideas, and the guidance questions to think about quantities and the meaning of numerical values. Throughout the *percentage problems* part, the task is getting difficult in terms of mathematical expressions and calculations.

After the percentage problems are given, pie charts and percentage bar graphs are introduced under the title of *Graphs Representing Rates*. The connection between representation of data, rate, and percentage is tried to be supported with the use of situations involving the comparison of quantities. In the teachers' guide, the place of representations in 5th grade Japanese mathematics education is expressed as "*Students should learn to make their purpose is clear such as: "show the size of quantities," "show the change of quantities," or "show the ratio of quantities." Help students select the appropriate table or graph for their purposes*." (Isoda, 2010a, p. 142). In relation to this sentence, there is an extension part named *Let's Think about Which Is a Better Deal!* in the textbook involving two rate problems. The problems in that part requires to use order of operation and compare two situations to find out which one is better. For instance, in the first problem, there is a toy having cost value of 2000yen. Two shops are selling the toy. One of the shops makes a discount by 20% and then adds 5% tax; whereas, the other adds 5% tax to original price and then makes a discount by 20%. Since both the amount of discount and tax are multiplied with original quantity, the cases are identical. Students are supported to work with the multiplicative situations by examining the size and change of quantity and comparing two different situations.

Furthermore, in this grade level, it is explicitly stated in the COS that students are expected to examine proportional relationships through working with covarying quantities. However, the definition of proportion is shared first time in this grade level while the concept of volume is presented. In particular, students are given with a task to examine the relationships between height and volume for a given cuboid which have constant width and length (i.e. 3cm and 5cm). Students are expected to examine the volume of the cuboid when the height is 1 cm, 2cm, 3cm, 4cm. They are asked to represent the proportional relationships of volume and height of a cuboid having a constant base: "Write a math sentence using a height of □cm and a volume of ◯cm^3." (Fujii and Iitaka, 2012, Grade 5, p. A20). Students are asked create a table to examine this proportional relationship and then introduced to the definition of *proportional relationships* as "Suppose there are two quantities, □ and ◯. If ◯ becomes 2, 3, … times as much while □ becomes 2, 3, … times as much, we say that "◯ is proportional to □"." (Fujii and Iitaka, 2012, Grade 5, p. A20). I claim that sharing this definition in this grade level within the volume concept might be a scaffold for the direct proportions. Since it is not directly presented in a functional relationships unit, I do not share a detailed analysis on this volume task in regard to quantitative and covariational reasoning.

<u>6.2.2.3. The units of textbooks regarding covariation.</u> As I explained at the beginning of the presentation of the textbook units, the extending mathematical thinking part is designed for the objective which is about covarying quantities. In the following paragraphs, I share my analysis about tasks that are introduced in the extending mathematical thinking parts.

<u>Extending Mathematical Thinking (1).</u> The first content related extending mathematical thinking part is named as *Looking for Patterns: Thinking with diagrams, tables, and mathematical expressions*.

<u>Task3.</u> The task requires to recognize the pattern of an arithmetic sequence, use tables and algebraic expressions to work with the sequence, but it is not mentioned in the task that the pattern is actually a sequence. The figure 6.22 displays the task.

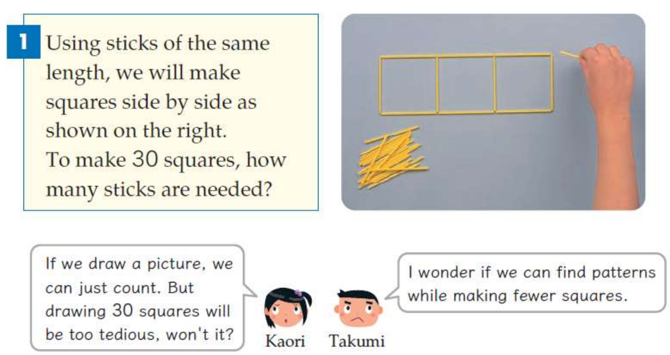


Figure 6.22. Stick and square task (Fujii and Iitaka, 2012, Grade 5, p. A103).

The first question is about finding number of sticks to make 5 squares. Then there are two students' ideas shared in response to the first question shown as below Figure 6.23.

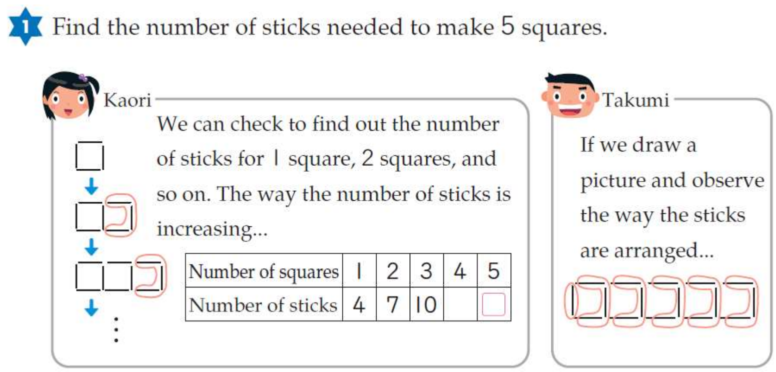


Figure 6.23. Students' ideas-1 on stick and square task (Fujii and Iitaka, 2012, Grade 5, p. A103).

The variables of number of squares and number of sticks used in this task are both *discrete* in nature. As it is shown in the Figure 6.23, Kaori is making a table and examine the pattern through table. She considers the changes between two consecutive values of number of sticks for each quantity, e.g. number of squares. However, Takumi examines the arrangement of sticks in the picture such that given the first stick his focus is on how many more sticks is needed for each square. That is, given the first stick for the first square he needs 3 more; for the second square he need 3 more and so on.

After the change in the number of sticks with the increase in the number of squares by 1 is asked in the second question and the explanation of Takumi's thinking is asked in the third question. Then, the number of sticks needed to make 30 squares are asked as fourth question. The second and third question are aimed to help students to recognize the *common difference* in the number of sticks for each new square. Students' ideas are shared in response to the fourth questions as given Figure 6.24.

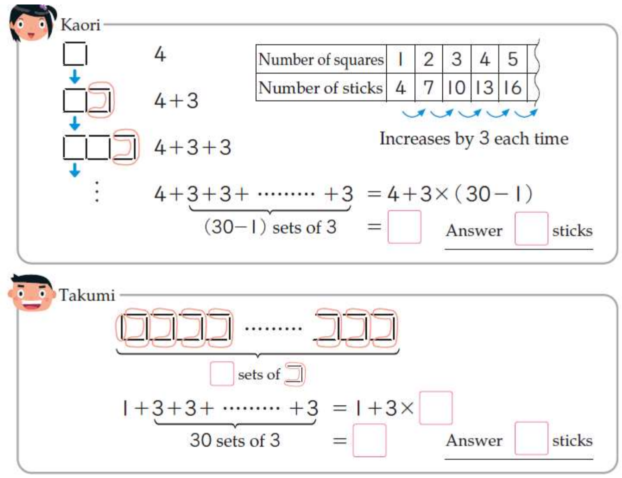


Figure 6.24. Students' ideas-2 on stick and square task (Fujii and Iitaka, 2012, Grade 5, p. A104).

Then, students are supposed to examine the differences between two students idea given above which improve their sense of mathematical operations. In response to the sixth question of the task, students are expected to verbalize what the values of 3, 4, 30, and (30-1) represent. Then, Kaori's method is supported to use to find out how many sticks needed for 50 squares and the mathematical sentence $4 + 3 \times (\square - 1) = \square$ is given already. At the end, students are supported to think about how to represent the situation with using Takumi's method. As it is emphasized in teachers' guide, this task provides students opportunities to examine the correspondence between two quantities and to foster their understanding of algebraic expressions. However, the task is worth analyzing further. From the covariational perspective, it seems that although the solutions of Kaori and Takumi differs, their reasonings are similar. In the following paragraphs, I examine each students' reasoning in terms of covariational reasoning.

Kaori presented quantities, i.e. number of squares and number of sticks, in chunks at her table such that number of squares is increasing by 1 while the number of sticks is increasing by 3 each time. Since her table illustrated the correspondence of increase in number of squares with the increase in number of sticks, and her explanation supports that she considers "the way the number of sticks increasing…" (Fujii and Iitaka, 2012, Grade 5,

p. A103), Kaori's behavior underscores the *MA3* of the covariational reasoning framework. That is, existence of MA3 indicate that Kaori's covariational reasoning level might correspond to *coordination of values*.

Nevertheless, although Takumi did not present through a table that for each new square there is 3 more sticks needed, it seems that he recognized the relationships of for each new square there has to be 3 more sticks. He considered not only the coordination of the direction of change, i.e. coordination of increases in both number of sticks and number of squares, but also the amount of change, i.e. increase in the number of squares by 1 and increase in the number of sticks by 3. However, he did not explicitly present the coordination of the amount of the increase in the number of squares by 1 requires the increase in the number of sticks as 3, as Kaori did. Therefore, it might be safe to claim that his behavior seems to underscore the *MA2* and the corresponding covariational reasoning level is *gross coordination of values*.

In sum, in the fifth grade, percentage is presented as the measure of multiplicative part-whole relationship. The use of number line and as a result forming a new part-whole expression, e.g. the Task2A, contributes students multiplicative reasoning. That is to say, students could compare quantities additively, but the use of number line has impact to move students' additive comparison to multiplicative comparison. Moreover, in regard to task variable, throughout the 5th grade discrete variables are intended to be used such as, number of rabbits, number of people, number of baskets, and price. However, in some tasks, continuous variables like area, weight, and volume of a liquid are used to be compared with discrete variables. It is given in the literature that students are compare two discreate and representable quantities more easily than the continuous and non-representable quantities, because students could use counting and matching with the discrete and representable quantities to solve the problems (Heinz, 2000). In addition, the numbers chosen for the tasks and problem situations are getting harder, e.g. in the introduction tasks whole numbers are used while the unit flows the decimal numbers and rational numbers are used. Likewise, although ratio is not explicitly presented to students in the fifth grade, students are engaged with multiplicative comparison in this grade level.

Regarding to covariational reasoning, students might be triggered to coordination of values and gross coordination of values level of covariational reasoning with their engagement with the tasks and problem situations.

6.2.3. Grade-6 Japanese Curriculum Materials

6.2.3.1. Content related objectives and the explanations given in the teachers' guide. In *the sixth grade*, the content related objective is stated as "*Students will understand ratio and direct proportion, use their understanding to consider quantitative relationships, and represent relationships in mathematical expressions by using letters.*" (Takahashi *et al.*, 2008, p. 15). This objective is further explained in the content section under the quantitative relations domain of the course of study that:

> (1) Students will understand ratio.
>
> (2) Students will be able to consider the relationship between two quantities varying simultaneously.
>
> > (a) To understand proportional relationships and to investigate their characteristics by using mathematical expressions, tables, and graphs.
> >
> > (b) To solve problems by using proportional relationships.
> >
> > (c) To become aware of inversely proportional relationships.
>
> (3) Students will deepen their understanding of mathematical expressions that represent quantitative relationships and be able to use them.
>
> > (a) To represent numbers and quantities in mathematical expressions by using letters such as a, x, etc. instead of using words, □ and Δ, and to investigate relationships by substituting numbers for the letters. (Takahashi *et al.*, 2008, p.16)

Regarding the above quotes, the teachers' guide expends the explanation of the content related objective regarding *ratio, proportional relationships,* and *algebraic expressions with letters*.

Students are expected to work with per-unit quantities to represent the ratio of two unlike quantities in 5^{th} grade and here in 6^{th} grade they are introduced a more formal definition of *ratio*. The definition is stated as "*When comparing the size of two quantities and representing their proportions, we sometimes use a pair of simple whole numbers to* *represent this without using a unit quantity. This is called ratio*." (Isoda, 2010a, p.154). Since this definition involves the comparison of two quantities to represent their proportions, I could say that it is compatible with Thompson's definition of ratio which mentions the multiplicative relationships between quantities.

In addition, the teachers are suggested to emphasize the meaning of the equality of the ratios and it is explained with an example of a mixture case involving 3 to 5 cups ratio as below:

> If you want to mix these two kinds of liquid to get a liquid of the same concentration, the ratio of the two kinds of liquid should stay the same- such as 6 cups for one and 10 cups for the other or 9 cups for one and 15 cups for the other. From this fact, help students to understand that 3:5 is equal to 6:10, 9:15 or 1.5:2.5. (Isoda, 2010a, p.154)

By emphasizing the equality of ratios and to form equal ratios in order to keep the quality of the mixture same, the *identical groups conception* might be triggered. Since students' having identical groups conception cannot cope with divisibility failure, if there are examples involving divisibility failure in the textbook, *within state ratio conception* might be triggered.

Concerning the *proportional relationships* part, as it is given in the above passage of the course of study that it is explicitly stated and anticipated students to investigate the relationships between covarying quantities through proportional situations. Unlike the goal for direct proportional relationships in which the words "understand" and "investigate" are used, the goal for inversely proportional relationships is to raise students' "awareness" of the situations having inversely proportional relationships. Moreover, the emphasis of *representing proportional relationships through mathematical expressions with the use of letters* implies the representation of linear relationships existing in proportional situations. In the teachers' guide, this idea is supported that for the direct proportional relationships as "*(c)The quotient of two corresponding quantities remains constant*" and "*An algebraic expression that represents the proportional relationship, if the quotient in (c) is k, is* $y = k \times x$." (Isoda, 2010a, p. 155). The rationale behind this expression is explained as below:

> Generally speaking, a graph that represents a proportional relationship is a straight line passing through the origin. This is an important characteristic used to distinguish proportional relationships. Here, it is important to teach students using concrete numbers, and through the activities such as representing on a graph two quantities that vary simultaneously so that they understand that if two quantities are in a proportional relationship, the graph representing this relationship is a straight line. (Isoda, 2010a, p. 155)

Representation of direct proportional relationships as $y = k \times x$ requires developed conception of proportionality. This representation contributes the correspondence and covariation meaning of functional relationships.

Moreover, the quote of "*the product of the two corresponding quantities remains constant*" (Isoda, 2010a, p.156) indicates the mathematical expression of $y = \frac{1}{k} \times x$ for the inversely proportional relationships. However, the mathematical expressions of $y = \frac{1}{k} \times x$ is not explicitly presented in the teachers' guide and I think this is consistent with the goals, because the goal is only to create awareness of inversely proportional relationships.

In relation to the *algebraic expressions with letters* part, the content related objective is stated as the third and the last item given in the course of study passage above. It is aimed to introduce formal algebraic expression to represent quantitative relationships, so that students' abstract thinking might be cultivated through quantitative relations. In the teachers' guide, teachers are suggested to use decimal numbers and fractions in addition to whole numbers to substitute into the algebraic expressions, so that students can improve their understanding of range of numbers.

<u>6.2.3.2. The units of textbooks regarding the concepts of ratio.</u> In the 6th grade textbook, the content related objective is introduced through four units and two extending mathematical thinking parts. The units are named respectively as *Let's use letters and write math sentences*, *Let's think about how to express proportions*, *Let's think about how to express speed*, and *Let's investigate proportional relationships*. Also, the one of the *Extending Mathematical Thinking* parts is named as *Fixing the whole: Thinking with diagrams* and it includes task to work with inversely proportional relationships. The other

one is named as *Looking for Patterns: Thinking with diagrams, tables, and mathematical expressions* and the purpose of this section is to provide students a task to work with quantities varying simultaneously. In this section, I share my findings about the units of *Let's think about how to express proportions* and *Let's think about how to express speed* in order to explain how the quantitative relations and concepts of ratio are given in the textbook. Then, I present my finding about the other units and extending mathematical thinking parts in which students are expected to reason covariationally.

Let's think about how to express proportions. This unit of textbook is specified on ratios and values of ratios. The unit is presented as three sub-units having titles of *Ratios and values of ratios, Properties of equivalent ratios,* and *Applications of ratios*. After I express the introductory part of the unit, I represent the tasks and the questions from the first sub-unit of the textbook and go forward with the others.

Providing four different recipes, namely noodle sauce, juice, milk coffee, and steak sauce, the introductory part of the unit starts with asking a question "In what proportions should we mix?" (Fujii and Iitaka, 2012, Grade 6, p. A60). Each recipe involves two substance and the ratio of each substance used to make the recipe is given. The first task of the unit is established on the last example, i.e. steak sauce.

Task1A. In the introductory part, a student Tadashi uses 2 cups of Worcestershire sauce and 3 cups of ketchup to make a steak sauce. Mika plans to prepare 2 servings by using Tadashi's recipe while Ken plans to prepare 3 servings by using the same recipe. In the explanation in the textbook it is stated that both Mika and Ken try to make their steak sauces' *tastes the same as* Tadashi's. The main goal of the task, displayed in Figure 6.25, is to introduce the concept of *ratio* to students.

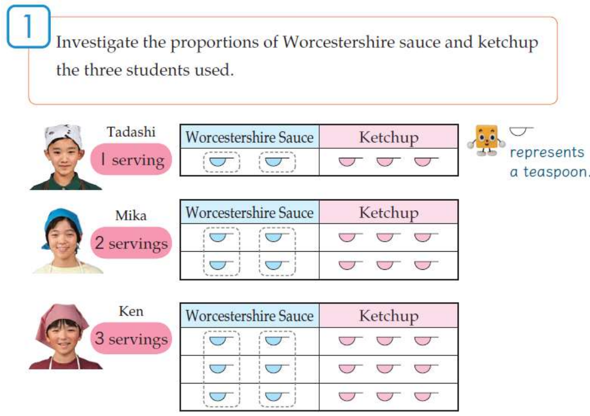


Figure 6.25. Steak sauce task (Fujii and Iitaka, 2012, Grade 6, p. A61).

First, the questions are asked to make students verbally express the amount of ketchup used by Mika and Ken in their servings. The main aim seems to highlight the extensive quantities that come together to make up a ratio of 2 to 3. Then, the expression 2:3 which is used to represent proportional relationships is introduced as ratio. In a student's idea balloon, it is given that "If you use ratios, you can express the same proportion in many different ways, can't you?" (Fujii and Iitaka, 2012, Grade 6, p. A62). It is important to ask this question because the same proportion indicates the sameness of the quality, i.e. the taste of the sauce. The same taste could be attained by using different amounts of ingredients if the proportion 2:3 is preserved.

The next task also uses the same context of steak sauce. The aim is to express the *value of the ratio* of 2:3 as $\frac{2}{3}$ and make a comparison of rate and ratio. The comparison is made as "A ratio is a way to express a proportion using 2 numbers while the rate we learned in the 5th grade expresses the proportion using 1 number" (Fujii and Iitaka, 2012, Grade 6, p. A63). Just after giving this comparison, the rate values for both Mika's and Ken's recipes (having the ratios of 4:6 and 6:9 respectively) are asked to calculate by making the division. Since students will get the results as the same value, it suggests that they are expected to investigate the invariant relationship of the situation defined as rate: That is, there can be different ratios such as 2:3, 4:6 and 6:9, representing the same relationships, i.e. having the same proportion such that the same (invariant) relationship refers to rate how many times 3 goes into 2 or how much of 2 is shared among 3. Similarly, the same invariant relationship exists for the ratios of 4 to 6 and 6 to 9 such that the invariant relationship is 0,6 cups of Worcestershire sauce per 1 cup of ketchup. These explanations seems to be compatible with Kaput and West (1994) definitions of particular ratio and rate-ratio.

In terms of *the task variables*, the task involves the quantities having integer multiples of each other as 2 to 3, 4 to 6, and 6 to 9. Use of integer multiples might help students to realize the multiplicative relationships and to conduct building up and abbreviated building up strategies more effectively (Kaput and West, 1994). Since students are introduced to the abbreviated building up strategy in the 5^{th} grade, they might realize that by using the abbreviated building up they are actually forming equivalent ratios. In addition, the task involves mixture situation, and it is claimed that students have difficulty dealing with mixture problems because of the original qualities of the quantities are lost (Kaput and West, 1994). However, the task does not involve any problem situation to compare ratios to find the unknown, rather it comprises all the values of the ratios and students are expected to realize the multiplicative relationships.

Task1B. The *Properties of equivalent ratios* sub-unit includes three tasks, and these tasks are prepared as a follow-up of each other. Thereof, I present all through Task1B.

On the top of the Task1A, the first task starts with the same ratios representing Tadashi's, Mika's, and Ken's servings of the steak sauce, which are 2:3, 4:6, and 6:9. It is given in the task as shown in Figure 6.26 that these ratios are equivalent.

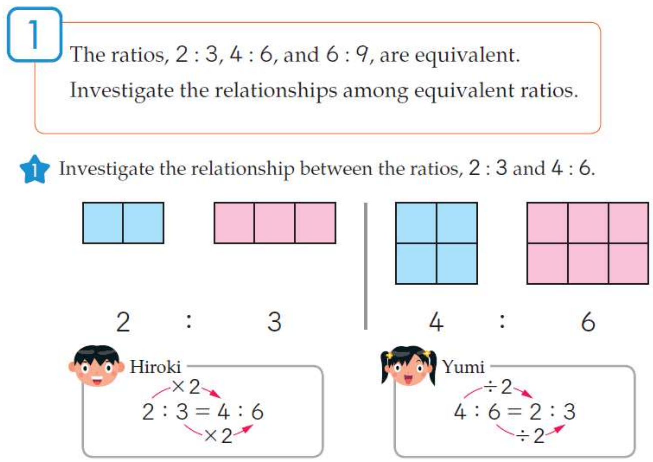


Figure 6.26. Equivalent ratios task (Fujii and Iitaka, 2012, Grade 6, p. A64).

Students are supposed to examine the relationships that equivalent ratios have. For this purpose, two of the three values of ratios are selected to provide students to make a comparison. At first 2:3 and 4:6, then 2:3 and 6:9, and lastly 4:6 and 6:9 are chosen to examine the relationships between them. As it is displayed in Figure 6.26 above, the relationships between 2:3 and 4:6 are demonstrated with rectangles having unit squares. Students are supposed to consider the relationships between two *representable* and *discrete* quantities, namely the blue rectangle and the pink rectangle. They might investigate the relationships by simply counting and matching without paying attention to multiplicative relationships. Still, what seems important is that students realize that the amounts change while taste of the steak sauce stays the same. Then the ideas shared by Hiroki and Yumi (in Figure 6.31) support students to be aware of the relationships in between-state ratios namely 2 cups of Worcestershire sauce and 4 cups of Worcestershire sauce; and, 3 cups of ketchup and 6 cups of ketchup. That is, in Task1A students were asked to think about the invariant relationship among the quantities in within-state ratios, namely cups of Worcestershire sauce and cups of ketchup, whereas in Task1B students are asked to think about quantities in between-state ratios. However, it is important to restate that students' use of multiplication and division as operations does not indicate that they realize the multiplicative relationship between quantities. It is also worth mentioning that the task might target to allow students recognize the situation involving multiplicative relationship by focusing on numerical properties. That is to say, the task might trigger students to use arithmetic operations but not quantitative operations. Though, the first two comparisons, namely 2:3 and 4:6 as well as 2:3 and 6:9, are integer multiples of each other. However, in the last one, the ratio of 6:9 is 1.5 times of the ratio of 4:6. This suggests that throughout the task, students are gradually introduced to integer multiples and then non-integer multiples.

In the next task, students are expected to examine whether the ratios of 4:10 and 6:15 are equivalent. There are three students ideas shared in response to the task as given below in Figure 6.27.

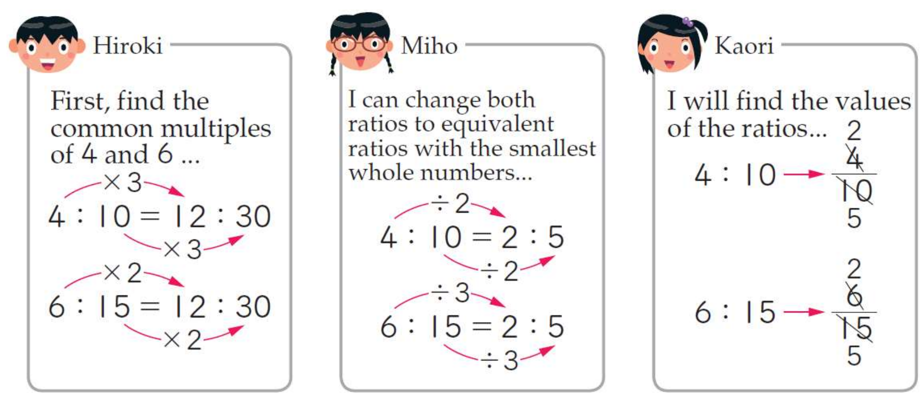


Figure 6.27. Students' ideas involving abbreviated building up (Fujii and Iitaka, 2012, Grade 6, p. A65).

The method used by Hiroki and Miho is compatible with *abbreviated building up*. To determine the equivalent ratios, Hiroki scaled up the actual ratio values, whereas Miho scaled down. The method used by Kaori is called *simplification* and is compatible with Miho's. The two representations (4:10 and 4/10) used in Kaori's explanation also suggests that students are expected to realize those two representations as compatible. In addition, simplification of ratios is defined in the textbook that the ratio is simplified to be expressed as the smallest integer values. Lastly, Kaori's statement "…the values of ratios…" suggests that there is an emphasis such that 4 and 10 are not only the extensive quantities involved in ratios but also they refer to those quantities' values (i.e., the magnitude).

In the last task of the sub-unit shown in Figure 6.28, equivalent ratios having non-integer multiples are provided for students. This suggests that students are expected to recognize that values of ratios could also involve decimals and fractions.

3 Which ratios in Ⓐ~Ⓒ are equivalent to 3 : 5?

Ⓐ 27 : 45 Ⓑ 0.9 : 1.5 Ⓒ $\frac{2}{3} : \frac{4}{5}$

Figure 6.28. Ratio values having decimals and fractions (Fujii and Iitaka, 2012, Grade 6, p. A66).

Since the cases (B) and (C) in Figure 6.28 are newly introduced situations, there are two students' ideas given to express methods to get equivalent ratios to their values. One of the methods is to transform the values of the ratios to the integer values. For instance, the ratio 0.9:1.5 is multiplied by 10 to make the ratio having integer values and then simplify to examine whether it is equivalent to 3:5. For the case of (C), ratio $\frac{2}{3}:\frac{4}{5}$ is multiplied by the common multiple of the denominators $\left(\frac{2}{3}\times 15\right):\left(\frac{4}{5}\times 15\right)$ to get a ratio involving integer values as 10:12, and then the ratio is simplified to 5:6. Students are encouraged to make the given ratios in the form of a ratio involving the values as integer numbers before they form the equivalent ratios. The other method is predetermining a unit so that the ratio can be expressed as a ratio having the values of integer numbers. For the case (B), the exemplified student idea is to determine 0.1 as the unit, so that the ratio can be expressed as 9:15. As I mentioned before, it is found easier for students to consider 9:15 rather than 0.9:1.5 to form equivalent ratios (Kaput and West, 1994). For the case (C), the fraction ratios are expressed in a way to have common denominator as $\frac{10}{15}:\frac{12}{15}$. Then, $\frac{1}{15}$ is decided as the unit, hence the ratio is stated as 10:15.

I claim that especially the use of units such as 0.1 and 1/15 in the cases (B) and (C) is strong: Students might realize that any quantity in two between-state ratios such as 0,9 (cups of Worcestershire sauce) and 9 (cups of Worcestershire sauce); and, 1,5 (cups of ketchup) and 15 (cups of ketchup) is a rational multiplier of the other respectively. Similarly, any quantity in a within-state ratio such as $\frac{10}{15}$ (cups of Worcestershire sauce) and $\frac{12}{15}$ (cups of ketchup) is a rational multiplier of the other quantity.

After some similar drill questions are provided, the *Application of ratios* sub-unit is placed as the next. The sub-unit involves two tasks, Task1C and Task1D, which are further comprised by mixture problems.

<u>*Task1C.*</u> The task displayed at the Figure 6.34, comprises a mixture problem.

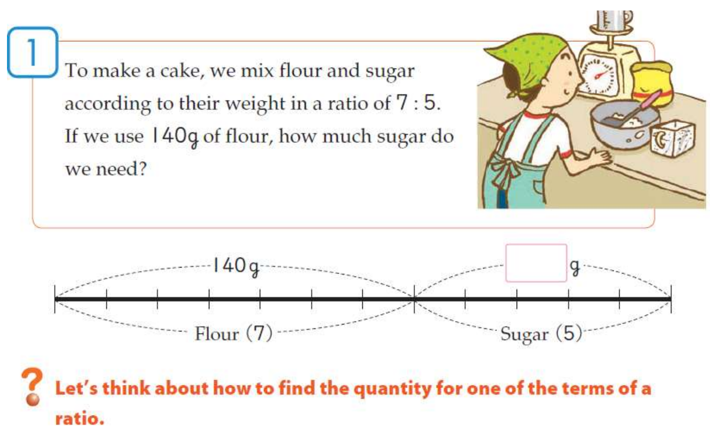


Figure 6.29. Making a cake task (Fujii and Iitaka, 2012, Grade 6, p. A67).

As it is illustrated in Figure 6.29 above, *the size of the quantities* are explicitly presented through a line segment. The whole segment represent the mixture and the whole is partitioned into parts corresponding to the quantities. By the use of this display, students might match the quantity 140 g with 7 and amount of sugar with 5 such that this image might also trigger the use of quotative division by measuring 140 grams by 7 and x grams by 5. That is, students might think that how many 7 there are in 140 holds for how many times 5 there are in x, which results in that x is 20 times as much of 5. Also, the use of the line segments might assist students to represent continuous quantities like weight. Research shows that students find questions of ratios involving non-representable and continuous quantities are harder than the ones involving discrete and representable quantities (Heinz, 2000). This is because representable quantities could assist students to match and count to solve the questions. Therefore, the use of line segments provide a representation for students while preserving the continuous nature of quantities.

In response to the task there are two student- ideas shared in the textbook. I share them in Figure 6.30 below. Students are asked to explain these ideas.

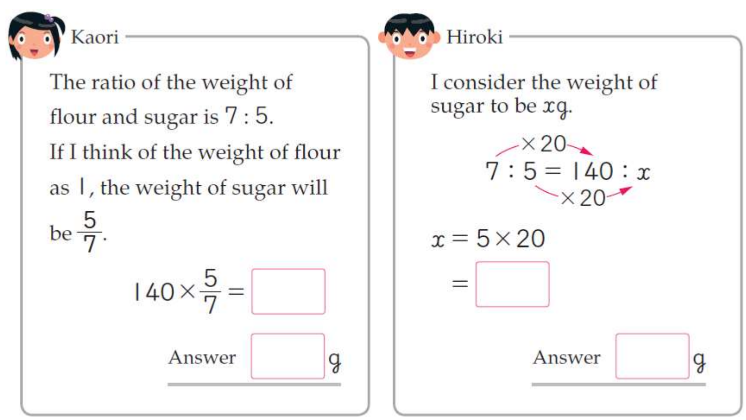


Figure 6.30. Students' ideas on making a cake task (Fujii and Iitaka, 2012, Grade 6, p. A67).

As it is seen from the Figure 6.30, Hiroki's method is comparing 7 with 140 to find out how much the quantity of flour increases. Since both quantities forming the ratio 7:5 have to be increased by same multiple, the amount of sugar is found by the multiplying 5 with 20. As mentioned earlier, Hiroki's explanation in fact aligns with the representation in Figure 6.34. More interestingly, Hiroki's idea actually fosters the consideration of *between state ratios*. The 20 corresponds to *increment multiple*. Again, it is hard at this point to claim that 20 is understood as the multiplicative relationship such that 20 is the amount by which 5 is increased. Though, Kaori's explanation triggers the use of rate idea learned in the 5th grade. Kaori actually is using the *within state ratios* such that the expression $\frac{5}{7}$ represents the *linear relationship* of the situation, i.e. invariant relationship. two quantities coming together to make up a ratio. That is, $\frac{5}{7}$ represents two extensive quantities coming together to keep the quality of interest--the taste of the mixture--the same. Though what is striking is that Kaori states "if I think of the weight of flour as 1, the weight of sugar will be 5/7". This is strong because Kaori thinks that given 7 units of flour there are 5 units of sugar and decreasing the amount of flour by $1/7^{th}$, also operates on the amount of sugar such that it decreases by $1/7^{th}$, resulting in 5/7. Such thinking is multiplicative in nature triggering the idea of *times as much*. And, since students know that the quality of interest, the taste, need to be kept the same, they know that they can multiply 140 with $\frac{5}{7}$ because 140 is made up of 140 times 1. Thus, Hiroki's idea seems to be thinking of the result of the division of quantities in *between-state ratio*, 140 by 7, as representing the *increment multiple* (the abbreviated building up strategy), while Kaori's idea of *within-state ratio*, $\frac{5}{7}$ as the invariant relationship, seems to be that $\frac{5}{7}$ is the amount by which140 is reduced. That is, in Kaori's explanation, $\frac{5}{7}$ seems to be targeted to be the operator acting as reducer. Though, there is no questions in the task to discuss about what 20 and $\frac{5}{7}$ represents. Rather, students are asked to explain the students' ideas provided in Figure 5.30. Then, there are two problems shared –a word problem and drill-and-practice exercises– to let students practice through similar questions.

Task1D. The task is similar to the Task1C in terms of task variables. However, in the Task1D shown in Figure 6.31, since students are provided with the whole amount of the mixture and asked to find the amount of the part quantity milk, they need to consider splitting the whole amount at first then finding the part. So, the idea is triggering the part-whole relationship.

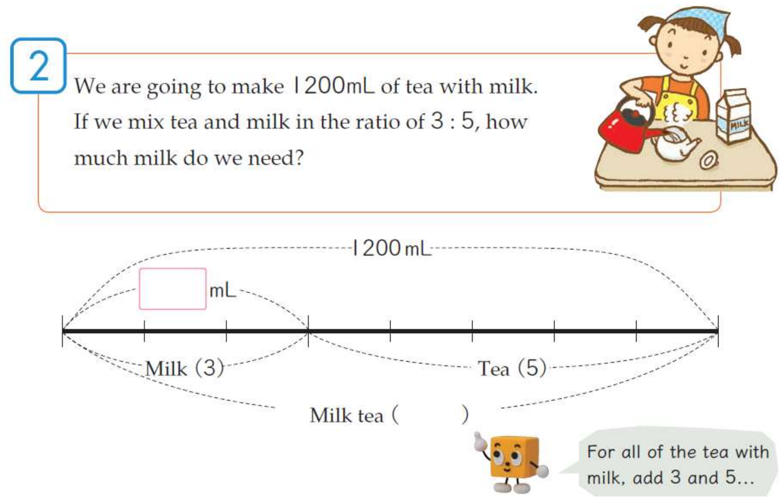


Figure 6.31. Tea with milk task (Fujii and Iitaka, 2012, Grade 6, p. A68).

The use of line segments again seems to help students to split the whole amount into its parts, namely tea and milk. There are also two students' ideas shared in Figure 6.32 to illustrate how to approach the task.

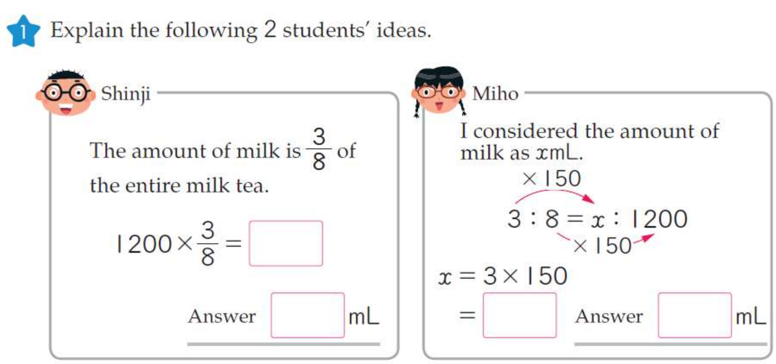


Figure 6.32. Students' ideas on tea with milk task (Fujii and Iitaka, 2012, Grade 6, p. A68).

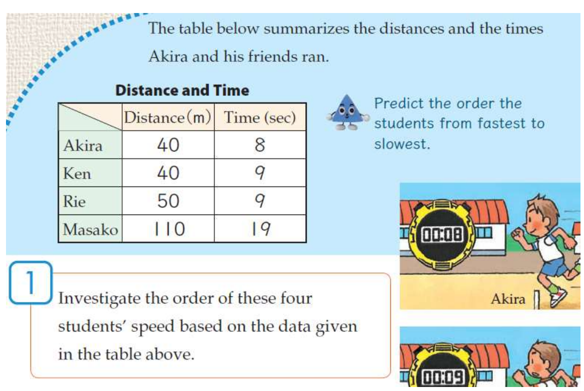
As Figure 6.32 shows, Shinji uses the part-whole ratio $\frac{3}{8}$ to represent the amount of milk to the whole mixture and do multiplication to find the unknown quantity. Although in the text, there is no question asking the meaning of 3/8, the expression of the ratio $\frac{3}{8}$ seems to involve the *times as large* meaning of multiplication: 3 is $\frac{3}{8}$ times as large as 8. Also, $\frac{3}{8}$ represents a *within state ratio* such that 3 represents the amount of milk in the whole mixture of 8. Again as in the Task2A, the case of Kaori's explanation in Figure 6.30, students might be expected to reason that "If I think of the whole mixture as 1, the amount of milk will be 3/8 as much". Then, multiplying 1200 with 3/8 makes sense because the amount of milk is 3/8 times as much of the whole mixture. For the conception of ratio as a multiplicative comparison, students need to understand multiplication as *times as large*. Though, it is important to re-state that such reasoning has not been explicitly stated as previously shown in Kaori's explanation in Figure 6.30. Regarding, Miho's explanation in Figure 6.32, data suggests that the use of *increment multiple* might be fostered on the quantities of *between state ratios.* On the other hand, Miho's strategy might be compatible with thinking of the result of division of the quantities, 150, in between-state ratios, in Figure 6.32 as *times as large.* Though this is not explicit. Still, together with Kaori's explanation in Figure 6.30, the two students' explanations in Figure 6.32 might be triggering the proportion equation strategy which is a multiplicative strategy involving both within and between state ratios.

Additional parts(1). At the end of the Task1D, there is a practice question similar to the task shared. Then, there is a *Math Story* section in which ratio of three numbers are exemplified with two situations: Japanese style salad dressing and triangle. The section express the dressing including 70mL vegetable oil, 50mL vinegar, and 30mL soy sauce with a ratio 70:50:30. Also it is given that the sides of a triangle could be expressed with a ratio of three numbers. In addition, there are two *Extension* parts designed to provide geometric situations requiring working with ratios, such as splitting the side lengths as well as areas of the regular polygons. The Japanese textbook explicitly states and exemplifies the inter-subject connections, such as ratio and geometry.

Let's think about how to express speed. The unit of textbook starts with an actual experience of tracking the time when the distance is fixed and distance when the time is fixed. Students are expected to use stopwatch and experience what going "slow" and "fast" means. This actual experience is compatible with the phenomenon *motion*. Thompson (1994) explained

Piaget's work on development of moment and speed concepts of children as "…*the emergence of concept of speed as a process wherein children first re-conceive motion as entailing changes in position simultaneously with changes in time, then coordinate the two dimensions of distance and time as changing in proportion to one another.*" (as cited in Thompson (1994) as Piaget, 1970, pp.279-280). Here in this unit of the textbook, students are supposed to be engaged in motion at first (through Task2A) and then introduced to speed. Through Task2A to Task2G students are expected to engage with strategies that they have learned so far to compare extensive quantities of distance and time in order to develop understanding for speed concept as an intensive quantity. I briefly talk about each of those tasks to exemplify how the quantity "speed" is constructed in the textbook and then I examine the Task2G in terms of the covariational relationships between extensive quantities, i.e. time and distance, and the invariant relationship among extensive quantities, i.e. speed. Finally, I share the Task2G which involves speed as an expression of the amount of work done in a unit of time.

Task2A. The task given below in Figure 6.33 is for students to compare the speed of four people. The two quantities are expressed through a table in which the distance values for Akira and Ken are the same and the time values for Ken and Rie are identical.



	Distance (m)	Time (sec)
Akira	40	8
Ken	40	9
Rie	50	9
Masako	110	19

Figure 6.33. Comparing speed task (Fujii and Iitaka, 2012, Grade 6, p. A83).

The first two questions are asked to compare the cases when one of the quantities' values are the same, i.e. Akira and Ken and Ken and Rie. Then in the third question students are asked to *compare* the speeds of Akira, Ken, and Masako. In the fourth it is asked to compare the speed of Rie and Masako; and, in the fifth question the easiest way of comparing the speeds of Akira, Ken, and Masako is asked. In response to this question, three different students' ideas are provided as displayed through Figure 6.34 below.

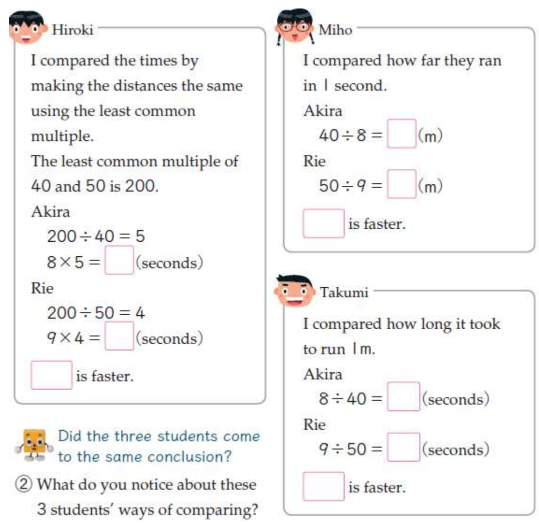


Figure 6.34. Students' ideas on comparing speed task (Fujii and Iitaka, 2012, Grade 6, p. A84).

First of all, asking students to compare the speeds when one of the quantities' values are the same requires students to focus on only one extensive quantity keeping the other as invariant. Then, comparing the speeds of Akira, Ken, and Masako; and Rie and Masako, could be established based on the first question and also reasoning on *between-state ratios* as students had developed engaging in tasks earlier. For the fifth question, Hiroki's use of common multiple to make one of the quantities the same is compatible with *abbreviated building up strategy*. The others are compatible with *per-one strategy* that Miho uses it to make time as the unit whereas Takumi uses it to make the distance as unit. The reversibility of time as unit (how much distance in one unit of time) and distance as unit (how many minutes in one unit of distance) might be important for students to realize that both determine the invariant relationship between one unit of a quantity for some units of another quantity. When students are asked to compare these three students' ideas, the textbook provides explanation that the method that Miho and Takumi used is compatible with the method students used in the 5th grade to determine crowdedness. Crowdedness and being fast or slow actually expresses intensive quantities and the use of per one strategy might contribute students' understanding of these qualities through intensive quantities (Kaput and West, 1994). Then, students are supported to use per-unit quantities once they compare speed in the last question.

Although both quantities are continuous and the speed comprises covariation, there is no indication about the covariation between quantities in this task. The aim seems to improve students' motion concept which is foundational for speed conception. There are also two further exercises given for students to practice motion.

Task2B. The task shown in Figure 6.35 is designed to express speed.

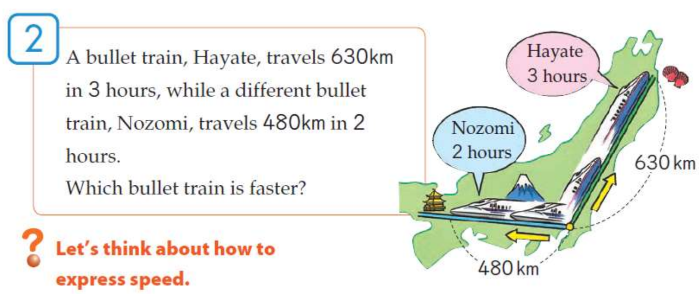


Figure 6.35. Speed of bullet trains task (Fujii and Iitaka, 2012, Grade 6, p. A85).

To represent the quantities and the match between quantities of distance and time, line segments are provided. In the line segments for both Hayate and Nozomi, students are expected to express the corresponding distance for the unit of 1hour. Then speed is introduced as "*distance travelled in a unit amount of time*" (Fujii and Iitaka, 2012, Grade 6, p. A86). In the textbook, different units of time, namely hour, minute and second, is also referred. In the task, students are also expected to state the speed of Nozomi in terms of km per hour and km per minute.

Task2C. Students are supposed to use the speed formula of *Speed* = *Distance* ÷ *Time* in the form of *Distance* = *Speed* × *Time* in some problem situations requiring to find the distance travelled with known values of speed and time as it is in the task given below in Figure 6.36.

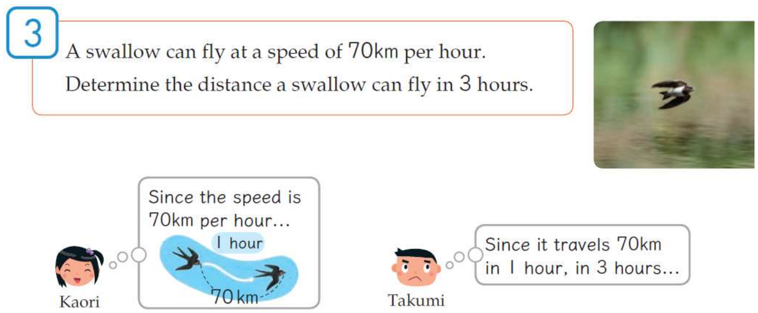


Figure 6.36. Speed of swallow task (Fujii and Iitaka, 2012, Grade 6, p. A87).

The representation of line segment is used again to express distance travelled in an hour, so that students could either iteratively find the distance travelled for 3 hours or use the speed formula. Students are asked to compare the speed formula and distance formula; thus, he multiplicative relationship between distance and speed might be discussed.

Task2D. Similar to Task2C, in this task (Figure 6.37) the main aim is to find the time passed when the speed and distance travelled is given.

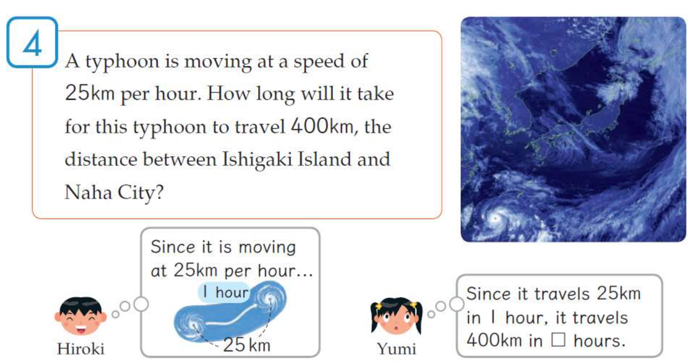


Figure 6.37. Speed of typhoon task (Fujii and Iitaka, 2012, Grade 6, p. A88).

The line segment representation is used in this task too. The provided answer for the task is expressed as $25 \times x = 400$ and students are asked to solve for x.

<u>*Task2E.*</u> From now on students are introduced to speed and they are supposed to consider the multiplicative relationships between speed, distance, and time to find out the value of a quantity when asked. Also, the textbook provides students to use different time units to express the speed by using the same distance travelled. The Task2E involves the quantity of time expressed with both minute and second. The task is stated as "*There is a moving sidewalk that is 214m long. It took 5 minutes and 21 seconds to go from the beginning to the end of the sidewalk. What is the speed per minute of this moving sidewalk?*" (Fujii and Iitaka, 2012, Grade 6, p. A89). Students are asked to express the value of the quantity of time with fractions and then find the speed of the sidewalk.

<u>*Task2F.*</u> Similar to Task2E, in this task given at Figure 6.38 below, students are expected to use fractions to express time.

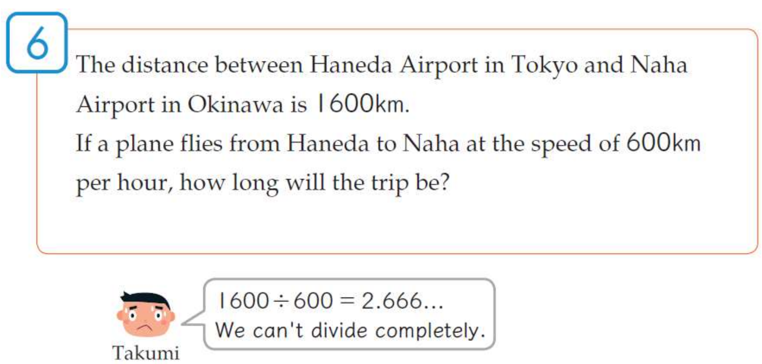


Figure 6.38. Airport task (Fujii and Iitaka, 2012, Grade 6, p. A89).

Since the time is found as infinite decimal number, students are supported with a shared student's ideas about expressing the result of the division 1600÷600 with a fraction. Since time is expressed with a fraction $2\frac{2}{3}$, 2 represents 2 hours and $\frac{2}{3}$ indicates 40 minutes. By use of fraction, the meaning of the quantity *time* might be supported. That is to say, the textbook might support students to pay attention to meaning of the numerical value that is found by the arithmetic operations.

Task2G. Through Task2G, the direct proportional relationship between distance and time is aimed to be examined.

7 Investigate the relationship between the time and the distance traveled by an airplane that flies at the speed of 13km per minute.

Figure 6.39. Airplane task (Fujii and Iitaka, 2012, Grade 6, p. A90).

The first question is stated as "Write a math sentence to calculate the distance, y km, traveled in x minutes." (Fujii and Iitaka, 2012, Grade 6, p. A90). As a response to this question, expressing the situation with $y = 13 \times x$ actually highlights the invariant relationship between distance and time which is speed. Then, the second question is stated as below in Figure 6.40.

2 As x changes 1, 2, 3, ... 6, how does y change? Summarize it in the table below.

Time traveled, x(min)	1	2	3	4	5	6
Distance traveled, y(km)						

Figure 6.40. Table showing changes in time and distance of the airplane task (Fujii and Iitaka, 2012, Grade 6, p. A90).

First of all, students are asked to examine the relationship between time and distance. Secondly, as Figure 6.40 represents, time is increased with 1 min intervals through the table.

Both variables x and y are continuous, and they are covarying quantities. Also, the table displays the change in y with the change in x by 1 min. Therefore, *MA1* is triggered through the table. Then, in the third question, students are asked to express whether the distance traveled is *proportional* to time passed. Students are supported to consider the fact "*When the time becomes 2, 3,… times as much, the distance…*" (Fujii and Iitaka, 2012, Grade 6, p. A90) which entails to the *MA2* because the proportionality indicates a straight line which is also expressed in the task by $y = 13 \times x$. In the fourth and fifth questions given in Figure 6.41 below, students are expected to consider correspondence between x and y.

4 Find the value of y corresponding to when x is 7.5.
Also, find the value of x when the corresponding value of y is 182.

5 As the number of minutes traveled increases by 1, by how much does the distance traveled increase?

Think about it by looking at the math sentence and the table.

Figure 6.41. Fourth and fifth questions of the airplane task (Fujii and Iitaka, 2012, Grade 6, p. A90).

Although the values of time is given as integers, in the 4th question time is given as 7.5 to find the corresponding y value for it. The intermediate values of time might contribute students' thinking about time changing chunkily. At the fifth question, coordination between the amount of change in the y variable (distance) with the amount of change in the x variable (time) is fostered. The mental action 3, MA3, which is described as "*Coordinating the amount of change of one variable with changes in the other variable*" (Carlson *et al.*, 2002, p.357) seems to be aimed to be triggered.

In conclusion, speed indicates a rate. The $y = 13 \times x$ indicates a line passing through the origin with a value of slope as 13. Throughout the task, mental actions at MA1, MA2, and MA3 levels are triggered. The corresponding covariational reasoning level involving these mental actions is named *coordination of values*:

> *Coordination of values* involves instances of coordinating the values of one variable or quantity with values of another by providing specific and discrete pairs of values without providing the opportunity for students to conceive two variables or quantities whose value varies together in between those pairs of values. (Taşova *et al.*, p.1530)

The explanation made in the quote is compatible with the Task2G. Through the task students are expected to consider the relationships between time and distance traveled to generalize an algebraic representation for them. Students are supposed to recognize the change in time such that it will lead the change in distance traveled or vice versa. However, it seems that the task is not triggering students to consider the continuous and simultaneous change in variables since the continuous variation of neither time nor distance traveled is highlighted in the task.

<u>*Task2H.*</u> The last task given below (Figure 6.42) is aimed to use speed to express the amount of work done in a unit of time.

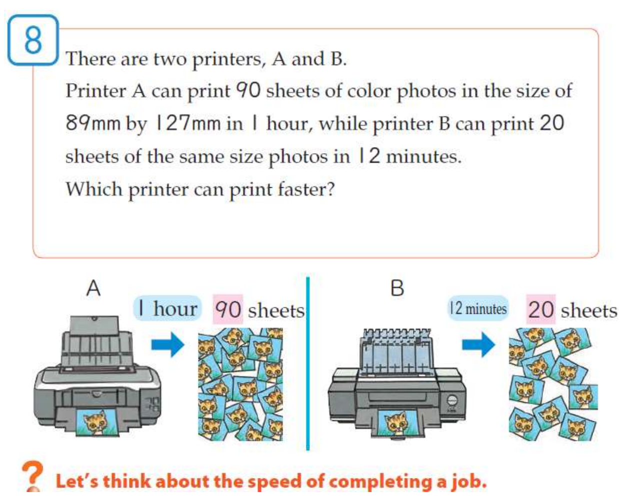


Figure 6.42. Printer task (Fujii and Iitaka, 2012, Grade 6, p. A91).

It is asked to write down the number of papers printed in a minute for each printer. *Unit factor approach* is triggered to use to compare printers' speed. Although the speed is not expressed as a within state ratio of number of papers printed in a specific time period and the invariance is not given a point, the problem might be defined as a rate problem if students conceptualize the time as an extensive quantity (Thompson, 1994a). Therefore, this task might contribute to students' rate conception depending on students' understanding of time and relationship between time and work.

<u>*Additional Part(2).*</u> The extension part, which is given at the end of the 6A book, has the title *Let's calculate the speed of a train!* and it involves two problems to improve students' understanding of speed. The first one is shared through Figure 6.43 below.

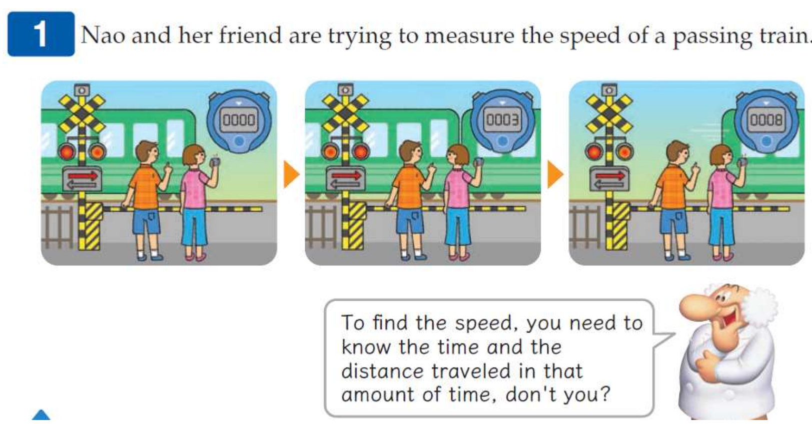


Figure 6.43. Speed of a train task (Fujii and Iitaka, 2012, Grade 6, p. A119).

With the guidance questions of calculating the total length of the train and the amount of time for the train totally passing the standing girl (named as Nao in the task and displayed in the front with a pink t-shirt in Figure 6.43), the main aim seems to improve students' understanding of the quantity of *distance travelled*. Also, in the second question, students are expected to consider the speed of a train passing through a bridge for a given number of cars and the length of each car that the train has, the length of the bridge, and amount of time that the train passing the bridge completely. In the second question, students need to consider both bridge length and total length of the train to determine the distance travelled. It is targeted for students to think about the quantity of distance travelled and amount of time, rather than calculate the speed for given quantities. For instance, if only the front of the first car of the train is considered to measure the amount of time passed for the train entering the bridge and the front of the train reaches the end of the bridge, students should use bridge's length as distance travelled and decide the speed by dividing the length of the bridge to the time passing. That is, students might improve their conception of extensive quantities like distance travelled and time through different problem situations. As Thompson (1994a) claimed, students with a developed understanding of extensive quantities, such as distance and time, might conceive the multiplicative relationships between them to form the intensive quantity, speed.

6.2.3.3. The units of textbooks regarding covariation. In this section, I present findings regarding to functional relationships highlighting covariational reasoning. In particular, tasks coming from the units of *Let's use letters and write math sentences, Let's think about how to express speed*, *Let's investigate proportional relationships,* and *Extending mathematical thinking* parts are presented.

Let's use letters and write math sentences. The first task presented as Task3A leads students to work with area of different rectangles.

Task3A. The task, given as Figure 6.44, involves a tape width 5cm and by cutting the tape by different lengths, there are rectangles formed having different areas.

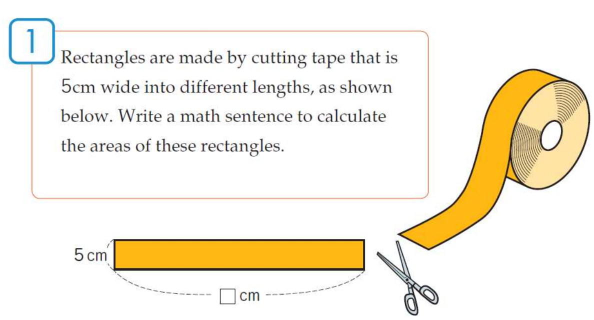


Figure 6.44. Area of rectangles task (Fujii and Iitaka, 2012, Grade 6, p. A17).

The first question canalize students to write down a mathematical sentence to represent the area and calculate the area of the tape pieces for lengths 10cm, 15cm, 20cm, 25cm… The next three questions of the task are presented at the below Figure 6.45.

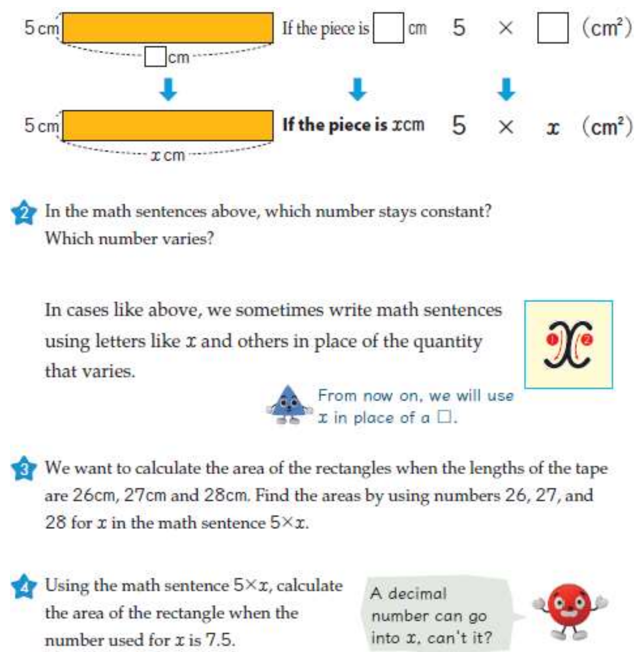


Figure 6.45. Area of rectangles task questions (Fujii and Iitaka, 2012, Grade 6, p. A18).

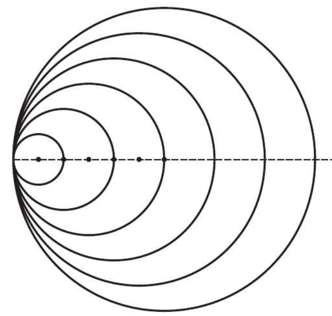
In response to the first question, mathematical sentence 5×□ is shared and the values are put into the □ to calculate area. By doing so, students could realize the pattern that each time the area is calculated through multiplying the length with 5.

By the second questions, students are expected to determine the constant and the variable. Third and fourth question is for students to practice calculating area by using the expression of $5 \times x$. The *length* and *area* are both continuous variables. The range of the length is chosen as 5cm in first question. In the third question by asking the lengths of 26cm, 27cm, and 28cm, the range of the length is reduced to 1cm. In the fourth question, the length of 7.5cm meaning that the range of the length is reduced to 0.5cm. In other words, the values of the variable length are asked in chunks. This might support the students to recognize the nature of the variable length is *continuous*. The change in the value of area is not highlighted in the task, while the change of the length is underlined. Therefore, I claim that although the main aim of the task is to introduce algebraic representation, the task might also trigger students' *variational reasoning*. Since it is intended through the first four questions to emphasize the continuous nature of the variable length, the task might contribute students' understanding of *continuous variation*. In particular, the tasks seems to be designed to support students chunky continuous variation; however, depending on the students' understanding the task, students might consider the variable length as smooth continuous variable.

Lastly, in the fifth question the area is given 135 and the corresponding length is asked. Since in the last question students are expected to calculate the length using a given value of area, they might recognize the correspondence relationships. However, students might also conduct arithmetic operation without realizing the correspondence between variables. Although the value of length is coordinated with the value of area and the change in both quantities are calculated, only the change in the variable *length* is highlighted throughout the task. In case the change in area is emphasized, and the values of length and area are considered as pairs in the instruction, then it could be inferred that *MA1* might be triggered by the task. Even though this claim is not well supported by the questions of the task, it is worth presenting it in the analysis, because the upcoming task is designed to examine relationships of two quantities which might trigger students' covariational reasoning. Having a variational reasoning task first, and then presenting a covariational reasoning task might entail an important pedagogical strategy used in Japanese textbooks.

Task3B. In the task shown in Figure 6.46, students are expected to examine the relationship between the diameter and the circumference for circles.

2 Represent the relationship between the diameter and the circumference of a circle in one math sentence.



1 Write math sentences that show the relationship between the diameter and the circumference of circles when the diameters are 1cm, 2cm, 3cm, ...

Figure 6.46. Circle task (Fujii and Iitaka, 2012, Grade 6, p. A19).

Differently from the Task3A, in this task both variables, diameter and circumference, are comprised. In the first question displayed in Figure 6.46, the corresponding value of the circumference is asked to be written down for the given diameters. In the earlier grades students are expected to use different symbols like ◯ and □ to represent different quantities. Here in the 6th grade, they are introduced to represent different variables with different letters as it is displayed in Figure 6.47.

If the diameter is □ cm	□	×	3.14	=	◯	(cm)
⬇	⬇				⬇	
If the diameter is $x$cm	x	×	3.14	=	y	(cm)

Figure 6.47. Use of the letters x and y as covarying variables on the circle task (Fujii and Iitaka, 2012, Grade 6, p. A19).

2 What number does y represent in the math sentence $x \times 3.14 = y$, when the numbers 10, 15, and 20 are used for x?

The 10 that was used for x is called the value of x. The resulting number that y represents, 31.4, is called the corresponding value of y when the value of x is 10.

3 Find the corresponding value of y when the value of x is 2.5.

4 Find the value of x for which the corresponding value of y is 47.1.

Figure 6.48. Questions of the circle task (Fujii and Iitaka, 2012, Grade 6, p. A20).

The following questions of the Task3B is given above (Figure 6.48). As in the explanations made in response to the second question, the coordination of values is explicitly asked. The expected students' reasoning resembles MA1, which is exemplified as "*y* changes with the changes in *x*" (Carlson *et al.*,2002, p.357), is compatible with the question. Similar to the Task1A both variables, diameter and circumference, are continuous and the correspondent *y* values are calculated for the diameter values varying by 5 unit, 1 unit, and 0.5 unit. Although both variables are varying chunkily, there is no indication for other mental action levels (e.g. MA2, MA3, and so on) other than MA1. Also, the task does not emphasize the constant multiplicative relationship existing in the situation, which is the ratio of circumference to the diameter, i.e. pi. Therefore, the task might activate the *precoordination of values* level of the covariational reasoning framework. A person in *precoordination of values* does not recognize the multiplicative relationships between the pairs of covarying quantities (Thompson and Carlson, 2017).

Also, after the task is given, there are 4 different situations provided to represent the situations by using *x* and *y*. As it is seen from the Task3A and 3B presented above, the unit, *Let's use letters and write math sentences*, relies more on algebra. However, the use of problem situations, e.g. proportional relationship between diameter and perimeter in Task3B, and questions are worth sharing due to their relation to covariation. Although only the MA1 might be triggered through the tasks of the unit, the unit is important in terms of examining covarying situations even with the heavy focus on algebra. Moreover, the next textbook unit is designed for ratio and proportions which I presented as follows.

Let's investigate proportional relationships. This unit of the textbook is quite long and therefore the unit is given through five sub-units, namely: *Math sentences for proportional relationships*, *Characteristics of proportional relationships*, *Graphs of proportional relationships*, *Applications of proportional relationships*, and *Inverse proportional relationships*, respectively. The unit starts with providing 4 cases for students to determine whether the quantity *y* is proportional to the quantity *x*. For each case, students are provided a table to fill for the values of *x* and *y*. Speed (distance travelled: amount of time), area of a parallelogram with a given fixed base (area: height) , and increase in the height of a water in a tank with the shape of rectangular prism (depth of water: amount of time) involve proportional relationships; however, the area of square to the length of 1 side does not indicate a proportional relationship. These proportional relationships are displayed in the beginning of the *Math sentences for proportional relationships* sub-unit as given in Figure 6.49.

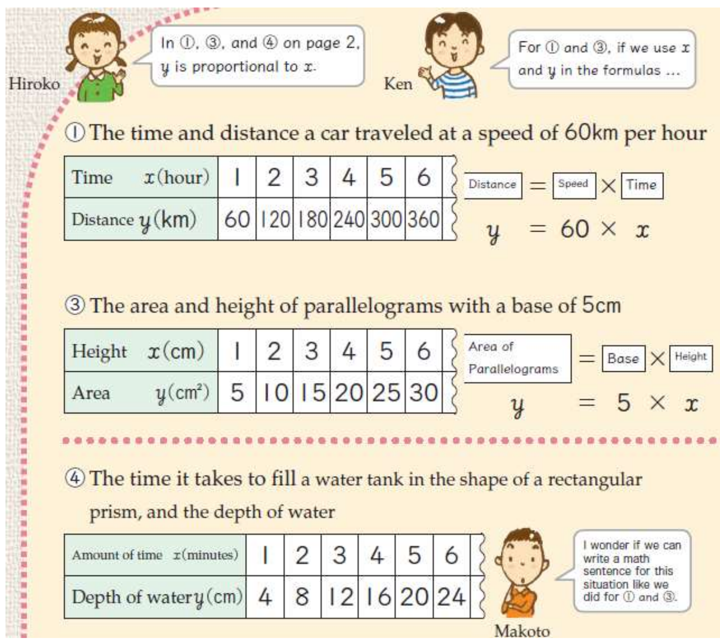
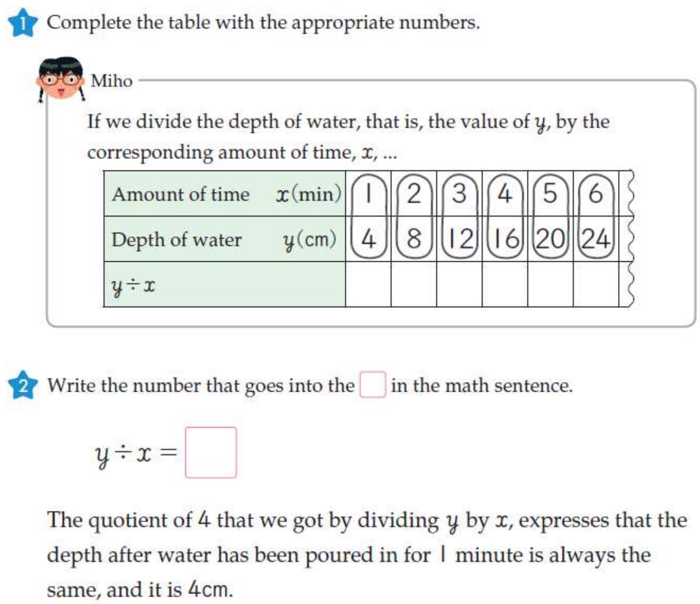


Figure 6.49. Math sentences for proportional relationships (Fujii and Iitaka, 2012, Grade 6, p. B4).

Through this introductory part, *MA1* might be aimed to be triggered by expecting students to form a table which displays the coordination of the values of y with the changes in *x*. *MA2* might be triggered through supporting students to focus on the proportional situation when *x* becomes 2, 3, 4,… times as big, the values of *y* becomes 2, 3, 4,… times as big too. Also, *construction of an increasing straight line* is classified as a students' behavior for *MA2* (Carlson *et al.*, 2002). Students are provided to write a math sentence representing an increasing straight line, but they are not asked for drawing the line representing the math sentence.

Task4A. The task involves the last case in the introductory part, of which the depth of water is proportional to the amount of time. Students are expected to write a math sentence to express this proportional relationship. The first question of the task shown in Figure 6.50 requires calculating the rate value of $y \div x$.



1 Complete the table with the appropriate numbers.

Miho

If we divide the depth of water, that is, the value of y, by the corresponding amount of time, x, ...

Amount of time x(min)	1	2	3	4	5	6
Depth of water y(cm)	4	8	12	16	20	24
$y \div x$						

2 Write the number that goes into the $\square$ in the math sentence.

$y \div x = \square$

The quotient of 4 that we got by dividing y by x, expresses that the depth after water has been poured in for 1 minute is always the same, and it is 4cm.

Figure 6.50. Filling a tank task (Fujii and Iitaka, 2012, Grade 6, p. B5).

As it is presented in the figure above, the *invariant relationship* between x and y is explained through the sameness of the quotient of $y \div x$: depth of water poured per 1 minute. Then, as a response to the third question of the task, which requires to write a math sentence for finding y values, $y = 4 \times x$ is shared. With the use of this math sentence, students are asked to find out corresponding y values for the x given values of 0, 7, and 8.5. Also, the corresponding x value for the $y = 40$ is asked in the last question. It is explicitly stated that "When y is proportional to x, the quotient of the value of y divided by the corresponding value of x always remains constant. In addition, the following match sentence can be written. $y = constant \times x$" (Fujii and Iitaka, 2012, Grade 6, p. B6). Similar to introductory part, *MA1* and *MA2* is aimed to be triggered so that students might coordinate the values of variables which are both continuous and so that students can focus on the direction of change. Since in the task, the constant is used to find out the amount of change in y with the change in the values of *x*, *MA3* might be triggered too. However, the *constant* actually is an expression of the multiplicative relationship within a situation, i.e. rate. There is no further examination supported to trigger students' mental actions about average and instantaneous rate of change in this task. Since first three mental actions are tried to be triggered through the task, the corresponding covariational level aimed for the task might be *coordination of values*. Students, having this level of covariational understanding, could realize that they are forming discrete collection of pairs of (*x, y*).

The same Filling a water tank task is used to introduce the next two sub-units, namely *Characteristics of proportional relationships* and *Graphs of proportional relationships*. In order to deepen characteristics of the proportional relationship existing in the filling a water tank task, the table of values of x and y is displayed. Then, students are presented with two students' ideas shared in Figure 6.51 below.

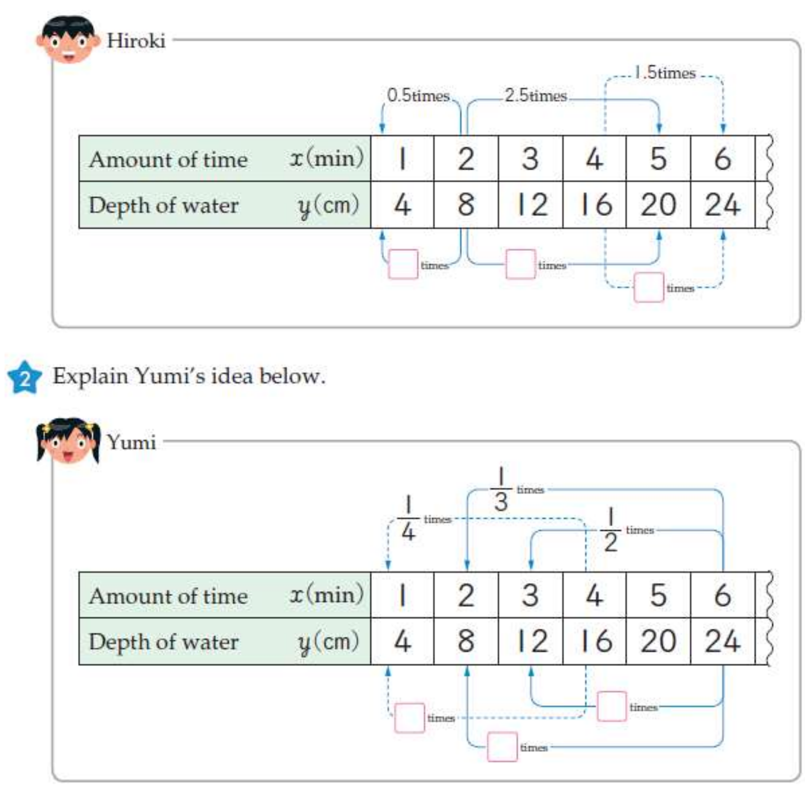


Figure 6.51. Characteristics of proportional relationships (Fujii and Iitaka, 2012, Grade 6, p. B7).

As it is seen from the Figure 6.51, students are expected to examine the amount of increase and decrease as the same for the quantities being in a proportional relationship. For instance, the depth of water is 8 cm in 2 minutes. If the time value of 2 min becomes 2.5 times as big, then the *y* value of 8 cm becomes 2.5 times as big. Actually, the amount of increase decrease represents a *between state ratio* and the expression 2.5 times as big is identical to say 250% of the initial value. It is worth to indicate that both non-integer and fraction values of multiples are also presented. Use of fractions to express increase/ decrease might be beneficial for students to also recognize inversely proportional relationships, because in the inversely proportional relationships, if *x* becomes 2 times as big, the corresponding *y* becomes $\frac{1}{2}$ times as big.

Then, in order to *graph the proportional relationships*, a plotting paper is given, and students are asked to place the coordinates of *x* and *y* from the table into the plotting paper at first. The first two values is placed by the textbook writer as shown in Figure 6.52.

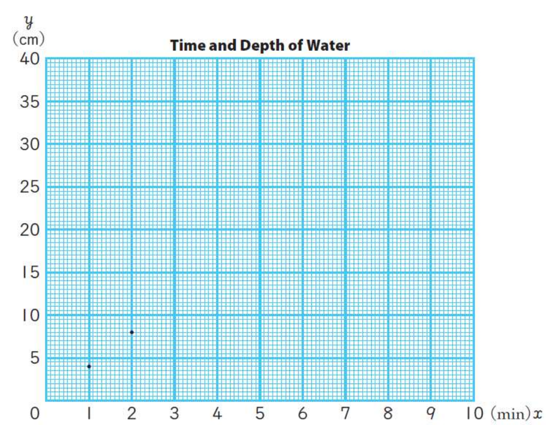


Figure 6.52. Placement coordinates of (*x, y*) pairs of filling water tank task (Fujii and Iitaka, 2012, Grade 6, p. B9).

Then, in the second question, it is asked to calculate the values of *y*, when the *x* is 0.5, 1.5, and 4.5 with using the math sentence $y = 4 \times x$ and place them into the plotting paper. This question might foster students to recognize the values of *x* and *y* varying between the values given at the table for the x as 1,2,3… and for the y as 4,8,12… Therefore, the continuous nature of the variation might be cultivated through the question. Since students in the *coordination of values* level of covariation are not be able to recognize the continuous variation, the task might encourage students to recognize the *continuous covariation* between quantities. This is especially because, students are asked to think about the corresponding values of y when x is increased in uniform increments of 0,5. In the third question, the value of *y* is asked, when *x* is 0. Then students are asked how the points are arranged in the graph and the *straight line passing through the origin* with the slope of 4 is introduced to be a representation of the proportional relations of the filling water tank task. The graphical representation of proportional relationships is explicitly stated as *straight line passing through the origin* without indicating the constant quotient as slope. It is also asked to find out *y* values for some *x* values with the use of the graph and vice versa. The corresponding *y* values are asked for the *x* values of 8 and 1.2, and students are supported to show the points on the graph.

<u>*Task4B.*</u> The task given below in Figure 6.53 is developed to examine graphs of proportional relationships. The questions of the task is shown in Figure 6.54.

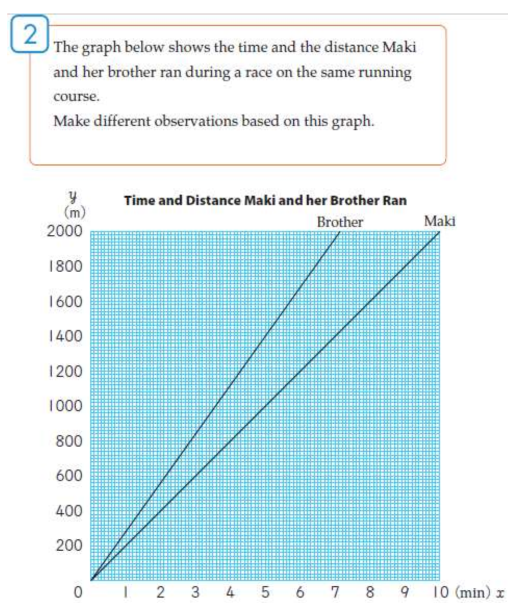
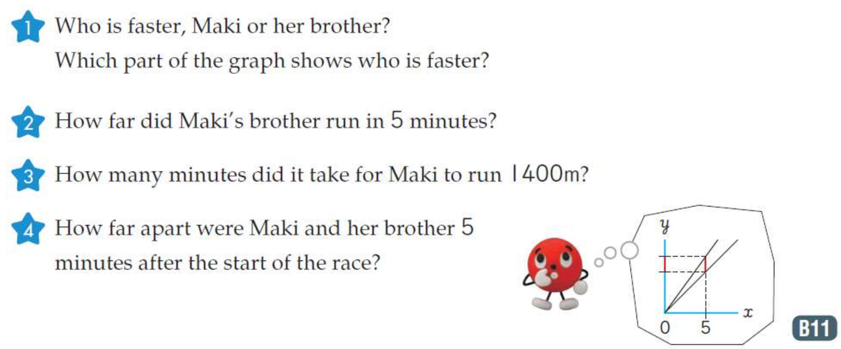


Figure 6.53. Who is faster task (Fujii and Iitaka, 2012, Grade 6, p. B11).



1 Who is faster, Maki or her brother?
Which part of the graph shows who is faster?

2 How far did Maki's brother run in 5 minutes?

3 How many minutes did it take for Maki to run 1400m?

4 How far apart were Maki and her brother 5 minutes after the start of the race?

Figure 6.54. Questions of the who is faster task (Fujii and Iitaka, 2012, Grade 6, p. B11).

First, the first question especially asking "Which part of the graph shows who is faster" (Fujii and Iitaka, 2012, Grade 6, p. B11) is very important: Students might respond to that by comparing the distances that Maki and her brother run for the same amount of time, by comparing the amount of time passed for Maki and her brother running the same distance, or by expressing their speeds which are represented via the slope of the lines. The value of slope is a *invariant* and gives the *speed*. For the second question the point (5, 1400) expressed the distance, 1400 m, that Maki's brother run in 5 min. For the third question the point (7, 1400) indicated the time passed as 7 min for Maki to run 1400m. The time difference between them to run 1400m is 2 min but it is not asked explicitly. Nevertheless, the distance between them is asked in the last question when they run 5 min. The red ball in Figure 6.54 indicates this difference on the graph. The task supports students to coordinate the amount of changes between variables which is again compatible with *MA3*. I infer that the task might support students to develop the *coordination of values* level of covariational reasoning.

At the end of the sub-unit of *graph the proportional relationships*, there is a math story section involving two situations in which x and y are not in a proportional relationship. Students are expected to draw the graphs for the situations and examine the lines that one is a decreasing straight line and the other one is an increasing straight line but not passing through the origin. This is very important because it provides students with non-examples of proportional relationships. That is, it provides an opportunity for students to differentiate linear relationships between two quantities from proportional relationships between two quantities.

Task4C. In the Applications of proportional relationships sub-unit, it is stated that the weight of construction paper is proportional to the number of sheets. Students are asked to consider how to get 300 sheets of construction paper without counting. In the first question, the weights for 10 sheets and 30 sheets of construction papers are shared through a table as shown in Figure 6.55.

1 10 and 30 sheets of the construction paper were weighed, and the results are shown in the table below. Think about how to get 300 sheets based on this information and write down your ideas using math sentences and words.

Number of Sheets and Weight of Construction Paper

Number of sheets x (sheets)	10	30	300
Weight y (g)	73	219	☐

Figure 6.55. Construction paper task (Fujii and Iitaka, 2012, Grade 6, p. B13).

There are four students' solutions each using proportional relationships to find the weight of 300 sheets presented at the table below. Students are asked to explain each student's idea, examine the similarities and differences between them, and group the similar ones.

Table 6.2. Students' uses of proportional relationships (Fujii and Iitaka, 2012, Grade 6, p.B14-15) and the findings about shared students' methods.

"Miho: $73 \div 10 = 7.3$ $7.3 \times 300 = 2190$ We need to get 2190 g of construction paper." (p.B14)	"Hiroki: $219 \div 30 = 7.3$ $7.3 \times 300 = 2190$ We need to get 2190 g of construction paper."(p.B14)	"Shinji: $300 \div 10 = 30$ $73 \times 30 = 2190$ We need to get 2190 g of construction paper." (p.B15)	"Kaori: $300 \div 30 = 10$ $219 \times 10 = 2190$ We need to get 2190 g of construction paper." (p.B15)
Miho found the weight of one sheet and then multiplied it with 300 to find the weight of 300 sheets	Hiroki found the weight of one sheet and then multiplied it with 300 to find the weight of 300 sheets	Shinji found the increment multiple 30 and then multiplied it with the 73 to find the increase in weight	Kaori found the increment multiple 10 and then multiplied it with the 219 to find the increase in weight
Miho and Hiroki compared the two unlike quantities exist *within a situation*.		Shinji and Kaori compared the values of the same quantity exist *between the situations*.	

As the table indicated, data suggests that students are expected to understand that dividing any pair of quantities in within state ratios or dividing any pair of quantities in between state ratios would yield to the invariant multiplicative relationship between the quantities in ratio.

In the last sub-unit of *Inverse proportional relationships,* there are two situations to examine whether they are inversely proportional or not. One of them has inversely proportional relationships, while the other does not. Nonetheless, both of them have one increasing quantity while the other quantity decreases. Then, math sentences for inversely proportional relationships, characteristics of inversely proportional relationships, and graphs of inversely proportional relationships are examined through the example having the inversely proportional relationship given before. Since the whole sub-unit is covered on the same example, I explain the sub-unit through this example which I present as Task4D below.

Task4D. The sub-unit starts with a problem sentence that is "Investigate how the length and width of the rectangles with a fixed parameter or a fixed area change." (Fujii and Iitaka, 2012, Grade 6, p. B18). Rectangles having area of 18cm^2 and rectangles having perimeter of 18cm are examined through tables involving length values increasing by 1cm and corresponding width values to the length values of 1, 2, 3, … For the rectangles with area of 18cm^2, students are asked to examine the changes in the width when the length becomes 2, 3, … times as long. In response to the question, the table given in Figure 6.56 is shared.

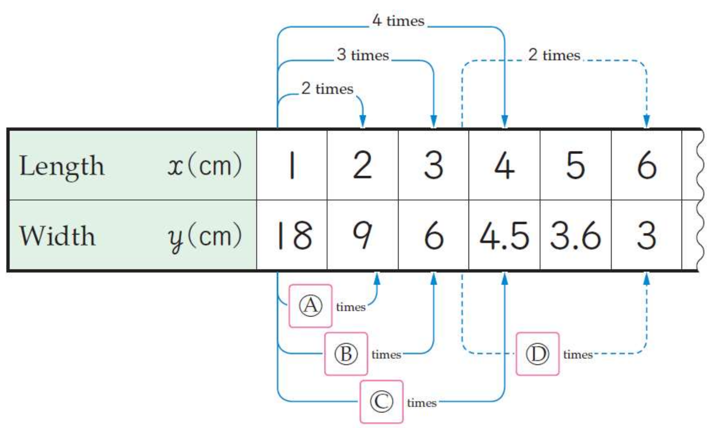
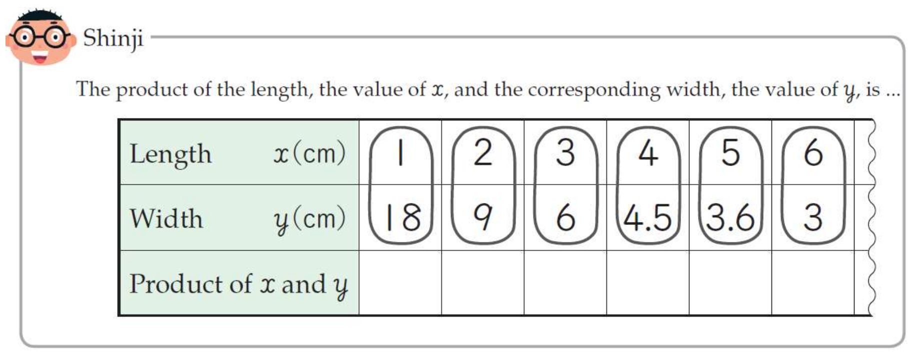


Figure 6.56. Rectangles with area 18cm^2 (Fujii and Iitaka, 2012, Grade 6, p. B20).

As it is seen at the table, when the length 1cm becomes 2cm, the length becomes 2 times as long; however, in relation to the change of the length, the width becomes 9 which is $\frac{1}{2}$ times as long as the length 18cm. Hence, when the length values (the *x* variable) become 2, 3, 4 … times as long, the corresponding width values (the *y* variable) become $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ … times as long. This relationships between *x* and *y* is stated as "*y is inversely proportional to x*" in the textbook (Fujii and Iitaka, 2012, Grade 6, p. B20). Then, students are asked to examine whether the width is proportional to the length for the example of rectangles with the perimeter of 18cm. It is not proportional because they do not have multiplicative relationship such that the product of the two corresponding quantities are not constant as it is in the inversely proportional relationships.

In order to write a *math sentence for the inversely proportional relationship* between the width and length of the rectangle with the area of $18cm^2$, students are expected to fill out the table given in Figure 6.57.


Shinji

The product of the length, the value of x, and the corresponding width, the value of y, is ...

Length x(cm)	1	2	3	4	5	6
Width y(cm)	18	9	6	4.5	3.6	3
Product of x and y						

Figure 6.57. Table for showing the constant product (Fujii and Iitaka, 2012, Grade 6, p. B21).

As it is indicated in the table above, the product of the quantities in inversely proportional relationships is constant. The product $x \times y = 18$ is also expressed as $y = 18 \div x$. Then, the values of y are asked for the values of x such as 2.5, 8, and 10. The math sentence $y = constant \div x$ is defined for the inversely proportional relationships in the textbook.

Moreover, there is an example given in the textbook for students to examine the inversely proportional relationships existing between speed and the amount of time. Since speed is the ratio of the distance traveled to the time, the distance and time have direct proportional relationship, while the speed and time have inversely proportional relationship. It is important to introduce the speed and time relationship, which is inversely proportional, because students have already introduced distance travelled and amount of time relationships, which is directly proportional. With the "quick review" parts presented through the Task4D, in which the definition and rules for directly proportional relationships are recalled when the definition and rules for inversely proportional relationships are introduced, to remind students how the relationships between quantities are defined for direct proportions, inversely proportional relationships are explicitly compared with direct proportional relationships as well as non-proportional relationships, e.g. the rectangles with fixed perimeter.

Next, characteristics of inversely proportional relationships are aimed to be investigated. For this purpose, the table shown in Figure 6.56 is redrawn with the inverse direction of the arrows. For example, the arrow is drawn from the length 6cm to the 3cm, which indicates the length becomes $\frac{1}{2}$ times as long, while the corresponding width value for the length 6cm is 3 cm which becomes 2 times as long. That is to say if the one of the quantities becomes $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$ times as long, the corresponding quantity becomes 2,3,4… times as long.

Lastly, students are expected to draw a *graph* to represent the inversely proportional relationships between the width (y cm) and the length (x cm). Then characteristics of the graph is asked to be investigated and the graph for a direct proportional relationship is recalled as a fact in this task to make a comparison. The direction and the amount of changes in the one quantity with the changes in the other is different for the inversely proportional relationships than the direct proportional relationships. Examination of inversely proportional relationships by following the similar procedure used in the presentation of the direct proportional relationships provides students to compare proportional and inversely proportional situations.

The sub-unit is also worth examining with respect to covariational reasoning framework. First of all, students are expected to coordinate the values of one variable, i.e. lengths of rectangles, with the changes in the other variables, i.e. widths of rectangles, with the use of not only tables but also graphs; thus *MA1* might be triggered. Moreover, they are explicitly asked to verbalize the direction of the changes in variables and express the amount of changes. For example when the length is 1 cm, the width is 18 cm and when the length is 2 cm, the width is 9 cm, so on. Examining and expressing the increases in length resulting the decreases in width might indicate the *MA2*. In addition, students are expected to realize and express the multiplicative link between the changes in variables, such that when the length becomes 2 times as large, the width becomes $\frac{1}{2}$ times as large, when the length becomes 3 times as large, the width becomes $\frac{1}{3}$ times as large, and so on. This corresponds to *MA3*, i.e. "coordinating the amount of change of one variable which the changes in the other variable" (Carlson *et al.*, 2002, p. 357). Since the first three mental action corresponds to coordination of values level of covariation, students' understanding of covariation as the *coordination of values* might be encouraged.

<u>*Additional part(3).*</u> At the end of the book, there is an *extension* part for *Let's investigate proportional relationships* unit which is named as *What Will the Graph Look Like.* The first question is given in Figure 6.58 below.

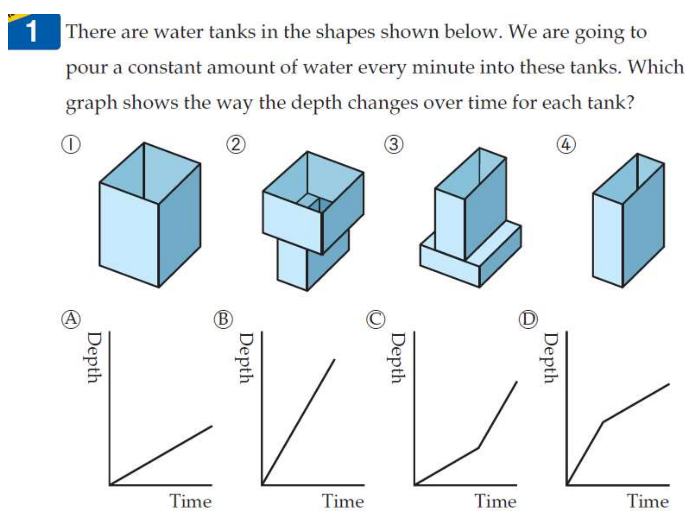


Figure 6.58. What Will the Graph Look Like (Fujii and Iitaka, 2012, Grade 6, p. B100).

The question is important to examine the covarying quantities and their graphical representations. The graphs (A) and (B) involve straight increasing lines which indicate uniform tanks, such as (1) and (4). Through examining the tanks (1) and (4), students are expected to consider the area of the bases and compare the increase in the depth of the water poured in the tanks in an amount of time. Since the base of the tank (4) is narrower, the depth of it will increase faster than the tank (1). Therefore, the graph (B) which is a straight line with the greatest slope is compatible with the tank (4). Then, the graph (A) is appropriate to express the depth of the water in the tank (1). To determine which of the graphs (C) and (D) correspond to the tanks (2) and (3), the amount of changes in depths for the bottom prisms should be considered. The amount of change in the depth in a unit of time for (2) is greater than (3). Thus, (D) corresponds to the tank (2) and (C) corresponds to the tank (3). Throughout the task, two continuous variables are compared in a multiplicative world. The graph actually represents the *rate of change* of depth with the changes in times. The action of coordination the rate of change with the uniform increments of inputs is *MA4*. Since the mental actions are developmental, meaning that existence of *MA4* indicates the existence of *MA3*, *MA2*, and *MA1*. Therefore, the task might trigger students' covariational reasoning of *continuous covariation*, which is defined as "*Continuous covariation* involves instances providing a simultaneous and continuous change in the values of two variables or quantities." (Taşova *et al.*, 2018, p.1530). However, it is worth stating that the numerical values of quantities are not examined with the task. Thus, it seems that the *MA4* is not explicitly aimed to be fostered. Instead, I could claim that the task might be targeted to raise students' awareness about *rate of change* as a scaffold for the upcoming grades to examine it.

The second question is shared in Figure 6.59 below. In this question the base $|AD|$ is constant while the height of the triangle is changing with the location of E, so that the area is changing.

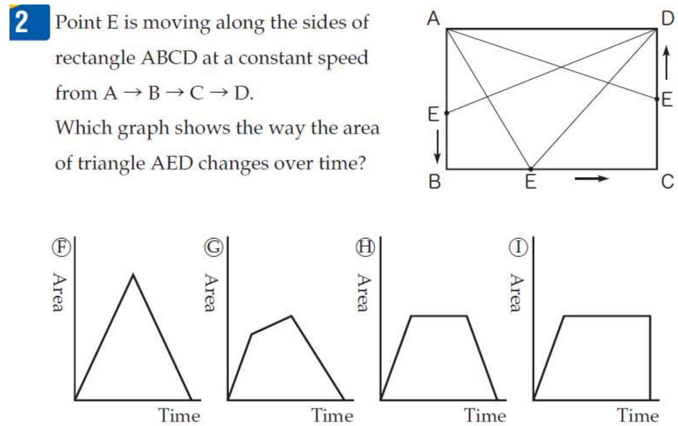


Figure 6.59. Area of a triangle with a changing edge task (Fujii and Iitaka, 2012, Grade 6, p. B100).

When the E travels form the edge A to the edge B, the height of the triangle AED is increasing. With the increase in the height, the area is increasing. The area is constant, greater and equals to the half of the area of the rectangle during the time E travels from B to C because the height is constant which is short side of the rectangle. The area is decreasing when D travels from C to D due to the decrease in the height of the triangle AED. The graph (H) corresponds to the situation. Again, in this example, both variables are continuous and rate of change in the area with respect to time is required to be examined. The level of *continuous covariation* again might be activated through the task.

Extending Mathematical Thinking (1). This part involves a task to tacitly raise an awareness about inversely proportional relationships. The part is given before the unit in which inversely proportional relationships are introduced. The task begins with the following problem situation "To pave a certain street, it will take machine A 15 days while machine B can finish it in 10 days. If both machines A and B are used, how many days will it take to pave this street?" (Fujii and Iitaka, 2012, Grade 6, p. A103). If both of the machines work together, it will be quicker to pave the street. The work done is inversely proportional to the amount of time. There are two students' ideas shared in the textbook to guide students about examining the situation. One of the students' ideas is to consider a hypothetical length for the street so that the length of the street for each machine paved in a day could be calculated.

Thus, the total length paved in a day with the use of both of the machines is expressed. For instance, the textbook offers to choose 30m as a hypothetical length of the street, so that the machine A could pave $30 \div 15 = 2\ m\ in\ a\ day$, while the machine B could pave $30 \div 10 = 3\ m\ in\ a\ day$. Both machine A and B could pave *5 m in a day*, so the 30m street could be paved in $30 \div 5 = 6\ days$ with the use of both of the machines. Students are also supported to use 90m as a hypothetical length to examine that the choice of hypothetical length does not affect the solution. Actually, the divisions students provided as 30÷15 and 30÷10 indicate *within-state ratios*, and the values of the division which are *2m in a day* and *3m in a day* express a rate, i.e. an intensive quantity of performance of the machine.

On the other hand, the other student's idea supports to thinking the length of the street as 1 unit. Students are supported to use line segments that each represent 1 unit. Then, one of them is partitioned to 15 equal parts to represent the length of the street that the machine A paved in a day. The other segment divided into10 equal pieces indicates the length of the street paved by the machine B in a day. If both of the machines work together, the length of the street paved in a day is shown at a line segment as shown below in Figure 6.60.

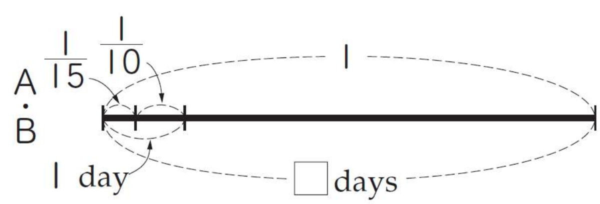


Figure 6.60. Line segment use in paving street task (Fujii and Iitaka, 2012, Grade 6, p. A103).

The Figure 6.60 displays the length of the street paved in a day with the use of both machines which is $\frac{1}{6}$. The length of 1 unit comprises 6 pieces of $\frac{1}{6}$ which corresponds to 6 days to complete the street. In this example, students are supported to use unit quantities which are indicating a *between- state ratio* such as $\frac{1}{15}$, $\frac{1}{10}$, and $\frac{1}{6}$. Then students are informed about machine C which completes the paving the same street in 12 days. In this case, students are asked to examine the same question with the use of machines A, B, and C.

Extending Mathematical Thinking (2). This part, namely, *Look for Patterns*, is intentionally designed for students to examine quantitative and covariational relationships between variables with the use of different representations such as tables, diagrams, and math sentences. The task is given below through Figure 6.61.

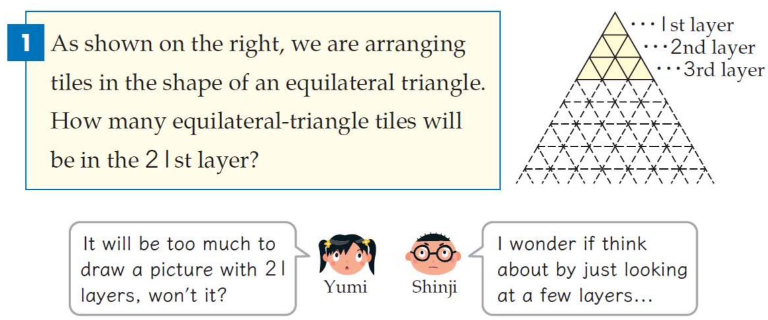


Figure 6.61. Tiles task (Fujii and Iitaka, 2012, Grade 6, p. B39).

It is asked at first to form a table for number of tiles in each layer for the first 5 layers. Then, students are asked to examine the table in two ways: One is to consider the change of number of tiles when the layer increases by 1; and the other is to compare the layer number with the number of tiles in that layer. With the consideration of patterns coming from these two examinations, students are asked to find out the number tiles in the 21st layer.

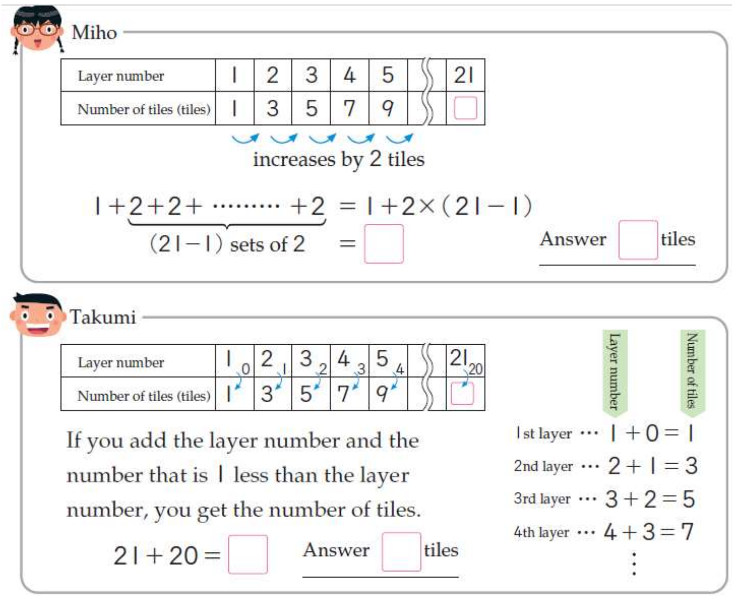


Figure 6.62. Clarifying the patterns in tiles task (Fujii and Iitaka, 2012, Grade 6, p. B40).

The students' ideas shared in Figure 6.62 indicated the result of the additive comparison made between two consecutive tiles numbers as well as the number of tiles and the layer number. Both tiles numbers and layer numbers are discrete, and they have additive relationships. Students are expected to benefit from the table to write down a math sentence for Miho's answer. Since it is stated in the task to accept layer number as *x* and the number of tiles as *y*, the math sentence becomes $y = 1 + 2 \times (x - 1)$. This sentence actually indicates a linear function with the *y* intercept as -1 and the slope as 2. In the task, students are expected to coordinate the values of number of tiles (*y*) with the values of layer number (*x*) which might trigger *MA1*. Also the direction of the change between variables is examined through tables, so that the *MA2* might be triggered by allowing students to examine the coordination of change in the number of tiles with the change in layer number. Similarly, they examine the change in the difference between the number of tiles and layer number with the change in layer number by 1. Thus, the task might be targeted to improve students' covariational reasoning to the level of *gross coordination of values*. Students having this level does not recognize the multiplicative link between the quantities rather they could realize the values of quantities increasing (or decreasing) together.

6.2.4. Grade-7 Japanese Curriculum Materials

6.2.4.1. Content related objectives and the explanations given in the teachers' guide. The *seventh grade* is the first grade of lower secondary level. On top of the 6th grade, students are expected to enhance their understanding of direct and inverse proportional relationships in real-life situations. It is stated explicitly in *the course of study* that:

> "*Through investigations of actual phenomena, students will deepen their understanding of proportional and inversely proportional relationships*." (Takahashi *et al.*, 2008, p.19).

This objective is explained further in the *Functions* domain as follows:

> Students will deepen their understanding of direct and inverse proportional relationships by examining correspondences and variations of two quantities in real-life situations; students will foster their ability to identify, represent, and examine functional relationships.
>
> a. To understand the meaning of functional relationships.
> b. To understand the meaning of direct and inverse proportional relationships.
> c. To understand the meaning of coordinates.
> d. To understand the characteristics of direct and inverse proportional relationships and be able to represent the relationships by using tables, mathematical expressions, and graphs.
> e. To be able to identify direct and inverse proportional relationships in concrete situations and be able to explain them. (Takahashi *et al.*, 2008, p. 21)

In the teachers' guide, the excerpt is further explained for the teachers with the sentences presented as follows:

> In the grade 1 of the lower secondary school, students are expected to understand what it means to say "…and… are in a functional relationship" or "… is a function of…" As a basic model of quantitative relationships, direct and inverse proportional relationship studied in elementary school will be re-conceptualized as functions. (Isoda, 2010b, p. 46)

The excerpt above enunciates that proportional relationships indicates functional relationships. The term *functional relationships* is explained in the teachers' guide as *"A functional relationship exists between two quantities when the value of one quantity is determined, the value of the other quantity will also be fixed to one and only one value*."

(Isoda, 2010b, p. 69). Although mathematical sentences to represent proportional relationships are presented to students before the 7th grade, it is not indicated that the relationships are actually representing *functions*. Mathematical sentences for direct proportional relationships are the simple form of *linear functions* with the expression of $y = kx$, while the mathematical sentences formed to represent inverse proportional relationships are the simple form of *reciprocal functions* with the expression of $y = k\frac{1}{x}$. In the teachers' guide, it is explicitly stated that the direct and inverse proportional relationships is planned to be re-examined through focusing on their algebraic expressions in the forms of $y = ax$ and $y = \frac{a}{x}$. Also, the domain of the proportional relationships is *non-negative numbers* in elementary grades, but it is expanded to *rational numbers* in the lower secondary grades. The aim of placing inversely proportional relationships in the 6th grade is stated as to deepen students' understanding on directly proportional relationships. However, in the 7th grade, variables of an inverse proportion and the relationships between them are aimed to be clarified. The graph of inversely proportional relationships, which comprises two *curves* do not cut the axes, is targeted to present formally in the 7th grade mathematics. The rationale of the use of proportional relationships to trigger *correspondence* and *covariation* meaning of functions is supported in the teacher's guide that proportional relationships exist in many phenomena in daily life. The characteristics of proportional relationships is appropriate to examine how the quantities change and correspond. Proportional relationships involve fruitful examples of concrete situations which enable students to focus on the range of values that the variables could take.

Examining the evolution of the concept of function, it is important to cultivate the understanding of functions in terms of *correspondence* since in modern algebra, it is the formal definition of functions which enables to deal with even advanced mathematical structures. However, the understanding of functions regarding the *covariation* between variables is also essential because the covariation view of functions fosters to examine dynamic situations and enable modelling. Although covariation view of functions does not facilitate to make sense of every mathematical situation, it is valid to examine functions through covariation held in the lower level mathematics. Also, research show that covariation view of function can initiate the development of correspondence view

(Thompson,1994b; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008; Smith, 2003; Cooney *et al.*, 2010).

In addition to functional relationships, students are expected to improve their ability to use linear equations in this grade level. Both arithmetic operations with linear equations and the meaning and the use of linear equations as a representation of real-life situations are held in the 7th grade.

6.2.4.2. The units of textbooks regarding the concepts of ratio. The textbook for the 7th grade involves the objective related units, namely *Letters in Algebraic Expressions*, *Equations*, *Direct and Inverse Proportions*. Particularly, findings regarding the concepts of ratio come from the sub-unit of *Equations*. Covariational reasoning seem to be highlighted in the unit of *Direct and Inverse Proportions*.

Letters in Algebraic Expressions. The unit is developed to represent quantitative relationships as algebraic equations and perform operations about these equations. As the title of the unit is *How many matchsticks do we need?* indicates, the task is writing an algebraic equation to represent the number of matchsticks used in the structure given in Figure 6.63 below.

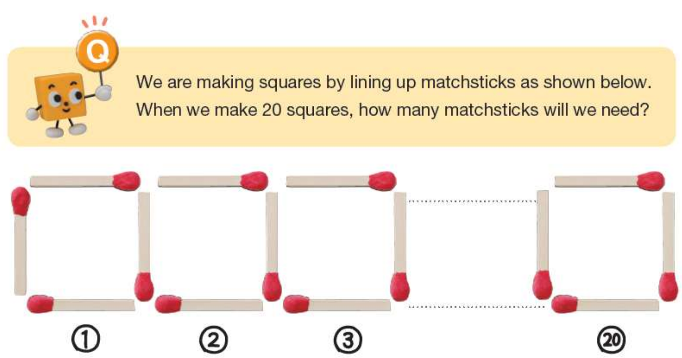


Figure 6.63. Matchsticks task (Fujii and Matano, 2012, Grade 7, p. 50).

A similar task was also presented in elementary school. However, the questions of the task are more sophisticated in this grade level. For instance, in the 5th grade students were triggered to generate the math sentence $3 \times x + 1$, where *x* stands for the number of squares. In this grade, the elimination of multiplication sign is introduced. In addition to 3*x*+1, different algebraic expressions are provided to students such as, $4x - (x - 1)$ and $x + (x + 1) + x$ in order to investigate the quantities represented by x. As it is in the task, the quantities and their relationships are emphasized throughout the unit which comprises equality, equations with one variable, and inequalities.

<u>*Equations.*</u> The unit is divided into 3 sub-units to present the contents, namely, (1) *Equations* and *how to solve them*, (2) *Utilizing linear equations*, and (3) *Proportions*. The first two sub-units are designed for objectives of algebra topics. The last sub-unit, named *proportions*, starts with an introductory question in which the context and the ratio of the quantities are similar to the Task1D given in the 6th grade. I first explain the introductory question of *proportions* sub-unit, then I present the Task1A to clarify the students' strategies supported in this grade level to approach proportional situations.

Similar to the Task1D in the 6th grade, the question is about making some milk tea with the milk tea ratio of 3:5. Differently from the 6th grade task, in which the total amount is given and students are expected to find the amount of milk, the amount of ingredients are given in this question as 200mL of milk and 600mL of tea. It is asked to find the additional amount of milk needed to make some milk tea by using all of the tea. Since students are introduced to use x for the unknown value, the supported approach to find the unknown quantity involves the equation $(200 + x){:}\,600 = 3{:}\,5$. Here, the textbook defines this equation of equivalent ratios as *proportion* and the mathematical property of proportions which is "if a: b = m: n then an = bm" (Fujii and Matano, 2012, Grade 7, p. 98) is given. The textbook presents examples for students to practice *proportion equation strategy*. Some of the examples shares the same problem context used in earlier grades, such as baking a cake and salad dressing. I explain an example and the supported students' strategies in the example through Task1A below.

<u>*Task1A.*</u> "We are thinking to divide 180 pieces of origami paper between an older and younger sister at a ratio of 3:2. How many pieces of paper should the older sister get?" (Fujii and Matano, 2012, Grade 7, p. 101). As a *thinking process* provided for the task, students are encouraged to consider the whole as 5 and form the *proportion equation* of $180{:}\,x = 5{:}\,3$. The equation involves two equavelent whole to part ratios, i.e., total number of sheet to the number of sheets that the older sister gets.

In addition to *proportion equation strategy*, there is a question in the task to foster students to consider alternative ways to find the answer. In response to the question, there are two students' ideas shared in the textbook which is shown in Figure 6.64 below.

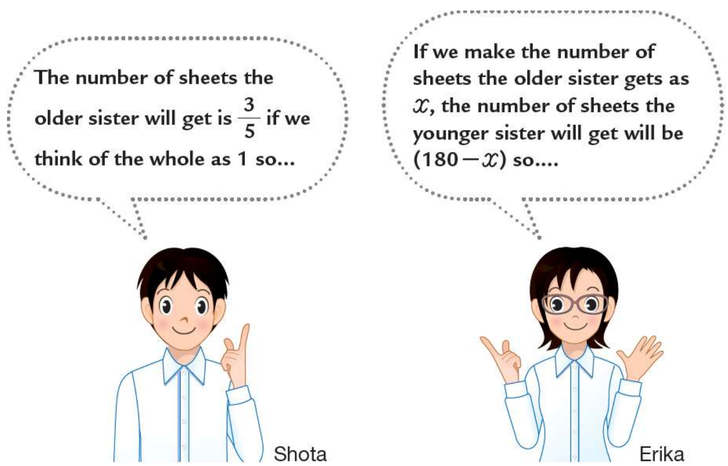


Figure 6.64. Alternative strategies for origami paper task (Fujii and Matano, 2012, Grade 7, p. 101).

If we accept the situation as expressing the number of papers the older sister gets relative to the numbers of paper the younger sister gets, the idea of Shota given above triggers the use of *between state ratios.* The ratio of $\frac{3}{5}$ could be used to determine what percent the older sister gets relative to the whole. That is, thinking of the whole paper as 1 unit, the ratio $\frac{3}{5}$ indicates the amount of paper the older sister gets relative to the amount of the whole or the amount of whole paper,1, is $\frac{5}{3}$ rd of the amount of paper the older sister gets.

The fraction value of percentages (i.e., $\frac{3}{5}$) indicates between- state ratios (Karagöz Akar, 2020). So, the solution of $\frac{3}{5} \times 180$ is aimed to be supported by Shota's idea. The idea of Erika underpins the use of proportion equation strategy by considering the part to part equivalent ratios, so the equation should be like $x: (180 - x) = 3: 2$. The ratios used in this proportion equation are *within state ratios* because each ratio indicates the size of paper that the older sister gets relative to the size of the paper the younger sister gets. That is, the extensive quantities in the ratio, $3: 2$, are the original quantities in the ratio situation. Although it is hard to claim that the proportion equation strategy used by Erika is multiplicative from the students' point of view, still Shota's explanation seems to be supporting between state ratios as a multiplicative comparison of two quantities. With the use of different multiplicative strategies students might be supported to reason multiplicatively in problem situations of missing values proportion problems and ratio comparison problems.

<u>6.2.4.2. The units of textbooks regarding covariation.</u> The textbook unit of *Direct and Inverse Proportions* seem to trigger students' covariational reasoning. I first introduce briefly the unit and present the tasks given in the unit as follows.

<u>*Direct and Inverse Proportions.*</u> The unit is for examination of the relationships between *two quantities changing together*. There are four sub-units namely, (1) *Functions*, (2) *Direct proportions*, (3) *Inverse proportions*, and (4) *Utilizing direct and inverse proportions*. To explain the sub-units, I present Task2A for the functions, Tasks2B and 2C for direct proportions, and Task2D for inverse proportions as follows.

<u>*Task2A.*</u> The task in the beginning of *Functions* sub-unit aims to introduce variables and functions. Rather than directly stating the values of quantities and variables, the task is given in Figure 6.65 invites students to think about what are changing, whether they are changing together, and how they are used to find the amount of time to fill the pool.

Q

Maki went to the pool inside the amusement park but she couldn't go in because they were refilling the pool. Maki was interested in how long it would take to fill up the pool with water.
In order to find out how long it takes to fill the pool with water, what do we need to know?

Figure 6.65. Pool in the amusement park task (Fujii and Matano, 2012, Grade 7, p. 108).

It is explicitly stated after the Figure 6.65 that the shape of the pool, which is cuboid, the rate of water poured in a unit of time and depth of the water are necessary to determine the time needed to fill up the pool. The first question involving this necessary information is stated as "When water was poured into an empty pool at a constant rate, after 2 hours the depth was 10 cm. In how many more hours will the depth of the water reach 100 cm?" (Fujii and Matano, 2012, Grade 7, p. 108). The depth of the water is determined as y and time is called x in the textbook. The letters y and x, which represent different values of depth in cm and time in hours respectively, are defined as *variables*. Then, in the second question, students are asked to form a table including the increasing values of x by 1 and the corresponding y values as shown below in Figure 6.66.

Prob. 2 In Prob. 1, using x as hours and y as the depth in cm, changing the value of x, determine the values of y corresponding to the various values of x and fill in the blanks of the table below.

x	0	1	2	3	4	5
y			10			

When there is one quantity that changes with another, each is represented with a variable, x and y. When the value of x is determined, and there is only 1 corresponding value of y, it is said that

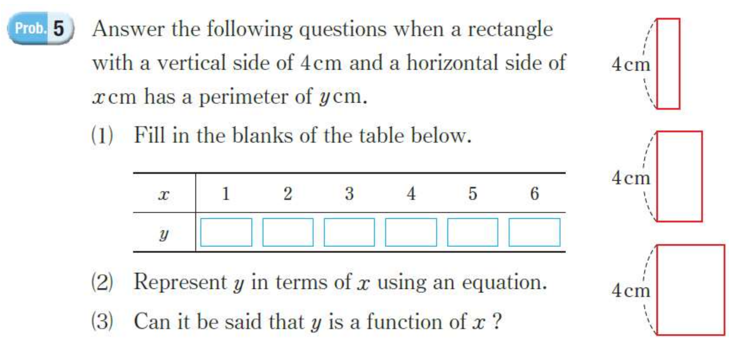
y is a function of x.

In the pool example on the previous page, if in x hours, the depth of water is $y$cm, y is a function of x.

Figure 6.66. Definition of function (Fujii and Matano, 2012, Grade 7, p. 109).

The table is used to illustrate correspondence between variables, and the definition of function is given. What is also important that “x and y variables” are emphasized as quantities changing together. The task involves proportional relationships, and the answer could be found through proportions. Since every proportion indicates a linear function in the form of $y = ax$, introducing functions on the top of proportions might not only improve students understanding of proportions but also ease the comprise the abstract concept like functions. Then there are problem situations provided in the textbook to express them as “… is function of …” The first two situations are the ones provided in the 6th grade, namely, area of a square with respect to its one side length and the circumference of a circle.

In addition to proportional relationships, there are examples having expressions of linear functions of the form $y = ax + b$. The question shared in Figure 6.67 exemplifies the use of linear functions with a non-zero y-intercept.


Prob. 5 Answer the following questions when a rectangle with a vertical side of 4cm and a horizontal side of $x$cm has a perimeter of $y$cm.

(1) Fill in the blanks of the table below.

x	1	2	3	4	5	6
y						

(2) Represent y in terms of x using an equation.

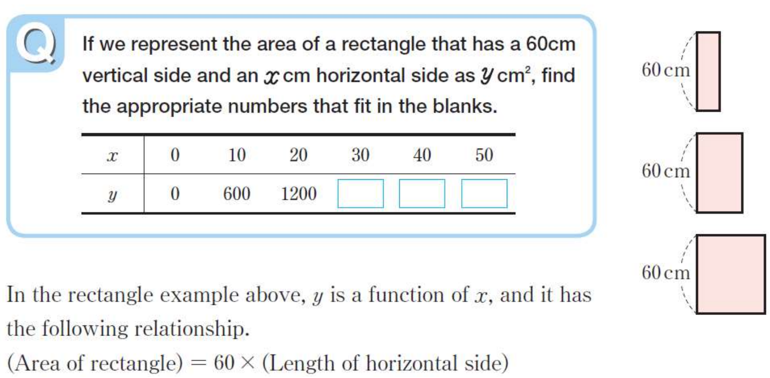
(3) Can it be said that y is a function of x ?

Figure 6.67. Function of perimeter of the rectangle (Fujii and Matano, 2012, Grade 7, p. 110).

As it is displayed in both Figure 6.66 and Figure 6.67, although both variables continuous in nature, only the integer values of x is considered. With the use of a table, students are expected to coordinate the change in the y values with the changes in x values and to write the function of $y = 2x + 8$. With this, *MA1* might be triggered. In addition, the function $y = 2x + 8$ indicates an increasing linear relationship, such that increase in horizontal length would lead the increase in perimeter. The representative pictures display this direction of change, hence the *MA2* might also be triggered. The corresponding covariational reasoning level that might be triggered through these two examples is *gross coordination of values*.

Furthermore, the nature of variables in Task2A is *continuous*. To highlight the continuous nature of variables existing in the Task1A, which are time to fill the pool and the depth of the water, the *interval of values* is introduced in the textbook just after the Task2B is presented. The amount of time to fill up the pool to make the height 100cm is found as 20hours. Since both time and depth are continuous variables, the time is expressed as $0 \leq x \leq 20$, while the depth is expressed as $0 \leq y \leq 100$. The representation of these intervals on the number line is presented too. That is to say, the continuous change in x in a chunk is coordinated with the continuous change in y in a chunk. With this emphasis, students might recognize the simultaneous and continuous covariation between the variables, but there is no further question provided to lead students to consider the continuous covariation existing in the situation.

<u>*Task2B.*</u> With the start of the *direct proportions* sub-unit, the task given in Figure 6.68 below is presented.



Q If we represent the area of a rectangle that has a 60cm vertical side and an x cm horizontal side as y cm², find the appropriate numbers that fit in the blanks.

x	0	10	20	30	40	50
y	0	600	1200			

In the rectangle example above, y is a function of x, and it has the following relationship.

(Area of rectangle) = 60 × (Length of horizontal side)

Figure 6.68. Function of area of the rectangle (Fujii and Matano, 2012, Grade 7, p. 111).

Every time the same vertical size of 60cm is multiplied with the horizontal side of x cm the area value of y cm^2 is obtained. The y changes with respect to the changes in x is 60 times as much because the vertical size is constant. The equation for this task is given as $y = 60x$. Similar to the example given in Figure 6.68, the first two mental actions, *MA1 and MA2,* seem to be targeted for this task too. That is, the table indicates the coordination of change in y with respected to change in x (i.e. *MA1*) as well as the direction of these changes (i.e. *MA2*). The task is given to share the definition of proportion with the representation of equation $y = ax$. The equation form is explained with a sentence of "*When y is a function of x and the relationships can be represented with the following equation, we can say that y is proportional to x. (Following equation:)* $y = ax$" (Fujii and Matano, 2012, Grade 7, p. 111). Then, students are expected to compare the equation of this task which is $y = 60x$ with the equation of $y = 2x + 8$, the question shared in Figure 6.67. Since $y = 2x + 8$ does not have the form of proportion equation, the relationships between the horizontal side length of the rectangle and its perimeter is stated as non-proportional. However, this conclusion is designed to do further explorations on the questions requiring the examinations of proportionality. The further questions require students to examine the change in y values when the x values become 2, 3, 4 times as much, as well as the change in y values when the x values become $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ times as much. The equation in the form of $y = ax$ sustain the proportional relationships which is when x becomes 2, 3, 4 times as large, y also becomes 2, 3, 4 times as large or when x becomes $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ times as large, y also becomes $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ times as large. Students are supposed to recognize the function form of $y = ax + b$ does not sustain such a proportional relationship.

Moreover, students are provided problems to examine proportionality with the negative values of variables. That is to say, if y is proportional to x, the x variable could take all real values and the corresponding y values could take all real values. In the next task presented below as Task2C, students are supposed to examine whether y could be proportional to x when the *proportion constant* is negative.

<u>Task2C.</u> Students are asked to examine the proportional relationships of the equation $y = -3x$ with constructing a table for demonstrating the negative and positive *x* values and the corresponding *y* values. After the table is formed, it is asked to examine the change in *y* values, when the *x* becomes 2, 3, 4 times as large. Students are targeted to come up with the fact that even for the negative proportion constant, the proportional relationship is sustained. In order to support students to verbalize what it means having proportion constant as a positive or a negative number, there is a question stated as "When *y* is proportional to *x*, how do the values of *y* change as the values of *x* increase? Compare when the constant of proportion is a positive number and a negative number and describe what you noticed." (Fujii and Matano, 2012, Grade 7, p. 114). The use of the expression *the values of x increase* is significant in the question, because if the proportion constant is a negative value, the *y* decreases when *x* increases and the amount of change in *x* and *y* is identical. Thus, the first three mental actions might be supported in the task. In the following paragraph, I explain which mental actions are targeted in the task.

With the use of table, students *coordinate* the *y* values with *changes* in *x* values and this targets the mental action in *MA1*. The examination of the values to express the *direction of change* as when the *x* values *increase,* the *y* values *decrease* is compatible with the mental actions in *MA2*. The verbalization of the *amount of change in x,* when the *x values becomes 2 times as much*, the *corresponding y values becomes 2 times as less* indicates the use of the *MA3*. Thus, students might realize that graphical representation of a proportion equation having a negative constant is a *decreasing straight line passing through the origin*.

Moreover, the *graphs of proportion equations* are introduced so that the properties of equations are fully examined. The coordinate system and axes are introduced prior to the graphs. The graphing is introduced with the representation of the proportion equation $y = 2x$ gradually as shown in Figure 6.69.

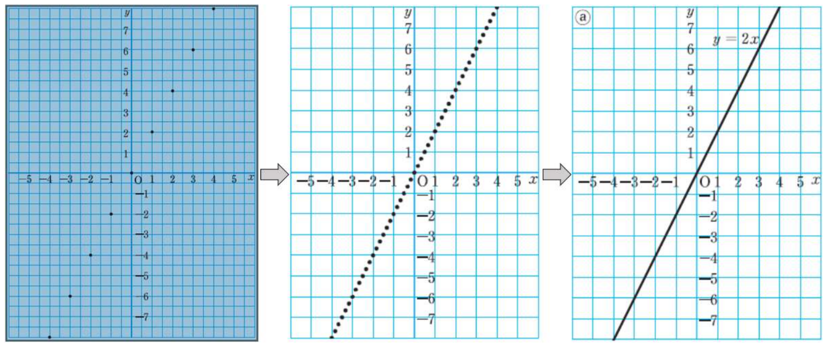


Figure 6.69. Graphing y=2x (Fujii and Matano, 2012, Grade 7, p. 117-118).

As it is seen in Figure 6.69, the interval for x value gets smaller and the line $y = 2x$ is formed. I think this representation is beneficial to indicate the *continuous covariation*. However, the rate of change is not explicitly questioned in this unit. Thus, I do not say that the continuous covariation level is triggered, rather I infer the continuous nature of variables are targeted to improve through graphical representations. Moreover, the direction of change is targeted for students to investigate through proportion equations having proportion constant as positive integers, negative integers, and rational numbers. The Figure 6.70 given below explicitly exemplifies that the direction of change is targeted to be examined in the textbook.

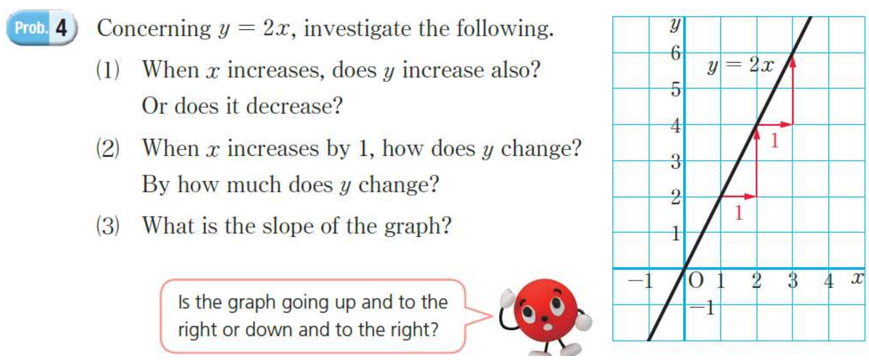


Figure 6.70. Direction of change (Fujii and Matano, 2012, Grade 7, p. 120).

The first question of the problem apparently directs students to think about the *direction of change* by asking the *how y changes with the increase in x*. In the second question, *the amount of change in y* when *the amount of change in x is 1* is targeted to be verbalized. Therefore, the first three mental actions, *MA1, MA2 and MA3*, seems aimed to be triggered. Although the variables are continuous, and the line indicates smooth continuous covariation, there is no question to indicate that simultaneous continuous covariation exists between x and y.

In the *Inverse Proportions* sub-unit, the task to investigate relationships between the width and height of a rectangle with the area 18cm^2, which was used in the 6th grade, is conducted again to recall students' knowledge of inverse proportions. It is expressed in the unit that if *y* is function of *x* and it is represented with the equation $y = \frac{a}{x}$, then it can be said that *y* is inversely proportion to *x*. Similar to the procedure of introducing direct proportions, in the inverse proportions sub-unit, students are expected to write an equation to the problem situations involving inversely proportional variables, investigate the inversely proportional relationship between variables through positive and negative values of *x*, as well as positive and negative values of proportion constant, so that the negative values of *x* and proportion constant also hold true for the inverse proportions.

Task2D. The example of $y = \frac{6}{x}$ is provided and the graphing is shown gradually as shown below in Figure 6.71.

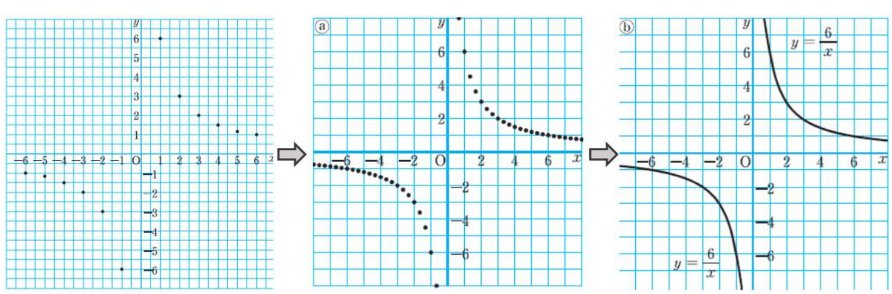
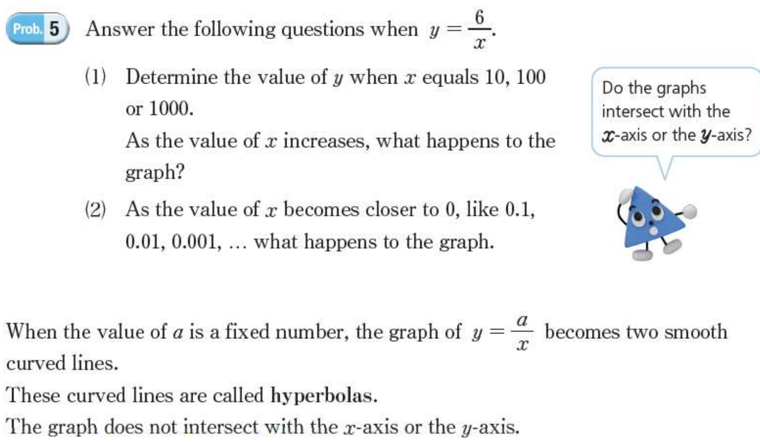


Figure 6.71. Graphing $y = \frac{6}{x}$ (Fujii and Matano, 2012, Grade 7, p. 127-128).

After students are expected to practice graphing various inverse proportion equations including negative values of x and negative values of proportion constant, questions requires to examine similarities and differences between direct proportion graphs shared. Then, stuents are expected to *recognize the continuous covariation between x and y values* through a problem. I present it below in Figure 6.72.



Prob. 5 Answer the following questions when $y = \frac{6}{x}$.

(1) Determine the value of y when x equals 10, 100 or 1000.
As the value of x increases, what happens to the graph?

(2) As the value of x becomes closer to 0, like 0.1, 0.01, 0.001, … what happens to the graph.

When the value of a is a fixed number, the graph of $y = \frac{a}{x}$ becomes two smooth curved lines.
These curved lines are called **hyperbolas.**
The graph does not intersect with the x-axis or the y-axis.

Figure 6.72. Hyperbolas (Fujii and Matano, 2012, Grade 7, p. 129).

Students are expected to examine the change of the graph with the increase in *x*. The graph will be close to the *x* axis but never will touch the axis. The increase in *x* causes the decrease in *y*. Then, students are supposed to narrow down the interval of values of *x*. When the values of *x* get smaller, the corresponding *y* values get bigger and the graph will be closer to the *y* axis but never touches the axis again. By increasing and decreasing the intervals of *x*, which results in decrease and increase in the intervals of *y*, students are triggered to focus on the *continuous nature of variables*. It is explicitly stated and expected of students to realize that *x and y variables vary continuously and smoothly*. In particular, students are asked to examine the values of x in chunks including 10, 100, and 1000 at first and then 0.1, 0.01, and 0.001. That is, they are expected to investigate the changes in y values in regard to the changes in x values when x values are getting 10 times as much and 1/10 times as much. Thus, the problem might trigger *MA4* which described as "coordinating the average rate of change of the function with uniform increments of change in the input variable" (Carlson *et* *al.*, 2002, p.357). The covariational reasoning level involving *MA4* is *chunky continuous covariation*.

In the last sub-unit *utilizing direct and inverse proportions,* the proportional relationships existing in perimeter and area of rectangles and squares are shared. Then, some word problems are provided in the sub-unit to deepen students' use of proportional relationships and graphs. Similar to the 6th grade, students are provided with a comparison of speed example. In this example, there are two people: Yumi and Takuya. Yumi uses moving walkway which has 60m length and moving with the speed of 0.5m/s and Takuya prefers to walk with the speed of 1m/s. Students are expected to graph both Yumi's and Takuya's movement. Then they are provided with questions to answer using the graphs. For example, they are asked to determine the distance between Yumi and Takuya 40 seconds later and the distance between them when Takuya reaches the end of moving walkway. Through these questions, students' use and understanding of graphs are targeted to be improved.

6.2.5. Grade-8 Japanese Curriculum Materials

6.2.5.1. Content related objectives and the explanations given in the teachers' guide. Thus far, students are expected to examine functional relationships through proportions. In the *eighth grade*, on the top of direct and inverse proportional relationships, *linear functions* are introduced. It is supported in the *teachers' guide* as follows:

> The study of linear functions is an extension of the study of proportional relationships. At the same time, it is an entrance to a more in-depth study of functions represented in algebraic expressions with letters, such as the focusing on the rate of change. (Isoda, 2010b, p.91)

The quote makes it clear by stating *rate of change*, that the covariation between the variables is highlighted in the 8th grade.

The content related objective for linear functions is given as

"*Students will understand linear functions through explorations of concrete phenomena while developing their ability to identify, represent and utilize function relationships in inquiries.*" (Takahashi *et al.*, 2008, p.22).

It is further explained in the content domain of functions as follows:

> Students will understand linear functions by analyzing the change and correspondences of two quantities identified in concrete situations, while developing their ability to represent and analyze functional relationships.
>
> a. To know that some phenomena may be characterized by linear functions.
> b. To understand and relate representations of linear functions, i.e., tables, mathematical expressions, and graphs.
> c. To perceive linear equations with 2 variables as representations of functions.
> d. To grasp and explain concrete phenomena by using linear functions (Takahashi *et al.*, 2008, p. 23-24).

It is explicitly stated and expected of students to examine the changes and correspondences of two quantities which might trigger mental operations of covariational reasoning. The change is examined in the earlier grades with the use of tables and ratios of quantities to verbalize the increase or decrease in variables. The 8th grade objective stated above is further explained in the teachers' guide that the meaning of change is improved to "*rates of changes of values of corresponding variables*" (Isoda, 2010b, p. 92). The rate of change for the linear functions, which have the form $y = ax + b$, is expressed as $\frac{y_2 - y_1}{x_2 - x_1}$. The rate of change equals to the constant a, and this indicates that graphs of linear functions are straight lines. Moreover, the following quote taken from the teachers' guide explicitly expresses that the *mental action 3* is targeted to activate through the tasks and questions:

> By examining rates of changes, help students understand that the coefficient of x, that is a, is how much y will increase when x increases by 1. Moreover, help students to understand that the amount of increase in y corresponding to the amount of increase in x can be determined based on the value of a. (Isoda, 2010b, p. 92)

The expected students' behavior compatible with *MA3* is identified as "*verbalizing an awareness of the amount of change of the output while considering changes in the input*" by Carlson and her colleagues (2002). When the covariational reasoning framework is considered, the level comprising the *first three mental actions* is named as *coordination of values*. The excerpt above supports the claim that students' understanding of covariation between variables is *explicitly* aimed to be improved to the level of *coordination of values* in the 8th grade.

Also, if the tasks are designed to examine the rate of change of *y* variable with the uniform increments of the *x* variable, then it might be said the tasks might activate the *mental action 4*. The teachers' guide indicates in the following paragraph that the conceptual understanding of the rate of change is aimed rather than only procedural understanding of it.

> The reason the phrase, "rate of change," is included in "Terms and Symbols" is so that its instruction will not simply focus on calculation of rates of change procedurally. In addition, it is important to enable students to use rates of change in examinations and explanations of phenomena appropriately. (Isoda, 2010b, p. 92)

As it is stated in the quote above, the meaning of rate of change for a situation is targeted for students to explain. I infer that students' explanations might be compatible with the students' "*verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input*" (Carlson *et al.*, 2002, p. 357). This is specified for the *mental action 4*. The level of covariation involving the first four mental actions is *chunky continuous covariation*. The excerpts from both the content objectives in the course of study and the teachers' guide suggests that chunky continuous covariation is aimed for students.

In addition, students are supposed to recognize the relationships between the expressions of equations with two variables and functions in the 8th grade. The objective is further explained in the teachers' guide that when the coordination of values of y and x is considered for an equation with two variables expressed as $ax + by + c = 0$, there is one and only *y* value for every *x* value if $b \neq 0$ (Isoda, 2010b).

Therefore, this relationship indicates that *y* is a function of *x* and the equation could be rewritten as $y = -\frac{a}{b}x - \frac{c}{b}$ to illustrate the functional relationship explicitly. Since the students are expected to realize that the equations with two variables actually represent a linear function, the systems of simultaneous linear equations could be solved with the use of graphs.

Moreover, the aim to provide students with opportunities for grasping and explaining phenomena with the use of linear functions is clarified with an example in the teachers' guide. The example involves the linear relationships between time and temperature of water when it is heated. Students are expected to observe the relationship between quantities and idealize it. The graph of the situation is supposed to be idealized and an algebraic representation is supposed to be used to find out the temperature at a specific time.

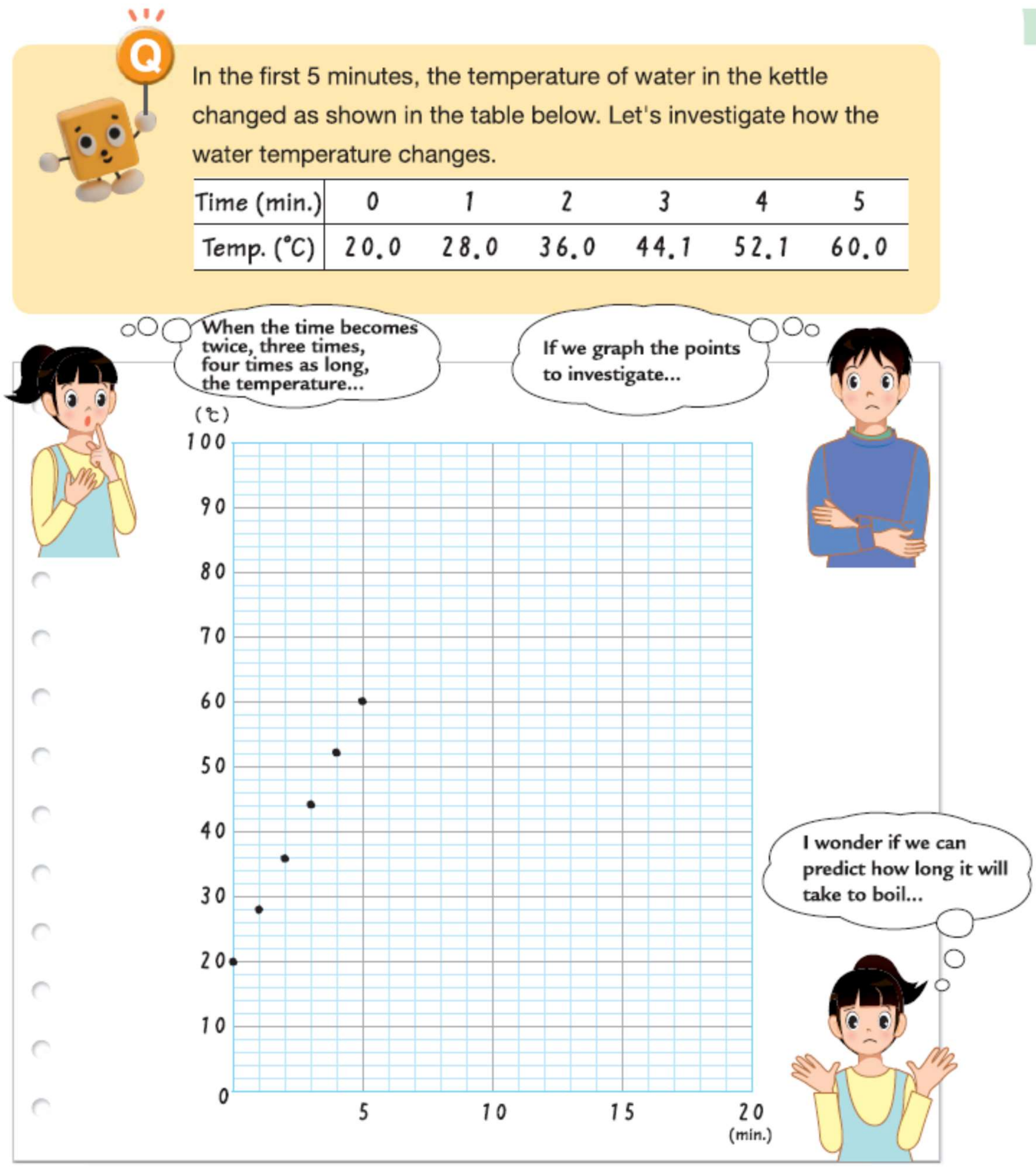
<u>6.2.5.2. The units of textbooks regarding covariation.</u> In this grade level content related objectives are covered in *Linear Functions* unit of textbook.

<u>*Linear Functions.*</u> The unit of linear functions is classified into two sub-units, namely, (1) *Linear functions*, and (2) *Linear functions and equations*. In the first sub-unit, definition of linear functions, changes in variables of them, graphic and algebraic representation of these functions are referred. In the second sub-unit, graphs of linear equations and the use of graphs for linear functions and systems of equations is presented.

The first sub-unit starts with the task that referred to a boiling water example of a linear relationship between time and temperature in the teachers' guide. The unit is introduced through examining the different characteristics of the Task1A, but there are some extra problems for students to practice and reinforce the intended learning. Therefore, throughout the presentation of the task given below, I refer some additional questions that are not specifically related with the task.

<u>*Task1A.*</u> In the task, a child examines the quicker method for boiling water either by electric water heather or stove. To examine the amount of time passed to boil some water

which initially is 20 °C, the child conducts an experiment. The records of the experiment is shared with a table as displayed in Figure 6.73 below.



Time (min.)	0	1	2	3	4	5
Temp. (°C)	20.0	28.0	36.0	44.1	52.1	60.0

Figure 6.73. Boiling water task (Fujii and Matano, 2012, Grade 8, p. 53).

As a first question of the task, students are asked and expected to write down an algebraic expression for the temperature change with the change in time in x minutes. The answer is provided as $y = 8x + 20$. Then, immediately after the question “When will the temperature of the water in the kettle reaches 80°C?” (Fujii and Matano, 2012, Grade 8, p. 54) is stated, the definition of linear functions are given as “*Given 2 variables x and y, if we can express y in terms of a linear expression in x, we can say y is a linear function of x. A linear function is generally written in the form,* $y = ax + b$*.”* (Fujii and Matano, 2012, Grade 8, p. 54). As it is seen in Figure 6.73, use of tables, the guidance to coordinate the change in *y* with the change in *x*, the placing the points and observing the pattern for graphing are designed in a similar way that proportions are presented in the earlier grades. The task is designed to introduce linear functions on the top of proportions by clarifying and explicitly stating that *proportions are simple form of linear functions* which does not have the *y*-intercept, i.e. *b*=0.

In order to provide students with opportunities for examining linear functional relationships and practice algebraic representation of linear functions, there are additional problems with various mathematical situations, such as water level in a tank with respect to time, gasoline consumption with respect to distance travelled, and, shortening of a stick, when it burned, with respect to time. In some of these examples the constant *a* becomes negative and non-integer numbers. For instance, in the gasoline consumption question, it is given that the car uses 1L gasoline to travel 17km and it starts to travel with the 40L of gasoline. Students are expected to express gasoline left (*y* L) when the car travels *x* km as a function of $y = 40 - \frac{1}{17}x$.

Then, the situation of the Task1A is considered again. Students are supposed to examine the graphs of linear functions in terms of *changes in variables* to realize the *proportional increase/decrease having a constant of y-intercept*. The question to form a table to examine the changes in variables in the boiling water task is re-stated in a way that the temperature of the water in the kettle increases 4°C in each minute. In response to the question, the explanation in Figure 6.74 is given.

Ex. 1 Let's investigate the changes in the values of x and y in the linear function, $y = 4x + 20$, as x changes from 2 to 5.

As the value of x increases from 2 to 5,

the amount of increase in x is $5 - 2 = 3$

the amount of increase in y is

$(4 \times 5 + 20) - (4 \times 2 + 20) = 12$

The amount of increase in y is 4 times as much as the amount of increase in x. In other words,

$$\frac{(\text{increase in } y)}{(\text{increase in } x)} = 4$$

3

x	$\cdots$	2	$\cdots$	5	$\cdots$
y	$\cdots$	28	$\cdots$	40	$\cdots$

12

Figure 6.74. Changes in variables in boiling water task (Fujii and Matano, 2012, Grade 8, p. 56).

The ratio of amount of increase in *y* to the amount of increase in *x* is actually *rate of change* and it equals to *a* of the general formula of linear functions, i.e., $y = ax + b$. In other words, the increase in *x* by 1 corresponds the increase in *y*, such that their proportion equals to *a*. Since the rate of change, i.e. the value *a*, is the result of coordination of the amount of change in y to the amount of increase in x, the *MA3* is aimed to be triggered. Before formal introduction of the rate of change is given, there are algebraic examples for students to examine the rate of change for the linear functions like the way presented in Figure 6.74. Some of the additional problems have negative values of the constant *a*. Students are expected to realize the negative values of *a* indicating a decrease. One of the additional problems is shared in Figure 6.75.

Prob. 3 When the temp is 25°C at sea level, the relationship between the altitude, x km, and the temperature, y °C, may be expressed as

$y = -6x + 25$

until about 10 km above sea level.

(1) What does the rate of change, -6, represent?

(2) If the altitude increases by 3 km, how much does the temperature decrease?

Snow covered valley of Hakuba mountains in the summer (Altitude 1560m, Nagano Prefecture)

Even though there is snow in the summer, I wonder if the temperature is below 0°C.

Prob. 4 In an inversely proportional relationship where $y = \frac{24}{x}$, calculate the rate of change when x changes as indicated below.

(1) from 2 to 6 (2) from 4 to 8

Figure 6.75. Additional problems for rate of change (Fujii and Matano, 2012, Grade 8, p. 57).

As it is seen from the Prob.3 displayed in Figure 6.75, students are expected to explain the meaning of the negative *a* value. Students are supported to verbalize the situation that *one quantity is increasing while the other is decreasing and the ratio of the change in quantities' values represents a rate having negative value*. Lastly, an example to examine the change in variables for inverse proportional relationships is given in Figure 6.75. Since students find the rate of change when x changes as from 2 to 6 is -2. However, for the change in x from 4 to 8, the rate of change is -$\frac{3}{4}$. Thus, the rate of change is not constant as it is in linear functions. Students are expected to realize that the change in variables for the functions involving inverse proportional relationships does not indicate a constant rate of change.

Then, *graphs of linear functions* are targeted to present. With the use of tables, students are expected to place the values of variables on the coordinate system and draw a line to represent linear functions. The linear functions (i.e., $y = ax + b$) are compared with proportional equations (i.e., $y = ax$) again and it is explained that *translation* of the graph of $y = ax$ on the *y* axis by the value of *b* represents $y = ax + b$. The value of *b* is also expressed as *y-intercept* which is the value of *y* when *x* is zero. After the graphs of linear functions are presented the meaning of rate of change is examined on the graphs and introduced as *slope*. For instance, the rate of change of the linear function $y = 2x + 3$ is 2, while the rate of change of the linear function $y = -2x + 5$ is -2. In the textbook, the graphs of these two functions are illustrated as shown below in Figure 6.76. The rate of change of the function on its graph is explained for the former example that moving to the right 1 unit makes the graph to go up 2 units and for the latter example that moving to the right 1 unit makes the graph to go down 2 units. Hereby, both *direction* and *amount* of the change in *y* values with respect to the change in *x* values are exhibited with the graphical representation. Regarding to covariational reasoning, since the direction of changes and amount of changes between x and y values are coordinated on the graph, the *MA2* and *MA3* might be triggered.

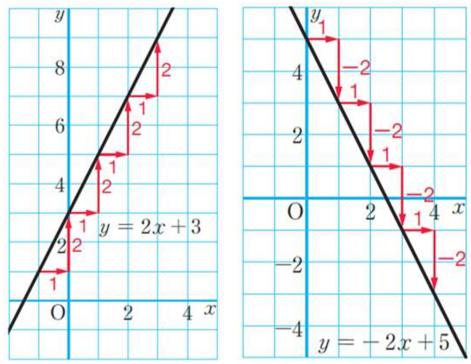


Figure 6.76. Slope of linear functions (Fujii and Matano, 2012, Grade 8, p. 61).

Moreover, the use of different types of representations are summed in the textbook as it is given in Figure 6.77 below. With the Figure 6.77, the relationships between tabular, algebraic, and graphical representations are explicitly presented in the textbook. Problems are provided for students for using a table to write down the equation, using a graph to write down the equation, using a table to draw the graph; and, using an equation to draw the graph.

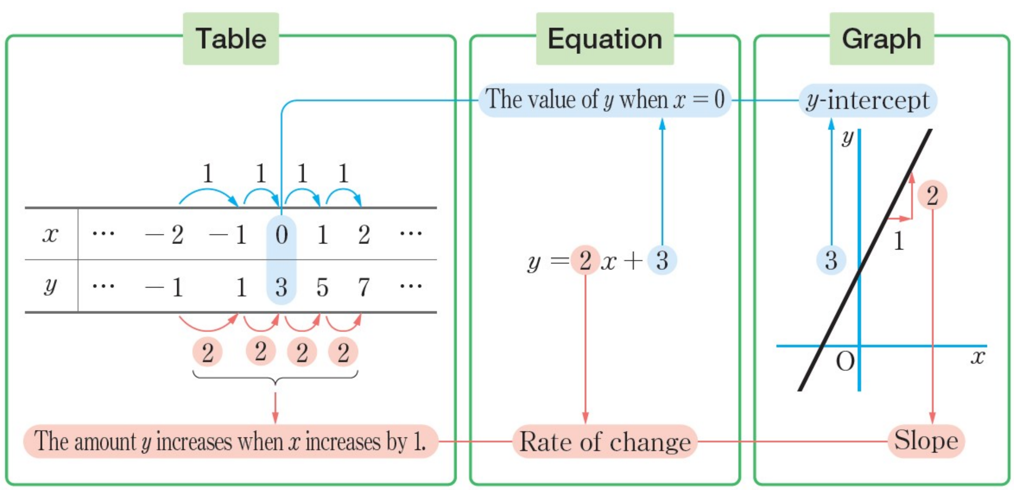


Figure 6.77. Use of different representations (Fujii and Matano, 2012, Grade 8, p. 62).

In addition, there is a question on the Task1A of boiling water that requires to consider interval values of variables on the graph. It is asked that "Between 5 and 15 minutes after we started heating the water, the temperature changed from what degree to what degree?" (Fujii and Matano, 2012, Grade 8, p. 66). The Figure 6.78 given below exemplifies how the interval values of variables shown on the graph in reply to the question is provided.

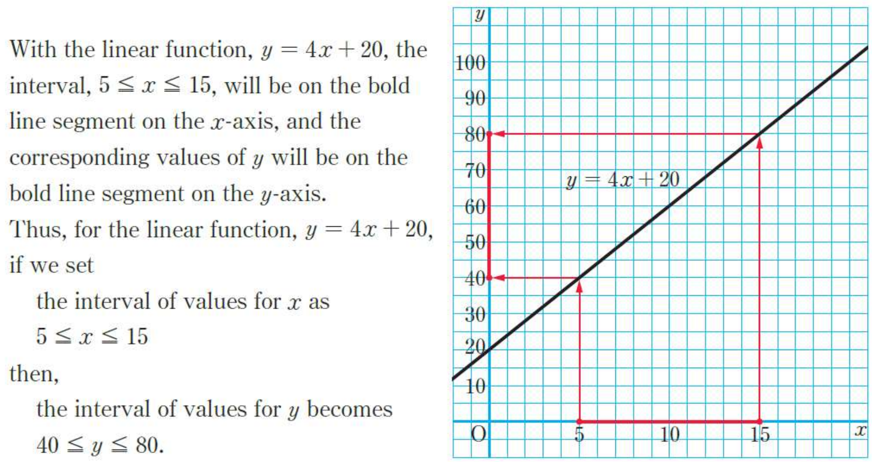


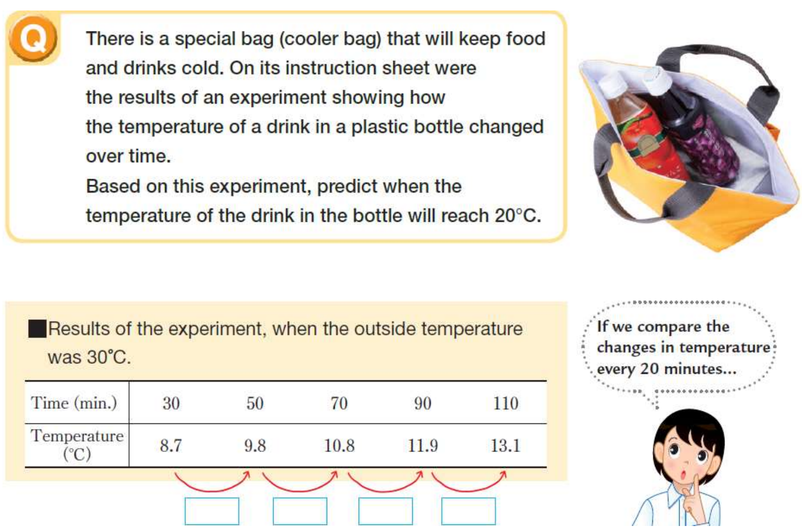
Figure 6.78. Interval values of variables (Fujii and Matano, 2012, Grade 8, p. 66).

Rather than expressing the change in the temperature as 40°C, the change is expressed as an interval value of $40 \leq y \leq 80$ to answer the question. The change in both variables is expressed as interval values. By doing so, I think the *continuous variation* of the variables is aimed to be enlightened. Since both *variables are varying simultaneously*, students' understanding of the situation as a *continuous covariation* between the variables might be triggered.

Furthermore, right after the Task1A, students are expected the write equations for linear functions: Firstly, they are provided with examples to write equations for linear functions with a given values of slope and y-intercept. Secondly, they are expected to write equations for a line, i.e. linear function, with a given point on a line and the slope. Lastly, students are expected to write an equation for a line with given two points on the line. Since these practices rely on algebra, I do not share them in detail.

Thus far throughout the task, first four mental actions are targeted. The use of tables to express the result of experimentation/values of variables and placing the coordinate values for each result on a coordinate system (e.g., shown at the Figure 6.73) might trigger the *MA1*. Then with the emphasis on *change in variables*, students are targeted to activate/develop *MA2* because they are expected to verbalize *the direction of change as increase or decrease*. The Figure 6.74 exemplifies *MA3* which is *coordination the amount of changes with each other*. Also, the aim of *examining the rate of change of the y with respect to the x when the increase in x is uniform*, is compatible with the *MA4*. Thus, *chunky continuous covariation* might be triggered through the unit.

<u>*Task1B.*</u> This task given below in Figure 6.79 is similar to the previous one, but in this task the values of *y* variables do not increase uniformly, and they are decimal numbers.



Q There is a special bag (cooler bag) that will keep food and drinks cold. On its instruction sheet were the results of an experiment showing how the temperature of a drink in a plastic bottle changed over time.
Based on this experiment, predict when the temperature of the drink in the bottle will reach 20°C.

■Results of the experiment, when the outside temperature was 30°C.

Time (min.)	30	50	70	90	110
Temperature (°C)	8.7	9.8	10.8	11.9	13.1

Figure 6.79. Temperature change of drinks task (Fujii and Matano, 2012, Grade 8, p. 70).

It is asked and expected of students to place the points on a coordinate system and represent the situation with a graph. Although the points cannot be on the same line, the placement of the points is appropriate to assume the relationships between variables as temperature is a function of time, i.e. *y* is a function of *x*. Since students are provided with the table in the Figure 6.79 which coordinate the values of temperature with the changes in time, the *MA1* might be triggered. Moreover, the table displays the increase in time variable and decrease in temperature. Thus, direction of changes is highlighted, which might activate the *MA2*. In addition, students are given with the amount of changes in time in chunks- i.e. 30 min., 50 min., 70 min., 90 min.- and asked to find the amount of changes in temperature measured every 20 minutes starting with 30 min. Therefore, the *MA3* might also be triggered. To find out when the temperature will be 20 °C, students need to find a linear equation representing temperature as a function of time. They are expected to consider the variables as continuous to find that equation. Thus, it is expected students of "coordinating the average rate-of-change of the function with the uniform increments in the input variables" (Carlson *et al.*, 2002, p. 357) which is defined as *MA4*. In case, students considers the rate of change as instantaneous rate of change, the *MA5* might also be triggered. Since continuous covariation depends on students' sense making and there is no questions to allow students to think about the values of variables between chunks (e.g. when time is 35 min. or 93 min.), I evaluate this task as designed to trigger first four mental actions. The corresponding covariational reasoning level for the first four mental actions is *chunky continuous covariation*.

In the second sub-unit *Linear Functions and Equations*, students are supposed to engage in drawing the graph of linear equations with 2 variables. With the help of the linear graph, the one to one correspondence relationship is tacitly expressed with the words of "…if we fix the value of *x*, the corresponding value of *y* will also be fixed" (Fujii and Matano, 2012, Grade 8, p. 73) which means and is explicitly stated in the textbook that *y* is a function of *x*. Also, to exemplify the use of linear functions to examine geometric figures, the problem presented in the 6^{th} grade in Figure 6.58 is re-stated for the 8^{th} grade mathematics. Students are expected to draw a graph for the same problem situation given in Figure 6.58 (in the 6^{th} grade). Then, to present the uses of graphs of linear functions, students are presented with two tasks presented below.

<u>Task1C.</u> The task is presented as the below Figure 6.80.

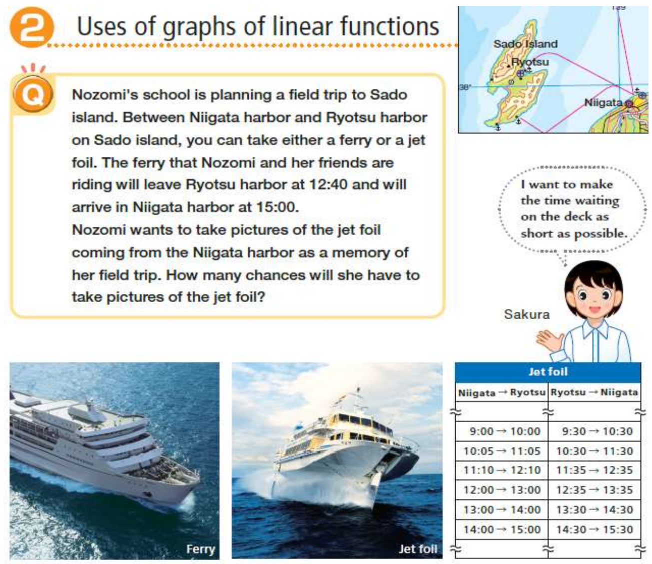


Figure 6.80. Ferry trip task (Fujii and Matano, 2012, Grade 8, p. 79).

In order to examine the situations, students are expected to draw and examine graphs as the way illustrated in Figure 6.81.

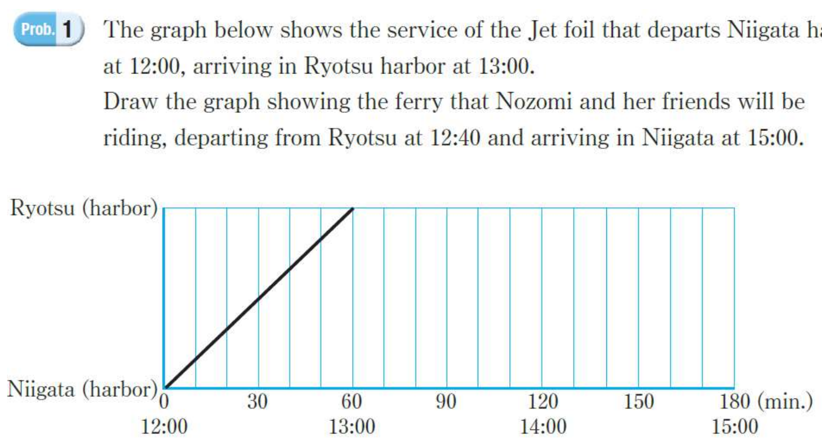
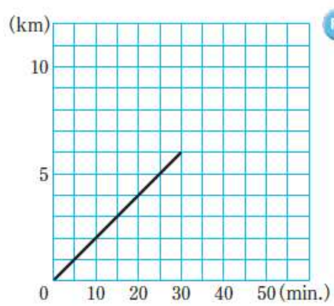


Figure 6.81. Use of graphs of linear fucntion for ferry trip task (Fujii and Matano, 2012, Grade 8, p. 79).

The distance travelled does not given as variables having numerical values, rather students are expected to focus on the differences in time and realize the distance travelled is the same for all the trips. As it is stated, it is expected to draw the line representing the ferry trip. In the next questions, it is asked to draw the graphs of jet foil and then examine the intersection points to count how many times the photo could be taken. Similar to the task1C, there is another problem situation for the use of graphs to answer the problem as presented below in Task1D.

<u>*Task1D.*</u> As Figure 6.82 displays, the problem is similar to the Task1C. However, in this problem, the movement is not uniform throughout the trip. Students are expected to consider the movement with respect to the different time intervals. That is to say, the speed is different for the different time intervals and the graph will be formed with three different line segments to represent the speed of Hiroko.



Prob. 4 Hiroko left home at 9 o'clock on her bicycle to go to a park that is located 12 km from her home. The graph on the left represents a part of her trip.

(1) How fast (in km per hour) was she traveling at first?

(2) She stopped for 10 minutes to rest 6 km from her home. Draw this situation on the graph.

(3) In order to arrive at the park at 10 o'clock, how fast will she have to travel the rest of the way?

Prob. 5 In Prob. 4, while Hiroko was resting, her sister passed her even though she left home after Hiroko. They arrived at the park at the same time. If we assume that Hiroko's sister was traveling at a constant speed and we know that she passed Hiroko after 5 minutes of resting, at what time did her sister leave home?

Figure 6.82. Biking task (Fujii and Matano, 2012, Grade 8, p. 80).

After the graph representing Hiroko's speed is drawn, it is asked to examine Hiroko's sister's speed to find out when she left their home in the Prob.5 in Figure 6.82 above. To be able to answer this question, students need to realize that the slope of a line is uniform and holds for any interval of variables. For instance, the slope of the line representing the speed of Hiroki's sister is uniform since her speed is constant. Since she started her travel and ended it with a constant speed and without any break, her speed for her travel is a line. From the given information in the problem, it is known that she travels 6km in 25 min; therefore, her speed is 14.4 km/h. Since she travelled the first half of her road, i.e. 6 km, with constant speed, she spent 25 min. before she passed Hiroki. Students are expected to coordinate not only amount of changes between variables (*MA3*) but also the rate of change, i.e. speed, with the changes in distance (*MA4*). Since both distance and time are continuous variables and they co-vary continuously, as well as students are not explicitly supported to examine the variables in chunks, rather they supported to consider variables varying smoothly, students' reasoning might be triggered to *smooth continuous covariation*. Students are expected to use the graph, the value of the multiplicative comparison of the quantities, and numerical operations to find that the sister left the home at 9:10. It is interesting that in the textbook, the examples are presented to examine the point of intersection tacitly, and then the solving systems of equations with the use of graphs is introduced. That is, students are expected to examine the situations and use graphs for linear functions first then use the same method for the equations.

6.2.6. Grade-9 Japanese Curriculum Materials

The *ninth grade* is classified as the last grade in lower secondary mathematics in Japan. The functional relationships is discussed in the ninth grade through the functions having the form of simple polynomials as $y = ax^2$. The content related objective is stated in the course of study as

> "*Students will understand functions in the form $y = ax^2$ of through explorations of concrete phenomena, while extending their ability to identify, represent and analyze functional relationships*." (Takahashi *et al.*, 2008, p. 24).

It is explicitly stated that students understanding of functions is aimed to be improved to quadratic functions, so that the relationships between variables are no longer directly proportional. Since the focus of this study is to examine functional relationships involving ratio, rate, proportions, and linear functions, the analysis regarding ninth grade is not necessary for this study. Also, the grade 9 is considered as the first grade of secondary level mathematics in Turkey and the corresponding objective of ninth grade Japanese mathematics education is held in secondary level mathematics in Turkey. Therefore, I do not share findings of 9th grade Japanese mathematics topics, because they are not in the scope of this study.

6.3. Findings Coming from Turkish Curricula Materials

6.3.1. Grade-5 Turkish Curriculum Materials

6.3.1.1. Content related objectives. The concept of ratio is mentioned the first time in the *fifth grade* in Turkish mathematics education. Ratio is introduced under the learning area named *numbers*. As already stated, the learning areas are named as *domain* in the Japanese curriculum. So, in this section, instead of using the term *learning area* as defined in Turkish curriculum, I prefer to use the term domain in order to be more consistent in presentation of findings.

In Turkish curriculum, there are two content related objectives in the fifth grade. The first one expects students to "*express the relationships between two quantities as a ratio"* (MoNE, 2009a, p. 281). It is explained that the ratio might indicate the comparison of a part to the whole or a whole to the part. The quantities chosen for comparison are restricted to have the same unit of measure. As an activity example for the first objective stated in the course of study, there is a mixture problem of which a lemonade recipe, involving 4 glass of water and 1 glass of lemon juice, is shared. By forming a ratio, students are expected to make comparisons of "the amount of lemon juice to the amount of water", "the amount of water to the amount of lemon juice", "the amount of lemon juice to the amount of lemonade", and "the amount of water to the amount of lemonade". For instance, students are expected to express that the ratio of "the amount of lemon juice to the amount of water" is 1:4, but there is no specific explanation for the meaning of this ratio which indicates that the amount of lemon juice is one fourth of the amount of water. I will depict the examination of the textbook later in terms of whether the meaning of the multiplicative relationships are supported.

The other objective states "*use tables to express and solve ratio problems"* (MoNE, 2009a, p. 281). For instance, the lemonade example shared above is used to serve lemonade to 2 people and students are asked to find the recipe to serve lemonade for 3, 4, 5, ... people. It is supported in the course of study for students to form a table to solve the problem. With the use of table, students might be supported to consider keeping the quality of a situation the same. It seems like the *building up strategy* might be highlighted with the use of tables.

The percentages are also planned to be introduced in the fifth grade; however, the percentages are connected with decimal representations of fractions. And, in the course of study, there is no indication that percentages actually represent ratios.

<u>6.3.1.2. The units of textbooks and the explanations given in the teachers' guide regarding the concepts of ratio.</u> In the textbook, the concept of ratio is connected with the concept of fractions. In the *Fraction* unit, students are planned to be introduced with improper fractions, equivalent fractions, decimal fractions, and percentages. Then, decimal fractions are further explained as a separate unit named as *The world of decimal fractions* and ratio is mentioned the first time as a sub-unit in here. In this part, I share my findings of the sub-units of *percentages* and *ratio*.

<u>Fractions.</u> In the sub-unit of *percentages*, the scores Turkish National football team from the FIFA 2002 World Cup Final and the percentages of the game watched are presented in a table as an introductory task. Students are expected to compare the percentages of games watched and the percentages of games watched of which the Turkish team won the game. The aim of the task is explained in the *teachers' guide* as to gain students' interest and to check their skills of reading tables.

Then, the images (e.g. Figure 6.83) are shared to compare and express the part to whole ratios as percentages. Students are asked to express the percentages of orange area to the whole image. The expressions of $\frac{35}{100} = 0.35 = 35\%$ are shared as a response. Still, there is no explicit statement that the percentage actually expresses a ratio.

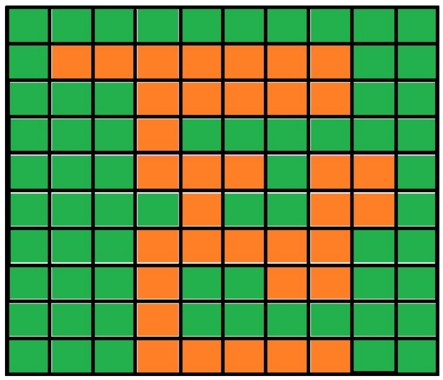


Figure 6.83. Hundred chart (Aktaş *et al.*, 2007, İlköğretim Matematik 5, p. 58).

In the following section, I share my analysis about the unit of *The world of decimal fractions* in which the ratio concept is appeared the first time in elementary school mathematics.

The world of decimal fractions. It is planned for the sub-unit named *Ratio* (within the decimal fractions) and is stated in the textbook that students are expected to learn ratio, perimeter of circles, and probability though studying the *ratio* sub-unit. I mainly focus on the ratio part of the sub-unit.

The sub-unit starts with the introduction of Miniatürk which is a model park involving miniature versions of historical and cultural heritages of Turkey, such as Hagia Sophia, Galata Tower, Aspendos Antique Theatre, Temple of Artemis, etc. Each model in Miniatürk is created with a scale to represent $\frac{1}{25}$ of the real version of the constructions. It is expressed in the teachers' guide that the term $\frac{1}{25}$ *of real version* might be emphasized and a discussion might be started to talk about its meaning. Then, the Task1A given below is presented.

<u>*Task1A.*</u> Students are asked to find the ratios of $\frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{d}{b}, \frac{d}{c}$ of the pattern blocks represented with the letters a, b, c, and d as illustrated in Figure 6.84.

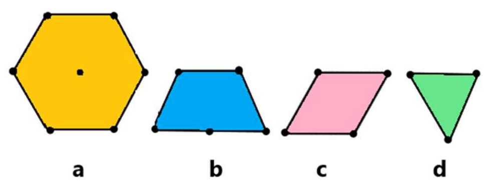


Figure 6.84. Pattern blocks (Aktaş *et al.*, 2007, İlköğretim Matematik 5, p. 112).

The first one $\frac{b}{a}$ is given as $\frac{1}{2}$ in the textbook. The aim of the task is explained in the teachers' guide as to introduce ratio concept with associating regular polygons. With this task students are introduced to *compare the size of two quantities*, which seem to be the areas of the polygon. Then, the lemonade recipe given in the course of study is stated in the textbook. The recipe is shared as involving 4 glass of water, 1 glass of lemon juice, and some sugar. There are two questions: "Can we get more lemonade without spoiling the taste? How?" and "How many glasses of water should we use to make a lemonade with the same taste by using 2 glasses of lemon juice?" (Aktaş *et al.*, 2007, İlköğretim Matematik 5, p. 112). The first question is aimed to canalize students' thinking towards how to preserve the quality, i.e., taste. By keeping the quality of the quantity the same, students are expected to express the amount of water as 8 glasses in response to the second question. Although there is no explanation about answers to the questions either in the textbook or in the teachers' guide, *building up strategy* might be targeted to support. The task requires to double the original amounts to keep the quality the same and students might approach the questions by drawing or matching the corresponding quantities. That is to say, students might conduct additive reasoning to deal with these type of ratio problems.

<u>*Task1B.*</u> The name of the task is *The mysterious ratio of the circle*. Students are expected to measure the perimeter of circular materials- such as, coins, bracelets, glasses- with the use of a rope and ruler. Then, they are supposed to put each circular material between two books to measure the diameter of each which could be found by measuring the distance between the books. The ratio of the perimeter to the diameter is asked to calculate with the use of calculator and students are expected to discuss the results. Then, the ratio is introduced as the number "π" briefly. The main aim of the task, which is stated in the teachers' guide, is to reinforce the targeted ratio understanding of students of this grade level and provide students with an opportunity to realize the number "π". It is also explicitly suggested for the teachers in the teachers' guide not to use the term *proportions* throughout this unit. It seems that the task is designed for students to engage with numerical operations.

<u>*Additional Parts.*</u> The *Exercises* part involves 5 questions. The first one is to verbalize the meaning of ratios to express the real versions of each figure. For instance, there is a car photo, and the ratio is given as 1:200. Students are expected to express that the actual size of the car is 200 times as bigger than the photo. For the first question verbalization of the meaning of ratios is targeted. In the second question, students are provided with a mixture problem involving a recipe of cherry juice to serve to 5 people which includes 500g of cherry, 250 g of sugar, and 1L water. Students are supposed to determine the amounts of cherry and sugar for serving to 10 people and then they are expected to find what is the amount of cherry needed to make the juice with using 1 kg of sugar. To answer the second question, students are expected to consider equivalent ratios but neither any strategy nor any arithmetic is shared or suggested in any curricular material. Also the problem seems to target at *building up strategy* again with increasing the amount of sugar 4 times as much of the original amount, 250 g. The third question presents the number of yellow, red, and blue chalk in a box of chalk. Students are supposed to express the ratio of the number of yellow chalks to the whole box, the number of blue chalks to the whole box, and the number of red chalks to the number of the blue chalks. Through the third question, it seems that the target is that a ratio could represent the part- whole relationship. For the fourth question, students are expected enlarge the size of a rectangular shape photo given with side lengths by the ratio 1:3, so that they could find the enlarged side lengths. At the last question, a table of ingredients of halvah desert is shared as in Table 6.3 and students are asked to fill in the blanks in the table. It is

stated in the teachers' guide that this last question is provided for students to use tables with ratios.

Table 6.3. Ingredients of flour halva (Aktaş *et al.*, 2007, İlköğretim Matematik 5, p. 114).

Ingredients (a glass of)	Number of people		
	4 people	8 people	… people
Oil	1		
Flour	4		12
Sugar	3	6	
Milk	2		6
Water	2		

Again it seems that the *building up strategy* is targeted in this problem. For instance, once the amount of people is doubled for which the amount of sugar is also doubled requires doubling the amount of other ingredients to keep the taste of halva the same. In addition in the teachers' guide, there is a possible project homework for students to draw a plan of their home. The aim of the homework is to provide a practice of ratio and measuring for students. Also, the problem situations provided to students in the students' workbook are similar to the 5 questions presented in Exercise section, which I have already explained above.

In sum, in the fifth grade, the building up strategy is aimed to be highlighted with the tasks and questions. With the examples of the photos compared to actual size of objects, e.g. the size of a car photo to the actual size of the car, the ratio of stretching and shrinking is mentioned. Thus, *the identical groups conception* might be supported in this grade level. In terms of the task variables, the variables have the values of integer multiples of each other. For instance, in the Taks1A the recipe of lemonade involves 4 glass of water and 1 glass of lemon juice. The questions state to find unknown amount for 2 glass of lemon juice and 8 glass of water. Moreover, the contexts of the tasks comprise of geometric figures and recipes, i.e. mixture problems. In addition, it is suggested for teacher not to use the term percentages in fifth grade. It is interesting that non proportional situations are not used in fifth grade.

6.3.2. Grade-6 Turkish Curriculum Materials

6.3.2.1. Content related objectives. In the 2005 curriculum, the *sixth grade* is classified as the first grade of the lower secondary school. In addition to the learning domains of Numbers, Geometry, Data Analysis, and Measurement, the domain of Algebra is also started to be given place. Also, probability is attached to Data Analysis domain. Therefore, I express the functional relationships through examining *Numbers* domain for ratio, rate, and proportions, and *Algebra* domain for linear functions.

Similar to the elementary mathematics, ratio is held under *Numbers* domain in the sixth grade. The content related objective expects students to "*use ratio to compare quantities and express ratio in different ways"* (MoNE, 2009b, p. 153). Just after the ratio is discussed, *proportion* is planned to be introduced in this grade level. The connections between fractions, percentages, ratio, proportion, and rational numbers are targeted to be established. It is claimed in the course of study that students' understanding of proportions will improved with the development of students' understanding of rational numbers (MoNE, 2009b). It is also referred in the course of study that the meaning of proportions embrace not only the equivalence of ratios, but also the relationships between quantities. The content related objective for proportions is stated as "*Students will explain proportion and the relationships between the directly proportional quantities*" (MoNE, 2009b, p. 154). This objective is further explained for teachers to provide their students with examples to examine the relationships between directly proportional quantities through the use of tables, so that students might be able to investigate patterns displayed by the table and make sense of direct proportions. In parallel to this explanation, it is also explicitly stated in *Numbers* domain of the course of study that students are expected to recognize the quantities that are proportional and examine the relationships between proportional quantities with the use of different types of representations, i.e. tables, graphs, and algebraic expressions.

Moreover, the objectives of the *Algebra* domain is classified into three categories: These categories involve Patterns and Relationships, Algebraic Expressions, and Equality and Equations. The first category involves tasks and exercises to examine the relationship between two quantities varying simultaneously; however, the other two categories are designed for promoting algebraic reasoning. Since the scope of this study involves the examination of quantitative and covariational reasoning, I share only the objective of the first category, i.e. patterns and relationships, as follows: "*Model the number patterns and use letters to express the relationships existing in the patterns*" (MoNE, 2009b, p. 206). In the explanation of the objective, it is stated that the term *variable* is introduced. Also, the use of tables and algebraic expressions is supported in the course of study.

Through the objectives of Algebra domain, generalization and algebraic representation are planned to be introduced in the sixth grade. It is explicitly stated in the course of study that the aim of the objectives of the domain is to develop some prior knowledge which might be beneficial for students' conception of functions through expressing generalizations of patterns and algebraic representations of simple equations with one unknown in the sixth grade. It is also stated that these generalizations are to be used in later grades for investigating the relationships between two variables varying simultaneously.

Although the algebra domain is introduced earlier than the contents of ratio and proportions in the 6^{th} grade textbook, I first share the unit designed for ratio and proportions, and then share the unit of algebra in order to be consistent with my presentation of findings. It is worth saying that there is no specifically expressed connection between those units to affect one another. Thus, I assume that presenting ratio and proportions first sustains the coherence of the findings.

6.3.2.2. The units of textbooks and the explanations given in the teachers' guide regarding the concepts of ratio. There are two units of textbook involving tasks to address these objectives: *Reflection from numbers to probability* and *Mathematics and art*. The former provides learning opportunities for the concepts of ratio, while the latter seems to aim triggering students' covariational reasoning.

<u>*Reflection from numbers to probability.*</u> The unit involves some topics of fractions, ratio, proportion, length measuring, and probability. The sub-unit named *Ratio and Proportion* starts with a task identical to the Task1A of the 5^{th} grade. It is stated in the teachers' guide that the aim of the introductory task involving the examination of ratio with the use of pattern blocks is to examine students' prior knowledge. After the introductory task, students are provided with three examples of mathematical situations involving ratios. Firstly, I share these three examples and secondly, I explain the examples in terms of ratio concept in the following paragraphs. Then, I present the task.

The first example states the ratio of 1:4 expressing the length of the head of a newborn baby to the length of the body of a newborn baby. It is explicitly stated in the teachers' guide that students are asked to explain the meaning of 1:4 and form a table to examine the different lengths of the heads and bodies of newborn babies. The second example involves the situation in which a car has constant speed and travels 160km in 2 hours. The ratio of distance travelled to the amount of time passed is expressed 160km/2h (i.e., 80 km/h) and this ratio is directly defined as *unit rate*. The last one is similar to the scaling problems that students were introduced in the 5^{th} grade. In the last example, two line segments $|KL|$ and $|MN|$ shared with a length of 4cm and 5cm respectively. It is provided that the line segments are drawn in 1/1800 reduction and the actual lengths of the segments are asked.

In the first example, students might verbalize that the body length is 4 *times as large* as the head length. The ratio used in here is to compare two extensive quantities. In the second example, there is an intensive quantity, i.e. speed, which is expressed as a result of the comparison of the two extensive quantities, i.e., distance and time. However, neither the speed as an *intensive quantity* nor the invariant relationship between the variables are highlighted. In the last one, students are expected to multiply the given lengths with 1800 to find out the actual lengths. To do so, as explicitly stated in the textbook, students are encouraged to use the knowledge that 1 cm (in the reduced figure) corresponds to 1800cm (in the actual length), to find the actual lengths for the line segments with 4cm and 5cm length. Neither the textbook nor the other curricular materials involve questions or strategies to prompt students' multiplicative reasoning for this question. The *building up strategy* might seem to be activated.

Task1A. Wet cake recipe is shared in the task. The amount of the ingredients are shared in the recipe for 4 servings. Then students are asked to draw a table to express the amount of ingredients with respect to the servings. The table shared in the task is given below.

Table 6.4. Wet Cake Task (Aktaş et al, 2006, İlköğretim Matematik 6, p. 166).

Servings / Ingredients	4 people	8 people	12 people	16 people
Egg	4			
Sugar	2 water glasses			
Cacao	2 tablespoons			
Milk	1 water glass			
Oil	1 water glass			
Flour	3 water glasses			
Baking powder	1 pack			
Vanillin	1 pack			

First, students are asked to write down the ratios of ingredients for the serving of 4 people. Then they are asked to examine whether these ratios are changed for other servings or stay same. Proportions are introduced as an answer to a question of the task. It is explicitly stated that "Let's even up the ratio of the amount of cacao to the number of people for the serving of 4 people with the ratio of the amount of cacao to the number of people for the serving of 16 people. Use this equivalence to find out the amount of cacao." (Aktaş et al, 2006, İlköğretim Matematik 6, p. 166). It is obviously targeted for students to use the equivalence of the ratios, i.e. proportion, to find out the unknown quantity. They are asked to find the unknown values of the quantities given in the table for other servings. It seems that students are supported to use proportions in their calculations, since only way to approach the questions given in the textbook is the use of proportions. It is clearly stated in the teachers' guide that the table supports to examine the patterns between quantities to realize the properties of the proportional quantities. However, there is no further explanation made for properties of proportions, especially for direct proportions, either in the textbook or in the teachers' guide as in the Japanese curricular materials. That is, the direct proportional relationships is not expressed as when the one quantity becomes 2, 3, 4, etc. times bigger, the other quantity also becomes 2, 3, 4, etc. times bigger or when the one quantity becomes $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc. times bigger, the other quantity also becomes $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc. times bigger. Thus, students might not be explicitly fostered to recognize the multiplicative relationships between quantities. Although they might recognize the multiplicative relationships between quantities, questions are not designed to foster multiplicative reasoning. Instead, students might use such as the *building up strategy* based on additive reasoning to fill the table and answer the questions.

Moreover, the numerical values asked in the task are integer multiplies of the values of the quantities of the first serving. The amounts of quantities are also expressed with simple integers such as 4 egg, 3 water glasses of flour, 2 tablespoons of cacao, 1 pact of vanillin. Thus, students might easily deal with the table by drawing, counting and/or matching the quantities, rather than using numerical operations. For instance, they could draw 4 people and match people with eggs, e.g. an egg for each person. Then they could draw 4 more people to find out how many eggs needed. They could match the amount of ingredients with number of people in servings or use drawing to show those matches.

Just after the Task1A is presented, there is an example shared in the textbook which I explain as Task1B below.

Task1B. There is a square divided into 100 smaller squares with the colors such as blue, white, red, and green as illustrated below in Figure 6.85.

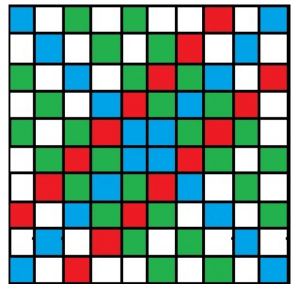


Figure 6.85. A big square involving hundred small squares (Aktaş *et al.*, 2006, İlköğretim Matematik 6, p. 167).

In relation to the task there are three questions shared. First, it is asked to express the number of blue squares to the number of squares of the whole shape which involves a hundred small squares. Second, the ratio of the number of red squares to the whole shape, when the whole shape involves 1000 small squares with the same coloring, is asked. Third, students are expected to fill a table with the use of ratios for expressing the number of small squares of each color for the whole squares involving 10, 100, 1000, and 10000 small squares.

In response to the first question, students are introduced for simplifying the ratios. Students are explicitly asked to find the greatest common divisor (GCD) for the quantities compared, then divide the quantities to get the simplified ratio. For instance, the ratio of the blue squares to the whole shape is expressed as $\frac{20}{100}$. Since the GCD of 20 and 100 is the number 20, the values of the quantities of the ratio is divided by 20 and the simplified version of the ratio, i.e. $\frac{1}{5}$, is found. The arithmetic operations are highlighted for the answer of the first question rather than the quantitative meaning of the simplification, which is preserving the quality of the situation. Just after the answer of the first question is given, students are asked to find the ratio of green squares to the whole shape. That is, the textbook supports an identical question for students to conduct same procedure, which heavily relies on arithmetic operations.

In response to the second question, as provided in the textbook, the ratio of the number of red squares to the whole shape of 100 squares is given as $\frac{16}{100}$. Since the coloring is same for the shapes of 100 squares and 1000squares, the *proportion equation* is explicitly stated as $\frac{16}{100} = \frac{a}{1000}$. In addition to the *cross multiplication strategy*, the relationship of the quantities in the between state ratios such as 16 and a is introduced in increments of 10. Although it is written as “x10” in the Figure 6.86, which might trigger the 10 times as much meaning, still there is no indication of such meaning other than the numerical explanation of “x10”. There is no questions or explanations given to highlight between state ratios, i.e. the total amount of small squares 100 and 1000. It is interesting that, in the Japanese textbooks, the quantities values are placed on a line segment to trigger students’ multiplicative strategies, such as between state ratios. Moreover, as the use of arrows on the top and the bottom of the proportion equation displayed in Figure 6.86, there is a mistake in the use of direction of the below arrow. It should have been written as ÷*10* rather than ×*10.*

$$\underbrace{\overbrace{\frac{16}{100} = \frac{a}{1000}}^{\times 10}}_{\times 10} \quad \text{then } a = 160$$

Figure 6.86. Use of the arrow sign (Aktaş *et al.*, 2006, İlköğretim Matematik 6, p. 167).

Finally, students are expected to use proportions to fill the table involving the number of red, blue, green, and white squares for each shape involving 10, 100, 1000, and 10000 small squares.

<u>6.3.2.3. The units of textbooks and the explanations given in the teachers’ guide regarding covariation.</u> The textbook unit of *Mathematics and art* involves tasks and questions that might trigger students’ covariational reasoning.

<u>*Mathematics and art.*</u> Functional relationships are included in the sub-unit having the title *Everyone should learn algebra*. It starts with the examination of simple situations involving

two unknowns with the use of tables and algebraic expressions (e.g. Task2A). Then, there are examples of number patterns of which the use of tabular and algebraic representations are supported (e.g. Task2B and Task2D). The images representing patterns are also shared to provide students for examining relationships through the use of different representations (e.g. Task2C). I explain each task in the following paragraphs separately.

Task2A. The aim of the task is to provide students to calculate the perimeter of the shape given in Figure 6.87 below.

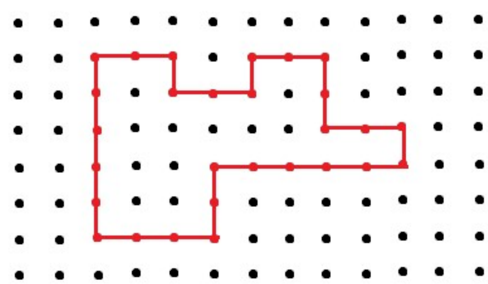


Figure 6.87. Perimeter of the shape (Aktaş *et al.*, 2006, İlköğretim Matematik 6, p. 125).

It is asked in the task to accept the distance between two points, which forms a line segment, as 3cm, 4cm, 5cm, 6cm, … and form a table to represent the relationships between the change in distance between two points and the change in the perimeter of the shape. Then, students are expected to generalize the relationship between variables by using letters. The variables, i.e. the length of the line segments and the perimeter of the shape, are continuous in nature. The task also involves values of variables changing in chunks. Students are expected to *coordinate the values of variables* with the use of tables such that the task seems possibly to activate the *MA1* on the part of students. Also, students are expected to coordinate the direction of change by examining the *values of variables changing in chunks*. In particular, students are expected to coordinate the direction of changes in the perimeter with the direction of changes in the length of line segments as 3cm, 4cm, 5cm, and so on. Therefore, the *MA2* might be triggered through the task too. Thus, the task might seem to trigger the covariational reasoning level called *gross coordination of values* which involves

MA1 and *MA2* levels as expected of students' behaviors. A student who conducts reasoning level of gross coordination of values, does not consider the multiplicative link existing between the changes in variables. Moreover, in the teachers' guide, there is no emphasis on the quantitative relationships between variables, rather the meaning and the use of algebraic expressions are highlighted.

Task2B. This task is shared as an example in the textbook. But I share it as Task2B to emphasize the use of tables while number patters are supposed to be examined. The number pattern 2, 4, 6, 8, … are given. Then, the fifth and sixth numbers of the pattern are asked. It is stated as a response to finding the fifth and the sixth numbers of the pattern that "It is seen that the number in each step is equal to the number which is 2 times as large as the step number, thus the fifth number is found as 2.5=10 and the sixth is 2.6=12" (Aktaş *et al.*, 2006, İlköğretim Matematik 6, p.127). In the second question, students are expected to find an algebraic expression for this pattern. To answer the second question, students are asked to use a table, given in Table 6.5.

Table 6.5. Table for the number pattern (Aktaş *et al.*, 2006, İlköğretim Matematik 6, p. 127).

Step number	The number corresponding the step	The rule of the pattern
1	2	$2 \times 1 = 2$
2	4	$2 \times 2 = 4$
3	6	$2 \times 3 = 6$
…	…	…
n	…	$2 \times n = 2.n$

After the table is presented, students are asked to examine the number pattern 3, 6, 9, ... as in the same way used to examine the pattern of 2, 4, 6, … Similar to the Task1A, the Task2B might seem to target at activating *MA1* and *MA2* on the part of students. Yet, there is no further questions or explanations shared in the unit of textbook to indicate the first two mental actions might be targeted. The task seems to rely on arithmetic operations. Although with the proper questioning the task might be used to activate the *MA1* and the *MA2*, I could not infer that the task might be designed to activate those MAs by looking at the presentation of it in the unit of textbook. Thus, if students could grasp a gross image of covariation existing in the pattern, by indicating the changes in both variables without verbalizing the multiplicative link (i.e. *MA2*), the task might support their covariation reasoning as gross *coordination of values* level, which includes the first and second mental actions.

<u>Task2C.</u> The name of the task is *Aesthetic Shapes* in which students are expected to examine the canonical shapes. The canonical shapes involves equilateral triangles as illustrated in Figure 6.88.

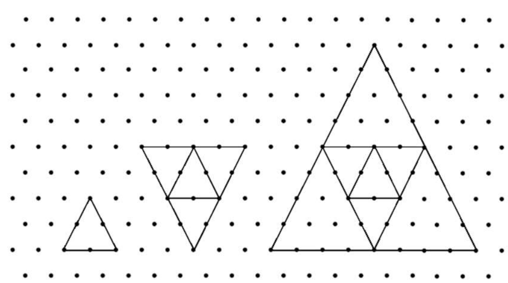


Figure 6.88. Aesthetic shapes task (Aktaş *et al.*, 2006, İlköğretim Matematik 6, p. 128).

Students are asked to find the number of equilateral triangles for each step. For instance, the first step has 1 equilateral triangle, the second has 5 equilateral triangles, and the third has 9 equilateral triangles and so on. Then they are asked to form a table to make their examination of the mathematical relationships between variables, i.e., the step number and number of equilateral triangles. Then the number of triangles in the 10^{th} step is asked in the task. In response to this question, they are also asked to find a rule for the pattern. Students are expected to provide both verbal explanation of the rule and then the mathematical expression of the rule. This task seems to trigger that students might use *MA1* to answer the first question. This is because they are asked to express the correspondence between the step number and the number of equilateral triangles. When they form a table and investigate the relationship between the variables, they are also supposed to express that the number of triangles becomes 4 more than the previous step. When the steps are ordered as 1, 2, 3, …, the number of equivalent triangles are 1, 5, 9, … correspondingly. The *direction of change* in variables (*MA2*) and the *amount of change* of each corresponding variables (*MA3*) are also expected to be verbalized and written with an algebraic expression by students. Students might come up with the formula of the number of equivalent tringles in any step, of which is expressed with the unknown *n*, as $1 + 4 \times (n - 1)$ or $4n - 3$. They are also supported to consider different ways of algebraic representation for the pattern given. Although the variables are discrete in nature, the task might trigger the first three mental actions. The corresponding covariational reasoning level is *coordination of values*.

Task2D. The name of the task is *From Numbers to Shapes* and the aim of it is to represent the pattern existing in the number series with some canonical shapes. In the task, students are expected to model the number pattern given as 3, 6, 9, 12, … with the use of matchsticks. Similar to the Task1C, students are expected to form a table to examine the relationships between variables and come up with an algebraic expression, i.e., $3 \times n$ with the *n* as the step number.

After this task, students are asked to examine and model the number pattern of 4, 7, 10, 13, … as an example. In response to the modeling, the Figure 6.89 given below is shared in the unit of textbook.

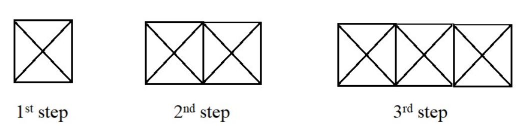


Figure 6.89. Modelling the pattern of 4, 7, 10, and so on (Aktaş et al, 2006, İlköğretim Matematik 6, p. 130).

As it is seen in the Figure 6.89, there are sticks used to form squares as in Figure 6.22 given in the 5th grade Japanese textbook. In the case of the number of steps expressed as *n*, the rule of the pattern is explicitly stated in two sentences: one is $4 + (n - 1) \times 3$ and the other is $n \times 2 + (n + 1)$. Although the tasks in Japanese and Turkish textbooks are almost identical, there are no guided explorations either in the textbook or in teachers' guide for the Turkish task. Although I infer the first three mental actions might be triggered through the tasks in the *Algebra* domain of the 6th grade in Turkish textbook, my claim is not as strong as the ones made for the Japanese textbooks. This is because, the Japanese textbook involves examples of students' ideas and strategies. Specifically for the Task3 in the 5th grade Japanese textbook, there are two different students' ideas and strategies explicitly explained. For instance, in one of those students' ideas, the expression of "the way the number of sticks increasing…" (Fujii and Iitaka, 2012, Grade 5, p. A103) and visual representation to show the increase in each time guides students explicitly to MA3.

In sum, it seems that multiplicative reasoning might not be fostered explicitly with the tasks and questions provided in the 6th grade Turkish textbook, regarding to ratio and proportions. Students are explicitly introduced and expected to use *building up* and *proportion equation strategies*. In terms of the task variables, the context of mixture problem is used in Task1A. However, students are asked to compare the quantities from ingredients and number of servings, rather than comparing an ingredient and the mixture. Next, they used part-whole ratios and introduced *cross multiplication strategy.* In terms of the numerical features of the task variables, the tasks involve whole numbers only. Moreover, in terms of the covariational reasoning, *MA1* (i.e. coordination of values of variables) and *MA2* (i.e. coordination of direction of changes of variables) might be triggered though the tasks and questions of the units in the 6th grade textbook. Thus, students' reasoning might be supported to the level of the *gross coordination of values*. Students are expected to use tabular and algebraic representations in this grade level.

6.3.3. Grade-7 Turkish Curriculum Materials

<u>6.3.3.1. Content related objectives.</u> There are two ratio and proportion related objectives at the *seventh grade*. The first objective targets that students "*explain the relationships between direct and inversely proportional quantities*" (MoNE, 2009b, p. 227). Teachers are expected

to provide their students to examine the relationships between daily life situations, such as number of workers and time to finish a work, the perimeter of a circle and its diameter, etc. It is explicitly stated in the course of study that the relationships between these situations are asked to students and expected them to verbalize the relationships by declaring increment and decrement. In other words, students are expected to coordinate "the direction of change of one variable with the changes in the other variable" (Carlson *et al.*,2002, p. 357). Thus, I infer that the *MA2* might be activated through the textbook parts designed for this objective. In addition, the objective is further explained in the course of study that the proportion constant, which is the quotient of two quantities in direct proportions and the product of two quantities in inverse proportions, is expected to be highlighted and represented through algebraic expressions. The situations not involving proportional relationships are also aimed to be presented for students. Similar to Japanese curricular materials, the relationships between the varying side lengths for a rectangle having fixed area is provided as an example for teachers to use in classroom teaching. I also infer that since the examination of proportional quantities varying simultaneously and algebraic expressions of proportions is targeted, students' understanding of proportions as a simple version of linear functions might be triggered.

The other objective is stated and expected that students "*form and solve problems of direct and inverse proportions*" (MoNE, 2009b, p. 228). In the explanation of the objective, it is expressed that "Students are provided to develop strategies to find out the unknown term of a proportion, such as cross multiplication, equivalence of fractions, patterns, and so on." (MoNE, 2009b, p. 228). Although it is not elucidated in detail, I infer that students might be supported to use multiplicative strategies to deal with proportion problems. For instance, they might use between state or within state ratio strategies when they consider equivalence of fractions or patterns: For a proportion expressed as $\frac{a}{b} = \frac{c}{d}$, if students consider relationship between a and c or b and d, they might recognize the between state ratios. Whereas, if they examines the relationships between a and b or c and d, they might realize the within state ratios.

Moreover, there are a number of objectives prepared for the *Algebra* domain. On the top of students' practice in the 6th grade, students are expected to use letters to express the relationships between number patterns and form a model for them in this grade level. Through the examinations of number patterns, students' functional understanding might be enhanced. In addition, it is explicitly stated in the course of study that students are expected *to "(1) explain linear equations, (2) draw graphs of linear equations.*" (MoNE, 2009b, p. 278). In the following paragraph, I share explanation of each of these objectives.

To explain the linear equations, students are expected to examine linear relationships between two variables through forming a table and writing algebraic expressions in the light of information in the table. Then they are expected to draw the graph of their linear equations and recognize that the equation represents a linear relationship. It is explicitly stated in the course of study that *the explanation of the change in one variable with the changes in the other variable is to be emphasized* (MoNE, 2009b). There are also provided examples of mathematical activities related to the objective shared in the course of study. In one of the examples, it is asked to find "How many trees needed to absorb carbon dioxide emission of 450 automobiles, if a tree can absorb carbon dioxide emission of 26 automobiles?" (MoNE, 2009b, p. 285). In response to the problem, a table is shared, involving the number of automobiles, the number of trees, and the relationship between these two variables, as well as the graph of the relationship displayed by the table. The relationships between variables is further expressed in the table as *the number of automobiles is 26 times as much as the number of trees*. Expressing the relationship for each change in the number of trees with respect to the number of automobiles, the invariant might be highlighted, so that students might recognize the linear relationship and make sense of the algebraic expression of the relationship. In addition, the second objective is clarified in the course of study that students are expected to use different strategies to represent the draw the graph for a given equation. Students are expected to find the variables, form a table for the values of variables, indicate ordered pairs as points on the graph, and draw the graph. However, finding out an algebraic expression for a given values of variables through a table or a graph is not included in this grade level. To sum, these two objectives seems to indicate that the actions of *coordinating the values of variables*, *coordinating the direction of change*, and *amount of changes in variables* might be supported. That is to say, I infer that students' understanding of covariation might be enhanced to the level of *coordination of values*, if these objectives effectively implemented in textbooks and classroom teachings.

<u>6.3.3.2. The units of textbooks and the explanations given in the teachers' guide regarding the concepts of ratio.</u> There two textbook units involving tasks addressing these objectives: *We started with proportion* and *Algebra and probability.* Since covariation is a necessary construct for the concept of ratio, there are some tasks and questions seem to trigger some mental actions of covariational reasoning in the former unit; however, the main focus of the unit is on improving students' understanding of concepts of ratio.

<u>*We started with proportion.*</u> The unit involves three sub-units: *Ratio and proportion*, *Polygons*, and *Graphs*. The ratio and proportion sub-unit starts with an introductory problem to examine the relationship between speed of a car and fuel consumption as well as the speed of a car and the driver vantage point. The former involves direct proportional relationship, whereas the latter represent inverse proportional relationship. It is emphasized in the teacher's guide that the purpose of the introductory problem is to provide students with an opportunity to recognize the relationship between quantities, one of which increases as the other decreases, and one increases as the other increases (Aygün *et al.*, 2007). That is, the direction of change in one quantity is supported to coordinate with the direction of change in the other quantity, which corresponds the *MA2*. Thus, the introductory problem seems to be designed for students to trigger the *MA2*.

After the class discussion of the introductory problem is finished, students are expected to examine proportions through a task involving a making and using a scale of a balance which is presented below as Task1A. Also, Task1B and Task1C involves examples shared in the textbook that I examine to explain possible supported student strategies.

<u>*Task1A.*</u> The task named *Ratio in the scale of a balance* is presented in two different parts; one of which is concentrated on direct proportions while the other is focused on inverse proportions. In the first part of the task students are expected to build a scale of a balance

with the use of plastic plates, rope, and a stick. Then they are expected to place marbles on one of the plates and sugar cubes on the other plate to form a balance. After they set a balance, they are asked to examine the number of marbles needed to preserve the balance when the number of sugar cubes become 2 and 3 times as much. It is asked in the task to form a table to represent the relationship between marbles and sugar cubes. Also it is asked that "what changes in the number of marbles is it needed to preserve the balance when there is a decrease or increase in the number of sugar cubes? Explain." (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 96). This question seems to provide an opportunity for students with engaging in *MA2* such that they might explain the direction of change in one quantity with respect to the change in the other. Then students are expected to write the ratio of number of sugar cubes and the number of marbles in the balance situation and use the ratio to find the number of marbles needed to preserve the balance if the number of sugar cubes is increased 20 times as much. This last question seems to intend that students express the common quotient of directly proportional quantities; so that the relationship between quantities could be expressed with an algebraic expression of $y = ax$. However, neither the equation of direct proportion, which is $y = ax$, nor any question to express the proportional relationship with an equation is supported in the unit of textbook, although there is an objective requiring to write algebraic expressions of direct proportional and inversely proportional relationships.

The second part is designed for students to work with inversely proportional quantities. Students are expected to build a scale of a balance and put some amount of the sugar cubes in each plate to get a balance state. Then, they are expected to add sugar cubes to one of the plates one by one, while keeping the amount of sugar cubes on the other plate the same. To preserve the balance, they are asked to move the rope on the stick and in a table to record the number of sugar cubes and the distance of the rope to the balance point of the beginning. It is asked to examine the relationship between the number of sugar cubes and the distance to the first balance point to emphasize the inversely proportional relationships. In the last question, they are expected to answer "When you bring the rope back to its initial balance point, what change should you make to the numbers of sugar cubes in the plates to preserve the balance? Discuss." (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 97). Since students are expected to track and then verbalize that the balance could be kept by shortening the distance in comparison to the first balance point with the increase in the number of sugar cubes, I infer that the task might trigger the *MA2*. That is, the task might facilitate students to engage in verbalizing the direction of changes in one variable regarding the changes in the other variable, which corresponds to the *MA2* and practicing arithmetic operations. Though, there is no further question to engage students writing algebraic equations of the proportional relationships existing in the given situation. It is interesting that there is a statement in the teachers' guide as

> The proportion should not be perceived as using the equality of cross products to find the term that is not given. The concept of proportion entails recognizing proportional quantities and being able to examine the relationships between quantities using numbers, tables, graphs, and equations. (Aygün et al, 2007, İlköğretim Matematik 7 Öğretmen Kılavuz Kitabı, p. 97)

Thus, although explicitly emphasized in the teacher's guide, unlike in the Japanese curricular materials, it seems that the task and the related textbook part is not powerful enough to underpin the functional relationships existing in proportional situations. Hence, I infer that the excerpt given above is not strongly supported through textbook contents.

Moreover, there are examples presented in the textbook to exemplify how to examine proportional situations. I explain some of them through Task1B below to exemplify how the student strategies might be triggered and how the representations are presented in the textbook.

<u>*Task1B.*</u> After the task is introduced, there are four examples shared in the textbook. I share my findings for the first two of them to examine possibly triggered student strategies.

For the first example, students are expected to determine whether the two given situations represent direct or inverse proportional relationships. First situation represents direct proportional relationships which exists between the fuel consumption of a car and distance travelled. The second situation indicates inverse proportional relationships which is between the number of taps and the time to fill out the pool.

For the second example, an example of shape of a wire of 70 cm long is given, which is divided into two parts in proportion to 2 and 5 respectively. Students are expected to find the length of each part. There is a solution shared in the textbook for this problem. I share the solution and explain the underlying student strategies as follows. In response to the problem, the equation $\frac{x}{2} = \frac{y}{5} = k$ is formed and $x = 2k$ and $y = 5k$ is gotten. Since it is known that $x + y = 70\ cm$, the values of *x* and *y* is substituted in this equation as $2k$ and $5k$ respectively. Thus, the value of *k* is found as 10 and the length of the parts are 20 cm and 50 cm. The strategy provided in the textbook is prepared to reinforce arithmetic and algebraic skills of students. The common quotient *k* is provided but there is no detailed explanation of the proportion equation. In addition, unlike Japanese curriculum materials, in Turkish curricular materials, students neither are asked to find nor are introduced different strategies to approach the problem: In particular, as shown earlier, there is a similar example in Japanese curriculum materials. As a response to the example (i.e., the strategy presented in Turkish textbook), the strategy shared in Japanese curriculum materials is explained in detail. Similarly, proportion equation strategy is further conducted with the use of part-whole ratios, i.e. $\frac{2}{7} = \frac{x}{70}$ as well as $\frac{5}{7} = \frac{y}{70}$ in Japanese textbook.

For the third example, the same strategy presented in the second example is conducted for a situation involving inversely proportional relationships. Also, cross multiplication is conducted in the last example to find the missing information. Since the structure of the problems and provided answers are similar to the first two examples, I do not explain the third and the fourth examples.

Additional Parts. In Turkish curriculum materials, the graphical representation of proportional relationships is introduced differently from Japanese curricula materials. There are three graphs shared in the textbook that are displayed in Figure 6.90 below. Since the graphs are shared at the *Application Problems* section of the textbook, which is at the end of the sub-unit, I present my findings as an Additional Part.

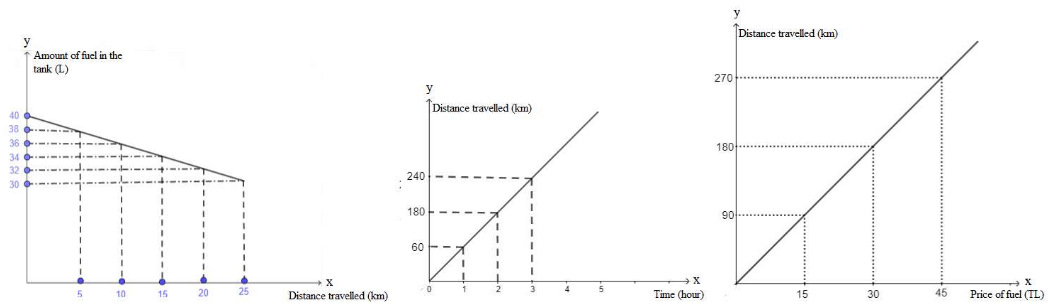


Figure 6.90. Graphical representations of proportions (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 99).

Students are expected to determine the type of proportions between the variables as displayed on the graphs. Though, there is an improper use of graphical representation of inverse proportions. It is stated in the teachers' guide that the first graph represents an inversely proportional relationship. However, the graph represents a proportional relationship such that while the amount of fuel decreases in increments of 2 the distance increases in increments of 5. That is, the graph could not be classified as direct or inverse proportional graph, although the situation itself involves a proportional relationship. Since the meanings of direct proportions and inverse proportions are not shared as detailly as they are in Japanese curricula materials, this mistake might lead a misconception for both teachers and students: The meaning of inverse proportions should be explained such that whenever one quantity increases m times as much, the other becomes $\frac{1}{m}$ times as much resulting in their product staying the same. For instance, given the same area of a rectangular shape, the size of the length and the width might change in such a way that the area keeps the same. The shared definition of inverse proportion in textbook is "If one of the two quantities increases when the other decreases at the same rate, or if one decreases when the other increases at the same rate, these quantities are inversely proportional." (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 98). Although the definition and explanation in the Turkish curriculum materials mathematically true, I think it is abstract and implicit for students, because students might not recognize the meaning of "increase/ decrease at the same rate (or ratio)". It is possible that they are not be able to recognize the multiplicative relationship implied by the expression of *at the same rate*. Similarly, the definition of direct proportion is not detailed as in the Japanese materials. Direct proportion is defined in the Turkish textbook that "If one of the two quantities increases when the other also increases at the same rate, or if one decreases when the other decreases at the same rate, these quantities are direct proportional." (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 97). Compared to these definitions of proportional relationships, I think the definitions and examples shared in Japanese curriculum materials are more comprehensive regarding covariational relationships between variables.

Thus, so far, in the 7th grade textbook, the teacher's guide and the course of study, I infer that the Turkish materials might trigger the *MA2* by providing opportunities for students to express the direction of changes in variables. However, the Japanese materials also support students to examine the amount of changes in one variable with the changes in the other variable; thus, the *MA3* might seem to be triggered in their presented materials of proportions.

Task1C. This task is actually presented as a proportion problem in the textbook. At the beginning of the problem, to gain students' interest, there is some information about climbing and a mountain climber Ali Nasuh Mahruki, who is the lead of the Rescue Team for natural disastrous situations. The problem is stated as follows:

> Climbers take their vehicles and food that they will use during the climb. The ratio of the mass of vehicles to the mass of food is usually 10:3. How many kilograms of food is in a climber's bag if s/he has a 20-kilogram vehicle in his/her bag? (Aygün et al, 2007, İlköğretim Matematik 7, p.102)

There is a strategy provided in the textbook to approach the problem. It starts with naming the mass of food in the bag as *x*. Then the cross multiplication is presented as shown in Figure 6.91.

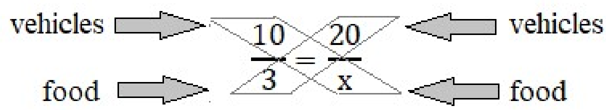


Figure 6.91. Cross multiplication display (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 102).

The proportion equation $10.x = 3.20$ is written and the equation is solved. In addition to proportion equation strategy, between state ratios are supported in the textbook to check the result. It is stated that 20 equals 2 times 10, so x has to be 2 times 3. Although students are expected to examine the relationship between the quantities in between state ratios, the sophisticated meaning of between state ratio involving the explanation of the amount of change is not supported. So, neither the meaning of 2 nor the preservation of the initial situation is implied. Hence, I infer that students might engage in only arithmetic operations rather than engaging in the mental operations of such as the change in the size of one quantity with respect to the change in the other quantity and the resultant arithmetic operations. In sum, *proportion equation strategy* seems to be the mainly targeted strategy introduced to the students throughout the sub-unit.

In sum, students are expected to use proportion equation strategy to find the unknown term for a given direct proportional situation. Although between state ratios are also mentioned in the textbook, the emphasized student strategy is the proportion equation strategy. In terms of the numerical features of the task variables, the problems generally involve whole numbers; however, since the unit involves the Task1A in which students get their values of quantities though measuring and experimenting, rational numbers and real number might also be used. Moreover, *MA2* (i.e. coordination of direction of changes in variables) and *MA1* (i.e. coordination of changes in variables) might be triggered through the unit designed for proportional relationships. There are some differences in terms of presentation of ratio and proportions in 7^{th} grade Turkish curriculum materials compared to Japanese curriculum materials: Firstly, the definitions of proportional relationships are not explicit. Secondly, the proportion equation strategy is heavily emphasized in the unit of ratio and proportions and other multiplicative strategies do not given elaborately on the textbook.

Lastly, although different types of representations are given a great emphasis in the COS, the textbook units do not provide proper and various examples to support different types of representations of proportional situations as it is in Japanese materials. I share the tasks and questions related to Algebra domain as follows.

6.3.3.3. The units of textbooks and the explanations given in the teachers' guide regarding covariation. In the algebra domain, students are introduced to linear equations and their graphical representation.

Algebra and probability. Functional relationships are covered with the sub-unit of *Linear Equations and Their Graphs.* This sub-unit is introduced under three categories: Linear Relationships, Cartesian Coordinate System, and Graphs of Linear Equations, respectively. There is a preparation part presented in the teachers' guide for teachers to use before they introduce Linear Relationships to examine students' prior knowledge. The preparation part involves a shape pattern and a number pattern. Students are expected to find the rules of the patterns and express the rules with algebraic expressions. After this preparatory part is conducted, Task2A, namely *From Table to Equation*, is presented.

Task2A. The task *From Table to Equation* involves a proportional relationship. It starts with sharing the fact that "In order to obtain 1 kg of raisins, it is necessary to dry an average of 5 kg of fresh grapes." (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 150). Then the relationship between the variables, i.e. the amount of raisin and the amount of fresh grape, is asked for students to explain. Students are expected to draw a table and then use the values on the table to express the relationship by an equation. Then they are asked to draw a broken-line graph by using the values on the table. It is asked "What geometric shape was formed in your graph?" (Aygün et al, 2007, İlköğretim Matematik 7, p.150). Then students are supposed to use the graph, table, and equation they formed to answer questions like "How many kilograms of fresh grapes are needed to obtain 12 kilograms of raisins?", "How many kilograms of raisins are obtained from 35 kilograms of fresh grapes?" etc. (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 150).

The variables in this task are not chosen as two unlike covarying quantities. Rather, one variable transforms to the other, i.e. the given amount of fresh grape transforms to some amount of raisin. Each variable is continuous in nature. The linear relationship existing in the task also represents a proportional relationship. Thus, the algebraic expression of the relationship is in the form of proportion equation, i.e. $y = 5x$ in which x represents the amount of raisin and y represents the amount of fresh grape. Since students are also expected to examine the changes in x with the changes in y through a table of which an example is given below (Table5.5), I infer that they might coordinate the values of variables and the direction of change. Hence, the *MA1*, i.e., "Coordinating the value of one variable with the changes in the other" (Carlson *et al.*, 2002, p. 357), and *MA2*, i.e., "Coordinating the direction of change of one variable with changes in the other variable" (Carlson *et al.*, 2002, p. 357), might seem to be activated through the task.

Table 6.6. The relationship between raisin and fresh grape (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 150).

Raisin (kg)	Amount of fresh grape to obtain raisin (kg)	The relationship
1	5	
2		
…		
a		

The variable assigned as the amount of raisin is provided in the textbook as integers such as 1, 2, 3, … a. Since students are expected to indicate the relationship for each value of variable, they might realize the rule of the pattern of linear relationship between variables easily. The constant value is overtly seen through the table. Moreover, it is expected for students to represent the linear relationship through a graph. Students might be aware of the amount of changes of raisins with the changes in fresh grapes through graphing. Also, they might realize the amount of changes of variables when they are expected to find the amount of fresh grape for a given amount of raisin with the use of graph in the last question.

Therefore, the *MA3,* which is described as "coordinating the amount of change of one variable with the changes in the other variable" (Carlson *et al.*, 2002, p. 357), might be triggered through this task. *The covariational reasoning level* involving the *first three mental actions* is called *coordination of values*. Thus, the task might seem to trigger students' understanding of covariation as coordination of values.

Task2B. This task is given in the textbook just after the Task2A is presented. The name of the task is *The Change in The Length*. It is given that "Nesrin and Serdar record the change in the length of a spring which is initially 7cm long by attaching objects of different masses to the end of the spring. For every 1 kg of mass attached, the spring extends by 3cm." (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 150). Similar questions, asked in the Task2A, are adopted in this task too, such that students are expected to examine the relationship between the length of the spring and the masses attached by using tabular, graphical, and algebraic representations. Since students are expected to go through the same processes as in the previous task, the same mental actions might be activated through the Task2B. Thus, I infer that the *coordination of values* level of covariational reasoning might be targeted for students. The difference between the Task2A and Task2B is that the relationship existing in the Task2B is not proportional, but linear. That is, if the mass of an object attached is represented by x and the length of the spring is represented by y, the relationship between variables is expressed as $y = 7 + 3m$. Hereby the graph does not pass though the origin, referring to proportionality, rather it touches the y-axis at the point (0,7) which is named as y-intercept, so referring to linearity. This suggests that, although not explicitly stated neither in the course of study, nor in the teacher's guide, the students are expected to examine both the proportional and linear relationships.

In sum, the unit starts with an activity, students are familiar with, to examine the relationship between variables. Then, the simple version of linear equation expressed as $y = ax$, and the graph of a line which students are familiar with is developed to the linear equation in the form of $y = ax + b$. The numerical values of variables are chosen as integers in these two tasks. However, after the tasks are introduced, there are examples which involves rational values of variables. As an example of variables chosen as rational numbers and the comparison of two linear relationships, I share the Task2C below.

Task2C. After the tasks 2A and 2B are introduced, there are practice problems for students to examine the concept, linear functions. This task actually is given as a practice problem. It involves the prices of phone calls with respect to the amount of time of the calls for two different tariffs of a telecommunication company. The prices in regard to the duration of calls are presented through tables. The values of the price variable for the first tariff are 0.9, 1.8, 2.7, … regarding the duration of the calls given as 1 min, 2 min, 3 min, … respectively. The second tariff has the price values of 9, 9.3, 9.6, 9.9, … for the duration of the calls given as 0 min, 1 min, 2 min, 3 min, … respectively. Students are expected to examine the relationships between variables for each tariff and determine whether they are linear or not. They are also expected to express algebraic and graphical representation of each tariff. Then, it is asked that "Which tariff should someone who can speak for 30 minutes select? Why?" (Aygün *et al.*, 2007, İlköğretim Matematik 7, p. 153). Students are supposed to compare the two situations to answer this question. They might compare the situations with the use of the graphs, i.e. the location of the line when x equals 30. The meaning of the point of intersection and the slope of the lines could be examined through graphical representations, too. Students might compare the situation through algebraic expressions which might involve substitution of the value of x as 30 in each equation. They might find and explain the point of intersection through algebraic representation, too.

In addition, the task might be used to emphasize the invariant relationship of linear equations where the invariant is the rate of change of the price with respect to duration of a call. If the task is also used to highlight the invariance, the *MA4* might be triggered. Nevertheless neither questions to direct students' attention to the invariance nor explanations given in teachers' guide to refer to the invariance are provided. Thus, I conclude that the sub-unit might be designed to activate only the first three mental actions, in turn *coordination of values*. Also, examining the other sub-units reveals that the tasks and questions are similar to the Tasks 2A, 2B, and 2C presented above. Therefore, I do not share the tasks given in the other sub-units.

In conclusion, the unit designed for linear equations in 7th grade Turkish curriculum materials involves the tasks and questions that might trigger students' covariational reasoning level to the *coordination of values.* The tasks support students' use of algebraic, tabular, and graphical representations to express mathematical situations originated in real life. In terms of the numerical values of the task variables, the tasks of the unit comprise not only whole numbers (e.g. Task2A) but also real numbers (e.g. Task2C). Although the variables, especially for the first two task both of the variables expressed in the situation, are continuous in their nature. However, neither the continuous nature of variables nor the continuous covariation existing between the variables are highlighted though the questions and explanations given in the unit of 7th grade textbook.

In the following section, I share the findings from 8th grade curriculum materials. In the 8th grade, the curriculum materials that I examined are designed for *Algebra* domain.

6.3.4. Grade-8 Turkish Curriculum Materials

6.3.4.1. Content related objectives. The objectives related to functional relationships in the *eighth grade*, which is the last grade in lower secondary school in Turkey, are based on *Algebra* domain. Algebra domain in the 8th grade involves patterns and relationships, algebraic expressions, equations, and inequalities as sub-domains. In regard to patterns and relationships sub-domain, students are expected to *"explain the relationships between numbers occupied in a special number patterns"* (MoNE, 2009b, p. 345), such as Fibonacci sequence, examples of arithmetic and geometric sequences, triangular and rectangular numbers. Through this objective, students might be triggered to engage in activities to coordinate the values and changes of two different variables. In addition, it is explicitly stated in the course of study that students are supposed to recognize the common difference for arithmetic sequences and common ratio for geometric sequences (MoNE, 2009). Through expressing common difference and common ratio, they would probably engage in coordinating the amount of change in one variable with the amount of changes in the other. Therefore, I infer that the goals might be used to trigger *the first three mental actions, i.e. MA1 (coordination of values of variables), MA2 (coordination of direction of changes in* *variables), and MA3 (coordination of amount of changes in variables),* of covariational reasoning framework which are necessary for students' understanding of covariation as *coordination of values*.

In addition to patterns and relationships, there is a sub-domain called Equations in which the geometric, algebraic, and tabular representations of linear equations are supposed to be taught. As an objective, students are expected to *"explain the slope of a line by using models"* (MoNE, 2009b, p. 352). They are also expected to "*determine the relationship between the slope and equation of a line"* (MoNE, 2009b, p. 353). As an example of mathematical activities given in relation to this objective, students are expected to form a table for values of variables and draw a graph for $y = 2x$ as illustrated in Figure 6.92.

x	2x	y
1	2.1	2
2	2.2	4
3	2.3	6
4	2.4	8

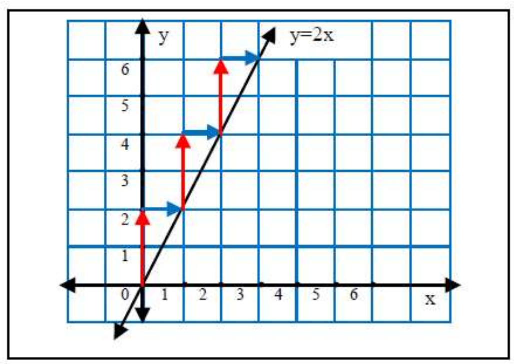


Figure 6.92. Changes and amount of changes highlighted in course of study (MoNE, 2009b, p. 353).

As it is seen in Figure 6.92, the coordination of changes in *x* as 1, 2, 3, 4 with the changes in *y* as 2, 4, 6, 8 is displayed through the table. The coordination of values of variables and direction of changes in variables indicate the *MA1* and *MA2.*Moreover, the arrows on the table show the amount of change for each variable. Thus, the table displays that 1 unit change in the variable x results in 2 unit-change in the other variable, y. In other words, the coordination of the amount of change, an expected behavior of *MA3,* is initiated through the example. The graph also displays the changes and the amount of changes as well as the ratio of the amount of changes, which is defined as slope. Therefore, it seems that *the* *first three mental actions* of covariational reasoning framework might be supported with the uses of tables and graphs throughout the unit.

Moreover, as it is seen form the example, it is also explicitly planned to encourage students to use different types of representations, especially algebraic and graphical representations, to solve systems of linear equations. Although functions are not mentioned in lower secondary level mathematics in Turkey, it is planned for students to use tables and graphs to examine the concept of slope which might guide students to coordinate the values/ direction of changes/ amounts of changes in the variable, x, regarding the values/ direction of changes/ amounts of changes in the variable, y. That is to say covariational understanding of functions up to the level named as *coordination of values*, might be initiated through these objectives.

6.3.4.2. The units of textbooks and the explanations given in the teachers' guide regarding covariation. In this grade level there two textbooks units involving the tasks addressing these objectives: *Triangles and algebra* and *Starting point and end point: Geometry.*

Triangles and algebra. Functional relationships are presented in the sub-unit of *Number Patterns and Identities*. As a preparation to the sub-unit, it is explicitly stated in the teachers' guide that students should be engaged in number patterns and shape patterns to find a rule. The textbook sub-unit starts with referring Fibonacci number. The reflection of the Fibonacci number on nature, architecture, and paintings are also presented with specific examples in a paragraph. Then the task "Square Numbers" is presented as in Task1A below.

Task1A. It is explicitly stated in the teachers' guide that this task is created to provide students with opportunities for examining square numbers through modelling. To begin with, students are expected to line up the squares, having side lengths of 1 unit, 2 units, 3 units and so on, on a grid paper. Then, they are asked to calculate the area of each square and write the values below the squares. Right after they calculate the areas, they are asked to explain and algebraically express the rule of the number pattern formed by the areas of squares. The last question of the task requires students to write down each term of the number pattern as the addition of odd numbers, such as 2^2 as 1+3, 3^2 as 1+3+5 and so on. Students are then asked to explain the pattern that is formed through adding the odd numbers. Although the task involves two covarying quantities (i.e. the side lengths and the areas of the squares), there is no question specifically guiding students to coordinate the change or amount of change in one variable with respect to the changes or amount of changes in the other variable. The task seems to be designed to embrace algebraic reasoning rather than (co)variational reasoning such that students' expressing the relationship between the numbers as symbols is emphasized. Also the last question might be used to foster students to deepen their investigation about the pattern ; however, there is no guidelines or further explanation shared neither in the textbook nor in teachers' guide.

Interestingly in the teachers' guide, there is a task named *Function Machine* in which students are expected to find the definition for a function. Nevertheless, there is no tasks or even questions about function machines in the textbook. I present what is given in the teachers' guide about Function Machine task shown in the Task1B as follows.

<u>Task1B.</u> It is indicated in the teachers' guide that the task could be conducted in a computer environment. It is explained that there are numbers on the left side of the function machine which are inputs. When an input enters the machine, the machine gives an output number which is written on the right side of the machine. Students are expected to write down the input and output values on a table and then state the definition of the function in their own words. Since there are no specific questions nor any design of a task presented, I could only infer that this task might have had an impact on students' functional understanding if it had been stated explicitly. However, I could not examine which meaning of functions (i.e., correspondence or covariation), have been intended to be improved throughout the task.

Moreover, there are exercises to examine arithmetic and geometric sequences in the following part of the unit. Similarly, the presentation of arithmetic and geometric sequences seem to be created to support students' algebraic expression such that students' construction of the algebraic expressions given the table of values is emphasized. Thus, I conclude as a result of my analysis that the tasks and questions of the unit might not be designed in a way to trigger students' covariational reasoning.

In the following paragraph, I introduce the unit of *Examination of a Line* which was designed for the Equations sub-domain.

Starting point and end point: Geometry. The sub-unit *Examination of a Line* includes the functional relationships. The sub-unit starts with a daily life example of the concept of slope: The traffic sign boards created to warn drivers about the inclined roads as exemplified in Figure 6.93. There is also a question to initiate the class discussion on daily life examples of models of slope.

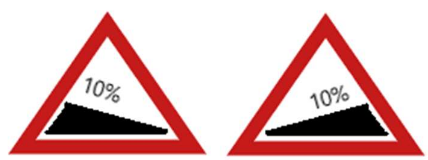


Figure 6.93. Traffic sign boards to highlight slope of a road (Aygün *et al.*, 2009, İlköğretim Matematik 8, p. 189).

Task2A. The task of *Let's Explore Slope* had the scenario of a car climbing 4 different inclined roads as shown in Figure 6.94 below; and, students are expected to compare to determine which road is harder to climb.

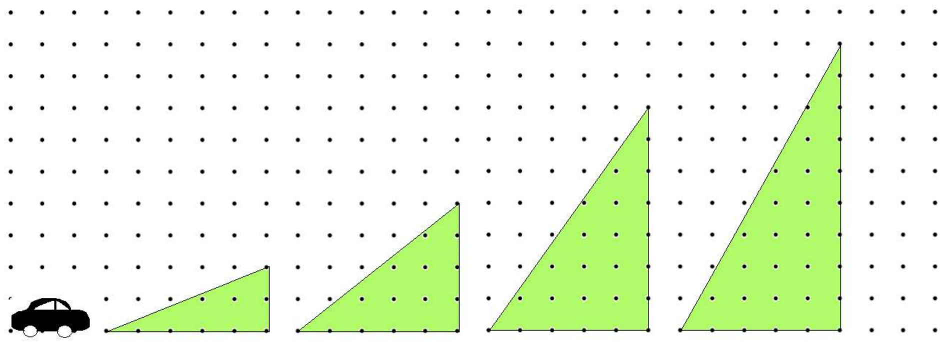


Figure 6.94. Let's explore the slope task (Aygün *et al.*, 2009, İlköğretim Matematik 8, p. 189).

As seen in the Figure 6.94 above, the horizontal length are the same for all triangles. The only variable in this task is the vertical lengths. Students are then asked to fill out the table illustrated in Table 6.7 as follows.

Table 6.7. Examination of the steepness of roads (Aygün *et al.*, 2009, İlköğretim Matematik 8, p. 189).

	1st triangle	2nd triangle	3rd triangle	4th triangle	5th triangle
Vertical Length	2				
Horizontal Length	5				
Vertical ÷ Horizontal	2÷5				

The next question in the task is "Compare the ratio of Vertical ÷ Horizontal in the table. What is therelationship between which slope is harder to climb and the values of these ratios? Explain." (Aygün *et al.*, 2009, İlköğretim Matematik 8, p. 189). Students are expected to compare the values of ratios to make sense of the steepness of the roads. After the slope is introduced through the Task2A, slope is shown in the graphical representation of a line as shown in Figure 6.95 below.

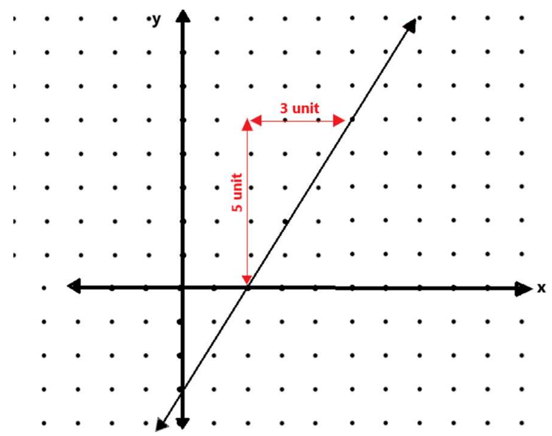


Figure 6.95. Representation of slope on a line (Aygün *et al.*, 2009, İlköğretim Matematik 8, p. 191).

Nevertheless, the invariance relationship represented by the slope is not highlighted explicitly through the task. Also, the tasks and questions of the unit do not guide students to coordinate changes or amount of changes of one variable regarding the changes or amount of changes of the other variable. However, the objective of "*determine the relationships between the slope and equation of a line*" (Aygün *et al.*, 2009, İlköğretim Matematik 8, p. 353) is exemplified in the course of study in a way to support teachers and students to examine the changes or amount of change. Differently from what is given in the course of study, the textbook unit does not involve questions to guide students to use tables and graphs as it is displayed in the course of study. Although the concept of slope itself means that the ratio of amount of change in y with respect to the amount of change in x, students need to coordinate the amount of changes. However, they are not supported to think about covariation such that verbalizing the amount of changes with regard to each other which corresponds the *MA3*. Thus, examining slope might trigger the *MA3*, but the textbook unit seems to be not designed to active students' covariational reasoning.

In sum, the 8th grade Turkish COS seems to support the first three mental actions and thus triggering the *coordination of values* level of covariation, but the units of textbook might not be sufficient to trigger students' covariational reasoning level up to coordination of values. Throughout the units of textbook of 8^{th} grade, different kinds of representations, i.e. algebraic, tabular, and graphical representations, are supported.

7. DISCUSSION AND CONCLUSION

This study investigated the similarities and differences between Japanese and Turkish curriculum materials in terms of quantitative reasoning and covariational reasoning depicted in the concepts of functional relationships including ratio, rate, proportion, and linear function. The curriculum materials involves course of study, teachers' guides, and textbooks for both Japanese and Turkish curricula. The curriculum materials of Japan are created in regard to 2008 standards. In this study, I used English translated version of course of study, teachers' guide, and Tokyo Shoseki's textbook series. The Turkish curriculum materials are designed for the 2005 reform movement in elementary and lower secondary mathematic education. Since relatively small editions are made in 2009 to the 2005 reform curriculum, I analyzed the course of study published in 2009, in this study. The teachers' guides and Turkish textbooks, which I analyzed in this study, were published between the years 2006-2009 by the Turkish Ministry of National Education and delivered as free to every student.

In order to address micro and macro analysis as suggested in the textbook research literature, the first research question is conducted for displaying the overall structures of textbooks regarding topic placements, grade placements for functional relationships, number of pages allocated for functional relationships, and total number of pages of the textbooks. The second question is conducted to depict the learning opportunities of functional relationships in regard to quantitative and covariational reasoning presented in curricula materials. While examining quantitative and covariational reasoning regarding functional relationships concepts (i.e. ratio, rate, proportion, and linear function) for a micro level analysis, the focus of analysis was on the tasks and problem situations, the questions asked of students and the different representations used in the curriculum materials.

7.1. The Overall Structures of Japanese And Turkish Textbooks

While examining the overall structures of the Japanese and Turkish textbooks, the focus on analysis was on the topic placement within the 4 to 8 grade levels, grade placement of the topics, number of units involving functional relationships and total number of units, number of pages allocated to functional relationship topics, and the total number of pages of textbooks. The findings pertaining the first research question illustrated that functional relationships started to be introduced in 4th grade in Japanese textbooks but in 5th grade in Turkish textbooks. In the Turkish textbooks, the names of the units seems abstract and broad in a way that involving the goals of different learning domains (e.g. the 8th grade unit of Turkish textbook *Triangles and Algebra* involves the learning areas of geometry, algebra, and numbers). As a result, the number of units in Turkish textbooks are less than Japanese textbooks' number of units in general. In terms of the page numbers allocated to functional relationships in textbooks, Japanese textbooks have more pages to introduce and develop functional relationships than Turkish textbooks regardless of grade level. Moreover, there is no mention on functions in Turkish textbooks while Japanese textbooks have a unit for linear functions in 8th grade.

7.2. The Learning Opportunities in Terms of Functional Relationships Depicted in Japanese And Turkish Curriculum Materials

In order to examine the learning opportunities depicted in these curricula, quantitative and covariational reasoning frameworks were used. The findings of the study portray that Japanese elementary and lower secondary curriculum has a spiral nature in terms of covariational reasoning, ratio conception, and task variables. In particular, Japanese elementary and lower secondary curriculum materials have a clear focus on supporting learners' quantitative reasoning by emphasizing quantities and their relationships in situations; likewise, covariational reasoning up to *continuous covariation* through the units of functional relationships gradually. On the other hand, results depicted that for the Turkish elementary and lower secondary textbooks and teachers' guides, the focus on quantitative and covariational reasoning has differences compared to Japanese materials. In particular, the objectives of the course of study (COS) assigned in each grade level are almost identical, the Turkish curriculum materials seem to support students to reach *coordination of values* level of covariational reasoning. Moreover, Japanese curriculum materials started with supporting students' additive strategies for tasks of ratio and proportions and gradually introduced multiplicative strategies. Nevertheless, Turkish curriculum does not cover all multiplicative strategies and it does not seem to have spiral nature in terms of ratio conception. Interestingly for both curricula, the complexity of task variables in terms of numerical, contextual, and semantic features are increasing gradually. Therefore, to highlight and explain the spiral nature of Japanese curriculum and compare it with Turkish curriculum, I discuss about the findings of similarities and differences in Japanese and Turkish curriculum materials in terms of covariational reasoning, quantitative reasoning and task variables, and the mathematical contents, i.e. ratio, rate, proportions, and linear functions in the following parts. Lastly, I deliberate the findings regarding the implications to teaching, further research, and teacher knowledge, and the limitations of the study.

7.2.1. Curriculum presentation of functional relationships, especially proportions and linear functions in terms of covariational reasoning

Findings, pertaining to the second research question which requires an examination of curriculum to interpret learning opportunities in them with respect to quantitative and covariational reasoning, shows that starting from the 4th grade different stages of <u>*covariational reasoning*</u> are triggered incrementally in Japanese materials. In contrast, although incremental again, gradual progression of covariational reasoning start at the 6th grade in Turkish curriculum materials.

In particular, in Japanese curriculum materials, *MA1*[1] and *MA2*[2] are supported with the tasks and questions in the 4th grade; *MA1*, *MA2*, and *MA3*[3] in the 5th grade; *MA1*, *MA2*,

[1] "Coordinating the value of one variable with the changes in the other" (Carlson *et al.*, 2002, p. 357).
[2] "Coordinating the direction of change of one variable with changes in the other variable" (Carlson *et al.*, 2002, p. 357)
[3] "Coordinating the amount of change of one variable with changes in the other variable" (Carlson *et al.*, 2002, p. 357)

MA3, and *MA4*[4] in the 6th grade and the 7th grade; and *MA1*, *MA2*, *MA3*, *MA4* and *MA5*[5] in the 8th grade. That is, results point that *gross coordination of values* level of covariational reasoning seems to be aimed in the 4th grade; *coordination of values* level seems to be triggered in the 5th grade; and *continuous covariation* seems to be activated in the 6th, 7th, and the 8th grade. It is also worth mentioning that *chunky continuous covariation* seems to be aimed for the 6th and 7th grade students, while *smooth continuous covariation* might be triggered in the 8th grade. All these suggest that the Japanese curriculum gradually provides opportunities in an inclusive and iterating way for students to reason covariationally. In other words, in terms of covariational reasoning, Japanese curriculum has a spiral nature such that starting from the 4th grade, it might possibly cover all mental actions and trigger covariational reasoning of learners up to the highest level, the smooth continuous covariation, at the 8th grade.

Regarding Turkish curriculum, results have shown that the curriculum materials seem to support covariational reasoning in functional relationship concepts beginning with the 6th grade. Results point that in the 6th grade at first, by triggering *MA1* and *MA2*, *gross coordination of values* and then by underpinning *MA1*, *MA2*, and *MA3*, *coordination of values* might be supported through the tasks and questions. In the 7th grade, *coordination of values* level seems to be supported but the variables get more complex in terms of having non integer values and being continuous. In the following paragraphs, in comparison to each other, I discuss the Japanese and Turkish curriculum in terms of covariational reasoning for each grade level.

Particularly, in Japanese *4th grade* curriculum, in Task2, students are expected to examine the relationships between variables of temperature, which is continuous, and months, which is categorical but presented with the numbers 1, 2, 3…, 12 through a tabular representation. These variables are introduced with broken line graphs by starting with reminding students the representation of bar graphs and creating need for the broken line

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[4] "Coordinating the average rate-of-change of the function with uniform increment of change in the input variable." (Carlson *et al.*, 2002, p. 357)

[5] "Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function." (Carlson *et al.*, 2002, p. 357)

graph. According to Peck (2020), mathematical artifacts that students have already been introduced come up together and create new artifacts. Mathematical artifacts refer to cultural and historical objects playing the main role in learning processes (Peck, 2020). “Mathematical artifacts include tools, like function tables; techniques, such as algorithms for calculating slope; symbols, such as algebraic notation; and concepts, such as the notion of a rate of change.” (Peck, 2020, p. 438). Thus, in the task, for the objectification of new artifact, i.e. the broken line graph, in order to highlight the covarying quantities such as temperature and month, a bar graph is used as an earlier artifact in the task. So, the main aim in this grade level is to introduce students with a covariational situation in which changes in one of the two variables, the temperature and months, are questioned. Here one important issue needs further attention. As Thompson (1994) stated, whenever the learner thinks of a measurable quality of an object, s/he thinks of such measurable attribute as a quantity. Thus, students’ being asked to think about different values of temperature within different geographical places (i.e. Tokyo and Sydney) through different months suggest that students’ focus is taken on temperature and months as quantities and possible relationships between them. Moreover, at this grade level, with the questions such as “Between which months is the temperature going up?” the change in temperature, i.e. the vertical change in the broken line graph, is emphasized while at the same time mentioning the months in the question to emphasize covariation. These suggest that, in Task 1, coordination of changes in temperature with the changes in month (i.e. *MA1*) and coordination of direction of changes in temperature with the changes in month and verbalizing it as increase, decrease, or steady (i.e. *MA2*) are supported such that students’ covariational reasoning level of *gross coordination of values* seems planned to be activated. In addition, the representation and investigation of relationships between two covarying quantities is explicitly stated as an objective (Takahashi *et al.*, 2008). Similarly, teachers are supported to use “*activities to find two quantities in everyday life that vary in proportion to each other, and to represent and investigate the relationships of numbers/ quantities in tables and graphs.*” (Isoda, 2010a, p. 116) through this quote shared in teachers’ guide.

Regarding Turkish curriculum, on the other hand, the broken line graph is introduced as an objective of “Data Analysis” domain of the COS with having the objectives for students “(1) To create broken line graphs; (2) To interpret the given broken line graphs; and (3) To explain the convenience of using (broken line) graphics" (MoNE, 2009a, p. 314). Even though the unit of textbook, designed for these objectives, includes tasks (e.g. experiment results for boiling water at sea level presented both with tabular and graphical representations) similar to the Japanese textbooks, none of the curriculum materials (i.e. COS, teachers' guide, and textbook) emphasize covariation.

All these results suggest that the functional relationships through covariation of quantities with the representation of broken line graphs are connected with Data Analysis units in the Japanese curriculum. However, they are not connected in the Turkish curriculum such that there is no specific attention to the covariation of quantities depicted in the problem situations. Thus, the interconnectedness of these topics (the broken line graphs in data analysis units, functional relationships and covariation of quantities) in Japanese mathematics curriculum at 4th grade level seems to be more developed than Turkish curriculum.

In the *5th grade* Japanese curriculum, in the textbook, with the presentation of two different students' strategies on a task presented as Extending Mathematical Thinking (1), students are asked to find out how many sticks needed for making 30 squares if they use equal length sticks to make side by side squares. This suggests that students are supported to activate the first three mental actions; however there is also a shared student's strategy promoting the first two mental actions. In particular, it is interesting that one of the shared students' strategies (i.e. Takumi's strategy) corresponds to *gross coordination of values* level while the other (i.e. Kaori's strategy) is tuned with *coordination of values* level. More specifically, Kaori displayed the relationships between number of squares and number of sticks in chunks at her table by explicitly focusing on "the way the number of sticks increasing…" (Fujii and Iitaka, 2012, Grade 5, p. A103). Thus, her behavior underscores the amount of change in one quantity with corresponding to the amount of change in the other quantity (i.e. *MA3*). Therefore, her covariational reasoning level is *coordination of values*. However, Takumi focused on the coordination of the direction of change (i.e. *MA2*) by coordinating the increases in both number of sticks and number of squares. Although Takumi expresses the increase in the number of squares by 1 and increase in the number of sticks by

3, he did not explicitly mention the coordination these amounts of increases. Therefore, it might be safe to claim that his covariational reasoning level is *gross coordination of values*. I argue that in the 5th grade, presenting different levels of covariational reasoning strategies might be beneficial to engage students who have not reached the *coordination of values* level but stayed at *gross coordination of values* level which was supported in the 4th grade. Moreover, this task expects students to generalize and formulate the arithmetic sequence existing in the situation. Thus, the correspondence perspective of functions also seems to be supported through the task. It is emphasized in the literature that connection between covariation and correspondence perspectives of functions is important and there has to be opportunities for students to be engaged in tasks and questions to develop a fruitful understanding of functional relationships through both perspectives (e.g. Peck, 2020; Cooney et al, 2010).

Similar to the tasks in the Japanese 5th grade textbooks, in Turkish curriculum in the 6th grade, the tasks expect students to examine the covarying relationships existing in number patterns and shape patterns presented such as in Task2B, Task2C, and Task2D. Especially, the Task2D in the 6th grade Turkish textbook is identical to the task presented as Extending Mathematical Thinking (1) in the Japanese 5th grade textbook. In Task2D, students are expected to use matchsticks to model the number pattern of 3, 6, 9, 12, … However, the Turkish curriculum focused more on algebraic formulation of the given pattern such that students are supported to form a table for examining the relationships between variables and come up with an algebraic expression, i.e., $3 \times n$ with the *n* as the step number. Whereas the Japanese curriculum task focused more on the covariation through sharing two different students' reasoning as I shared above (i.e. strategies of Takumi and Kaori). Specifically, in the Turkish tasks in 6th grade (i.e. Task2B, Task2C, and Task2D) students are expected to realize the rule of the given pattern, provide verbal explanation and mathematical expression for the rule. Since the step number and numbers of the pattern placed according to each step number on a table, the coordination of changes of values of these two variables (i.e. *MA1*) seems to be supported. Also, by asking students to verbalize the pattern's rule and write a mathematical expression for the rule, students might coordinate the direction of change in pattern in each step, which seems to trigger *MA2*, and the *amount of change* in pattern for each step, which might support *MA3*. However, my claim, which the first three mental actions might be triggered through the tasks in the *Algebra* domain of the 6^{th} grade in Turkish textbook, is not as strong as the ones made for the Japanese textbooks, because the Turkish textbooks do not involve examples of students' ideas and strategies like the Japanese textbooks.

In the *6^{th} grade* Japanese curriculum, starting with the c*hunky variational reasoning* (e.g. Task3A) in which only one variable varies chunkily, students are expected to examine covariational relationships between variables with an increasing complexity. That is to say, students are provided with an example of *chunky continuous variation* at first (in Task3A) and then the *covariation* between two continuous variables (i.e. diameter and circumference via Task3B), in order to introduce algebraic representation first time in 6^{th} grade. However, in Task3B, the main aim seems to introduce algebraic representation without emphasizing the continuous and multiplicative relationships between variables. Although it seems contradictory to support *pre-coordination of values* level via Task3B in 6^{th} grade due to the support of covariational relationships in earlier grade levels, I claim that the Japanese curriculum seems to reduce cognitive load for students by only focusing on the algebraic representation which is a new and abstract mathematical artifact, i.e. algebraic representation, for students.

It is interesting that within the scope of tasks examined through covariational reasoning in this study, first time in 6^{th} grade both of the variables, in which a situation they covary simultaneously, are selected as continuous variables. More specifically, although covariational reasoning is supported in 4^{th} and 5^{th} grade, only one of the variables is continuous. For example, in the 4^{th} grade through Task1, students are expected to examine the covariation between a continuous variable of temperature and a categorical variable of month which is presented discretely. In 5^{th} grade, both variables are discrete (e.g. number of sticks and number of squares). The selection of continuous variables in covariational situations in 6^{th} grade might support students' meaning making of variation and covariation. Starting with the use of continuous variables to support students *coordination of values* level of reasoning in Task4D, in which students are expected to examine the relationships between lengths and widths of rectangles having fixed area and fixed perimeter though tabular, algebraic, and graphical representations, students are also supported to focus on *continuous covariation* without examining numerical values of the quantities (i.e. area and time) in the tasks of “What Will the Graph Look Like” shared in Additional Parts. In particular, in this grade level the first three mental actions of covariational reasoning are supported with the examination of proportional situations (e.g. in Task4D) in which students are expected to realize and express the multiplicative link between the changes in variables, such that when the length becomes 2 times as large, the width becomes $\frac{1}{2}$ times as large, when the length becomes 3 times as large, the width becomes $\frac{1}{3}$ times as large, and so on. This corresponds to MA3, i.e. “coordinating the amount of change of one variable which the changes in the other variable” (Carlson *et al.*, 2002, p. 357).

Moreover, continuous covariation seems to be supported implicitly through graphical representation without assigning any numerical values to the variables through the tasks shared in *Extensions* part (i.e. the tasks are named as What Will the Graph Look Like). These tasks might be aimed to raise students’ awareness about *rate of change* as a scaffold for the upcoming grades. This is because, students are expected to select the graph representing the rate of change in depth with the changes in times for the first one and the rate of change in the area with respect to time in the second one.

Similar to the Japanese curriculum, algebraic representation is also introduced beside the use of tables and graphs starting with the 6th grade in Turkish curriculum. Particularly, in Turkish curriculum covariational relationships are involved in the algebra unit through tasks requiring students to examine number and shape patterns. As I explained above just after the 5th grade Japanese curriculum is discussed, in the 6th grade Turkish textbooks covariational reasoning is supported starting with *gross coordination of values* and up to the *coordination of values* level. However, covariation relationships are supported with tasks and questions in the units of proportions and additional parts in Japanese curriculum in 6th grade. Although proportions are also presented in 6th grade in the Turkish curriculum, covariational relationships are not mentioned in proportions unit but only involved in algebra units.

In the *7th grade* Japanese curriculum, covariational reasoning is integrated with the proportional and inversely proportional relationships chapter which includes functions, direct and inverse proportions. At this grade level, on top of the coordination of values level of covariational reasoning, *chunky continuous covariation* level is supported through focusing on quantities, graphs, tables, and equations in the presented tasks. It is stated in the COS that "Students will deepen their understanding of direct and inverse proportional relationships by examining *correspondences* and *variations* of two quantities in real-life situations; students will foster their ability to identify, represent, and examine functional relationships." (Takahashi *at al.*, 2008, p. 21). Moreover, the meaning of functional relationships is explicitly highlighted in the teachers' guide. Specifically, the focus on quantitative relationships are emphasized in the teachers' guide with the sentence that "Basic model of quantitative relationships, direct and inverse proportional relationship studied in elementary school will be re-conceptualized as functions" (Isoda, 2010b, p. 46). These supports the claim that both correspondence and covariation meaning of functions is highlighted in Japanese curriculum. In particular, the graph of y=6/x shown in the Figure 6.71 and the task itself is designed to activate *the continuous covariation* between x and y values. That is, the task initiates focusing on intervals of x values and corresponding intervals of y values by getting those intervals smaller and smaller to display the smoothness of the covariation on a graphical representation. Specifically, examination of uniform increments of chunky changes in the values of x variable (see Figure 6.72 for the questions) and the corresponding changes in y variable, and then coordinating and verbalizing these changes by forming a graph, the task might trigger *MA4* which described as "coordinating the average rate of change of the function with uniform increments of change in the input variable" (Carlson *et al.*, 2002, p. 357).

Moreover, at this grade level, students are expected to be introduced with function definition which highlights the correspondence relationships (see Figure 6.66). Similarly, in the teachers' guide, functional relationship are stated as *"A functional relationship exists between two quantities when the value of one quantity is determined, the value of the other quantity will also be fixed to one and only one value*." (Isoda, 2010b, p.69). These definitions are static and emphasize correspondence meaning of functions (Cooney *et al.*, 2010). In addition to previous examinations of covarying quantities and rate of change in elementary school, the 7th grade mathematics might have supported students' covariational perspective of functions. Researchers said that students' use of covariational perspective can lead to the development of correspondence view of functions (Thompson, 1994b; Carlson *et al.*, 2002; Oehrtman *et al.*, 2008; Smith, 2003; Cooney *et al.*, 2010). Thus, beginning with providing students with opportunities for situations involving covarying quantities and expecting students to examine rate of change of quantities, and then, introducing the correspondence definition of function fits with formal definition of function as suggested in the literature (Cooney *et al.*, 2010). Interestingly, only the form of linear functions of $y = ax$, which indicates direct proportional relationships, is covered in this grade level. The rationale of the use of proportional relationships to trigger *correspondence* and *covariation* meaning of functions is supported in the teacher's guide that proportional relationships exist in many phenomena in daily life. Moreover, it is easier to find concrete examples of proportional relationships, thus, in the teachers' guide it is explained that proportional situations are preferred to be used in tasks to examine how the quantities change and correspond. Similarly, Ellis (2009) suggested teachers to provide students opportunities with examining the relationships between quantities to create ratios which at the end promote students' generalizations about linear functions.

In contrast to the integrated view of proportions and linear functions in Japanese curriculum materials, in the Turkish curriculum, proportions and linear equations are discussed in separate units. Particularly, neither the definition of proportional relationships nor the questions in the tasks are explicitly emphasized the coordination of the amounts of changes in variables as it has been in the Japanese curriculum. That is, the proportional relationships, which are introduced in *We started with proportion* unit, are not explicitly asked students to represent algebraically as $y = ax$. Also, even some tasks shared in algebra units have proportional relationships, there is no explicit mention to proportionality. In addition, in the Turkish curriculum, although covariational relationships are emphasized in the COS (see (MoNE, 2009b, p. 285)) and the textbook unit of linear equations and their graphs by focusing on the *coordination of values* level of covariation, rather than defining functions and expressing constant rate of changes in the problem situations, Turkish curriculum relies more on linear equations. Unlike the Japanese curriculum, the definition of linear functions is not given in lower secondary level in Turkish curriculum. Linearity and graphs of linear functions are given through linear equations. In terms of the contexts of the tasks, Turkish 7th grade textbook relies heavily on inexact data (e.g. Task1A and Task2B) to express linear equations. According to Ellis (2009), tasks having context of inexact data are best qualified for students who have already conceptualize linearity as having constant rate of change. Nevertheless, constant rate of change of linear functions is not explicitly presented in 7th grade Turkish curriculum.

In the *8th grade* Japanese curriculum, linear functions are further developed through the tasks and questions together with introducing of the concept of slope. Different representations of linear functions, i.e. tabular, graphical, and algebraic, are included. At this grade level, linear functions having the form of $y = ax + b$ is compared with directly proportional situations which have an equation of $y = ax$. Particularly, the constant a is defined as the rate of change, the slope, such that it refers to the amount of change of y when x is increased by 1 (see Figure 6.77). Also, the means of *b* as a unit of translation of the graph of direct proportional relationships along the y-axis and as y-intercept are expressed in the textbook. Cooney *et al.* (2010) emphasized that it is important to differentiate proportional relationships from linear functions for the developed understanding of functions. Furthermore, in the literature, researchers agreed on the idea that connection between algebraic and graphical representations is significant for the examination of change and relationships of variables (Cooney *et al.*, 2010). Cooney and his colleagues further stated that the rate of change for functions is an important aspect to classify functions; and linear functions are characterized as having a constant rate of change. So, results point that in the Japanese curriculum, not only the relationships between different representations but also the focus on the aspect of the constant rate of change are explicitly presented (e.g. see Figure 6.71). In fact, in the teacher's guide, it is further mentioned as follows:

> By examining rates of changes, help students understand that the coefficient of x, that is a, is how much y will increase when x increases by 1. Moreover, help students to understand that the amount of increase in y corresponding to the amount of increase in x can be determined based on the value of a. (Isoda, 2010b, p. 92)

Moreover, five subconstructs of slope is covered in Japanese textbook: which are rate of change, physical property, algebraic ratio, geometric ratio, and parametric coefficient (Peck, 2020). Only trigonometric ratio and derivative of functions subconstruct are not included which are also not in lower secondary level curriculum. Similar to the learning trajectory found by Peck (2020), the five subconstruct included in Japanese curriculum starting with the rate of change and followed by parametric coefficient, algebraic ratio, geometric ratio, and physical property, respectively. Particularly, Peck (2020) found that even though researchers expected students of getting geometric ratio after students engaged with rate of change and algebraic ratio, students did not come up with geometric ratio, rather some students used algebraic ratio and the other used tables to bring out points from the graphs given to them. Thus, to allow students to get to geometric ratio, they used number lines with arrows as a new artifact which helped students to see the changes in a graph (Peck, 2020). It is interesting that instead of introducing slope as a geometric ratio by mentioning it as *rise over run*, Japanese curriculum includes number lines with arrows to correspond geometric ratio subconstruct.

Similarly, in the Turkish curriculum, the concept of slope is also introduced in the 8^{th} grade. However, linear functions are not included in the 8^{th} grade. In the Turkish textbook, physical property subconstruct and geometric ratio subconstruct, as a ratio of rise over run, are introduced in the same task (i.e. Task2A in 8^{th} grade). Then, parametric coefficient and algebraic ratio subconstructs are covered, respectively. Yet, rate of change is not included although Peck found that "rates of change can serve as the central artifact through which other subconstructs can emerge" (Peck, 2020, p. 460). Thus, researchers promoted the use of rates for a comprehensive understanding of all subconstructs of slope (Peck, 2020).

7.2.2. Curriculum presentation of functional relationships in terms of quantities and quantitative reasoning

In terms of the *quantitative reasoning*, results point that in Japanese curriculum, it is expected for students at first to determine the quality of interest and second focus on the quantities. Specifically, in the teachers' guide it is stated

> It is important to enable students to represent and to investigate quantities and their relationships by using numbers, algebraic expressions, diagrams, tables, and graphs, and to express and investigate them with words, to make decisions about them, and to explain own ideas. (Isoda, 2010a, p. 43)

In a similar vein, questions focus on the identification and determination of quantities and relationships between quantities regardless of the grade level. In addition, students are introduced with quantities in real life situations and are asked to determine what quantities they need to consider in making their interpretations and/ or comparisons, which is also recommended to teachers to pay attention in teachers' guide. Thus, they are expected to both identify and determine quantities and use quantitative operations to make sense of the situation, i.e. reason quantitatively (Thomson, 1990). For example, in the Task1A in the 5th grade textbook, students are expected to first determine which quantities are needed to compare crowdedness of rabbit cages and then multiplicatively compare two extensive quantities, i.e. rabbit population and space, to define an intensive quantity, which is population density (Schwartz, 1988). It is also clarified in the teachers' guide that some intensive quantities, such as population density, could be represented through multiplicative comparison of two quantities, i.e. the ratio of two quantities. This shows that multiplicative comparison of two quantities might indicate referent transforming composition, i.e. a quantitative operation for Thompson (1994a), which results in a quantity indicating quality of the situation, i.e. crowdedness or population density (Schwartz, 1988). Promoting the determination and use of different types of quantities, asking students the differences between quantities and to compare them, and highlighting the quantitative operations in mathematical situations is claimed to sophisticate learners' mathematical conceptions (Schwartz, 1988; Thompson, 1990, 1994a).

Similarly, the teachers' guide has emphasis for teachers to allow students to determine quantities and engage in their relationships. For instance, in the 4th grade teachers' guide, on top of explaining what quantity is and what types of quantities there are in mathematics education, the examination and generalization of quantities is highlighted: "It is important to represent quantitative relations using formulas and to interpret formulas in concrete situations. Consideration should be taken so that students understand the merit of formulas that generalize quantitative relationships through these activities." (Isoda, 2010a, p.118). Generalization is important for scheme construction and thus for quantitative reasoning (Thompson, 1990, 1994a).

Teachers are suggested to create situations for students to compare extensive quantities with intensive quantities that might contribute to students' understanding of different types of quantities and which in turn might enable them to differentiate proportional situations from non-proportional situations. Similarly, in Task2C in the 5th grade, students are expected to form a mathematical sentence and to perform calculation for quantity being compared. Even though *arithmetic operations* are aimed to calculate the quantity being compared; students are also required of considering quantities and the multiplicative relationship between quantities. By the same token, expressions of multiplicative relationships between quantities with mathematical sentences, such as "Since 20% of 300mL is 0.2 times as much of 300mL, 300×0.2= □, Answer □mL" (Fujii and Iitaka, 2012, Grade 5, p. B57), might trigger the *quantitative operation* of "comparing quantities multiplicatively". Thompson (1994) explained that "A quantitative operation… has to do with the comprehension of a situation" (p.13). Thus, results points that in Japanese textbooks, arithmetic operations are supported with their quantitative underpinnings in the quantitative situations. Moreover, "times as much" (Thompson and Saldanha, 2003) meaning of multiplication, multiplicative reasoning, and "relative sizes" (Byerley and Thompson, 2014; Thompson and Saldanha, 2003) meaning of division are supported in overall Japanese curriculum units which are examined in this study in relation to the functional relationships.

In comparison to Japanese Curriculum, in Turkish curriculum, "times as much" and "multiplicative reasoning" are covered especially starting with the 6th grade regarding functional relationships. However, "relative size" meaning of division (Byerley and Thompson, 2014; Thompson and Saldanha, 2003) seems not to be highlighted. In Turkish textbooks, since students are told the quantities in the tasks, it is not planned for students to determine the quantities in tasks. Rather, students are only expected to identify quantities by measuring, counting, or calculating to assign numerical values to quantities. In addition, multiplicative comparison, which is a quantitative operation used to create a ratio (Thompson, 1990), is not emphasized strongly in Turkish curriculum. Rather, arithmetic operations are highlighted more than quantitative operations. By the same token, compared to how they are explained and emphasized in the Japanese curriculum materials, in Turkish teachers' guide, classification of quantities and quantitative operations are not specifically referred to and emphasized to take students' attention to. Thus, I propose that Japanese curriculum's definition of ratio matches with the Thompson's definition (1994a) whereas the definition of ratio in the Turkish curriculum does not directly match Thompson's definition.

In the following sub-section, I discuss the results on the concepts of ratio, rate, and proportions in regards with quantitative reasoning framework.

7.2.3. Curriculum presentation of functional relationships, especially ratio, rate, and proportions, in terms of quantitative reasoning

In this section I discuss the general and some of the grade based findings coming from the analysis of the concepts of ratio, rate, and proportions by comparing the Japanese curriculum and the Turkish curriculum. I do this since multiplicative relationships between quantities prevail in the curriculum in regard to the concepts of ratio, rate, and proportions. In the following paragraphs, first I discuss some important characteristics of the Japanese curriculum materials embedded in all grade levels gradually; and then, continue the discussion with respect to 5^{th} and 6^{th} grade level specifically.

As I shared in the previous part, the Japanese curriculum involves tasks and questions emphasizing quantities; in other words, the identification and determination of quantities and the relationships between quantities exist at every grade level. Starting with the additive reasoning in grade 5, students' multiplicative reasoning gains more emphasis in the curriculum at advancing grade levels. Moreover, part-part and part-whole situations is covered in Japanese curriculum. Specifically, part-whole relationships are included both additively and multiplicatively. Interestingly, line segments are used to express part-whole relationships which entails the use of line segment as a measure to compare quantities and form a ratio (Thompson and Saldanha, 2003). Regarding the explanations of Thomson and Saldanha (2003), use of line segments to compare ratios might support students' conception of proportion as multiplication. Moreover, line segments are also used to express part-part ratios. It seems that line segments are used through ratio and rate concepts as didactic objects (Gravemeijer, 2020) which corresponds to "… "a thing to talk about" that is designed with the intention of supporting reflective mathematical discourse." (Thompson, 2002, p. 198). So, multiplicative relationships are addressed with the use of line segments throughout all grade levels.

In particular in the 5th grade, strategies of abbreviated building up and per-one (e.g. in Task1A) are mainly highlighted strategies in Japanese curriculum. However, in addition to informal strategies of students, they are provided with tasks and strategies that support their multiplicative reasoning (e.g. Task2A). Thus, *identical groups conception* (Heinz, 2000) seems to be accessible to students in the 5th grade. Lobato and her colleagues (2010) suggested to use building-up strategies to support students' multiplicative strategies. On the other hand, at the 5th grade, with the use of line segments, "times as much" and "relative sizes" are also highlighted which requires students' multiplicative thinking (e.g. in the Task2A).

At the 6th grade, students are further provided with multiplicative strategies such as between-state ratios, within-state ratios, and proportion equation strategy (e.g. Task1D). Furthermore, results showed that at this grade level, students' *ratio as measure* conception might have been triggered with the use of proportional situations, differentiation of rate from ratio, and introduction of speed concept. Moreover, *between state* (e.g. Task4A) and *within state* (e.g. Task4B) ratio conception might be triggered at this grade level. Finally, results showed that the comparison of direct proportional, inverse proportional situations as well as non-proportional situations are aimed to be discussed in 6th grade (e.g. Task4D). In the 7th grade, students' proportional reasoning and multiplicative reasoning are further emphasized. Proportion equation strategy is highlighted in 7th grade in the Japanese curriculum while at the same time students are fostered to think alternative ways of approaching the task to find the unknown (e.g. Task1A). In this grade level students are also supported to examine relative sizes and express the within situation via ratios as measures. Proportional situations are used to introduce linear functions in Japanese curriculum. Nonetheless, the connection between proportional situations and linear functions is not explicitly stated in Turkish curriculum. Moreover, proportion equation strategy seems to be the mainly targeted strategy introduced to the students throughout the 7th grade Turkish curriculum.

In Japanese curriculum, students are expected to think proportional situations through relative sizes which requiring multiplicative reasoning. Since slope and linear functions necessitates proportional understanding (Lobato *et al.*, 2010), and proportionality is covered fruitfully in Japanese curriculum by supporting students' covariational and quantitative reasoning, Japanese curriculum seems helpful for students' developed understanding of slope and linear functions in the 8th grade. Moreover, covering multiplicative comparison of relatives sizes might support students' understanding of rate of change and derivative (Byerley *et al.*, 2012).

On the other hand, the Turkish curriculum focuses more one additive reasoning and involves only proportion equation strategy as a multiplicative strategy. At the 5th grade, students are introduced with building up strategy and supported to reason additively. At the 6th grade, cross multiplication and proportion equation strategies are provided but students' multiplicative reasoning is not supported as explicitly as in Japanese curriculum with the use of different multiplicative strategies. At the 7th grade, proportion equation strategy is highlighted as a student strategy. These further suggest that tasks and questions in Turkish curriculum do not address triggering of students' between state and within state ratio conceptions. In general, it seems that students' *identical groups conception* of ratio is supported in Turkish materials.

However, use of between-state and within-state ratios are important (Lobato *et al.*, 2010) since first of all these are multiplicative strategies students might use. As the results have shown, Japanese curriculum has brought up these strategies explicitly to the attention of students. This important because teachers might have an opportunity to discuss those strategies within classroom with students who have already acknowledge and use those strategies and haven't yet established those strategies. Secondly, as the literature has suggested (Heinz, 2000; Karagoz Akar, 2009, 2010, 2017; Lobato *et al.*, 2010) these strategies seem to be administered by students at the intermediate stages of the concept of ratio. Thus, students' reasoning about and making sense of these strategies might further allow them to progress towards a full understanding of ratio (Lobato *et al.*, 2010).

In the following paragraphs, I discuss the similarities and differences of Japanese and Turkish curriculum materials with respect to the 5th and 6th grade levels due the main focus on ratio and rate in these grades.

With respect to the underlying concepts of ratio, rate, and proportion, at the *5th grade* Japanese curriculum, students are expected reason on tasks and questions that have the possibility to trigger their partitive and quotative division schemes, which do not require students to reason multiplicatively (Byerley *et al.*, 2012). Particularly, partitive and quotative division schemas in regard to abbreviated-building up and per-one strategies are supported, in 5th grade. Similarly, multiplicative relationships such as "times as much" and "percentages referring to parts being times as much of a whole" get place in the textbooks beginning with the 5th grade. For instance, in Task2A, students are expected to compare successful shots and the total number of attempts made for four basketball games. Since students, given any whole (the base quantity), can think of 1 in terms of that quantity (the whole) such that 1 is always (1/the value of the base quantity) times as much of the whole, they can take this multiplicative relationship as an operator. Thus, students might be triggered to think percentages as the multiplicative relationship between the part and the whole and relative sizes as such that it is the result of comparing the first quantity in terms of the second quantity. Moreover, part-whole relationships are introduced and represented through number line segments to provide students with opportunities to have discussions about situations triggering multiplicative reasoning. In fact, even discrete quantities are expressed on the number line segments to ease multiplicative comparison of quantities. Rate is also defined at this grade level as "*The number that expresses how many times as much a quantity is compared to the base quantity is called the rate*." (Fujii and Iitaka, 2012, Grade 5, p.B53). However, this definition of rate seems not totally match with a sophisticated rate definition accepting rate as a reflectively abstracted constant ratio by Thompson (1994a). In addition, it is interesting that at the 5th grade, students are also introduced with "relative sizes" which requires multiplicative reasoning (Byerley *et al.*, 2012).

Similar to the Japanese curriculum materials, in Turkish curriculum, students are introduced to ratio based on their additive thinking. However, only building-up strategy is supported in the tasks. Part- whole relationships are also involved (e.g. Task1A in 5th grade) to allow students to make a comparison of the size of quantities. However, questions for determining the quantities are not asked in the Turkish curriculum.

In the *6th grade* Japanese curriculum, with the use of between-state and within-state ratios, students are expected to use multiplicative reasoning. Students are supported to consider the amount of increase or decrease in quantities between situations in which increment or decrement multiple is not an integer. They are also expected to recognize the invariant relationship and representing it by multiplicatively comparing quantities within a situation though forming a line. Karagöz Akar (2017) found that dealing with divisibility failure existing in between-state ratios, even the person does not make sense of such as per-one in within-state ratios, indicates that the person's understanding of ratios extends identical groups conception. As Heinz (2000) study showed students having identical groups conception are not able to deal with the divisibility failure although they can make sense of per-one strategy. Moreover, in this grade level students are expected to express ratios not only as an association between quantities but also as an indication of invariant proportional relationships. Thus they are expected to think multiplicatively within situations starting from lower levels of reasoning. Karagöz Akar (2010) found that Mark as a student who uses within-state ratios, realized the invariant relationships between quantities and expressed ratio as an intensive quantity which qualifies the situation. Starting from within state ratios, he conceptualized ratio as a measure (Karagöz Akar, 2010). Thus, introducing speed as an intensive quantity, providing students with non-integer values of ratios for both within and between-state ratio situations, and providing students' strategies involving both within and between state ratios, their multiplicative reasoning seems to be supported gradually in Japanese curriculum. Also, the difference between ratio and rate is specifically mentioned in the 6^{th} grade such that "A ratio is a way to express a proportion using 2 numbers while the rate we learned in the 5th grade expresses the proportion using 1 number" (Fujii and Iitaka, 2012, Grade 6, p. A63). These explanations seems to be compatible with Kaput and West (1994) definitions of particular ratio and rate-ratio. In this grade level students are also introduced to use line segments for representing part-part ratios. In addition, between and within state ratios, which are used in the task in Extending Mathematical Thinking part, highlight the relative size meaning of division as displayed in Figure 6.60 (Fujii and Iitaka, 2012, Grade 6, p. A103). All these results show that, Japanese 6^{th} grade curriculum support students' multiplicative reasoning in proportional situations. In contrast, additive reasoning seems to be supported in Turkish curriculum at the 6^{th} grade. In addition to building-up strategy, proportion equation strategy is introduced at this grade level. That suggests that, in contrast to the Japanese curriculum, multiplicative strategies are supported by highlighting mostly arithmetic operations in the Turkish curriculum. Thus, it seems that within the realm of identical groups conception, some strategies are mainly the ratio conception focused on the Turkish curriculum.

With respect to the *task variables*, I discuss similarities and differences between Japanese and Turkish curriculum materials considering covariation and ratio concepts. Specifically, in the comparison based on ratio concepts, I discussed the compared findings in terms of *numerical, semantical,* and *contextual features* (Kaput and West, 1994).

Findings regarding covariational reasoning suggest that the numerical values of variables and categories of variables get more complicated for both Japanese and Turkish curriculum. In particular, for Japanese textbooks, in the 4^{th} grade only one of the variables in the provided covarying situation is given as continuous while the other is categorical, in the 5^{th} grade covarying variables are given as discrete; and in the 6^{th} grade both variables are not only given as continuous, but also are non-integer numbers (e.g., decimals). Moreover, the domain of the proportional relationships is expanded to *rational numbers* in the lower secondary grades, i.e. 7^{th} and 8^{th} grade, from *non-negative numbers* which are covered in elementary grades. Similarly, in the 7^{th} grade, tasks and problems (e.g. Task2A and Task 2C) are presented to support *simultaneous and continuous covariation of variables* such that interval representation of variables are expected to be used. This might indicate that students' understanding of continuous variables is aimed to be strengthened. In the 8th grade, with the content of speed and use of graphical representations, smoothly varying variables are further supported. Thus, this indicates that first starting at the 7th grade, continuous variables are presented chunkily, and then in the 8th grade, students' understanding of variables continuing smoothly are further emphasized and triggered.

Compared to the Japanese curriculum materials, in Turkish curriculum, students are introduced with variables with integer values in the 6th grade. In the 7th grade, after the tasks having integer values of variables are presented, tasks having non-integer values of variables (e.g., decimals) also are shared. In the 8th grade, although rational values are used for variables, which might indicate that continuous variables are emphasized, there is no introduction for smooth continuous variables like what Japanese materials did with the use of interval values. By the same token, continuous variables are presented chunkily in Turkish curriculum. Although, in the 8th grade, the graphs of linear equations involves smooth continuous x and y variables, there is no question about the values existing between chunks. However, the smooth continuous variables have a place in the 8th grade under the unit of Inequalities in which neither variational nor covariational reasoning is intended but interval values are covered.

In addition to *numerical features*, the familiar multiples are used in the beginning of the ratio units in both Japanese and Turkish curriculum materials, which facilitate students' multiplicative reasoning (Heinz, 2000). In Japanese curriculum, non-integer values are used in ratio comparison examples but not in tasks (see Figure 6.33); whereas, in Turkish curriculum only integer values are used in ratio tasks. In particular, tasks, giving values of quantities to ask students compare the quantities and form a ratio, are presented with integer values of quantities in both curricula. Though this, students' use of additive thinking might be prevented (Heinz, 2000).

In terms of *semantic features*, in Japanese curriculum, "for every/for each" statements are shared in tasks and questions. For example, some tasks about speed concept are stated as "km per hour" (e.g. Task2C and Task2D in 6th grade) and the task about crowdedness of cages (i.e. Task1A in 5th grade) involves to examination of the amount of space for each rabbit. Kaput and West (1994) said that "for every/each" statements highlight the rate-ratio conceptualization of the situation. Similarly, the rate definition shared in Japanese curriculum resonates with rate-ratio conception. Nevertheless, in Turkish curriculum neither the definition of rate nor for every/each statements are presented.

Regarding with the *contextual features*, both curricula involve mixture problems of which students have difficulties (Kaput and West, 1994; Heinz, 2000). In Turkish curriculum, part-whole relationships are mainly focused on tasks. Lamon (1993) stated that students tend to use building up strategy in part-part-whole situations and building up is the mainly emphasized strategy in Turkish curriculum. It is interesting that number line segments are used to highlight relative sizes in the part-part-whole situations which seems to trigger students' multiplicative reasoning.

In the following section, I first share implications of this study for teaching, research, and teacher knowledge. Second, I discussed about limitations of the study.

8. IMPLICATIONS AND LIMITATIONS

As the findings suggested, functional relationships presented in the Japanese elementary and lower secondary curriculum materials might be used to examine students' development of covariational reasoning. Moreover, the tasks might be used for further research by conducting a teaching experiment or a classroom teaching experiment to examine the development of students' covariational reasoning. Previous research also suggested to examine the various curricular interventions in terms of their effectiveness in students' covariational reasoning (Carlson *et al.*, 2002; Thompson and Carlson, 2017). In addition, Taşova and his colleagues (2018) found little evidence in calculus textbooks which seems to support students' sophisticated quantitative and covariational reasoning. Thus, the Japanese materials might be examined by textbook authors in order to create textbooks promising the sophisticated quantitative and covariational reasoning of students. Also, teacher educators might use the tasks to examine pre-service teachers' covariational reasoning.

Furthermore, there seems to be learning trajectories supported in the Japanese curriculum materials in terms of ratio, rate, proportion, and linear functions. For instance, the learning trajectory of slope found by Peck (2020) resonated with the presentation of slope in Japanese curriculum materials. Therefore, these topics, i.e. functional relationships topics, might be examined regarding learning trajectories.

It is shown in the mathematic education literature that preservice, and even in-service, mathematics teachers have lack of knowledge in regard to ratio, rate, and proportion (Bahar, 2019; Ekawati *et al.*, 2015). Especially, working with 590 prospective teachers from Turkey including all secondary school mathematics and science education departments, Bahar (2019) found that pre-service teachers are unaware of students' strategies of ratio and proportion including between state ratios. Since textbooks are potential sources for teachers to plan their lessons, study contents, select classroom activities, and assign homework (e.g., Son and Kim, 2015; Ma, 1999; Ball and Cohen 1996; Remillard, 2005; Özgeldi, 2011), they have impact on teachers' knowledge. With the explanations of contents, mentioning students' strategies, and detailed interpretation of objectives in teachers' guide, and with providing examples from possible students' thinking and strategies in textbooks like in the Japanese curriculum, teachers' knowledge on ratio, rate, and proportion seems to be supported and improved. Furthermore, in Japanese textbooks not only different strategies of students are presented for the same task but also the differences between those strategies are asked to students. Teachers might also reflect on those students' strategies and differences between those strategies which at the end might develop their knowledge. For instance, some studies (e.g. Bahar, 2019) displayed that teachers did not select that the graphical representation of ratio relationships is represented by $y = mx$. Instead, some teachers selected $y = mx + b$ and $y = c$ types of relationships as representing a ratio relationship. On the contrary, the graphical representation of proportional relationships is explicitly stated in the 7th grade Japanese textbooks as *straight line passing through the origin* without indicating the constant quotient as slope. Furthermore, Task1A (Fujii and Matano, 2012, Grade 8, p. 53-54), which is shared in the 8th grade Japanese textbook, is designed to introduce linear functions on the top of proportions by clarifying and explicitly stating that proportions are simple form of linear functions which does not have the y-intercept, i.e. b=0. Hence, these suggest that teachers might develop their understanding of graphical representation of ratio relationships through engaging with the Japanese textbooks. Therefore, I propose that Japanese curriculum materials might be examined from the perspective of teacher knowledge on functional relationships including the concepts of ratio, rate, and proportion.

In particular, Ball and her colleagues (2008) classified mathematical knowledge for teaching as subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK involves three sub-categories as: Common content knowledge (CCK), horizon content knowledge (HCK), and specialized content knowledge (SCK). Since they defined SCK is a knowledge specific for mathematics teachers, type of decompressed mathematical knowledge including knowledge of analyzing students' errors mathematically, use of mathematical language and representations effectively (Ball *et al.*, 2008), Japanese curriculum materials seems to support SCK of teachers. Moreover, HCK includes how the mathematical topics in the curriculum related with each other. Since the findings of this study shows the interconnectedness of topics in Japanese curriculum, HCK might be supported too. In addition, PCK consists of knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). KCS is about knowing the students in terms of what motivates them, what they will find confusing, how they think about a task/topic/example, and familiarity about students' mathematics (Ball *et al.*, 2008). KCT includes knowing about teaching and mathematics, so it contains planning for teaching, examples, representations and tasks used in teaching (Ball *et al.*, 2008). KCC includes knowledge of curriculum and mathematics. Thus, with the explanations giving in the COS and teachers' guide, as well as the examples of students' ideas in textbooks, teachers' PCK seems to be supported in Japanese curriculum. As a result of all these, the Japanese curriculum might have some characteristics to contribute teachers' knowledge. Nevertheless, I did not conduct a framework to examine the curriculum materials in terms of teacher knowledge in this study. Therefore, future research might be worth to examine Japanese curriculum with the lenses of teacher knowledge.

There are some limitations in terms of the selection of curriculum materials in this study. Particularly, I examined Tokyo Shoseki's MI textbook series to make a comparison between Japanese and Turkish curriculum materials. However, there are 6 different textbooks series also got acceptance to be published and used in schools. It is worth mentioning that Tokyo Shoseki's textbooks is expressed as most widely used textbook series. Following Tokyo Shoseki's textbooks series, KeirinKan's textbooks are second mostly used in Japan. Thus, functional relationships might be examined with the use of other textbooks series, especially by using KeirinKan's textbooks, published and used in Japan.

Moreover, the Turkish curriculum materials used in this study are not the current materials. However, since there is not a drastic change made in the reformed curricula in Turkey since 2005 curriculum regarding functional relationships topics including ratio, proportion, and linear functions, still use of curriculum materials designed based on 2005 reform is valuable. The reason of 2005 curriculum selection is that only in 2005 curriculum have teachers' guides which allows triangulation of data. Nonetheless, there is a curriculum developed for gifted students in 2018 in Turkey. Future studies might use the gifted curriculum to examine similarities and differences between the gifted curriculum and Japanese curriculum.

Furthermore, quantitative reasoning and covariational reasoning are constructs for teachers and researchers to reason about how students reason. Although textbooks are called potentially implemented curriculum, in order to examine those constructs in the curriculum materials I formed an analytical lens (see Appendices A and B). This analytical lens might be used to be developed to create a framework to analyze written curricula.

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APPENDIX A: (CO)VARITIONAL REASONING

Table A.1. Covariational reasoning framework (Carlson *et al.*, 2002, p. 357-358).

Mental Actions		**Levels of Covariational Reasoning Framework**		
Name	**Descriptions of MAs**	**Name**	**Supported mental actions**	**Descriptions of levels**
MA1	"Coordinating the value of one variable with changes in the other." (p. 357)	Coordination	MA1	"The images of covariation can support the mental action of coordinating the change of one variable with change in the other variable." (p. 358)
MA2	"Coordinating the direction of change of one variable with changes in the other variable." (p. 357)	Direction	MA1 MA2	"The images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable." (p. 358)
MA3	"Coordinating the amount of change of one variable with changes in the other variable." (p. 357)	Quantitative Coordination	MA1 MA2 MA3	"The images of covariation can support the mental actions of coordinating the amount of change in one variable with change in the other variable." (p. 358)

Table A.1. Covariational reasoning framework (Carlson *et al.*, 2002, p. 357-358) (cont.).

Mental Actions		**Levels of Covariational Reasoning Framework**		
Name	**Descriptions of MAs**	**Name**	**Supported mental actions**	**Descriptions of levels**
MA4	"Coordinating the average rate-of-change of the function with uniform increments of change in the input variable." (p. 357)	Average Rate	MA1 MA2 MA3 MA4	"The images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable." (p. 358)
MA5	"Coordinating the instantaneous rate of change of the functions with continuous changes in the independent variable for the entire domain of the function." (p. 357)	Instantaneous Rate	MA1 MA2 MA3 MA4 MA5	"The images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change. It also includes an awareness that the inflection point is where the rate of change changes from increasing to decreasing, or decreasing to increasing." (p. 358)

Table A.2. Variational Reasoning Framework (Thompson and Carlson, 2017, p. 434).

Levels	Description
Smooth continuous variation	"The person thinks of variation of a quantity's or variable's (hereafter, variable's) value as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously. The person might think of same-size intervals of variation, but not necessarily." (p. 434)
Chunky continuous variation	"The person thinks of variation of a variable's value as changing by intervals of a fixed size. The intervals might be the same size, but not necessarily. The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, and so on, "come along" by virtue of each being part of a chunk—like numbers on a ruler—but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values. Chunky continuous variation is not just a person thinking that changes happen in whole number amounts. Thinking of a variable's value going from 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that entailed intervals "come along") is just as much thinking with chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on." (p. 434)
Gross variation	"The person envisions that the value of a variable increases or decreases, but gives little or no thought that it might have values while changing." (p. 434)
Discrete variation	"The person envisions a variable as taking specific values. The person sees the variable's value changing from a to b by taking values a1, a2, …, an but does not envision the variable taking any value between ai and ai + 1." (p. 434)
No variation	"The person envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario." (p. 434)
Variable as a symbol	"The person understands a variable as being just a symbol that has nothing to do with variation." (p. 434)

Table A.3. Major levels of developed version of covariational reasoning framework (Thompson and Carlson, 2017, p. 435).

Covariational reasoning framework		
Level	Description	Corresponding mental action
Smooth continuous covariation	"The person envisions increases and decreases (thereafter, changes) in one quantity's or variable's value, and the person envisions both variables varying smoothly and continuously." (p. 435)	MA1 MA2 MA3 MA4 MA5
Chunky continuous covariation	"The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation." (p. 435)	MA1 MA2 MA3 MA4
Coordination of values	"The person coordinates the value of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x,y)." (p. 435)	MA1 MA2 MA3
Gross coordination of values	"The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values." (p. 435)	MA1 MA2
Precoordination of values	"The person envisions two variables' values varying, but asynchronously- one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects." (p. 435)	MA1
No coordination	"The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values." (p. 435)	-

APPENDIX B: CURRICULUM MATERIALS

Table B.1. The trend of the change in number of objectives of Turkish curriculums over the years.

Grade	Learning Domains	2005	2009	2013	2017	2018
5	Numbers (and operations)	46	46	33	33	33
	Geometry (and measurement)	22	23	20	20	20
	Measurement	16	16	-	-	-
	Probability (and data analysis)	9	9	6	3	3
	Algebra	-	-	-	-	-
	Total	**93**	**94**	**59**	**56**	**56**
6	Numbers (and operations)	32	31	35	32	32
	Geometry (and measurement)	19	17	22	19	19
	Measurement	19	18	-	-	-
	Probability (and data analysis)	11	11	6	5	5
	Algebra	6	6	6	3	3
	Total	**87**	**83**	**69**	**59**	**59**
7	Numbers (and operations)	15	15	23	25	25
	Geometry (and measurement)	23	23	19	12	12
	Measurement	20	20	-	-	-
	Probability (and data analysis)	12	12	4	4	4
	Algebra	9	9	7	7	7
	Total	**79**	**79**	**53**	**48**	**48**
8	Numbers (and operations)	12	12	17	16	16
	Geometry (and measurement)	21	21	17	16	16
	Measurement	15	15	-	-	-
	Probability (and data analysis)	10	8	7	7	7
	Algebra	13	13	13	13	13
	Total	**71**	**69**	**54**	**52**	**52**
In Total		**330**	**325**	**235**	**215**	**215**

Table B.2. Topic placements and number of pages of the textbooks.

	Japanese textbooks		**Turkish textbooks**	
Grade level	Topic placement	Number of pages of *the units* / total pages of the textbook	Topic placement	Number of pages of *the units*/ total pages of the textbook
4th grade	**4A:** 1. Structure of large numbers 2. Size of angles 3. Division algorithm 4. Perpendicular/ parallel lines and quadrilaterals **5. Broken line graphs** 6. Abacus 7. Structure of decimal numbers 8. Division algorithm (2) **4B:** 9. How to organize data **10. Properties of operations** 11. How to measure and express area 12. Fractions 13. Investigating changes 14. Approximate numbers 15. Multiplication and division of decimal numbers 16. Cubes and cuboids	5. Let's represent how the quantities change in graphs (p.A76-A84/ A135) 10. Let's investigate the properties of operations (p. B8-B17/ B131)	-	-

Table B.2. Topic placements and number of pages of the textbooks (cont.).

	Japanese textbooks		**Turkish textbooks**	
Grade level	Topic placement	Number of pages of *the units* / total pages of the textbook	Topic placement	Number of pages of *the units*/ total pages of the textbook
5th grade	**5A:** 1. Whole numbers and decimal numbers 2. Volume of cubes and cuboids 3. Multiplication of decimal numbers 4. Division of decimal numbers 5. Congruent shapes 6. Even and odd numbers, multiples and factors **7. Per unit quantity** 8. Fractions and decimal numbers **5B:** 9. Angles and geometric figures 10. Addition and subtraction of fractions 11. Area of quadrilateral and triangles **12. Percentages and graphs** 13. Regular polygons and length around circles 14. Multiplication and division of fractions 15. Prisms and cylinders	7. Let's think about how to compare (1) (p. A85-A101/ A129) **Extending Mathematical Thinking (1):** Looking for Patterns: Thinking with diagrams, tables, and mathematical expressions (p.A103-104). 12. Let's think about how to compare (2) (p.B50-B69 /B129)	1. Geometric shapes **2. Fractions** **3. The world of decimal fractions** 4. Geometric objects 5. Numbers in our lives 6. Multiplication and division operations	2.Percentages (p. 57-60/ 224) 3.Ratio (p.112-114 /224)

Table B.2. Topic placements and number of pages of the textbooks (cont.).

	Japanese textbooks		**Turkish textbooks**	
Grade level	Topic placement	Number of pages of *the units* / total pages of the textbook	Topic placement	Number of pages of *the units*/ total pages of the textbook
6th grade	**6A:** 1. Area of circles **2. Letters and math sentences** 3. Multiplication of fractions 4. Division of fractions 5. Symmetric figures **6. Ratios and values of ratios** 7. Enlarged and reduced drawings **8. Speed** 9. Volume of prisms and cylinders 10. Approximate area **6B:** **11. Direct and Inverse proportional relationships** 12. How to analyze data 13. Number of cases 14. The system of units of measurement	2. Let's use letters and write math sentences (p. A16-A21/ A121) 6.Let's think about how to express proportions (p. A60-A70 /A121) 8.Let's think about how to express speed (p. A82-A93/ A121) **Extending Mathematical Thinking (1).** Fixing the whole (p.A103-A104) **Extension.** Let's calculate the speed of a train! (p. A119) 11.Let's investigate proportional relationships (p.B2-B25 /B105) **Extending Mathematical Thinking (2).** Look for Patterns (p. B39) **Extension.** What Will the Graph Look Like (p.B100)	1. Staring a nice journey 2. From numbers to geometry **3. Mathematics and art** **4. Reflection from numbers to probability** 5. A gateway to measuring from decimal fractions 6. The story of measuring from area to volume	3. Everyone should learn algebra (p.125-131/255) 4.Ratio and Proportions (p.165-169/255)

Table B.2. Topic placements and number of pages of the textbooks (cont.).

	Japanese textbooks		**Turkish textbooks**	
Grade level	Topic placement	Number of pages of *the units* / total pages of the textbook	Topic placement	Number of pages of *the units*/ total pages of the textbook
7th grade	1. Positive and negative numbers **2. Letters in algebraic expressions** **3. Equations** **4. Direct and inverse proportions** 5. Plane figures 6. Spatial figures 7. Variational data and representative values	2.How many matchsticks do we need? (p.50-51 /267) 3.Proportions (p.98-103 /267) 4.Direct and inverse proportions (p.106-137 /267)	1. From integers to rational numbers 2. The collaboration of rational numbers, algebra, and circle **3. We started with proportions** **4. Algebra and probability** 5. Mathematics in our lives 6. Geometry and measurement	3.Ratio and proportions (p.96-106/238) 4.Linear relationships (p.150-153/238) and Graphs of linear equations (p.158-161/238)
8th grade	1. Calculations with algebraic expressions 2. Systems of equations **3. Linear functions** 4. Parallelism and congruence 5. Triangles and quadrilaterals 6. Probability	3.Linear functions (p.52-87 /216)	1. From canonical figures to canonical numbers 2. Probability, statistics, and numbers **3. Triangles and algebra** 4. Journey in mathematics 5. Measurement in geometric figures and perspective **6. Starting point and end point: Geometry**	3.Number patterns (p.86-88/224) 6.Examination of a line (p.189-197/224)

APPENDIX C: ANALYSIS TABLES

Table C.1. Quantitative reasoning and the concepts of ratio, rate, and proportion.

Quantity	**Types of quantities**	**Concepts**	**Conceptions**
"A quantity is a quality of something that one has conceived as admitting some measurement process. Part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit." (Thompson, 1990, p.4)	Extensive Quantity: *"An extensive quantity is a quantity that may be measured directly or is a combination of directly measurable quantities.* Put another way, extensive quantities are quantities whose measures admit normal arithmetic (Cohen and Nagel, 1939)." (as cited in Thompson, 1990, p.5)	-	Identical group conception: The identical groups conception involves students' mental actions of the collection of sets of the extensive quantities or breaking down the ratio into equal parts; i.e. association of two extensive quantity (Heinz, 2000) Between-state ratio: a student having between state ratio conception might understand ratio as the amount of increment (Karagöz Akar, 2007).
	Intensive Quantity: *"Intensive quantities are quantities whose measures are non-additive as in normal arithmetic* (Cohen and Nagel, 1939)." (as cited in Thompson, 1990, p.5)	Ratio: *"A ratio is the result of comparing two quantities multiplicatively."* (Thompson, 1994, p.17) Rate: "A rate is reflectively abstracted constant ratio (Thompson, 1994, p. 18); i.e., *A rate is a quantity that may be analyzed into a multiplicative comparison between two other quantities—where one quantity's value is conceived as varying in constant ratio with variations in the value of the other." (Thompson, 1990, p. 7).*	Ratio-as- quantity: Ratio- as- quantity conception also requires multiplicative reasoning, but this conception requires understanding the invariant relationship between quantities indicating amount of change rather than defining a measure (Heinz, 2000). Ratio-as- measure: Ratio-as-measure conception requires the understanding of ratio as an intensive quantity such that the ratio itself indicates a measure that cannot be directly measured (Simon and Blume, 1994). Within-state ratio: Comparing the quantities existing in a situation. Within state ratio conception both includes and surpass the identical groups conception. Yet, at the early stages of within state ratios, students might not necessarily understand between state ratios as amount of change or enlargement. Therefore, ratio as measure conception surpass and include within state ratio conception (Karagoz Akar, 2007) Between state ratio: ratio is an enlargement/ amount of change in a quantity (Karagöz Akar, 2007).

Table C.1. Quantitative reasoning and the concepts of ratio, rate, and proportion (cont.).

Quantitative Operations	**List of Quantitative Operations**	**Examples of Quantitative Operations**	**Related Strategies Students use to approach ratio problems**	**Students' conceptions of ratio**
"A quantitative operation is the conception of two quantities being taken to produce a new quantity." (Thompson, 1990, p.9) There are 8 quantitative operations given in the next column.	Combine quantities additively	"Unite two sets; consider two regions as one." (Thompson, 1990, p.10)	Incorrect addition	Identical group conception
	Compare quantities additively	"How much more (less) of this is there than that?" (Thompson, 1990, p.10)	Incorrect addition	
	Combine quantities multiplicatively	"Combine distance and force to get torque; combine linear dimensions to get regions; combine force applied and distance travelled to get work." (Thompson, 1990, p.10)	Building up Abbreviated building up Ratio-unit/ build up	Identical group conception
	Compare quantities multiplicatively	"How many times bigger is this than that?" "This is (multiplicatively) what part of that?" "How many of these in those?" (Thompson, 1990, p.10)	Unit factor Ratio-unit/ build up Within strategy Between strategy Proportion equation strategy	Identical group conception Ratio-as-measure Ratio-as- quantity Between-state ratio Within-state ratio
	Generalize a ratio	"Suppose this comparison applies generally (i.e., suppose it were to continue at the same rate)." (Thompson, 1990, p.10)		Ratio-as-measure Within-state ratio
	Compose ratios	"Jim has 3 times as many marbles as Sally; Sally has 4 times as many marbles as Fred. Jim has so many times more marbles than Fred." (Thompson, 1990, p.10)		
	Compose rates	"A German mark is 75.53 Japanese yen. A US dollar is 1.88 marks. A dollar is some number of yen." (Thompson, 1990, p.10)		

Table C.1. Quantitative reasoning and the concepts of ratio, rate, and proportion (cont.).

Numerical Operations	**Types of numerical operations**	**Related quantitative operation (Thompson, 1990, p. 24)**	**Related Strategies Students use to approach ratio problems**	**Students' conceptions of ratio**
"Arithmetic operations are *numerical operations that are used to calculate a quantity's value*; there is no direct correspondence, except in a canonical sense, between quantitative operations and the arithmetic operations actually used to calculate a quantity's value in a given situation." (Thompson, 1990, p.10) There are 4 numerical operations: Addition, subtraction, multiplication and division.	Addition	A quantity is the result of an additive combination of two quantities	Incorrect addition (Repeated addition) Building up Ratio-unit/ Building up	Identical group conception
	Subtraction	A quantity is the result of an additive comparison of two quantities	Incorrect addition	
	Multiplication	A quantity is the result of a multiplicative combination of two quantities A quantity is the result of an instantiation of a rate A quantity is the result of a composition of ratios A quantity is the result of a composition of rates	Abbreviated building up Unit factor Within Between Proportion equation	Identical group conception Ratio-as-measure Ratio-as- quantity Between-state ratio Within-state ratio
	Division	A quantity is the result of a multiplicative comparison of two quantities	Unit factor Within Between Proportion equation	Identical group conception Ratio-as-measure Ratio-as- quantity Between-state ratio Within-state ratio

Table C.2. Covariational reasoning and linear functions.

Categories	**Sub-categories**	**Corresponding mental action(s)**	**Corresponding conception**
Static: "It refers to any instances of narratives or worked examples that do not reference quantities and relationships among them in ways that entail those quantities varying. For example, we code things as *static* when they entail instances that provide students images of variables and formulas based on perceptual associations among visual shape, analytic form, and perceptual features." (Taşova, Stevens, and Moore, 2018, p.1528)	Perceptual association: "This category has subcategories (i.e., form-name, form-shape, shape-name, and property-shape associations)" (Taşova, Stevens, and Moore, 2018, p.1528)		Action conception: • "A function is tied to a specific rule, formula, or computation and requires the completion of specific computations and/or steps. • A student must perform or imagine each action. • The "answer" depends on the formula. • A student can only imagine a single value at a time as input or output (e.g. x stands for a specific number). • Composition is substituting a formula or expression for x. • Inverse is about algebra (switch y and x then solve) or geometry (reflect across y=x). • Finding domain and range is conceived at most as an algebra problem (e.g. the denominator cannot be zero, and the radicand cannot be negative). • Functions are conceived as static. • A function's graph is a geometric figure." (Oehrtman, Carlson and Thompson, 2008, p.10)
	Variable as unknown: "*variables as unknown*, involves presenting a variable as having a fixed unknown value of a quantity or being only a visual symbol that is not varying in the way Thompson and Carlson (2017) categorized as "no variation" and "variable as symbol."" (Taşova, Stevens, and Moore, 2018, p.1529)		
	Correspondence: "the narratives provide instances in which there exists an established static link among numbers in sets, but there is no consideration of either the covariation of variables or the dynamic relationship between number of sets (Cooney and Wilson, 1993; Vinner and Dreyfus, 1989). We also coded instances as correspondence when they simply provide a rule for students to calculate a unique value of a variable or quantity by using any given value of another variable or quantity" (Confrey and Smith, 1994) (Taşova, Stevens, and Moore, 2018, p.1529)		Process conception: • "A function is a generalized input-output process that defines a mapping of a set of input values to a set of output values. • Inverse is the reversal of a process that defines a mapping from a set of output values to a set of input values. • Domain and range are produced by operating and reflecting on the set of all possible inputs and outputs." (Oehrtman, Carlson and Thompson, 2008, p.10)

Table C.2. Covariational reasoning and linear functions (cont.).

Categories	**Definition**	**Sub-categories**	**Definition**	**Corresponding mental action(s)**	**Corresponding conception**
Emergent: *"emergent,* identifies the narratives and worked examples representing various levels of varying and covarying quantities or variables based on Thompson and Carlson's (2017) outline of levels of reasoning variationally and covariationally. We adjusted those levels to fit a textbook analysis and used them as our sub codes under *emergent* to determine the level of opportunities provided in a written curriculum for students to develop quantitative and covariational reasoning." (Taşova, Stevens, and Moore, 2018, p.1530)	Variation	Continuous "C*ontinuous variation* involves presenting a variable or quantity whose values increase or decrease continuously" (Taşova, Stevens, and Moore, 2018, p. 1530- 1531)	Smooth: "The person thinks of variation of a quantity's or variable's (hereafter, variable's) value as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously. Chunky: The person thinks of variation of a variable's value as changing by intervals of a fixed size." (Thompson and Carlson, 2017, p. 434)		
		Gross *"Gross variation* involves presenting a variable or quantity whose values increase or decrease without mentioning the specific values of the variable or quantity while increasing or decreasing in the narratives." (Taşova, Stevens, and Moore, 2018, p. 1530)	"The person envisions that the value of a variable increases or decreases, but gives little or no thought that it might have values while changing." (Thompson and Carlson, 2017, p. 434)		

Table C.2. Covariational reasoning and linear functions (cont.).

Categories	**Definition**	**Sub-categories**	**Definition**	**Corresponding mental action(s)** (Carlson *et al.*,2002, 157-158).)	**Corresponding conception**
Emergent	Variation	Discrete *"Discrete variation* involves presenting a variable or quantity in the narratives as taking specific values, but without providing the opportunity for students to conceive the variable or quantity whose value varies in between those specific values." (Taşova, Stevens, and Moore, 2018, p. 1530)	"The person envisions a variable as taking specific values. The person sees the variable's value changing from a to b by taking values a1, a2, …, an but does not envision the variable taking any value between ai and ai + 1." (Thompson and Carlson, 2017, p. 434)		

Table C.2. Covariational reasoning and linear functions (cont.).

Categories	**Definition**	**Sub-categories**	**Definition**	**Corresponding mental action(s)** (Carlson *et al.*,2002, 157-158).)	**Corresponding conception**
Emergent	Covariation	Continuous *"Continuous covariation* involves instances providing a simultaneous and continuous change in the values of two variables or quantities." (Tașova, Stevens, and Moore, 2018, p.1530)	Smooth: "The person envisions increases and decreases (thereafter, changes) in one quantity's or variable's value, and the person envisions both variables varying smoothly and continuously." (Thompson and Carlson, 2017, p. 435). Chunky: "The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation." (Thompson and Carlson, 2017, p. 435).	"MA1- Coordination (Coordinating the value of one variable with changes in the other) MA2- Direction (Coordinating the direction of change of one variable with changes in the other variable) MA3- Quantitative Coordination (Coordinating the amount of change of one variable with changes in the other variable) MA4 – Average rate (Coordinating the average rate-of-change of the function with uniform increments of change in the input variable)" (Carlson et al., 2002, p. 357) (MA5, only in smooth)	Process conception: • "A student can imagine the entire process without having to perform each action. • The process is independent of the formula. • A student can imagine all input at once or "run through" a continuum of inputs." (Rest of them on the following part) (Oehrtman, Carlson and Thompson, 2008, p.10)

Table C.2. Covariational reasoning and linear functions (cont.).

Categories	**Definition**	**Sub-categories**	**Definition**	**Corresponding mental action(s)** (Carlson *et al.*,2002, 157-158).)	**Corresponding conception**
Emergent	Covariation	Coordination of values *"Coordination of values* involves instances of coordinating the values of one variable or quantity with values of another by providing specific and discrete pairs of values without providing the opportunity for students to conceive two variables or quantities whose value varies together in between those pairs of values." (Taşova, Stevens, and Moore, 2018, p.1530)	"The person coordinates the value of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x,y)." (Thompson and Carlson, 2017, p. 435).	"MA1- Coordination (Coordinating the value of one variable with changes in the other) MA2- Direction (Coordinating the direction of change of one variable with changes in the other variable) MA3- Quantitative Coordination (Coordinating the amount of change of one variable with changes in the other variable)" (Carlson et al., 2002, p. 357)	Process conception: • "Inverse is the reversal of a process that defines a mapping from a set of output values to a set of input values. • Domain and range are produced by operating and reflecting on the set of all possible inputs and outputs. • Functions are conceived as dynamic." (Rest of them on the following part) (Oehrtman, Carlson and Thompson, 2008, p.10)

Table C.2. Covariational reasoning and linear functions (cont.).

Categories	**Definition**	**Sub-categories**	**Definition**	**Corresponding mental action(s)** (Carlson *et al.*,2002, 157-158).)	**Corresponding conception**
Emergent	Covariation	Gross coordination *"Gross coordination of values* involves representing two variables or quantities whose values increase or decrease together without mentioning the individual values of variables as varying together in the narratives." (Taşova, Stevens, and Moore, 2018, p.1530)	"The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values." (Thompson and Carlson, 2017, p. 435).	"MA1 - Coordination (Coordinating the value of one variable with changes in the other) MA2- Direction (Coordinating the direction of change of one variable with changes in the other variable)" (Carlson et al., 2002, p. 357)	Process conception: • "A function's graph defines a specific mapping of a set of input values to a set of output values. • A function is a transformation of entire spaces. • Composition is coordination of two input-output processes; input is processed by a second function." (Oehrtman, Carlson and Thompson, 2008, p.10)
		Precoordination	"The person envisions two variables' values varying, but asynchronously- one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects." (Thompson and Carlson, 2017, p. 435).	"MA1 - Coordination (Coordinating the value of one variable with changes in the other.)" (Carlson et al., 2002, p. 357)	