THE NOTICING SKILLS OF NOVICE MATHEMATICS TEACHERS WHO HAVE DIFFERENT TEACHER PERSPECTIVES

by

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ABSTRACT

THE NOTICING SKILLS OF NOVICE MATHEMATICS TEACHERS WHO HAVE DIFFERENT TEACHER PERSPECTIVES

The aim of this study was to investigate novice teachers' perspectives and noticing skills simultaneously in order to determine the relation between them. To examine the research question "What are the noticing skills of novice teachers who hold different teacher perspectives?", a multi-case qualitative research study was conducted. Two participants were selected from twenty-six pre-service teachers who had taken a professional development methods course in 2016. For the analysis, these two participants' two- real classroom-teachings both as a pre-service teacher in 2016 and as a novice teacher in 2018, the pre-interviews prior to and the post-interviews upon completion of the teachings were analysed by using the characteristics of teacher perspectives. In addition, the pre-interviews and post-interviews were further analysed for determining the teacher noticing by using the codes of learning to notice frameworks. Results of the analysis showed that one of the participants had the characteristics of Progressive Incorporation Perspective (PIP) and the other had the characteristics of Perception Based Perspective (PBP) both as pre-service teachers and novice teachers. Similarly, results showed that the participant with PIP perspective were at the extended level of teacher noticing and noticed more events and explained the events in more detail, in addition, results from the pre-interviews showed that both participants noticed and explained significant aspects of their teaching in the lesson planning process albeit having different reasons. The consistency between the results from the data in 2016 in which the participants were pre-service teachers and in 2018 in which the participants were novice teachers further has suggested that there is a correspondence between teacher perspectives and teacher noticing skills.

ÖZET

FARKLI ÖĞRETMEN BAKIŞ AÇILARINA SAHİP DENEYİMSİZ MATEMATİK ÖĞRETMENLERİNİN FARK ETME BECERİLERİ

Bu çalışmanın amacı, deneyimsiz öğretmenlerin bakış açılarını ve fark etme becerilerini araştırmak ve aralarındaki ilişkiyi incelemektir. Araştırma sorusu, "Farklı öğretmen bakış açılarına sahip deneyimsiz öğretmenlerin fark etme becerileri nelerdir?" olan çalışma, çoklu-vaka yöntemi ile yapılan nitel bir araştırmadır. 2016 yılında öğretim yöntemlerine özgü bir ders alan yirmi altı öğretmen adayı arasından iki katılımcı seçilmiştir. Katılımcıların, hem 2016 yılında öğretmen adayı iken hem de 2018 yılında deneyimsiz öğretmen iken yaptıkları iki sınıf için öğretimleri, bu öğretim öncesi ve sonrası görüşmeleri öğretmen perspektifi özelliklerinden yararlanılarak analiz edilmiştir. Ayrıca, katılımcıların öğretim öncesi ve sonrası mülakatları, Fark Etmeyi Öğrenme çerçevesi de yer alan kodlar ile analiz edilmiştir. Veri analizi sonuçları, bu iki katılımcıdan birinin hem 2016 yılında öğretmen adayı iken hem de 2018 yılında deneyimsiz öğretmen iken ilerlemeci bakış açısı özelliklerine sahip olduğunu, diğerinin ise her iki yılda algı tabanlı bakış açısına sahip olduğunu bulgulamıştır. Ayrıca, sonuçlar, ilerlemeci bakış açısına sahip katılımcının Fark Etmeyi Öğrenme çerçevesinin dördüncü seviyesinde fark etme becerisine sahip olduğunu, algı tabanlı bakış açısına sahip olan katılımcının ise ikinci seviyede fark etme becerisine sahip olduğunu; ek olarak ilerlemeci bakış açısına sahip katılımcının algı tabanlı bakış açısına sahip katılımcıya göre öğretimler sırasında daha fazla noktayı fark ettiğini bulgulamıştır. 2016 yılı ve 2018 yılı verilerindeki tüm bu tutarlılıklar ise, öğretmen perspektifi ile öğretmen fark etme becerileri arasında anlamlı bir ilişki olduğunu göstermektedir. Buna ek olarak, her iki katılımcının da ders planlama sürecinde öğretimlerinin önemli yönlerini fark ettiğini ve farklı sebepler ile açıkladığını bulgulamıştır.

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LIST OF ABBREVIATIONS

A	Alin
CBP	Conception-Based Perspective
CCK	Common Content Knowledge
E	Elisa
H	How Teachers Notice
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KQ	Knowledge Quartet
MKT	Mathematical Knowledge for Teaching
PBP	Perception-Based Perspective
PCK	Pedagogical Content Knowledge
PIP	Progressive Incorporation Perspective
R	Researcher
S	Student
SCK	Specialized Content Knowledge
TP	Traditional Perspective
W	What Teachers Notice

1. INTRODUCTION

Many researchers agree on the significance of teacher knowledge for effective teaching (Ball, Thames and Phelps, 2008; Darling-Hammond, 2009; Hill, Rowan and Ball, 2005). Two important components of teacher knowledge, the pedagogical content knowledge and content knowledge, acts as the two sides of the balance of teaching; teachers need both of them to teach the content in appropriate ways for the students with different learning levels and different grades (Ball, Thames and Phelps, 2008; Hill, Rowan and Ball, 2005; Rowland, Huckstep and Thwaites, 2005; Shulman, 1986). Also, teacher knowledge is required to see, understand and select some events to respond to during teaching in the classroom.

Seeing, understanding and selecting events refers to noticing skills of teachers. While some researchers consider noticing as only for students' mathematical thinking (Barnhart and Van Es, 2015; Jacobs, Lamb and Philipp, 2010), some researchers consider noticing focusing on every aspects of teaching such as students' behaviours and thinking, classroom environment and mathematical content (Star and Strickland, 2008; Van Es and Sherin, 2002; Jacobs, Lamb and Philipp, 2010). Particularly, Jacobs and her colleagues (2010) defined the professional noticing of students' mathematical thinking with three features; attending to students' strategies, interpreting students' understanding and responding to students' understanding. In addition, Van Es and Sherin (2002) determined three features of noticing skills as; "identifying what is important or noteworthy about classroom situation, making connections between the specifics of classroom interactions and broader principles of teaching and learning they represent, using what one knows about the context to reason about classroom interactions". Extending the investigation of noticing skills of teachers, Van Es and Sherin (2002) developed learning to notice framework by pointing to "how" a teacher notices in addition to "what" a teacher notices in the classroom events. Therefore, as part of teaching act, noticing is accepted as a complicated and hard decision-making process such that both the attention and the choices of teachers for the events in classroom play a significant role in the quality of learning (Van Es and Sherin, 2002; Jacobs, Lamb and Philipp, 2010; Mason, 2002) .

In fact, previous research has indicated that mathematical knowledge of teachers and noticing skills are highly related with each other such that teachers with high mathematical knowledge for teaching show higher level noticing skills (Van Es and Sherin, 2002; Van Es and Sherin, 2008). Similarly, through improving mathematical knowledge for teaching and belief about teaching and learning, development of pre-service and novice teachers' noticing skills is emphasized by many researchers (Jacobs, Lamb and Philipp, 2010; Jacobs, Lamb, Philipp, Schappelle and Burke, 2007; Kılıç, 2016; Sherin and Van Es, 2009; Star and Strickland, 2008; Van Es and Sherin, 2002).

The relationship between improved mathematical knowledge and beliefs regarding teaching and learning and teachers' noticing skills necessitates the discussion on the relationship between knowledge and beliefs. In this study, I acknowledged that knowledge and belief are inseparable (Fennema, 1992; Leatham, 2006; Wilson and Cooney, 2002), this is because knowledge can be inherited only through the filter of an existing belief system (Kane, Sandretto and Heath, 2002; Richardson and Placier, 2001) such that one has knowledge of teaching mathematics effectively if s/he has reasonable evidence to support the structure of it and holds belief in it (Wilson and Cooney, 2002). Having such view on knowledge and beliefs, in this study I investigated the relationship between teacher perspectives and their noticing skills. Simon and Tzur (1999) defined perspective as "a conglomerate that cannot be understood by looking at parts split off from the whole (i.e., looking only at beliefs or methods of questioning or mathematical knowledge)".

Particularly, Simon *et al.* (2000) provided the field with the teacher perspective framework as '...an attempt to go beyond understanding particular knowledge and beliefs in the context of practice of teachers in transition? (p.580). Particularly, aligning with the Ernest's (1989) description of knowing, learning and teaching, the teacher perspectives framework includes Traditional Perspective (TP), Perception-Based Perspective (PBP), and Conception-Based Perspective (CBP) (Heinz *et al.*, 2000; Simon *et al.* 2000; Tzur *et al.*, 2001) together with the Progressive Incorporation Perspective (PIP)

(Jin and Tzur, 2011). Previous research also has shown that teacher perspectives are related with mathematical knowledge of teachers such that pre-service teachers holding PIP perspectives shows higher-level mathematical knowledge for teaching (Bukova *et al.*, 2019; Karagöz, 2016). As mentioned earlier, research also has shown that teachers who have higher mathematical knowledge for teaching have high noticing skills (Jacobs *et al.*, 2007; Star and Strickland, 2008; Van Es and Sherin, 2008). Similarly, professional development influences teachers' mathematical knowledge for teaching (Ball *et al.*, 2008; Smith, 2001), which also influence teachers' noticing skills (Jacobs *et al.*, 2010; Van Es and Sherin, 2002). When analysing the characteristics of the teacher perspectives (Bukova *et al.*, 2019) and the codes of the teacher noticing in the learning to notice framework (Van Es and Sherin, 2002), both of the two frameworks showed some similar characteristics such as students' understanding, teachers' focus, teachers' attending and teachers' decisions during teaching. In addition, participants of this study took a professional development methods course that was prepared to develop the progressive incorporation perspective on the part of pre-service teachers during their practicum year. Talking about nature of mathematics, conceptual understanding, quantitative reasoning, quantitative operations and numerical operations during the professional development methods course could have afforded the development of different teacher perspectives. Therefore, in this study, I hypothesized that through the multi-case design with the same participants both during their preservice education and during the first years of teaching, investigation of the relationship between teacher noticing and teacher perspectives of novice teachers was significant: Studying the relationship between their perspectives and noticing skills might provide valuable information to the field in terms of what, how and why they notice in classroom events as well as how to prepare pre-service teachers during preservice education.

2. LITERATURE REVIEW

In this section, first, I will explain teacher knowledge by discussing different frameworks. Then, I explain the main framework this study was dwelt on namely teachers' noticing skills. By pointing to previous research, I also discuss in detail the factors affecting teacher noticing.

2.1. Teacher Knowledge

By industrial and technological development, teaching became a significant factor for countries success. The first investigations were made about students' achievement to understand the effectiveness of teaching before the 1980s (Cochran-Smith and Fries, 2005; Darling-Hammond, 2009). By the expansion of studies, teachers and teachers' characteristics were seen another main source in addition to students' achievement informing about effective teaching (Shulman, 1986). Teaching that is the significant process for effective learning is a complex task. Teaching requires various knowledge types rather than only one variable. Results of many studies indicated that teacher knowledge is the corner stone of effective teaching and learning (Ball *et al.*, 2008; Hill *et al.*, 2005; Shulman, 1986). Shulman (1986) argued that although teachers' strong content knowledge of mathematics is crucial for effective teaching, the pedagogical content knowledge of teachers cannot be overlooked and seen unnecessary for students' learning. Shulman (1986) believed that teachers' content knowledge and pedagogical knowledge must be observable together in teachers' actions and choices before, during and after the instruction. Therefore, for effective teaching, teacher must combine both content knowledge and pedagogical knowledge in order to transfer the content in pedagogically powerful ways (Shulman, 1986).

Shulman asserted that teacher knowledge constitutes three aspects: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. He stated that subject matter content knowledge refers to the knowledge of subject in the mind of the teacher. The facts, concepts, and procedures of a domain as well as understanding the underlying structures of a domain are seen within the realm of subject matter knowledge of teachers. Though, the pedagogical content knowledge refers to "dimension of subject matter for teaching". Specifically, two dimensions of pedagogical content knowledge are "knowing the ways of representations of subject matter" and "knowing students' learning". The first dimension, as part of pedagogical content knowledge, includes preferring appropriate examples, representations or materials to make the subject easier for students' understanding. The second dimension includes knowing misconceptions, preconceptions, difficulties or strategies of students and handling these difficulties. Finally, the last knowledge type, the curricular knowledge, refers to the knowledge about the content in curriculum (Shulman, 1986).

Building on Shulman's (1986) pioneering ideas, researchers focused on different dimensions of teacher knowledge. Peculiar to the mathematics education field, two significant frameworks, Mathematical Knowledge for Teaching and Knowledge Quartet were developed.

Particularly, agreeing with Shulman about the investigation of both teachers and students rather than solely focusing on students' achievement to examine the effectiveness of teaching. Ball, Thames, and Phelps (2008) expanded teacher's knowledge types and developed Mathematical Knowledge for Teaching (MKT) framework. The framework was comprised of four domains (See Figure 2.1): 1) Common Content Knowledge (CCK), 2) Specialized Content Knowledge (SCK), 3) Knowledge of Content and Students (KCS) and 4) Knowledge of Content and Teaching (KCT) (Ball, Thames and Phelps, 2008).

Similar to Shulman (1986), common content knowledge refers to the general mathematical knowledge used in different areas in addition to teaching settings. Therefore, this knowledge is not specific to only teachers but is for all adults in different professions. Ball and her colleagues also agreed with Shulman (1986) about the nature of the knowledge of content and curriculum. Though, they emphasized that specialized content knowledge is the kind of mathematical knowledge only specific to teaching since it includes paying attention to and understanding what students' need and based on that using different ways to make content clear and learning easier. The third knowledge type includes knowledge about both content and students so that teachers can motivate and engage students in the content by listening to and interpreting their thoughts. The fourth type of knowledge, knowledge of content and teaching, refers to teachers' use of knowledge about mathematics and teaching such that they can determine and organize the instructional design of the lesson, the sequence of the content and the representations (Ball, Thames and Phelps, 2008).

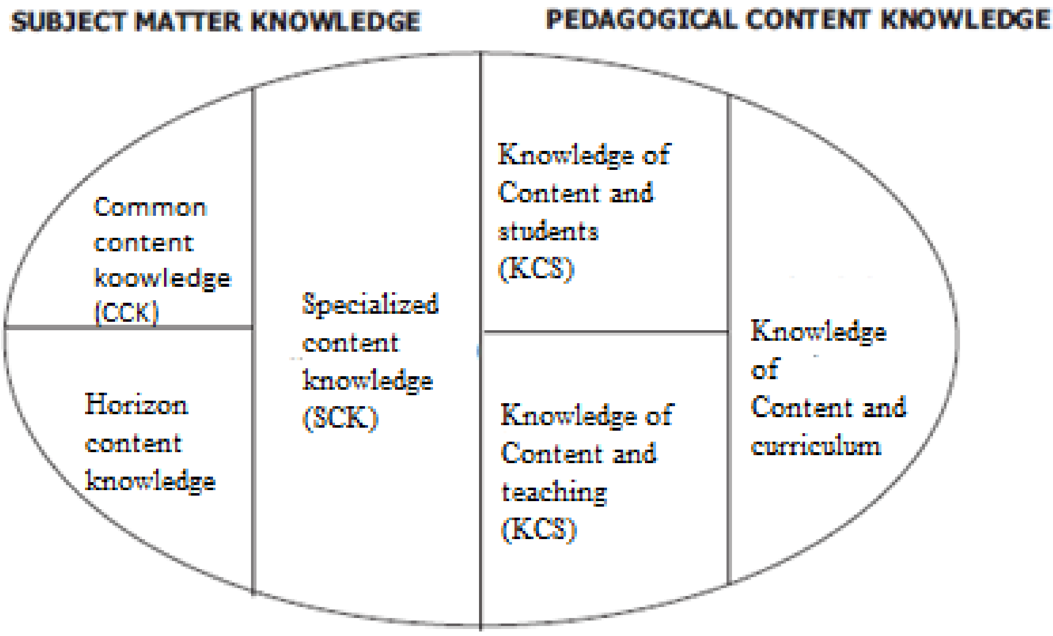

Figure 2.1. Domains of Mathematical Knowledge for Teaching.

Another framework that was built on Shulman's (1986) subject matter knowledge and pedagogical content knowledge dimensions of PCK model was the Knowledge Quartet (KQ) framework. Rowland, Huckstep and Thwaites (2005) examined teaching and learning processes through observations of pre-service teachers' actions in classrooms, so that what teachers have or do not have in terms of mathematical knowledge for teaching could be reflected. Knowledge for teaching were divided into four categories: foundations, transformation, connection and contingency (Rowland, Huckstep and Thwaites, 2005) Table 2.1.

Table 2.1. The codes of Knowledge Quartet.

Categories	**Codes**
Foundation	Awareness of purpose
	Adheres to textbook
	Concentration on procedures
	Identifying errors
	Overt display of subject knowledge
	Theoretical underpinning of pedagogy
	Use of mathematical terminology
Transformation	Choice of examples
	Choice of representation
	Use of instructional materials
	Teacher demonstration
Connection	Anticipation of complexity
	Decisions about sequencing
	Making connections between procedures
	Making connections between concepts
	Recognition of conceptual appropriateness
Contingency	Deviation from agenda
	Responding to students' ideas
	Use of opportunities
	Teacher insight during instruction

The foundation category includes knowing the nature of mathematics, having a rationale behind "why" to teach mathematics and the beliefs about teaching and learning mathematics. Therefore, not only the content knowledge but also pedagogical knowledge is emphasized in the foundation category. The second category, transformation refers to the knowledge of representations of the content so that transferring the topic could be easier for students' understanding. Therefore, knowledge of materials, illustrations, analogies and questions are included in the category. In addition to different representations, explanations and demonstrations of a teacher are also seen as significant specific to this category. The connections category then includes making connections between the sequence of the instructional topics, tasks and concepts. Therefore, determination of the appropriateness of the sequence and the complexity of the instruction are important aspects. Finally, contingency category refers to the unexpected events during instruction such as students' answers for a particular question or a question from the students. Researchers have emphasized that decisions of a teacher for unexpected events are significant because deviation from agenda or using unexpected events as an opportunity during instruction influence the effectiveness of teaching (Rowland, Huckstep and Thwaites, 2005). Therefore, teacher knowledge constitutes the roots of effective teaching (Ball, Thames and Phelps, 2008; Hill, Rowan and Ball, 2005; Shulman, 1986)which also influences teachers' noticing skills (Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2002).

2.2. Teachers' Noticing

Teachers' noticing skills is one of the new focus for research in teacher education. Though, there is no consensus on the definition, what was included in, extension and measurement of noticing skills (Jacobs, Lamb and Philipp, 2010; Sherin *et al.*, 2008; Star and Strickland, 2008; Van Es and Sherin, 2002). Sherin, Jacobs, and Philipp (2011) defined teachers' noticing skill as "attending to particular events in an instructional setting and making sense of events to act according to these events" (p.5). Teacher noticing can be seen as the relationship between what teachers' pay attention during instruction and how they act based on these specific events (Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2002). In the following paragraphs, by pointing to previous research, I explain the development of different aspects of teachers' noticing frameworks (Jacobs, Lamb and Philipp, 2010; Santagata, Zannoni and Stigler, 2007; Sherin and Van Es, 2009; Star and Strickland, 2008; Van Es and Sherin, 2002).

The base of teachers' noticing depends on the Goodwin's (1994) concept of professional vision. Goodwin (1994) described the professional vision as a way to see and understand the events to respond to the interests of a particular social group.

In mathematics education, this concept can be adapted as the teacher's organizations of their actions for their instruction. The concept of professional vision is used as a root for teacher observation and interpretation of events by Van Es and Sherin (2002). The foci of these researchers were the mathematical thinking of students, pedagogy of instruction, climate of the classroom and management in the instructional setting.

Particularly, mathematical thinking of students that includes students' thinking, ideas and understanding is the most significant aspect of the teacher noticing. Similarly, pedagogy of instruction involves the investigation of the methods, strategies or manipulatives that are necessary and appropriate to teach content. In addition, climate of classroom refers to the environment of the class and students' emotional conditions seen and controlled by the teacher. Finally, management refers to teacher's actions for the unexpected conditions in class so it is significant for teachers' noticing (Van Es and Sherin, 2002; Van Es and Sherin, 2008).

Considering all these aspects as important and investigating them, Van Es and Sherin (2002) created "Framework for Learning to Notice Student Mathematical Thinking" to show the progression of teachers' noticing skills during a professional development program. The three parts of this framework were:

a) Identifying the significant events in a teaching situation, b) using knowledge from one's context to reason about these events, c) making connections between specific events and broader principles of teaching and learning (Van Es and Sherin, 2002; Van Es and Sherin, 2008).

The first component refers to teachers' need to attend to what students think, understand and do in an instructional setting, and which materials, representations, analogies or experiences are beneficial for students learning. The second characteristics refers to the ability of creating the relationship among the specific events such as students' ideas, learning types, misconceptions and teaching. The last step, understanding the reasons of these specific situations through the filters of teacher's knowledge about students' characteristics, content or context is also vital for teachers to attend to for effective teaching (Van Es and Sherin, 2002).

Regarding teacher actions, Sherin and Van Es (2009) further divided teacher noticing in two distinctive categories: teachers' selective attention and teachers' knowledge - based reasoning. Teachers' attending to noticing events during instruction shows selective attention such that includes teacher noticing of what is significant to respond to or insignificant to ignore during instruction (Sherin and Van Es, 2009). Attending to events in instruction also refers to "situational awareness". On the other hand, the knowledge-based reasoning shows teachers' knowledge for noticing all parts of the instruction. It is in this respect that teachers' MKT and experience becomes significant due to the fact that teacher noticing based on knowledge-based reasoning gives more reliable results rather than their focusing on events haphazardly during instruction. Still, researchers prefer investigating teachers' selective attention and teachers' knowledge-based reasoning separately to see teacher noticing but distinguishing noticing types during the instruction is difficult due to the speed of lesson and students. To distinguish these types and examine see teacher noticing, based on "learning to notice" framework and by changing some aspects of it, many researchers have studied various aspects of teacher's noticing (Miller and Zhou, 2007; Santagata, Zannoni and Stigler, 2007; Star and Strickland, 2008).

Particularly, similar to Van Es and Sherin's (2002) study, the noticing skills of pre-service teachers' are investigated in different aspects (Star and Strickland, 2008). Star and Strickland (2008) divided teacher noticing of various events into five pieces: classroom environment, management, tasks, mathematical content and communication. The results of the study showed that by attending to the method course pre-service teachers' observation skills increased especially in teachers' ability to notice features of the classroom environment, mathematical content of a lesson, and teacher and student communication (Star and Strickland, 2008). Similarly, taking into consideration the effect of the cultural nature of teaching and learning on the noticing skills of teachers, Santagata, Zannoni, and Stigler (2007) developed The Lesson Analysis Framework in Chinese teaching context by focusing on the learning goals of instruction, different teaching strategies and students' learning regarding these goals. Such framework allowed to observe the differences between noticing skills of the United States teachers and Chinese teachers such that researchers found out that Chinese teachers pay attention to and notice mathematical content of the lessons rather than pedagogical aspects of the lessons. In contrast, teachers in the United States focus mostly on pedagogical aspects of the instruction (Miller and Zhou, 2007).

Differing from the aforementioned research focusing on all aspects of the instruction or instructional setting, Jacobs, Lamp, and Philipp (2010) focused solely the "professional noticing of children's mathematical thinking". They regarded mathematical thinking of students as the most significant issue in instructional setting considering that students constitute the most valuable source for investigating teacher noticing. They also argued that focusing on one aspect of instructional setting might have provided more detailed information with more reliable results rather than investigating many issues. Therefore, their main concern was "how teachers notice" and "to which extent" because they are not curious about only "what teachers notice" in instruction (Jacobs, Lamb and Philipp, 2010). In-the-moment noticing indicated to teacher noticing special events of instruction in order to decide on the responses for these events. Therefore, regarding in-the-moment noticing, Jacob and his colleagues developed "professional noticing of children's mathematical thinking" framework. This framework included three skills a) attending to students' strategies, b) interpreting students' understanding and c) deciding on how to respond students' understanding. Researchers argued that teachers first attend to and interpret students' understanding and then upon such noticing for taking initiative for the further steps during teaching. Taking these actions are important because teachers' decisions to respond to students' understanding and scaffolding based on such interpretations influence students' further thinking and understanding (Jacobs, Lamb and Philipp, 2010; Kılıç, 2016).

2.2.1. Previous Research on Teachers' Noticing Skills

In this section, I report on previous research on teacher noticing skills in two ways: by pointing to techniques used to examine teacher noticing skills and by pointing to the factors affecting teachers' noticing skills.

Teachers' noticing skills can be observed by using different techniques (Ainley and Luntley, 2007; Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2002). Particularly, the video analysis method (Chao, Murray and Star, 2016; Levin, Hammer and Coffey, 2009; Van Es and Sherin, 2002), the written scenarios (Dreher and Kuntze, 2015; Jacobs, Lamb and Philipp, 2010)and the retrospective analysis are the most used techniques to investigate teachers' noticing skills (Ainley and Luntley, 2007; Borko and Livingston, 1989; Rosaen *et al.*, 2008).

Many researchers concur that the most useful method to observe teachers' noticing skills is the video analysis (Chao, Murray and Star, 2016; Kılıç, 2016; Levin, Hammer and Coffey, 2009; Sherin and Van Es, 2009; Van Es and Sherin, 2002). In video analysis, teachers might either watch a video record on their own instruction (Jacobs, Lamb and Philipp, 2010; Kleinknecht and Groschner, 2016) or another teacher's instruction (Kılıç, 2016; Star and Strickland, 2008). Lesson Study videos of Japanese teachers showed that video analysis gives a significant opportunity to teachers to improve themselves for noticing skills about students' mathematical thinking (Stigler and Hiebert, 1999). Though, there is still no consensus about whether teachers should watch their own instruction or other teachers' instruction. Some argue that teachers who knows the characteristics of students and instructional conditions attend more to special events and interpret them more easily during instruction (Jacobs, Lamb and Philipp, 2010; Kleinknecht and Groschner, 2016). In addition, they can attend to more details in the instruction and interpret the events more profoundly; however, when teacher watch and interpret another teacher' instruction video recording, they can miss some points due to lack of knowledge about the teacher, students and instructional setting in the video recording (Sherin, Russ and Colestock, 2011). Still, some others argue that watching video clips of another teacher's instruction for video analysis might also be beneficial (Kılıç, 2016; Star and Strickland, 2008; Van Es and Sherin, 2008).

Video analysis is not the only method for investigation of teachers' noticing. Some researchers evaluated teachers' noticing skills by giving scenarios or dialogues of a teacher and students in written form without any video recording (Dreher and

Kuntze, 2015; Jacobs *et al.*, 2007). By reading the written material of the teacher and students' dialogues including questions of the teacher, students' answers or students' mathematical thinking, teachers can notice special events in the scenario and interpret these events based on their own knowledge. Therefore, what teachers attend to in those can be collected in written or asking teachers to talk about the scenarios. Though, teachers' oral reflections on the scenarios give more reliable results about teachers' noticing skills rather than their written reflections since teachers talk about more details of special events (Jacobs *et al.*, 2007).

The third and the oldest techniques to conceive teacher noticing is retrospectively recalling of teachers' own instruction. By asking the interpretations of teachers about their own instruction either without giving any visual reminder (Borko and Livingston, 1989) or giving their instruction video (Ainley and Luntley, 2007; Rosaen, Lundeberg, Cooper, Fritzen and Terpstra, 2008) significant information regarding what teachers see and think about special events in their instruction could be gathered. The interpretation of teachers' own instruction could be obtained through post-interviews with either the teacher of the specific instruction or with a group of teachers discussing one of the instructions (Sherin, Russ and Colestock, 2011).

2.2.1.1. Factors Affecting Teachers' Noticing Skills. Previous research has pointed that teachers' knowledge (Jacobs, Lamb and Philipp, 2010), teaching experience (Levin, Hammer and Coffey, 2009), and beliefs about teaching and learning (Star and Strickland, 2008) influence teacher noticing. In addition to these, the teacher education program and professional development of teachers play a key role in the development of teachers' noticing skills (Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2002). Following, I discuss each of them in detail.

Teacher knowledge is the most significant aspect of teacher noticing since both content knowledge and pedagogical knowledge affect teacher noticing of special events in instructional setting (Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2002; Thomas *et al.*, 2017). This is because teachers attend to and evaluate situations by using their knowledge (Ball, Thames and Phelps, 2008). Although Ball and Bass (2009) did not name teachers' noticing skills as a framework, they particularly mentioned noticing skills among teaching responsibilities of a teacher:

Making judgments about mathematical importance; Hearing mathematical significance in what students are saying; Highlighting and underscoring key points; Anticipating and making connections; Noticing and evaluating mathematical opportunities (Ball and Bass, 2009) (p.6).

Some other researchers also pointed that teacher noticing and teacher knowledge have a mutual relationship since the development of each improves the other's characteristics (Dreher and Kuntze, 2015; Jacobs, Lamb and Philipp, 2010; Thomas *et al.*, 2017). Specifically, knowledge of i) students' misconceptions, ii) different teaching strategies, iii) content of the lesson or instructional setting requires a significant knowledge to notice special events in instruction (Schoenfeld, 2011; Sherin, Russ and Colestock, 2011). Knowledge of these areas refer to teachers content knowledge, specialized content knowledge, knowledge of content and students and knowledge of content and teaching (Ball, Thames and Phelps, 2008). In addition, some researchers pointed that MKT scores of teachers gives information about teachers' noticing skills since teachers having low scores on the tests measuring teachers' MKT have difficulty in noticing and interpreting special events during instruction (Schoenfeld, 2011; Thomas *et al.*, 2017). Thomas and his colleagues hypothesised that noticing of pre-service teachers can be improved by the development of teachers' MKT. The results of the study indicated that there is a significant correlation between the MKT scores and the professional noticing scores of pre-service teachers (Thomas *et al.*, 2017).

Dreher and Kuntze (2015) mainly focused on different types of teacher knowledge to investigate teachers' noticing skills. Their findings showed that in-service teachers attended to more specific events about students' thinking during instruction more than pre-service teachers. Therefore, they pointed to necessity of professional development for the development of teacher noticing (Dreher and Kuntze, 2015). In fact, research has shown that in professional development studies aiming at developing knowledge of teachers, attending to students' strategies, understanding reasons behind students' thinking and the response of teacher on students' thinking has been improved, which in turn has improved teachers' noticing skills (Jacobs, Lamb, Philipp, Schappelle and Burke, 2007; Jacobs *et al.*, 2008). Particularly, Jacobs and her colleagues (2010) worked with four groups of teachers: pre-service teachers, teachers who began to attend to professional development program, teachers who attended to professional development program for 2 years, and teacher who attended to 4 or more years. Jacobs, Lamb, and Philipp (2010) study showed that although not on the same level, pre-service, novice or expert teachers who attended to professional development programs improved both their knowledge and noticing skills. In addition, as founders of the video analysis methods to show teachers' noticing skills, Van Es and Sherin (2002) examined the video clubs as a professional development program. Results of the study with video club showed that discussing with a group of teachers about noticing skills of a teacher in the video is beneficial for each teacher's content knowledge and pedagogical knowledge (Van Es and Sherin, 2002; Van Es and Sherin, 2008).

Researchers have also pointed that teaching experience of teachers plays a key role in teacher noticing (Barnhart and Van Es, 2015; Choppin, 2011; Dreher and Kuntze, 2015; Kılıç, 2016; Levin, Hammer and Coffey, 2009; Taylan, 2014; Thomas *et al.*, 2017). Specifically, researchers have asserted that the years of experience change teachers' observation and noticing since spending more time with students, the noticing skills of teacher also increase (Dreher and Kuntze, 2015; Jacobs, Lamb and Philipp, 2010). Investigating the effect of teaching experience on teacher noticing some studies focused either on one pre-service (Barnhart and Van Es, 2015; Kılıç, 2016; Van Es and Sherin, 2002) or in-service teachers' noticing skills (Choppin, 2011; Taylan, 2014; Van Es and Sherin, 2008), some studies focused both on pre-service and in-service teachers (Dreher and Kuntze, 2015; Jacobs *et al.*, 2007; Jacobs, Lamb and Philipp, 2010; Star and Strickland, 2008) or novice teachers' noticing skills (Levin, Hammer and Coffey, 2009; Jacobs, Lamb and Philipp, 2010; Taylan, 2014)

Results have shown that pre-service teachers face with significant problems on noticing of students' mathematical thinking or special events during instruction (Barnhart and Van Es, 2015; Star and Strickland, 2008; Van Es and Sherin, 2002). Similarly, Van Es and Sherin (2002) study showed that a group of pre-service teachers who did not have any formal teaching experience although taking a secondary mathematics methods course had difficulty in attending to and explaining special events. In addition, Barnhart and Van Es (2015) study indicated that pre-service teachers who had attending skills did not guarantee analyses or responses to the students' thinking. Similarly, although some pre-service teachers attend to students' thinking and special events during instruction, their explaining and responding skills to these events was not observed by researchers (Barnhart and Van Es, 2015).

Compared to research on pre-service teachers, research on in-service teachers' noticing skills showed different results due to the years of experience (Berliner *et al.*, 1988; Star and Strickland, 2008). Berliner focused on both three teacher groups; pre-service, novice and in-service teachers. The results of their study showed that more experienced teachers attended to special events especially students' thinking and interpreted in detail the noticing event during instruction (Berliner, *et al.*, 1988). Though, novice teachers might also be seen as pre-service teachers since one- or two-years of experience might not be sufficient for noticing compared to an expert teacher (Kagan, 1992). Therefore, novice and experienced teachers might have differences in terms of teacher knowledge and some teaching skills (Rodríguez and McKay, 2010).

Researchers also pointed that differences in pre-service and in-service teachers' noticing skills might be due to their belief systems since in-service teachers who spend more time with students in lessons or teach a topic several times might have developed many beliefs about students' learning or their own teaching (Dreher and Kuntze, 2015; Star and Strickland, 2008; Thomas *et al.*, 2017). In particular, results of the studies have shown that development of a belief structure helps teachers to notice more significant details and to act on noticed events (Star and Strickland, 2008; Thomas *et al.*, 2017). Though Levin, Hammer, and Coffey (2009) and Taylan (2014) studies indicated that novice teachers could attend to students' thinking and features of instruction by some development. Particularly, Levin *et al.* (2009) stated that the professional relationship with other teachers, co-planning, and co-teaching or teacher education programs are seen necessary for improvement of novice teachers' noticing skills because teachers need encouragement and learning of noticing special events during instruction. Similarly, Taylan (2014) study showed that novice teachers could notice students' thinking who attended once teacher education programs focus on understanding students' thinking.

In sum, using different techniques such as video analysis, written scenario analysis and retrospectively- recall, researchers not only developed different aspects of teachers' noticing skills but also determined factors effecting noticing skills. Particularly, teacher knowledge (Jacobs, Lamb and Philipp, 2010), teaching experience (Berliner, *et al.*, 1988) and beliefs about teaching and learning mathematics (Star and Strickland, 2008) were found to be effective on teachers' noticing special events during instruction. Similarly, some research showed that professional development and teacher education programs were significant sources for improvement of pre-service, novice and in-service teachers' knowledge and beliefs about teaching and learning mathematics, which in turn had effects on their noticing skills (Berliner *et al.*, 1988; Jacobs, Lamb and Philipp, 2010; Star and Strickland, 2008; Taylan, 2014).

3. CONCEPTUAL FRAMEWORK

In this section, I explain the two frameworks this study was based on, namely, the Learning to Notice Framework and the Teacher Perspectives Framework. I also point to the rationale behind the relationship between these two frameworks.

3.1. Learning to Notice Framework

Before explaining Learning to Notice Framework, it is important to emphasize the three significant characteristics of teacher noticing Van Es and Sherin (2002) elaborated on:

(a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom interactions (p.573).

They argued that due to the complex structure of teaching in the moment of instruction, teachers might not be able to easily attend to significant issues regarding both students' thinking or understanding, the appropriate representations or materials and teaching strategies for the students' engagement simultaneously. That is why teachers' noticing is needed to attend and select significant events for responding appropriately. Therefore, teachers need some "check points" during instruction such that they can determine the next step in the instruction by identifying what is significant among all variables to decide how they can be act based on special events (Van Es and Sherin, 2002). Yet, attending to special events is not sufficient since solely attending to any event or describing the events without any connection do not show any noticing skills (Jacobs, Lamb, and Philipp, 2010; Van Es and Sherin, 2002). That is, teachers should also understand and be able to describe the possible reasons behind those events by considering the relationships between those specific events and principles of teaching and learning (Van Es and Sherin, 2002). Interpreting specific events with teachers' using their own knowledge and teaching and learning principles is considered as the cornerstone of noticing (Jacobs, Lamb and Philipp, 2010; Putnam and Borko, 2000) because teachers who notice the reasons behind specific events can explain and interpret their thoughts in more detail (Jacobs, Lamb and Philipp, 2010). In addition, all events are connected to each other within the realm of instruction as a whole such that explaining specific events through a principle is essential to acknowledge the general picture of the instruction (Shulman, 1986). Therefore, teachers need to be able to connect what they know and the specific events in the instruction. Specifically, using their content knowledge, knowledge of content and students or knowledge of content and teaching they need to be able to determine the reasons behind specific events (Van Es and Sherin, 2002).

Based on the aforementioned characteristics, Van es and Sherin (2002) developed Learning to Notice framework to show teachers' noticing skills attending to a professional development program. By the development of framework, Van Es (2011) focused on both "what teachers notice" and "how teachers notice" in an instructional setting. The target of "what teachers notice" question was to understand the focus of teachers in terms of students and the topic specific to the teaching process; while the target of "how teachers notice" question was to show teachers' "analytic stance" and "depth of analysis" by investigating their evaluating and interpreting skills of significant events in the instructional process. Four levels of teacher noticing were determined for learning to notice framework: 1) Baseline Level, 2) Mixed Level, 3) Focused Level, 4) Extended Level (See Table 3.1) (Van Es and Sherin, 2002).

Some significant issues indicated according to the results of the study. First, the levels of what teachers noticed and the levels of how teachers noticed were the approximately same. Teachers who attended all issues and topics in the class cannot explain their observations with detail and evidence. What is noticed and how is noticed is highly related to what teachers observed in the lesson so by learning to attend specific events or students' mathematical thinking, teachers can explain their observation and reason behind their actions. Second, the attending video club affects teacher noticing skills positively and increase the teachers' level from Baseline Level (Level 1) to

Extended Level (Level 4). The reason behind this development is not examined by researchers but they refer to some possible reasons for teachers' development such as videos that are watched by teachers, working together, discussion among teachers and the teachers' analyses for the video. Lastly, the participants of the project were a group of teachers and the results showed their development. The researchers emphasized that another group or an individual teacher can be examined to see the progress or alternative changes in their noticing skills because of different dimensions.

In the following paragraphs, I explain the characteristics of the levels for what teachers notice and how teachers notice from the learning to framework. I give the details of the levels and some specific examples from the codes to indicate how I used the codes of learning to Notice framework in this study.

Table 3.1. Framework for Learning to Notice Student Mathematical Thinking - What and How Teacher Notice (Van Es and Sherin, 2002).

	Level 1 Baseline	Level 2 Mixed	Level 3 Focused	Level 4 Extended
What Teachers Noticed (W)	1-A Attended to whole class environment, behaviour, and learning and to teacher pedagogy	2-A Primarily attend to teacher pedagogy Begin to attend to particular students' mathematic al thinking and behaviours	3-A Attend to particular students' mathematical thinking	4-A Attend to the Relationship between particular students? mathematical thinking and between teaching strategies and student mathematical thinking

Table 3.1. Framework for Learning to Notice Student Mathematical Thinking - What and How Teacher Notice (Van Es and Sherin, 2002) (cont.).

	Level 1 Baseline	Level 2 Mixed	Level 3 Focused	Level 4 Extended
How Teachers Notice (H)	1-A Form general impressions of what occurred	2-A Form general impressions and highlight noteworthy events	3-A Highlight noteworthy events	4-A Highlight noteworthy events
	1-B Provide descriptive and evaluative comments	2-B Provide primarily evaluative with some interpretive comments	3-B Provide interpretive comments	4-B Provide interpretive comments
	1-C Provide little or no evidence to support analysis	2-C Begin to refer to specific events and interactions as evidence	3-C Refer to specific events and interactions as evidence	4-C Refer to specific events and interactions as evidence
			3-D Elaborate on events and interactions	4-D Elaborate on events and interactions
				4-E Make connections between events and principles of teaching and learning
				4-F On the basis of interpretations, propose alternative pedagogical solutions

As the Table 3.1 shows, at Level 1, Baseline Noticing, teachers focus on everything in class without any specific selection such as students' behaviours, classroom climate, participation, teacher pedagogy and students' learning. The teachers' main focus is the whole class rather than particular students, so the whole class is called they, the group, or children? "They were on task" or "They all wanted to volunteer". In how they notice part, the participants give general impressions about their observation from the class. They only describe and evaluate the class such as "They were not making disruption". "The students are so focused", or "I was amazed by the vocabulary". The teachers provide little or no evidence from the classroom events to support their comments.

In Level 2, Mixed Noticing, teachers focus mainly on the teacher pedagogy in the class, but they also began to attend to students' mathematical thinking. The teachers' comments refer to the whole class, but some specific students' thinking can be attended in the class. Although the teachers give general impressions from the class like Baseline Level, some the noteworthy events can be noticed. For instance, one teacher can refer a student and say "He may not understand tenths and hundredths". They attend specific events or students' thinking in class, but they cannot explain in detail or giving evidence for their observation from the class ("He's a visual learner" or "They do not get it").

Level 3, Focused Noticing shows a significant shift from first two levels, because teachers began to notice specific students and noteworthy events in the class. The mathematical thinking of specific students can be attended and interpreted in detail. The teachers can give specific examples from students' thinking and support their observation about students' thinking by evidence from the transcripts or video record of the lesson. In Van Es work, teachers discuss a student's answer and one of them said that "she was trying to estimate the answer" and the other teacher attend another aspect of the student's solution and said, "It looks like she is mixing it up with another method". The most important feature that separates Focused Noticing from Level 2 is that the teachers can notice and interpret the methods which are used by particular students ("She started with the conventional method of division and then got stuck, and so she's going to this other method, partial products").

In Extended Noticing, Level 4, the teachers show the same characteristics with Level 3, so examine particular students' mathematical thinking, explain their observations by interpreting in detail and give evidence from the class. In addition to these focus points, the teachers not only focus on students' thinking and also notice teacher pedagogy to analyse these two aspects of the lesson together. The main difference from Level 3 is that the teachers try to make connections between special events and teaching approaches used in the lesson. They explore the teacher moves which help students' thinking or create opportunities to students for new thinking approaches. For instance, when they discuss with a group of teachers, one of the teachers said that "When you asked to her to share her solution again, it seemed like she could better explain it." The main purpose of the teachers in Level 4 is the development of alternative teaching approaches according to students' mathematical thinking, so their interpretations refer to teacher moves in the lesson. In addition, the teachers make connection between the teacher actions that they observed in the class and the teaching and learning principles, so their discussions based on broader issues such as assessment, or equity in class. One of the teachers in the discussion said that "So maybe we need to really rethink our assessments of students". to see their effects on students' learning. In the following paragraphs, I will give the teacher perspectives framework in detail.

3.2. Teacher Perspectives Framework

Teacher perspectives framework points to the reasons behind the teacher knowledge that teachers might possibly show during instruction. Particularly, this framework includes three different perspectives namely, the traditional perspective (TP), perception-based perspective (PBP) and a conception-based perspective (CBP). Later, Jin and Tzur (2011) have included the Progressive Incorporation perspectives (PIP) between the PBP and CBP (Table 3.2). In the following paragraphs, I point to the differences and similarities among TP, PBP and CBP first. Then, due to its importance for this study, I explain PIP contrasting it to other perspectives specifically to PBP. Regarding each perspective, in Table 3.2 also share teacher behaviours before, during and after teaching.

Table 3.2. Teacher Perspectives.

	View of Knowing	**View of Learning**	**View of Teaching**
Traditional Perspective (TP)	Independent of the knower, out there	Learning is passive reception	Transmission, lecturing instructor.
Perception-Based Perspective (PBP)	Independent of the knower, out there	Learning is discovery via active perception	Teachers as explainer (points out)
Progressive Incorporation Perspective (PIP)	Dialectically independent and dependent on the knower	Learning is active (mentally); focus on the known required as start, new is incorporated in to known.	Teacher as guide and engineer of learning conducive conditions.
Conception-based Perspective (CBP)	Dynamic; depends on the knower's assimilatory schemes	Active construction of the new as transformation in the known (via reflection)	Engaging students in problem solving; Orienting reflection; facilitator

Particularly, CBP is different from TP and PBP in terms of both the nature of mathematics, mathematics learning and mathematics teaching since teachers having CBP believe in radical and social constructivist epistemology (Tzur *et al.*, 2001). Therefore, while teachers having TP and PBP holds the belief that knowledge is out there and independent of the knower, teachers having CBP acknowledge that knowledge is dynamic and changes from one person to another based on their assimilatory schemes (Simon *et al.*, 2000). That is Platonic view of knowledge is observed in teachers holding both TP and PBP, teachers with TP view mathematics as static and certain knowledge. However, different from teachers holding TP, teachers with PBP can focus both on mathematics and the learner. That is, teachers with PBP acknowledge mathematics as meaningful and interconnected set of ideas such that they believe that students need to connect these ideas by discovery of knowledge through active participation (Heinz *et al.*, 2000; Karagöz Akar, 2016; Tzur *et al.*, 2001).

Therefore, believing that knowledge is independent of the learner, teachers with TP acknowledge learning as passive transmission of knowledge from teacher to students. In contrast to TP, teachers with PBP explains the learning as discovery of knowledge from outside by an active participation and first-hand experiences of learners. Though active participation of students is also significant for teachers holding CBP. The difference lies in the fact that for teachers holding CBP, learning is not only the accumulation of new knowledge but is the transformation of learners' assimilatory schemes by new ideas through their mental activities. Therefore, each learner's learning change according to their modification of knowledge in their already existing schemes (Jin and Tzur, 2011; Tzur *et al.*, 2001).

Based on their views on mathematics and mathematics learning holding different perspectives, teachers' views regarding mathematics teaching also differ. The role of the teacher in TP is only a knowledge source for transmission of knowledge to students, while the teachers holding PBP gives importance to learners' active participation for discovery of knowledge outside of the knower. Therefore, while the teachers having TP use lecturing strategy, teachers having PBP prepare the instruction appropriate to learners' characteristics and understanding. So, teachers having PBP or CBP have responsibility for preparation of the instructional materials, questions, examples or explanations according to learners' pre-existing knowledge and needs. However, teachers having CBP acknowledges that there is flexibility for learners to construct new knowledge by transforming their existing knowledge (Tzur *et al.*, 2001). That is, while teacher both having PBP and CBP might determine the sequence of teaching within their lesson plans prior to teaching act, teachers holding PBP focus on the understanding of the content to pass the next step during instruction, teachers having CBP focuses mainly on the understanding of learners and uses the learners' reflection to pass to next step during instruction (Heinz *et al.*, 2000).

Different from teachers having these two perspectives, teachers holding PIP acknowledges mathematics as having two parts: dialectically independent including the facts and concepts outside of the learner and dependent on the learner including their ability for problem solving. Therefore, the mathematical knowledge being dialectically independent from the learner resembles TP and PBP views of knowing, the mathematical knowledge being dialectically dependent on learner resembles CBP view of knowing (Jin and Tzur, 2011).

What such view suggests is that, teachers with PIP acknowledge that newly constructed knowledge and the already existing knowledge independently of the knower have commonalities and differences yet it is the learners who need to realize and make sense of such relationship between the mathematical ideas. Therefore, different from PBP, PIP requires to acknowledge that mathematics is not obvious to students in materials or tasks such that students learn through their mental processes actively working and experiencing with materials or tasks. That is, learning of students require their active mental processes albeit the nature of mathematical knowledge being independently existing.

It is in this respect that, in contrast to CBP, teachers having PIP ask more questions to increase the participation of students for explaining their ideas (Jin and Tzur, 2011). The goal of questioning is not only to determine students' prior knowledge because they add new knowledge on their pre-exiting knowledge but also to take them through the next step in their learning. Therefore, teachers having PIP perspective have a dual view of mathematics while teaching, the one that needs to be reached and the one that is already existing in the mind of the learner. Teachers' evaluation of students' prior knowledge through questioning is also important to determine their misconceptions and difficulties (Heinz *et al.*, 2000; Tzur *et al.*, 2001).

Based on these characteristics, in the following subheading, I share teacher behaviours prior to, during and after teaching. In the following paragraphs, I explain these characteristics also pointing to how I used them in this study and how they relate to the noticing skills.

3.2.1. Teacher Behaviours Based on Teachers Perspectives

Using the codes in Table 3.3, previous research (Bukova Güzel *et al.*, 2019; Karagöz Akar, 2016) have investigated the relationship between pre-service teachers' perspectives and their mathematical knowledge for teaching. Results of these studies have shown that pre-service teachers having PIP perspective have depicted all the codes in Knowledge Quartet. This showed that pre-service teachers having PIP perspective also held mathematical knowledge for teaching. Results of these studies together with the previous research on teachers' noticing further suggest that there could be a relationship between teacher perspectives and the noticing skills of teachers.

In the following paragraphs, dwelling on previous research, I explain the codes in Table 3.3 showing teachers' behaviours before, during and after instruction as well as pointing to such possible relationship between teacher perspectives and their noticing skills. For that, I mainly discuss the codes from PBP and PIP since these are the ones that would be used mostly for the analysis in this study. This is particularly because participants in this study have taken a professional development program during methods course within their teacher education through which they have developed these perspectives.

Table 3.3. Teacher Perspectives and Teacher Behaviours.

Teachers' role and what they engage in and how they view mathematics learning and teaching	**Before teaching**	**During teaching**	**After teaching**
Traditional Perspective	1-A Traditional approaches include a platonic view of knowing	2-A Lecture, transmit ideas (Jin & Tzur, 2011)	
Perception-based Perspective (PBP) Perception-based Perspective (PBP)	1-A Creates a learning trajectory for students "...designed to make the mathematical relationships as apparent as possible" (Heinz *et al.*, 2000) 1-B Mathematics as a connected set of ideas" (Tzur*et al.*, 2001) His identification of division into the world around him was a perception of one aspect of objective reality- thus he assumed that division could be seen by all in the distribution of physical objects in the group. Further for him the connection between division in physical representations and the steps of the algorithm was perceivable and apparent (p.242). That is learning meant coming to see a first-hand particular aspect of mathematical reality and the connections among them. Learning is a gradual process of coming to see additional aspects of the web of mathematical relationships-aspects seen previously serve as points of reference for what is yet to be seen.	2-A Provides first-hand activities/. 2-B Focuses on what students do not know (Heinz *et al.*, 2000). 2-C Assumes the learners see what xsee in the instructional setting (Heinz *et al.*, 2000; Tzur *et al.*, 2001) 2-D "Listens to students to find out what they are perceiving? determine if they have progressed to the next stage" (Heinz *et al.*, 2000) 2-E Becomes at points "...increasingly directive, or even tell..." (Heinz *et al.*, 2000)	3-A When the lessons did not go as intended, "what is problematic is creating a situation in which students can readily see the mathematics" (Simon *et al.*, 2000)

Table 3.3. Teacher Perspectives and Teacher Behaviours (cont.).

Teachers' role and what they engage in and how they view mathematics learning and teaching	**Before teaching**	**During teaching**	**After teaching**
	1-C "Create environment conducive to students' perception ('discovery') of those ideas via engaging them in activities that allow 'seeing' and connecting the intended ideas (Jin and Tzur, 2011). Teacher decomposes his mathematics into perceivable pieces and connections. He creates conditions that affords students' perceptions of these pieces and connections. He monitors whether students perceive the mathematics by determining what they report match the mathematics as he knows (Tzur *et al.*, 2001). ...The teacher focused on students' active involvement in the exploration of concrete representations of the intended mathematics so they would discover the mathematics rather than being told or shown (Tzur *et al.*, 2001) 1-D Assumes "the potential to foster a particular understanding is a property of the activity and would try to create the same experiences for ... students" (Heinz *et al.*, 2000) 1-E A view of mathematics teaching as creating situations that reveal the mathematical ideas and as orienting students' attention to key aspects of those situations. Thus, the teachers' role is to create opportunities for students to perceive, first-hand, intended aspects of mathematics (Tzur *et al.*, 2001)		

Table 3.3. Teacher Perspectives and Teacher Behaviours (cont.).

Teachers' role and what they engage in and how they view mathematics learning and teaching	**Before teaching**	**During teaching**	**After teaching**
	1-F Interprets interviewing students as "being in service of determining to what extent the students have seen the mathematics as it exists rather than as a means of gathering information about students' thinking and conceptions? (Heinz *et al.*, 2000). Assessment activity includes two related key features: Carefully listening to the students' responses and analysing what they learned of the intended mathematics. Secondly, the teacher's (Nevil) listening is oriented to whether students' contributions matched his way of understanding the mathematics (Tzur *et al.*, 2001) Nevil's decisions about what and how to teach next were based on what he had identified as lacking. Thus, he monitored students to determine what they still needed to learn and to identify what they did know (Tzur *et al.*, 2001).		
Progressive Incorporation Perspective (PIP)	1-A "...create entire lesson plans on one's own or by adjusting available curricular materials" (Jin and Tzur, 2011) 1-A-1 Hypothetically plans the lesson based on cognitive process (Jin & Tzur, 2011). 1-A-2 Hypothetically plans the lesson based on students' behaviours (Jin & Tzur, 2011). 1-B "...assess what students already know not only or mainly as indication for 'time to move on', but rather as a conceptual anchor that must be judiciously re-activated in students for the intended learning to take place" (Jin & Tzur, 2011)	2-A Continually questions "... for helping students learn the intended (new) mathematics by first re-activating what they already know" (Jin & Tzur, 2011) because "...the 'old' and the 'new' include constituents that exist independent of the learner" (Jin and Tzur, 2011). 2-A-1 The old and new include not only mathematical constitutes but also cognitive process-mind activities. Therefore, it is not only independent of the learner but also dependent of the learner. Old and new dependent of the learner, since necessary mind activities are reactivated during lesson.	3-A "Assess students known mathematics, and plan? (Jin and Tzur, 2011) for the next days' lesson "...to fit with this assessment and with the curricular objectives" (p. 17).

Table 3.3. Teacher Perspectives and Teacher Behaviours (cont.).

Teachers' role and what they engage in and how they view mathematics learning and teaching	**Before teaching**	**During teaching**	**After teaching**
		2-B That is, "closely monitoring students' responses to..problems, the teacher finds if the required known mathematics has been established as an anchor to which the new will be linked" (Jin and Tzur, 2011) 2-C In other words, teacher questions because "...much of students' learning of 'what is' draws on negation of 'what is not', which is accomplished via learner-learner or teacher-learner social exchanges (answers and justifications) about students' actual or potential errors" (Jin and Tzur, 2011) 2-D Listens to the students because "...there are always many ways to solve or approach a problem" (Jin and Tzur, 2011) 2-E At the beginning of the lesson, poses reviewing problems that "...reflect what students learned in previous lesson (or two) and hopefully can bring forth as prerequisite for the day's lesson" (Jin and Tzur, 2011)	

Table 3.3. Teacher Perspectives and Teacher Behaviours (cont.).

Teachers' role and what they engage in and how they view mathematics learning and teaching	**Before teaching**	**During teaching**	**After teaching**
Conception-based Perspective (CBP)	1-A Interprets interviewing students as "....a means of gathering information about students' thinking and conceptions" (Heinz *et al.*, 2000) 1-B Mathematics is thought of as a web of conceptions that humans abstract through reflection (Tzur *et al.*, 2001) 1-C Mathematical learning is the building up and the continual transformation of one's conceptions (Tzur *et al.*, 2001) 1-D Mathematics teaching promoting intended conceptual advances, entails two major components: The first is the creation of learning tasks for which students set goals and use activities available to them to accomplish the tasks. The second is orienting students' reflections to identify patterns in their activity and the effects of that activity (Tzur *et al.*, 2001).	2-A Listens to students to understand how they think (Heinz *et al.*, 2000) because she is aware of the fact that "assimilatory schemes afford and constrain what learners perceive, interpret and may re-organize of the teachers' intended learning? (Jin and Tzur, 2011)	3-A When lessons did not go as intended, "they interpret the results as indicating a problem in either the teachers? understanding of the students or in their anticipation of how the concepts might develop (hypothetical learning trajectory)" (Simon *et al.*, 2000)

Among all the codes, the code, PBP.1A points that prior to teaching the focus of the teachers' hypothetical learning trajectory for their students is on the mathematical relationships rather than the students' thinking. That is, although prior to the instruction teachers might hypothetically have ideas about how mathematical ideas might be constructed, their focus is more on the mathematical ideas in the materials or tasks being as apparently accessible as to students. This is again because teachers holding PBP views mathematics as outside of the learner obviously apparent to everyone. Similarly, the code PBP.1B focuses on the idea that mathematics constitutes connected set of ideas. The codes PBP.1C, PBP.1D and PBP.1E mainly focus on the ideas that providing first-hand activities for students, they are given opportunities for learning. That is, teachers create situations such that going through those situations will allow students to learn because those situations have salient aspects making mathematics apparent to students. By the same token, the code PBP.1F focuses on teachers' intention for assessment prior to teaching such that their goal is to identify if the students have seen the mathematics obvious in the materials or the tasks engaged in rather than what and how students really think. Also, teachers' goal is to determine what students do not know rather than what they know. Therefore, they monitor whether the students did know or did not know the intended learning rather than what they already conceive in their assimilatory scheme as a precursor to the intended learning.

Having such view, the code PBP.2A focuses on providing students with first-hand tasks to foster the intended learning (the new knowledge). First-hand tasks are important because teachers with PBP view that students discover new knowledge embedded in the materials or tasks provided to them. Similar to their assessing prior to teaching, their goal for assessment during teaching is again to figure out what students do not know yet. Therefore, the code PBP.2B focuses on the extent to which students have or have not yet reached the intended learning. Similarly, the code PBP.2C focuses on the intention of teachers' while listening to their students such that they would like to determine if the students have perceived the mathematics obviously apparent in the materials or tasks as seen by the teachers. That is why teachers becomes at points "...increasingly directive, or even tell..." (Heinz *et al.*, 2000, p. 101) .In other words, teachers' focus is not on how students think but is on whether they have perceived the mathematics in tasks or materials. Therefore, even if teachers might determine and realize that students do not think similarly to what they think, they might not be able to come up with ways to respond to students' ideas (Karagöz Akar, 2016).They then choose either ignoring the situation in which students might ask questions or have difficulties or they might choose to tell students what they would like to hear from them. Teachers with PBP might also have difficulty in helping their students to make connections between concepts and procedures or determination of students' misconceptions (Karagöz Akar, 2016). Many researchers emphasized that attending to students' ideas or mistakes, responding to these ideas and helping students to make connections among the concepts are among noticing skills of teachers (Barnhart and Van Es, 2015; Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2002). Therefore, as the code PBP.3.A points to rather than focusing on how students think and how to respond to their ideas, when the classes did not go as intended, teachers holding PBP might think that they need to change the tasks to make mathematics under consideration as more apparent to students. Aligning with previous research pointing to pre-service teachers' having difficulty in attending to and explaining special events (Van Es and Sherin, 2002), these characteristics suggest that teachers with PBP perspective might have noticing skills at lower levels . Similarly, even if they have attending skills, this might not guarantee analyses or responses to the students' thinking (Barnhart and Van Es, 2015). Moreover, Karagoz Akar (2016) proposed that it is more likely that teachers with PBP might not depict all the codes in Knowledge Quartet showing mathematical knowledge for teaching. Similarly, previous research has shown that teachers having low scores on the tests measuring MKT have difficulty in noticing and interpreting special events during instruction (Schoenfeld, 2011; Thomas *et al.*, 2017).

In comparison to the code from PBP.1A, the codes PIP.1A, PIP.1A1 and PIP.1A2 focus on teachers' hypothetical learning trajectory taking into consideration of students' thinking. That is, teachers believing that intended learning dwells on their students' active mental processes based on what they already know, they hypothetically determine the learning trajectory students might possibly go through. Therefore, the code PIP.1B focuses on teachers' assessment of what their students already know prior to the teaching as a precursor of the intended learning to take place. During teaching teachers also continually use questioning, the codes PIP.2A, PIP.2A1 and PIP.2B, not only to understand what their students already know to activate the necessary knowledge base for the construction of new knowledge since both have commonalities but also to mentally engage students in the problem solving process to construct the link between them. Therefore, teachers' goal is not solely on passing to next step for the intended learning to take place but to understand how students really think. Similarly, the code PIP.2C focuses on teachers' monitoring students' misconceptions, errors or difficulties to create effective discourse during instruction by pointing to alternative approaches or different solution strategies. Based on such view, the code PIP.3A focuses on teachers' planning the next day lesson according to the students learning. The goal of the teacher again is to link the curricular objectives with students' current knowledge base. This suggests that teachers holding PIP perspective are both attentive to students' thinking and also are able to respond to their ideas even if they need to deviate from their agenda (Bukova Güzel *et al.*, 2019; Jin and Tzur, 2011; Karagöz Akar, 2016). Similarly, they are able to consider possible reasons behind how their students think (Bukova Güzel *et al.*, 2019; Jin and Tzur, 2011; Karagöz Akar, 2016). These characteristics also align with previous research results indicating that teachers having developed beliefs about teaching and learning might allow for noticing more significant details as well as acting on those noticed events (Dreher and Kuntze, 2015; Star and Strickland, 2008; Thomas *et al.*, 2017). Similarly, previous research has shown that teachers with PIP have mathematical knowledge for teaching (Bukova Güzel *et al.*, 2019; Karagöz Akar, 2016). By the same token, some researchers have pointed that MKT scores of teachers gives information about teacher noticing skills such that there is a significant correlation between the MKT scores and the professional noticing scores of pre-service teachers (Thomas *et al.*, 2017).

Also, researchers emphasized that once pre-service and in-service teachers including novices attend to professional development programs, their noticing skills might develop due to the increases in their content knowledge and knowledge for teaching (Jacobs, Lamb and Philipp, 2010; Star and Strickland, 2008). Therefore, in this study, hypothesizing that there might be a relationship between teachers' perspectives and their noticing skills, I investigated novice teachers' noticing skills once they held different perspectives.

4. SIGNIFICANCE OF THE STUDY

Teachers' noticing significant events or situations to respond during lessons is necessary for students' learning (Van Es and Sherin, 2002). It is in this respect that in this study novice teachers' perspectives and noticing skills will be investigated simultaneously to examine the relation between them. Rationale follows:

Previous research on teacher knowledge mainly focused on "what" teachers know and do not know about mathematics and pedagogy (Ball, Thames and Phelps, 2008; Rowland, Huckstep and Thwaites, 2005). The teacher perspectives framework, on the other hand, allowed for examining "why" teachers know "what" they know through depicting the reasons behind their actions (Simon *et al.*, 2000; Tzur *et al.*, 2001). Similarly, previous research examining the relationship between teacher knowledge and teacher perspectives showed that there is a correspondence between especially the progressive incorporation perspective and mathematical knowledge for teaching (Bukova Güzel *et al.*, 2019; Karagöz Akar, 2016). Similarly, previous research also has shown that teachers who have higher mathematical knowledge for teaching have high noticing skills (Jacobs *et al.*, 2007; Star and Strickland, 2008; Van Es and Sherin, 2008). However, previous research has indicated that pre-service and novice teachers are disadvantaged in terms of noticing skills due to the lack of teaching experience (Berliner *et al.*, 1988; Kagan, 1992; Star and Strickland, 2008). On the other hand, researchers argued that professional development might influence pre-service and novice and expert in-service teachers' mathematical knowledge for teaching (Ball, Thames and Phelps, 2008; Smith, 2001), which in turn also might influence teacher noticing skills (Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2002). In addition, it is also possible that despite the lack of teaching experience, either pre-service teachers or novice teachers having especially progressive incorporation perspective due to their involvement in a professional development study might depict noticing skills due to their mathematical knowledge for teaching. Therefore, in this study, I hypothesized that novice teachers having different perspectives in terms of the nature of mathematics, mathematics learning and mathematics teaching might have mathematical knowledge for teaching at different levels. By the same token, novice teachers with different perspectives might notice different events in instruction. Therefore, each novice teacher might have different noticing skills due to their perspectives and so due to their mathematical knowledge for teaching. It is in this respect that through multi-case study in a two-year-period with the same participants both during their preservice education and during their first years of teaching, I investigated the relationship between teacher noticing and teacher perspectives of novice teachers. Studying the relationship between their perspectives and noticing skills might provide valuable information to the field in terms of what, how and why teachers being pre-service and novice notice in classroom events as well as how to prepare future teachers during preservice education. In particular, previous research has pointed to "what" and "how" teachers notice. Though depicting the factors affecting noticing skills of teachers there was an effort in terms of explaining why teachers notice what they notice. Though, analysing the data using two different frameworks, learning to notice and the teacher perspectives, in-juxtaposition to each other might provide the reasons behind what and how teachers notice. Secondly, results of this study might point to the fact that once teachers, pre-service or novice, hold at least progressive incorporation perspective they might hold noticing skills recommended in the literature (Van Es and Sherin, 2002). Thus, in turn this might allow for the elaboration of both the curriculum and the content of methods courses in which pre-service teachers attend to particular professional development program to develop progressive incorporation perspective.

5. STATEMENT OF THE PROBLEMS

This study is part of a larger scale research Project, BAP 11220, conducted in and founded by one of the best universities in Turkey. Data were collected in 2016 in which participants were pre-service teachers and in 2018 in which the same participants were novice teachers at different secondary schools. The aim of this research study was to investigate noticing skills of novice teachers with different teacher perspectives when they were pre-service, and when they were in-service teachers. The research question in line with this aim was as follows:

- What are the noticing skills of novice teachers who hold different teacher perspectives?
- What are the teacher perspectives of participants as pre-service teachers and novice teachers?
- What are the noticing skills of pre-service teachers who hold progressive incorporation perspectives?
- What are the noticing skills of pre-service teachers who hold perception-based perspective?
- What are the noticing skills of novice teachers who hold progressive incorporation perspective?
- What are the noticing skills of novice teachers who hold perception-based perspective?

6. METHODOLOGY

In this section, I first explain design of the study. Then, I mention the selection of the participants for the larger project and this study. Following, I explain data collection tools and procedures as well as the data analysis process.

6.1. Design

This study was a qualitative research study. Specifically, a multi-case study was conducted. Case studies gives in-depth description and detailed analysis of a bounded system (Merriam and Tisdell, 2015). As a bounded system (Yin, 2014), in this study two novice teachers with PBP and/or PIP were selected to be the participants. Therefore, the special features of the bounded system in this study was having the characteristics of PBP and PIP. Similarly multi-case studies can explain the phenomenon under investigation in a more valid way than single case-studies since the results of multi-case studies is verified with data from different participants (Merriam and Tisdell, 2015; Yin, 2014).

6.2. Participants

6.2.1. Participant Selection

I selected the participants of this study from the participants of the larger research project. The larger research project was conducted between the years 2016 and 2018. The participants of the larger research project were six out of the twenty-six secondary school pre-service mathematics teachers at one of the high rate universities in Turkey in which the medium of language was English. Both two participants graduated from the university in 2016. Both in the larger research project and this study, purposeful sampling was used.

For this study, two teachers were selected among those six novice teachers within the larger research project. There is no consensus on novice teacher definition because of the years of experience. Some researchers stated that teachers with 2 or less years of experience are novice teachers such that they define novice teachers as teachers with little or no mastery experiences (Gatbonton, 2008). Novice teachers differ from experienced teachers based on their knowledge base, ability of integrating different kind of knowledge, ability of solving teaching problems, ability of understanding students' needs and students' learning, awareness on learning context or instructional objectives (Richards and Farrell, 2005). Therefore, two participants being novice teachers who have two years teaching experience were selected for this study.

In this study, I used purposeful sampling strategy and I had some reasons to select two participants. Reasons were as follows: they have taken a methods course in which as pre-service teachers, they attended to a particular professional development program through which the development of progressive incorporation perspective on their part was targeted. Therefore, they were rich in data about teacher perspectives and teacher noticing. Secondly, three teachers were convenient for the second data collection. One of the six participants of the research project did not work as a teacher. Two teachers cannot get permission for video recording of their instruction from school administration. After the data analysis, two of three teachers showed the same characteristics of teacher perspectives and the same codes of teacher noticing. To explain the characteristics of PIP and the extended level noticing more detailly, the teacher who represented the level was selected. Although three teachers' data were analysed, the data of two teachers that showed characteristics of different level for both teacher perspective and teacher noticing was given in the results part. Therefore, two teachers were selected from six participants of the research project.

6.2.2. Participants

The participants of the study, Alin and Elisa, were graduated in 2016 from the Department of Mathematics and Science Education in a public university in Turkey where the medium of instruction in English. The university accepts students who are in the top 1% of students up to 1 million taking the National Standardized University Exam Test. During their education, both participants took some compulsory mathematics courses such as Calculus, Logic, Introduction to Mathematical Structures, Complex Analysis; pedagogical courses such as Fundamentals of Guidance and Counselling, Educational Psychology; and a methods course such as Learning and Teaching Methods in Science and Mathematics. In addition, both of the participants attended to a professional development methods course, Teaching Methods in Mathematics, in 2015 Fall semester and following that, they attended to a practicum course, Seminar on Practice Teaching in Mathematics, in 2016 Spring semester. The data in 2016 were collected during their internship when they attended to the Seminar on Practice Teaching in Mathematics course.

In 2018, the first participant, Alin, had been working at one of the best private secondary schools in İstanbul in Turkey. The medium of language is English at this private school which includes a year-long preparation class and 4 years of high school education. The school accepts students with the highest scores (within 1%) in the National Entrance Exam for High Schools to which more than one billion students attend. Students attending to this school have high socio-economic status. Teachers in this private school have to use a special curriculum in addition to the National Curriculum. In addition, the school management supports their teachers to use different teaching strategies and their own lesson plans.

In 2018, the second participant Elisa also had been working at a private school that was among the well-known private secondary schools in Istanbul. This private school also offers a four-year-long education along with accepting students with high scores in the National Entrance Exam for High Schools to which more than one billion students attend. Students attending to this school have high socio-economic status, too. In this school, The IB curriculum and the National Curriculum were also used. Similarly, the school management supports teachers to prepare their own lesson plans, but completing the curricula is their priority.

6.3. Data Collection

6.3.1. Procedure

The data of this study came from the larger research project that was conducted between 2016 and 2018. Participants of the research project took the Teaching Methods in Mathematics course that prepared them to develop the Progressive Incorporation Perspective. So, the content of the methods course was prepared by focusing on teacher perspectives' development unlike the other methods course during the same term. Particularly, in this professional development methods course, pre-service teachers focused on the nature of mathematical understandings, mathematical learning, conceptual analysis and interviewing. Specifically, the nature of mathematics, conceptual understanding through quantitative reasoning consisting of quantitative and mathematical operations, mathematics learning through one's own logical mathematical operations and high cognitively demand tasks were some topics in the course (For more information see Appendix B). During the methods course, as well as collecting data through whole class discussions from twenty-six pre-service teachers, six of them were selected to collect data for further examination of how the development of teacher perspectives occurred. Also, data depicting these six pre-service teachers' practicum teaching were collected during the 2016 Spring Semester within the Seminar on Practice Teaching Mathematics Course. In this study, two participants were selected from these six pre-service teachers who attended to the methods course. This was because these participants volunteered for the collection of further data. Therefore, in this study, data collected in 2016 from the practicum teachings of these two pre-service teachers were used together with the data collected in 2018 when the participants were novice teachers and worked in private schools.

Data collection process in this study included two phases with various forms of data sources.

In the literature, the main data source for determining the noticing skills is he post-interviews after the videotaped instructions of novice teachers when they were pre-service teachers and also when they were in-service teachers. On the other hand, researchers argued for the importance of lesson plans to investigate noticing skills since lesson plans has potential to provide information on teachers' knowledge and prior noticing skills for the students or instructional setting (Jacobs, Lamb and Philipp, 2010). Therefore, the main data source for determining the noticing skills was both the videotaped and transcribed pre- and the post-interviews of the participants together with the lesson plans of the instructions when they were pre-service teachers and also when they were in service teachers. For determining the teacher perspectives, the main data sources were the same with addition of videotaped and transcribed teachings. As auxiliary data sources, written artifacts such as participants' written notes from the interviews and their written reflections on the teachings (if provided), their students' worksheets during instruction and the field notes of the researchers both from the interviews and the observation of the instructions were used.

6.3.2. Data Collection

As already mentioned, data collection process included two phases: The first phase of the data collection occurred upon completion of the methods course and during the practicum in 2016 Spring Semester. For the data collection in the first phase, all participants' two practice teachings were videotaped and transcribed. Also, interviews were conducted (See Appendix A) with them to talk about their lesson plans prior to the teachings, observed their teachings and conducted interviews upon completion of the teachings within the same week. In this regard, a modification of account of practice methodology is used (Simon and Tzur, 1999). In addition, all the participants wrote self-reflection papers after watching their videotaped lessons. They wrote self-reflection papers based on reflection-tasks (Oner and Adadan, 2011).

The second phase of the data collection occurred in 2018 Fall Semester. All participants' one teaching sessions were videotaped and transcribed. Also, interviews were conducted (See Appendix A) with them to talk about their lesson plans prior to the teachings, observed their teachings and conducted interviews upon completion of the teachings within the same week. In this regard, a modification of account of practice methodology were used (Simon and Tzur, 1999).

Table 6.1. Data Collection Times and Data Sources.

<table>
<tr><th colspan="2">2016</th><th colspan="2">2018</th></tr>
<tr><td colspan="2">Lesson Plans</td><td colspan="2">Lesson Plans</td></tr>
<tr><td colspan="2">Pre-interviews</td><td colspan="2">Pre-interviews</td></tr>
<tr><td colspan="2">Observation of Instruction</td><td colspan="2">Observation of Instruction</td></tr>
<tr><td colspan="2">Post-interviews</td><td colspan="2">Post-interviews</td></tr>
<tr><td rowspan="3">Written Artifacts</td><td>Students' Worksheets</td><td rowspan="3">Written Artifacts</td><td>Students' Worksheets</td></tr>
<tr><td>Participants' Written Notes</td><td>Participants' Written Notes</td></tr>
<tr><td>The Field Notes of The Researchers</td><td>The Field Notes of The Researchers</td></tr>
<tr><td colspan="2">Self-Reflection of Participants</td><td colspan="2"></td></tr>
</table>

Interview questions were prepared to measure the teacher perspectives but the literature of the teacher noticing showed that approximately the same questions asked teachers to measure their noticing skills. These questions can be seen in different research about teacher noticing such as Jacobs *et al.* (2007), Jacobs *et al.* (2010) or Van Es, (2011). The first question in post-interview is "How was your lesson?" to direct teachers thinking about special events in their lesson. Van Es (2011) asked a general question to teachers in group discussion that "What did you notice?". These two questions' main purpose is to determine the teacher's general observations and abilities of attending special events in the lesson.

The second and third questions in the interview that is "Can you give examples of two or three students you think they have learned" and "Can you give examples of two or three students you think you have not learned" measure the teacher perspectives and also teachers' noticing of students' mathematical thinking. Jacobs *et al.* (2010) asked that "Please describe in detail what you think each child did in response to this problem" and "Please explain what you learned about these children's understandings". Questions in the post-interview that is appropriate to investigate teacher noticing of students' mathematical thinking according to previous research.

The last question in the post-interview that is "When you think of the activities that you have prepared in your lesson plan and implemented in the class, what do you change or leave the same? Why?" intend to see teachers' attending and actions that planned for the next lesson or the same lesson plan for the next year. In Jacobs *et al.* (2007) research, participants watched a lesson video of a teacher and analyzed teacher's actions and students' thinking so the "Pretend that you are the teacher of these children. What problem or problems might you pose next?" is asked them to measure their deciding how to respond students' thinking ability. The next action of teacher is significant for teacher noticing because they decide the next action according to their attending and understanding the special events in the lesson.

6.4. Data Analysis

For the analysis, I transcribed the videotaped pre- interviews, teachings and the post-interviews. Considering the post-interview as the main data source, I analysed all these transcriptions using both the teacher perspectives and learning to notice frameworks. The unit of analysis was chunks of data such as a sentence or a cluster of sentences (Merriam, 1998) and dialogues from the teaching sessions depicting what, how and why the participants noticed during the teachings.

For the analysis in terms of the teacher perspectives, I used the codes in Table 3.3 As already mentioned, these codes were prepared based on the literature (Heinz *et al.*, 2000; Jin and Tzur, 2011; Simon *et al.*, 2000) considering before teaching, during teaching and after teaching (Table 3.3) (Bukova Güzel *et al.*, 2019). Similarly, for the analysis of the noticing skills, I used the codes from the Learning to Notice framework (Van Es and Sherin, 2008).

I analysed the data using coded analysis that is defined as "focuses on observations that are assigned to predefined categories by a coder, usually from relatively small segments of a transcript" (Clement, 2000). I also used constant comparative method (Glaser and Strauss's (1967) to compare the different data sources with each other. First, I read the transcripts from the post-interview line by line to determine the noticing skills of the participants. Then reading the transcripts from both the pre-interview and the teaching sessions as well as examining lesson plans and written artifacts including self-reflections, I also looked for further data evidencing noticing skills of the participants. Then, comparing these different types of data constantly that might provide an opportunity for determining both the similarities and differences in all data sets, I finalized the level of the noticing skills of participants. Following, I analysed all of the data to determine the teacher perspective of the participants. For that, I also read all the transcribed data line by line by again using the constant comparative method. I also asked another researcher to examine the data as a second coder as well as discussing plausibility of the interpretations. The consensus was significant for the elimination of the biased coding and for revealing more reliable results.

6.5. Trustworthiness of the Study

The findings and interpretations of data must give an accurate picture about data both during the data collection and analysis of the data process (Creswell, 2009). In qualitative research, trustworthiness is used as a general term for reliability and validity. Thus, establishing the trustworthiness of a qualitative research is significant for the quality of the research. In qualitative studies trustworthiness of the study was examined in terms of credibility, transferability, dependability and confirmability (Lincoln and Guba, 1985).

The first criteria in trustworthiness, the credibility, refers to internal validity. Internal validity "...deals with the question of how research findings match reality" (Merriam, 1998). The credibility in this study ensured with the triangulation of the four different data sources such as transcripts of the videotaped teaching sessions, pre and post interviews, lesson plans of the instructions and collected documents. The second criteria, transferability, refers to external validity evaluating the generalizability of the research findings (Merriam and Tisdell, 2015). Since I provided thick explanations regarding the choice of participants, design of the study, data collection processes, analysis and the findings of the study, the research can be applicable in other contexts by different researchers.

The dependability refers to the reliability of the study and it measures the extent of which research findings could be replicated (Creswell, 2009). First, I analysed data from the same participants both in 2016 and 2018. This might allow for the consistency of the results from one cohort to the other. Secondly, the use of data from both the pre and post interviews might allow for determining the similarities in these data sets. Similarly, I asked another researcher for analysing the data using the same codes as well as discussing the plausibility of the interpretations. The final term, confirmability refers to the objectivity of the research in terms of researcher, participants or data gathering and analysing methods (Creswell, 2009). To avoid the researcher bias, I transcribed the video recordings of teachers' instructions and interviews. I also made the analysis of the data with unnamed files to avoid the participant bias. By taking into account trustworthiness issues such as credibility, transferability, dependability and confirmability before the data collection and for analysis procedures, the high-quality research is attempted to be conducted (Lincoln and Guba, 1985).

7. RESULTS

In this study, I investigated the relationship between teacher perspectives and teacher noticing. For that purpose, data were collected from the pre-service teachers in 2016 and from the same novice teachers in 2018. For the analysis, lesson plans of the teachings, the transcripts of the pre-and post-interviews and the actual teachings were used as data sources. Data were coded with the codes that came from literature on teacher perspectives and teacher noticing frameworks.

There are four levels in the teacher perspectives framework, namely, traditional perspective (TP), perception-based perspective (PBP), progression incorporation perspective (PIP) and conception-based perspective (CBP). In this study, I benefited from the characteristics of the perception-based perspective (PBP) and the progressive incorporation perspective (PIP) mostly since the participants, Alin and Elisa, in this study showed the features of these perspectives. Similarly, there are four levels in the teacher noticing framework namely baseline level (Level 1), mixed level (Level 2), focused level (Level 3) and extended level (Level 4). In this study, Alin and Elisa showed the codes of the extended level and the mixed level. Therefore, in the following sections, I depict the results from the data in both 2016 and 2018. Firstly, I show the results regarding the teacher perspectives the participants hold and then show the results regarding the noticing levels of participants.

7.1. Alin: Case of 2016

7.1.1. 2016 Results for Teacher Perspectives

Results from the analyses of the data from Alin's lesson plan, pre-and-post interview, and the teaching pointed that before, during and after the teaching, Alin depicted the characteristics of progressive incorporation perspective more than once (See Table 7.1).

Table 7.1. The frequency of the characteristics of PIP before, during and after Alin's teaching in 2016.

Alin-2016		PBP	PIP			CBP
Before	Lesson Plan		1A	1	11	
			1A1	6		
			1B	3		
			3A	1		
	Interview		1A	1	16	
			1A1	10		
			1A2	2		
			1B	3		
During	Teaching		2A	3	10	
			2A1	2		
			2B	2		
			2C	1		
			2D	1		
			2E	2		
After	Interview		3A	3	3	

As the table indicated, Alin depicted the characteristics of progressive incorporation perspective prior to the teaching 24 times, during teaching 10 times and after teaching 3 times, In the following sub-sections, I share data to show how Alin exhibits these codes in her lesson plan, during interviews, and also during teaching.

7.1.1.1. Alin's Interviews Before and After Teaching. In this section, I share data from 2016 including both the lesson plan, the pre-interview and the post interview since while talking about her lesson after the teaching, Alin also pointed to why she planned the lesson as she did. Data from the lesson plan and the pre-interview specifically pointed that Alin depicted PIP.1A, 1A1, 1A2, 1B and 3A. Also, the data from post-interview further showed that she depicted the characteristics of PIP.1A and PIP.3A.

Particularly, previous research has shown that teachers holding PIP plans their lessons either by modifying curricular materials or creating their own (PIP.1A). Similarly, they plan lessons to fit with their prior assessments of student knowledge and with the curricular objectives (PIP 3A). Moreover, in order to assess students' prior knowledge and to activate such knowledge for the intended learning of the lesson, they also acknowledge the need for frequent questioning in the lesson plan (PIP.1B). In addition to these, with this data, I also empirically show that (pre-service) teachers might construct hypothetical learning trajectories based on their students' possible cognitive processes (PIP.1A1) or behaviours (PIP.1A2) including possible misconceptions, difficulties etc. All these characteristics were observed in data both from Alin's lesson plan and pre-interview.

In particular, Alin had designed her lesson for students' making sense of the graph of quadratic functions, f(x)=ax^2+bx+c, with real coefficients and its link with the coefficients, a, b, and c. That is, her objective was for students to make sense of the relationship between the algebraic and the geometric (graphical) representation of quadratic functions. Considering such objective, in her lesson plan, she stated:

Students will be able to understand what effects the coefficients a, b, and c have on the graph of a standard quadratic equation of the form y=ax^2 +bx+c.

During the pre-interview Alin also stated that she constructed a task sequence on her own (PIP. 1A). First, when asked how she had chosen such an objective she stated:

R: Now, you just said your objective. How did you decide on that objective? A: They already learned the quadratic equations. For example, they always said something, for example, when it was +, they put up smiley face, when it was -, they put a sad face, but they did not think why. But, they did not talk about the reason, they knew how to draw the graph. They know that delta cut the x axis at two points when it is greater than 0. They know that it is tangent to the x axis and touch the x axis in one point when it (delta) is equal to 0. They also know that when it (delta) is less than 0 [parabola] was upward. But a student, for example, could not draw it and understand it and asked about it. When he drew the graph, he did not fill some the points where the graph cut the x axis. I thought there was something that they didn't understand about the graph. I decided on this subject so that learning how the coefficients are changing will be good to understand the meaning of the complex numbers, to understand the meaning of the *delta, the meaning of the axis of symmetry. They will also be interested (10).*

As the excerpt indicated, although, Alin had not taught this class before, based on her observations of the class prior to her teaching, she planned to design her next lesson based on what the students already have known. For instance, she was aware that students knew that when the leading coefficient, “a”, was positive, then the graph would be looking up wards and when “a” was negative then the graph would look down wards. Though, she was aware that students had not thought of the reason behind it. In addition, she has taken into consideration what students will learn in the future (e.g. complex numbers), As the data indicated she thought that once students reasoned on the relationship between the graphs and the algebraic representation of quadratic functions, it would construct a foundation to link new ideas (e.g. complex numbers) on their old knowledge as well as a motivation for them to learn. All these suggested that she planned the lesson to fit with her assessment and with the curricular objectives (PIP.3A).

Therefore, along with these awareness, both in her lesson plan and the pre-interview, Alin mentioned to start the lesson first asking students to reflect on quadratic functions. For instance, in her lesson plan, she wrote: “What is the standard formula for a quadratic equation? (Lesson plan, p.2)”; and, explained during the pre-interview how she would proceed with that question:

I am asking the first general formula, what is this standard formulation, I want them to give “$ax^2 + bx + c$”, I can ask directly how do we write in the definition of quadratic functions, what is the domain and range of f, real numbers and real to real. What are the “a”, “b” and “c”? I can ask that where do I choose those numbers from? (Interview p.24).

As the data indicated, asking the standard formula of the quadratic functions and the definition, Alin wanted to assess what her students knew.

Alin's explanations in her post-interview further supported her ideas before the lesson. Alin explained the main goal of her lesson plan as follows:

When I was writing this lesson plan, the students in that class had already spoken on the quadratic equations, they knew how to draw the shape of the graph, they are aware that when they gave value to x, the values of y change with respect to x value. So, I prepared a lesson plan according to it. (after 48).

Creating the lesson plan according to students' prior knowledge on the basis of curricular objectives showed that the main goal of the Alin was to link the curricular objectives with students' current knowledge base. She determined her students' prior knowledge and the intended learning together to plan her lesson. In addition to both students' cognitive processes and the students' behaviours, students' knowledge base and curricular materials were significant in her lesson plan, so she showed PIP.1A characteristic. In addition, knowing the students' prior knowledge and needs for the new knowledge was also significant for planning the next lesson. By talking about the curriculum and students' knowledge base, she planned next lesson, so Alin also showed the PIP.3A characteristic.

Though, the reason behind her questioning at the beginning of the lesson was more apparent once she commented on why she planned to continue with the graph of $y=x^2$ at the beginning:

R: So, you say that you will start the lesson with the graph of $y=x^2$. Why would you do that? A: Because, $y=x^2$ will be an easy thing for them to understand and play on, that is, they will be able to simulate something by thinking it, for example, because I did not add $bx + c$ in the first place, to avoid the confusion. So, $y=x^2$ to the axis of full symmetry. The graph was what they know best. They will not struggle to find their roots, but it will be easy for them to observe the change in value of "a" (25-26).

As the data showed for Alin, the students' already known knowledge was one of the most significant sources for the lesson planning process (PIP.1B). Her emphasis on the students' linking what they already know (i. e, the graph of y=x^2) and what they need to learn (i.e., the effect of the changes of "a" on the graph) suggested that she acknowledged that students could construct new knowledge by using their prior knowledge.

She further stated:

In a statement like $ax^2 + bx + c$, first of all, it is necessary to observe the change of the value of "a" one by one, it is necessary to keep "b" and "c" constant, only be aware of the change in "a". So, in my first example, I will accept "b" and "c" as 0, so let me say this is my thing, my preference. My goal there is that, they must be aware of $y=x^2$, let me say the amount of increase in y, rate of change of y, the amount of change of y with respect to x. I said amount of change rather than amount of increase because it would be negative also. They need to determine how the amount of change in y has changed. They will compare the changes of y when the value of "a" changes and the value of x changes one unit. So that, they may have an idea about the looking of the graph, as the amount of change in y increases with respect to one-nit change of x. This is not a linear but a quadratic function.

First of all, data suggested that she knew that keeping the coefficients "b" and "c" as invariant, students would be able to realize more easily the effect of the change of the values of "a" on the graph. Secondly, Alin was aware that students needed to focus on the changes of the variables x and y simultaneously. Her statement ". I said amount of change rather than amount of increase because it would be negative also?? suggested that she expected students not to focus solely on the values of y (the increase or decrease) but to focus on x and y relatively such that they would realize how the rate of change of y with respect to x occur. Such realization then would allow them to realize the effect of the values of "a" on the graph. In other words, she hypothetically envisioned that given y=x^2, if students reason that y values changed increasing at an increasing rate while x values changed one by one, they could conclude that such relationship between x and y would refer to a relationship different from a linear one. This suggested that she planned to base her lesson plan on students' cognitive processes

(PIP.1A1) in two successive ways. First, she planned her lesson so that students first would recognize the relationship between x and y in quadratic functions. Second, they would use such knowledge as a foundation in constructing the effect of the change in the values of "a" on the graph of quadratic functions, which would in turn be used as a foundation on reasoning on the effect of the change in the values of b and c respectively.

In fact, further data in her lesson plan pointed to more evidence for this claim: She articulated how students might possibly reason on the questions in the task sequence.

A: By giving "a" different values and obtaining related y values, this time students compare the respective rate of changes in y values for different "a" values. For the same change in x values, the rate of change in y gets bigger as absolute value of a increases. Meanwhile, in students' minds width of the parabola gets steeper and hence the width of the opening of the parabola gets narrower (2).

First, Alin's emphasis on how she expected students to reason was important. Particularly, after working on y=x^2, she expected students to compare the respective rate of changes for different "a" values. That is, she hypothetically envisioned that for different values of "a", if students simultaneously compare the change in values of x, which was the same in increments of one, and the changes in the values of; so the rate of change of y with respect to x, would allow them to realize that the rate of change of y with respect to x would be bigger as the absolute value of "a" increases. This would also enable them to imagine in their minds that "width of the parabola get steeper and hence the width of the opening of the parabola gets narrower". This suggested that she was hypothetically envisioning how students might reason given the different values of "a". That is, she would expect students to go through the mental activity, the simultaneous comparison of the change in the variables x and y, so that they could know the reason behind the effects of the changes in the values of "a" on the graph (PIP.1A1).

In addition to the explanation in her lesson plan, during the pre-interview Alin stated:

At first, they looked at the change of $y=2x^2$ *by* x*,* $4x^2$ *looked, let me talk about the two of them, then they compare the difference between these two. I think they will understand what* x^2*,* $2x^2$ *and* $4x^2$ *are, and that the width of the parabola are narrowed with the operation in their mind. Because they first thought that* $y=2x^2$*, they know that the graphic is upwards, that is, it is related to the shape of the graph. So, I think they will realize the growth and decrease in the width of the graph in their mind by playing with these numbers (a), I want them to do (44).*

Her explanations during the interview further indicated that Alin's main purpose was for students to construct and imagine the effects of the changes in the value of "a" on the graph. In fact, allowing students to reason first on the positive integer values of "a" such as 1, 2, and 4 showed that Alin chose those values on purpose. She wanted her students to imagine and realize that the arms of the graph get closer and narrowed. Though, one important issue is that she expected students not to only observe but to reason on why such change (the arms of the parabola getting closer and narrower) occur. So, the data again suggested that she focused on students' cognitive processes during the lesson planning (PIP.1A1).

For further triggering students' cognitive processes for understanding the effect of the different values of "a" on the graph, at the pre-interview, Alin also mentioned and emphasized how she expected students to reason once the values of "a" was between "0 and 1" and smaller than 0. She even further pointed that once students knew the effect of the changes in the values of "a" for positive values, they might have anticipated the effect of the changes in the values of "a" for negative values. She emphasised during the pre-interview that:

In positive real numbers, I will examine the increase. Then, when it is between 0 and 1, because the amount of change decreases between 0 and 1, the graph's arms need to open, so they have to explain it as two separate cases (28).

Alin focused that students can only create ideas about the structure of the graph by passing through these mental processes (PIP.1A1). By thinking about the values of “a” between 0 and 1, students can think that “a” is decreasing, so the change of rate in y values with respect to x values decrease. Alin expected that students can reach the result that the width of parabola widens when the “a” is between 0 and 1. In addition, Alin considered that students need to think about the negative value of “a” to understand the reason behind the direction of the parabola. Alin stated that:

When I make “a” negative at first, so now if we only think of x^2. x^2 will always be a positive thing when I multiply with a negative value, it goes down. But even if this bx + c exists, that is, even if the graphic starts with positive values at the very beginning, since x^2 always grows, it comes down to 0 and goes down to negatives. That's why it's always down to my quadratic function (50).

Alin explained the direction of the curve according to negative “a” value in her pre-interview. She expected that by passing through mental processes, students can understand and apply the effects of minus on the graph by thinking with together bx+c (PIP.1A1).

Alin not only emphasized focusing on the different values of “a” but also the different values of “b” and “c”. In her lesson plan she stated:

A: To be able to understand the effect of change in variable "b", students should know as a prior knowledge that the symmetry axis of a parabola passes through line. By changing b values students minds should move the line x= -b/2a on the x-axis. Also, as b changes depending on the discriminant value, the graph should also move up and down with the same y intercept?.In order to understand the effect of change in the "c" value, students should have conceptual understanding of an equation. In other words, they should know that in the equation $y=ax^2+bx+c$, y takes values dependent on the right side of the equation. Therefore, changing variable "c" directly changes y value (2).

And during the pre-interview she commented:

When speaking "b" and "c", knowing where the two roots are make them aware of how the roots change when they keep "a" and "c" constant and "b" change (16).

Reasoning on different values of "b" keeping the values of "a" and "c" as invariant, Alin's purpose for students was to construct cognitively that the placement of the symmetry axis of the parabola and the roots would change resulting in the movement of the graph up and down keeping the y intercept the same. Similarly, she knew that once the values of "a" and "b" were kept the same, the changing values of "c" would have an impact on the values of y.

In sum, all these data suggested that Alin planned her lesson based on students' knowledge and cognitive processes in successive paths (PIP. 1A1): First, she planned the lesson in such a way that only after the students focused on the changes of the values of "a" she mentioned engaging her students on the changes of the values of "b" and "c" by focusing on them one at a time. She even mentioned that once "a" and "c" were kept constant, students would realize the change of the roots with respect to each other. As the data earlier showed, this might have been also because she considered that students might have get confused if they had studied all variables together (PIP.1A2). In addition, Alin emphasized that students needed to consider "a" and "b" as constant in ax^2+bx, so that they could reason on the effect of the different values of "c". That is, such reasoning would allow them to realize that the point where the graph intersects y-axis would change depending on the different values of c. Data altogether also suggested that the moves in her planning was not only because she wanted to move forward in the lesson but also, she wanted to use what students' already know as a conceptual anchor already activated for the intended learning to take place (PIP.1B). This was even further evidenced from the pre-interview: when she explained the rest of the lesson plan progressively developing with a focus on the students' prior knowledge:

In total, data prior and after the teaching suggested that one of the most significant and salient aspects of Alin's lesson planning was her focus on the cognitive processes of students. She hypothetically envisioned how students might mentally operate on in order to reach the intended learning goal(s). This in fact pointed to two other important aspects of progressive incorporation perspective. First, in almost all parts of her lesson plan and during the pre-interview, Alin's emphasis was on how she hypothesized how students might possibly reason and what difficulties they might encounter. This suggested that, from her point of view, development of mathematical ideas was dependent on the students. In addition, her determination of the learning trajectory students might possibly go through suggested that she acknowledged that students learn mathematics mentally operating on their already established knowledge.

7.1.1.2. Alin's Teaching and Interview After Teaching. In this section I depict data from Alin's 40 minute- practicum teaching in 2016 as well as data from the post-interview since again during the post-interview, Alin pointed to why she acted the way she did during teaching. Data from her teaching and the post-interview were also coherent with the data from per-interview and the lesson plan. As she planned, Alin first started the lesson activating students? prior knowledge on quadratic functions and then continued with the graph of y=x^2. Following, she focused on the effect of the changes of the values of "a" on the graph. Though, since the lesson was a 40-minute lesson, she could not continue with the effect of the change of the values of "b" and "c". Specifically, data from the teaching showed that Alin depicted the characteristics such as PIP.2A, 2A1, 2B, 2C, 2D and 2E.

Particularly, Alin started re-activating students' knowledge about functions and equations asking:

First let's write the domain of the function. what is my domain? Where do I take my f values and where they go to? But let's first just write f(x). What was you saying? (4-6).

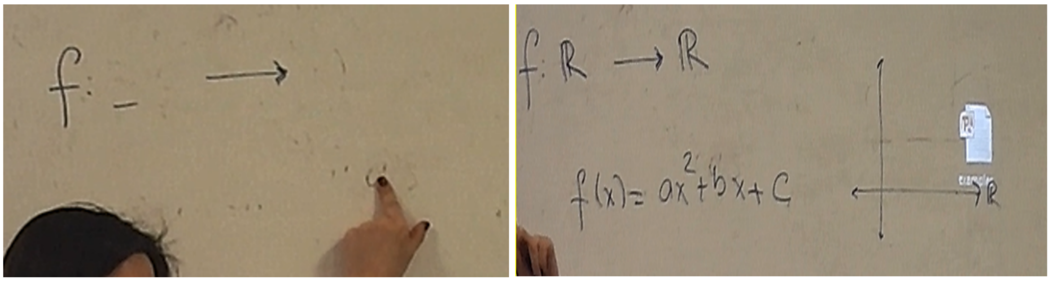

Figure 7.1. Beginning of Alin's teaching in 2016.

Answering the question for the rule of f, one student stated that the rule of f would be f(x)=ax^2+bx+c. Then some students stated that the domain of the function would be Real Numbers. Alin even asked "So what is values of numbers a, b, and c? Are they integers? Real? Rational?" Again, some students stated that the values of a, b, c were real numbers. Alin further questioned the value of "a" "So what if a is equal to zero?". Then, some students stated that the function would be a linear one. Then, Alin again asked "Why?", one student stated that when "a" is equal to zero than the function rule becomes bx+c, so, a linear one, concluding that the value of "a" should be different from zero. All these showed that acknowledging the intended learning goal, the relationship between the algebraic and the graphical representation of quadratic functions, Alin was aware that students needed to recall some features of quadratic functions such as the domain, range, the rule and the restrictions on the coefficients. Her asking questions when starting the lesson suggested that she reviewed the topic from earlier lesson with the goal of assessing what students had already learned in previous lesson (PIP.2E).

Discussion got interesting when one student asked:

- *S3: Are there any complex number roots or can x be complex numbers?*
- *A: But where do I choose my x values? But where do I take my x values?*
- *S3: If we find the roots of the equation, then place them the x values?*
- *A: ... Yes we can find roots as complex numbers in quadratic functions. (29-32) but in designing the quadratic functions,I specify the domain where do I choose my x values. First, I said that I take my x values then their image is in the x*

values (She showed the values of domain), so far, we do not work on complex numbers. Yes, there are complex numbers in quadratic functions? but we do not consider yet the complex numbers (29-32).

Alin's providing an explanation based on the S3's question might seem that she went back to direct teaching. Still, data was important: First, Alin's response to S3 indicated that she was open to listen to different students' ideas. Listening to students is significant to catch different ideas or solution strategies (PIP.2D). Secondly, data suggested that Alin was aware of what S3 was thinking such that she realized her confusion: S3 was talking about the range values (the complex roots) although she was asked to consider the domain of quadratic functions. Alin took the student's focus back to what they have been discussing: the values of x in the domain and the corresponding values in the range as being restricted to the Real Numbers. She even further commented that quadratic functions could be considered within the domain of complex numbers, but this was restricted by the definition discussed. This suggested that she did not ignore student's thoughts and questions such that she realized that students might think and ask questions differently from what the teacher might expect (PIP. 2D).

Then, Alin distributed the activity sheet which included a table. She provided 5 minutes for students to focus individually on the first question: I share the data in four parts: First, Alin started the discussion focusing on the graph of y=x^2:

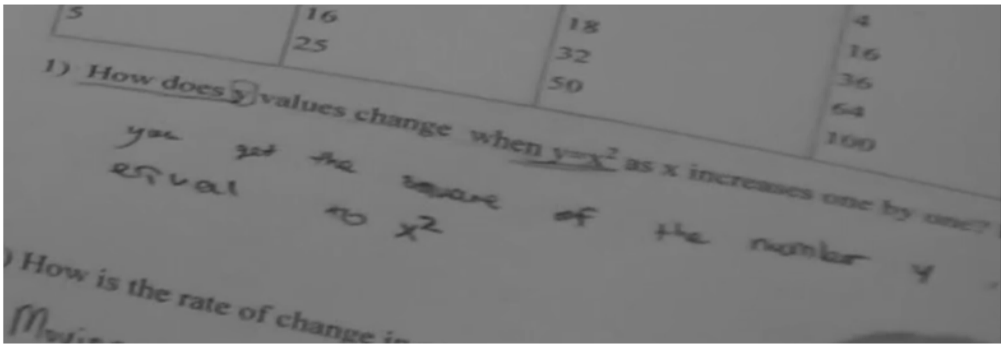

Figure 7.2. Answer of a students for the first question in the task.

Part I:

- *A: I want you to analyse the data for x and y=x^2. How does y values change when y=x^2 as x increases one by one, so when you consider the change of x…When you consider one-unit change in it from 0 to 1, how does my y values change? From 1 to 2, how does y values change? (waiting for thirty seconds). Okay, let's consider linear functions; y= 2x. (Drawing on the board (see Figure 7.3)). In the linear function like this,*
- *when x goes through 1 to 2. When x is 1; y is 2. and x is 2 which value of y?*
- *S1: 4*
- *A: 4. When x is 3, y?*
- *S1: 6*
- *A: 6. What will you say about the change of y? When x is increases one by one, how these are changing? (Pointing to the y values)*

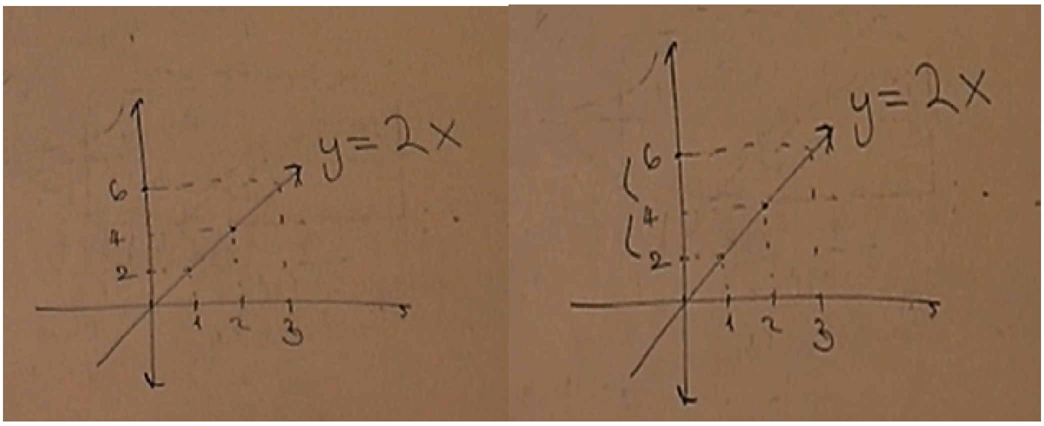

Figure 7.3. Drawing of Alin about the y=2x.

- *S1: y increases by 2. y increases by 2, x increases by 1.*
- *A: In this case (showing the linear one) is the change of rate for y, is it the same?*
- *Someone: yes*

As the data showed, students paused for almost thirty seconds when Alin asked "How does y values change when y=x^2 as x increases one by one? how does my y values change?" I claim that this was important for two reasons: First, Alin did not revert back to direct teaching. Instead, she made a change in her lesson plan and asked further questions regarding a specific linear function. Although she first asked corresponding values of y given the specific value of x, her second question focused on how y values changed correspondingly given x values changed in units of one. This suggested that realizing students' difficulty, she continued questioning to guide students by re-activating what they already know. I further claim that this was also important because from her point of view, the prior knowledge and the intended knowledge had common constituents existing independent of the knower (PIP.2A). That is, the two examples, y=2x and y=x^2, had common constituents (e.g., x and y values change simultaneously) such that with further questioning she believed that she could activate the reasoning required of students. This was further evident in the data from the post-interview.

In fact, the data above showed that Alin responded to students' ideas even if she needed to deviate from their agenda: Although she started the lesson with examining the function y=x^2, once she could not take any thoughts from her students, she decided to focus their attention on the function y=2x. She commented on why she acted the way she did:

I then drew a linear function graph. I chose the graph y=2x, I did not want y=x because x and y would always go one to one. I gave it 2x, so I said, for example, although I don't remember the numbers I gave here very clearly, I asked them what the values of y would be when they x change from 2 to 3 or 3 to 4. I said once 2. Then I said what is y at 3? What is y at 4? What is the increase here in 2x, will this always be like this? Then I asked what happens at 50 and 51, always the same. So, let's talk about this with x^2. They already said that after thinking the values of x, the value of y increased by sequence. Well, when I said how I show this on the graph, a few children already got the answer. They said it was a parabolic or exponential increase. I did something there, are they saying their answer by heart, or did they understand the rate of change because they didn't calculate too much? I asked are you sure? Why are you sure? Let's say what your friend says again. Come to draw the graph on the board? What will happen now? I pushed them like that, but in the end, everyone agreed on the whole thing. They were already drew correctly. As the coefficients increased, they drew the graph correctly. But I also asked the reasons because they might already know. Perhaps *they have heard or guessed. But they always said the increase in y value is more. For example, just a child said something, the difference between y values is increasing, and I said the rate of change is the same with the difference between y values? A girl said no. No, no, I asked her to come, then. The girl said, we know we can't explain. I said, you say it in your own words. She said that the increase in y values with respect to one unit of x. Then when I said "what's going on when x is (inaudible)", they showed it by hand, that it will be downward. They showed below by saying it is reflecting. When I said what would happen when they increased the coefficient of the x, they said it was getting narrower and wider. I said them to draw on the board.*

R: You said I started with y=2x, what was the reason for starting with a question like that?

A: I mean I have to reach rate of change. What they know is the constant increase in linear function, but y=x was misleading there. Always increasing one to one. But in y=2x, y value always increases by 2 and 2. So the amount of increases was fixed in 2. I thought they would understand it better. That's why I selected it.

As the data showed, Alin's main purpose was taking students' attention on the reasoning behind the different values of "a" for $y=ax^2$. So, she began with students' already known knowledge, y=2x, in order to trigger on their part, the similarities between the graph of y=2x and $y=x^2$. That is, she emphasized that students had the knowledge about y=ax, the graph of linear functions, to trigger the reasoning. Activating such knowledge on the part of students would allow them to reason on how the values of y change with respect to the values of x given the function $y=x^2$.That is, from her point of view, students would use such existing knowledge (the knowledge of the rate of change in y=ax) to reason on the new knowledge, $y=ax^2$ (the knowledge of the rate of change in y with respect to x given $y=x^2$). This provided evidence for the PIP.2A characteristics.

This was also evident in Alin's further questioning during the lesson:

- *A: So when you consider this function (showing y=x^2), when x increases one unit, how does y values changes? Do they change at same rate or increase / decrease? What is happening?*
- *S: The rate increases as well.*
- *A: Does everybody agree with him? Why do you agree or please paraphrase what he says. What is increasing rate of change in y mean? He said that when x increases one unit, the rate of change increases so what does it mean?*
- *S5: Y values. Yeah, Gap between them actually also increases.*
- *A: Yeah. Am I understanding right? You are saying that when x increases 1 unit, the change in y is increasing, right? yeah. Am I right? Or does everybody agree with him? Someone to paraphrase? (Alin drew the graph in Desmos)*

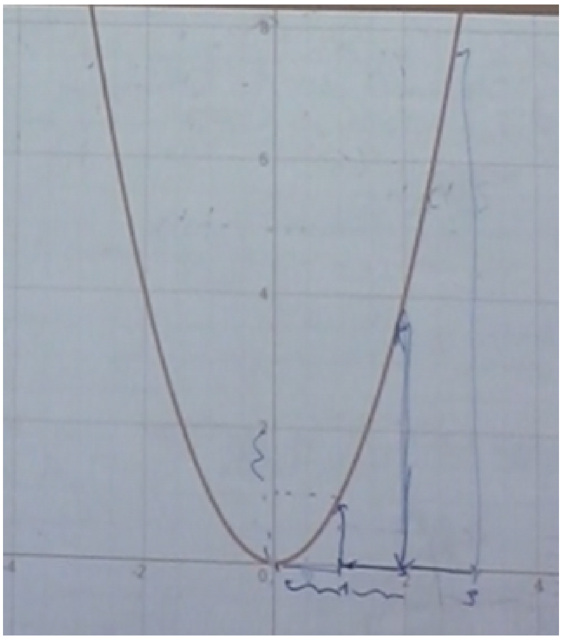

Figure 7.4. Drawing of Alin in Desmos.

S5: When both is on the same plane, you will see that on the graph,, eee, the quadratic graph actually goes higher like, it is almost like plane taking off like as straight and on the other hand, is going like upwards. You can see of rates of its increase, it also increases because it is like, it doesn't stand the same line, it is almost like always *increases to go up to like straight line.*

As the data showed, right after Alin directed her students' focus on the change in the y values given the change in the x values for the function y=2x, she questioned them on the behaviour of x and y values for the function y=x^2 (PIP. 2A). Her probing different students and asking different students to paraphrase how others reasoned also suggested that she paid attention to the different students' cognitive processes. Specifically the statements "gap between them actually also increases?" and "...rates of its increase, it also increases because it is like, it doesn't stand the same line..." showed that S5 was able to explain that the y values increased with an increasing rate given the x values changed in one unit. S5' statement "...it doesn't stand the same line..." even seemed to suggest that she might have compared the rate of change in a line, y=2x, to the rate of change in the graph of y=x^2. From the learning point of view, all these data suggested that Alin focused on students' reasoning process on how the variables x and y changed simultaneously (PIP.2A1).

During the post interview, Alin pointed to the limitations of how some other students thought about the changing of x and y values:

He said when x increases, y increases but you know that it is same in the linear function and in so. So, something was not changing. So he did not give us a reason for "a". There was only another boy who said the difference between the y values. There was no rate of change in the answer of the boy. We are looking for the one that corresponds to one-unit change in x. Actually, I can increase the difference between the values of y by taking bigger things. So, if I look at the difference between the two distant y 'values, it will increase, I am looking at the ratio of the change in y with respect to one unit of x. That boy who did not tell that.

Alin focused students' answers to examine their thoughts about the changing of values of y with respect to x. She noticed that one student gave a limited answer because he thought that the difference between y values increased in each trial without thinking the differences of y values with respect to x values. Therefore, Alin asked further question to create a discussion environment in the groups. Although the answers of some students were not wrong, they were limited for Alin because she expected students to think about the changes of y values with respect to x. This suggested that in her explanations in her post-interview, Alin showed PIP.2C characteristic.

Alin's paying attention to students' reasoning was further evidenced when she questioned on different values of "a":

Part II: *A: when I made for example, now have y=* x^2*, if I made y=* $2x^2$*, and y=4x , 10* x^2 *(writing on the board as:)*

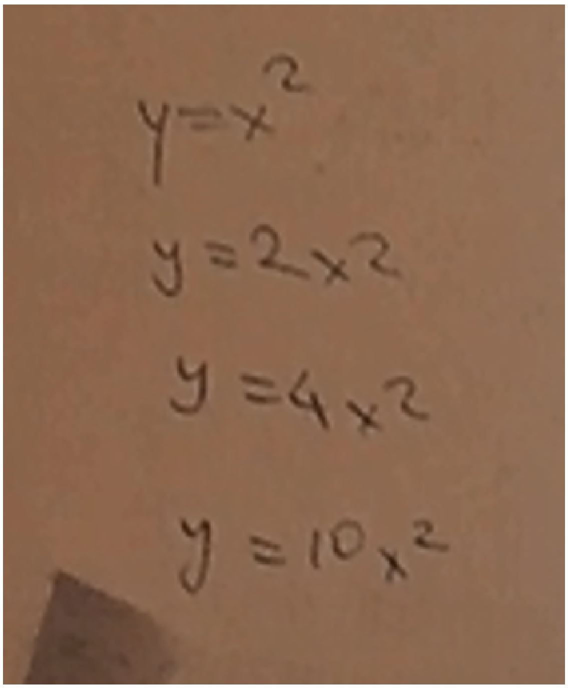

Figure 7.5. Writing of Alin for positive values of "a".

So what do you expect from graph? What it looks like?

S7: It is going to be nearer to the origin. (S7 draws the narrower parabola in blue on the graph see Figure 7.8) If a value increases, it is going to be closer to the y axis.

Figure 7.6. Drawing of S7 for graph of y=10x^2.

A: Does everyone agree with him?

Class: (class say yes) thank you.

A: Why do you think that this is happening? What's changing when a increases? What is the reason that the graph change?

S9: Because y value increases so x can goes same but y values increase so it gets narrower. Y value is bigger now, so its become narrower.

S8: Like outcome is 10 times bigger now (referring to y=10x^2) (77-80).

Data is important in the following way: First, Alin knew the importance of contrasting the graphs of y=x^2, y=2x^2, y=4x^2, y=10x^2 towards the understanding of the relationship between the graph and the coefficients in the algebraic representation based on the conditions comprised of the different values of "a". Her choice of values for "a" such as 1, 2, 4 and 10 suggested that she wanted students first to focus on the condition a'1. Though her focus was not solely on the numbers, 1, 2, 4, and 10. Her questions "Why do you think that this is happening? What's changing when a increases? What is the reason that the graph change?" showed that she wanted her students to focus on the rate of change of the different y and x values. In other words, as the data showed students were able to compare the graphs by fixing the differences in x values (1 unit in each graph) compared to the differences in y values. By comparing the rate of change among the graphs, students were able to determine that the bigger the difference between y values the bigger the rate and the narrower the graph is. In addition, as the data showed, without exactly drawing the graphs, she wanted students to answer "So what do you expect from graph? What it looks like?". This showed that she first expected students to imagine/anticipate how the graphs would look like compared to the graph of $y=x^2$. All these pointed that acknowledging that students already knew the graph of $y=x^2$ she wanted them to determine how and why the graphs of other functions have changed. More importantly data suggested that by asking students to anticipate how the graphs would look like compared to each other, Alin focused on if they were able to already assimilate the question into their already existing schema. That is, her asking students to imagine without exactly showing or drawing the graphs suggested that her focus was on students' assimilatory schemes (i.e., their current reasoning) (PIP.2A1).

Part III

As in her lesson planning, Alin continued the lesson with the values of "a" between 0 and 1:

A: For now, consider this type and y= $^1/_2\, x^2$*. I also want you to think what is happen when I increase and decrease my x value?*

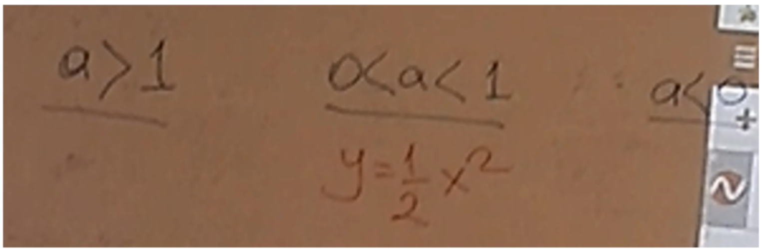

Figure 7.7. Writing of Alin for the value of “a” between 0 and 1.

S3: Is it lower than 0?

A: I have just aimed “a” values between 0 and 1. For example, y= $^1\!/_2\ x^2$*. So what will be different between y=* x^2 *and... What will happen to the graph? Could you draw what is on your mind?*

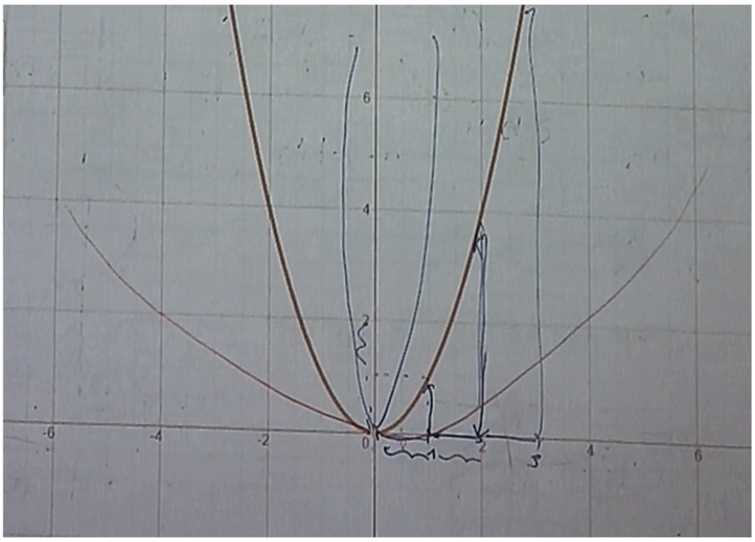

Figure 7.8. S6’ drawing on the board what she imagined for y=1/2x^2.

A: So what do you say about this graph now? She (referring to S6) drew the graph of y? When I take a value between 0 and 1, the leaves spread. What does this mean? Do you agree with her or who agrees with her? Can you raise your hand? Who says that when I take a value is between 0 and 1, the leaves get wider? How many people agree with her and how many does not? (Students raise their hands, and no one disagrees.) Is there anyone who does not? Then why do you agree with her?

S3: Because the difference between two y values decreases. So the leaves of the function get wider.

A: What is decreasing? Difference between y values?

S6: Rate of change.

A: She said rate of change. Are they the same rate of change, and different between the y values?

S6: How they change according to y values.

(Someone says something in the classroom, but it is not understood because they all talk together.)

A: Yes. Because when you say only the difference between the y values, you just take the difference. But you need to ratio of difference between two y values, for example y2 minus y1 and corresponding x values (wrote on the board what she said as). So, I can get the changes of x from the changes in y values.

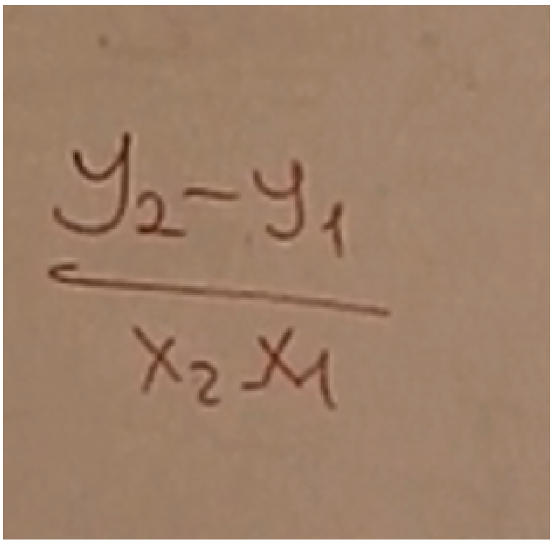

Figure 7.9. The formulation written by Alin on the board.

First, data pointed that Alin asked students to use what they already knew about $y=x^2$ to compare and explain how the graph $y=\frac{1}{2}x^2$ would look like. Her first asking students to imagine how the graph of $y=\frac{1}{2}x^2$ would look like and then draw on the board justifying their reasoning again suggested that Alin's focus was on students' assimilatory schemes (i.e., their current reasoning). This suggested that activating the already established knowledge, Alin wanted her students to construct new knowledge by pointing to the links between the two (PIP. 2A). Also, Alin's further questioning why the graph would look like as drawn Table 7.8 suggested that she wanted her students to justify their reasoning focusing on the correspondent change in the values of y with respect to the change in the values of x (e.g., rate of change). This further suggested that her focus was on students' cognitive processes during teaching (PIP. 2A1) rather than superficial aspects regarding the external representations of graphs (i.e., the graphs change because the leading coefficient "a" changes). That is, rather than solely observing the graphs with respect to the change in the values of "a", she wanted that students knew the reason behind the graphical representation of quadratic functions through thinking about the link between the algebraic and graphical representations. Specifically, she expected students to reason that since the rate of change of y with respect to x changes from one graph to the other once the values of "a" changes, the graphs change.

Alin continued the lesson with the negative values of "a" such as a=-10, -4, -2:

Part IV

A: Now, this time we have coefficients smaller than zero, negative coefficients. When you compare the two tables, table A and B, what do you observe about the y values? And how can this affect the shape of the graph?

S7: Reflected on the x axis.

A: Can you draw? What is going to happen when a is negative? How does the graph look like?

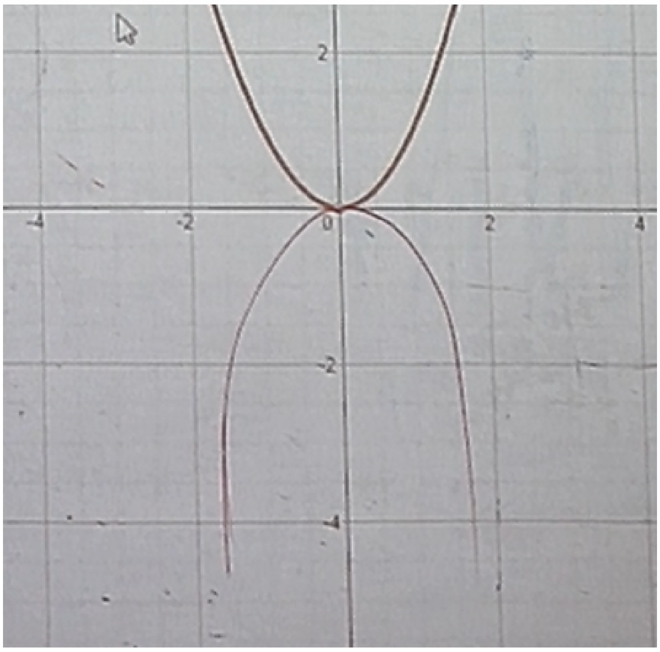

Figure 7.10. S9's drawing what she imagined given that the value of a is negative.

A: Okay, the most important question comes now! Okay, now I am only using negative a values, and this time, what happens when I decrease a value? What is going to happen the graph? what do you expect? (she wrote on the board as:)

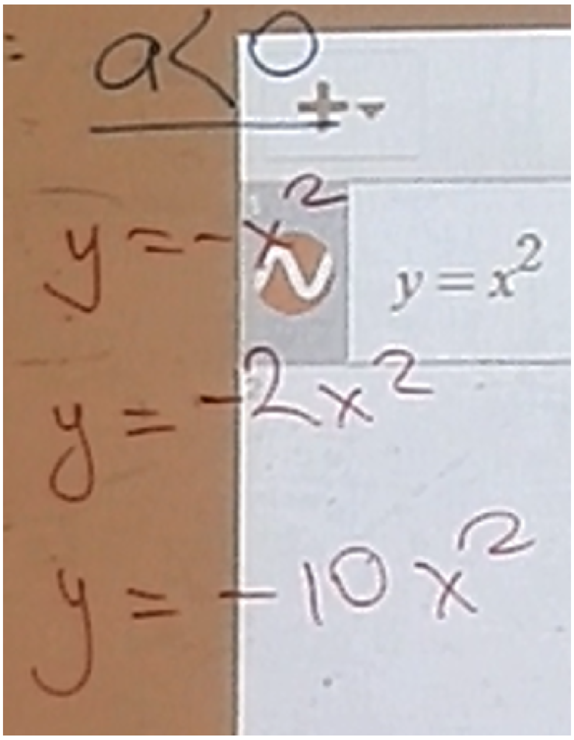

Figure 7.11. Negative values of "a" written by Alin.

A: You can just consider the data.

S6: We know it but we don't know how to say. (Alin gives S6 the board marker and asks her to draw on the board what she had in mind.) ... You can use colors (104).

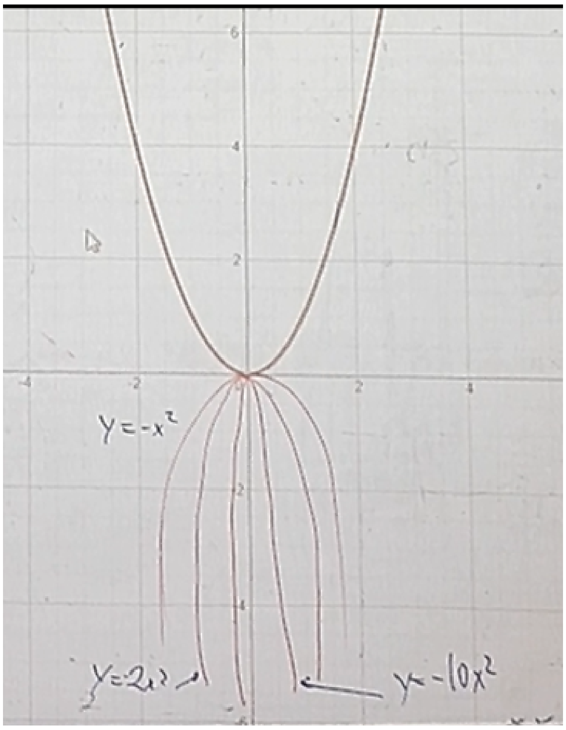

Figure 7.12. S6's drawing what she imagined given the different negative values of "a".

A: Okay, he says that the y= -10x^2 will be the narrower one. What might be so? I mean what was the reason of his thinking. What is the effect of negative sign here? And what is the effect of 10 here?

S10: 10 makes the line bigger, and minus makes closer.

S8: It is same. The a is same as previous section, but minus sign just makes parabola opposite direction.

A: So when I increase the absolute value of a value, what should I expect? Should the graph get narrower or wider? when I increase the absolute value of a?

Alin's first asking the effect of the negative values of "a" on the values of y suggested that Alin wanted to take her student's' attention on the reasoning behind the reflection of the graph, $y=x^2$, with respect to the x axis (PIP.2A1). Data further suggested that Alin assessed, using their prior engagement about the graph of the quadratic functions given the specific values when $a \geq 1$ and $0 \leq a \leq 1$, whether students would be able to explain the changes on the width of the graphs when $a \leq 0$. Her first asking students to imagine and then to draw also further suggested that she was after their current reasoning (i.e., assimilatory schemata). That is, had she first asked them to draw, they might have only focused on the graphs externally. Rather, asking them to imagine first, she wanted to assess whether students were able to mentally construct the image of the graphs focusing on why the graphs would look like as they do. This again suggested that from her point of view mathematics was dependent on the knower and the students learn mathematics through their own mind activities. This further indicated that from her point of view the graphs had similarities and differences when $0 \leq a$ and $a \leq 0$ (PIP 2A). Therefore, she wanted her students to realize such characteristics of parabola so that they could come to some generalization for the intended learning. That is, she wanted her students to internalize that once the absolute value of "a" increases, due to the increase in the rate of change of y values with respect to the x values, the graphs would be narrower. All these data pointed to the characteristics of PIP.2B such that she closely monitored students' responses to determine if the required known mathematics has been established for the newly learned to be linked. Particularly, Alin's goal for the learning was to bring students to a generalization that for different values of "a", such as $1 \leq a$, $a \leq -1$ and $-1 \leq a \leq 1$, the graph of the quadratic functions would behave similarly albeit some differences.

Alin's post-interview provided some more data regarding some significant changes in her current lesson and planning for future. She focused on the questions she asked during teaching and stated:

I would definitely make the questions here more clear, perhaps I would have started directly together. I would say take the differences between the values of y and examine how these changed according to the change in x. I think I had to give the directive more *clearly. they did not see the rate of change, so they couldn't focus on what I wanted at the beginning because the question I asked was not understood. so I thought a little bit about what I would do to focus them on what I wanted (42).*

Data indicated that Alin interpreted her students' difficulty in understanding her questions as their lack of knowledge of the terminology of "rate of change". So, she indicated that she might design her new lesson plan according to her inferences from this teaching. Particularly, she mentioned a change of some questions for the lesson planning in the future:

A: I would leave my questions the same, or, for example, I would leave my table the same. So, I just changed the directives in my question. I would have made it clearer, so, I would make the same things more understandable. I would not change the questions I prepared, or the questions I asked during the lesson, for example, why do you think so.

R: Well, what is the reason you did not change them?

A: Because what I planned for my lesson can activate their mental actions. So, when they go through those mental operations, I have to ask them because I want them to learn this. Therefore, only I can ask questions more effectively, what can I relate to the topic rather than making a change in my lesson plan (50-52).

Again, the data pointed that Alin's main purpose was to examine students' cognitive processes and activation of prior knowledge for the intended learning to take place. So, all the questions she asked were to serve this purpose. Some questions such as "why do you think..." were tools for her to examine and interpret students' mental activities during teaching. Planning to change and also holding some of the questions the same for the lessons in the future showed the characteristics, PIP.3A. That is, data pointed that Alin's planning for future lessons depended on her interpretations of how her students reasoned and what difficulties they have encountered.

In sum, as well as the data from Alin's lesson plan and the pre-interview, data from her teaching and post-interview also pointed to some significant and salient aspects: First, by continuously monitoring students' reasoning through questioning, Alin paid close attention to her students' assimilatory scheme (i.e. what and how they currently reason and know) to active for the intended learning to take place. Second, her focus was on the cognitive processes of the students while requiring them of their justification. Again, all these pointed to two other important aspects of progressive incorporation perspective. First, in almost all parts of the teaching, Alin's emphasis on how students reasoned and what difficulties they encountered suggested that she considered mathematics as dependent on the knower. In addition, her not only focusing on the intended learning goal but also how students reasoned suggested that she acknowledged that students learn mathematics mentally operating on their already established knowledge. Therefore, all these showed that Alin held all characteristics of PIP. In the next part, I further analyse the data for teacher noticing stage of Alin.

7.1.2. 2016 Results for Noticing

As the data showed from previous section Alin had PIP perspective. In order to determine the stage of Alin's noticing, I further analysed the interviews conducted before and after the practicum teaching with the codes of Learning to Notice Framework (Van Es, 2011). Table 7.2 below shows the frequency of codes.

Table 7.2. The frequency table of the codes of noticing before and after of Alin's teaching in 2016.

		What Teachers Notice							How Teachers Notice					
		1-Baseline		2-Mixed		3-Focused	4- Extended		1-Baseline	2-Mixed	3-Focused		4-extended	
Before	Interview						4a	5			3a		4a	1
											3b		4b	1
											3c		4c	1
											3d		4d	1
													4e	5
													4f	2
Total														11
After	Interview	1a	1	2a	1		4a	5			3a		4a	4
											3b		4b	3
											3c		4c	7
											3d		4d	3
													4e	6
													4f	4
Total			1		1			5						27

As the data indicated, Alin depicted mostly the codes from the extended level albeit rare codes from the baseline and mixed levels. Particularly, data showed that Alin started to talk about the lesson she taught similar to the teachers at the baseline level attending whole class behaviours; and, teachers at mixed level attending teacher pedagogy. While data showed that at the baseline and mixed level the codes referred to what teachers notice, Alin did not depict any codes at the baseline and mixed levels for how teachers notice. In addition, data mostly showed that Alin depicted each of the codes at the extended level more than once adding up to 11 times during the pre-interview and 27 times in total during the post-interview. In particular, similar to the teachers at the extended levels, Alin attended to particular students' mathematical thinking, interpreted these thoughts deeply and noticed and explained the relationship between these thoughts and the teaching strategies.

In the following paragraphs, first I share data showing what Alin has noticed and then share data to show how Alin has noticed at different levels. For that, I provide data from the interviews conducted after the teaching session and before the teaching session.

<u>7.1.2.1. What Alin Noticed.</u> During the post-interview, as already mentioned, Alin started to talk with the general views about her class attending to whole class learning.

I think I focused the students on the points I wanted, because at least the answers they gave and the things they wrote were in that direction. So, it wasn't too bad for the first lesson because they are not my students. I only taught them for 35 minutes. Because of that, they tried to help as much as they could. I think it was nice (2).

As the data indicated Alin pointed to the effectiveness of the lesson providing some general descriptions. Particularly, emphasizing the classroom climate, she mentioned that the whole class participated in the lesson eagerly. She also interpreted the lesson stating, "I think the lesson was good". Though talking at the beginning of the interview with general impressions (1A), she continued with the general teacher pedagogy and strategies. Alin stated:

First of all, I left them alone, I walked around and observed them, they did not do much. So, the questions were not understood. I asked if the questions were not understood. I said, let's start the discussion together, we have already drawn $y=x^2$ in desmos (4).

Alin noticed that students were not able to answer the questions asked in the task sequence. She interpreted that the questions might have been complicated for students and she decided to create a discussion with the whole class (2A). Though, she mentioned the whole class and did not give any example from particular student's thinking. She also mentioned her preparation of a graph using Desmos application. Alin's attending to her teaching strategies (ie. creating discussion and the use of technology) and the problematic issues in her teaching without any evidence from particular student's thinking showed that, at the beginning of the interview, she showed the 2A code at the mixed level. Yet, for the rest of the interview, Alin focused on particular students' mathematical thinking and talked about the relationship between these and the teaching strategies.

As the coefficients increased, they drew the graph correctly. But I also asked the reasons because they might already know. Perhaps they have heard or guessed. But they always said the increase in y value is more. For example, just a child said something, the difference between y values is increasing, and I said the rate of change is the same with the difference between y values? A girl said no. No, no, I asked her to come, then. The girl said, we know we can't explain. I said, you say it in your own words. She said that the increase in y values with respect to one unit of x. Then when I said "what's going on when x is (inaudible)", they showed it by hand, that it will be downward. They showed below by saying it is reflecting (10).

As the data indicated, students' thinking, and understanding was important for Alin because she had determined the teaching strategies paying attention to the difficulties that students might encounter. Particularly, data showed that Alin noticed a student's thinking such that the student only mentioned the values of y changing increasingly. Though, she further mentioned questioning as a teaching strategy to take her focus to the difference between the rate of change rather than a sole increase in y values. Similarly, she mentioned another student answering her question correctly. Further, although the student was not expressive enough, Alin encouraged her to justify the reasoning behind her answer. Alin further explained how that student reasoned "the changes in the values of y need to correspond with the changes in x values that are in increments of one". All these data suggested that Alin noticed and related both the teacher pedagogy and particular students' thinking. That is how she decided how to respond to these special events during the lesson. By using different strategies such as asking probing questions, creating discussion among students and also drawing the graphs on the board, Alin attended to particular students' different solutions and thinking ways. Even she mentioned a particular student's using her hands to show the direction and the shape of the graph. Alin's attending both to the particular students' thinking and linking those with different strategies she has used during the lesson showed the code 4A, at the extended level.

In addition, Alin planned hypothetically to notice particular students' thinking in her lesson plan and explained during the pre- interview:

They already learned the quadratic equations. For example, they always said something, for example, when it was +, they put up smiley face, when it was -, they put a sad face, but they did not think why. But, they did not talk about the reason, they knew how to draw the graph. They know that delta cut the x axis at two points when it is greater than 0. They know that it is tangent to the x axis and touch the x axis in one point when it is equal to 0. They also know that when it is less than 0 [parabola] was upward. But a student, for example, could not draw it and understand it and asked about it. When he drew the graph, he did not fill some the points where the graph cut the x axis. I thought there was something that they didn't understand about the graph. I decided on this subject so that learning how the coefficients are changing will be good to understand the meaning of the complex numbers, to understand the meaning of the delta, the meaning of the axis of symmetry (pre-10).

In lesson planning process, Alin noticed some important aspects about the intended knowledge. So, she emphasized her attending pointing to three significant aspects: First, Alin planned to attend to students' thinking in general about the direction of the graph according to the sign of "a" in $y=ax^2$.Though, later she justified why she has developed such a lesson plan linking it with a particular student's question (to understand how the graph of a quadratic function once the delta was smaller than zero look like). Secondly, Alin attended that students' prior knowledge was very important in the lesson planning process. She noticed that students already knew how to find the roots of quadratic equations with three conditions of delta, how to graph quadratic functions and how they look like when the leading coefficient "a" was positive or negative. So, she constructed her new plan based on students' already established knowledge. Lastly, Alin mentioned the link between what students will learn (ie., the intended goal) and the related topics such as complex numbers to be learned in the future because there were significant connections among the intended learning and the related topics. Her mentioning how students' thinking about the different values of "a" for $y=ax^2$ might help them in reasoning about the axis of symmetry and complex numbers further showed that she noticed the relationship between the intended learning goal and the related topics.

<u>7.1.2.2. How Alin Noticed.</u> In this part, I share data from both Alin's pre-interview and post-interview. She showed all the codes of extended level noticing during the pre-interview and the post-interview. I divided the codes 4A, 4B, 4C, 4D and the codes 4E and 4F to show why Alin was at the extended level (Level 4) although the focused level (Level 3) had also the same codes (4A, 4B, 4C, 4D) with extended level.

Data for 4A, 4B, 4C, 4D from the Post-interview and the Pre-interview

Data showed that Alin not only attended to different aspects of the lesson and different students' thinking but also explained and interpreted these attending to situations detailly by providing evidence based on the reasons behind the students' thoughts. Particularly, Alin started talking about the lesson providing some general impressions as below:

I think I focused the students on the points I wanted, because at least the answers they gave and the things they wrote were in that direction. So, it wasn't too bad for the first lesson because they are not my students. I only taught them for 35 minutes. Because of that, they tried to help as much as they could. I think it was nice (2).

She explained her lesson with evaluative comments such as "it wasn't too bad" and "I think it was good". In addition, she pointed to the students' answers as providing some evidence that she reached her learning goal for them. These explanations showed that Alin depicted the 2A and 2B codes at the mixed level for the how teachers notice.

Although at the beginning of the interview, Alin mentioned the whole class learning with few sentences, she continued with important events in her teaching. She highlighted a limited answer by the students as below:

Difference in x values ... they did something, they looked at the increase in the initial x values, 2 increased, so they thought that it was always adding 2, but they didn't think that the increase here increased. They didn't understand that I was asking them or because of my question. They always said that, for example, when x increases y increases also, so they gave an incorrect answer, or if they tried to find a pattern of change of x values. More precisely, they did not say there was an increasing rate of change, they said the changes in y values are increasing (6).

Alin attended that students understood the relationship between the variables, x and y, but they could not explain the rate of change of y with respect to x. She emphasized students' limited understanding and missed point about the rate of change. In addition, she attended and emphasized other students' thinking and answers as below:

As the coefficients increased, they drew the graph correctly. But I also asked the reasons because they might already know. Perhaps they have heard or guessed. But they *always said the increase in y value is more. For example, just a child said something, the difference between y values is increasing, and I said the rate of change is the same with the difference between y values? A girl said no. No, no, I asked her to come, then. The girl said, we know we can't explain. I said, you say it in your own words. She said that the increase in y values with respect to one unit of x. Then when I said, "what's going on when x is (inaudible)", they showed it by hand, that it will be downward. They showed below by saying it is reflecting. When I said what would happen when they increased the coefficient of the x, they said it was getting narrower and wider. I said them to draw on the board (10).*

For Alin, students' making sense of the rate of change of y with respect to x was one of the significant mathematical ideas in the lesson. Therefore, Alin attended both the whole class and the particular students' thought processes about the simultaneous change of the values of x and y. Highlighting both the limited and the correct ideas of students showed that Alin did not focus solely on the expected answers. That is, her focus was not only on evaluating particular students' answers but was also on how they reasoned. These data from the post-interview showed that Alin depicted the code 4A for how teacher notice.

In addition to the results from the post-interview, Alin noticed some significant events during the lesson planning process:

R: So, you say that you will start the lesson with the graph of $y=x^2$. Why would you do that?

A: Because, $y=x^2$ will be an easy thing for them to understand and play on, that is, they will be able to simulate something by thinking it, for example, because I did not add bx + c in the first place, to avoid the confusion. So, $y=x^2$ to the axis of full symmetry. The graph was what they know best. They will not struggle to find their roots, but it will be easy for them to observe the change in value of "a" (25-26).

Alin explained whys she would start the lesson with y=x^2 function: students knew this function and the graph of it. This suggested that Alin noticed the prior knowledge of students in the lesson planning process. In addition, she emphasized the possibility of students facing with some difficulties. For example, she hypothesised that if students were asked to think about the effects of different values of “a, b, and c” all together, they might not have realized each parameter’s effect. Therefore, she planned to start the lesson asking them only to reason on the different values of a. All these suggested that Alin noticed students’ expected difficulties and planned her lesson by paying attention to these difficulties. Attending both to students’ prior knowledge and difficulties and explaining these aspects in lesson planning process showed that Alin highlighted significant events in her lesson plan.

During the post interview, Alin also pointed to specific students answers and gave evidence from specific students’ thoughts in her explanations. Particularly, during the lesson, when Alin first asked students to reason on the rate of change of y with respect to x for the function y=x^2, students could not explain that. Then, she asked them to think about the function, y=2x. Her discussion of this specific event during the post-interview follows:

I then drew a linear function graph. I chose the graph y=2x, I did not want y=x because x and y would always go one to one. I gave it 2x, so I said, for example, although I don’t remember the numbers I gave here very clearly, I asked them what the values of y would be when they x change from 2 to 3 or 3 to 4. I said once 2. Then I said what is y at 3? What is y at 4? What is the increase here is 2x, will this always be like this? Then I asked what happens at 50 and 51, always the same. So, let’s talk about this with x^2*. They already said that after thinking the values of x, the value of y increased by arrow. Well, when I said how I show this on the graph, a few children already got the answer. They said it was a parabolic or exponential increase. I did something there, are they saying their answer by heart, or did they understand the rate of change because they did not calculate too much? I asked are you sure? Why are you sure? Let’s say what your friend says again. Come to draw the graph on the board? What will happen now? I pushed them like that, but in the end, everyone agreed on the whole thing. they* *were already drew correctly. As the coefficients increased, they drew the graph correctly. But I also asked the reasons because they might already know. Perhaps they have heard or guessed. But they always said the increase in y value is more. For example, just a child said something, the difference between y values is increasing, and I said the rate of change is the same with the difference between y values? A girl said no. No, no, I asked her to come, then. The girl said, we know we can't explain. I said, you say it in your own words. She said that the increase in y values with respect to one unit of x. Then when I said, "what's going on when x is (inaudible)", they showed it by hand, that it will be downward. They showed below by saying it is reflecting. When I said what would happen when they increased the coefficient of the x, they said it was getting narrower and wider. I said them to draw on the board (10).*

Data provides important evidence with regards to her noticing skills: First, Alin noticed that given the function $y=x^2$, students had difficulty about reasoning on the values of y as increasing at an increasing rate. Then, attending to what they knew earlier, she provided them with the function y=2x. Her mention of not using y=x but y=2x as an example with her justification also suggested that she noticed what students needed to reason on the simultaneous change of y with respect to the changes in one-unit increments of x. That is, it would be harder for students to pay attention to the simultaneous change in the values of y and x if the function was y=x. Using their reasoning of the change in values of y with changing values of x in one unit- as a step for her main purpose, Alin passed to the discussion on the function $y=x^2$ and the graph of it. Alin's emphasizing the specific events in the lesson, exemplifying both the specific students' thoughts and interactions with them as evidence showed that Alin also depicted the 4C code for how teachers notice.

In addition, Alin's comments about the students' thoughts were interpretive rather than evaluative and descriptive because she targeted to show not only the general learning of the whole class but also each specific students' thoughts in detail. During the post-interview, she further provided a specific student's thought and her explanation about it as follow:

A: For example, we can look at that, he said that the y values in $y=x^2$ increase as exponential increases as x increases one by one. He already calculated one here, for example. The child who wrote this was already verbally explicit in the lesson, so he wrote less. They have difficulty on expressing themselves. They say that changes of y with respect to each unit change in x was more, so differences in y values were increasing. But obviously they had a hard time expressing it as a proportion representation.

R: As if he was telling you the thing there, he said "... increases the multiplication of that number x, for each time x increases by one..." what do you understand?

A: So, I thought of what they said here. All of them did not said exponential. So in $y=2x$, we normally multiply x by 2, so x values increase one by one. Here ($y=x^2$), when it is multiplied by two x, it always starts to multiply as x.x, 3.3, 4.4, 5.5. That's why the y values are increasing more. I increase it more. Since they increased the rate of change in y, they said it was increasing, but they could not express it exactly at the beginning. But at last, they realized the reason for the increase in the graph. For example, when we made $\frac{1}{2}$, they said that the arms of the graph would open towards x. This time it is decreasing because we are multiplying x with $^1/_2$. (18-24)

Alin noticed that while students had difficulty with the rate of change of y with respect to x, by taking their attention to understanding the change in the values of y with respect to x for the function y=2x, students could manage to focus on the changes in the values of y by increasing of the values of x in one unit. She compared students' development by using the students' thoughts at the beginning of the lesson and during the lesson, so she could analyse students' progress. Alin explained her noticing both by discussing how specific students talked and how she interpreted them. Alin's explaining her noticing referring to specific events and specific students' development with interpretive comments rather than only pointing to general thoughts about students' understandings showed that she depicted 4B code for how to notice.

Not only during the post interview, but also in lesson planning process, Alin pointed to specific events prior to the lesson. She interpreted students' thinking about the values of a as follows:

They might make comparisons, i.e. when x increases by one, how many changes in y values, how much change in y. They might say that these rates are always increasing. Now at the beginning they looked at the change of $y=2x^2$ *by different x values, they looked* $y=4x^2$*. Let me talk about the two of them, then they compare the difference between these two. They might realize what x,* $2x^2$ *and* $4x^2$ *are. They might understand that the arms of the graph are narrowing with the operations that occur in their mind. Because they thought it was the first thing, so let me say the slope at* $y=2x^2$*, not say the amount of increase. Let me say the slope, they know that the arms are upward. That is, they are related to the shape of the arms. So, I think they might realize the growth in the arms of the graph as they play with these numbers such as "a". I want them to be. So, if they say it with their reasons, I will say okay (44).*

Data showed that in her lesson plan, Alin envisioned how students might possibly think about the different values of "a" and explained such thought during the pre-interview. Particularly, she interpreted that students might imagine how the graph might change given that the values of "a" would be 2 and 4. She hypothesized that students might envision the rate of change of y with respect to x for such values of "a" and then they might conclude that the graph would be closer to the y axis when the value of "a" was 4 compared to 2. Her attending both to students' thinking and also her own views about students' thinking and changes about the width of the graph showed that Alin noticed students' thoughts and explained these thoughts with interpretive comments.

Alin not only provided evidence from particular students' ideas or interpreted their ideas detailly but also elaborated on the explanations by pointing to some other specific events and different aspects of the lesson. She mentioned examples from two students' thoughts about the rate of change and the graph of $y=x^2$:

He said when x increases, y increases but you know that it is same in the linear function and in so. So, something was not changing. So he did not give us a reason *for "a". There was only another boy who said the difference between the y values. There was no rate of change in the answer of the boy. We are looking for the one that corresponds to one-unit change in x. Actually, I can increase the difference between the values of y by taking bigger things. So, if I look at the difference between the two distant y 'values, it will increase, I am looking at the ratio of the change in y with respect to one unit of x. That boy who did not tell that... There is an answer, he said, "parabola will be thicker", but what do you mean by thicker we drew this on the board, but did he write it first or later (32)?*

Data showed that Alin elaborated on her explanations about students' learning pointing to different students' ideas. She talked about a particular student who could think about the changes of y, but did not consider the rate of change of y. She attended that this student was able to interpret the differences between y values for different x values. However, for Alin, the student's reasoning on just the differences were not sufficient for understanding the rate of change of y with respect to x.

In addition to this, during the pre-interview, Alin referred to some expected events she mentioned in her lesson plan. She planned to attend to particular students' reasoning and particular students' answers based on her questions during the lesson to guide students' thinking further. She even prepared some hypothetical alternative conditions to observe and examine students' thinking. Her mention of attending to particular students' thinking about the graph of $y=ax^2$ follows:

Why did you think so, did you just look at y, don't you see a relationship between x and y? Or what do you see? Why do you think or why not? I'm sure they have the right answer. For example, even if few children give the correct answer, I think to ask, does everyone agree? Can you explain why you agree? Do you think it is right? or do you explain in another way? If the answer does not come, for example, if they cannot say the rate of change, for example, I will ask, which one is faster when you compare the changes in x and y? While x is increasing one by one, y is faster with 2,4, or 5 or something. I will ask them to understand how I am showing this on the graph. If we really get stuck here, if they don't understand $y=x^2$ on the graph, it's one of the *different scenarios, I will ask them to mark the changes in y with respect to one unit of x at* $y=x^2$ *graph on the board. That way we'll at least get to that point, but I think they're going to give an answer already (24).*

Data showed that attending to special events in detail, she further elaborated on her explanations pointing to different students' views. For instance, she had one particular student in mind whom she expected to respond to her questions. She even mentioned what probing questions she would ask if she had not had any student to reason on the rate of change. Her interpretations of how students might reason and her planning to attend those thinking processes indicated that Alin had the 4D code for how teachers notice. In addition, in lesson planning process, Alin prepared a lesson with a task sequence but she also planned the alternative solutions for the unexpected conditions in her teaching. In planning a discussion with students, she prepared some questions but if Alin could not attend students' thinking, she prepared some alternative ways for noticing students' thinking in her lesson. Proposing some alternative pedagogical solutions hypothetically showed that Alin depicted the code 4F for how teacher notice in her lesson planning process.

<u>7.1.2.3. Data for 4E and 4F codes from the Post-interview and the Pre-interview.</u> Considering the link between students' thoughts and teacher's actions, Alin planned her lesson according to her students' characteristics such that the task, questions, applications and the discussion depended on students' thought processes. Although she thought so many aspects of the lesson, she faced some difficulty during the lesson. Further data from the post interview indicated how she planned to provide some solutions to act on for the future lessons:

I would definitely make the questions here more clear, perhaps I would have started directly together. I would say take the differences between the values of y and examine how these changed according to the change in x. I think I had to give the directive more clearly. They did not see the rate of change, so they couldn't focus on what I wanted at the beginning because the question I asked was not understood. So I thought *a little bit about what I would do to focus them on what I wanted. There would not be many changes in the next questions about the values of "a", in other parts, because it is easy to discuss them after the first one over. If I had time, they didn't want to make calculation there too much, I thought if I made them calculate, they would make calculation. But again, the most important thing I will change is to take the differences between these and calculate how they change with respect to each other. So, what you have done? that is, what is the process you are doing? I could ask something like that. I could make my questions clearer (42).*

Attending that students had difficulty in thinking about how the x and y values change in the function $y=x^2$, Alin mentioned using different questions in the future. Proposing alternative pedagogical solutions showed that Alin depicted the code 4F for how teacher notice.

Attending the events is significant for teacher noticing but linking with the teacher's actions or pedagogical ideas of teacher was more necessary for Alin. She stated an example of this connection as follows:

R: What is the reason why you want them look at the differences between the y values according to the differences between the x values?

A: The reason for the shape of the graph, that is, when they compare the variance between the two at the same time and then do this for different "a" values, they need to do mentally how the variance changes as the values of the "a" increase. I will ask a question for them to go through that mental process, but they should do calculations first. Then, I can say that I am asking a very nice question now. I have to ask the question that will enable them to make sense of what they are doing while doing that calculations, so my goal is to activate that mental action.

Data showed that for Alin students' cognitive processes were significant. Alin particularly prepared questions so that the students would compare and contrast the changes in the values of x and y. This was important for Alin because she believed that there is a significant relation between her questions and the students' thought processes and learning. In particular, data suggested that she believed that learning is dependent on the knower's activities both mentally and externally. Also, by asking probing questions, she planned to notice students' thinking and understanding. This indicated that Alin linked her actions with pedagogical ideas such asking probing questions and students' learning through their own actions both mentally and physically. In addition to her questions, planning to use multiple representations such as graphical, algebraic, verbal and tables, Alin further noticed and showed the link among principles in teaching and students' thinking and learning. In her plan, she attended to the significance of questions in the task, materials in the task and different representations of the graphs.

Alin further commented on how she linked teacher principles and students' thinking.

I want them to carry out quantitative operations. I need to ask the questions for learning of them, thinking about performing those operations. If I notice the places where they are confusing, I will ask questions what they will associate with. Some children may go too quickly, and some may not. I can try to change my questions that they can understand. I can ask questions to correct what they misunderstand, that is, I will act according to their reactions (46).

Data showed that Alin mentioned that her questions would guide students' reasoning. She prepared her questions before the lesson in such a way that it would allow her to attend students' thinking and reasoning. She planned to continue her lesson depending on how students might possibly think. This suggested that she planned to notice students' thinking and reasoning during the lesson. This connection showed that Alin had 4E code for how teacher notice both at the pre- interview and the post-interview.

Aforementioned data showed that Alin depicted the codes 3A and 4/A, 3B and 4/B, 3C and 4/C; and, 3D and 4/D. These suggested that she had attained both the focused and the extended levels of noticing. Though, data further indicated that

Alin also depicted the codes, 4E and 4F, differentiating the focused (3) level from the extended (4) level: attending the events was significant for teacher noticing but linking with the teacher's actions or pedagogical ideas of teacher was more necessary according to Alin, Alin interpreted attending events for her teaching during the post interview so she showed each code at the 4th level at least once. Also, the data from her pre-interview supported the post interview results. The results of both the pre-interview and the post-interview were shared together to display that Alin showed the 4^{th}, extended level for teacher noticing.

7.2. Alin: Case of 2018

7.2.1. 2018 Results for Teacher Perspectives

The analysis of Alin's pre- and post-interview and teaching in 2018 depicted hat Alin showed all characteristics of PIP at least once (See Table 4.2.1). In the following paragraph, I share data to show how these codes revealed during interviews, and also during teaching.

Table 7.3. The frequency table of the characteristics of PIP before, during and after Alin's teaching in 2018.

Alin-2018		PBP	PIP			CBP	Noticing
			1a	2			
Before	Interview		1a1	9			
			1a2	1			
			1b	3			
During	Teaching		2a	3			
			2a1	3			
			2b	4			
			2c	2			
			2d	1			
			2e	1			
After	Interview		3a	5			

7.2.1.1. Alin's Interviews Before and After Teaching. In this section, I share data from both Alin's pre-interview and post-interview in 2018. She did not provide her lesson plan in written form, though she had the planning envisioned. Alin showed the characteristics of PIP.1A, 1A1, 1A2, 1B and 3A. Previous research has shown that teachers holding PIP prepare their lessons according to the curricular objectives or they create lessons on their own (PIP.1A). In addition, teachers having PIP assess their students' already known knowledge to use as a conceptual anchor for their intended learning (PIP.1B). Alin's explanations from pre-interview included examples of all these characteristics at least once. Data also suggested that she hypothetically planned for the lesson based on her students' cognitive processes (PIP.1A1) and their behaviours (PIP.1A2). In addition, she planned her next lesson based on her observations that came from the previous lesson (PIP.3A).

Alin's objective for teaching for the next couple of classes was that her students would understand exponential functions. Therefore, she first wanted them to focus on the meaning of exponential growth and decay. She planned her lesson for students' making sense of the rate of change in exponential growth and decay such that she prepared two tasks in a sequence (i.e., exponential growth and decay). She stated during the pre-interview that her objective was as follows:

Now, the objective of the lesson is.. we will teach this lesson exponential functions, in fact, 2-3 lessons. The objective of this course is actually what are the exponential growth and decay mean before switching to exponential function. 1. 2. 4. They are always multiplied by a constant factor?. In the exponential function, each time it is multiplied by the same number. There is a common factor, I expect them to realize it. If this factor is greater than one, this is exactly the definition of the exponential function. We do not examine that a and b are x over a, and the value of b is, and we will do it the next lesson. Before I do, they will realize that if the value of the factor is greater than 1 there is some increasing, if the value of the factor is less than 1, there is some decreasing. They will determine whether the amount of change increases or decreases each time, that is, actually, they will learn. I want them to realize that multiplying each time with a value that is more than one increases the amount of increase. Or if it is *multiplied by a value less than one, I expect them to realize, learn and explain that the first amount gradually decreases and the amount of decrease decreases and the amount of change decreases (7).*

Alin stated that she expected her students to reason that for example for exponential growth there is a constant factor the output is always multiplied with. So, she wanted her students to make sense that when the common factor is bigger or smaller than one then the function is an exponential function. But, she would like them to consider that later. For this class especially she wanted her students to explain that the amount of change between the two values of the function increases once the common factor is bigger than one and decreases once the common factor is smaller than 1.

Alin explained how she decided on this objective:

R: So how did you decide on the objective of the lesson? To the learning goal.

A: Normally, the goal of the subject was to learn about exponential functions and to make appropriate applications. I expect them to construct the relationships in the applications such as an increase in the population, a decrease in the number of bacteria, after defining exponential functions.? Let me show you something and turn it towards the camera. Our lesson plan, our curriculum (She was trying to find it from the computer.). At the beginning of the second semester, by teaching the functions? During the first term, we actually learned quadratic functions, their graphs, polynomials, how y and x change with respect to each other. We learned basic parent functions, basic graphs of the parent functions and how they shaped? So children know that each function consists of x and y ordered pairs, each value of x and y ordered pairs shows a point in the graph, and when we combine all of the points that are matched with these functions' domain and range, we get the graph. While we were drawing graphs of third-degree polynomials, they actually learned to draw graphs of functions of rational numbers (7) (37).

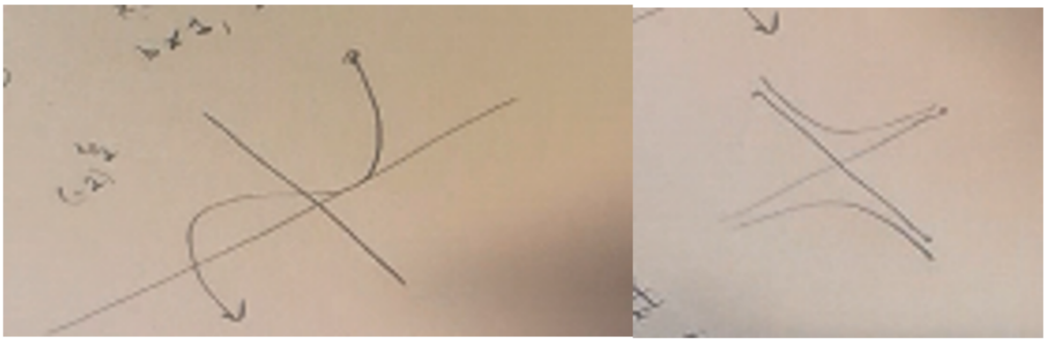

Figure 7.13. Drawing of Alin to show example from prior knowledge of students.

Alin pointed that in lesson planning process, she considered both the curricular materials and her own views about her students' background knowledge. She pointed to the sequence of the topics in the curriculum to show the link between these topics and the intended topic. Particularly, she mentioned that students learned about parent functions such as for quadratics and polynomials to the third degree. They also knew that functions represent the relationships between the variables x and y such that the graph of any function would represent the correspondence between the ordered pairs of the elements of domain and range. Therefore, she knew that students already knew characteristics of some parent functions as well as how to graph them and what the graphs represented. So, emphasizing what students knew regarding functions and their graphs, she further explained how she planned the tasks on the students' prior knowledge as well as taking into consideration of the curricular materials (PIP.1A).

Alin further explained that her students were ready to learn the intended topic. She explained:

They normally saw quadratic functions, polynomials, linear functions. So, they know what is when there is a graph with constant increase or a quadratic graph. They have also learned how the values change, but each time it multiply by "b?, and if this "b" is greater than one, I am expecting them to examine how the values of the function change, or how the shape of the graph will be...So children know that each function consists of x and y ordered pairs, each value of x and y ordered pairs shows a point in the graph, and when we combine all of the points that are matched with these functions' *domain and range, we get the graph. Because, we give the x inputs to the function. They can say that we examine how y outputs behave, so they are ready to learn. So, they could give the set of domain. They could either define the set of range or tell if it was 0 or not. Does it get closer a certain value, is there an asymptote? They can read all the graphs now (49).*

Again, one of the most important aspects of Alin's lesson planning was her focus on previous knowledge of students. She stated students' already known knowledge as a conceptual anchor for her intended topic (PIP. IB). Particularly, Alin knew that students knew that for any function, the input values (i.e., x) in the domain and the output values (i.e., y) in the range have a correspondence such that all the ordered pairs (x,y) in relation to the domain and range construct the graph of any function. In addition, students also have known how quadratic functions, polynomials and linear functions behave in terms of the relationship between the variables x and y. Therefore, she conjectured that students were ready to analyse how the relationship between x and y values change once the y values is multiplied with a constant b. So, she expected them to analyse such relationship as well as examine the graph. Thus, she expected students to construct the meaning of exponential functions and draw the graph of these functions by using their already known knowledge (PIP.1B).

For that purpose, Alin prepared two tasks for her lesson and explained the tasks. The tasks involved both exponential decay and growth. One of Alin's task was about increasing the population of a bacterium by using the chips that include numbers on one side. Students would increase the number of chips by adding the chips according to the chips with the number. The second one was decreasing the number of the bacterium on a plate with beans. Students would decrease the number of beans according to the beans in the area on the plate. She also said that she would divide the class into two groups working on exponential growth and decay. Then, she commented further:

...Here, they actually observe the amount of change, counting it with hand, even though they did not focus on it. They note it and write the amount here. Likewise, for the increase, they start with two. The chips with the numbers increased and added

more each time. Finally, the chip is sometimes not enough, they might want from other tables. They understand that the increase is increasing, multiplying by two. It has such an effect. And at the same time, they can draw the graph, and this is not a linear, it is not also a quadratic. They also have an idea of what the graph will look like (61).

First, Alin knew that once students did the experiment with beans or chips to record the changes in the numbers of bacterium in each ten trials, they might not have focused on the amount of change. Still, in the long run, once they record the amount put in the plates and plot the recorded data on the graphs, they would have realized that the newly recorded amount would be two times as much of the previous amount. They would also realize that the amount of change increases each time. She even stated that given their prior knowledge students would have recognized that the function would be different from a linear or a quadratic function. This data suggested that she also focused on her students' behaviours (e.g., recording data, drawing graphs) (PIP. 1A2).

She further commented on her planning. Particularly, she stated that she would expect each student to read and understand what they are expected to do for the tasks. This was important for Alin as she commented "...if I read and explain the task at the beginning, I know the class. I don't want that while somebody start the task, somebody sit". Then, she further commented on how she planned to assess her students' thinking progress by asking questions during teaching:

If they don't understand nothing, I plan to ask like which one is the x variable, which one is the y variable, which changes with respect to which, we look which one is changing. I can ask them if they get stuck at this point. The trial number and number of beans were given in x. I want to ask them this too. Which one changes with respect to which? Why? Because it is writing here? Or, I might ask, can the trial number change according to the number of bacteria? At least there is a change with respect to each other, so they will understand that we're looking at their change. I can ask another thing to measure if they understand there. The rate of change is not in the objective of the lesson. I expected them to already know...(91).

As the data indicated, first, Alin planned to focus her students' attention on the problem context to determine the variables, x and y. Then, she expected her students to determine which of these values change with respect to the other and how they change and why. She even commented that she would ask if the number of trials might change with respect to the number of bacteria since she wanted them to realize that there is a change in these quantities with respect to each other. All these suggested that Alin planned to ask questions to focus on how students think (PIP.1B). Alin also realized that students may face difficulties about doing the experiment or thinking about the relationship between the number of trials and the number of bacterium. Thus, Alin's consideration of students' difficulties and confusion before teaching allowed her to prepare some alternative questions. This further suggested that she was ready to pay attention to students' misconceptions or difficulties during teaching as she prepared questions to examine students' thoughts from different views (PIP.1A2).

Alin emphasized the students' thinking process during the experiment as follow:

Normally when I say how does it grow? Exponentially,.. Actually, if they know English exactly, they will say exponentially because it is increasing. I make students to think there. I want them to think, so I will ask even if they don't think it themselves. Or I ask it when I say how the amount of change changes here. I want them to go through that stage. Likewise, since I want them to formulate this, it is my final product and I expect them to reach it. To reach it, I make them to think in steps here?. I will ask them, how is the graph decreasing or how is it increasing, how did you decide? I expect them to say something, either decreasing or increasing, we reduced it every time. Here too, they may not be able to focus directly on the amount of change. Only some of them showed but it may not be very important because I will ask more questions in the next sections. And I want them to do something. Let them find the percentage of the amount of change each time. 50% has changed, 30% has changed, 30%, 40%. While doing these, I said that each time, yes it changes differently because we make an experiment. It may not be the same, but what is the percentage of the amount of change. They have an idea about increasing, like that increased by 50%, increased by 30%. Then they take an average. For the next part, use the average that you find (63).

Doing the experiment and focusing on the questions she planned to ask, Alin first expected students to determine the difference between each trial and to think about the amount of change. Together with the how the change occurred, she also would like them to think about the percentage of change in each recorded trial. She also emphasized that students might calculate the percentage of change in each trial. Though, Alin wanted students to use the average of all percentages in ten trials to reach the constant factor. In this process, Alin pointed that she planned to ask questions continually to observe students' thinking and changes in students' thinking (PIP.1A1). She continued to explain how she hypothesized about how students might possibly think:

I increase 50% each time and this means multiplying by 1.5. I want them to reach it here, then they create a patten that I increase it by 50% each time, multiply by 1.5. What happens in this pattern? They say that, yes, I multiply by 1.5 and I want them to realize that my amount is increasing, and my amount of change is always increasing. I want them to explain this. By saying they can explain this, I mean that they understand, by comparing the change of x and y (77).

Data was important: Alin's main purpose was for students to make sense of the multiplicative growth in exponential functions. She wanted students to understand the reason behind why they multiply with a constant number to determine the amount of increase in exponential functions. Particularly, in her lesson plan, Alin focused on the number of bacterium with an increase of 50% from one state to the other for each trial such that 50% meant multiplying the initial value with 1.5.

Alin had commented that she would prefer at most three students per group to understand how they thought. She then commented further that understanding the meaning of multiplying with 1.5 was significant:

If it were like: children multiply with 1.5. It multiplies, multiplies. The friends made in the group. He saw all of them 1.5, 1.5, he wrote here, he multiplied it. When I ask why you multiply it seven times, there is no answer, it is only numerical. Here, he *says 2 in 2 and 3 in 3. But if when I ask why is the 1.5to the power 7, they stated that I multiplied it 7 times with 1.5 every year. So, if they give 7 times the multiplication, then it means they understand. So, my goal is to measure what they understand. So, the answers they give in this paper may not be very descriptive for children's understanding. So, I will try to ask questions...So what's going on here? How did you find this? Why is this happening? How is the change changing every time? How can you show this from the graph? So, for example, they explain in here, but I couldn't be sure about their understanding here, so I can ask that can you give me an example with numbers? The explanation sounds right, but I want it to be explained in a different way so I'm sure. Yes, he explained in one way, this can be said in another way, that means that he has a grasp of the topic. This child understands the topic. I am thinking of doing analysis there. In fact, when I visit the groups every time, asking the questions means doing analysis for me (91).*

Alin emphasized a significant point in her planned activities: analysing each students' thinking in a group work. Her statement focusing on students' thinking on what the meaning of 1.5 and 7 means in the expression $(1.5)^7$ suggested that she did not want students to solely find out the pattern procedurally but expected them to know what they referred to and why the expression $(1.5)^7$ came through. Alin also planned to collect students' worksheets at the end of the class for analysing their understanding and planning the next lesson. Still, she believed that worksheets would be not enough for observing students' thinking and understanding. Therefore, she planned to ask further probing questions to analyse students' thinking and cognitive processes during the lesson (PIP.1A1).

At the post-interview, Alin pointed that she had prepared the lesson plan last year, but she made changes in specific parts due to some difficulties of students. She explained the changes by comparing lesson plan of two years as follow:

So, let's assume the percentage change rate is the same and let's calculate the number of bacteria in the second try with the previous one or the number of chips or the number of bacteria. In the first year, here, this was very clear.

Trial #	Number of Beans	Final amount of Beans in terms of B_0
0	$B_{0=}$	------
1	$B_1 = B_0 - B_0(\quad) = B_0(\quad)$	----- ----
2	$B_{2=} B_1 - B_1(\quad) = B_1(\quad)$	$B_2=$

Figure 7.14. The task prepared for last year by Alin.

When I say calculate the number of bacteria in second year by using the first year, the student writes B1 in by using the B0 each time. While doing this, I subtract 50% from B1.. Likewise, the percentage of B0 minus B0 is this. And, here, they examine what their meanings are. They do the same operation every time and multiply by the same number.

$$B_0 - B_0(\quad) = B_0 x(\quad)$$

Figure 7.15. A question in the task from last year.

Trial number	Number of Chips/Beans	Final amount of chips/bean in terms of C_0
0	$C_{0}= 2$	
1	$C_1 = C_0 + 0.41 C_0$	
2	$C_2 = C_1 + 0.41 C_1$	$C_2 = C_0(1+0.41)^2$

Figure 7.16. The question that was changed in the task.

Last year, for example, students used 0.5. When they multiplied 0.5 by 0.5, so they wrote 0.25. They increased 0.5 every time. I'm reducing it here too. When they said I will multiply by 1.5 but did not write 1.5 as the power of 1.5, they could not realize that there is a constant factor, they multiplied with a constant factor, they are going to find exponential function. Because the number (0.5) is a nice number but when they calculate 0.13 to the power of 2 and 3, they cannot able to analyse the multiplication by 0.13 each time. So this time, I wrote this step by step... (21).

Alin expected that students find the average percentage of changes and use it as a constant number in each trial. In the last year lesson, Alin had thought that students faced a difficulty understanding the meaning of decay or growth factor due to the provided task sheet that was too open ended ($C_1=C_0+C_0$*(growth or decay rate)). Therefore, she interpreted that students had not gone through thinking processes about the growth or decay factor and the function exponentially. Such assessment enabled her to change the activity so that students could reason based on the initial value and the growth or decay factor (e. g, $B_1=B_0+B_0$*(growth or decay rate) = B_0*(growth factor)) (PIP.1A). Particularly, Alin's observation of students thinking processes about the percentage of change, growth or decay factor and increasing/ decreasing amounts exponentially in the last year lesson with same task (PIP.2A1) provided her with ways to change some problematic points in the lesson plan this year (PIP.3A).

7.2.1.2. Alin's Teaching and Interview After Teaching. In this part, I share data from Alin's teaching in 2018. Data from her teaching was also coherent with the data from her pre-interview. As she planned, she made groups of 3 students and gave the tasks and the materials for doing the experiments. Students started to do the experiments with the beans and coins. Alin asked same questions to different groups to observe students' thinking about how the values of x and y changed. Alin focused on the independent and dependent variables in the experiments. She asked that "Which one is dependent which one is independent variable" to analyse students' thoughts about the values of x and y with respect to each other. Students considered that while the trial numbers were independent variable, number of chips were the dependent variable, because they stated that they thought the trial numbers as x values in the function. Although Alin got the correct answer from students, by asking "How did you decide?" and "Why did you write to x?" questions, she wanted to examine students' thinking and prior knowledge about the values of x and y in an experiment. This was important because as Alin had stated during the pre-interview, she wanted students to remember the meanings of x and y in any function formula and how these variables change with respect to each other. This suggested that, Alin expected students to start the experiment with remembering their prior knowledge (PIP.2E). In addition, asking questions to help students activate their prior knowledge for learning of the intended knowledge is significant. Therefore, Alin continually questioned to re-activate students' already known knowledge to connect students' prior knowledge and the experiments' conditions (PIP.2A).

Then, Alin asked about the amount of change in these variables with respect to each other:

- *Alin: Can you show how did you get the amount of increases? How did you decide?*
- *S2: We add every time we see number as.*
- *S11: After that it constantly increase but it 34. How much is it? 49.*
- *Alin: Why do you need this operation? (She asked the amount of increase that written under the operations). Why do you need this to understand what?*
- *S2: Understand the pattern.*
- *Alin: Pattern of What?*
- *S2: Growth.*
- *Alin: Growth, Growth of What? What is growing?*
- *S11: Number of Chips.*
- *S2: Number facing up. (12.50)*
- *Alin: Number of chips is increasing. Why are you adding these numbers? (12.56)*
- *S11: Number of chips pacing up.*
- *S2: Number of. Numbers pacing up.*
- *Alin: Numbers pacing up, what else. If the numbers pacing up, what do you do?*

- *S11: The number of things added.*
- *S11: Growth.*
- *S11: the number.*
- *S2: Two*
- *S11: The growth of two. Yes.*
- *Alin: In how many trials?*
- *S2: Ten.*
- *S11: Ten.*
- *S2: Within one trial.*
- *Alin: Okay, which one is independent variable?*
- *S2: Hmm. Independent (is) trial.*
- *Alin: Independent variable comes from what?. you can use your TI. Make it easy (26-54).*

Data showed that students' strategies to find the amount of change were significant for Alin's objective. Alin knew that students needed to record the difference between each trial. However, since students were conducting an experiment, the amount of change of y was different in each trail for each group. Alin's students knew that thy needed to find patterns and the possibility of variables, so she directed her students to think about the increase by asking them "how did you get the amount of increase". By asking questions continually, Alin again targeted to re-activate students' prior knowledge to reason on the intended learning goal (PIP.2A).

During the discussion of a group, some students had difficulty in determining the relationship between the number of trails and the number of beans:

- *Alin: What can be growth? It increases. What is the independent variable? Which one is dependent?*
- *S4: Dependent variable? (He is looking to groupmates)*
- *S8: Independent is total count.*
- *S4: Total is dependent. Independent is*
- *Alin: What do you think? (She is looking the third student in the group)*

- *S3: Independent is the changing ones. So, the number of chips.*
- *Alin: So, which one is the dependent on the other one?*
- *S3: Number of chips dependent on the number of trials.*
- *Alin: What is your dependent variable?*
- *S3: Trials.*
- *S8: No.*
- *S4: Total of chips.*
- *Alin: Total number of chips. You can plot the data in here or you can use TI if you want. After plot the data, I will come with more question. (93-103)*

In the discussion about the dependent and independent variables of the function, two students gave different conflicting answers to Alin. Since Alin wanted to observe each students' thought processes, she directed the question to the third student in the group. So, at first, although S3 and S8 thought of the total number of chips as the independent variable, through Alin's questioning students could determine that was dependent on the trials. Therefore, data showed that Alin did not revert back to direct teaching; instead, she asked questions so that students could understand their errors on their own and come up with the relation discussing with their groupmates (PIP.2C).

Students had been asked to determine the percentage of change in each trial by using the formula given in the task as below:

To calculate the percentage, we will calculate the percent change for each trial using the formula:

$$\frac{\text{\# of beans in trial 1 - \# of beans in trial 0}}{\text{\# of beans in trial 0}} = \frac{\text{new amount - old amount}}{\text{old amount}}$$

Figure 7.17. The formula to calculate the percentage.

Again, after the calculation of percentages, Alin discussed with the groups about the real-life data and why there were different values in each trial for each group. After finding the percentage change in each trail in their experiments, Alin asked each group to find the average percentage of all percentages so that they would reach the constant factor in exponential functions. Alin wanted students to reason in two significant ways: First, students needed to think that dividing by 4 meant multiplying with 0.25. However, since students found the percentage in each trial by using the formula that was given in the worksheet (see above), they first considered dividing with a number rather then, multiplying with a constant factor. Though students started sharing different ideas and Alin listened to all students' approaches: dividing by 4, finding 25% of a number or multiplying with 0.75. Secondly, Alin knew that multiplying with 0.25 was not enough for the experiments: students needed to calculate the decrease in the number of beans by finding 0.25 of the number of beans in each trial and then subtract it from the previous number. To reach 0.75 as answer, Alin asked students "How do you find the amount when you decrease by 0.25, 25% B0?". Students filled the blanks in the worksheet such that the decrease of 25% of B0 might be written as B0*0.75. Alin wanted students to reach the idea that a decrease in the 25% of the number of beans would refer to 75% of that number. All these data suggested that Alin was open to listening to different students' ideas (PIP.2D)

During the post-interview, Alin explained why her students had difficulties about the rate of change and how she handled it by further questions:

While doing here, the rate of change changes all the time, so they can be stuck at the beginning. The amount of change is always decreasing or increasing, but they cannot realize this because it does not change at the same rate in every time. Then let's assume to them that it is constant now, because we will average it, and we will fit it on a normal function. They think that when discussing the real data regression.. I think that if I do something directly. How to increase a bacterial population with an initial population of 2 and an increase of 50%. But then, I realized that when I asked the children how the rate of change changes, most of them stated that they are throwing less bacteria each time, so the number is decreasing. Because they counted it with one hand, how many we took out each time? then we took out 30, then we took out 20. They go through that counting process because they are counting. When I ask, they can remember and respond directly from the experiment they did. When I say 50% and increase 2 every time, it will be more difficult to imagine both visually and focus on the change in between (47).

During the teaching, students in some groups had difficulty on reasoning about the rate of change due to the real-life data in the experiments. Therefore, Alin planned for students to reach the growth or decay rate by calculating the average of the percentages of change in each trial. Still, instead of calculating the average percentages, some students used different percentages in each trial and had difficulty in explaining the meaning of the constant percentage of change (i.e, the constant factor) in the exponential growth or decay. Thus, although providing a constant percentage rate for each group at the beginning of the task could have been more beneficial for students' carrying out the task, Alin has chosen to let students handle such difficulty. This was because Alin realized that acting as part of a group experiencing the whole experiment students' thinking processes were more significant for students' realization of the meaning of the constant factor in the exponential growth or decay (PIP.2A1).

Similarly, as the data below showed, she used questioning to pay attention to her students' cognitive processes (PIP.2A1).

In another group, Alin faced with the same thinking process:

- *A: What is the average percent change? What is your average percentage change in decimals?*
- *S7: We have that 34.26*
- *S18: Average?*
- *A: Let make it 34 to make it easy. 34 percent.*
- *S18: Average percent.*
- *A: Yes. Because you assume from now on that it will change by 34% in every trial.*
- *S18: Okay.*
- *A: You just took the average. The question is. We want to find the number of trials in first trial. The number of beans in the first trial. Decreased by 34%. How can you calculate 34% of initial amount?*
- *S6: We multiplied by 34 and divided by 100.*
- *S18: 0.34*

- *S7: Try to find the value that we should have*
- *A: Know we assume that is our assumption that is our data. Ok. You need to write B2 in terms of B1 and then B0. B3 in terms of B2 and B0. B4 in terms of B3 and B0 (276-308).*

Again, Alin wanted students to make sense that finding 34% of a number means multiplying the number with 0.34. Alin listened carefully to each student's answer because she knew that there was always different approaches and ways to solve a problem (PIP.2D). By again considering the meaning of multiplying with 0.66, students could reason on the constant factor in exponential decay. Therefore, data suggested that discussions in the groups and Alin's questions were significant for Alin to observe students' thinking during the lesson (PIP.2A1). Again, after all the groups have worked with their data and thought about the initial number (B0) and the average percentage of B0, Alin asked how they could find B1, B2 and B3 in terms of B0. In addition, students had understood that subtracting 25% of a number from itself had the same meaning of multiplying that number with 0.75 or subtracting 34% of a number from itself had the same meaning of multiplying that number with 0,66. This also suggested that Alin continually questioned students not only to assess how they reasoned but also to determine if students were ready to move to the next stage towards the intended learning goal (PIP. 2B).

At post-interview, Alin explained how the activities she prepared, questions she asked during teaching and the groupwork might trigger students' reasoning:

If I say only worksheet in his hands, only 50% increase in paper, increase 2 every time, it will not be a groupwork. They will try and discuss by looking at them. Everybody here, every member of the group, was holding the plate, someone was writing down the notes, someone was counting. So, everyone was involved in the activity. So, when I asked what this activity was, everyone was discussing somehow. Otherwise, they will try to understand individually. They may ask their friends if they don't understand it, but it's very simple here, count beans, or count round rings of chips. This is something everyone can do. The experiment is also visual. Or they can make a work sharing.

They also catch the connection with a groupwork because they discuss what happened in the activity. Therefore, everyone can understand by drawing data (53).

As the data indicated, Alin first mentioned that instead of asking students to fill out the table in the worksheet individually, working with a group helped students to make sense of the links between the numerical data they collected and the graphs they drew. They also had a chance to have discussions on the possible questions they had. Then Alin continued with what she expected students to focus on while doing the task:

I want them to focus on: how does the decrease in the number of beans or the increase in chip numbers change, while the trial numbers change one by one. I want them to focus on how the two variables change with respect to each other. At the same time, when I write this as a function, I want them to think about how it multiplies by the same number and how this multiplication relates to the amount of change. They can do this from the table, they can do this from the graph. Apart from that, they can also do numerical things in the function in the second part of task, or what B1, B2 does. He can do this while answering the questions. That is, they can experience the same cognitive process in different parts. They can experience this from very different points, so there are multiple entry points (53).

Alin focused on students' reasoning about how the two variables (the number of trials and the number of chips or beans) change with respect to each other. Also, she expected them to understand that once they write the relationship between the variables as a function, they would have realized to multiply the initial values with a constant number and think about how growth or decay factor were related to the amount of change. She stated that students had multiple entry points to go through these cognitive processes engaging in the task sequence: constructing table of values with experiment results, drawing the graph based on the number of trials and the number of chips/beans; and also calculating and writing algebraic expressions of B1 and B2 in terms of B0. All these suggested that Alin realized that what students already knew (such as constructing and reading table of values; drawing graphs, writing algebraic expressions) but also what they need to learn (such as the relationship between the variables as exponential growth or decay) had commonalities. She also knew that students reasoning on how the two variables (the number of trials and the number of chips or beans) change with respect to each other would realize that the subsequent values of the number of beans/chips would be a multiple of the initial values once the number of trials increase in increments of one. As well, they would have understood that such multiple is invariant for different subsequent values (i.e., growth or decay factor) such that they would have related to the amount of change in those values (PIP.2A1).

Alin was also opened to listen to students' difficulties. A student in one of the groups had a confusion about whether to use 0.34 or 0.66 as a common factor. Discussion followed:

Part 1:

- *S18: We should find the 34 percent of 170. (Percentage change 34%, first trial 170)*
- *S7: No, we don't write 170 for this. This also have 170. You are writing 170*0.34.*

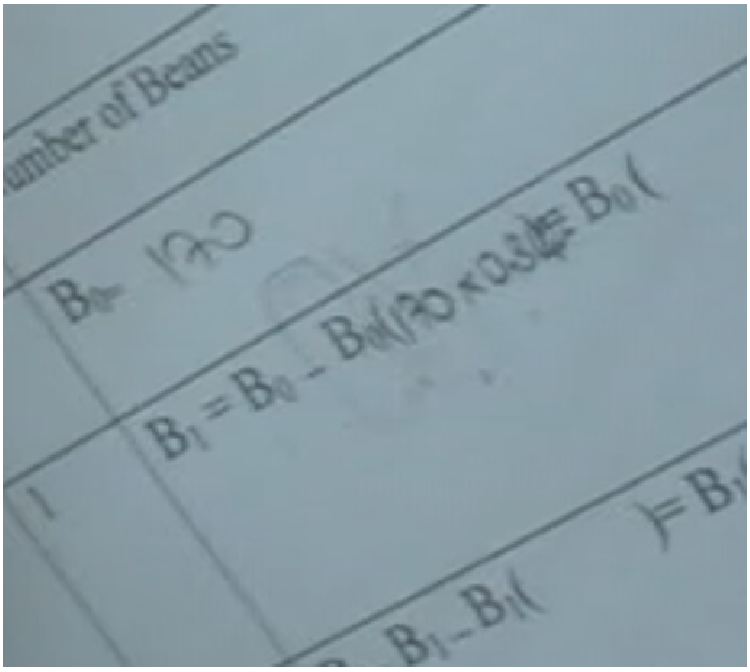

Figure 7.18. Written work of a student.

- *S7: Miss Alin. Isn't like that? In every experiment from the previous one we move 0.34*
- *A: What percent?*
- *S7: 0.34 so 34%. And then. This means that from the previous numbers it would be decreased in this percent.*
- *A: How did you represent that percent in your equation with decimal number?*
- *S7:0.34*
- *A: What is the result if you find B0?*
- *S7:0.66 percent*
- *A: Okay. The question is how we can find B2 in terms of B1.*
- *S7: Okay. It would be*
- *S18: We subtract*
- *S7: Isn't again 0.34?*
- *A: Why.*
- *S7: Because the average percentage of decreasing?*
- *S18: We should take 34% or 66.*
- *S7: It's in terms of B1*

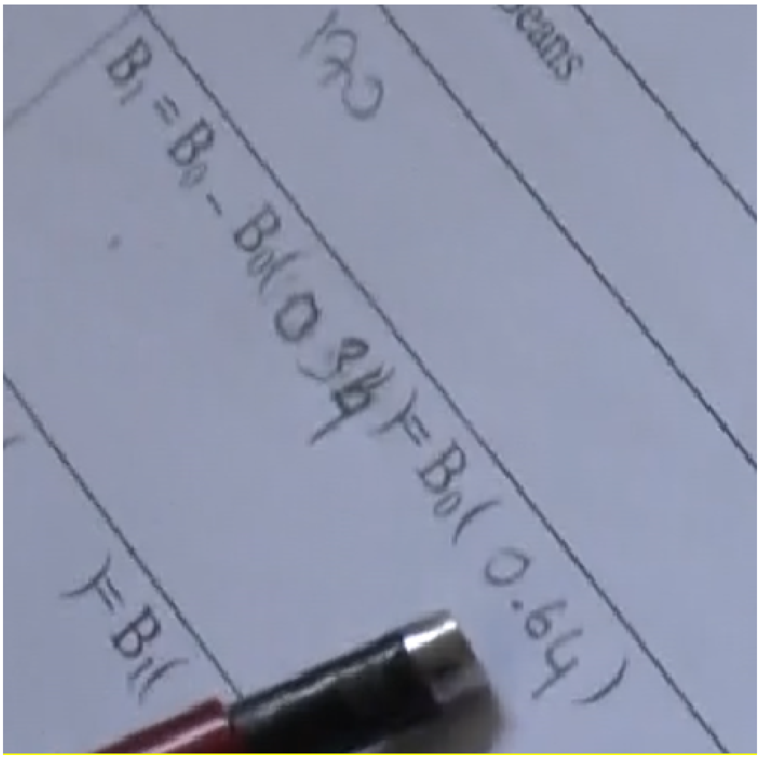

Figure 7.19. Written work of a student.

- *A: What does 66 means? What this represent (pointing to 0,66)? You are saying that B1 is 0.66.*
- *S7: May I. Same thing 0.34 and 0.66 we want.*
- *A: Multiplying 0.66 means what?*
- *S18: Shouldn't we take 34% for 66%.*

As the showed, Alin did not revert back to direct telling nor that she answered students' questions directly. Instead, redirecting students' attention to what the written questions asked, she probed students further asking such as "what does 66 mean? or Why?". This suggested that Alin was after students' thinking rather than expecting them to finish the task properly (PIP. 2B). At that point, realizing that the other group members have not been paying attention to, Alin questioned again and stayed silenced listening to the group discussion:

A: (Group members are interested in other things, drawing attention to other members by touching them) Everybody should we take the 34 percent of 66?

Part 2

- *Alin: Explain your.*
- *S7: In terms of B0 since the experiments goes to 0. As you said, percentage will be decreased nearly reach to 0 by multiplying this value.*
- *S6: You already took the average. Already took the average.*
- *S18: Okay, but you take this 1 in each time. 34% of 1. 34% of 1.*
- *S6: In terms of B1. We already took the average. It is always average.*
- *S18: Okay. No. The percentage is same, so decreasing number will be the same.*
- *S6: Yes, Yes.*
- *S7: Decreasing number. The rate of decrease is same.*
- *S18: E. Okay. You have to calculate 34% of 66.*
- *S6: No brother. How?*
- *S7: In terms of B0.*
- *S6: You find from the rest of difference.*

- *S18: Okay. The rest is 116.*
- *S7: Rest is 170.*
- *S7: Yes, Yes rest.*
- *S18: The rest is not 170, it is 116.*
- *S7: You said that we have only 116. We have B2 and B1.*
- *S6: You subtract from 170 but it is not true.*
- *S18: Yes, I said this. I said that the 34% of 170 and 34% of 116 are not same.*
- *S7: Because the amount of decrease decreases.*
- *S6: The 66% of 170 and the 66% of 110 are not equal. The 66% of 110 is less.*
- *S18: I said same thing.*
- *S6: Okay, You said same thing as we said.*
- *S18: Okay, then, how do we do it?*
- *S7: This always 0.66 but when we write according to B0 then multiple by...*
- *S6: If we pass to next step and need to do in terms of B0, we need to do what she says. Because initial number of beans is changing.*
- *S7: Yes, Yes. We have to multiplied if we find 66% of 66.*
- *S18: But we get 66 because it wants us to write it as B1. (314-358)*

As the data showed, at the beginning of the group discussion, passing to the next step in her lesson plan sequence, Alin focused students' attention on the questions in the task sequence and asked how they could find B2, B3 and B4, using average percentage. In spite of the miscalculations of students for finding B2 by using the B1 and average percentage, Alin wanted to examine students' reasoning for calculation of B2, B3 and B4 by using the initial value in the experiment (PIP.2B). At that point, some students had not realized that decrease in the values were not at the same rate since they multiplied the number of bacteria for each trial (e.g., 170,116...) by a constant number (0.66). Similarly, some of them had not made sense of the multiplication by 0.66, either. In fact, some students made sense that subtracting 0.34 of a number was the same as multiplying with 0.66; yet, one of the students in the group were confused because instead of multiplying 0.66 with B1 she multiplied 0.34 with 0.66. Realizing that student's difficulty Alin asked unprepared questions to that student and groupmates to create a discussion environment in the group (PIP.2C). By Alin's questions, students started to talk about the change in the initial value and decrease in the values for each trial. This then allowed all students in the group to agree on the fact that a constant number such as 0.66 for the multiplication of initial values was utilized. They further reasoned that the amount of decrease also had decreased due to decreasing in the previous values. That is, they realized that the amount of change decreased at a decreasing rate. All these suggested that realizing that what students had known before (such as function behaviours decreasing at a decreasing rate or increasing at an increasing rate; average etc.) and the intended learning goal constituted commonalities, Alin expected students to not only pay attention to such commonality but also reason on the quantities' simultaneous change given in the problem contexts (PIP.2A1).

At that point in the group discussions, Alin realized that students came to an agreement such that they realized that they needed to multiply the initial value by a constant number. One group talked about calculations in their worksheet as below:

- *A: This is what you have. You have (03.15). In every trial you have 0.31. What is the amount and check the amount of change okay? Do you think the amount of change is the same?*
- *S10: Percent of the change is the same*
- *S9: Haaa.*
- *S10: The amount decreases*
- *A: How do you show that decreases? How do you decide?*
- *S5: This calculated.*
- *S9: Compare*
- *A: Comparing what? How do you compare it? (361-370)*

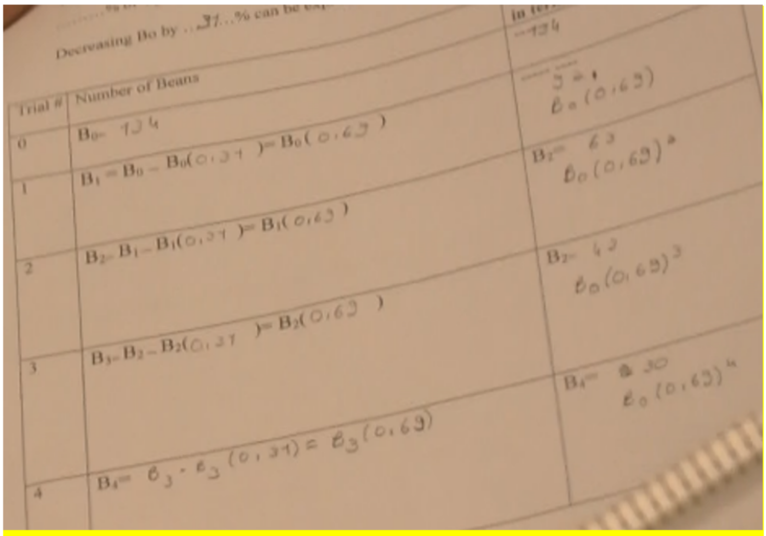

Figure 7.20. The paper of S5 about the amount of decrease.

As the data indicated, Alin expected students to understand that the amount of change was not the same, even though they multiplied by the same number. She asked her questions by using the established knowledge (e.g. 0.31) during the lesson as a base for understanding of difference between the amount of change and percentage of change (PIP.2B). The calculations in the worksheet supported students' explanations because students reasoned multiplying the initial values with 0.66 by thinking the average percentage of change. In the last step of the lesson plan, students calculated the exponents of the same number in each trial; therefore, they reached the purpose of the lesson. All these suggested that Alin focused on students' thinking processes: the simultaneous change in the values of the number of bacteria and each trial as well as finding the rate of the amount of change. In addition, she expected them to differ the percent of change from amount of change. That is, she expected them to realize that although the amount of change from one value to the other has decreased and the amount of change decreased at a decreasing rate, this was because the invariant relationship, the percent of change, was the same. Therefore, data pointed that Alin not only realized the commonalities of what students already have known and what they need to learn but also, she focused on their mind activities. That is, data suggested that from her perspective old and new knowledge were dependent on the learner such that during the lesson their mind activities were triggered to realize such commonality

(PIP.2A1).

At the post-interview, due to students' difficulty on the calculations of B1, B2, B3.. by using the B0 (initial value), Alin planned to make some changes in the task rather than totally giving up on it for future use:

I asked them how much the average? Of course, I just asked the thing here, how much did the average change? It is 30%. I said write the amount of x. Then I said something, let's find B1. I can do highlight bold or something "Changing by the same rate". Or I can ask this part that... that was missing on me. They thought it was a repetition of this question (30%) that I gave here (first part). I can say that What is the average? Now Let's suppose that every trial has changed 30%. I can ask that if we calculate B1 with 30% change. Because it was perceived as what is your average change and how much did you have first? After what I said at the beginning, the task asked about the percentage and chip amount. They didn't think of it that way. I think I need to change the sequence of these... (63).

For this part of the activity, let's assume that from the beginning the percentage of the change is the same for <u>each trial</u> and it's equal to the average of all percents.

- **What is your percentage change in decimals and initial amount of beans?**

…………..%

Trial number	Number of Beans	
0	$B_0 =$	

Complete the table below. For each trial, express the final amount of beans in terms of the previous amount of beans and then in terms of the initial amount of beans (Co). DO NOT EXPAND the factors.

Figure 7.21. Questions in the task about the average percentage.

As the data earlier have shown, the main problem students had faced in the task was using different percentages rather than the average percentage. Alin had realized the difficulty students encountered and planned to take students' attention on the average percentage by making it bold in the task or asking them more clearly.

This was important for Alin because focusing on the average percentage, students might have reasoned on the relationship between the amount of change and the rate of change. In addition, Alin realized another difficulty students had about the sequence of the questions in the task. Particularly, students first were asked to answer the question regarding the average percentage by using the changing percentage for all trials. The second task was about finding the values of B1, B2, B3, B4.. by using B0 and average percentage, but students used the values that came from each trial in the experiment and each percentage. Therefore, for future classes, Alin planned to change the sequence of the directions in the task, so that students could use the average percentage and initial value (B0) for their calculations of B1, B2, B3, B4... Although what Alin planned for was for the future classes possibly with other students but her focus on students' thinking, what they understood and what difficulties they encountered, to make changes on the task for future classes suggested that she made plans based on her assessment of students known mathematics (PIP.3A).

In sum, data from Alin's pre-interview, teaching and post interview from her teaching in 2018 showed that Alin depicted all characteristics of PIP. After the determination of Alin's teacher perspective, I analysed the same set of data (the pre and the post interview) for teacher noticing of her. I share the results from teacher noticing in the following section.

7.2.2. 2018 Results for Noticing

As shown in the previous section, Alin showed all characteristics of PIP in pre-interview, teaching and post interview from her teaching in 2018. I also analysed the pre and post interviews with the codes from "learning to notice framework" to show the stage of her noticing. Table 7.4 shows the frequency of the codes.

Table 7.4. The frequency table of the codes of noticing before and after Alin's teaching in 2018.

		What Teachers Notice					How Teachers Notice				
		1-Baseline	2-Mixed	3-Focused	4- Extended		1-Baseline	2-Mixed	3-Focused	4-extended	
Before	Lesson Plan and Interview				4a	3				4a	4
										4b	3
										4c	3
										4d	-
										4e	3
										4f	1
After	Interview				4a	4				4a	5
										4b	6
										4c	7
										4d	3
										4e	5
										4f	4

As the table shows, the results of the analysis depicted that Alin showed all the codes at the extended level in her interviews both for what teacher notice and how teacher notice. She attended mostly students' thinking and her teaching strategies. She explained her attendings in detail by giving evidence from the teaching. The connection between the special events in teaching and Alin's principles were also salient in her interviews. In her explanations, Alin noticed some problematic issues in her teaching and proposed alternative pedagogical solutions to the issues she pointed. Finally, she depicted all the codes more than once as shown in Table 7.4.

In the following paragraphs, first I share data showing what Alin has noticed and then share data to show how Alin has noticed. For that, I first provide data from the post-interviews and then, the pre-interview.

<u>7.2.2.1. What Alin Noticed.</u> Alin attended to whole class learning and her teaching strategies. She started to talk about her general idea about her teaching. She stated some noticed events in her teaching by combining students' thinking and her teaching strategies.

I think my lesson was good because when I gave this activity last year, I realized that the activity led students to learn?. Yes, I received the data I wanted, but since the *questions I asked on the second page are not very correct, in this part. That's what I asked last year,*

Trial number	Number of Chips/Beans	Final amount of chips/bear in terms of C_0
0	$C_0 = 2$	
1	$C_1 = C_0 + 0.41\,C_0$	
2	$C_2 = C_1 + 0.41\,C_1$	$C_2 = C_0(1+0.41)^2$
3	$C_3 = C_2 + 0.41\,C_2$	$C_3 = C_0(1+0.41)^3$
4	$C_4 = C_3 + 0.41\,C_3$	$C_4 = C_0(1+0.41)^4$

Figure 7.22. The task Alin used last year.

Trial #	Number of Beans	Final amount of Beans in terms of B_0
0	$B_0 =$	-------
1	$B_1 = B_0 - B_0(\qquad) = B_0(\qquad)$	----- ----
2	$B_2 = B_1 - B_1(\qquad) = B_1(\qquad)$	$B_2 =$
3	$B_3 = B_2 - B_2(\qquad) = B_2(\qquad)$	$B_2 =$
4	$B_4 =$	$B_4 =$

Figure 7.23. The task Alin used in 2018.

That is what I asked this year (pointing to Figure 7.23). I had more difficulty here in last year and I had to guide students. So, actually, I think the activity was well. I can say that it was well because each group was talking about how chips or *beans changed, at least when I observed them. They considered that how the amount of change had changed or where they could show it. Even if they couldn't show anything, they started to discuss the question of how they would show it. So, I think it was good.*

As the data indicated, Alin made some changes in her lesson plan this year taking into consideration of students' understanding and difficulties from the teaching last year. The differences in the task in two years showed that Alin interpreted that students had problems understanding the meaning of growth or decay factor, the constant number in exponential functions. So, Alin changed some parts of the task accordingly. Particularly, she noticed that students discussed the amount of change or percentage of change with their groupmates based on the questions in the task. Therefore, she focused on not only students' thinking but also on her teaching strategy such as using groupwork, discussion and questioning. She further explained that even if students could not reach the learning goal in the lesson, the groupwork and the discussion among students were significant to observe their thought and difficulties.

Therefore, having evidence from the students' understanding in her lesson, Alin thought that she reached her purpose. In addition, she knew that although the entire class could not reason on the meaning of exponential decay or growth, the majority of the class understood the growth factor or percentage of decreasing and increasing. She also noticed some particular students' thinking and problematic points in her lesson:

There were those who did not understand. I have a very weak student. I think it was a little late at the beginning of the lesson, if I remember correctly. While I was solving examples at the end, I select four students to solve on the blackboard. He was confused. At that point, we have already discussed what is the amount of change, I multiply by the same number every time, bigger than 1 always increased. How the amount of increase has changed, how we have seen this? We discussed all of them. Despite that, he didn't understand why we were multiplying 0.66 each time (43).

Alin made groups of three students and selected some students to solve the problems on the board. So, in the excerpt above, she mentioned how students discussed the amount of change in the increasing values and multiplying by the growth factor. This suggested that Alin attended students' thinking about the increasing rate or multiplying by the growth factor in her lesson. She estimated that a particular student missed the beginning of the lesson but joined the discussion about the growth factor in the group. She further commented that the student could not reason on the main aspects of the lesson. Also, Alin focused on both students' solving the questions on board and discussion with them as teaching strategies.

She continued with another students thinking as follows:

There he was confused, I see that he did not understand. He is not aware of the fact that he will find 34% and reduce the same amount each time, that is, the amount of change there, and the meaning of multiplying with a constant rate. Because he wasn't able to multiply by 0.66 when going from 1 to 2. I realize that he was confused there. I realized it, so I asked how you think? He said something like 66% remained, we'll multiply it by 0.34 again. I asked, "What does it mean to multiply by 0.34?". Calculate 34% of it. No, when you say you found the next amount when you find 34% of this. Already his group mates were discussing. I guess that his friends helped him, not my questions. If I see one of them explaining, I don't interfere much. If they explain it wrong, I am starting to ask questions (43).

Alin's main purpose was that all of her students understand the main ideas in the lesson. So, she had her attention on each student to notice their thinking. When she noticed any student's difficulty on understanding the multiplication of the initial values with the growth or decay factor in the task, she benefited from questions to direct students to reason further on the ideas. Creating discussion environment in the group allowed students to help each other because they thought together to find a solution to the problem. In fact, Alin noticed a particular student's thinking and difficulty such that she utilized particular strategies such as asking probing questions to handle the student's problem. Noticing both students' reasoning and her teaching strategies as well as the relationship between them showed that Alin depicted the code 4A at the extended level of the noticing framework.

In addition to the post-interview, Alin also showed the code 4A at the extended level during her pre-interview. Particularly, Alin planned to notice both students' thinking during the teaching and also her teaching strategies:

Mathematical learning is like the quantitative operations that I want children to experience, how the y values change when the x values change one by one, and what causes this is multiplying by the same number every time. Multiplication with the same number causes this. Because I want them to think and realize that the number I multiply every time decreases the main number. It might happen here when they compare it. Here, yes, we multiply by 1.5 each time, but it may be because I ask, it can be when discussing themselves. Each time, we multiply by the same number, but the amount of decrease decreases because the amount of B1, B2, B3 may change. They may be thinking of it here. When I came back to the last page, they may have thought before, very nice. I may have asked, or they may have argued themselves. If they cannot think there, I will be asking on the last page, how the amount of change has changed, how the numbers have changed, what pattern is there. I expect them to think about it. I will assess these (85).

As the data showed, Alin focused on what she expected from students and her purpose in her teaching. She emphasized that she wanted to observe students' thinking processes rather than solely focusing on what they will learn finally. This was because students' thinking processes during teaching would be an important evidence of how they learned and what they learned as well as informing about the efficiency of the lesson, the role of the teacher and difficulties of students. Particularly, she planned to attend to students' thinking about the changes of the variables, x and y, with respect to each other such that the amount of decrease or increase in the values and the multiplication by the growth or decay factor in every trial would be salient for them. She expected that students would learn because of their own mind activities they will possibly go through. Similarly, she expected that group discussion or her questioning would also trigger and act as catalyser for their learning. Therefore, data suggested that Alin hypothetically planned to notice her strategies and students' thinking in her teaching. This pointed to the code 4A at the extended level of the noticing framework.

7.2.2.2. How Alin Noticed. The results from Alin's pre-interview and post- interview showed that Alin not only attended to some special events about students' thinking and her teaching strategies but also explained in detail what she noticed by providing evidence from students' thinking. In this part, I share data from both Alin's pre-interview and post-interview. She showed all the codes of the extended level noticing during the post-interview. The data analysis of Alin's pre-interview depicted that she showed all codes except 4D during her pre-interview. I divided the codes 4A, 4B, 4C, 4D and 4E, 4F to show why Alin was at the extended level (Level 4) although the focused level (Level 3) has the same codes (4A, 4B, 4C, 4D) with the extended level.

Data for 4A, 4B, 4C, 4D from the Post-interview and the Pre-interview

Alin started the lesson providing the materials of the experiment and the task and giving the instructions of both the experiment and the whole teaching. Although she read the instructions to all students before starting the groupwork, all groups were confused about the sequence of the experiment and continued with the wrong page after the first one. She noticed their confusion of the pages and elaborated on this during the post-interview:

At the beginning of the lesson, I say that when you finish the first page, do not go back and start the other page directly. Everyone started on the back page. I can directly delete that second page and do something. Because it was a good sign that the children did not receive the instructions very well (49).

Students' confusion about the sequence of the paper caused them to misunderstand the task. They passed a step towards thinking about the changes of the number of beans/chips, the amount of changes and the calculation of the next trial using the previous trial. Therefore, students had difficulty in reasoning about the average percentage and the growth or decay factor in the related exponential functions. Thus, as Alin mentioned above, she decided to delete the second page to direct students' thinking on the questions on the third page. Alin's noticing and highlighting this confusion at her post-interview showed the code 4A at the extended level in the noticing framework.

Furthermore, Alin noticed one of the significant discussions about the graph of the data in the experiment. In particular, students had thought that all bacterium died at the end of the experiment so that the graph would intersect the x axis. Alin noticed students' thinking and stated:

A group asked me this. Does graphic touch x axis or not? How should we do? I said why do you think it touches? Because if all the beans fall on the shaded area, they will all die. What do you mean then? Antibiotic is 100% effective. This child said that if we look at the equation. Of course, since I am talking about this, theoretically, according to the equation, it will be impossible to touch. And at the point where it touches, it will show the trial number that there are no beans left. So, we see that he knows the meaning of the x axis, meaning of the independent input variable. But he didn't explain why it was impossible here, he didn't say it here. The first amount is always decreasing? he always takes half of something, half, half, get closer to zero, but not zero. He tries to say it, but he couldn't explain it so much here (7).

Alin attended to the discussion of a group about the graph intersecting the x axis. She interpreted that students believed that when all bacterium was dead the graph would intersect the x axis since the number of bacterium was zero. So, she asked some probing questions to that group to examine their thoughts. Alin noticed that a student explained that when he took half of the bacterium in each trial, the number of bacterium would be closer to 0 but would never be 0. Since for Alin students' thinking was the most significant events in the teaching, noticing and highlighting students' thinking about the graph of the function, Alin again showed the code 4A at the extended level noticing.

Alin also noticed some significant aspects of the lesson planning and highlighted these aspects during her pre-interview:

Exponential functions are very much related to daily life. At the first exit, exponential functions start with a question that, if we deposit a certain amount of money with 30 percent interest, how much will it accumulate or progress? How the money will progress is important because the same interest rate or rate of increase is applied each time. But the amount of money is increasing because total money is increasing every time. Since the increase amount of money to which the increase amount is applied increases, the total amount is increasing gradually. They associate with daily life... It is an important concept to learn logarithms, an important concept to learn how the features of functions changes. After that, when they normally learn about the concepts of limit or derivative, when they think about how a function changes, how y values change ith respect to x values. It is an important concept because we do all this while we are teaching exponential functions. Our lesson plan, our curriculum (She was trying to find it from the computer.). At the beginning of the second semester, by teaching the functions? During the first term, we actually learned quadratic functions, their graphs, polynomials, how y and x change with respect to each other. We learned basic parent functions, basic graphs of the parent functions and how they shaped? So children know that each function consists of x and y ordered pairs, each value of x and y ordered pairs showed a point in the graph, and when we combine all of the points that are matched with these functions' domain and range, we get the graph (37).

First, Alin planned the lesson based on her noticing of three significant aspects: the use of exponential functions in real life situations, the importance of what students know about functions; and the link between what students know about functions (i.e. the prior topics) and exponential functions (i.e., the new topic) as well as the link to the subsequent topics (i.e., the logarithm functions). In particular, answering the question "why this topic is significant to teach", she pointed to the use of real-life examples for the exponential decay or growth. She attended that teaching of exponential functions was vital for students because functions were to a solution to the needs of humans in daily life. Similarly, Alin pointed to the prerequisite knowledge for the exponential functions. Alin highlighted that students learned quadratic functions, their graphs, basic parent functions, polynomials and how y and x change relative to each other. Alin noticed that these mathematical ideas were significant to think about and understand the exponential function. So, she interpreted these topics as the prerequisites of exponential functions during the pre-interview. Lastly, Alin attended to the relation of the exponential functions and the following topics. She highlighted that students need to learn this topic to construct the logarithm functions. But more importantly, Alin noticed that students' thinking about the relative change in the values of y with respect to the changes in the values of x (i.e., rate of change) was not only important in the learning of exponential functions but also was vital in the learning of limit for determining the behaviour of functions and derivative. Thus, data showed that Alin noticed the necessity of the learning of the subject (i.e., exponential functions) for daily life, the prerequisite of the topic and the subsequent topics in the lesson planning process and highlighted the significance of these aspects at her pre-interview (H.4A).

As the data earlier have pointed to, one group of students had mentioned the death of the bacterium and therefore the graph's intersecting the x axis. Alin further commented on students' thinking evidencing the code 4C:

A: When asked whether the chart touches the x axis or not, a child interprets it according to the function he wrote. He decreases a little each time. But since he always reduces the same thing, always divide by two, divide by two, that amount will not be 0 because he has some left. Does the graph have to touch then? Here, a girl asked, while we do the rational functions, it does not touch to the y axis, now, does it touch the y-axis? I asked him if I remember correctly. Does it touch? What does it mean to touch? This means that the bacteria fall in the shaded area. So, what does this mean? The antibiotic was 100% effective. Could anything like this happen in this experiment? I asked them could it be when we divide every number?

R: What did they get at the end of the day, what did they get? A: Okay, well, I think they weren't sure whether it would touch on the normal graph. Because we did not discuss in a very detailed way. Yes, it never touches the x-axis and always get closer to 0. They had an idea that it touches in this situation (85-93).

Alin explained that she asked questions to a group discussing the graph's intersecting x axis to probe students to think about this experiment and the other examples of exponential functions. She noticed that students needed to think that while in this experiment, all bacterium could be death at the end of the experiment in real life; modelling it with the exponential functions they also needed to think that the value of y might get closer to 0 though never intersecting the x axis. During the post interview, when she explained her noticing of the groups' discussion, she pointed to particular student's ideas and a student's question about the graph's intersecting the x axis as evidences of students' thinking. This suggested that Alin attended to students' thoughts in the discussions and depicted such interpretation as evidence for the code 4C at the extended level of noticing framework.

Alin also focused on students' thinking and answers before her teaching and hypothetically reasoned on some possible student-answers. The evidences of possible questions and answers are as follow:

If it were like: children multiply with 1.5. It multiplies, multiplies. The friends made in the group. He saw all of them 1.5, 1.5, he wrote here, he multiplied it. When I ask why you multiply it seven times, there is no answer, it is only numerical. Here, he says 2 in 2 and 3 in 3. But if when I ask why is the 1.5to the power 7, they stated that I multiplied it 7 times with 1.5 every year. So, if they give 7 times the multiplication, then it means they understand. So, my goal is to measure what they understand. So, the answers they give in this paper may not be very descriptive for children's understanding. So, I will try to ask questions (89).

Understanding the growth or decay factor in exponential functions was the main purpose of Alin's teaching. So, Alin planned to assess students' thinking and understanding by asking probing questions. She also hypothetically created possible answers from the students so as to assess whether students understood why they multiply the initial values with a constant number in each trial to determine the subsequent values, and, how this common factor explains the amount of change in the y values with respect to the amount of change in the x values. That is, she hypothetically envisioned possible student answers whether they have understood the exponential decay or growth or if they only procedurally have utilized multiplication without any explanation. She even commented on some possible student answers as an evidence for her assessment during teaching. In addition to such evidence, she mentioned collecting the tasks of students to examine their thinking. Yet, as she mentioned above, she considered that students' writings might not be enough to interpret their thinking. So, she hypothetically planned to attend to the students' explanations during the lesson as the most significant evidences to observe, examine and interpret their understanding. This data therefore depicted that Alin showed the code 4C at the extended level during pre-interview.

Using the worksheets of students, Alin continued to provide evidences from students' thinking during the post interview. She noticed that one of the students explained his thoughts in the worksheet as follows:

A: When I asked how the amount of change increases and how it changes, he wrote these numbers.

R: Can you tell us the numbers

A: One of them is 2, the other is 3.08 or 7. Increasing is 1.01 in first. In second 1.36 or 86. The other 4.74. Then it increased 2.57. Then it increased 3.94. He calculated how much he increased each time. From this, I can say that he focused on the amount of bilateral change. Then I can see it by looking at the function he wrote? He made the explanation here. He said "how does the number of chips change at each trial". It increases because we said something like multiple it by its percentage?. But of course, there are very little explanation. Then I am asking "using the table of data explain how the amount of change changes at each trial". He wrote "the amount of change always increases. Since we use a constant increase in percentage amount of chips always increase and it will always be higher last trial chips". Here, as I said before, I tried to get verbally, because I know my own students that they have difficulty in writing their own sentences. As the number of chips in his hand increases each time *and he said "I multiplied by the same number every time, the increase in my amount also increases". The child tells us that has already calculated it. So, we see that he did it. When I have a class discussion in the next class, I will expect that comment when I ask his opinion. I will expect him to say that the amount of chips is increasing every time. Although I multiply by the same number, I can say that this will increase because the amount I multiply increases. Or if someone gives this answer, I will probably ask him would you do something different. Maybe I do not ask what he did, I can tell him to produce something new (17).*

As the data showed, Alin attended to particular students' thinking about the amount of change and the exponential function gathered from the experiment. For that she used both the student's explanations during the teaching and the student's writing in the worksheet. She interpreted the student's thought in detail to show how he reasoned about the meaning of multiplication by a constant number and how he constructed the function. She explained the student's answer on the worksheet according to her own observation and noticing. So, she interpreted the particular student's reasoning by elaborating on and adding her own consideration of what she had examined. Alin's interpretive comments depicted that Alin showed the code 4B at the extended level of the noticing framework.

Again, since students' making sense of the growth or decay factor in exponential functions were the main purpose of Alin's teaching, she planned for some steps in the lesson planning process to reach her purpose. She hypothetically developed some possible questions and some possible answers from students as well:

A: At the beginning, as the students separate the chips each time aside and the number of chips decreases, I ask how does this amount of change? I will ask them why it changes every time. There are possible answers I expect from them. They may just say that it is decreasing, they may not look at any amount of change. So, I will ask that decrease or increase, do you observe a difference in the amount of decrease? In this way, while trials, x values or our inputs change one by one, I expect them to observe the amount of change. There is something that is getting smaller with decreasing rate or *greater with an increasing rate. They're not Americans, exponentially means growth, so that they understand exponential growth. Normally when you say how does it grow, it is exponentially. If they know English exactly, the children will say exponentially because it is actually, increasing. I want them to think, so I will ask even if they don't think it myself. Or when I ask how the amount of change changes here, I ask it. I want them to go through that process. Likewise, since I want them to formulate this, it is my final product, I expect them to reach it.*

As the data indicated, Alin hypothetically planned to notice students' thinking in her lesson planning process. She explained that students might focus solely on the changes of x and y; so, she prepared some questions to direct students' thinking about the amount of change in the values of y with respect to the amount of change in the values of x. In addition to possible students' answers, Alin interpreted her expectations about students' thinking and cognitive processes for making sense of exponential functions. Specifically, she emphasized that students might go through a thinking process to make sense of exponential growth or decay. She also pointed to the importance of teacher questioning in this process. Alin's interpretations before teaching about possible students' thinking and prepared questions depicted that she showed the code 4B code at the extended level of the noticing framework.

Alin's evidences from students' worksheets and her interpretations about the special events or expected students' thinking were significant for showing Alin's noticing. She continued to elaborate on some specific interactions with students and their ideas about the multiplication with the growth or decay factor:

There was no problem doing the first one here. Then while doing the second, he wanted to multiply 0.66 by 0.34 again. What is 0.66 times 0.34? Because there is a meaning of multiplying B0 by 0.66 here. B0 minus 0.34, that is, we reduced 34 percent of it. What do the meaning of getting 34 percent of 66 percent? He wanted to get 34 percent of the amount left in his hand, but he needs to multiply by 0.66. So, the student was not aware of the thing, he needs to find 34 of the previous amount, but he has to subtract it. What he finds is only 34%. As I find 34% of the previous part. There he *was confused; I see that he did not understand. He is not aware that he will find 34% and reduce the same amount each time, that is, the amount of change there, and the meaning of multiplying with a constant rate. Because he wasn't able to multiply by 0.66 when going from 1 to 2. I realize that he was confused there. I realized it, so I asked how you think? He said something like 66% remained, we'll multiply it by 0.34 again. I asked that "What does it mean to multiply by 0.34?". Calculate 34%of it. No, when you say you found the next amount when you find 34% of this. Already his group mates were discussing. I guess that his friends helped him, not my questions. If I see one of them explaining, I don't interfere much. If they explain it wrong, I am starting to ask questions. And their friends explained it, I did not. After that, they got the right answer (110).*

As explained before, Alin attended to particular students' difficulties during teaching and interpreted students' difficulties as evidences. As the data above showed, Alin elaborated on her attending to the student's confusion regarding the multiplication by 0.66 (decay factor). She noticed that the student calculated the first multiplication for B0 but the students had difficulty in making sense of the multiplication with the same factor, 0.66, for the rest of the trials. Alin interpreted the student's difficulty such that from her point of view the student was not able to make sense that multiplying with 0.66 would be the same as subtracting 34% of a number from itself. Alin noticed the confusion and decided on how to respond this special event: She asked pre-prepared questions to that particular student to focus his attention on the meaning of multiplication of a number with 0.34 and subtracting that from the number itself. In addition to student's confusion, Alin further explained how the groupwork was advantageous for students: She emphasized that students in the group discussed the problem together to find a solution such that they were able to find the meaningful answer thinking together. Therefore, Alin noticed both the interactions between her and the students and also the interaction among the groupmates. She elaborated on the special events in teaching by pointing to and elaborating on these interactions. All these suggested that Alin showed the code 4D at the extended level of noticing.

7.2.2.3. Data for 4E and 4F codes from the Post-interview and the Pre-interview. Alin's plan for the lesson was asking probing questions to guide students to think about the amount of change and the growth and decay factor in the exponential functions. Moreover, she planned to form groupwork in her lesson plan so that students would have opportunity to think together. As the data before have shown, Alin stated that "Already his group mates were discussing. I guess that his friends helped him, not my questions. If I see one of them explaining, I don't interfere much. If they explain it wrong, I am starting to ask questions. And their friends explained it, I did not. After that, they got the right answer". This data showed that Alin noticed students' difficulties but did not get involved in the discussions because she knew and believed that talking together, students could explain their reasoning about the topic and reach a conclusion. She further explained the advantages of groupwork on the students' understanding as follows:

Everybody here, every member of the group, was holding the plate, someone was writing down the notes, someone was counting. So, everyone was involved in the activity. So, when I asked what this activity was, everyone was discussing somehow. Otherwise, they will try to understand individually. They may ask their friends if they don't understand it, but it's very simple here, count beans, or count round rings of chips. This is something everyone can do. The experiment is also visual. Or they can make a work sharing. They also catch the connection with a groupwork because they discuss what happened in the activity. Therefore, everyone can understand by drawing data (53).

Data showed that Alin thinks that as a member of the group, students might take responsibility at any stage of the experiment such that they might observe and engage in all the steps to think about the function of exponential decay or growth. Particularly, she thought that students counted the beans or chips, drew table of values with the data from each trial, calculated the amount of change and the average percentage to find the decay or growth factor in the function rule and discussed how they made sense of the generalizations they came up with their groupmates. Therefore, Alin thought that students had different entry points to the task to make sense of exponential functions. Alin noticed that both her teaching strategies such as allowing groupwork and asking questions and teaching principles focusing on students' mind activities while engaging in the tasks allowed her to attend to the connections between the principles and students' thinking (particular and the group).

Such evidence showing the connection between her questions and the groupwork depicting students' thinking was also salient in Alin's lesson plan and pre-interview:

A: What are they doing as a group when I visit the groups? The questions I ask will be measuring their knowledge. Let everyone read and try to understand what the task is, without explaining it to them. Get everyone involved. Because at the very beginning, if I read and explain, I know the students in the class. I don't want that while someone start doing the experiment, somebody sit down. I say everyone try to understand the task?. How can you show this from the graph? So, for example, I couldn't be sure about it here, so can you give me an example with numbers? The explanation sounds right, but I want it to be explained in a different way so I'm sure. Yes, he explained in one way, this can be said in another way, that means that he has a grasp of the topic. This child understands the topic. I am thinking of doing analysis there. In fact, when I visit the groups every time, asking the questions means doing analysis for me.

R: Okay. What is the focus of those questions when analysing?

A: So, how does the child think? What did he focus on every time? What is he doing in his mind, is he doing something operational? If he is on the same page with his friends or is doing something wrong, why does he multiply something with a different numbers and did not do anything right? What did he think? I have a focus like this (91-93).

As the data showed, Alin planned her lesson in such a way that her focus would be on students' thinking. This was evident in her lesson plan with possible probing questions and hypothetical answers from students. She planned to examine her students' reasoning by creating opportunities for students to think more deeply. The questions Alin had prepared, the worksheets from each student and the discussions of the groups were the significant resources to notice students' thinking. Alin emphasized the connection of her teaching strategies and students' thinking by explaining her lesson plan at her pre-interview. Her explanations depicted that Alin showed the code 4E at the extended level of noticing both during the post-interview and the pre-interview. In addition, Alin hypothetically prepared some solutions for unexpected conditions that might have occurred during teaching. She stated that if students could not provide answers to her questions, she would ask further pre-prepared probing questions to guide students to think more profoundly. Her main purpose was attending to students' thinking and acting based on their ideas. So, prior to the teaching, she considered possible events that might have occurred during teaching as well as difficulties students might have encountered. The lesson plan depicted that she considered some alternative solutions even before teaching so that she would be prepared for students' different ideas (H.4F).

Alin also noticed some difficulties students had encountered during teaching. So, she explained some changes in her lesson plan for future classes and elaborated on some alternative ways to solve students' difficulties:

I asked them how much the average? Of course, I just asked the thing here, how much did the average change? It is 30%. I said write the amount of x. Then I said something, let's find B1. I can do highlight bold or something "Changing by the same rate". Or I can ask this part that... that was missing on me. They thought it was a repetition of this question (30%) that I gave here (first part). I can say that What is the average? Now Let's suppose that every trial has changed 30%. I can ask that if we calculate B1 with 30% change. Because it was perceived as what is your average change and how much did you have first? After what I said at the beginning, the task asked about the percentage and chip amount. They didn't think of it that way. I think I need to change the sequence of these. First ask the percentage change and what is the initial number of beans and then "let assume change is this percent for all trials" make capital bold. I can put it there in a way that they will never miss. If we say, let's *analyse how is the change, there is a constant change every time, they can focus on y. They can even see it. It would be nice too. Every time, by accepting the common amount of change, we look at the changes of the number of beans and the number of amount of change. Okay then they might think I should focus on them too, even better (63-70).*

As the data indicated, Alin noticed that students faced with difficulty using the average percentage in each trial. Alin interpreted such difficulty as students' not making sense of the relation between subtracting average percentage of a number from itself and multiplying the number with the decay or growth factor. Therefore, she considered that she might change the sequence of the questions and write them in bold to take students' attention to first think about the amount of change and the initial amount and then the average percentage. All these data suggested that Alin noticed significant problems/difficulties students encountered during teaching and that she proposed alternative solutions for her next lesson (H.4F).

Results of the analysis of data from Alin's pre-interviews and post-interviews depicted that she showed all the codes at the extended level of noticing more than once. She attended specially students' thinking and her teaching pedagogy as she was aware of the significant relation between them (W.4A/ H.4E). She interpreted the noticed events by providing evidences from students thinking individually or in group discussions and elaborated on the events by giving some examples (H.4A/H.4C/H.4B/H.4D). She even provided suggestions pointing to some alternative solutions about her future lesson planning to create opportunities for students to think more profoundly (H.4F).

7.3. Elisa: Case of 2016

7.3.1. 2016 Results for Teacher Perspectives

Analyses of the data of Elisa's lesson plan, pre-interview, teaching and post-interview, depicted that Elisa showed the characteristics of the perception-based perspective more than once (See Table 7.5).

Table 7.5. The frequency of the characteristics of PIP before, during and after Elisa's teaching in 2016.

<table>
<tr><th colspan="2">Elisa-2016</th><th colspan="3">PBP</th><th>PIP</th><th>CBP</th></tr>
<tr><td rowspan="13">Before</td><td rowspan="7">Lesson plan</td><td>1A</td><td>2</td><td rowspan="7">7</td><td rowspan="13"></td><td rowspan="13"></td></tr>
<tr><td>1B</td><td>1</td></tr>
<tr><td>1C</td><td>1</td></tr>
<tr><td>1D</td><td>1</td></tr>
<tr><td>1E</td><td>1</td></tr>
<tr><td>1F</td><td>1</td></tr>
<tr><td>1A</td><td>4</td></tr>
<tr><td rowspan="6">Interview</td><td>1B</td><td>4</td><td rowspan="6">22</td></tr>
<tr><td>1C</td><td>6</td></tr>
<tr><td>1D</td><td>2</td></tr>
<tr><td>1E</td><td>3</td></tr>
<tr><td>1F</td><td>3</td></tr>
<tr><td>2A</td><td>5</td></tr>
<tr><td rowspan="4">During</td><td rowspan="4">Teaching</td><td>2B</td><td>2</td><td rowspan="4">28</td><td rowspan="4"></td><td rowspan="4"></td></tr>
<tr><td>2C</td><td>5</td></tr>
<tr><td>2D</td><td>6</td></tr>
<tr><td>2E</td><td>10</td></tr>
<tr><td>After</td><td>Interview</td><td>3A</td><td>5</td><td>5</td><td></td><td></td></tr>
</table>

As the Table 7.5 depicted, Elisa showed the characteristics of the perception-based perspective before teaching 29 times, during teaching 22 times and after teaching 28 times. In the following sections, I share some parts of data from Elisa's lesson plan, pre-interview, teaching and post-interview to show how Elisa depicted the characteristics of the perception-based perspective (PBP).

<u>7.3.1.1. Elisa's Before Teaching.</u> In this section, I share data from 2016 including both the lesson plan and the pre-interview of Elisa. Data showed that Elisa depicted PBP.1A, PBP.1B, PBP.1C, PBP.1D, PBP.1E, and PBP.1F characteristics. According to previous research, teachers who hold PBP create a learning trajectory to make mathematical relationships more apparent for students (PBP.1A). Teachers who hold

PBP acknowledge that mathematics is formed by the combination of many interrelated ideas independent of the knower, so learning is seen as a gradual process for students by attending to those interrelated ideas (PBP.1B). In addition to the view of the nature of mathematics and learning, teachers who hold PBP believe that teachers have a key role to create an environment for active engagement and discovery for students (PBP.1C and PBP.1E). In this discovery environment, teachers direct their students to think about the intended goal by using the materials, tasks or questions to provide an opportunity for students to have some experiences with them (PBP.1D). That is, since teachers holding PBP acknowledge that mathematics is independent of the knower such that they think that mathematics has been established already outside of the knower, they think that materials, tasks etc. are tools to convey the mathematics obvious in those materials, tasks to students. Therefore, assessment of students during teaching is seen as significant to pass to the following topic or determine the lack of knowledge in students' understanding (PBP.1F). In other words, embracing the idea that the intended learning goal is outside the knower, the teachers ask questions to determine whether the students achieved such goal rather than determining how they reason.

Elisa designed her lesson for an introduction to the subject of sequences for her 11th-grade students in a 35-minute lesson. She mainly planned to focus on the main properties of sequences and types of sequences. She stated her objectives in her lesson plan as follows:

Students will be able to; define sequence, know terminology and symbols about sequences (term, element, an...), explain the relationship between concept of sequence and the concept of function, define finite and constant sequences, explain equality of two sequences (1).

Data showed that Elisa planned to focus on both the definition and the features of sequences, different types of sequences such as finite and constant sequences and the relationship with other concepts (i.e., functions) in a 35 minute in a lesson. When she explained her plan during the pre-interview, the researcher asked her how she would be able reach all these objectives in a 35-minutes-lesson: She stated:

Yes, it was little much, but ... let's see, because I am doing it through the examples. It seems like it can happen. Even though it is not, I just think about the sequences, make lecture and I will leave it. If I have time, I will continue (4).

Elisa pointed that her main purpose was the construction of the concept of sequences by using different examples of sequences. So, although she had written more than one learning goal, she would focus mostly on the definition of sequences. Elisa's emphasis on the use of examples for her students to understand the definition of sequences seems to indicate that she thinks of examples making mathematical ideas apparent for her students (PBP.1A).

Elisa further clarified her reasons for the selection of objectives and the plan of the lesson as follows:

R: So how did you decide on your learning goal?

E: Once I looked at what the sequence means, what do I need to know to understand the sequence, I read a little bit, so the sequence domain positive natural numbers is a special function. Actually, my other learning goals are not difficult. By using the examples, I did my lesson. I think that it would be better because they don't know the subject (23-24).

Elissa pointed that she did some investigation about the concept of sequences and the prerequisite knowledge required of students. Therefore, she knew that sequences were a special function and that domain of sequences consisted of Natural Numbers. In addition, Elisa was aware that her students did not have any knowledge about the sequences, so she planned to discuss with students the examples she prepared earlier in the lesson planning process. This suggested that her main concern was making the concept of sequence and its relationship with the concept of function as apparent as possible for her students. That is, preparing examples to discuss with students, Elisa intended to both direct the students towards what she wanted and also made the subject clearer and more understandable to the students. In addition, although Elisa stated that the students did not know anything about the concept of sequences and she did some search for the pre-requisite knowledge possibly required of students for the introduction to the concept, in her statements she did not mention at all about what her students knew prior to the lesson. Nor that she mentioned how she assessed and determined what her students knew. This suggested that she interpreted the learning objective as outside of the knower such that her plan was for her students to reach such objective with the help of examples progressively. All these suggested that data from the lesson plan and the pre-interview indicated that she showed the PBP.1A characteristics.

Thus, the objectives of the lesson showed that Elisa considered mathematics as a connected set of ideas because her focus was on the construction of mathematical relationships inherent within the sequences and functions. During the pre-interview, Elisa further explained how she thought that the examples she prepared would help students to connect the concept of sequence with functions:

Because by looking so many examples, they are seeking something common. They will realize that sequence is going as 1,2,3,4,5,6,7 positive natural numbers here. I think they will reach the learning goals themselves, because we do it with discussion and we don't leave it here. They said function, okay we said domain is natural numbers. We are reinforcing every time by the next examples, isn't this sequence here? Why not? Here are two terms here. Then, what kind of connection is there provided that it is a function, we always talk. I think they will get the reasons and students learn why.

Data pointed to three significant characteristics of PBP: First, Elisa considered that students could reach the objectives by the help of examples in the task such that she expected them to observe and think about the similarities and differences about the sequences and functions (PBP.1A). From her perspective, creating a learning environment through questions and discussion with active engagement of students was necessary to accomplish the objectives (PBP.1C and PBP.1E). Second, to show that sequences were a special function, Elisa created discussion to talk about what it means to be a function and the domain of such function for each question in the task. That is, Elisa thought that learning was a gradual process such that students would consider the similarities and differences among the different examples building on the previous ones (PBP.1B), so they would deduce the relationship between the sequences and functions depicted in the examples (PBP.1A). Third, data indicated that mathematics was independent of and accessible to all her students: Elisa believed that once she provided the examples to make mathematics accessible to each student, students would reach the objective that was given in the task through their own effort (PBP.1B). That is, through answering the questions in the task the teacher prepared (PBP.1D) students would construct the knowledge of whether a given example was a sequence or not (i.e., a function with the domain of Natural numbers) in their mind (PBP.1B).

It is interesting that although Elisa knew that the concept of functions was a prerequisite for making sense of sequences, she did not start the lesson activating what her students knew about functions. She stated:

First of all, after we say let's go to a function, they will always think about why we go back to the functions, that is, I will give the link as a hint directly. They will say the function directly, they will not notice, or I will try to make them say without passing the function word. One or two of them maybe say the function, as I will talk to them again by extending the topic. So, doing it in the middle is always better to rather than in the beginning (32).

Elisa believed that instead of starting the lesson with functions, she might use examples and ask questions to create a thinking process for her students to think about the connection between the features of sequences shown in the examples and functions. Therefore, she stated that if she provided opportunities by using examples and questions in the task to discover the main features of sequences, students could construct the knowledge themselves. In addition, Elisa decomposed the intended mathematics in the teaching into understandable pieces such as features of sequences, connection with functions and types of sequences. By the active engagement of students to the task in the teaching, Elisa thought that students could construct the intended knowledge in their mind by seeing these pieces and establishing a connection between them. That is, by creating an active engagement environment for students' discovery, Elisa showed PBP.1C characteristic of perception-based perspective (PBP.1C).

Elisa focused on not only students' learning but also teacher's role in the lesson planning process and during teaching for creating opportunities for students' discovery. The sequence of the task, the criteria for selection of examples and the questions to direct students or materials used in the lesson were all dependent on the teacher. Elisa further explained her reasons for choosing the examples in the task:

This is not the sequence when the domain is the subset of the positive natural numbers. Sequences always have to start from 1 and stop somewhere or go infinite. There will be no missing terms in the terms. Order is important, so I put it in task, so this is [8^{th} example] an examples of a finite sequence.

8. (sin30°,sin60°,sin90°)

9.

1	2	3	4
25		75	100

Figure 7.24. 8^{th} and 9^{th} questions in the task of Elisa.

This [9^{th} example] I also put it as an example that is not a sequence. Because after I put this in the task, they think that I can make a sequence from any subset of positive natural numbers. We cannot do it because it has to start from 1 and go somewhere. We should not have missing terms (18).

As the data indicated, Elisa showed that every example given in the task was chosen for a reason. By using each question, Elisa wanted to attract students' attention to the key aspects of examples of sequences so that they would see the properties of the sequences and how they relate them to functions. For instance , she explained that the eighth question was a finite sequence example and the ninth question was a non-sequence example, so she emphasized that she selected these examples to create an opportunity for students to think deeply and construct the knowledge about the different features of sequences. The aim of Elisa was to create situations that can generate mathematical ideas and direct students to these ideas, so she depicted the PBP.1E characteristic of perception-based perspective (PBP.1E).

Furthermore, Elisa prepared purposeful questions to ask during the teaching and examples in the task for students' discovery such that she planned to create the same experience for all students. She prepared a lesson plan with the following teacher roles and possible students' answers as follow:

Teacher's Activities	**Possible Students' Answers**
Can we say that every function is a sequence?	No.
What makes sequence special?	Domain is set of positive natural numbers.
If the answer doesn't come;	
Remember, a function has domain and range. Can you realize anything special about domain or range of sequences?	Domain is always set of positive natural numbers.
So, how can we define sequences?	Sequences are function whose domain is positive natural numbers.
Okay. We show every term of sequence with a_n. First term is a_1, second term is a_2, n th term is a_n.	

Figure 7.25. Teacher activities and expected students' answers in the lesson plan.

As the data showed, Elisa prepared some questions to direct students to think about the functions with the domain of Natural Numbers. Her main purpose was for students to reach the objectives of the lesson and to provide the same experiences to all students in this process. The questions Elisa planned to ask and the expected answers from her students indicated that her focus was on the correct answers that students might have provided. This also might seem to indicate that she was not much open to different answers from students. Similarly, she planned to ask new questions only if the expected correct answer did not come from students. Therefore, she expected students to reach the sequences by using the idea of functions, not considering much different views (PBP.1D). In addition, during the pre-interview, she further explained that given the lesson plan, all her students would go through the same thinking process:

I say that do these sequences remember anything you already know? The answer may not come, what I asked may not be understood here. For each value, I see one correspondence value. There is a correspondence for me. What does it look like for me? Does it remind you? .... My questions are always on this way and for that purpose. If we look at the questions, or for example, what do you think about the 5th example? I think they can say that it is a sequence. Why do you think it's a sequence? By always asking questions and attributing the feature of being a function, my questions are always in this way (42).

Elisa mentioned that her questions would direct students to think about the functions. Specifically, once she asked if the correspondence between the values in the domain and the resultant value in the sequence remind them of anything they knew before, then students would be able to realize that the correspondence apparent in the provided examples would refer to functions. Her statement "My questions are always on this way and for that purpose ... By always asking questions and attributing the feature of being a function, my questions are always in this way" further suggested that her questions were all related to the function attribute inherent in the examples. This showed that she planned to create the same environment for all students to reach the objective rather than focusing on their individual ways of thinking differently. So, the data suggested that Elisa depicted the PBP.1D characteristics of perception-based perspective.

In the lesson planning process, Elisa also determined some criteria for assessing students' reasoning about sequences. She planned to evaluate students reasoning by observing whether the intended knowledge was in the students' repertoire. She explained what she expected from students:

I made a schema in their mind. They think based on this schema, and say there was a box for each box, but that is not like that. They might say that this is not a sequence. But I want them to tell that is not a function by considering the feature of being a function, so my questions based on it. So, when they see this (9th example), they will say to me that this is not a sequence. Because a term is missing, but they should not tell me according to the schema in their mind. They should tell me that it is not a sequence because there should be no matching elements in the domain of the function. I always ask my questions to hear this. It is proof that they have learned that they can define it, define it as a function that the domain with positive integers, and can say a sequence or anything that I gave them...(36).

As the data showed, Elisa planned to listen to the students' answers and discussions in the lesson carefully with the goal of determining to what extend her students learned of the intended mathematics. Elisa focused on students' distinguishing of whether an example was a sequence or not rather than focusing on their thinking. She emphasized that her questions in her teaching were to assess students' making sense of sequences as functions, but, she wanted and expected that students would explain the examples in the task by using the idea of functions instead of their already existing schema. That is, data seemed to suggest that Elisa interpreted and expected that the students would perceive the existing mathematical relationship in the examples. Therefore, her questioning would target to find out how much the students could perceive the mathematics inherent in the examples. Also, the answers expected from the students would allow her to determine to what extend she achieved her intended learning goals for the lesson. In addition, by listening to students' thinking, she could observe their misconceptions or deficiencies in their explanations. Thus, preparing an assessment for the evaluation of how students progressed was necessary to determine what was learnt in terms of the intended learning goals and what students further need to learn in the next lesson.

Consequently, data in total showed that Elisa prepared her lesson plan by considering students' making sense of the definition and types of sequences with regards to the idea of functions. In other words, she hypothetically planned a lesson sequence through which students might reach the intended learning goals. However, her planning suggested that she planned to foster a particular understanding of students and planned to generate the same experiences for all students. Her main purpose for asking questions was to observe and determine whether the students reached the intended learning goals rather than listening to them focusing on their thinking. Similarly, she thought that teacher played a significant role in creating an environment for students' active engagement. Though, she seemed to believe that mathematics was outside and independent of the learner such that by asking questions, she wanted to create an environment for students' discovery of the knowledge already inherent in the examples she had chosen. Therefore, from her perspective learning seemed to occur through the active involvement of students such that they discover the mathematics already existing in the provided tasks or examples.

<u>7.3.1.2. Elisa's Teaching and the Post-Interview in 2016.</u> In this section, I depict data from Elisa's 35-minute- practicum teaching in 2016. Data from her teaching was also coherent with the data from the pre-interview and the lesson plan. At the beginning of the lesson, Elisa gave a task to the students with different examples of sequences and asked questions to them so that they would perceive the definition of sequences and their relationships with functions. However, once Elisa realized that her students did not see the relationship between sequences and functions as she expected, she began to talk about the properties of functions:

- *E: Let's talk about this first. What does a relation need to be a function? What do we call a function?*
- *S3: Being bijective (one to one and onto).*
- *E: If there is no one-to-one and no onto, is it a function?*

- *S3: Yes.*
- *E: What could be the condition to be a function?*
- *S4: Everything in set X matches something in set Y.*
- *E: This is our first conditions, right? So, every element in our domain has to go to an element in the range. There will be no element that does not match. This is our first condition to be a function. 2nd?*
- *S5: An element in set X cannot go to two elements in set Y.*
- *E: Yeah. Can another person repeat? Did you hear your friend? Can you tell me a little louder?*
- *S5: An element in set X cannot go to two elements in set Y.*
- *E: An element in set X cannot go to two elements in set Y. Okay, let's take a look at these four (examples in the task) again by the light of the rule we talked. Does it all give a function.*

As the data indicated, once Elisa noticed that her students could not make the connection between sequences and functions, she directed her students to think about the features of functions. Particularly, in order to show that sequences form a special function, the injective and bijective functions were discussed in terms of their domain and range. Data in particular pointed that Elisa saw mathematics as a connected set of ideas such that to connect the idea of sequences and functions in her lesson, she intended to focus on such link between them. Though her focus was on what students did not know (PBP.2B). That is, she knew that students could not perceive what she expected them to perceive what is apparent to her in those examples. Therefore, instead of further probing them in terms of what they understand and how they reason, she directed their focus to what she really wanted them to perceive. In other words, using questions, Elisa provided an opportunity for her students to remember the injective and bijective functions so that they could make the connection between sequences and functions. The last question she asked directing her students' focus to whether the last four examples determine a function further suggested that Elisa expected her students to perceive what she apparently realized in the examples of sequences; the function property (PBP.2A).

Similarly, Elisa continued to ask many other questions to create a learning environment for her students to think about the features of sequences. In addition, rather than asking students to say in their own words what they deduced she stated the following and then faced that a student had a misconception:

- *E: Okay, then let's say: Each function whose domain set is positive natural numbers is a sequence?*
- *Some of them: Let's say.*
- *S7: It has to have a certain rule.*
- *Elisa: What do you mean when you say rule?*
- *S7: For example, they are all 50 and 50 each. Here we could not say if there was not such a constant sequence.*
- *Elisa: Is there a rule in two here?*
- *Class: None.*
- *E: No. Can't I say that this is a sequence if there isn't a rule?*

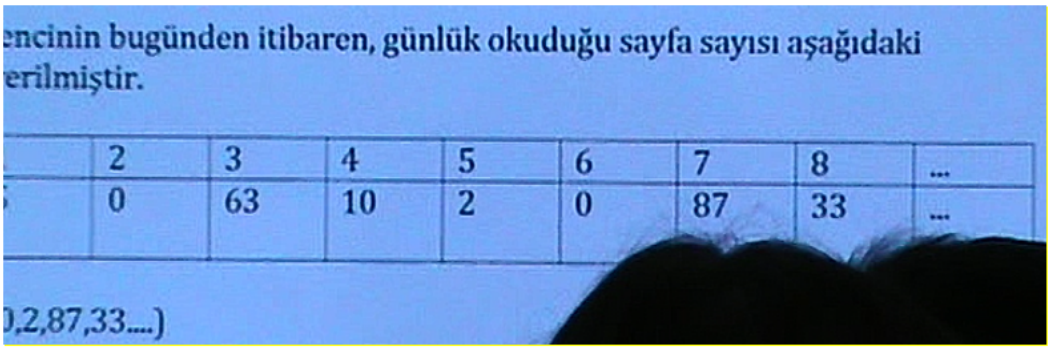

Figure 7.26. Students' difficulties on sequences.

- *S9: It is sequence but not arithmetic sequence.*
- *E: Does anything have to have a rule to be a function; Does it have to have a mathematical rule?*
- *Class: No:*
- *E: You say no rule is required; Okay. Then I'll ask again. Can I call a sequence for any function whose domain is made up of positive integers?*
- *S10: No.*
- *E: Why?*

- *S10: Sequences must have an arithmetic or geometric rule.*
- *E: Each sample on the paper I gave you is an example of sequence. In each of them, for example, do you see the rule of two and a third?*
- *S: No.*
- *E: We don't see. Guys, we do not need a rule to be a function or a sequence. It is enough to fulfil the two conditions we just mentioned. What we said: There will be no matching elements in the domain set. Each element will match one element from the range of values, and each will go to only one element. And so, if the domain set is integers, we call it a sequence.*

In the data, there are several issues that needs attention. First, as mentioned above, rather than asking students how they make sense of the provided examples (i.e. the examples showing a correspondence between the set of positive Natural Numbers and the functional values), Elisa herself stated "Okay, let's say: Each function whose domain set is positive Natural Numbers is a sequence". Then, once she spotted that some students had a misconception such that any sequence had to have a function rule either with arithmetic or geometric growth, rather than asking and determining how they could make such an interpretation and asking them why they say so, she started asking directive questions. Although some of the students agreed on the idea that sequences don't have to have a rule, S7, S9 and S10 still believed in the necessity of sequences? having a rule. At this point, Elisa's behaviour was worth examining from three aspects: First, as already mentioned, Elisa focused on only handling the misconception students held rather than listening to their thinking. Some of the students? (e.g. S9 and S10) answers "It is sequence but not an arithmetic sequence.?," Sequences must have an arithmetic or geometric rule" suggested that students not only thought that sequences have to have a rule but also that such rules need to indicate either arithmetic or geometric growth. However, Elisa did not show any concern about how and why students could think that sequences need to have a rule with either arithmetic or geometric growth. Rather, she depended on the main materials, the pre-prepared questions in the task, such that she believed that the questions would allow students to perceive that not all sequences have to have functional rules. In fact, she had an opportunity to create a class discussion using some of the counter examples in the task sequence she prepared; though again her main purpose was for the students to perceive what was apparent in the examples rather than digging into how they reasoned (PBP.2A). Third, Elisa listened to her students' talking to determine what students perceived rather than listening to them how they made sense of what they have been engaging in during teaching. Her main purpose in determining what students perceived was to pass to the next stage in the teaching rather than evaluating their ways of understanding. That is, listening to students' misconceptions or difficulties in understanding the topic was significant to only pass to the next stage in the sequence of the lesson (PBP.2D). In addition, when she realized that some students still believed that there should be a rule for calling a mapping as sequences, she explained the objective of the lesson. Data showed that either asking directive questions or telling the answer directly, Elisa focused to reach the learning goals rather than focusing on students' thinking or reasoning (PBP.2E).

Although during the teaching, Elisa could not focus on students' reasoning to help them overcome their difficulty, she further discussed the issue during the post-interview. In fact, Elisa mentioned that she had not known that the students she taught that day were introduced to the concept of sequences by their real classroom teacher earlier in the week. Therefore, she mentioned that if she had known that the students were introduced to the concept earlier, she would have acted in the following way once she realized their misconception:

E: I wouldn't accept it that way. First of all, I would try to bring them to the point I wanted. Because I read it in the book. I am sure, I looked at the book of MEB. How I dealt with it, I would probably return to the example of the function, the definition of this comes from the function, did it have to be a rule in the function. Because this is where it comes from? It comes from the function, it could be a 5-10 minute return to the function (48).

As the data showed, Elisa mentioned that she would probe students to think about the definition of functions. In other words, she would have asked them to think about whether all functions have to have rules or not. This way, she would have allowed them to think that sequences do not have to have rules either. This suggested that Elisa thought that the main problem was in students' seeing the mathematical relationships in the task (PBP.3A).

As the data from the lesson plan and the pre-interview indicated earlier, asking questions, Elisa expected her students to provide the answers that she determined in the lesson planning process. She believed that students needed to perceive the objective as she thought and planned. Elisa's expectations and students' answers follow:

- *S: There are no unmatched elements left.*
- *E: There are no unmatched elements left in the set of domain, okay. Let's look at example 2, does this indicate a sequence?*

1. $(a_n) = 2n$

2. $(a_n) = 1/(n-3)$

Figure 7.27. 1^{st} and 2^{nd} questions in the task Elisa used.

- *S11: This does not indicate a sequence.*
- *E: Why doesn't it indicate?*
- *S11: Because it doesn't contain all the positive integers.*
- *E: It takes the value of 3 and is undefined. So, it does not indicate. That is not any unmatched element in the domain set. This broke the condition of being a function. So we said that there was no sequence (70-79).*

As the data showed, rather than asking students (e.g., S and S11) to further determine how they reason, Elisa interpreted what they stated and further explained what was not mentioned by the students. For instance, Elisa explained specifically why an=1/n-3 determined a function; but not a sequence due to the value of “3”. Though, the student had not stated clearly whether the value she thought of was in the domain or range. Elisa, instead, emphasised that since the domain of the function did not include the value of “3”, the example did not correspond to a sequence. That is, rather than asking for instance “could you say more about” could you explain what you mean by “Because it doesn’t contain all the positive integers”, she interpreted what the student meant rather than listening to his reasoning. Elisa’s explanations suggested that she thought that her students had perceived everything she had in mind regarding the objective of the lesson (PBP.2C). In addition, Elisa’s explanations showed that she was too directive in giving the answers. By talking about the values in the domain and the relationship between sequences and functions, she directed her students to think like herself.

To sum up, data from Elisa’s lesson plan, the pre-interview, teaching and the post-interview depicted that Elisa showed the characteristics of PBP many times. After determining Elisa’s perspective, I further analysed the data for determining her teacher noticing. Results from her noticing are shared in the next part.

7.3.2. 2016 Results for Noticing

In order to determine the stage of Elisa’s noticing, I analysed the pre and post interviews from 2016 by using the codes of Learning to Notice Framework (Van Es, 2011). Table 7.6 below shows the frequency of the codes.

Table 7.6. The frequency table of the codes of noticing before and after Elisa’s teaching in 2016.

Elisa		What Teachers Notice					How Teachers Notice				
		1-Baseline	2-Mixed		3-Focused	4- Extended	1-Baseline	2-Mixed		3-Focused	4-extended
Before	Lesson Plan and Interview		2a	4				2a	3		
								2b	4		
			2b	4				2c	3		
After	Interview		2a	5				2a	1		
								2b	4		
			2b	2				2c	3		

As the Table 7.6 indicated data from both the pre-interview and the post-interview indicated that Elisa showed all the codes at the mixed level of Noticing, 18 times and14 times respectively. Particularly, Elisa attended to her teaching and learning pedagogy and strategies without considering students' thinking. She noticed some aspects of teaching and learning but she explained noticing events with general and evaluative comments both during the pre and the post-interviews.

In the following paragraphs, I share data in detail, first showing what Elisa noticed; and then, share data to show how Elisa explained her noticing at different levels. For that, I provide data from her pre-interview and post-interview.

<u>7.3.2.1. What Elisa Noticed.</u> Elisa described her lesson plan and the understanding of the whole class at the end of her teaching with using general impressions as follow:

E: The beginning of the lesson was nice. Since the lesson was 35 minutes, I just wanted them to distinguish between the definition of the sequence and whether something was a sequence. They started the sequence in the last lesson, but I didn't know, so it was fast, what I planned as 35 minutes is over 20 minutes. The first 20 minutes were beautiful, so everything ended as well as I thought. But the students reached the answer that all the functions that have all positive numbers, natural numbers in the domain. When I asked whether being a sequence, they could explain by making connections with examples. So, the first 20 minutes were nice, but the last 15 minutes were too much for me. I did not have a plan B, so it was not written form, yes, I was going to teach the sequences, I looked at the geometric sequence and the arithmetic sequence, but I did not look detailed enough to explain in the lesson. I wasn't too sure of myself there. It was a little unplanned there (Post-2).

As the data indicated, Elisa noticed two significant aspects of the lesson: Teacher pedagogy and students' thinking during her teaching. First, she mentioned that she prepared her plan for a 35-minute long lesson to introduce the topic, sequences; and, its relationship with functions. But she noticed during teaching that the actual teacher of the class had already introduced the concept of sequences during an earlier session that week. So, she claimed that students were able to comment easily on the examples in the task she prepared. Thus, the lesson ended before her expected time. Then, she mentioned that she continued with arithmetic and geometric sequences without any planning although she had inquired into those types of sequences prior to the lesson. Secondly, she noticed that her students reached the objective of the lesson such that the students in the class could determine if a given relationship forms a sequence or not by examining the values in the domain and range. Though as the data indicated, Elisa attended to all students' understanding of the topic, rather than providing any specific example from particular students' thinking. This suggested that data depicted the code (W.2A) at the mixed level of noticing.

During the post-interview, Elisa further explained the questions she asked during teaching. She particularly referred to the questions she had written in the lesson plan:

Do sequences make you remember something that you already know? (If they don't say function, the teacher asks additional questions to push them through function) For every value, I have a correspondence. What does it look like (Lesson Plan)?

She claimed that asking these questions during teaching, she directed students to perceive the first column given in the examples as a set and the second column as another set so that students would make the connection that the values in the first set (the values in first column) refers to the domain and the values in the second set (values in the second column) refers to the range of the function. In addition, Elisa noticed that she took students' focus on the domain set consisting of positive integers to show the features of sequences (i.e., that the domain of sequences consists of the elements of positive Natural Numbers). Though again, only attending to mainly her teaching pedagogy depicted that Elisa showed the W.2A code at the mixed level of learning to notice framework (W.2A).

Regarding what Elisa noticed prior to the teaching, data from the lesson plan and the pre-interview indicated that Elisa hypothetically planned what she needed to ask to students during the lesson to reach the learning goals of her lesson. Particularly, Elisa had written the teacher actions and students' expected answers in her lesson plan. Although she mentioned all the steps of the lesson and the questions she prepared as if the learning progression would occur the way she had expected, for unexpected situations, she also planned to ask some additional questions. Though, again the goal of her questioning was to direct students to the objectives of the lesson, rather than listening to how they possibly might reason. Regarding her hypothetical planning, Elisa had further commented during the pre-interview:

Yes, I mean, I'm trying to make students say that functions whose domain is positive integers or positive natural numbers are called sequences. Well, I wrote these discussions here for a long time. Then, after I talked about them, I give one more task. Okay, so we defined the sequences as a function with domain contains positive natural number. Let's discuss the questions to whether these are sequence (Pre-8).

As the data showed, although Elisa prepared expected student answers and evolving understandings, she explained her plan by pointing to her own actions. For instance, she stated that in the second part of the lesson, students would discuss whether the examples she provided would refer to a sequence based on the definition they would deduce. Thiss further evidenced that Elisa showed W.2A code at the mixed level of what teacher notice (W.2A).

Similarly, data from the pre-interview also pointed that Elisa showed the code 2B. Particularly, for Elisa, it was important for the students to reach the learning goals. Though, she wanted the students to follow the path that was hypothetically determined by her. At the pre-interview, she attended and explained how students might think during teaching as follows:

I made a schema in their mind. They think based on this schema, and say there was a box for each box, but that is not like that. They might say that this is not a sequence. But I want them to tell that is not a function by considering the feature of being a function, so my questions based on it. So, when they see this (9th example), *they will say to me that this is not a sequence. Because a term is missing, but they should not tell me according to the schema in their mind. They should tell me that it is not a sequence because there should be no matching elements in the domain of the function. I always ask my questions to hear this (48).*

As the data indicated, Elisa intended for students to construct the knowledge about the domain and the range of a sequence by using the table of x and y values provided in the examples. Elisa did not attend to particular students' thinking but she explained the possible thinking one might have regarding the features of a sequence. She noticed that given the example that did not refer to a sequence because of the void in the domain values students could determine such requisite and determine that the given example would not be a sequence. Though again she expected that students would reason how she hypothetically planned such that she wanted them to explain the given example in connection with functions. This suggested that while planning for the lesson, she hypothetically noticed what students need to think, even if not focusing on a particular student. Therefore, data showed that she depicted the W.2B code at the mixed level of learning to notice framework (W.2B).

Further data from the post interview also showed that Elisa thought that particular students reached the objectives given in the lesson plan. She stated her noticing providing explanations from students' thinking as follows:

I wanted them to think pairs, so I gave them in a table. They often saw this matching better in the general term of the sequence. For example, a1 is something, a2 is something, but it couldn't be understood here, but I think they realized it because we always talked that way, this is the domain, this is the range, this is a function. For example, we are discussing why there is no sequence here (9^{th} question), because there is no second term. there is no match, that is, there is no equivalent of 2, so it is as if they thought the second term here (post-16).

9.

1	2	3	4
25		75	100

Figure 7.28. The 9^{th} question in the task Elisa used.

As the data indicated, Elisa noticed students' thinking about why given a relationship represents a sequence such that she mentioned that since students knew the requisite condition they were able to determine that the example did not represent a sequence. She mentioned that she directed students to think about the domain and range of a function to decide whether the provided examples represented sequences. She attended to what students realized in the given examples such that they knew that the given examples did not refer to a sequence. Although Elisa generally noticed and attended to the thinking of the whole class, she was able to provide an example from particular students' thoughts in her teaching. Therefore, Elisa showed the W.2B code at the mixed level of what teacher notice (W.2B).

<u>7.3.2.2. How Elisa Noticed.</u> The analysis of data from Elisa's pre-interview and the post interview further showed that Elisa both noticed her own actions and few students' ideas and also explained these special events at the mixed level of teacher noticing.

Particularly, in the lesson planning process, Elisa noticed the significance of functions in making sense of sequences and students' prior knowledge about functions to make the connections between sequences and functions. During the pre-interview, she further explained:

I do not want to use the word function, I am waiting to hear it from students, if they do not say, I will ask whether it indicates a function, but maybe they can see themselves first. I will ask a few questions and wait a little bit. I will say the function myself. I will never leave without making it clear (Pre-58).

As shown in her explanation, Elisa noticed that the features of functions were important to understand sequences as connected with functions and students should and could perceive this relationship. Her focus was reaching the objective rather than noticing students' thinking and acting according to their thought processes. In addition, Elisa stated that if there was no response from the students, she could revert back to telling the relation between functions and sequences. She hypothetically planned to notice students' thinking about the features of functions and its relationship with sequences. Though, she highlighted noteworthy events with general impressions. Therefore, data showed that she depicted the H.2A code at the mixed level of how teacher notice (H.2A).

Data from the post interview further pointed to the same code. She mentioned that the lesson ended early and she noticed some students' misconceptions. She explained her noticing as follows:

- *E: They knew that the sequences had to have a rule, because their teachers gave it that way. Because there is no sequence without rule in the textbook,*
- *R: What do you think?*
- *E: It doesn't have to have rule. But they learned it. So, when I first asked, of course, they did not say that the sequence is a special function, but they could say what is the geometric sequence, what is the arithmetic sequence, how can the general term be found?. Although they cannot give the general definition of the sequence, what is the arithmetic sequence, what is the geometric sequence, how can the general term be found? They were good at them.*
- *R: Though they couldn't give a general definition of a sequence, how did you understand it?*
- *E: Because when I was trying to get them to see this, this means the function that the domain contains the positive integers, I did not get it quickly. I had to ask a few questions. If they knew it, maybe I would get the answer in my first question. They understand it with a little push of me. I noticed there. but they were familiar with examples of geometric and arithmetic sequences. I thought they had no familiarity because I think I'll tell them from the beginning (8-10).*

As shown in the data, even though Elisa planned her lesson to start to talk about sequences, she noticed that students constructed the types of sequences and general term of an arithmetic and geometric sequence in an earlier lesson with their own teacher. Though she noticed that students were not able to define what a sequence is although they could think of the features of for instance arithmetic and geometric sequences. That is, Elisa noticed that although students learned the general features of sequences, they could not explain the definition of a sequence and could not state that any sequence was a special function. Therefore, Elisa needed to ask some additional questions to direct students to think about the functions and their relationship with sequences. Data thus showed that she attended to the significant events and explained them with general sentences. Therefore, Elisa showed H.2A code at the mixed level for how teachers notice (H.2A).

Data from Elissa's pre-interview further indicated that she hypothetically planned to notice students' answers during the teaching as follows:

I write the answers expected from students. If they couldn't give it, I ask other questions, for example I say, do these sequences remind you of something you already know? The answer may not come. What I asked may not be understood here. For every value, I see one correspondence value, mutually, there is a pair for each value. What does it look like to me? Expected answers may still not come. I say, then let's take a look at the first example.

1. Travelled distance of a car with constant velocity

Hour	1	2	3	4	5	...
Travelled Distance	50	100	150	200	250	...

(50,100,150,200,250,300, ...)

Figure 7.29. First example in the task Elisa used.

A car travelled 50 meters in one hour and 100 meters in two hours. Can you tell me how long the car will travel in 40 hours? The students can explain probably how *they find it? So, can you tell me how long to go in x hours? They tell. What does this look like? Over and over again, my questions always happen in this way. For example, what do you think about the 5th example?*

5. (7,7,7,7,7,7,7,7,7,7,7,7,7,......)

Figure 7.30. The 5^{th} example in the task Elisa used.

I think they can say that it is a sequence. Why do you think it's a sequence? By always asking and attributing the feature of being a function, my questions are always in this way (42).

As the data indicated, Elisa's lesson plan included many questions to direct students to think about the values in the domain and range. Similarly, Elisa determined some additional questions she would ask when she could not get the answer expected from the students. At the beginning of the pre-interview, her explanations about expected students' thoughts included some evaluative comments (I made a schema in their mind. They think based on their own schema, and say number in each box paired with a number in another box, but this example is not like that, so it is not a sequence) but she continued with some interpretive comments during the interview (The answer may not come. What I asked may not be understood here, "Probably they say", I think they say these are a sequence). Using interpretive comments together with evaluating students' thoughts depicted that data from the pre-interview showed that Elisa depicted the code H.2B at the mixed level of how teacher notice in learning to notice framework (H.2B).

Elisa generally focused on both teacher pedagogy and students' thinking in her teaching so in the post-interview, she explained what she noticed in teaching. When asked what she would change or leave the same if she taught the same lesson, she evaluated her lesson and elaborated on it as follows:

- *R: Well, when you think about this lesson plan, what would you change and what would you leave same to teach the lesson to a similar group?*
- *E: Now, I thought the students do not know anything, but they are a little familiar with the sequences.*
- *R: Yeah.*
- *E: My lesson plan would be a little longer. I would pass to geometric arithmetic sequences and I would a little more focus on the next subject. I would have kept this part a bit shorter and focused on a little more there (geometric and arithmetic). Why we call it geometric? The ratio of the two terms to each other is always fixed. In fact, it can be said by associating it with the function. In the geometric sequence, the ratio of consecutive terms is constant. It could be spoken a little nicer than that. I realized that it was not appropriate for the class because they finished it quickly. And I would focus on the arithmetic sequence, the general term finding. The geometric sequence is actually a concept, I would concentrate on both geometric sequence and operational part in the task by passing beginning of the lesson quickly (39-42).*

As shown in the data, Elisa evaluated her lesson and explained some important points for consideration to change for the next time. Elisa noticed that she could allocate less time for the definition of sequences because students were familiar with the features of a sequence and were able to distinguish the sequences in the examples. She attended to that she provided much time for students to recognize sequences, but she could have allocated more time to the features of geometric and arithmetic sequences. In addition, Elisa noticed and interpreted that students might have understood the relationship between sequences and functions more easily if she had started the lesson discussing the features of geometric sequences. Therefore, Elisa's explanations in the post-interview seem to be both evaluative and interpretive such that she depicted the code H.2B at the mixed level of how teacher notice (H.2B).

Data from the pre-interview further pointed to the code H.2C at the mixed level. As mentioned earlier, Elisa wanted her students to deduce the relationship between sequences and functions. During the pre-interview, she explained why she planned her lesson:

- *E: I am not testing it in the beginning. I do not always start like that, for example, I can say, okay what is the condition of being a function. But I do not assess what they know in the beginning.*
- *R: Why?*
- *E: Why? ... For example, I did not like it when my teachers used that way, because they started the lesson with a topic which was not the objective of the lesson, I believed that we will learn this topic. Then the topic got more complex so I cannot understand what I do in lesson. I will give that it is a sequence, after first saying sequence let's get back to a function, then they will think about why we are getting back to the functions. That is, I will give them a direct link. They will say directly function. They will not notice. I will try to make them say the word without talking the function first. Maybe one or two of them will say, then I will talk to them again. So, it is always better to do it in the middle rather than in the beginning. I don't know, maybe I have to do it in the beginning. I thought it but students might not think that an element in the domain of a function matched with an element in the range of function, This information might not come to me, then what would I say of course what are the necessary condition to be a function (31-32)?*

As shown in the data, although Elisa was aware of the significance of prior knowledge students needed to possess for their understanding of sequences, she did not prepare her lesson plan to activate such knowledge about functions on their part at the beginning of the lesson. Elisa explained the reason of her strategy by giving evidence from her learning process. She believed that that if she had started the lesson with activating their knowledge about functions, students might have got confused. Instead, she thought that when students examined some examples of sequences and discussed the general features of sequences, the relationship of sequences and functions would be more salient to them and they could construct the relationship more easily. As an alternative way, when discussing the domain and range of sequences with students , she thought that if they did not see the relationship between the values of x in the domain and y in the range, she could always ask questions about functions to direct them further. So, Elisa hypothetically planned to notice students' thinking and her possible interactions with them. Therefore, data indicated that, Elisa provided explanations both from her learning process and elaborated on possible interactions with her own students. Still, these elaborations were mostly not grounded in particular students' thinking nor they were about particular instances. Thus, data pointed to the code H.2C at the mixed level of how teachers notice during the pre-interview (H.2C).

Similarly, as Elisa emphasized during the pre-interview that she planned to focus on and ask questions about the relationship between the x and y values in the provided examples of sequences during teaching, during the post-interview she noticed that one of the students was confused about the x and y values in the sequence without a table.

A student could not see this (pointing to Question 5), that there was a match, I do not remember what he was saying right now... (16)

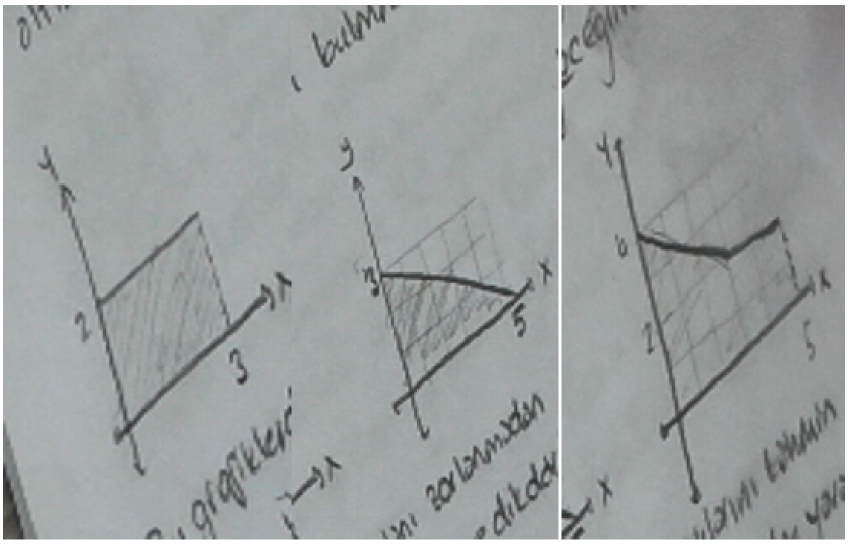

Figure 7.31. The 5^{th} example in the task Elisa used.

The girl in front could not see it the moment I wrote it. She said directly, was she *talking about positive natural number or the domain set, I don't remember the sentence right now. It was a bit of a personal feedback to her. 2-3 students in front of the class were students who spoke a lot. I was aware that they saw these binaries because they were constantly responding in that way. Even when I was talking about this (an = 2n), they were talking about the positive natural numbers and the values received from a set (post-23-24).*

Although in general during the post-interview, Elisa spoke about the understanding of the whole class, she noticed and gave evidence on some students' thoughts and difficulties. She mentioned that during teaching, she attended to a student's confusion about the domain of the sequence given in the example, but she emphasized that she gave the answer to the student directly. This suggested that instead of creating an environment of discovery, she reverted back to direct teaching again. Despite of students who had difficulties, Elisa realized that some students reached the intended goals because they answered all the questions correctly. This again suggested that her noticing of the students' answers were not towards how they thought but towards whether they acquired the intended goals. Therefore, this showed that she noticed her students' learning path towards the intended learning goals and provided some of their explanations as evidence of difficulties on their part and whether they overcame those difficulties.

In sum, data from both the pre-interview and the post-interview suggested that Elisa focused generally on her teaching strategies and questions rather than attending to students' thinking and difficulties. She noticed some special events in her teaching and lesson planning process, but she explained the noticed events with general and evaluative sentences rather than providing examples from particular events and particular students. She did not use an interpretive language to explain the evidence from her interactions with students either. All these data therefore suggested that Elisa showed all the codes at the mixed level more than once. So, she was at the Level 2 for noticing both during the pre-interview and the post-interview.

7.4. Elisa: Case of 2018

7.4.1. 2018 Results on Teacher Perspectives

Analyses of the data of Elisa's lesson plan, pre-interview, teaching and the post -interview, depicted that Elisa showed the characteristics of perception-based perspective (PBP) more than once (see Table 7.7).

Table 7.7. The frequency table of the characteristics of PBP before, during and after Elisa's teaching in 2018.

Elisa-2018		PBP			PIP	CBP
Before	Lesson plan	1A	2	7		
		1B	1			
		1C	1			
		1D	1			
		1E	1			
		1F	1			
	Interview	1A	4	22		
		1B	4			
		1C	6			
		1D	2			
		1E	3			
		1F	3			
During	Teaching	2A	5	28		
		2B	2			
		2C	5			
		2D	6			
		2E	10			
After	Interview	3A	5	5		

As shown in the Table 7.7, Elisa depicted the characteristics of the perception-based perspective prior to her teaching 29 times, during her teaching 22 times and after her teaching 28 times. In the following section, I share some parts of the data from Elisa's lesson plan, the pre-interview, teaching and the post-interview to show how Elisa depicted these characteristics.

<u>7.4.1.1. Elisa's Before Teaching.</u> In this section, I share data from 2018 from both the lesson plan and the pre-interview. Data showed that Elisa depicted the characteristics of PBP.1A, PBP.1B, PBP.1C, PBP.1D, PBP.1E, and PBP.1F. Previous research showed that, teachers who hold PBP create a learning trajectory to make the mathematical relationships as possibly apparent as for students (PBP.1A). Also, teachers who hold PBP believe that mathematics is formed by the combination of many interrelated ideas such that learning is acknowledged as a gradual process for students by making sense of the connections among these ideas (PBP.1B). In addition to the view on the nature of mathematics and learning, teachers who hold PBP believe that teachers have a key role to create an environment for active engagement and discovery of students (PBP.1C and PBP.1E). In this discovery environment, teachers direct their students to think about the intended learning goals with the help of materials, tasks or questions used during teaching (PBP.1D). Finally, assessment of students' knowledge base during teaching is seen significant as a tool to pass to the following step in the gradual process of learning, to pass to a new topic or to determine what lacks in students' understanding (PBP.1F).

Regarding Elisa's teaching in 2018, Elisa designed an 80-minute-lesson for 11th graders to talk about the Riemann Sum as the approximate calculation of the area under the graph of a function. Her main learning goals for the lesson were students' finding the (approximate) area under the graph by using the Riemann Sum and their showing and explaining the relationship between the summation and the integral using the calculation of the area under the graph. She stated her goals as follows:

What did I say the target? Students can estimate the area under a graph by Riemann Sum. This is the goal. Then, at the end of the lesson, the students should have information about the area under a curve that can be found by integral. I said only have information because the conceptual part of the work is where I will do it. This means integral. If we make infinite number of those small pieces, it is called an integral (5).

Elisa planned to start the lesson with dividing the area under the graph into pieces to calculate the approximate value by using the Riemann Sum. She wanted her students to learn the Riemann Sum first in order to show that the summation is related to the concept of integral. As the excerpt indicated, Elisa focused on whether the intended knowledge has been learned by the students at the end of the lesson rather than focusing on how her students might possibly mentally engage in the process (PBP.1A). In determination of the objective and the lesson planning process, Elisa considered her students' prior knowledge about the graph and the functions. She emphasized her thoughts as follows:

Students should be able to define what is a function. They should be able to draw and read the graph of a function. They should be able to analyse that where it increases, where it decreases, how it increases and how it decreases. They should analyse in different aspects. I said they should remember the demonstration of the series and some series formulas. It is not necessary to remember. I can give it at that moment ... They should be able to recognize the limit and apply the rules of taking limit. Since it is the 11th grade, I did not see it in the curriculum, only in the IB (Referring to the International Bachelor Program) ... They saw the derivative again in IB. They did not see the integral. In IB, there was a sequence like this let's reverse a derivative. Integral is just like the reverse of the derivative. They reversed expressions with coefficients like x^2, x^3. I was going to solve questions for the applications of the integral. We were right there. So, they know the derivative before this lesson. First, I thought they can find the areas of simple geometric shapes. Although they did not see the limit at a very deep level, they saw it in a simple sense. They know what it means, why they needed something like that. I thought they were ready because I believed this would come out of their synthesis (15).

As the data indicated, Elisa considered that in order to calculate the area under a graph, students needed to know and connect the concepts of functions, graph of the functions, limit, and derivative. She further commented that students in this class were from IB program in which students were expected to learn about integration as anti-derivative. Therefore, Elisa believed that students' prior knowledge about derivative, areas of some geometrical shapes and although limited some knowledge about the concept of limit would allow them to reach the meaning of Riemann Sum through the approximate calculation of the area under the graph. Elisa's explanations indicated that she saw mathematics as a connected set of ideas such that mathematics was independent and outside of the learner. Therefore, she thought that learning is a gradual process such that students could add new knowledge to their existing knowledge repertoire during teaching (PBP.1B).

After talking about what prerequisite knowledge students need to know, Elisa explained how she planned to get started her lesson:

I wanted to start the lesson with the areas they can calculate. A rectangle. I can ask this (showing the triangular space) too. Then I can ask this (showing the third figure) (5).

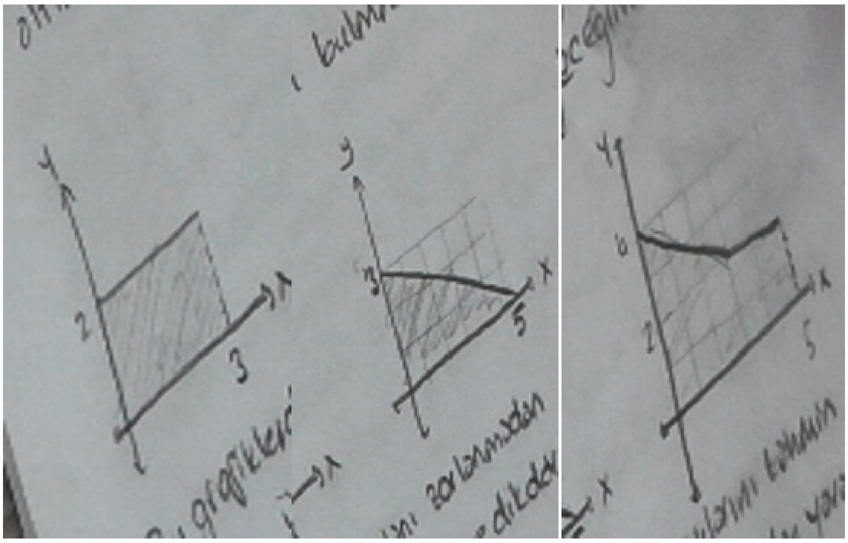

Figure 7.32. Beginning of the lesson planning Elisa prepared.

Then when asked why she would start the lesson with these graphs she further commented:

When I say how to find the area directly below, (pointing to the graph below in Figure 7.33), I will be very comprehensive, and my question will be very general. If I *start here suddenly, they might not be able to relate to anything they know. But if I slowly trigger a rectangle and a triangle they know first, then, if I trigger them to break into areas they know there, maybe they think to divide.*

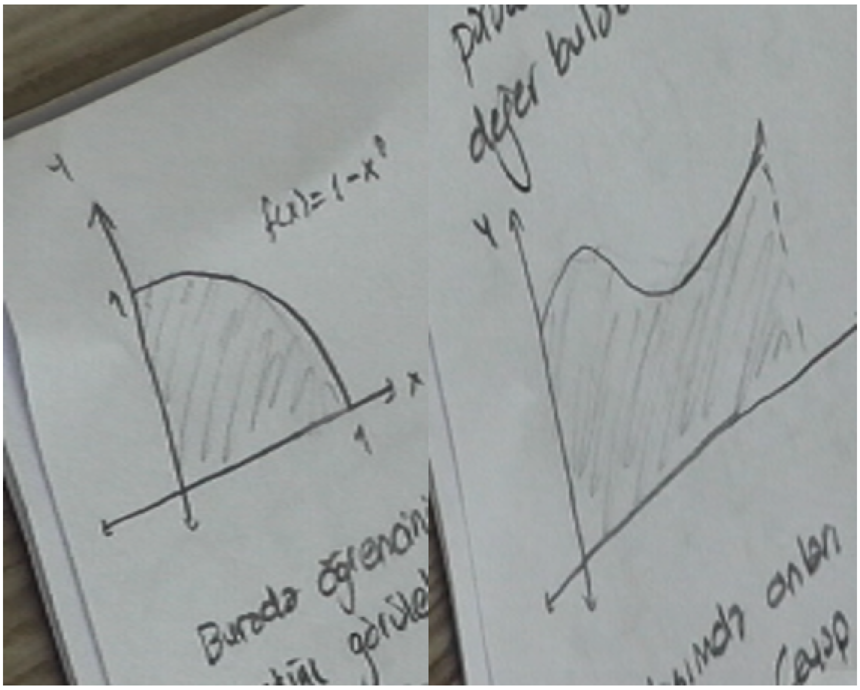

Figure 7.33. Examples Elisa thought to direct students to divide the area under the graph.

I gave them to think of approaching by dividing here. Because if I don't give it, they won't try to divide it into known shapes. They will say something that they have directly compared or say "I cannot find it?. Maybe there will not be a logical approach. I first gave these graphs to create the thought that they just divide the graph to some pieces maybe they can divide them into some unit squares or some other shapes". They can approach it that way (17).

First of all, data showed that Elisa determined graphs with the rectangular, triangular and a combination of both the rectangular and the triangular areas to start the lesson. This was because she expected that the students knew how to calculate the area of the first two shapes such that they would be able to calculate the area of the third shape by dividing it into the smaller ones whose areas they knew. By these examples, Elisa expected students to see that they can calculate the area by diving the shapes into smaller pieces whose areas they know. Therefore, they would be able to use such idea when they were asked to think about different functions and their graphs. In addition, she predicted that students might think to divide the areas under the provided graphs with different shapes and sizes. Thus, she considered that she could ask students to reach the approximate value of the area by using the summation of the areas of rectangular shapes. That is, she planned to direct her students to perceive that summation of the areas of the rectangular shapes are closer to the real value of the area under the graph than the summation of the areas of different shapes. In sum, the data indicated that Elisa benefited from students' existing knowledge to create a formula by using the sum of the areas of the rectangles under the graph. Though, her starting the lesson with the shapes students know and gradually adding further shapes to take their attention to further division of the shapes into smaller but manageable shapes suggested that Elisa created a learning trajectory for her students to make the intended learning goals and mathematical relationships as apparent as possible through her considerable effort (PBP.1A).

Secondly, data indicated that, if Elisa had asked the question "...How do we find the area under the graph..." for the first graph with the function rule, $f(x)=1-x^2$ and the second graph, this could have caused students to provide different explanations. In her explanation, she emphasized that she wanted to direct students' thinking towards dividing the area under the graph into further pieces that could be calculated easily. That is, Elisa explained that her reasons as to why she used the first three shapes known already by students and why she selected the given functions with the curvature nature were to trigger on their part the idea of the use of rectangular shapes toward the Riemann Sum. Specifically, she believed that by her selection of the examples and prepared questions, she could create an environment for her students' discovery of the idea of dividing the area under the graphs into smaller rectangular areas. That is, from her point of view, deciding the sequence of the lesson, the examples and the questions to be asked were the main responsibility of a teacher to direct her students to think about the area under the graph (PBP.1E).

Further, Elisa explained how she would guide students to reach the learning goal:

They realize that when I divide the graph with ten pieces, the areas that I lost much less and I get close much better to real value than dividing with four pieces. Because what I'm going to ask soon. What should you do to make an even better guess? I want to hear them to say more division. So, I wanted them to see that too. They have to say I can cut it into smaller pieces. We divided it into 4 pieces, we divided it into 10 pieces. Now, if we divide it into n pieces. The concepts will be abstract to them, but I will show them slowly. I will remind them the summation symbol (5).

Data showed that Elisa wanted students to gradually discover that dividing the area under the graph into 4 or 10 pieces would not be sufficient to approximate the area. So, through questions she wanted them to reach the idea that they needed to divide the area under the graph into more and more pieces (i.e., n pieces) so that they would reach the summation formula. That is, Elisa's aim for asking questions was to enable students to divide this area into as many pieces as possible to get closer to the real value of the area under the graph. This suggested that Elisa wanted all of her students to think about the same way that she pre-determined in her mind for the lesson planning to achieve the same objective. That is, she wanted to move in such a direction she had created in her mind that she ignored different ideas that students might possibly offer (PBP.1D). In addition, "I want to hear them to say more division? and "They have to say I can cut it into smaller pieces" showed that Elisa's emphasis while listening to her students was to know whether the students had reached the objective she pre-determined rather than focusing on how they reasoned. This further suggested that Elisa was aware that the intended goals were abstract such that she thought about breaking them into manageable concrete steps gradually allowing students to reach. In addition, the questions that Elisa planned to ask seemed to be prepared to show the deficiencies in students' understanding so that she would know whether to move to the next step during teaching. Thus, Elisa determined an assessment strategy to observe what students lack rather than gathering information about their thought processes or their already established conceptions (PBP.1F).

Data further showed that Elisa emphasized that she should create an environment for students' discovery since her expectation from students was to reach the intended knowledge by thinking, discussing and solving problems themselves. Given that the students perceived that they might divide the area under the graphs into more pieces in order to calculate the area as close as possible, she explained how she would guide the students to see the relationship between the summation formula and the integral as follows:

- *E: We're talking about the sum of things in a certain interval, not a certain number of things. This is fx and this is nx. So, I should direct them to think what is this? I can't say what's integral. They see it the first time. They see the integral sign, but I think they don't know much. They just saw it in the book. So, they will say it with my guidance.*
- *R: How do you plan to guide them?*
- *E: Can we relate these two? I said this (referring to Riemann Sum) as a total, but not a number, it's an interval. Well this f(x) is still here because we said when we take it infinite, f(x) is still there. So, what is this (f(x) in Riemann sum) in integral? How can we compare them? (97)*

Data showed that Elisa planned for her students to reach the summation formula by dividing the area under the graph into more pieces and collecting and summing the areas of all the pieces under the graph. By using the summation formula, Elisa targeted to show the relationship between the Riemann Sum and the concept of integral. Particularly, by using the similarities of the variables and formulas in Riemann sum and integral, Elisa aimed her students to discover that the integral can be used to calculate the area under the graph. When the researcher asked how she would guide the students, Elisa explained the questions she would ask to students She thought that further elaborating on which variables were related to each other could be a precursor for them to understand the relationship between the Riemann Sum and integral. Elisa's continual efforts by constantly asking questions and creating a discussion environment indicated that she planned a lesson sequence for her students to discover themselves the relationship between the Riemann Sum and integral instead of directly telling the relationship to them (PBP.1C). That is, data suggested that Elisa was aware that students needed to reach the intended learning goal through thinking actively on the questions or tasks they were given. Data further suggested that Elisa could hypothetically determine how students might possibly need to think to reach the learning goal she had in mind. However, the nature of what and how they were expected to think pointed to the fact that Elisa's focus was more on whether the students had completed the steps in thinking pre-determined by Elisa. Particularly, Elisa expected to hear from students what they specifically needed to tell given the progression in steps gradually developed toward the learning goal. Thus, her focus was not on how students possibly might think but rather on what they needed to tell about what they perceived apparent (from the point of view of Elisa) in the answers to the questions given the task sequence.

<u>7.4.1.2. Elisa's Teaching and Interview After Teaching.</u> In this section, I share data from Elisa's teaching and the post-interview in 2018. Data from her teaching and the post-interview was also coherent with the data from her lesson plan and the pre-interview. As she had planned before the lesson, Elisa started the lesson with the calculation of the area of the shapes already known by students. In order to calculate the area of the third shape, as Elisa expected, the students divided the shape into a triangle and a rectangle. Therefore, using the students' prior knowledge about the area of different shapes, Elisa was able to direct students to think to further divide the shapes into the pieces.

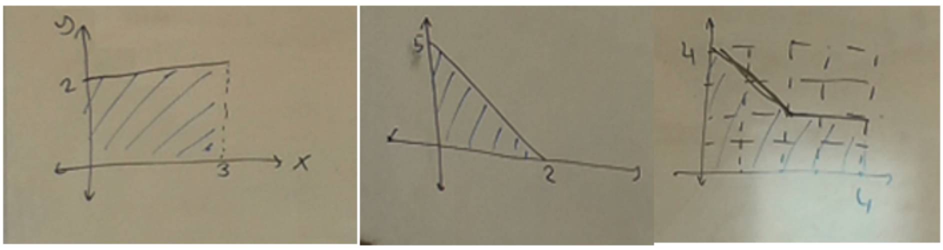

Figure 7.34. Beginning of the lesson Elisa taught.

Elisa continued the lesson with the graph of the function, f(x)=1-x^2. As Elisa stated in her pre-interview, students did not think directly to divide the area under the graph to find the area. Elisa asked some questions as follows:

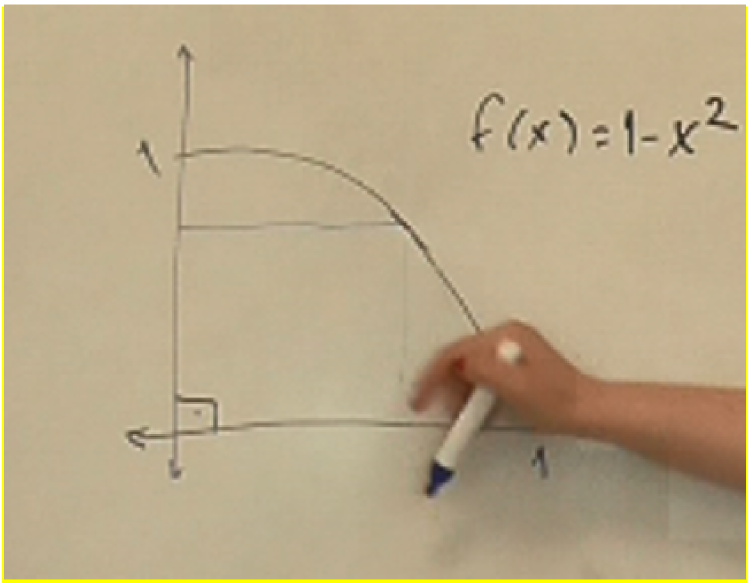

Figure 7.35. The example of dividing the area under the graph with the known shapes.

- *E: Now, I will ask again how will we find the area under the graph? I do not expect you to directly say the numerical value of the area under the graph. So, I want you, at least an estimate or a value. How can we approach? What can we do here?*
- *S1: If the axes are perpendicular, we find them from the area of a circle.*
- *E: Yeah. You said I could approach from a quarter circle. You can get pretty close. Is this a full quarter circle? S3: Yes*
- *S1: (Not Understanding)*
- *E: So, what's the graph of a quadratic function like y=x^2? If I don't give a piece of the graph because I have now given you the interval between 0- and 1, what if I give you the whole (referring to the graph of y=x^2 given the interval of Real Numbers). What would the parabola look like? (17-21)*

Elisa knew that her students did not know the calculation of the area under the graphs with non-specific shapes. She expected that the rectangular areas students had calculated might have triggered the idea that they might have divided the area under the graph of f(x)=1-x^2 into rectangular pieces, too. Instead, as Elisa had predicted during the pre-interview, the graph of f(x)=1-x^2 reminded some students of the idea of a known shape, such as a circle. Rather than focusing on how and why students could determine that the graph of the function was a circle, Elisa's emphasis that the given graph was a parabola rather than a circle depicted that she did not focus on how students thought. Rather her focus was to take students towards what they did not know (i.e., the learning goal). Therefore, the data showed that considering mathematics as a connected set of ideas and intending to trigger such connections on the part of students, Elisa focused on what students needed to perceive what was apparent in the graphs (from the point of view of the teacher) rather than focusing on how they thought (PBP.2B).

By discussing about parabola and the circle and asking new questions, Elisa tried to handle students' thinking about the graph of f(x)=1-x^2 and the quarter of a circle. One of the students drew a basic parabola on the board and then the discussion continued as follows:

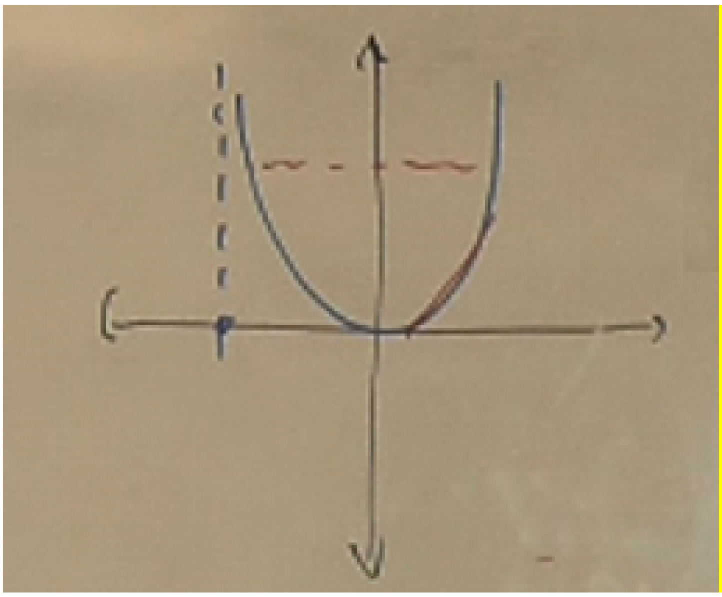

Figure 7.36. Drawing of a student of a parabola.

- *S1:Let's say it went like this.*
- *E: Okay, this is a parabola.*

- *S1: If we draw a line it will end at some point. I think it will go the same.*
- *E: Okay. Does it make a quarter circle when I cut it from anywhere? Is this a circle?*
- *S2: I think it is a circle.*
- *E: For example, why S4.*
- *S4: It is opening.*
- *E: It is opening. How is the circle?*
- *S4: Fixed.*
- *E: If I think of a quarter circle. What is meaning of the fixed? It looks a bit symmetrical here and there (showing the radii of the quarter of a circle on the x and y axis). Okay, actually, this is not a quarter of a circle, but you said I can approach it from a quarter of a circle. So, I want you to find a method like that, find a method that I can approach it too. I can approach the area under the graph in both. I want it.*

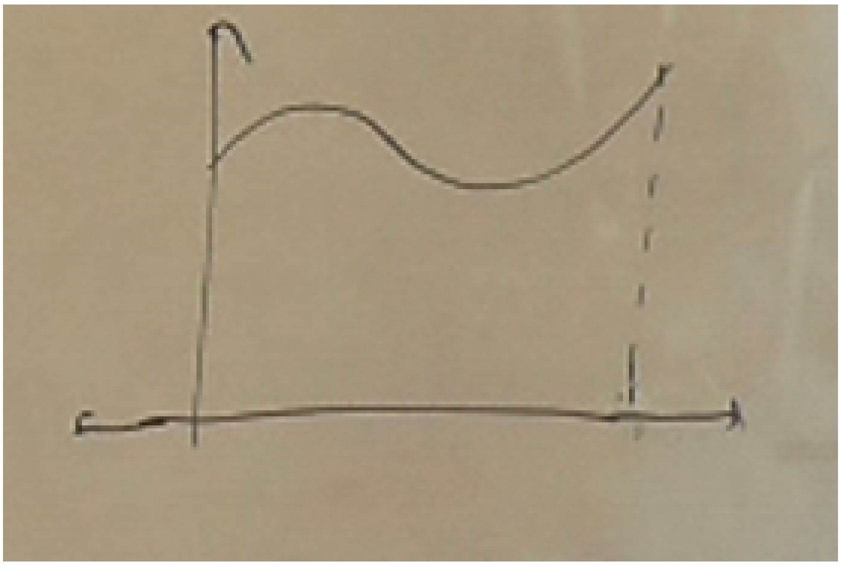

Figure 7.37. Another example Elisa had drawn on the board for finding the area.

- *E: For example, you said these (rectangles and triangles) directly. You couldn't say that (triangle + rectangle) directly, what did you do here?*
- *S1: I cut it in half.*
- *E: You divide. Maybe you can also divide them. But you can divide something*

you know. You don't have to say exact value, the approximate value is enough. Where's your problem? If I draw a straight line from here, and here (drawing a rectangle). You can find them. Your problem is finding area under the curve.

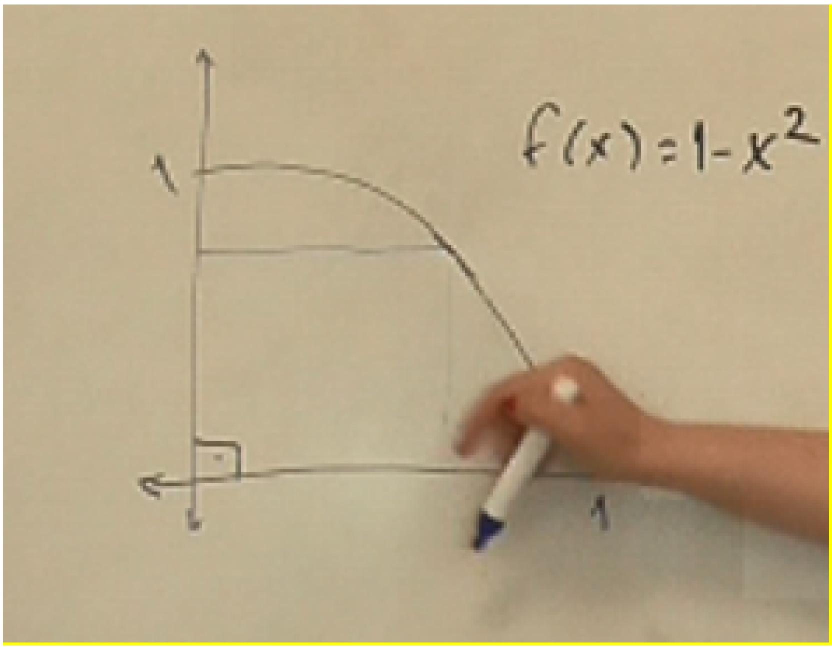

Figure 7.38. Elisa's showing dividing the area under the graph with known shapes.

As the data indicated, Elisa tried to show the differences of a circle and a parabola by asking questions. By asking questions, she tried to create a discussion environment in the class because while some students insisted on the quarter of a circle, some students realized that the example was not a quarter of a circle. She listened to her students' thoughts carefully and asked new questions to assess their understanding and to pass to the idea of dividing the area under the graph (PBP.2D). However, when she could not get any answers from students to her directive questions, she directly said the graph was not a quarter of a circle (PBP.2E). In addition, in the lesson planning, Elisa had predicted that students might have said the quarter of a circle , so she had prepared an alternative graph to direct students to the idea of dividing the areas under the graph rather than thinking a specific shape like quarter of a circle. By talking about the rectangular and triangular areas given at the beginning of the lesson, she directed her students to think a general strategy to use for non-specific shapes. The questions she asked, the examples she gave, and the discussion with the students showed that

Elisa directed students to think as she wanted (PBP.2E).

That is, given the particular examples at the beginning of the lesson, Elisa had expected that students would think of dividing the area under the graphs with rectangles. This idea was the key point of Elisa's lesson. However, the use of examples at the beginning of the lesson did not trigger such idea on the part of students. So, rather than acting on how students already have thought (the use of the area of quarter of a circle), and probing them further to trigger the idea of dividing the area under the shape into smaller pieces, Elisa directed their reasoning as follows:

I can fill the area under the graph. Now, I want to do some mathematical approach. I want to make some calculations. If I fill them up randomly, it will be very difficult for me to calculate them in a systematic way.

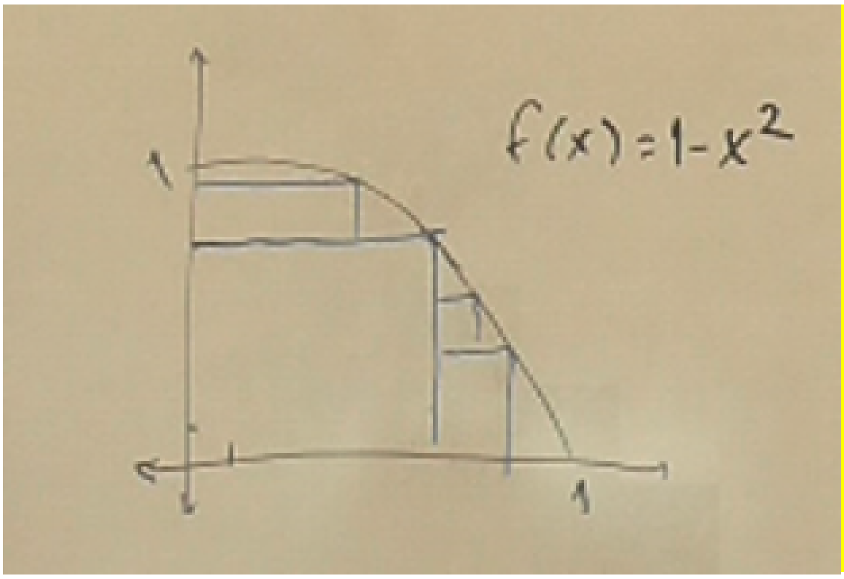

Figure 7.39. Elisa's drawing for dividing the area under the graph with known shapes.

So, I will do the following. Let's divide it into rectangles, but let's do all of them perpendicular. Let's divide it again into rectangles. I will divide it evenly like this with equal intervals. Let's get combine the end of the rectangles with the next rectangles. In this way, we can eliminate the areas that are left. It is a nice approach. I just take all of them perpendicular to make it more organized.

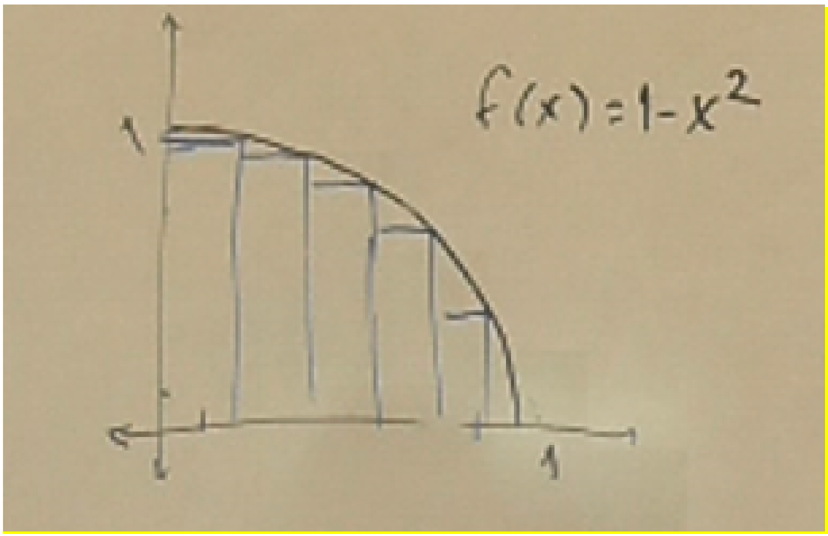

Figure 7.40. Elisa's drawing for dividing the area under the graph with rectangles.

Let's rewrite this. Let's divide it into 4 rectangles and try to calculate the area. Let's get close. Do you have your calculators? (Looking at the students) (49).

As shown in the data, once she did not get the expected answers from the students, Elisa went back to direct teaching and shared the idea of using rectangular shapes to divide the area under the graph. Also, instead of asking students to further think about and elaborate on how they could determine the size of the rectangular areas, she directly explained that dividing the area under the graph with shapes of different sizes would not be systematic. During the pre-interview, she had said that she would guide students by asking questions and creating a discussion environment to think about the use of rectangular areas smaller in size for finding the approximate area under the graph. On the contrary, during teaching, she directly told students that rather than using different shapes, they needed to use rectangular areas to systematically calculate the area under the graph. She even asked them to divide the area under the graph into four pieces. This suggested that Elisa's focus was more on reaching the intended learning goal rather than creating a learning environment to trigger discussion focusing on different students' ideas and thoughts. Thus, data showed that once Elisa realized that students did not perceive what was apparent in the examples from her point of view, she revert back to direct teaching and told the answer rather than focusing on how students made sense of what they have been engaged in. This showed that she depicted the characteristic of PBP.2E of teacher perspective during teaching.

Knowing that dividing the area under the graph into four rectangles would not be enough to reach the real value of the area, Elisa planned that students divide the area under the graph into10 rectangular areas. That is Elisa knew that students needed to think about the values of x and y to determine the horizontal and vertical lengths of the rectangular shapes. That is, plugging the value of x in the function rule, students could reach the y value so that they could calculate the rectangular area by multiplying the value of x and y. Discussing all these steps with her students in dividing the area under the graph into four rectangles was important to pass to the next step (dividing it into ten rectangular areas). She provided students with time to calculate the area under the graph by dividing it into ten rectangular areas:

- *E: Tell me what happened. 0.531. Okay, now I want something from you. Do this by dividing it by 10. Divide it into 10 pieces.*
- *S2: In 10 pieces?*
- *E: In 10 pieces.*
- *S3: Is it 1/10 things or 1/10?*
- *E: For example, yes. Start to do in your notebook. I'm coming. Let's again divide this graph into 10 equal parts this time. Let's calculate from 10 rectangles. (She gives time to students, examines their notebooks). Are you doing it together? We have time to do it yourself.*

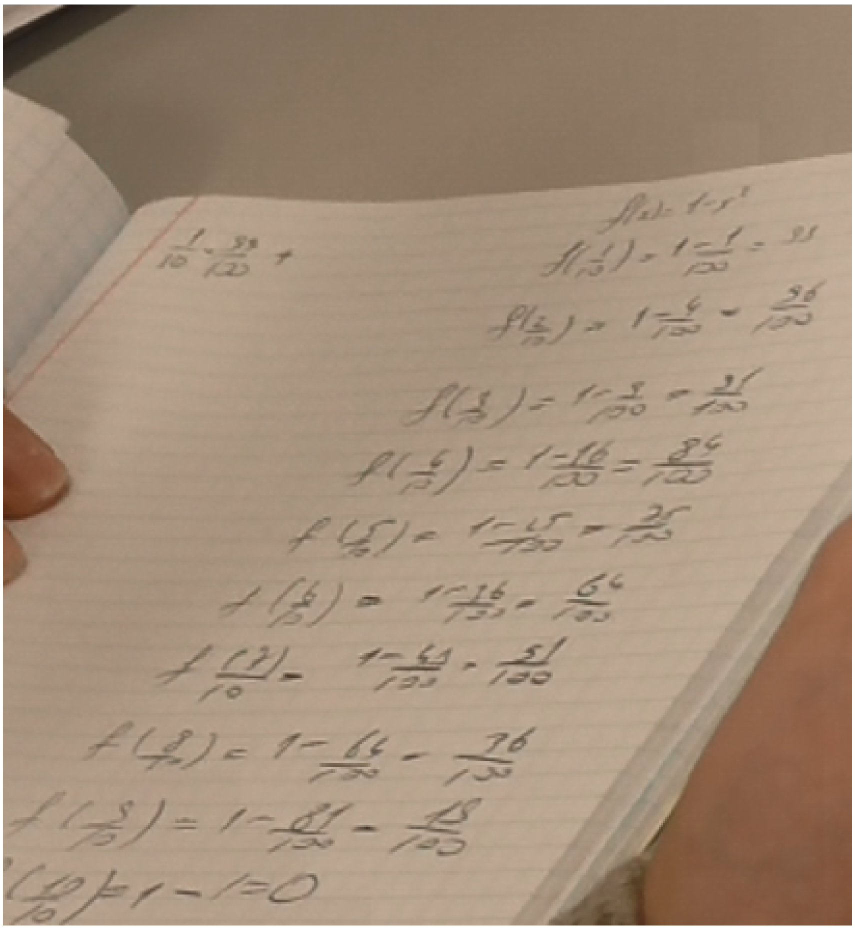

Figure 7.41. The notebook of a student about the area of rectangles.

For example, what did you get when you put 2/10 in f? (91-95)

As shown in the notebook from one of the students, the students calculated the y values of the function to find the area of the rectangular shapes. Elisa walked around the students and discussed about their calculations. For some problematic issues regarding students' calculations, she even asked questions to direct them to reach the correct answers. Active engagement of students in thinking and discovering the mathematical relationships, Elisa had provided opportunities for students towards the intended learning goal during teaching (PIP.2A). Again, her focus was more on that students attended to and perceived what was apparent in the examples and questions towards the learning goals rather than focusing on how students developed ideas and how they thought.

After students worked on the idea of dividing the area under the graph into ten rectangular pieces, discussion continued:

- *E: What have we done when we divided the area into more pieces?*
- *S1: We approached the area.*
- *E: I approached the area more accurately. Which area did I get smaller?*
- *S5: Rectangles.*
- *E: The rectangles became thinner rectangles, they became smaller.*
- *S3: The left over area became smaller.*
- *E: Yes, those triangular areas that are not counted shrink. I could calculate more areas. I got a better result. So, what should I do to get an even better result?*
- *S3: We can divide the area by 100.*
- *E: So, when we divide into more pieces, what will be happen?*
- *S1: We will approach the area.*
- *S3: We will get better...*
- *S5: We will guess more accurately.*
- *E: Okay, I want something like this, let's divide by n. Let's do a mathematical operation with n. So, we can divide it by 100, we can divide it by 1000. Let's call it n. We don't know what is n? It could be 5, it could be 10, it could be 100. But we will divide by n. So, let's suppose we divide by n. What about x values? (155-169)*

As the data indicated, students realized that both the rectangular areas and the area outside the ten rectangular areas got smaller once they divided the area under the graph into ten rectangular pieces rather than four rectangular pieces. This way they knew that they could get closer to the real value of the area under the graph. This suggested that students' answers and their inferences were important and necessary for Elisa to decide whether to pass to the next step. Though the nature of the dialogue between Elisa and students and among students suggested that Elisa's focus was on whether they were able to provide specific answers (such as the rectangular areas got smaller etc.). Her focus was not on how they thought. Had her focus was on how students had thought she might have asked how they determined that the rectangular areas got smaller or what made them think that. Instead, listening to students' answers and the explanations (i.e., the correct ones from the point of view of her), Elisa asked further questions to decide whether they were ready to pass to the next step (i.e, dividing the area under the graph into n pieces) in her lesson plan (PBP.2D). That is, Elisa listened to her students' thoughts and discussions not only to decide what they understood to pass to the next stage but also whether she heard what she expected from them.

After Elisa determined that students were ready to discuss the idea what if the area under the graph be divided into infinitely many partitions: Discussion continued:

- *E: First, we divided it in four. Then we divide it into ten. We made a better calculation and had smaller areas. We approached more and more correctly. Then I said to find it better. You said let's divide by hundred (showing S3). Maybe it would be better if we divide it by 1000. Then S2 said something there.*
- *S2: Let's divide it infinite.*
- *E: Infinite. So, what are we doing? What are we increasing?*
- *S2: The number we divide.*
- *S5: The number we divide.*
- *E: What we divide. What about the number of rectangles?*
- *S3: It will increase*
- *E: It will increase. How much will it increase, for example? We will divide into more pieces. What about the widths of the base of these rectangles?*
- *S5: It will be smaller.*
- *S3: It will get smaller.*
- *E: It will get smaller. How much will it get smaller?*
- *S3: As we have increased.*
- *E: How much do we increase? Is it too small like zero if we approach to the infinite? (245-255)*

Data showed that Elisa first reminded students of the idea that compared to the calculated area under the graph partitioned into four pieces, the calculated area under the graph partitioned into ten pieces were closer to the real value. Her focus on S2 and S3's ideas about dividing the area into more pieces was important. This in fact suggested that her focus was on what students thought: In particular, her questioning allowed students to further explain that dividing the area under the graph into infinitely many rectangular partitions would increase the number of rectangular pieces due to the decrease in the width of the shapes. This in turn would make it possible to calculate the real value closest to the area under the graph. Still, the questions she asked seemed to direct students' thinking to the idea of dividing the area under the graph to infinitely many rectangular pieces. That is, although her efforts attempting at gathering how students reasoned cannot be undermined, her focus was again more on what students thought rather than how they came to such deductions. Especially, her last sentence emphasizing the idea that the width of the rectangular shapes would be almost zero once the number of partitions were infinitely many suggested that Elisa listened to and examined her students' answers to her questions as determining if they corresponded to her own thoughts. Therefore, even though the answers came from students, by further taking students' attention to what she envisioned to hear from them, Elisa directed students to what she planned to reach. This suggested that Elisa believed that all students saw what she saw in her teaching through questioning, so her goal for listening to them was to confirm that their answers were the same as she had envisioned (PIP.2C).

At the post-interview, Elisa further explained why she asked the questions and why she created a discussion environment with the tasks she gave to students:

Instead of constantly presenting and explaining something on the board, while analysing it, drawing it, writing it on their own, they may establish the ideas themselves because they will see where it come from? That's why when I said they have to do it themselves. They have to do themselves, they have to establish a cause and effect relationship. Or they have to establish the connection with the limit (Post-27).

Elisa believed that the experiences students have during teaching were significant for their learning because once students made calculations, drew the graphs or made analysis in their calculations, they could come up with the cause-effect relationships towards the link between the Riemann Sum and the concept of integral. That is, instead of the teacher's (Elisa) giving the correct answer directly, she believed that students could reach the ideas by discussing with their classmates, answering questions and by doing the task themselves. Though again, the aim behind asking questions or using tasks was to provide students with an opportunity such that through their active engagement and discussion, they would discover from the point of view of the teacher what was apparent in the tasks (PBP.2A).

At the post-interview, Elisa stated that she faced with the following problem: During the lesson, Elisa just focused on the left Riemann Sum of the area under the graph, instead of using both the left and the right Riemann Sums of the area under the graph.

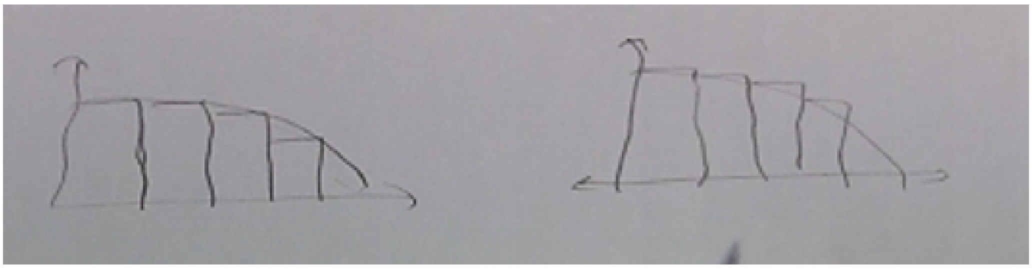

Figure 7.42. Drawing of Elisa about the right and the left Riemann Sum.

So, ultimately, when we divide both infinitely, both two (referring to the left and right Riemann Sums) converge to the area under the graph. And, if I don't divide the graph into infinite, if I divide it into a certain finite pieces, the area will be between these two value. I was very concerned about my time. Because I thought that it might not be finished in 70 minutes, I gave it up. I just talked about the Riemann Sum from the left. I thought I could give it conceptually because it would converge to the area while it approaches to infinite. But currently, I am thinking that I could give them both ideas (referring to the left and the right Riemann Sums) at the same time. Because when I think the infinite, when I passed to Riemann Sum, when I didn't get the intervals equal, they got confused about the rest of the area that is outside of the graph. Many questions *came from here. I had to spend 15 minutes for that (5).*

As the data indicated, Elisa noticed that students had problems about understanding how the summation of the area of the rectangular shapes in the left Riemann Sum of the graph could be used to calculate the area under the graph because they believed that the area of rectangular shapes in the left Riemann Sum of the area would have some remained area. While Elisa had planned the lesson, she expected students to understand that drawing rectangles from the right vertex or left vertex of the graph would give the area under the graph. She considered that students could see what she saw in the graph (PBP.2C). Elisa stated that had she had more time; she would have used both examples from the left and the right Riemann Sum of the area under the graph to avoid confusion among students. She also emphasized that she would have made changes in the plan due to the inability of students to see the relationship between the two graphs. Again, Elisa's interpretation that the lack of students' understanding was due to the fact that she did not ask students to simultaneously focus on both the left and the right Riemann Sums suggested that she did not take into consideration of how students thought. Rather, the reason behind students' difficulty in thinking about how the area under the graph was calculated by the right and the left Riemann Sums was how the task was provided to them (PBP.3A).

Again, once asked how she could help students overcome their difficulty, she explained her strategy as follows:

R: So, what do you mean by the intervention? You said I would intervene better.

E: For example, I am trying to convince him there, or they are now convinced by this (Right Riemann Sum) that whatever the rea will be close to what when he takes it to infinity. I have to intervene in this (Left Riemann Sum). I have to guide them. I mean by intervention, guidance, asking questions, getting something from friends or discussing with them, providing discussion environment. I mean, I will do something, I would do something for him if I knew why he thought that. I did not know his thoughts when I asked questions. I did something more general because I could not understand *his thoughts very clearly, and my talking was general rather than specific for him. In general, we tried to question again. In there, S4 understood before the other students. I asked a lot of questions to him. I tried to make them talk a little together too. S4 explained to him. S4 said to him. They talked themselves (13-15).*

As the data indicated, Elisa revealed that some students were convinced that the left Riemann Sum of the area under the graph would yield to the real area given infinitely many partitions. She commented that by asking questions and creating a discussion environment within the whole class further probing students who could and could not understand, she planned to direct her students to understand that the right Riemann Sum would also yield to the real area given infinitely many partitions. However, her focus was again on to direct students to what she intended to achieve in the lesson rather than providing opportunities for them to speak and think freely. All these explanations depicted that Elisa did not see any of her actions or decisions problematic such that the reasons as to why students could not perceive the mathematical relationships was because of how the task was provided (PBP.3A).

7.4.2. 2018 Results for Noticing

In order to determine the stage of Elisa's noticing, I analysed the data from both the pre- and the post-interviews by using the codes of Learning to Notice Framework (Van Es E., 2011). Table 7.8 below shows the frequency of the codes.

Table 7.8. The frequency table of the codes of noticing before and after Elisa's teaching in 2018.

		What Teachers Notice					How Teachers Notice				
		1-Baseline	2-Mixed		3-Focused	4- Extended	1-Baseline	2-Mixed		3-Focused	4-extended
Before	Lesson Plan and Interview		2a	5				2a	6		
			2b	2				2b	3		
								2c	3		
After	Interview		2a	3				2a	4		
			2b	4				2b	3		
								2c	8		

The results of the data from Elisa's pre-interview and the post-interview indicated that Elisa showed all the codes at the mixed level: 18 times during the interview before the teaching and 22 times during the interview after the teaching. Particularly, Elisa attended to her teaching and learning pedagogy and strategies without considering students' thinking. She noticed some aspects of teaching and learning but she explained noticing events with general and evaluative comments both during the pre-interview and the post-interviews.

In the following paragraphs, first I share data showing what Elisa noticed and then share data to show how Elisa explained her noticing at different levels.

<u>7.4.2.1. What Elisa Noticed.</u> During the pre-interview regarding the lesson planning process, Elisa commented on what her students' prior knowledge was to learn about the Riemann Sum of the area under a graph. She believed that if she started the lesson with the area under specific graphs which were known by students, in the following minutes during the lesson, they could have more easily thought about partitioning the area under the graphs. She explained why she selected the examples of such graphs as follows:

I wanted to start the lesson with the areas they can calculate. A rectangle. I can ask this (showing the triangular space) too. Then I can ask this (showing the third figure) (5).

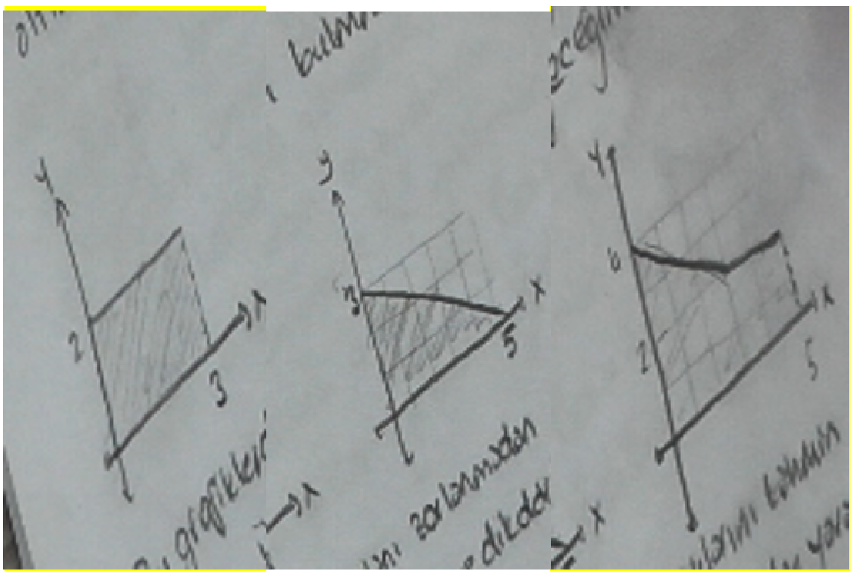

Figure 7.43. Beginning of the lesson in teaching of Elisa.

The reason I ask this here is that they say "I divide the graph". I want to trigger with those pieces that students think that "I can estimate the area under the graph by dividing it into pieces that I knew" (5).

During the pre-interview, Elisa explained her purpose in selecting the examples such that she was aware that her students knew how to calculate the area of a rectangular shape and a triangular shape. By using such shapes, she expected students to divide the third graph into rectangular and triangular pieces. Elisa was aware of why these questions were important and why she would ask them. Particularly, she noticed that once students' attention was taken to partition the third graph into familiar shapes, students might have mentally become ready to engage in the following steps in the task sequence and this would form the basis for students' answering the questions she would ask later. Attending to her teaching strategies and pedagogy depicted that Elisa showed W.2A code at the mixed level for teacher noticing.

Not only at the pre-interview, but also at the post interview she explained her teaching strategies. One of the most used strategies was to enable students to reach the intended knowledge. She noticed why she directed her students by asking questions, using tasks, and creating discussion in teaching:

I have to guide them. I mean by intervention, guidance, asking questions, getting something from friends or discussing with them, providing discussion environment. I mean, I will do something, I would do something for him if I knew why he thought that. I did not know his thoughts when I asked questions. I did something more general because I could not understand his thoughts very clearly, and my talking was general rather than specific for him. In general, we tried to question again. In there, S4 understood before the other students. I asked a lot of questions to him. I tried to make them talk a little together too. S4 explained to him. They talked themselves (13-15).

As the data showed, Elisa attended to one of the students' thinking who had a problem understanding the Riemann Sum of the area under the graph given the number of partitions were infinite. Elisa used different strategies to direct her students to think about the rectangular shapes as if they became a line due to the infinitely many partitions of the area under the graph. Elisa noticed that she could create a discussion environment within the whole class or only with some students or she could ask questions that might guide students to the intended knowledge. Although she mentioned particular students, she couldn't realize at which point the students were confused. So, she mentioned directing the students to discuss their ideas with their classmates. Noticing of her strategies and teaching pedagogy during her teaching showed that Elisa showed the code W.2A at the mixed level for teacher noticing.

Elisa believed that as well as the teacher's pedagogy and actions were important, the students' thoughts and behaviours were also significant for teacher actions. During the pre-interview, Elisa began to notice some mathematical thinking of students about the area under the graphs. In the lesson planning process, she predicted that her students could interpret the graph she planned to give as a quarter of a circle. During the pre-interview, she even commented on and interpreted how she would handle this problem as follows:

I will say them that if you don't say this exactly, guess. I expect them to divide here too, but here is an unexpected thing that they can approach like a quarter of a circle. And I will give the following example to them.

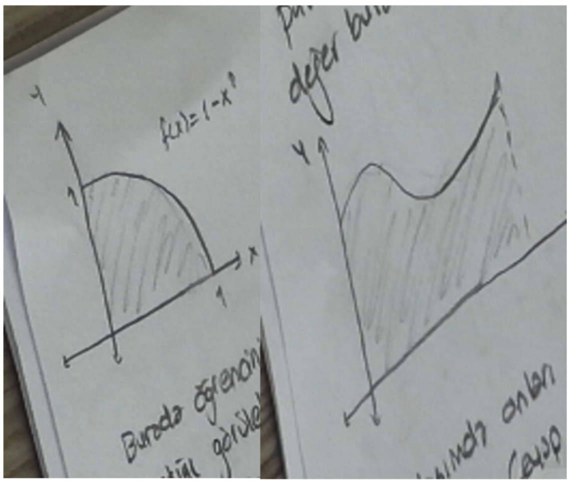

Figure 7.44. Examples Elisa thought to direct students to divide the area under the graph.

I will say to students that think for both of two graphs. Develop an approach to use both of two graphs that I can apply to both of them. I will handle the idea of quarter of a circle with using both graphs together ?. I gave them to think of approaching by dividing here. Because if I don't give it, they won't try to divide it into known shapes. They will say something that they have directly compared or say "I cannot find it". Maybe there will not be a logical approach. I first gave these graphs to create the thought that they just divide the graph to some pieces maybe they can divide them into some unit squares or some other shapes. They can approach it that way (17).

As the data indicated, Elisa hypothetically planned to notice her students' possible thoughts about both of the graphs. Since she knew her students' prior knowledge, she stated that students might interpret the graph as a quarter of a circle. Though, she mentioned that the main purpose she had in providing such an example was to direct students to think about dividing the area under the graph into the shapes they had already worked on (e. g, the rectangular shapes) rather than thinking about it as a whole (e.g., a known shape like a quarter of a circle). Planning to notice some possible answers or thoughts of students depicted that Elisa showed the W.2B code at the mixed level (i.e., hypothetically planning to notice students' mathematical thinking) (W.2B). In addition, she planned how she could handle students' thoughts if they deviated from her agenda. She prepared the second graph above to enable students to think that they need to divide the area under the graph into pieces. By noticing some possible students' answers and thought processes, she determined her teaching strategy and questions (W.2A). Elisa's explanations depicted that Elisa decided on her strategies according to her noticing of students' thoughts.

At the post-interview, Elisa noticed some students' thoughts about the area of the rectangular shapes under the graph. She mentioned different students' ideas as follows:

E: We were doing this (by taking the right Riemann sum) and I said, what do we do to get better approach, to get the better area? We divide more. Before I investigated "we divide more" one of them said that we will divide it into infinite pieces. I liked it *very much that they thought this. I think this is proof. Thinking of dividing these small little areas into pieces and where n goes as they increase the number of rectangles. Stating this sentence about limit was a good proof. I like it when they say height is getting ideal form. They stated that "as I have the width smaller, my height becomes the most ideal length". Maybe it's not a very mathematical sentence, but it almost turns into sticks, which was very good for me. Because the width is so small that it is not a rectangular area and it turns into tiny little sticks and traces the area under the graph. It is not a very mathematical sentence, but it is a good sentence. For example, he has internalized it that way. He put it in his mind that way. It was a good thing for me... (9).*

To find the area under the graph, students needed to sum the area of the rectangular shapes. Also, they should have imagined dividing the area under the graph into infinitely many pieces. Data showed that Elisa noticed that students considered dividing the area under the graph into more and more pieces so that the calculated area would be closer to the real value. Elisa pointed that she was satisfied with students' answers such that they mentioned that the rectangular shapes would be too thin and almost would look like a line. This way, she noticed that students were ready to understand the meaning of the use of limit given infinitely many partitions of the area under the graph. In addition, she attended to thoughts of students about the "x (the width)" and "y (the height)" values of the rectangular shapes to find the area of any of the infinitely many rectangular shapes. Elisa stated that students who were able to comment that the y value referred to the height of the rectangular shapes reached the most accurate value of the area under the graph as the x value of rectangular shapes decreased. Explanations of Elisa on the students' thoughts about the x and y values and the idea of limit showed that she listened to the students' answers and noticed their thoughts (W.2B). Though, Elisa did not explain students' ideas pointing to them individually by name. Even if she had noticed each student's thoughts pointing that Elisa started to realize students' ideas, the way she talked about and explained her thoughts and strategies and explained the reason for her behaviours showed that the general comments she made preceded particular students' thoughts.

7.4.2.2. How Elisa Noticed. Explanations of Elisa showed that she noticed significant aspects of the intended learning in the lesson planning process. Elisa used some general sentences while highlighting the important points she noticed in the lesson plan. Elisa noticed students' prior knowledge and prerequisite of the objective of the lesson and explained them with general impressions as follows:

Students should be able to define what is a function. They should be able to draw and read the graph of a function. They should be able to analyse that where it increases, where it decreases, how it increases and how it decreases. They should analyse in different aspects. I said they should remember the demonstration of the series and some series formulas. It is not necessary to remember. I can give it at that moment ... They should be able to recognize the limit and apply the rules of taking limit. Since it is the 11th grade, I did not see it in the curriculum, only in the IB (Referring to the International Bachelor Program) ... They saw the derivative again in IB. They did not see the integral. In IB, there was a sequence like this let's reverse a derivative. Integral is just like the reverse of the derivative. They reversed expressions with coefficients like x^2, x^3. I was going to solve questions for the applications of the integral. We were right there. So, they know the derivative before this lesson. First, I thought they can find the areas of simple geometric shapes. Although they did not see the limit at a very deep level, they saw it in a simple sense. They know what it means, why they needed something like that. I thought they were ready because I believed this would come out of their synthesis (15).

Elisa emphasized students' prior knowledge to learn about the Riemann Sum and the integral. Elisa stated that functions and the graph of functions, sequences, limit, derivative and the inverse of derivative were significant to understand the Riemann Sum, the integral and the relationship between them. Elisa's emphasis on how much of the topics the students have learned before depicted that she noticed and highlighted noteworthy aspects in the lesson planning process (H.2A). As seen in the explanations of Elisa, she mentioned the topics students were introduced earlier years. Though, her focus was on her actions and curriculum materials rather than what students understood from the past experiences. Therefore, while she explained the earlier learning experiences that would possibly form the basis for the intended learning goals, she formed general sentences about how she benefited from these prior knowledge base for her plan. Using general impressions and highlighting the noteworthy events depicted that Elisa showed the code H.2A at the mixed level for teacher noticing.

In addition to students' prior knowledge, during the interview, Elisa highlighted some significant points in determining her decisions for the lesson plan. She emphasized why she planned only talking about the left Riemann sum instead of focusing on both the left and the right Riemann sums during the lesson:

E: I can give the right Riemann sum and the left Riemann sum and I can say that the area is between these two value. Or I thought that after we divided the graph with ten pieces in the left Riemann sum, I could say that we could divide the graph from the right Riemann sum and then show it. I mean if we calculate from here (from the right), I could say that result is 2/3 again. Then I thought that this was not the main purpose. I asked myself that how did students calculate the area by left Riemann sum and right Riemann sum? How did students do this procedural part repeatedly? Is this the important thing? No. The important thing is how they approached this area. Whether they come from the left or the right vertex, they are approaching this area. They will reach to 2/3 in both cases. So, I thought that one of them (referring to the left or right Riemann Sum) is enough. I did not do the other because I would spend too much time in the operational part. I would do the same thing twice. What will happen is that they will take the height from here, not from here (not from the inside but from the outside) This is very procedural. The important thing is that I want them to perceive conceptually. What does it mean to get smaller with infinite pieces? So, what does it mean dividing infinite pieces? Since this part was what I gave importance to, I did not want to spend time in left or right Riemann sum (48-51).

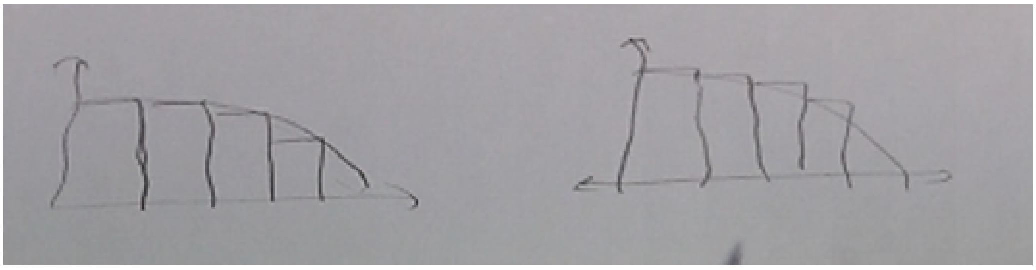

Figure 7.45. Drawing of Elisa about right and left Riemann Sum.

Elisa believed that the rectangular shapes touched the graph on the top-right vertex or top-left vertex, so both the right Riemann Sum and the left Riemann Sum of the areas under the graph were to be equal to each other. Therefore, due to the time restriction, she planned for students to calculate only the left Riemann Sum because she believed that students could deduce that both graphs had the same area. She emphasized the significance of the thinking of students about dividing the graph into infinitely many pieces. So, she considered the calculation of the area under the graph two times for both the right and the left Riemann Sums as time consuming. That is, she wanted to focus on how students would think about the area of the rectangular shapes when the area under the graph were divided into infinitely many pieces. Highlighting why she only used the right Riemann sum in her lesson plan and the one of the most significance points of the lesson depicted that Elisa showed the code H.2A at the mixed level of teacher noticing. At the post-interview, Elisa interpreted that students had difficulty about the right and the left Riemann Sums. She noticed her students' confusion due to the height of the rectangles inside of the graph. So, she planned to change this part of the lesson plan in the future as follows:

E: I regret somethings. When I do this again, I will not do something. I just gave students the right Riemann sum. Let me draw and show it. Of course, there is also some area from left Riemann sum, not only from the right. The areas in it show like this.

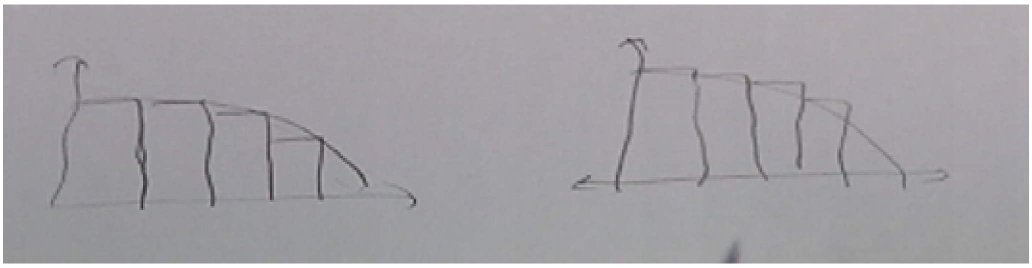

Figure 7.46. Drawing of Elisa for the right and left Riemann Sums.

So, ultimately, when we divide both infinitely, both two (referring to the left and right Riemann Sums) converge to the area under the graph. And, if I don't divide the graph into infinite, if I divide it into a certain finite pieces, the area will be between these two value. I was very concerned about my time. Because I thought that it might not be finished in 70 minutes, I gave it up. I just talked about the Riemann Sum from the left. I thought I could give it conceptually because it would converge to the area while it approaches to infinite. But currently, I am thinking that I could give them both ideas (referring to the left and the right Riemann Sums) at the same time. Because when I think the infinite, when I passed to Riemann Sum, when I didn't get the intervals equal, they got confused about the rest of the area that is outside of the graph. Many questions came from here. I had to spend 15 minutes for that. So, one of them said something, I don't remember the sentences exactly, but he said, "it cannot be the same with it?" so I said "why". He said, "There's area left in here,? he said. "Some area remains". I said, "there is some missing area here too (by showing the right Riemann Sum)". He said, "the missing area is filled". I started to talk "when the rectangles get smaller, do the remain areas disappear?" Someone said that the height of this (the graph of left Riemann Sum) does not give the most correct result (7).

As the data indicated, Elisa planned not to focus on the left Riemann Sum as she considered that focusing on only the right Riemann Sum of the infinitely many rectangular areas would be enough for students to understand that the same calculation can be done to calculate the area under the graph for the right Riemann Sum, too. However, students could not imagine how they could get close to the real value of the area under the graph by using the right Riemann Sum because of some residual areas of rectangular shapes. She noticed and highlighted that students did not understand why the two Riemann Sums might be used to find the approximate value of the real area under the graph. She noticed some students' answers such as "There are some area left in here, some area remains" and tried to handle by directing students to think about the left Riemann Sum. Therefore, she planned to simultaneously give the two Riemann Sums to restrain students' confusion in the future lessons. Noticing and highlighting the significant events in her teaching depicted that Elisa showed the code H.2A at the mixed level for teacher noticing.

Explanations of Elisa showed that she used an evaluative language at the beginning of the pre-interview, but she explained the significant events in teaching with an interpretive language at the rest of the pre-interview. She interpreted why she planned the task sequence to find the area under the graph as follows:

E: They can see that they are approaching in terms of procedures, with the procedures they can do. Firstly, they can calculate this in a concrete way and realize that they are getting closer to a number. They are experiencing this. We try to put a little more abstract concept in what they can do concretely. Actually, I think they can perceive this by starting from here (referring to the examples in the task where students divided the shapes into four and ten pieces), because they can do it. There is something they can imagine. From there, they are gradually trying to move up step by step (77).

As the data showed, Elisa believed that if students attended to and calculated some particular examples in the task sequence, they would experience the learning process on their own. That is, she noticed and interpreted that students could learn easily the abstract topics by constructing the ideas depending on the concrete steps. Particularly, she believed that when the students engaged in understanding how the area of the ten rectangular shapes under the graph got closer to the real value of the area compared to the area of the four rectangular shapes, they could start imagining more rectangular shapes under the graph to reach the real value. That is why in the task sequence she planned to allow students to concretely divide the area into 4 and 10 rectangular shapes, so that they could imagine the area under the graph divided into infinitely many pieces of rectangular shapes. Elisa evaluated students' possible reasoning and interpreted according to her viewpoint. Elisa showed the code H.2B at the mixed level in her pre-interview.

Elisa further emphasized active participation of students at the post interview. She explained why she gave the task sequence to students. She emphasized the effects of the different students' thinking and understanding on her teaching strategies:

What am I doing there? I am presenting how I reasoned in my head and how I realized it. My presenting ideas address to anyone's learning in this way, so one can get it this way. But the rest ninety percent may not think this way, because they may not construct the ideas like me, they may be thinking completely different. So, I have to appeal to everyone here not just someone who learn like me. When I come to the board and do something, maybe, one student who is very clever will understand quickly, so she gets it. She may internalize and create her own ideas. Or her reasoning resembles mine, so the student may construct the ideas directly without making any changes. So, if I give chance to them to do by themselves instead of constantly presenting and explaining something on the board, maybe, they will establish their original reasoning while thinking the reason, while analysing it, while drawing it, while writing it. That's why, I said they have to do it themselves. They must reach results themselves. They must establish a cause and effect relationship. Or they must establish the connection of this to the limit themselves (27).

As the data suggested, Elisa interpreted that students needed to deduce the ideas themselves by experiencing and engaging in the task during teaching. She noticed that in the lesson planning process that she considered her own reasoning about the task. That is she envisioned how the task might hypothetically enable a person to understand the reasoning behind the Riemann Sum. Therefore, instead of giving the information directly to the students, through engaging in the task sequence, she wanted students to construct their own reasoning. Though she attended to the importance of differences in students' thinking, characteristics and prior knowledge, she interpreted her actions in teaching rather than students' thinking. That is, when she analysed her teaching during the post interview, she began to use some interpretive language to explain some parts of her teaching. Therefore, Elisa's explanations in total depicted that she showed the code H.2B at the mixed level.

Elisa's attending to students' active participation and the importance of the task sequence was more apparent during the lesson planning process. In fact, one of the most significant parts of the task sequence for Elisa was to divide the area under the graph into infinitely many pieces and understanding the width of the rectangular shapes as approaching to zero. Elisa explained:

I don't want to say "width" for the infinitely smaller pieces because it will be wrong. It would be enough for me if they say it has no width because it is infinitely smaller. If they told me when I passed here, it would be a great proof for me that they learned (99).

The ultimate goal for the plan of the lesson was that students would understand the relationship between the Riemann Sum and the concept of integral. So Elisa focused on students' dividing the area under the graph into infinitely many rectangular shapes such that they would understand that given the width of the rectangular shapes approaches zero , once the limit value of the Riemann sum is found it would refer to the definite integral. Thus, Elisa determined some thoughts students could possibly state once she asked her questions and planned to notice students' learning and difficulties by determining their possible answers. So, providing some examples of the correct answers regarding the division of the areas under the graph into infinitely many pieces, she planned to attend to students' thoughts to assess their understanding. Although she did not show any evidence from particular students' thinking from the previous lessons, she provided envisioned possible answers that she expected from students as evidence. Therefore, at the pre-interview, she showed the code H.2C at the mixed level.

Some answers that Elisa had planned to catch from students actually were stated by them during teaching. Therefore, at the post-interview, she explained how some students made sense of the x and y values of the rectangular shapes and talked about the width and real value of the height of the shapes:

E: me sentences of students were good. We calculated the area (by taking the right Riemann sum) and I said, what do we do to get the better value for the real area? We divide more. Before I investigated "we divide more" one of them said that we will divide it into infinite pieces. I liked it very much that they thought this. I think this is proof. Thinking of dividing these small little areas into pieces and where n goes as they increase the number of rectangles. Stating this sentence about limit was a good proof. I like it when they say height is getting ideal form. They stated that "as I have the width smaller, my height becomes the most ideal length". Maybe it's not a very mathematical sentence, but it almost turns into sticks, which was very good for me. Because the width is so small that it is not a rectangular area and it turns into tiny little sticks and traces the area under the graph. It is not a very mathematical sentence, but it is a good sentence. For example, he One of them already said that "the width is so small that you do not have a lot of options to put the height of the rectangle". It is getting very smaller that when this area is large, you get the height from here, from here (showing some points where the given rectangle cuts off the graph). S4 used a sentence "as when it (referring to the width) get smaller, you wouldn't have any chance to get it (the height) from different points" (45).

As she had planned in her lesson planning process, Elisa noticed and showed evidence from students' thoughts to decide whether students understood partitioning the area under the graph into infinitely many pieces to find the real value. Particularly, data showed that she attended to students' thinking in three significant points: First, she noticed that when she asked students how to get closer to the real value of the area under the graph, students stated dividing the area into more pieces or even infinitely many pieces. Elisa satisfied with their answers deciding that they were ready to progress so that she continued the lesson to discuss about the width of the rectangles. Secondly, she attended to students' imagining the infinitely many rectangular shapes so that they were able to make sense of the "x (i.e., width)" value of rectangular shapes getting closer to zero. Students stated that rectangular shapes turned into a line with no width due to the partitioning of the area into infinitely many pieces. Third, Elisa noticed that students imagined the height of a piece when dividing the area into infinitely many pieces. Students understood that although they choose a height in the examples in which they divided the area into four and ten rectangular shapes (the first two examples in the task sequence), they had to take a point on the graph as the values of the height of the rectangular shapes. Elisa showed evidence from students' thoughts and pointed that students understood both the meaning of dividing the area under the graph into infinitely many pieces and the value of the width and the height of the pieces. These evidence depicted that, at the post-interview, Elisa showed the code H.2C at the mixed level.

In sum, results of the data both from the pre-interview and the post-interview in juxtaposition to each other displayed that Elisa showed the code at the 2nd the mixed level for teacher noticing. Particularly, the results of the data from the pre-interview supported the results from the post interview. Similarly, Elisa showed each code more than once during the pre-interview and the post-interview.

8. CONCLUSION AND DISCUSSION

The purpose of this study was to examine noticing skills of novice teachers holding different teacher perspectives. For this purpose, two different sets of data were used: firstly, the teacher perspectives of novice teachers were investigated by analysing the data from the lesson plans, the interviews before and after teaching and in-class teaching of the two novice teachers in 2016 and in 2018. Secondly, differing from previous studies that investigated and reported on teacher noticing only from the data after teaching, in this study, the interviews of the two novice teachers both from before and after teaching in 2016 and in 2018 were used to analyse their noticing levels. The characteristics of the teacher perspective framework (Bukova Güzel *et al.*, 2019; Heinz *et al.*, 2000; Jin and Tzur, 2011; Simon *et al.*, 2000; Tzur *et al.*, 2001) and the learning to notice framework (Van Es and Sherin, 2002; Van Es and Sherin, 2008) were used to analyse the data sets. After analysing the data sets separately, finally, perspective and noticing levels were re-examined from the same data sets. In this section, the levels of the teacher perspective and the teacher noticing of the two teachers are discussed. Therefore, in the following paragraphs, I focused on general results regarding the correspondence between the perspective and noticing levels of the participants. Particularly, the correspondence between progressive incorporation perspective (PIP) and the extended level noticing and the correspondence between perception-based perspective (PBP) and the mixed level noticing is explained in light of the results and the literature. In addition, limitations of the study and implications for further research are discussed.

Results showed that while one of the novice teachers, Alin, had the characteristics of PIP and showed all the codes of the extended level noticing both in 2016 and in 2018, the other participant, Elisa, had PBP and showed all the codes of the mixed level noticing in both in 2016 and in 2018. These results showed that there was a coherency between the pre-interview and the post-interview of participants in terms of the level of noticing both in 2016 and in 2018. Similarly, although the participants got experience in two years the consistency in the data depicted that teachers who held PIP and PBP perspectives did not change their perspective and noticed the same aspects of teaching with experience in two years. Though the difference between their levels of noticing suggested that teachers holding PIP might notice the special events before and after teaching and explain these events in detail.

Regarding the results prior to the teachings, although in general, Elisa showed the codes of the mixed (Level 2) noticing, she frequently showed the codes from the focused (Level 3) noticing depicted in the data before the teaching results. In addition, results prior to the teachings depicted that from these novice teachers' perspectives, the teaching is not separated from the lesson planning process in which they noticed some aspects of their teaching before the lesson. For example, both Alin who had PIP and extended level noticing and Elisa who had PBP and the mixed level noticing hypothetically planned to notice some aspects of teaching in their lesson planning process. In addition, in the lesson planning process, data showed that both Alin and Elisa attended to students' knowledge and difficulties, the content and the sequence of the task, the hypothetical learning progression of the content and the possible students' answers to teachers' prepared questions. Yet, data further indicated that the differences in their perspectives about the nature of mathematics, and the teaching and learning of mathematics changed the focus and rationale of them. Particularly, aligning with her perspective, while Alin planned to focus on students' current thoughts and progression of their thinking, Elisa targeted to reach the intended goals. This was also evident in their reasoning behind their frequent questioning. Alin stated that she would question frequently to determine how students were thinking and if they had any difficulties, Elisa rather planned to question if students perceived what she thought was apparent in the examples and the task sequence she prepared. Therefore, these results suggested that the differences in their noticing was due to why they noticed and how they noticed based on their rationale and focus on the teaching and learning processes. In the following paragraphs, I explain the results of the analysis of data from both Alin and Elisa in detail.

In particular, prior to the teaching, both in her written lesson plans and also during the pre-interviews, Alin stated that her objective in the lesson plan in 2016 was that her students would make sense of the graph of the quadratic functions; and her objective in the lesson plan in 2018 was that her students would make sense of the meaning of the decay and growth factor in exponential functions. In both of them, Alin planned her lesson based on what her students knew as well as taking into consideration of the curricular materials. Jin and Tzur (2011) emphasized that teachers who hold PIP create entire lesson plans on one's own or curricular materials based on students' prior knowledge. This suggested that Alin depicted PIP.1A. For instance, data in 2016 showed that Alin benefited from the national curriculum and curriculum of IB program and also students' prior knowledge about quadratic functions such as the features of a graph of a function and the rate of change of y with respect to x in $y=x^2$. Similarly, during the pre-interview in 2018, Alin pointed to the sequence of the topics in the curriculum to show the link among the intended knowledge (exponential functions) and parent functions such as quadratics and polynomials to the third degree.

In addition, Alin prepared both of the lesson plans sequencing the tasks from basic to complex such that the task sequence allowed students to reason on the concrete examples based on what they already know from which they progressively moved to the generalizations of the intended knowledge (PIP.1A1). Particularly, the salient characteristic of both of the lesson plans was that Alin hypothetically envisioned how students might reach the intended knowledge based on what they already knew. That is, she specifically mentioned what mental operations students might go through so that they possibly would reach the intended learning objective. For instance, in the pre-interview in 2016, Alin hypothesized on possible students' mental processes about the changes of the graph based on the different the values of "a" such as $1 \leq a$, $a \leq -1$ and $-1 \leq a \leq 1$ (PIP.1A1). Alin also hypothetically planned to focus on and attend to students' thinking about the graph of a function and different values of the coefficients of "a", "b" and "c" by observing and examining group work and asking probing questions. Similarly, during the pre-interview in 2018, Alin pointed that she planned to ask questions continually to observe students' thinking and changes in students' thinking during teaching (PIP.1A1). Focusing of both students' thinking and cognitive development about both for the quadratic functions in 2016 and the exponential functions in 2018 and her actions, questions, and task planning in the lesson planning process further depicted that Alin showed the W.4A and H.4E codes of the extended level of noticing. She acknowledged and interpreted that her questioning about the values of "a" and its effect on the values of y value in teaching 2016 and the group discussions about the growth factor or decay factor in teaching 2018 could be beneficial for students' making sense of the main ideas during teaching (H.4B).

Similarly, in her lesson plan in 2016, Alin emphasized the importance of what students know about functions; and the link between what students know about functions to the subsequent topics. For instance, she pointed to what students know such as their knowledge about $y=x^2$ and their knowledge about functions regarding the domain and the range and the graph of the functions, so that they could link such knowledge to the intended knowledge (i.e., the rate of change of y with respect to x depending on the different values of the coefficients "a", "b" and "c" which result in the different graphs). Jin and Tzur pointed that teachers who hold PIP assess students' prior knowledge for both indications for 'time to move on', and a conceptual anchor for the development of future topics (PIP.1B). Noticing and explaining the significant aspects of lesson was emphasized in previous research (Star and Strickland, 2008; Van Es and Sherin, 2002) By the same token, in 2018, in her lesson planning process, Alin emphasised the use of exponential functions in real-life situations, the importance of what students know about functions; and the link between what students know about functions with the subsequent topics (i.e., the logarithm functions). For example, Alin stated that students' already established knowledge how quadratic functions, polynomials and linear functions behave in terms of the relationship between the variables x and y would afford for their understanding of exponential growth and decay. Therefore, data further showed that Alin noticed the necessity of the learning of the subject (i.e., quadratic functions; exponential functions) for daily life situations as well as the prerequisite of the topic and the subsequent topics in the lesson planning process and highlighted the significance of these aspects (H.4A).

Moreover, during the lesson planning process in both 2016 and 2018, Alin took into consideration of possible students' difficulties, misconceptions and procedural knowledge about functions to prepare alternative questions to handle students' difficulties if occur during teaching (PIP.1A2). For instance, during the pre-interview in 2018, she stated that she also would focus on her students' behaviours in the experiment with beans/chips to record the changes in the numbers of the bacterium and draw graphs with data. So, she planned to ask some questions to guide them (i. e, what are the independent and dependent variables? for drawing the graph). In addition, data showed that Alin realized that students might have faced difficulties in doing the experiment or thinking about the relationship between the trial numbers and the number of bacterium during the teaching in 2018. She also hypothetically created possible answers of students so as to assess whether students understood how the common factor explains the amount of change in the values of y with respect to the amount of change in the values of x to understand the exponential decay and growth (H.4C). Thus, Alin's consideration of students' difficulties and confusion before teaching allowed her to prepare some alternative questions (H.4F). Van Es and Sherin (2002) stated that teachers who propose alternative pedagogical solutions is at the extended level noticing. Alin's preparing some questions and hypothetically interpreting how students might make sense of the common factor of the amount of change; and, how they might draw the graph of the number of the bacterium with respect to the trial numbers suggested that she attended to and proposed alternative pedagogical solutions (H.4F).

Results from the teaching and the post-interview also supported the pre-interview results such that Alin showed all the characteristics of PIP in the teacher perspective framework and all the codes at the extended level noticing in learning to notice framework. Particularly, Alin had started the lesson with the experiment about the relationship between the trial number and the number of chips/ beans (e.g., in 2018) and questioning what students knew about functions (e.g., in 2016) for students to re-activate their prior knowledge (PIP.2E). Similarly, during the post-interview, Alin stated that continually asking probing questions during the lesson was important for students to make connections with their already known knowledge and the intended knowledge (PIP.2A). For instance, during the teaching in 2016, she questioned the students on the effects of the changes of the values of x on the changes of the values of y for the function $y=ax^2$. Though, the only purpose of the questions Alin asked was not to reactivate students' prior knowledge but also was to listen to students so that she could examine the cognitive processes (such as what they think given different values of "a", $1\leq a$, $a\leq -1$ and $-1\leq a\leq 1$) students were going through (PIP.2A1). Therefore, discussions in the groups and Alin's questions were significant for Alin to detect and examine students' thinking. Similarly, Alin closely monitored her students to decide that her students could use their prior knowledge as an anchor for the intended knowledge to take place; and, link the old and the new to each other (PIP.2B). As Van Es and Sherin (2002) emphasized, by asking questions to observe students' thinking and responding to the students' answers with new questions, Alin showed that she made a connection between her teaching principles and students' learning (W.4A and H.4E). She stated during the post-interview that she listened to students' talking and assessed their writings in the worksheets and observed their body language to notice if they had difficulties, what their mental activities were and how they understood the intended knowledge (PIP.2A1, W.4A and H.4E).

As Jin and Tzur (2011) highlighted, Alin emphasized that she tried to catch students' confusions and errors by their answers to the questions (PIP.2C). Jin and Tzur (2011) suggested that learner-learner or teacher-learner exchanges can be solutions for students' errors, so to examine students' wrong answers. For instance, in 2018, Alin created group discussions and asked new probing questions focusing on students' confusion or errors when two students gave different conflicting answers about the independent and the dependent variables of the function (the number of trials and the total number of beans). As the emphasis of Van Es and Sherin (2002) on the evidences for teacher noticing, Alin showed evidences from students thinking about independent and dependent variables of the function such that these evidences depicted that she showed H.4C code of the extended level noticing. In addition, in her post-interview, she preferred an interpretive language rather than evaluation of students' difficulties or errors (H.4B). Another reason that Alin listened to students' answers was to detect different approaches of students. During the post-interview, she explained that each student had a different thinking strategy and approach about the average percentage of changes in the number of beans/ chips such that she needed to observe and listen to students' different thoughts such as dividing with 4, finding the 25% or multiplying with the 75% of the first trial (PIP.2D And H.4E). During the post-interview, she also stated that listening to students was significant to assess their understanding of the intended knowledge for the planning of the next lesson (PIP.3A). All these data suggested that Alin listened to and observed students and noticed significant problems/difficulties students encountered, noticed different approaches of students and students' cognitive development during teaching and that she proposed alternative solutions or some changes for her next lesson (PIP.3A and H.4F).

Based on all the aforementioned results, I propose that results suggests that once pre-service teachers develop an awareness on the link between what students know and what they need to learn by envisioning hypothetically how they might progressively reach the intended learning through their own mental activities might allow them to attend to students' thinking at higher levels. This is because hypothetical envisioning of how students progressively might reach the intended learning might allow pre-service and novice teachers to focus on what knowledge base students need to already have at different moments and stages in teaching progress, what difficulties students might encounter, and, what misconceptions they might possibly develop. Therefore, such awareness might allow them to both elaborate on and interpret particular student's thinking by attending to their thinking.

Contrary to the results of data from Alin, results of the data analysis from Elisa showed that Elisa depicted all the characteristics of PBP in the teacher perspective framework and all the codes in the mixed level noticing.

In fact, as mentioned earlier, although in general, Elisa showed the codes of the mixed (Level 2) noticing, she frequently showed the codes from the focused (Level 3) noticing depicted in the data before the teaching results. Thus, data showed that similar to Alin, Elisa also hypothetically planned her lesson based on the students' prior knowledge. For instance, in 2016, Elisa had prepared a lesson plan about the sequences and the relationship of sequences with functions; and, in 2018, about the Riemann sum and the relationship between the Riemann sum with integral. In fact, researchers suggest that teachers' determination of possible ways of thinking of students during the lesson planning process (Thompson and Silverman, 2008) is important. However, data showed that in the hypothetical learning progression, while the focus of Alin was students' current cognitive processes based on their mental activities so as to determine what they know and how they think during teaching, Elisa planned to observe and examine students' thinking to determine if they reached the intended goals. Similarly, by asking questions, in contrast to Alin whose focus was to examine how students think, Elisa wanted to make mathematical relationships such as sequences and functions as apparent as possible for students (PBP.1A). In addition, while Alin hypothetically planned to attend the relationship between students' thinking and teacher principles, Elisa focused on both her teaching strategies, actions, questions and plans for her teaching and students' thinking separately (W.2A). Particularly, Elisa was aware of and highlighted during the pre-interview why she prepared the task, how she designed the sequence of the lesson and which conditions were taken into consideration to reach the intended goal both for the topic of sequences in 2016 and the Riemann sum in 2018 (H.2A). However, as Tzur *et al.* (2001) stated that teachers who hold PBP see mathematics as a connected set of ideas and divide gradual steps to teach, Elisa divided the intended knowledge into some pieces and provided opportunities for students to discover and combine these knowledge pieces in gradual steps (PBP.1B). For instance, in 2016, Elisa thought that learning was a gradual process so, students would deduce the relationship between the sequences and functions by detecting the similarities and differences among different examples building on the previous ones in the task (PBP.1B). Similarly, in 2018, Elisa believed that students' prior knowledge about derivative, areas of some geometrical shapes and although limited some knowledge about the concept of limit would have allowed them to reach the meaning of Riemann Sum through the approximate calculation of the area under the graph. Therefore, as Jin and Tzur (2011) showed that teachers who hold PBP create an environment for students' discovery, in 2018, by using questions, directives and discussions, Elisa also provided opportunity for students to think about dividing the area under the graph into infinite pieces and connect the Riemann sum with the definite integral (PBP.1C and W.2B). Elisa further considered that for the active involvement of students, students' seeing and connecting different ideas or many pieces of mathematics, all materials, questions, in-class activities should be organized by the teacher in accordance with the characteristics of the class (PBP.1E). For instance, by using the summation formula, Elisa targeted to show the relationship between the Riemann Sum and the concept of definite integral. In addition, in 2016, Elisa's questions in her lesson plan showed that Elisa planned to direct her students to think about the sequences directing their focus to functions rather giving them a chance to share different approaches. Heinz et.al. (2000) pointed that teachers who hold PBP try to create the same experiences for students (PBP.1D). That is, in her lesson plan, Elisa's questions, such as "Can you realize anything special about domain and range of sequences?" and their particular answers, such as "Domain is always set of positive natural numbers" indicated that she expected that students would go through the same process as she envisioned. In 2016, her writing the expected students' answers about the sequences and the functions in the lesson plan to act during teaching based on these expected answers further indicated that she hypothetically showed evidence from students' expected answers in pre-interview (H.2C). In addition, Elisa decided to use these questions as an assessment strategy during teaching because she needed to measure students' understanding of the meaning of sequences in 2016 and Riemann sum in 2018 to act based on them. That is, as opposed to Alin who planned to listen to students to determine their different approaches and how they think, she planned to listen to students' answers to decide students' understanding level to pass to the following part or to determine their deficiencies (PBP.1F).

Similar to the results from data prior to the teaching, data from during the teaching and the post-interview both in 2016 and 2018, showed that Elisa's focus was on the link between the intended knowledge and students' prior knowledge. In particular, to direct students to think about the features of a sequence and the relationship between sequences and functions; and similarly, to direct students to think about the calculation of the area under a graph and the relationship between Riemann sum and definite integral, Elisa asked many questions that focused on intended goals (PBP.2B). For example, in 2018, she asked mostly leading questions and created discussion during teaching so that students would divide the area under the graph into more and more pieces. This way she believed that she had created a discovery environment for her students. As Heinz *et al.* (2000) study showed that teachers with PBP provide opportunities for students with first-hand activities, in both 2016 and 2018, by asking questions, using tasks and holding whole class discussions, Elisa provided a chance for her students to think about and realize the mathematical ideas themselves (PBP.2A). Yet again, compared to Alin who focused on students' mental processes based on which they would reach the intended leaning goal , Elisa focused on the features of the tasks such that as long as students observed the patterns in the task sequence. Elisa considered that they reached the intended learning goal. Similarly, during the post-interview in 2016, Elisa pointed to students' prior knowledge, their thinking about the features of sequences, their answers for questions in the task and their difficulties about the types of sequences. This suggested that Elisa noticed both her teaching strategies and students' mathematical thinking (W.2A and W.2B). Van Es and Sherin (2002) emphasized that teachers who are in the mixed level noticed their own teaching principles that the code of baseline level and noticed students' thinking and understanding that is the code of the focused level. However, in contrast to Alin's purpose, Elisa listened to students' answers or discussions carefully in her teaching to decide to ask new questions or direct students to what she wanted to hear (PBP.2D). Similarly in 2018, by listening to students' answers, she highlighted some thinking of students about their dividing the area under the graph into infinitely many pieces and evaluated students' understanding about the relationship between the Riemann sum and definite integral (H.2A). Though again, as Heinz et.al. (2000) emphasized that teachers who hold PBP focus on students' thinking and understanding to assess if the intended learning had taken place, Elisa's focus while listening to her students during teaching was to determine if they reached the knowledge piece they gradually needed to pass towards the intended learning goal and if she heard from them what she saw in the tasks and the same way of thinking she had in mind. That is, although Elisa stated prior to the teaching that she planned to asked questions to observe and examine students' cognitive processes, Elisa's questions during her teaching were too directive towards students' thinking such that they targeted hearing from the students of the particular thinking required. Research showed that when unexpected answers come from students, teachers who hold PBP do not deviate from the way they plan their lessons, instead either they ask questions to direct students to the point to what the teacher had determined in mind or they go revert back to direct telling (Heinz et. al. 2000) (PBP.2E). As the data showed, once the novice teachers, Alin and Elisa, had not heard the expected answers from their students, while Alin created a discussion environment and asked new probing questions for overcoming students' difficulties (PIP.2C), Elisa gave the answers directly (PBP.2E). For instance, during the post-interview in 2016, Elisa showed some evidence from students' answers about their difficulty of such as the order of sequence and whether a sequence need to have a function rule. She stated that she asked questions directing students to discuss whether a sequence have to have a rule but then gave the answer directly once she had not heard from them what she expected they would perceive in those examples (H.2C). That is, when Elisa asked questions and created whole class discussion, she directed her students to her purpose and expected answers from them rather than providing an opportunity for them to create and share their own ideas as Alin did. Elisa believed that students perceived the mathematical relationships within the topics as the teacher saw. Thus, although students shared different ideas, she did not pay attention to them and did not pass to the next step without hearing the answer in her mind (PBP.2C).

Thinking of all these results in juxtaposition to the results from previous research is important. First, previous research has shown that pre-service teachers' perspectives have a mutual relationship with their mathematical knowledge for teaching (Bukova Güzel *et al.*, 2019; Karagöz Akar, 2016; Tzur *et al.*, 2001). Similarly, results from this study showing that Alin had all the characteristics of PIP and depicted all the codes at the extended level of teacher noticing suggest that Alin might also have the mathematical knowledge for teaching. In addition, results from this study showed that Elisa had PBP and despite generally she showed the codes of the mixed (Level 2), she also showed the characteristics of the focused (Level 3) of teacher noticing. These further suggest that Elisa also might have the mathematical knowledge for teaching. These results further align with the previous research that has shown that teachers who have higher noticing show higher mathematical knowledge for teaching (Thomas, Jong, Fisher and Schack, 2017).

Particularly, the results from the pre-interviews showed that both Alin and Elisa as pre-service and novice teachers were aware of their purpose to reach the intended learning goal and the characteristics of their students such as prior knowledge, difficulties or learning types. Awareness of purpose and students' characteristics are among foundational knowledge teachers need to have theoretical underpinning of pedagogy and using of mathematical terminology in appropriate way (Rowland, Huckstep and Thwaites, 2005). Similarly, they both developed the subject knowledge and pedagogical knowledge. For instance, as a pre-service teacher and as a novice teacher, both Alin and Elisa knew the particular aspects of mathematics they needed to teach. In addition, their choice of examples paying attention to the difficulty levels; their choice of representations paying attention to the link among the algebraic, graphical and the tabular representations; their use of questions towards their intended learning goals in both 2016 and 2018 can be thought as evidence of their pedagogical knowledge of these two novice teachers. Particularly, these could be considered evidence of their knowledge of transformation (Rowland, Huckstep and Thwaites, 2005). Furthermore, both Alin and Elisa in 2016 and 2018, were able to explain the connections of the intended topic within the sequence of their lessons and the other concepts since they planned their lessons from basic to complex and from concrete to abstract. That is, during the task sequence, they focused their students on the particular examples of functions regarding the intended learning goal; but, they also expected students to generalize based on those examples. This therefore could be considered evidence of connections (Rowland, Huckstep and Thwaites, 2005). Yet, the two novice teachers differed from each other in terms of their focus and rationale behind their actions during teaching. For instance, to reach the intended learning goal, while Alin having PIP focused on students' current thinking and cognitive processes based on their own mental activities, Elisa's focus was to see if the students had the intended knowledge. In fact, these characteristics were both salient in 2016 and in 2018, in their lesson planning where they both hypothetically determined the learning progress towards the intended learning goal; and, were salient in their teaching. Similarly, while listening to students' Alin having PIP focused on students' mental processes based on which they would reach the intended leaning goal. However, Elisa having PBP had a focus on students for perceiving the intended goals. Finally, another difference was deviation from their plan during the lesson. Particularly, while Alin made some changes and paid attention to students' questions and confusions during the lesson, Elisa wanted only to reach the intended learning goal in her plan. Although Elisa listened to her students' ideas; yet, once she did not get the answers in her mind, either she ignored their confusions or gave the answer to them directly. Despite her actions during teaching, Elisa's mention of particular students' difficulties and her future planning for overcoming such difficulties during the post interviews indicate that she also depicted the contingency dimension of teacher knowledge (Rowland, Huckstep and Thwaites, 2005). In total, the mathematical and pedagogical knowledge of these teachers both as pre-service teachers and novice teachers, seemed to indicate that both might have mathematical knowledge for teaching. By the same token, results seem to suggest that if MKT levels of novice teachers with PIP and PBP perspectives could be determined, they may show the same MKT codes but though at different levels and with having different reasoning.

Previous research has also shown that compared to expert teachers, novice teachers mostly show low level noticing skills (Berliner *et al.*, 1988; Kagan, 1992). Similarly, previous research also has shown that once having an opportunity to attend to professional development courses and workshops, both expert and novice teachers' noticing skills might have develop together with their mathematical knowledge for teaching (Levin, Hammer and Coffey, 2009; Taylan, 2014). Particular to this study, data showed that, teacher perspectives and teacher noticing focused on the same aspects of teaching such as paying attention to what student know and how they think. Specifically, data showed that both Alin having PIP and the extended level noticing in 2016 and 2018, and Elisa having PBP and the mixed level noticing in 2016 and 2018, were able to focus on what students know and how they think albeit differences in their intentions and focuses. Thus, given that both of these novice teachers attended to same professional development methods course in 2016, results suggest that these teachers might have developed their mathematical knowledge for teaching during the professional development methods course taken within the practicum year during their undergraduate education

8.1. Limitations and Implications for Future Research

To investigate teacher perspective and teacher noticing skills of novice teachers, the qualitative method was appropriate for this research study. Data came from two novice teacher's teaching in their practicum year in 2016 and in 2018. Yet, the main limitation is about the generalizability in the present study such that the results of this study cannot be generalizable. That is, these two novice teachers cannot be representative for all pre-service teachers graduated in the same year because these teachers attended to a special methods course in the practicum year. Still, triangulating the data using the lesson plans, the interviews before and after the teachings; and the teaching of these two teachers in addition to using all these data both from 2016 and 2018 provided an opportunity to determine the similarities and differences between the same pre-service and novice teachers' perspectives and noticing skills. Therefore, although the results cannot be generalizable to all pre-service and novice teachers, results provide important information about correspondence between teacher perspectives and noticing skills.

For the further studies, there might be two significant implications of the present study. First, previous studies investigating teacher noticing focused only on the explanations of teachers after teaching (Jacobs, Lamb and Philipp, 2010; Van Es and Sherin, 2008). However, in this study, in addition to the post-interview, I also analysed the pre-interview data to investigate novice teachers' noticing skills in the lesson planning process. Results suggested that there was a coherency between the noticing skills of these novice teachers prior to and after teaching. Therefore, I suggest that the lesson plans and the pre-interviews with the teachers in juxtaposition to the post-interviews might be examined for investigating teacher noticing skills. Particularly, in determining noticing skills of both pre-service and in-service teachers using data prior to the teaching, researchers might focus on the properties of the lesson planning process such as the focus of teacher, the content of the task, and teaching principles.

Results from previous research also have shown that pre-service teachers' perspectives have a mutual relationship with their mathematical knowledge for teaching

(Bukova Güzel *et al.*, 2019; Karagöz Akar, 2016). In addition, previous research has shown that teachers who have higher noticing skills have shown higher mathematical knowledge for teaching (Thomas *et al.*, 2017). Particularly, participants in this study have taken a special professional development methods course that focused on developing teacher perspectives. Results from this study have shown that the teacher perspectives and teacher noticing skills have correspondence. This seems to suggest that development of teacher perspectives during the method course might have had contributed to the teacher noticing skills as well as mathematical knowledge for teaching. Therefore, professional development studies both with pre-service and in-service teachers might be conducted to examine taking into consideration of these three frameworks together to analyse the relationship between teacher knowledge, teacher perspectives, and noticing skills. In addition, results further suggest to examine the characteristics of, the tasks used and the core practices in the professional development methods course these teachers have taken to determine how it afforded for the development of teacher perspectives and thus, for the noticing skills.

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APPENDIX A: PRE-INTERVIEW AND POST-INTERVIEW QUESTIONS

Tarih:
Yer ve zaman:
Görüşmeyi yapan:
Katılımcı:

Geldiğiniz için teşekkür ederiz. Size daha önce de bildirdiğimiz üzere, burada bulunmamızın sebebi, sizlerin üzerinde çalıştığımız konular hakkındaki düşüncelerinizi incelemek. Bu sebeple, sorduğumuz sorularda, sizin nasıl düşündüğünüz ve kavramlara nasıl yaklaştığınız benim için önemli olacak. Sorulara doğru veya yanlış cevap vermiş olmanız benim için önemli değil. Amacımız sizin nasıl düşündüğünüz hakkında bilgi sahibi olmak. Dolayısıyla, size bazı ek (sonda) soruları sorabilirim. Bu sorularımın amacı sadece ne düşündüğünüzü daha net anlayabilmek. Ayrıca, cevap veremeyeceğinizi düşündüğünüz sorularda lütfen bildirin. Bir diğer soruya geçiş yapabiliriz böylelikle.

Görüşmeyi istediğiniz zaman sonlandırma hakkına sahipsiniz. Bir sorudan rahatsız olursanız da lütfen bildirin. Geldiğiniz için tekrar teşekkür ederim. Bu bizim için çok önemli ve katkınızdan dolayı çok mutluyum.

1- **Ders planınızı inceleyin lütfen. Bu derste ne öğreteceksiniz?**
   (sonda soruları)
   a. Ders planınızdaki öğrenme hedefine nasıl karar verdiniz
   b. Bu kavramı öğretmek neden önemli?
2- **Öğrencilerinizin bu kavramı öğrenmeye hazır olduklarını dair kanıtınız ne?**
   a. Bunu nereden biliyorsunuz?
3- **Öğrencilerinizin bu kavramı başarılı olarak öğrendiklerinin size düşündüren ne olur?**
   (sonda soruları)
   b. Öğrencilerinizin bu kavramı öğrendiklerine dair kanıtınız ne olur?
   c. Bu kavramı bilen bir öğrencinin nasıl düşüneceğine/ne yapacağına dair örnek verebilir misin?
4- **Dersi nasıl öğreteceksiniz?**
   (Sonda soruları):
   a. Bu planın neden öğrencilerinizi öğrenmeye götüreceğini düşünüyorsunuz?
5- **Ders planında sorduğunuz soruların amacı ne? Neden bu soruları sormayı planladınız?**

Figure A.1. Mülakat Ders Anlatımı Öncesi ve Sonrası.

Ders anlatımı sonrası

1- **Dersiniz nasıldı?**
   a. Bunun kanıtı ne?
2- **Öğrendiğini düşündüğünüz iki veya üç tane öğrenciden örnek verebilir misiniz?**
   a. Sizce öğrendiklerinin kanıtı ne?
3- **Öğrenmediğini düşündüğünüz öğrencilerden örnek verebilir misiniz?**
   a. Sizce öğrenemediklerinin kanıtı ne?
4- **Ders planınızda hazırladığınız ve uyguladığınız etkinlikleri düşündüğünüzde (dersinizde olanlar üzerinden)**
   a. Neleri değiştirir veya aynı bırakırsınız? Neden?

Figure A.2. Mülakat Ders Anlatımı Öncesi ve Sonrası.

APPENDIX B: THE SYLLABUS OF TEACHING METHODS IN MATHEMATICS

Table B.1. The Syllabus Of Teaching Methods In Mathematics.

# of Weeks		**Topic**	**Readings and Assignments**	**PROJECTS**
Week 1 29-Sep		Introduction	Syllabus Assessment (to be completed in class)	
Week 1 1-Oct			Assessment (to be completed in class)	
Week 2 6-Oct (Tue)	The Nature of Mathematics	Mathematical Understanding	Van de Walle (2007)	Lesson Plan 1
Week 2 8-Oct		A special topic from Middle School Mathematics: Fractions	Philipp and Vincent -2003	Journal 1
Week 3 13-Oct		Two special topics from High School Mathematics: Slope and Differentiation		
Week 3 15-Oct Week 4 20-Oct		Differentiation (Continued) QUANTITATIVE REASONING	Thompson (1994) Moore (2011)	No homework
Week 4 22-Oct		QUANTITATIVE REASONING: A special topic from High School Mathematics: Complex Numbers		Journal 2 Lesson Plan 2 (revised lesson plan1)

Table b.1. The Syllabus Of Teaching Methods In Mathematics (cont.).

# of Weeks		**Topic**	**Readings and Assignments**	**PROJECTS**
Week 5 27-Oct	The Nature of Mathematics Learning	TASK ANALYSIS	Stein et.al. -1999	
Week 5 29-Oct		NO CLASS Cumhuriyet Bayramı		
Week 6 3-Nov		TASK ANALYSIS		
Week 6 5-Nov		TASK ANALYSIS		Journal 3
Week 6 6-Nov		Professional Development: Technology		Friday 13:00 pm-17:00pm Associate Prof. Tolga Kabaca from Pamukkale University
Week 7 10-Nov		LOGICO-MATHEMATICAL VERSUS EMPRICIAL LEARNING	Simon (2006)	
Week 7 12-Nov		LOGICO-MATHEMATICAL LEARNING A special topic from High School Mathematics: Conics (Parabola)		Journal 4
Week 8 17-Nov		Questioning An example: Case study on Multiplications of fractions Clinical Interviewing	Manauchehri and Lappan (2003) Herbel-Eisenmann and Breyfogle -2005	Mathematical Task Analysis Project and Lesson Plan 3 (revised) Due 19 November 2015
Week 8 19-Nov		Conceptual analysis continued An example: Teaching episodes-2 (Area of a circle)	Teaching Episode	
Week 9 24-Nov		Before-During-After Plan	Wilburn and Peterson -2007	