THE FIBERED BURNSIDE RING OF A FUSION SYSTEM

by

Mert Seviniş

B.S., Mathematics, Boğaziçi University, 2006

M.S., Mathematics, Boğaziçi University, 2010

Submitted to the Institute for Graduate Studies in

Science and Engineering in partial fulfillment of

the requirements for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Boğaziçi University

2019

ACKNOWLEDGEMENTS

First of all I want to express my special gratitude to my supervisor Prof. Dr. Olcay Coşkun, without his help and guidance this thesis would be a mission impossible. I am also thankful to the jury members for their valuable advice and evaluations. I also owe special thanks to my master supervisor Prof. Dr. Selçuk Demir for his valuable part at my graduate instruction. I should mention that financial assistance provided by TÜBİTAK was greatly appreciated. I wish also to acknowledge the critical help provided by Demet Ilgın at getting this scholarship. I would like to offer my special thanks to many friends which happened to be with me in some way or other during my studies. To name some of them, Engin, Alper for their enjoying company, Murat for nice conversations and chess games, Erdi and Paul for their hospitality at the last moments, and Tolga for printing the thesis. I would also like to thank Boğaziçi University Mathematics Department for their assistance with my studies, my thanks of course extend the students, staff and friends from the university. At last but not at least I am very thankful to my family for their great patience and support during this period. I am especially thankful to my father and mother for their very valuable existence by me. My sisters Billur and Burcu and with them Murat and İbrahim, their company always make me feel happy and peaceful. And finally I want to dedicate this thesis to my luck bringer, Lea Güneş. You are welcome among us with all your lovely being.

ABSTRACT

THE FIBERED BURNSIDE RING OF A FUSION SYSTEM

J. S. Reeh, in his PhD thesis [1], described the Burnside ring and its free basis for the fusion-stable sets. In this thesis we will extend his results for fibered sets. Basically we will use the same technics J.S. Reeh developed in his thesis and we will show that these work with little modifications for the fibered case as well. In the end we hope to provide the reader with a similar description of the fibered Burnside ring and also to write a free basis for fusion-stable fibered sets in general.

ÖZET

FÜZYON SİSTEMLER İÇİN FİBER KÜMELERİN BURNSIDE HALKASI

J.S.Reeh daha önce doktora tezinde [1] füzyon-değişmez kümeler için Burnside halkasını ve onun serbest bazını tasvir etmişti. Biz de bu tezde onun elde ettiği bu sonuçları fiber kümeler için devam ettireceğiz. Temel olarak J.S. Reeh'in tezinde geliştirdiği teknikleri kullanarak bunların fiber-kümeler durumunda da çok az değişiklikle çalıştığını göstereceğiz. Sonuçta okuyucuya füzyon-değişmez kümeler için benzer bir Burnside halkası tasviri sunmayı ve bu halka için genel bir serbest baz yazmayı umuyoruz.

TABLE OF CONTENTS

LIST OF SYMBOLS

Ab	Category of abelian groups
Ring	Category of rings
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Numbers
$\mathrm{Aut}(X)$	Symmetric group on the finite set X
$[X]_G$, $[X]$	The isomorphism class of a $G-$set
$\mathrm{Stab}_G(x)$	The stabilizer in the finite group G of an element x of a G-set X
$X \times Y$	Cartesian product of the sets X and Y
$X \sqcup Y$	Disjoint union of the sets X and Y
$X \otimes Y$	Tensor product of the fibered sets X and Y
$\mathrm{Hom}(H, A)$	The set of group homomorphisms from the finite group H to the abelian group A
$B(G)$	The Burnside ring of G
${}_G\mathrm{set}^A$	Category of A-fibred G-sets
$B^A(G)$	The fibered Burnside ring of the group G over the abelian group A
(V, ν)	Stabilizing pair for the isomorphism class of the transitive fibered G- set corresponding to the subcharacter $\nu : V \to A$,
(V, ν)	Equivalently (V, ν) may denote the subcharacter $\nu : V \to A$ of G itself or the corresponding fibered G-set when there is no confusion
$\mathcal{M}_G^A$	The set of all stabilizing pairs (H, λ), $H \leq G$ and $\lambda \in \mathrm{Hom}(H, A)$
$\mathcal{M}_G^A/G$	A complete set of representatives of $G-$conjugacy classes in $\mathcal{M}_G^A$
$[V, \nu]_G$	The G-conjugacy class of the fibered set (V, ν)
$c_{H,\lambda}(X)$	The coefficient of the A-fibered G-set X at the basis element $[H, \lambda]_G$
D_n	The dihedral group of order n
S_n	The symmetric group over n letters

$\mathfrak{B}(G)$	A ghost ring of the finite group G
$\mathfrak{B}^A(G)$	An A-fibered ghost ring of the finite group G over the abelian group A
$\chi^G_{H,\lambda}$	Characteristic function of the conjugacy class of the pair $(H, \lambda) \in \mathcal{M}^A_G$
$\beta_{H,\lambda}(X)$	The number of A-orbits in the A-fibered H-set $X^{(H,\lambda)}$
β_G or β^A_G	The (A-fibered) mark homomorphism for the finite group G
$A(H)$	The ring of $A-$characters $\nu : H \to A$
$A_+(G)$	The generalized Burnside ring of G
$A^+(G)$	The generalized ghost ring of G
$c^g_H(\nu)$, ${}^g(H, \nu)$	The conjugation of a character ν by an element g of G
$\mathrm{Res}^H_I(\nu)$, $\nu\|^H_I$	The restriction of a character $\nu \in A(H)$ to a subgroup I of H
$\pi_K(X)$	The projection of an $A-$ fibered K-set X to its $A(K)$-component
$\sigma_H(f)$	The Mobius inversion of an element f of the ghost ring of H
$\mu_{L,K}$	the Mobius function of the partially ordered set of subgroups of a finite group G evaluated at (L, K)
$\mathcal{B}_H$	A free basis of the A-characters $A(H)$
$\mathcal{B}$	A G-stable basis of the restriction functor A
$\mathcal{M}_H$	The union set of all the A-subcharacters in $\mathcal{B}_K$ for $K \leq H$
$f_{K,\psi}$	The $(K, \psi)-$component of an element f of the fibered ghost ring of G
$N_G(H, \phi)$	The stabilizer of (H, ϕ) in the finite group G
$\mathcal{M}^*_G$	The set of the basis elements $(H, \phi) \in \mathcal{M}_G$ such that $H \neq N_G(H, \phi)$
$\mathcal{R}_G$	A set of the representatives of the G-orbits of $\mathcal{M}^*_G$
$\mathrm{Obs}^A(G)$	The A-fibered obstruction group for G
$\pi_{H,\phi}$	The projection map from the ghost ring to the $(H, \phi)-$ component of the obstruction group
$\mathcal{F}$	A fusion system over a p-group S
$\mathrm{Hom}_{\mathcal{F}}(P, Q)$	The set of morphisms in the fusion system $\mathcal{F}$ between the subgroups P and Q of the p-group S

$c_g : H \to G$	The injective homomorphism from $H \leq G$ induced by conjugation by $g \in G$
$\mathrm{Iso}^{\phi}_{Q,P}$, c^{ϕ}_{Q}	Isogation map for $\phi : Q \to P$
${}^{\phi}(V, \nu)$	Isogation of a sub-character of S by ϕ in $\mathcal{F}$
N_{ϕ}	The subgroup $\{g \in N_S(Q) : {}^{\phi}c_g \in \mathrm{Aut}_S(P)\}$ for an isomorphism $\phi : Q \to P$ of subgroups of S
$\mathcal{F}_S(G)$	The fusion system over a p-group S with morphisms induced by conjugations by elements of an ambient group G
$\mathrm{Aut}(Q)$	The group of automorphisms of the group Q
$\mathrm{Aut}_{\mathcal{F}}(Q)$	The automorphisms of Q in the category $\mathcal{F}$
$[P]_{\mathcal{F}}$	The $\mathcal{F}$-conjugacy class of $P \leq S$
$\mathrm{Aut}^{K}_{S}(Q)$	$= \mathrm{Aut}_S(Q) \cap K$ for a subgroup K of $\mathrm{Aut}(Q)$
$\mathrm{Aut}^{K}_{\mathcal{F}}(Q)$	$= \mathrm{Aut}_{\mathcal{F}}(Q) \cap K$ for a subgroup K of $\mathrm{Aut}(Q)$
$C_S(P)$	The centralizer of P in S
$\mathrm{mod}_{\mathcal{F}}$, ${}_{\mathcal{F}}\mathrm{mod}$	Category of $\mathcal{F}$-modules
$\Gamma_{\mathcal{F}}$	The category algebra of the fusion system $\mathcal{F}$
$\underline{\mathbb{Z}}$	The constant $\mathcal{F}$-module
$[H\backslash G/K]$	A complete set of double coset representatives of H and K in G
$S_{Q,V}$	The simple $\mathcal{F}$-module induced by the simple $k[\mathrm{Aut}_{\mathcal{F}}(Q)]$-module V
S_x	Splitting functor at the object $x \in \mathrm{Ob}(\mathcal{F})$
E_x	Extension functor at the object $x \in \mathrm{Ob}(\mathcal{F})$
$M(\mathcal{F})$	$\mathcal{F}$-stable elements of an $\mathcal{F}-$module M
$\varprojlim_{\mathcal{F}} M$	The projective limit of the functor M over the fusion system $\mathcal{F}$
$X^{(H,\lambda)}$	The union of the A-orbits of X with stabilizing pair containing (H, λ)
$\beta_{H,\lambda}(X)$	The number of A-orbits in the A-fibered H-set $X^{(H,\lambda)}$
$B^A(\mathcal{F})$	The A-fibered Burnside ring of $\mathcal{F}$
$(P, \lambda) \leq_{\mathcal{F}} (Q, \kappa)$	(P, λ) is sub $\mathcal{F}-$conjugate to (Q, κ)
$\alpha_{P,\lambda}$	The basis element of the $A-$fibered Burnside ring of $\mathcal{F}$ for the fully stabilized pair (P, λ)

$\mathfrak{B}^A(\mathcal{F})$	The A-fibered ghost ring for $\mathcal{F}$
$\mathrm{Obs}(\mathcal{F})$	The obstruction group of $\mathcal{F}$
$\mathrm{Obs}^A(\mathcal{F})$	The $A-$fibered obstruction group of $\mathcal{F}$
$\mathcal{M}_S^A/\mathcal{F}$	A complete set of representatives of the $\mathcal{F}$-conjugacy classes in $\mathcal{M}_S^A$
$[P,\lambda]_\mathcal{F}$	The $\mathcal{F}$-conjugacy class of the fibered set $(P,\lambda) \in \mathcal{M}_S^A$

1. INTRODUCTION

The thesis will consist of three chapters: the first two, Fibered Sets and Fibered Burnside Rings, and Fusion Systems will introduce the theory and in the Main Results we will conclude.

To begin with, we recall that a $G-$action on a finite set X is a group homomorphism $G \to \mathrm{Aut}(X)$ where $\mathrm{Aut}X$ denotes the symmetric group on the set X. Such sets are called $G - sets$ and the maps between $G-$sets which preserve the G- action are called G-set morphisms. The isomorphism class of a $G-$set X is denoted by $[X]$. We have a natural G-action on the disjoint unions of $G - sets$ and we can define a $G-$ action on the cartesian products componentwise. Hence, the disjoint unions and the cartesian products of $G-$sets have again G-set structures. The irreducible $G-$sets are in the form G/P for some $P \leq G$ and any G-set can be written as a disjoint union of irreducible G- sets.

The Burnside ring $B(G)$ for a finite group G is known to be an algebraic construction that encodes the $G-$action on finite sets. Formally speaking, under the operations of the disjoint union (written additively) and the cartesian product (written multiplicatively) isomorphism classes of $G - sets$ form a semi-ring whose Grothendieck Group (the set of formal differences) we call the Burnside ring $B(G)$ of G. It is a free abelian group generated by the irreducible $G-sets$, that is it is isomorphic to the direct product of copies of $\mathbb{Z}$ over the index set of the isomorphism classes of irreducible G-sets. The product on the Burnside ring is reflected by Mackey formula in terms of the irreducible G-sets:

$$[G/P][G/Q] = \sum_{PgQ \in P\backslash G/Q} [G/P \cap {}^{s}Q].$$

Besides decomposing a G-set into the irreducibles an alternative characterization is to determine the number of its fixed points under different subgroups of G. That is to consider the maps $\Phi_P(X) = |X|^P$ that gives the number of P-fixed $(P \leq G)$ points of a G-set X. Each Φ_P is a ring homomorphism from the Burnside ring of G into $\mathbb{Z}$. Compounding all such ring homomorphisms over subgroups $P \leq G$ we obtain "*the homomorphism of marks*" denoted by β_G. It is well known that β_G maps the Burnside ring of G injectively into a direct product of copies of $\mathbb{Z}$ over $P \leq G$ as rings. We call it the ghost ring of G and denote by $\mathcal{B}(G)$. Having multiplication componentwise, and easier restriction maps these rings are easier for calculations and the injective homomorphism of marks $\Phi = (\Phi_P)_P$ $(P \leq G)$ provides us with a short exact sequence:

$$0 \longrightarrow B(G) \xrightarrow{\Phi} \mathcal{B}(G) \xrightarrow{\Psi} \mathrm{Obs}(G) \longrightarrow 0$$

where the *obstruction group*

$$\mathrm{Obs}(G) = \prod_{\substack{[P]_G \textit{ is an } G-\textit{conjugacy} \\ \textit{class of subgroups}}} \mathbb{Z}/(|N_G P|/|P|)\mathbb{Z}$$

is the finite cokernel of Φ and Ψ is a group homomorphism.

A useful extension of $G-$sets in the study of complex, modular and integral group representations are fibered G-sets. Let A be a fixed Abelian group. We call an $A-$free $A \times G$-set with finitely many orbits, an $A-fibered\ G-set$. The category of A-fibered $G-$sets has finite co-products (disjoint union) and products (tensor products over A) which are preserved under the isomorphisms. Hence using this semi-ring structure on isomorphism classes of A-fibered $G-$sets we get its Grothendieck ring called "*the monomial Burnside ring for G with fiber group A*" by Barker [2], we denote it by $B^A(G)$. In particular if $A = 1$ then $B^A(G) = B(G)$, that is the Burnside ring of G.

We can write $B^A(G)$ as a free Abelian group with basis the transitive A-fibered $G-$sets. A further description can be given in terms of the $A-$characters of the subgroups $V \leq G$; i.e. the group homomorphisms $(V, \nu) : V \to A$. If $[V, \nu]_G$ denotes the G-conjugacy class of the transitive A-fibered G-set such that V is the stabilizer of a fibre Ax, and $vx = \nu(v)x$ for all $v \in V$ then we can write

$$B^A(G) = \bigoplus_{(V,\nu)} \mathbb{Z}[V,\nu]_G$$

where (V,ν) runs over a complete set of representatives of the isomorphism classes of the transitive A-fibered G-sets. Here a version of the Mackey formula would describe the product on the transitive A-fibered $G-$sets [2]: given A-subcharacters (V,ν) and (W,ω) of G we have

$$[V,\nu]_G[W,\omega]_G = \sum_{VgW \in V\backslash G/W} [V \cap {}^gW, \nu.{}^g\omega]_G$$

where the product character $\nu.{}^g\omega$ is defined on $V \cap {}^gW$ in obvious manner. In Chapter 2 we will introduce the basic terms of the theory of the fibered sets following [2] and [3] and we will write the fibered versions of the ghost ring, the obstruction group and the integrality conditions we mentioned above. Later in the following chapters our main objective will be to write these structures for fusion systems.

The concept of fusion in the Finite Group Theory has been introduced by Brauer in 1950s (cf. [5]). Given any group homomorphism $H \to A$ with A abelian we can define in a canonical way a homomorphism $G \to A$ where $H \leq G$, called $transfer$ of G into A via $H \to A$. This transfer map defined on the commutative cosets of the kernel requires information as to the $fusion$ of g in H, that is information on the orbit's elements (conjugates) $g^G \cap H$. This task is accomplished by the normal p-complement theorems of Burnside and Frobenius, which showed that, under suitable hypotheses on fusion, G possesses a normal $p-complement$: for a Sylow-p subgroup S in G we have a complement normal subgroup N such that $NS = G$ and $N \cap S = 1$. So constructed this notion of fusion was used by representation theorists to calculate the transfer maps. Later more complicated results on fusion appeared; e.g. Alperin's Fusion Theorem shows that the family of normalizers of suitable subgroups of S control fusion in S, that is if any pair of elements of S which are conjugate in G are also conjugate under H.

In nineties Puig [6] described this notion of fusion in a more abstract setting in terms of category theory. Let p be a fixed prime number, G be a finite group with a Sylow p-subgroup S, then the subgroups $P \leq S$ with isomorphisms $P \to S$ induced by the conjugations in G form a category. Puig abstracted this category by dropping the ambient group G and instead asserting axioms on the morphisms of the category, satisfied in particular by the above one. This abstract category was called "Frobenius category" by Puig. Later in 2000s, Broto-Levi-Oliver [4] following Puig refined this notion furthermore and provided axioms of what we call today a "fusion system " over a p-group S.

By further axioms on the morphisms and objects of a fusion system, one can have a more concrete structure on which we can imitate/emulate Sylow theory. Such fusion systems are called "saturated", which in fact coincides with the original construction of the Frobenius Category of Puig.

So constructed in categorical terms the theory of fusion systems is quite a new theory and the main task of the fusion theorists at the beginning was to extend the primary results on fusions in group theory (such as mentioned above the normal p-complement theorems of Burnside and Frobenius or Alperin's fusion theorem) to the categorical setting. So for example a correspondent of Alperin's Theorem above would state that $\mathcal{F}_S(G)$ is generated/determined by automorphism groups of certain subgroups of S. Nevertheless we should note that its usage and importance is not limited to the representation theory. In fact, Puig created Frobenius categories as a tool for the modular representation theory but later homotopy theorists like Broto, Levi and Oliver [4] applied Puig's ideas to p-completed classifying spaces $BG^{\wedge}_{p}$ of finite groups (i.e. a space which allows us to focus on the properties of the classifying space BG "at the prime p"). Actually, it turned out that there is a close connection between the homotopy theoretic properties of the p-completed classifying space $BG^{\wedge}_{p}$ for a finite group G and the structure of its fusion system over a Sylow p-subgroup S: two p-completed classifying spaces $BG^{\wedge}_{1p}$, $BG^{\wedge}_{2p}$ are conjectured to be homotopy equivalent if and only if their fusion systems are equivalent as categories.

In Chapter 3 we will provide the reader with a short introduction to the theory of fusion systems. Besides the basics of the theory we will introduce the concept of $\mathcal{F}-$modules and in terms of them we will write the fibered Burnside and ghost rings of a fusion system and the mark homomorphism between them. Later in the main results we will also define the fibered obstruction group for $\mathcal{F}$ and will show that it is the cokernel of the mark homomorphism.

As mentioned in the abstract our main objective will be to generalize S. P. Reeh's results on the Burnside ring of fusion stable S-sets for fibered S-sets over a commutative group A. S.P. Reeh (2015) [1] extended the short exact sequence on p.2 for the fusion systems, that is we have an injective ring homomorphism with finite cokernel from the Burnside ring of a fusion system into a direct product of the copies of $\mathbb{Z}$ as rings, and he showed that the Burnside ring has a free basis consisting of the irreducible $\mathcal{F}-$stable sets. The details how Reeh constructs these irreducible sets are a little bit technical to mention here but we will exclusively benefit from his ideas in our own problem with slight adaptations for fibered sets.

Before moving on let us recall restriction and conjugation maps on Burnside and ghost rings. We will need them in defining the fusion stable elements. If $H \leq G$ we have in general a ring homomorphism $\mathrm{Res}_H^G : B(G) \to B(H)$ defined by

$$\mathrm{Res}_H^G([G/P]_G) = \sum_{HgP \in H \backslash G/P} [H/H \cap {}^gP]_H.$$

Similarly, if $H \leq G$ we have a ring homomorphism $\mathrm{Res}_H^G : \mathfrak{B}(G) \to \mathfrak{B}(H)$ defined by $\mathrm{Res}_H^G f(P) = f(P)$ for $P \leq H$.

On the other hand if $\mathcal{F}$ is a fusion system over a p-group S and if $\phi : H \to K$ is an isomorphism in $\mathcal{F}$ we can define another ring homomorphism $c_H^\phi : B(H) \to B(K)$ by $c_H^\phi([H/P]_H) = [K/\phi(P)]_K$ for $P \leq H$.

Similarly we define $c_H^\phi : B(H) \to B(K)$ by $c_H^\phi(f)(P) = f(\phi^{-1}(P))$ for $P \leq K$.

We say that an $S-$set X is *$\mathcal{F}$-stable* if it satisfies the condition $c_H^\phi \circ \mathrm{Res}_H^S(X) = \mathrm{Res}_K^S(X)$ for any $H \leq G$ and for all isomorphisms $\phi : H \to K$ in $\mathcal{F}$.

We know that $\mathcal{F}$-stable elements in $B(S)$ form a semiring, by taking its Grothendieck group we obtain a subring of $B(S)$ which we call *the Burnside ring of the fusion system* $\mathcal{F}$. We will denote it by $B(\mathcal{F})$. Similarly we can obtain a *ghost ring of* $\mathcal{F}$ which consists of $\mathcal{F}$-stable functions in $\mathfrak{B}(S)$. We denote it by $\mathfrak{B}(\mathcal{F})$.

S.P. Reeh showed that if $\mathcal{F}$ is a saturated fusion system over a p-group S then the underlying abelian group of $B(\mathcal{F})$ is a free group over the $\mathcal{F}-$conjugacy classes of subgroups of S and by restricting the mark homomorphism β_G to $B(\mathcal{F})$ we obtain a short exact sequence:

$$0 \longrightarrow B(\mathcal{F}) \xrightarrow{\beta_{\mathcal{F}}} \mathfrak{B}(\mathcal{F}) \xrightarrow{\pi} \mathrm{Obs}(\mathcal{F}) \longrightarrow 0$$

where $\beta_{\mathcal{F}}$ is the restriction of β_G and $\pi : \mathfrak{B}(\mathcal{F}) \to \mathrm{Obs}(\mathcal{F})$ is a group homomorphism onto the group $\mathrm{Obs}(\mathcal{F})$; i.e. the cokernel of $\beta_{\mathcal{F}}$. We call $\mathrm{Obs}(\mathcal{F})$ *the obstruction group of* $\mathcal{F}$.

In the final chapter of the thesis we will continue the concept of fibered Burnside ring to saturated fusion systems as Reeh did for $G-$sets. The first two sections will introduce the necessary terms to adapt Reeh's original idea to fibered sets and we will prove some basic facts on them. In the final section we will conclude like Reeh by giving a proof of the fact that fibered version of the Burnside ring of a fusion system is also free on the $\mathcal{F}$-conjugacy classes of the transitive A-fibered sets.

A possible continuation of the thesis could be the calculation of the idempotents of the fibered Burnside ring using its embedding in the relatively simpler ghost ring of $\mathcal{F}$ where they amount to be characteristic functions on $\mathcal{F}-$conjugacy classes of the coordinates.

2. FIBERED SETS and FIBERED BURNSIDE RINGS

This chapter will consist of two sections. In the first we will introduce fibered permutation sets besides some notation and preliminaries of the theory which we will use in the latter chapters. We will first define the fibered Burnside ring and then will introduce the *generalized fixed points* and define the mark homomorphism and the ghost ring in terms of them. At the end of 2.1 we will give the fibered versions of the integrality conditions we mentioned in the introduction. A proof of them will be given at the end of the second section.

In Section 2.2 we will follow R. Boltje [3] to write the theory in a more general setting which can be interesting for the reader. In the first section we will write the fibered Burnside ring as a functor B^A from the category of the subgroups of G into the category of **Ring**. In this setting we will write Boltje's proof of the integrality conditions in particular for the fibered sets.

2.1. Fibered permutation sets

In this section we review the construction of fibered Burnside rings which can also be found in [7], [2], [8] or [9] with further details.

2.1.1. Fibered Burnside Ring

Let G be a finite group and A be an abelian group. We construct the product $A \times G = \{ag : a \in A, g \in G\}$. An A-free $A \times G$-set with finitely many A-orbits is called an $A - fibered\ G - set$ where A-orbits are called $fibers$. We can use a finite set X/A of representatives of fibers to express an A-fibered G-set in the form AX, which can be seen as a collection of the fibers Ax, $x \in X/A$.

Given A-fibered G-sets X and Y, an $A \times G$-equivariant function $f : X \to Y$ is called a *morphism* of A-fibered G-sets. Together with these morphisms the class of all

A-fibered G-sets becomes a category, denoted by ${}_G\mathrm{set}^A$

For A-fibered G-sets X and Y *the coproduct* $X+Y$ is defined to be the A-fibered G-set of the disjoint union $X \sqcup Y$. We denote the Grothendieck group of ${}_G\mathrm{set}^A$ (i.e. the group generated by the formal differences of isomorphism classes $[X]-[Y]$) by $B^A(G)$ and call it the *A-fibered Burnside group of* G. An A-fibered $G-$set is called *transitive* if the $G-action$ on A-orbits is transitive. We can decompose any fibered set into a sum of the transitive A-fibered G-sets. Hence the fibered Burnside group is freely generated by the isomorphism classes of the transitive A-fibered G-sets. To understand the latter sets we need to talk in terms of group homomorphisms (H,λ); $\lambda : V \to A$, called $A-subcharacters\ of\ G$. Given an A-subcharacter (H,λ) a G-action is given by ${}^g(H,\lambda) = ({}^gH, {}^g\lambda)$ for $g \in G$. We denote by $[H,\lambda]_G$ the isomorphism class of a transitive A-fibered G-set such that H is the stabilizer of the fibre Ax and $hx = \lambda(h)x$ for all $h \in H$ where $x \in X/A$. We call (H,λ) a *stabilizing pair* for $[H,\lambda]_G$. Note that any other stabilizing pair for $[H,\lambda]_G$ is conjugate to (H,λ) since $[H,\lambda]_G = [K,\kappa]_G$ if and only if ${}^g(H,\lambda) = (K,\kappa)$ for some $g \in G$, moreover every transitive set is in that form. We denote the set of all pairs (H,λ) with $H \leq G$ and $\lambda \in \mathrm{Hom}(H,A)$ by $\mathcal{M}_G^A$. The group G acts on $\mathcal{M}_G^A$ and we write $\mathcal{M}_G^A/G$ for a complete set of representatives of G-conjugacy classes in $\mathcal{M}_G^A$. With this notation, $\mathcal{M}_G^A/G$ forms a free basis for the fibered Burnside group $B^A(G)$, that is, we have

$$B^A(G) = \bigoplus_{(H,\lambda)\in\mathcal{M}_G^A/G} \mathbb{Z}\cdot[H,\lambda]_G.$$

Elements of $B^A(G)$ are called *virtual A-fibered G-sets* and given such an element X, we have the following coordinate decomposition of X with respect to the above basis.

$$X = \sum_{(H,\lambda)\in\mathcal{M}_G^A/G} c_{H,\lambda}(X)\cdot[H,\lambda]_G$$

On the other hand A acts diagonally on the cartesian product :$a(\xi, \eta) = (a\xi, a^{-1}\eta)$ for any $a \in A, (x, y) \in X \times Y$.we denote by $X \otimes Y$ the se set of the orbits $\xi \otimes \eta$. The *product* XY of the A-fibered $G-$sets is defined to be the tensor product $X \otimes Y$ with the $A \times G$-action $ag(\xi \otimes \eta) = ag\xi \otimes g\eta$. It is a well-defined product on $B^A(G)$. For two basis elements $[H, \lambda]_G, [K, \kappa]_G$ in $B^A(G)$, the product becomes

$$[H, \lambda]_G \cdot [K, \kappa]_G = \sum_{x \in [H \backslash G/K]} [H \cap {}^xK, \lambda|_{H \cap {}^xK} \cdot {}^x\kappa|_{H \cap {}^xK}]_G$$

where $[H \backslash G/K]$ denotes a complete set of double coset representatives of H and K in G. We call this *Mackey product formula.* Together with this multiplication, $B^A(G)$ becomes a commutative associative ring with identity $[G, 1]_G$. It is called the *A-fibered Burnside ring* of G. After identifying the isomorphism class $[G/H]$ of a transitive G-set with stabilizer H and the basis element $[H, 1]_G$, the A-fibered Burnside ring extends the Burnside ring $B(G)$.

At last we will try to visualize fibered sets over a simple example, namely $\mathbb{C}$-fibered sets on the dihedral group D_8 of order 8. We will later return to this example in the following chapters when we introduce the fusion systems on D_8.

Example 2.1 ($\mathbb{C}$-fibered D_8-sets)**.** *D_8 can be embedded in the symmetric group S_4 by taking the generators $x = (13)$ and $a = (1234)$. It has three normal subgroups of order 4: two Klein 4-subgroups $V_1 = \{e, ax, a^2, a^3x\}$ and $V_2 = \{e, x, a^2, a^2x\}$ and a cyclic subgroup $C_4 = \{e, a, a^2, a^3\}$. Moreover each Klein 4-group contains two subgroups of order 2 which are D_8-conjugate: $C_2^1 = \{e, ax\} \sim_{D_8} C_2^2 = \{e, a^3x\} \leq V_1$ and $C_2^3 = \{e, x\} \sim_{D_8} C_2^4 = \{e, a^2x\} \leq V_2$ and the center of D_8, $Z = Z(D_8) = \{e, a^2\}$ is the other subgroup of order 2.*

A character table for D_8 is given as follows (cf. [10, Ch. 5]):

	a^k	xa^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

for $k = 1, 2, 3, 4$. *So we write* $\mathrm{Hom}(D_8, \mathbb{C}) = \{\psi_1, \psi_2, \psi_3, \psi_4\}$.

On the other hand, in general for cyclic groups C_n we have n irreducible representations of degree 1 *whose characters $\chi_0, \ldots \chi_{n-1}$ are given as $\chi_h(a^k) = e^{2\pi i h k/n}$ [10, Ch.5]. Hence* $\mathrm{Hom}(C_4, \mathbb{C}) = \{\chi_0, \chi_1, \chi_2, \chi_3\}$. *It also follows that for Klein $4-$subgroups V_i, being the direct product of its cyclic subgroups the character tables are given by*

V_1	e	ax	a^2	a^3x
$\psi_5 = 1$	1	1	1	1
$\psi_6 = \psi_2\vert_{V_1}$	1	-1	1	-1
ψ_7	1	1	-1	-1
ψ_8	1	-1	-1	1

V_2	e	x	a^2	a^2x
$\psi_9 = 1$	1	1	1	1
$\psi_{10} = \psi_2\vert_{V_2}$	1	-1	1	-1
ψ_{11}	1	1	-1	-1
ψ_{12}	1	-1	-1	1

Hence, considering D_8-conjugacies $(V^1, \psi_7) \sim_{D_8} (V^1, \psi_8)$, $(V_2, \psi_{11}) \sim_{D_8} (V_2, \psi_{12})$, $(C_4, \chi_1) \sim_{D_8} (C_4, \chi_3)$ and similarly conjugacies for characters of C_2^i, we have a whole list of the isomorphism classes of $\mathbb{C}-$fibered D_8-sets, or the $\mathbb{C}$-fibered Burnside ring of D_8 :

$$B^{\mathbb{C}}(D_8) = \bigoplus_{i=1}^{4} \mathbb{Z}[D_8, \psi_i]_{D_8} \oplus \bigoplus_{i=5}^{7} \mathbb{Z}[V_1, \psi_i]_{D_8} \oplus \bigoplus_{i=9}^{11} \mathbb{Z}[V_2, \psi_i]_{D_8} \oplus$$

$$\bigoplus_{i=0}^{2} \mathbb{Z}[C_4, \chi_i]_{D_8} \oplus \mathbb{Z}[C_2^1, 1]_{D_8} \oplus \mathbb{Z}[C_2^1, \psi_2|_{C_2^1}]_{D_8} \oplus \mathbb{Z}[C_2^3, 1]_{D_8} \oplus$$

$$\mathbb{Z}[C_2^3, \psi_2|_{C_2^3}]_{D_8} \oplus \mathbb{Z}[Z,1]_{D_8} \oplus \mathbb{Z}[Z,\chi_1|_Z]_{D_8} \oplus \mathbb{Z}[1,1]_{D_8}.$$

2.1.2. Integrality Conditions for fibered sets

A ghost ring of a group G is in general the codomain of *the mark homomorphism* which we will define in the following as an embedding of the Burnside ring. Having a componentwise product and simpler restrictions the ghost ring has a simpler structure to study G-sets, e.g. the idempotents of the Burnside ring will appear in the form (a_i) with $a_i = 0, 1$ in the ghost ring. The embedding is subject to some modularity, or integrality conditions which we calculate by the cokernel of the mark homomorphism for the Burnside group. We call this cokernel the *obstruction group* of G.

A ghost ring for fibered Burnside groups together with a mark homomorphism is introduced by Boltje [8]. Actually he introduced plus constructions which include fibered Burnside groups as a special case. We will recall basic definitions together with Boltje's integrality conditions. Later we will deal Boltje's construction with more detail in Section 2.2 and will give Boltje's proof of the integrality conditions at the end of the chapter.

For any finite group G, we denote by $\mathfrak{B}^A(G)$ the ring of functions $\mathcal{M}_G^A \to \mathbb{Z}$ which are constant on G-conjugacy classes. As an abelian group, $\mathfrak{B}^A(G)$ has a basis consisting of characteristic functions $\chi_{H,\lambda}^G$ of conjugacy classes of pairs $(H,\lambda) \in \mathcal{M}_G^A$ and hence there is an isomorphism of rings

$$\mathfrak{B}^A(G) \cong \prod_{(H,\lambda)\in\mathcal{M}_G^A/G} \mathbb{Z}$$

where the right hand side is a ring under coordinate-wise multiplication. Given any function $f \in \mathfrak{B}^A(G)$, we write

$$f = \sum_{(H,\lambda)\in\mathcal{M}_G^A/G} f_{(H,\lambda)}\chi_{H,\lambda}^G$$

where we put $f_{(H,\lambda)} = f((H,\lambda))$.

In order to introduce mark homomorphism, we first consider generalized fixed points. Given an A-fibered G-set X and let $(H,\lambda) \in \mathcal{M}_G^A$. We define the set $X^{(H,\lambda)}$ of (H,λ)-*fixed points in* X as the set

$$X^{(H,\lambda)} = \{x \in X \mid g \cdot x = \lambda(g) \cdot x \,\text{for all}\, g \in H\}.$$

It is clear that $X^{(H,\lambda)}$ is an A-fibered H-set via restriction of the G-action on X and moreover its class $[X^{(H,\lambda)}]$ in $B^A(H)$ satisfies

$$[X^{(H,\lambda)}] = c_{H,\lambda}(\mathrm{Res}_H^G X) \cdot [H,\lambda]_H.$$

We write $\beta_{H,\lambda}(X)$ for the number of A-orbits in the A-fibered H-set $X^{(H,\lambda)}$, that is the coefficient $c_{H,\lambda}(\mathrm{Res}_H^G X)$ at the basis element $[H,\lambda]_H$. Note that if $(K,\kappa) \in \mathcal{M}_G^A$ is G-conjugate to (H,λ), say $(K,\kappa) = {}^g(H,\lambda)$ for $g \in G$, then the function

$$X^{(H,\lambda)} \to X^{(K,\kappa)}, x \mapsto g \cdot x$$

is a bijection. In particular $\beta_{K,\kappa}(X) = \beta_{H,\lambda}(X)$.

With this notation the assignment $[X] \mapsto ((H,\lambda) \mapsto \beta_{H,\lambda}(X))_{(H,\lambda)\in\mathcal{M}_G^A}$ extends linearly to a map $\beta_G : B^A(G) \to \mathfrak{B}^A(G)$, called '*the mark homomorphism*'. Hence β_G is a ring homomorphism and given $x \in B^A(G)$, we write

$$\beta_G(x) = \sum_{(H,\lambda)\in\mathcal{M}_G^A} \beta_{H,\lambda}(x) \cdot \chi_{H,\lambda}^G.$$

In Section 2.2 we will see by [8, Prop.2.4] that we have an almost inverse morphism to β_G and we will see that the mark morphism is injective.

Finally, $\mathcal{M}_G^A$ is a partially ordered set with respect to the following ordering:

$$(L, \mu) \leq (H, \lambda) \text{ iff } L \leq H \text{ and } \lambda|_L = \mu.$$

Integrality conditions for fibered sets in terms of our definitions will be

Proposition 2.2. *[3, Theorem 2.2] Let*

$$f = \sum_{(K,\psi)\in\mathcal{M}_G^A/G} f_{(K,\psi)}\chi_{K,\psi}^G \in \mathfrak{B}^A(G).$$

We have that $f \in \mathrm{im}(\beta_G)$ *if and only if for every* $(H,\phi) \in \mathcal{M}_G^A/G$ *one has the congruence*

$$\sum_{(H,\phi)\leq(I,\sigma)\in\mathcal{M}_{N_S(H,\phi)}^A} \mu_{H,I}.f_{(I,\sigma)} \equiv 0 \quad mod[N_S(H,\phi):H]$$

in $\mathbb{Z}$ *where* μ *denotes the Möbius function of the partially ordered set of subgroups of* G*; and* $N_S(H,\phi) = \{s \in S|^s(H,\phi) = (H,\phi)\}$ *is the stabilizer of the pair* (H,ϕ) *in* S*.*

A proof of this proposition besides the calculation of the cokernel of the mark homomorphism will be given at the end of the next section after we introduce the restriction functors.

2.2. Restriction Functors

In this section we will see Burnside ring in a more abstract setting as restriction functor [3] and at the end we will adopt Boltje's proof of the integrality conditions for Proposition 2.2.

2.2.1. Burnside ring as restriction functor

For each subgroup $H \leq G$ we have the ring of subcharacters $A(H) = \mathbb{Z}\{\nu : H \to A\}$. Conjugation $c_H^g(\nu) = {}^g\nu$ and restriction $\mathrm{Res}_I^H(\nu) = \nu|_I$ to $I \leq H$ furthermore satisfy the axioms:

- $c_H^h = \mathrm{Res}_H^H = id_{A(H)}$ for $h \in H$
- $\mathrm{Res}_J^I \circ \mathrm{Res}_I^H = \mathrm{Res}_J^H, \quad c_{{}^gH}^{g'} \circ c_H^g = c_H^{g'g}$,
- $c_I^g \circ \mathrm{Res}_I^H = \mathrm{Res}_{{}^gI}^{{}^gH} \circ_H^g$,

for all $J \leq I \leq H \leq G, \; h \in H$ and $g, g' \in G$. Such a family of the rings $A(H)$ is called *a* $(\mathbb{Z}-)$ *restriction functor on* G [3].

Then in terms of the above definitions, we will show below that the generalized Burnside ring of G will be the family of rings

$$A_+(H) = \Big(\bigoplus_{K \leq H} A(K)\Big)_H$$

for $H \leq G$, where $\bigoplus_{K \leq H} A(K)$ is a $\mathbb{Z}H-$ module under the action of conjugations c_K^h, $K \leq H$, and for any kH-module M, $M_H = M/\langle m - hm \mid m \in M, h \in H\rangle$ is the ring of H-cofixed points of M. We will write $(H, \lambda) \in A(H)$ to denote the subcharacter $(\lambda : H \to A) \in A(H)$ like a stabilizing pair when there is no confusion.

This module has the ring structure with the multiplication of fibered sets or in case of transitive sets, the Mackey formula:

$$[K, a]_H.[L, b]_H = \sum_{KhL \in K \backslash H/L} [K \cap {}^hL, \mathrm{Res}_{K \cap {}^hL}^K(a).\mathrm{Res}_{K \cap {}^hL}^{{}^hL}({}^hb)]_H.$$

Before moving on let's show that the generalized Burnside ring for G we defined above actually coincides the main definition we gave at Section 2.1:

$$B^A(G) = \bigoplus_{(H,\lambda)\in \mathcal{M}_G^A/G} \mathbb{Z}\cdot [H,\lambda]_G.$$

Proposition 2.3. *$B^A(G)$ and $A_+(G)$ are isomorphic as rings.*

Proof. We define a ring homomorphism $f : B^A(G) \to A_+(G)$ on the basis elements by the rule

$$[H,\lambda]_G \in B^A(G) \mapsto (H,\lambda) + \langle X - {}^gX : X \in \oplus_{I\le G} A(I),\ \ g\in G\rangle \in A_+(G).$$

For simplicity, let us write $\mathcal{X} = \langle X - {}^gX : X \in \oplus_{I\le G} A(I),\ \ g\in G\rangle$. One can then easily check that f is well-defined; i.e $(K,\kappa) \sim_G (H,\lambda)$ in $\mathcal{M}_G^A$ then $[H,\lambda]_G = [K,\kappa]_G$ and $(H,\lambda)+\mathcal{X} = (K,\kappa)+\mathcal{X}$ in $A_+(G)$ as co-fixed points. Moreover f is an isomorphism since $(H,\lambda)-(K,\kappa)\in\mathcal{X}$ only if $(H,\lambda) = {}^g(K,\kappa)$ as fibered sets, or $[H,\lambda]_G = [K,\kappa]_G$ and it is surjective as the conjugacy classes $f([H,\lambda]_G) = (H,\lambda)+\mathcal{X} \in A_+(G)$ generate all co-fixed points. $\square$

Henceforth we will also denote by $[K,a]_H$ the isomorphism class of an element $a\in A(K)$ in $B^A(H)$. Besides the multiplication we defined above, the conjugation ${}^g[K,a]_H = [{}^gK, {}^ga]_{{}^gH}$ by $g\in G$ and the restriction

$$\mathrm{Res}_I^H([K,a]_H) = \sum_{IhK\in I\backslash H/K} [I\cap {}^hK, \mathrm{Res}^{{}^hK}_{I\cap {}^hK}({}^ha)]_I$$

for $I\le H\le G$ satisfy the above axioms. Hence the Burnside ring B^A is a restriction functor on G.

2.2.2. Mark Homomorphism

As mentioned before, in a dual way we can obtain a more manageable image of the Burnside ring. The $H-$fixed points

$$A^{+}(H) = \Big(\prod_{K \leq H} A(K)\Big)^{H}$$

for $H \leq G$ form a restriction functor with the componentwise multiplication, the conjugation ${}^{g}(a_K)_{K \leq H} = ({}^{g}a_K)_{{}^{g}K \leq {}^{g}H}$ and the restriction :

$$\mathrm{Res}^{H}_{I}(a_K)_{K \leq H} = (a_K)_{K \leq I}.$$

This functor is called *a generalized ghost ring* of G. We will show below that the $A-$fibered ghost ring $\mathfrak{B}^{A}(G)$ we defined in Section 2.1 and $A^{+}(G)$ are isomorphic as rings:

Proposition 2.4. *$\mathfrak{B}^{A}(G)$ and $A^{+}(G)$ are isomorphic as rings.*

Proof. We define a ring homomorphism $\phi : \mathfrak{B}^{A}(G) \to A^{+}(G)$ by sending the characteristic functions

$$\chi^{G}_{H,\lambda} \in \mathfrak{B}^{A}(G) \mapsto \phi(\chi^{G}_{H,\lambda}) = \Big(\sum_{\kappa:K \to A} a_{K,\kappa}\kappa\Big)_{K \leq G} \in A^{+}(G)$$

where

$$a_{K,\kappa} = \begin{cases} 1 & \text{if } [K,\kappa]_G = [H,\lambda]_G \\ 0 & \text{otherwise} \end{cases}$$

We first check whether the image $\phi(\chi^{G}_{H,\lambda})$ is a G-fixed point:

$${}^{g}\Big(\sum_{\kappa:K \to A} a_{K,\kappa}\kappa\Big)_{K \leq G} = {}^{g}\Big(\sum_{\kappa:K \to A} \chi^{G}_{H,\lambda}(K,\kappa)\kappa\Big)_{K \leq G} = \Big(\sum_{\kappa:K \to A} \chi^{G}_{H,\lambda}(K,\kappa).{}^{g}\kappa\Big)_{{}^{g}K \leq G} =$$

$$\Big(\sum_{\kappa:K\to A}\chi^G_{H,\lambda}({}^{g^{-1}}K,{}^{g^{-1}}\kappa)\kappa\Big)_{K\le G}=\Big(\sum_{\kappa:K\to A}\chi^G_{H,\lambda}(K,\kappa).\kappa\Big)_{K\le G}=\Big(\sum_{\kappa:K\to A}a_{K,\kappa}\kappa\Big)_{K\le G}$$

since $\chi^G_{H,\lambda}$ is constant on conjugacy classes.

Then we check that ϕ is injective: for any $f=\sum_{(K,\psi)\in\mathcal{M}^A_G/G}f_{(K,\psi)}\chi^G_{K,\psi}\in\mathfrak{B}^A(G)$, $\phi(f)=(\sum_{\kappa:K\to A}f_{K,\kappa}\kappa)_{K\le G}=0$ implies that $\sum_{\kappa:K\to A}f_{K,\kappa}\kappa=0$ for each $K\le G$. This in turn implies that $f_{K,\kappa}=0$ since the characters $\kappa:K\to A$ are linearly independent in the group ring $A(K)$. Hence $f=0$.

Finally, ϕ is surjective since any G-fixed point $a=(a_K)_{K\le G}\in A^+(G)$ satisfies ${}^ga_K=a_{{}^gK}$ where $a_K=\sum_{\kappa:K\to A}f_{K,\kappa}\kappa$ for some $f_{K,\kappa}\in\mathbb{Z}$. Now we define $f=\sum_{(K,\kappa)\in\mathcal{M}^A_G/G}f_{(K,\kappa)}\chi^G_{K,\kappa}\in\mathfrak{B}^A(G)$ and $\phi(f)=a$ as desired. □

One can obtain an image of the fibered Burnside ring in its ghost ring by *the mark homomorphism*. It is a natural transformation $\beta:B^A\to\mathfrak{B}^A$ defined as

$$\beta_H=(\pi_K\circ\mathrm{Res}^H_K)_{K\le H}:B^A(H)\to\mathcal{B}^A(H)$$

where $\pi_K:B^A(K)\to A(K)$ is the projection map:

$$[L,a]\mapsto\begin{cases}a & \text{if } L=K\\ 0 & \text{if } L<K\end{cases}$$

It is easy to see that β_G coincides with the mark morphism introduced in Section 2.1; i.e. we have the equality

$$\pi_H\circ\mathrm{Res}^G_H[P,a]_G=\pi_H\Big(\sum_{HgP\in I\backslash H/K}[H\cap{}^gP,\mathrm{Res}^{{}^gP}_{H\cap{}^gP}({}^ga)]_H\Big)$$

$$= \sum_{\lambda : H \to A} \beta_{H,\lambda}([P,a]_G))\lambda \in A(H)$$

where we recall from Section 2.1 that $\beta_{H,\lambda}([P,a]_G))$ is the coefficient of $\mathrm{Res}^G_H[P,a]_G$ at the basis element $[H,\lambda]_H$.

Conversely, for each $H \leq G$ there is a natural transformation $\sigma_H : \mathfrak{B}^A(H) \to B^A(G)$ for each $H \leq G$:

$$(a_K)_{K \leq H} \mapsto \sum_{L \leq K \leq H} |L|\mu_{L,K}[L, a_K|_L]_H$$

where $\mu_{L,K}$ is the Mobius function of the partially ordered set of subgroups of G evaluated at (L,K). It is almost an inverse to β_H [8, Prop.2.4]:

$$\sigma_H \circ \beta^A_H = |H| id_{B^A(G)} \quad \text{and} \quad \beta^A_H \circ \sigma_H = |H| id_{\mathfrak{B}^A(G)}.$$

Since $B^A(G)$ is a free abelian group, or it has trivial $|G|-$torsion this proves at once that the mark homomorphism β_G is injective as we mentioned in the previous section.

Using the functor σ we will also be able to write the integrality conditions for the image of the Burnside ring in $\mathfrak{B}^A(H)$ for $H \leq G$. To begin with we must introduce some terms to express them.

The collection $\mathcal{B} = (\mathcal{B}_H)_{H \leq G}$ of sets with $\mathcal{B}_H = \mathrm{Hom}(H,A)$, the canonical basis of the group ring $A(H)$, is in general called *a $G-$ stable basis of B^A* since it satisfies that ${}^g\mathcal{B}_H = \mathcal{B}_{{}^gH}$. We note that with our choice of $\mathcal{B}_H$ we have $\{\psi|_H : \psi \in \mathcal{B}_G\} \subset \mathcal{B}_H$ for $H \leq G$.

We can associate with them the sets

$$\mathcal{M}_H = \{(K,\psi) \mid K \leq H, \psi \in \mathcal{B}_K\}$$

for $H \leq G$. We note that the sets $\mathcal{M}_G$ actually coincide with the sets $\mathcal{M}_G^A$ we introduced in Section 2.1, $\mathcal{M}_G$ is a G-set as before by the conjugation of A-subcharacters ${}^g(K, \psi) = ({}^gK, {}^g\psi)$. With this basis we can write the restriction of characters as projection maps

$$\mathrm{Res}_K^H(\phi) = \psi = \phi|_K \in \mathcal{B}_K.$$

As in Section 2.1 we can furthermore order the A-characters in the basis $\mathcal{M}_G$ with respect to the partial ordering

$$(K, \psi) \leq (H, \phi) \Longleftrightarrow K \leq H \quad \text{and} \quad \phi|_K = \psi.$$

With this notation the elements of the ghost ring can be written in the form:

$$f = \Big(\sum_{\psi \in \mathcal{B}_K} f_{K,\psi}.\psi \Big)_{K \leq H}$$

with unique coefficients satisfying $f_{K,\psi} = f_{{}^hK,{}^h\psi}$ for all $(K, \psi) \in \mathcal{M}_H$ and $h \in H$. Thus as in previous section we may see the ghost ring as the set of functions $f : \mathcal{M}_H \to \mathbb{Z}$ that are constant on H-orbits. Hence in particular it is a free abelian group. [3]

In the sequel we will write the integrality conditions of the Burnside ring of G and its cokernel, *the obstruction group.*

2.2.3. A Proof of the Integrality Conditions

In this section we will follow [3] to give a necessary and sufficient condition for an element $f \in \mathfrak{B}^A(G)$ to be in the image of the mark homomorphism. In the following we denote by $N_G(H, \phi)$ the stabilizer of (H, ϕ) in G as in Section 2.1.

Proposition 2.5. *[3, Theorem 2.2] An element* $f = \big(\sum_{\psi \in \mathcal{B}_K} f_{K,\psi}.\psi \big)_{K \leq G} \in \mathfrak{B}^A(G)$ *is in the image of the Burnside ring, i.e.* $f \in \mathrm{im}(\beta_G^A)$ *if and only if for every* $(H, \phi) \in \mathcal{M}_G$

and one has the congruence:

$$\sum_{(H,\phi)\leq(I,\psi)\in\mathcal{M}_{N_G(H,\phi)}} \mu_{H,I} f_{I,\psi} \;\equiv\; 0 \quad \bmod [N_G(H,\phi):H].$$

Proof. Here we adopt R. Boltje's original proof at [3, 2.2 Theorem], we make little changes to suit our case in particular. Let $x \in B^A(G)$ and $f = \beta_G(x)$. We know that β_G commutes with restrictions;i.e $\mathrm{Res}_P^S \beta_S = \beta_P res_P^S$ for $P \leq S$ and moreover $\sigma_Q \beta_Q = |Q| id_{B^A(S)}$ where $Q = N_G(H,\phi)$. We have

$$|Q|\mathrm{Res}_Q^G(x) = \sigma_Q(\beta_Q(\mathrm{Res}_Q^G(x))) = \sigma_Q(\mathrm{Res}_Q^G(\beta_G(x))) = \sigma_Q(\mathrm{Res}_Q^G(f)) =$$

$$\sum_{L\leq I\leq Q} |L|\mu_{L,I} \sum_{\psi\in\hat{I}} f_{I,\psi}[L,\psi|_L]_Q = \sum_{(L,\lambda)\leq(I,\psi)\in\mathcal{M}_Q} |L|.\mu_{L,I} f_{I,\psi}[L,\lambda]_Q.$$

Now $[L,\lambda]_Q = [H,\phi]_Q$ if and only if (L,λ) is $Q-$conjugate to (H,ϕ) or $(H,\phi) = (L,\lambda)$. Hence the coefficient of the basis element $[H,\phi]_Q$ at the right hand side becomes

$$\sum_{(H,\phi)\leq(I,\psi)\in\mathcal{M}_Q} |H|.\mu_{H,I} f_{I,\psi}.$$

Comparing this with the left hand side proves the necessary part of the condition.

Conversely, let $S_f = \{(K,\psi) \in \mathcal{M}_G \mid f_{K,\psi} \neq 0\}$ be the support of f. If S_f is not empty set we set $m(f) := max\{|K| \mid (K,\psi) \in S_f\}$ otherwise if S_f is empty we set $m(f) = 0$. We will prove by induction on $m(f)$ that $f \in \mathrm{im}(\beta_G)$ if it satisfies the integrality condition in the proposition. If $m(f) = 0$ then $f = 0 \in \mathrm{im}(\beta_G)$. Now let's assume that $m(f) > 0$ and take a stabilizing pair (H_i,ϕ_i) $i = 1,\ldots,n$ for each G-conjugacy classes $[H_i,\phi_i]_G \subset S_f$ with $|H_i| = m(f)$. The integrality condition implies that $f_{H_i,\phi_i} = \alpha_i[N_G(H_i,\phi_i):H_i]$ for some $\alpha_i \in \mathbb{Z}$. Now

$$f' = f - \sum_{i-1}^{n} \alpha_i \beta_G([H_i,\phi_i]_G).$$

By definition of the mark homomorphism $(\beta_G[H_i,\phi_i]_G)(K,\psi) = 0$ for all $m(f) < |K|$ and on the conjugacy class of (H_i,ϕ_i) it takes the value $\alpha_i[N_G(H_i,\phi_i) : H_i]$. Hence $m(f') < m(f)$. Since both f and the sum $\sum_{i-1}^{n} \alpha_i\beta_G([H_i,\phi_i]_G)$ satisfy the integrality condition in the proposition, by induction hypothesis we conclude that $f' \in \mathrm{im}(\beta_G)$ and hence f is in the image too as desired. $\square$

In the next proposition we'll determine the cokernel for the mark homomorphism, we begin by introducing the necessary terms. We set $\mathcal{M}_G^* = \{(H,\phi) \in \mathcal{M}_G \mid H < N_G(H,\phi)\}$. Since $\mathcal{M}_G^*$ is stable under $G-$action and the conditions in Proposition 2.5 are equivalent for conjugate pairs, it suffices to check the conditions for a set $\mathcal{R}_G$ of representatives of the G-conjugacy classes in $\mathcal{M}_G^*$. We will see that these conditions for $\mathcal{R}_G$ are minimal. Before we give the proposition we need to define the following map into the cokernel: For each $(H,\phi) \in \mathcal{M}_G$ we set the projection maps

$$\pi_{H,\phi} : \mathfrak{B}^A(G) \to \mathbb{Z}/[N_G(H,\phi) : H]\mathbb{Z}$$

by the rule

$$\Big(\sum_{\psi\in\mathcal{B}_K} f_{K,\psi}.\psi\Big)_{K\leq G} \mapsto \sum_{(H,\phi)\leq(I,\psi)\in\mathcal{M}_{N_G(H,\phi)}} \mu_{H,I}.f_{I,\psi} \; + \; [N_G(H,\phi) : H]\mathbb{Z}.$$

We note that $\pi_{^g(H,\phi)} = \pi_{(H,\phi)}$ for $g \in G$, $(H,\phi) \in \mathcal{M}_G$ and we denote by $\pi = (\pi_{H,\phi})_{(H,\phi)\in\mathcal{R}_G}$ the product of the projection maps into *the fibered obstruction group* of G:

$$\mathrm{Obs}^A(G) = \bigoplus_{(H,\phi)\in\mathcal{R}_G} \mathbb{Z}/[N_G(H,\phi) : H]\mathbb{Z}.$$

Proposition 2.6. *[3, Proposition 2.4] With the above notation one has the short exact sequence*

$$0 \longrightarrow B^A(G) \longrightarrow \mathfrak{B}^A(G) \longrightarrow \mathrm{Obs}^A(G) \longrightarrow 0$$

where the middle maps are the mark homomorphism $\beta_G : B^A(G) \to \mathfrak{B}^A(G)$ *and the projection map* $\pi : \mathfrak{B}^A(G) \to \mathrm{Obs}^A(G)$.

Proof. Here we adopt R. Boltje's original proof for [3, 2.4 Proposition]. We already know that β_G is injective. On the other hand we also know by the integrality conditions in Proposition 2.5 that $f \in \mathrm{im}(\beta_G)$ if and only if $\pi(f) = 0$. Hence we only need to show that π is surjective.

We define functions $e_{H,\phi} \in \mathrm{Obs}^A(G)$ by

$$e_{H,\phi}(K, \psi) = 1 + [N_G(H, \phi) : H]\mathbb{Z}$$

if and only if $(K, \psi) = (H, \phi) \in \mathcal{R}_G$ and 0 otherwise. These functions form the canonical basis of the obstruction group $\mathrm{Obs}^A(G)$. Hence we will prove by induction on $|H|$ that the functions $e_{H,\phi} \in \mathrm{im}(\pi)$ for each $(H, \phi) \in \mathcal{R}_G$. If $|H| = 1$ then $\pi(\chi^G_{(1,1)}) = e_{1,1}$ where the stabilizing pair $(1, 1)$ denotes the trivial character on the trivial group 1. Now let $(H, \phi) \in \mathcal{R}_G$ with $|H| > 1$. By definition we have $\pi_{K,\psi}(H, \phi) = 0$ unless $(K, \psi) \leq {}^g(H, \phi)$ for some $g \in G$. Moreover, $\pi_{H,\phi}(\chi^G_{H,\phi}) = 1 + [N_G(H, \phi) : H]\mathbb{Z}$. Hence $e_{H,\phi} - \pi_{H,\phi}(\chi^G_{H,\phi})$ has nonzero components only for $(K, \psi) \in \mathcal{R}_G$ with $|K| < |H|$. Therefore by induction $e_{H,\phi} - \pi_{H,\phi}(\chi^G_{H,\phi}) \in \mathrm{im}(\pi)$ and hence $e_{H,\phi} \in \mathrm{im}(\pi)$ as desired. $\square$

3. FUSION SYSTEMS

3.1. Preliminaries on fusion systems

In this chapter we will generalize the fibered Burnside ring for fusion systems and then in the final chapter we will conclude by writing the fibered version of the integrality conditions we gave at Chapter 2.

Let p be a prime number and S be a finite p-group. In this section we introduce (saturated) fusion systems over S together with their representations. If G is any group and $P, Q \leq G$ we denote by $\mathrm{Hom}_G(P, Q)$ the set of all injective homomorphisms $\phi : P \to Q$ induced by conjugations in G.

Definition 3.1. *Let $\mathcal{F}$ be a category where the objects are subgroups of S and if $P, Q \leq S$ then $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ is a set of injective group homomorphisms containing the set $\mathrm{Hom}_S(P, Q)$ together with the composition given by composition of homomorphisms. Then $\mathcal{F}$ is called a fusion system over S if any morphism in $\mathcal{F}$ is a composition of an isomorphism in $\mathcal{F}$ followed an inclusion.*

Let $\mathcal{F}$ be a fusion system over S and let $P \leq S$. In $\mathcal{F}$, every morphism $\phi : P \to P$ must be an isomorphism. Hence the set of endomorphisms of P in $\mathcal{F}$ consists only of automorphisms of P. We denote the group of automorphisms of P in $\mathcal{F}$ by $\mathrm{Aut}_{\mathcal{F}}(P)$. Since conjugation by elements of S is always contained in $\mathcal{F}$, the group $\mathrm{Aut}_S(P)$ of automorphisms of P induced by conjugations in S is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. The subgroup P is called *fully automized in $\mathcal{F}$* or *fully $\mathcal{F}$-automized* if $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

Given two subgroups $P, Q \leq S$, we say that P is *$\mathcal{F}$-conjugate* to Q if P and Q are isomorphic in $\mathcal{F}$. In this case, we write $P \sim_{\mathcal{F}} Q$ and denote the $\mathcal{F}$-conjugacy class of P by $[P]_{\mathcal{F}}$. We sometimes consider $[P]_{\mathcal{F}}$ as an S-set via conjugation action of S and hence regard it as a union of S-conjugacy classes of subgroups $\mathcal{F}$-conjugate to P. In particular, if Q and P are $\mathcal{F}$-conjugate but are not S-conjugate, their normalizers in S may not be S-conjugate to each other. We call a subgroup $P \leq S$ *fully normalized in* $\mathcal{F}$ or *fully* $\mathcal{F}$*-normalized* if the order of its normalizer is maximal among those of its $\mathcal{F}$-conjugates.

Example 3.2. *[A Fusion System on D_8] The dihedral group D_8 under conjugations of the symmetric group S_4 is a fusion system, say $\mathcal{F}$. For subgroups of $P \leq D_8$ we take the set of morphisms*

$$\mathrm{Hom}_{\mathcal{F}}(P, D_8) = \mathrm{Hom}_{S_4}(P, D_8) = \{\phi \in \mathrm{Hom}(P, D_8) \ : \ \phi = c_s \text{ for some } s \in S_4\}$$

where c_s denotes the conjugation map by $s \in S$. The subgroups of D_8 were listed in Example 2.1. Subgroups of order 4, V_1, V_2 and C_4, have no $\mathcal{F}-$conjugates, hence each constitute a conjugacy class itself. The center $Z(D_8) \sim_{\mathcal{F}} C_2^1$ via conjugation $c_{(43)} \in \mathcal{F}$ and considering the cycle types of the elements we see that $C_2^3 \nsim_{\mathcal{F}} C_2^1$. Hence $[C_2^1]_{\mathcal{F}} = \{C_2^1, C_2^2, Z(D_8)\}$ and $[C_2^3]_{\mathcal{F}} = \{C_2^3, C_2^4\}$ are the two $\mathcal{F}-$conjugacy classes of subgroups of order 2. Finally the trivial group form a conjugacy class itself.

Since $N_{S_4}(D_8) = D_8$ the outer and inner automorphisms of D_8 in $\mathcal{F}$ coincide, that is we have $\mathrm{Aut}_{\mathcal{F}}(D_8) = \mathrm{Aut}_{D_8}(D_8) = D_8/Z$. On the other hand for subgroups of order 4 we get $\mathrm{Aut}_{\mathcal{F}}(V_1) = \{\mathrm{id}_{V_1}, c_{(13)}, c_{(14)}, c_{(23)}, c_{(24)}, c_{(34)}\}$, $\mathrm{Aut}_{\mathcal{F}}(V_2) = \mathrm{Aut}_{D_8}(V_2) = \{\mathrm{id}_{V_2}, c_{(12)(34)}\}$ and $\mathrm{Aut}_{\mathcal{F}}(C_4) = \mathrm{Aut}_{D8}(C_4) = \{\mathrm{id}_{C_4}, c_{(13)}\}$. Each of them being normal subgroups of D_8 are fully normalized and since $\mathrm{Aut}_{D_8}(V_1) = \{\mathrm{id}_{V_1}, c_{(13)}\}$ is a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(V_1)$ they are also fully automized. Finally the subgroups of order 2 all have trivial automorphism groups in $\mathcal{F}$ and hence they are all fully automized. However since $N_{D_8}(C_2^i) = V_1 < D_8 = N_{D_8}(Z(D_8))$ only C_2^3, C_2^4 and $Z(D_8)$ are fully normalized.

Given an isomorphism $\phi : Q \to P$ in $\mathcal{F}$, we write

$$N_\phi = \{g \in N_S(Q) : {}^{\phi}c_g \in \mathrm{Aut}_S(P)\}$$

where c_g denotes the conjugation by $g \in G$. With this notation, the subgroup $P \leq S$ is called *$\mathcal{F}$-receptive* if for any $Q \leq S$ and any isomorphism $\phi : Q \to P$, there is a morphism $\bar{\phi} : N_\phi \to S$ in $\mathcal{F}$ extending ϕ.

Now we define saturation following [11].

Definition 3.3. *A fusion system $\mathcal{F}$ over S is called saturated if each subgroup of S is conjugate in $\mathcal{F}$ to a fully $\mathcal{F}$-automized and $\mathcal{F}$-receptive subgroup.*

A natural example of a saturated fusion system is the category $\mathcal{F}_S(G)$ defined as follows. Let G be a finite group which contains S as a Sylow p-subgroup. We define $\mathcal{F}_S(G)$ as the fusion system with morphisms induced by conjugations by elements in G, that is, we put

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q) = \{\phi : P \to Q | \exists g \in G : \phi = c_g\}.$$

It is straightforward to check that $\mathcal{F}_S(G)$ is a fusion system and it is proved by Puig that $\mathcal{F}_S(G)$ is saturated.

An alternative definition of saturated fusion systems, which we will use in the next chapter, can be given in terms of Sylow p-subgroups of $\mathcal{F}-$automorphisms and extension property of morphisms:

Definition 3.4. *[12, Definition 1.37] Let S be a p-group. A fusion system $\mathcal{F}$ over S is called saturated if it satisfies the following axioms:*

(i) Sylow axiom: $\mathrm{Aut}_S(S)$ *is a Sylow p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(S)$

(ii) Extension axiom: Every morphism $\phi : Q \to S$ with Q fully normalized extends to a morphism $\bar{\phi} : N_\phi \to S$.

There are several other equivalent definitions of saturated fusion systems. We refer to [11] for further details.

Example 3.5 ($\mathcal{F}_{D_8}(S_4)$ is saturated)**.** *This fusion system on the dihedral group D_8 is the same as in Example 3.2. As we mentioned above we already know it is a saturated fusion system by construction; i.e. such fusion systems are in general proven to be saturated by Puig (cf. [12][Theorem 1.39]). Nevertheless we can easily check the Sylow axiom since the inner and outer automorphism groups coincide:*

$$\mathrm{Aut}_{\mathcal{F}}(D_8) = \mathrm{Aut}_{D_8}(D_8) = D_8/Z(D_8).$$

Moreover, we can also check the extension axiom on the fully normalized subgroups with some calculations: First of all it holds for the identity automorphisms for obvious reasons. Otherwise $c_{(12)(34)}$ is already defined on $N_{c_{(12)(34)}} = V_1$, similarly for $c_{(14)}, c_{(23)}, c_{(43)}$ we have $N_{c_{(14)}} = N_{c_{(23)}} = N_{c_{(43)}} = V_2$ and the extension axiom is satisfied. On the other hand for $c_{(13)}$ and $c_{(24)}$ we have $N_{c_{(13)}} = N_{c_{(42)}} = D_8$ but we already know that $(13), (24) \in D_8$ and extension is no problem.

Such saturated fusion systems with an ambient finite group G are called *realizable*. Realizable fusion systems reflect the notion of fusion in its original setting of the finite group theory. However, as mentioned above the fusion systems in their categorical setting are more abstract and general structures than the original notion of fusion in groups. That is we expect that not all saturated fusion systems over S are realizable, i.e. there are *exotic* fusion systems defined to be non-realizable. One of the simplest examples of exotic fusion systems are constructed by Ruiz and Viruel [13]:

Example 3.6 (Exotic Fusion Systems)**.** *We recall that a group of prime power order (or, more generally, any p-group) is termed extraspecial if its center, derived subgroup and Frattini subgroup all coincide, and moreover, each of these is a group of prime order (and hence, a cyclic group). If S is extraspecial of order p^3 and exponent p then S contains $p+1$ subgroups $V_1, \ldots, V_{p+1}$ of order p^2 each of which is isomorphic to C_p^2. It is known by Alperin's Fusion Theorem that each saturated fusion system $\mathcal{F}$ over S is determined by the outer automorphisms $\mathrm{Out}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Inn}(P)$, together with the groups $\mathrm{Aut}_{\mathcal{F}}(V_i)$ for those V_i which are $\mathcal{F}$-radical. Starting with this idea and using the saturation axioms Ruiz and Viruel gave a complete classification of all saturated fusion* *systems over S for any odd prime p. They so constructed an example of a saturated fusion system $\mathcal{F}$ over S for $p = 7$ with the outer automorphisms $\mathrm{Out}_{\mathcal{F}}(S) = D_{16} \times C_3$ which have 4 $\mathcal{F}$ - conjugacy classes containing 4 $\mathcal{F}-$radical subgroups among V_i with $\mathrm{Aut}(V_i) = SL_2(p) \rtimes C_2$ and which is not realizable. The proof of this relies on the classification of the finite simple groups, cf. [12, Table 9.2], the details can be found in the original article of Ruiz and Viruel [13].*

To speak further on saturated fusion systems we will also need the concept of being *fully centralized*:

Definition 3.7. *Let $\mathcal{F}$ be a fusion system over a finite p-group S and $P \leq S$. Then the subgroup P is said to be fully centralized if for every isomorphism $\phi : P \to Q$ we have*

$$|C_S(Q)| \leq |C_S(P)|$$

where $C_S(P)$ denotes the centralizer of P in S.

A generalization of fully normalized and fully centralized subgroups is given in [11]. We recall this notion of fully K-normalized subgroups. Let $\mathcal{F}$ be a fusion system over S and Q be a subgroup of S. Also let $K \leq \mathrm{Aut}(Q)$ be a group of automorphisms of Q and let $\phi : Q \to S$ be a morphism in $\mathcal{F}$. We put

(i) $\mathrm{Aut}^K_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q) \cap K$,

(ii) $\mathrm{Aut}^K_S(Q) = \mathrm{Aut}_S(Q) \cap K$,

(iii) $N^K_S(Q) = \{x \in N_S(Q) : c_x \in \mathrm{Aut}^K_S(Q)\}$,

(iv) ${}^{\phi}K = \{{}^{\phi}\chi : \chi \in K\} \subseteq \mathrm{Aut}(\phi(Q))$.

With this notation, we have the following definition.

Definition 3.8. *The subgroup Q is said to be fully K-normalized in $\mathcal{F}$ if for any* *morphism* $\phi : Q \to S$ *in* $\mathcal{F}$, *we have*

$$|N_S^K(Q)| \geq |N_S^{\phi K}(\phi(Q))|.$$

Clearly a subgroup Q is fully normalized in $\mathcal{F}$ if and only if it is fully $\mathrm{Aut}(Q)$-normalized in $\mathcal{F}$. Also Q is fully centralized in $\mathcal{F}$ if and only if it is fully $\{\mathrm{id}_Q\}$-normalized in $\mathcal{F}$.

Fully K-normalized subgroups are fundamental to construct normalizer subsystems. The following proposition introduces two more characterizations when $\mathcal{F}$ is saturated.

Proposition 3.9. *[12, Lemma 4.35] Let* $\mathcal{F}$ *be a saturated fusion system over* S *and* $Q \leq S$. *Also let* $K \leq \mathrm{Aut}(Q)$. *Then the following are equivalent.*

(i) Q *is fully* K*-normalized in* $\mathcal{F}$.
(ii) Q *is fully centralized in* $\mathcal{F}$ *and* $\mathrm{Aut}_S^K(Q)$ *is a Sylow* p*-subgroup of* $\mathrm{Aut}_{\mathcal{F}}^K(Q)$.
(iii) For each $P \leq S$ *and for each isomorphism* $\phi : P \to Q$ *in* $\mathcal{F}$, *there are homomorphisms* $\chi \in \mathrm{Aut}_{\mathcal{F}}^K(Q)$ *and* $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(P \cdot N_S^{K^\phi}(P), S)$ *such that* $\bar{\phi}|_P = \chi \circ \phi$.

Similarly Reeh gives the following extension property for fully normalized groups.

Lemma 3.10. *[1, Lemma 2.3] Let* $\mathcal{F}$ *be a saturated fusion system and* $P \leq S$ *be a fully normalized subgroup. Then for any* $\phi : Q \to P$ *in* $\mathcal{F}$, *there is a lifting* $\tilde{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ *such that* $\tilde{\phi}|_Q = \phi$.

He uses this lemma in the proof of the main theorem to write a free basis of the Burnside ring in terms of fully normalized representatives of the $\mathcal{F}$- conjugacy classes of $P \leq S$. Later by Propositon 3.9 we will prove and use a fibered version of this technical lemma in Chapter 4.

3.2. Modules over fusion systems

In this thesis, we are interested in fusion stable elements in Mackey functors. For this aim, we introduce modules over a fusion system.

Definition 3.11. *Let $\mathcal{F}$ be a fusion system over S. A contravariant functor $F : \mathcal{F} \to Ab$ is called an $\mathcal{F}$-module. Given two $\mathcal{F}$-modules F and G, an $\mathcal{F}$-homomorphism $F \to G$ is a natural transformation from F to G. We denote the category of all $\mathcal{F}$-modules by* $\mathrm{mod}_{\mathcal{F}}$.

In other words an $\mathcal{F}$-module F is a collection of abelian groups $(F(P))_{P \leq S}$ together with group homomorphisms $F(\phi) : F(Q) \to F(P)$ for each morphism $\phi : P \to Q$ in $\mathcal{F}$ such that if $\psi : Q \to R$ is another morphism in $\mathcal{F}$, then $F(\psi \circ \phi) = F(\phi) \circ F(\psi)$. Note that each evaluation $F(P)$ becomes a $\mathbb{Z}[\mathrm{Aut}_{\mathcal{F}}(P)]$-module with the action of $\mathrm{Aut}_{\mathcal{F}}(P)$ given by $F(\phi)$ for each $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$.

It is possible to define an $\mathcal{F}$-module as a module over the category algebra of $\mathcal{F}$. We include this definition. Let k be a commutative ring with unity. Consider the free algebra over k generated by the following list of symbols.

(i) Res^P_Q for each $Q \leq P \leq S$,
(ii) $\mathrm{Iso}^{\phi}_{Q,P}$ for each isomorphism $\phi : Q \to P$ in $\mathcal{F}$.

Then we define the category algebra $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ over k as the quotient of this algebra by the ideal generated by the following relations.

(i) $\mathrm{Res}^Q_R \mathrm{Res}^P_Q = \mathrm{Res}^P_R$ for each $R \leq Q \leq P \leq S$.
(ii) $\mathrm{Iso}^{\psi}_{R,Q} \mathrm{Iso}^{\phi}_{Q,P} = \mathrm{Iso}^{\phi\psi}_{R,P}$ for all isomorphisms $\phi : Q \to P$ and $\psi : R \to Q$.
(iii) $\mathrm{Res}^P_Q \mathrm{Iso}^{\phi}_{P,R} = \mathrm{Iso}^{\phi}_{Q,\phi^{-1}(Q)} \mathrm{Res}^R_{\phi^{-1}(Q)}$ for each $Q \leq P$ and for each isomorpism $\phi : R \to P$.
(iv) $\mathrm{Res}^P_P = \mathrm{Iso}^{\mathrm{id}}_{P,P}$ for each $P \leq S$.
(v) $1 = \sum_{P \leq S} \mathrm{Res}^P_P$.

It is clear that any element in $\Gamma_{\mathcal{F}}$ is a linear combination of elements of the form $\mathrm{Iso}^{\phi}_{R,Q}\mathrm{Res}^{P}_{Q}$ with $Q \leq P \leq S$ and $\phi : Q \to R$ an isomorphism in $\mathcal{F}$. Since any morphism in $\mathcal{F}$ is the composition of an $\mathcal{F}$-isomorphism followed by an inclusion, any morphism in $\mathcal{F}$ is represented by such an element in $\Gamma_{\mathcal{F}}$ and vice versa, any such element in $\Gamma_{\mathcal{F}}$ corresponds to a morphism in $\mathcal{F}$. Thus $\Gamma_{\mathcal{F}}$ can also be considered as the algebra generated by all morphisms in $\mathcal{F}$. As shown in [21, 2.1 Proposition], the module category of $\Gamma_{\mathcal{F}}$ is equivalent to the category ${}_{\mathcal{F}}\mathrm{mod}$ of modules over $\mathcal{F}$. Thus an $\mathcal{F}$-module can be constructed by specifying an abelian group for each subgroup $P \leq S$ together with restriction maps Res^{P}_{Q} for each $Q \leq P \leq S$ and isogation maps $\mathrm{Iso}^{\phi}_{P,Q}$ for each isomorphism $\phi : Q \to P$ in $\mathcal{F}$ subject to the above conditions. Stated that way we could alternatively define $\mathcal{F}-$module in terms of a restriction functor satisfying some extra conditions:

Definition 3.12 (*Restriction functors* over a fusion system (see Section 2.2))**.** *A restriction functor over a fusion system $\mathcal{F}$ can be defined as a restriction functor on the p-group S with conjugation maps c^{ϕ}_{H} for each $\phi : H \to S$ in $\mathcal{F}$ which further satisfy the following axioms:*

(i) $c^{\phi}_{H} \circ \mathrm{Res}^{H}_{I} = \mathrm{Res}^{\phi H}_{\phi I} \circ c^{\phi}_{H}$ *for* $I \leq H \leq S$, $\phi : H \to S$ *in* $\mathcal{F}$

(ii) $c^{\phi}_{H} \circ c^{\psi}_{K} = c^{\phi\psi}_{K}$ *for* $\phi : H \to S$ *and* $\psi : K \to S$ *in* $\mathcal{F}$ *such that* ${}^{\psi}K = H$.

In general we see that a restriction functor over $\mathcal{F}$ as we defined above determines an $\mathcal{F}-$module and vice versa, where we take the symbols Res^{P}_{Q} the restrictions for $Q \leq P$ and $Iso^{\phi}_{Q,P} = c^{\phi}_{Q}$ for $\phi : Q \to P$ in $\mathcal{F}$.

3.3. Examples of $\mathcal{F}-$modules

Example 3.13. *(Constant $\mathcal{F}$-module). Let $\underline{\mathbb{Z}}$ denote the $\mathcal{F}$-module where $\underline{\mathbb{Z}}(P) = \mathbb{Z}$ for each $P \leq S$ and $\underline{\mathbb{Z}}(\phi) = \mathrm{id}$, the identity homomorphism, for each $\phi : Q \to P$ in $\mathcal{F}$. Clearly $\underline{\mathbb{Z}}$ is an $\mathcal{F}$-module. We call it the constant $\mathcal{F}$-module.*

Example 3.14. *(Globally-defined Mackey functors for $\mathcal{F}$) [14, pp.23-24] [15, Mackey functors for fusion systems]. Given a globally defined Mackey functor M for $\mathcal{F}$ over*

$\mathbb{Z}$ with respect to the trivial sets of finite groups $\mathcal{X} = \mathcal{Y} = \{1\}$, M specifies an abelian group $M(P)$ for each subgroup $P \leq S$ together with restriction maps $\mathrm{Res}_Q^P = \iota^*$ for each inclusion $\iota : Q \to P$ and isogation maps $Iso_{P,Q}^{\phi} = \phi^*$ for each isomorphism $\phi : Q \to P$ in $\mathcal{F}$ subject to the above conditions [14, page 24]. Hence a globally defined functor M is in particular an $\mathcal{F}$-module.

Example 3.15. *(Simple $\mathcal{F}$-modules) [14, 9.1 Theorem] Let k be a field, F an $\mathcal{F}$-module and let Q be a subgroup of minimal order such that $F(Q) \neq 0$. Consider the submodule $\langle V \rangle$ of $\mathcal{F}$ generated by any non-zero $\mathrm{Aut}_{\mathcal{F}}(Q)$-submodule $V \subseteq F(Q)$. Then $\langle V \rangle$ is proper whenever V is proper or there is a subgroup $P \leq S$ not $\mathcal{F}$-conjugate to Q such that $F(P)$ is non-zero. Therefore simple $\mathcal{F}$-modules are of the form $S_{Q,V}$ where $Q \leq S$ and V is a simple $k[\mathrm{Aut}_{\mathcal{F}}(Q)]$-module, $S_{Q,V}(Q) = V$ and $S_{Q,V}(P) = 0$ is zero if P is not $\mathcal{F}$-conjugate to Q.*

Example 3.16. *(Projective or free $\mathcal{F}$- modules) [16, 9. Modules over a category and a splitting of projectives] Let $\mathrm{Ob}(\mathcal{F})$ denote the objects of a fusion system $\mathcal{F}$. The objects $\mathrm{Ob}(\mathcal{F})$ with identity morphisms form a category. An $\mathcal{F} -$ set is a functor $\mathrm{Ob}(\mathcal{F}) \to \mathbf{Set}$, and maps between $\mathcal{F}-$sets are natural transformations. An $\mathcal{F}$-module M has the underlying $\mathcal{F}$-set $\mathrm{Ob}(\mathcal{F})$. Following [16], we denote this $\mathcal{F}$-set by B. An $\mathcal{F}$-module M is free with an $\mathcal{F}$-set B as basis if for any $\mathcal{F}$-module N and any map $f : B \to N$ of $\mathcal{F}$-sets there is exactly one $\mathcal{F}-$module homomorphism $\bar{f} : M \to N$ extending f. And the following statements are equivalent for an $\mathcal{F}$-module P:*

(i) P is projective

(ii) Every exact sequence $0 \to M \to N \to P \to 0$ splits.

(iii) $\mathrm{Hom}_{\mathcal{F}}(P, -)$ is exact.

(iv) P is a direct summand of a free $\mathcal{F}$-module.

A description of the projective $\mathcal{F}-$modules can be given in terms of their representations in suitable group-ring modules [16, 9]; i.e. a projective $\mathcal{F}$-module P is *locally represented by an* $\mathbb{Z}[\mathrm{Aut}(x)]-$*module for each* $x \in \mathrm{Ob}\mathcal{F}$*. Let* x, y *be two objects of* $\mathcal{F}$*, like [16], let* $P(x)_s$ *denote the abelian group generated by the images of* $P(f) : M(y) \to M(x)$ *for non-isomorphic* $f : x \to y$ *in* $\mathcal{F}$*. We define the splitting functor* $S_x : {}_{\mathcal{F}}\mathrm{mod} \to \mathrm{mod}\ \mathbb{Z}[X]$ *by* $M \mapsto M(x)/M(x)_s$*. On the other hand any abelian group can be extended to an* $\mathcal{F}-$*module; we define the extension functor* $E_x : \mathbb{A} \to {}_{\mathcal{F}}\mathrm{mod}$ *by* $M(y) \mapsto M \otimes_{\mathbb{Z}[x]} \mathbb{Z}\mathrm{Hom}_{\mathcal{F}}(y, x)$*. By [16, Corollary 9.40] we know that*

$$P \simeq \bigoplus_{[x] \in \mathrm{Iso}P} E_x \circ S_x P.$$

where $\mathrm{Iso}P$ *is the set of* $\mathcal{F}$*-isomorphism classes* $[x]$ *such that* $S_x P \neq 0$.

At last we note that later, after we define the fibered-Burnside ring, we will show that it is a free $\mathcal{F}-$*module on an '*$\mathcal{F}$*-basis' .*

3.4. Fusion stable elements

In this section, we introduce fusion stable elements.

Definition 3.17. *Let* $\mathcal{F}$ *be a fusion system over* S *and* M *an* $\mathcal{F}$*-module. We define the group* $M(\mathcal{F})$ *of* $\mathcal{F}$*-stable elements in* M *by*

$$M(\mathcal{F}) = \{x \in M(S) | \mathrm{Res}^S_P x = \mathrm{Iso}^{\phi}_{P,\phi(P)} \mathrm{Res}^S_{\phi(P)} x \text{ for any } \phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\}.$$

Equivalently we could see $M(\mathcal{F})$ as the projective limit of the functor M over the category $\mathcal{F}$:

Lemma 3.18. $M(\mathcal{F}) = \varprojlim_{\mathcal{F}} M$ *as groups.*

Proof. The projective limit amounts to be a subgroup of the product $\prod_{P\leq S} M(P)$:

$$\varprojlim_{\mathcal{F}} M = \{(a_P)_{P\leq S} \in \prod_{P\leq S} M(P) \mid a_P = \phi_{PQ}(a_Q)\}$$

where $(P, Q \leq S,\, \phi_{PQ})$ is the inverse system of all objects directed under $\mathcal{F}-$ subconjugation and the morphisms $\phi_{PQ} = \mathrm{Iso}_{P,\phi(P)} \circ \mathrm{Res}^{Q}_{\phi P} : M(Q) \to M(P)$ for $\phi : P \to Q$ in $\mathcal{F}$. To get the equality of groups we see $\mathcal{F}-$stable elements $\mathbf{x} \in M(\mathcal{F})$ as vectors $(x_P)_{P\leq S} \in \prod_{P\leq S} M(P)$ where $x_P = \mathrm{Res}^S_P \mathbf{x} \in M(P)$. Now the result follows easily: $\mathbf{x} \in M(\mathcal{F})$ if and only if

$$x_P = \mathrm{Res}^S_P \mathbf{x} = \mathrm{Iso}^{\phi}_{P,\phi(P)} \mathrm{Res}^S_{\phi(P)} \mathbf{x} = \mathrm{Iso}^{\phi}_{P,\phi(P)} \mathrm{Res}^Q_{\phi(P)} x_Q = \phi_{PQ} x_Q$$

for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ as desired. $\square$

Another characterization for the limit in the above definition uses the constant $\mathcal{F}$-module. Given an $\mathcal{F}$-module M, we have an isomorphism of abelian groups

$$\varprojlim_{\mathcal{F}} M \cong \mathrm{Hom}_{\mathrm{mod}_{\mathcal{F}}}(\underline{\mathbb{Z}}, M).$$

An isomorphism is given by associating a natural transformation $\eta : \underline{\mathbb{Z}} \to M$ to $\eta_S(1) \in M(S)$. Since $\underline{\mathbb{Z}}$ is generated, as an $\mathcal{F}$-module, by $1 \in \underline{\mathbb{Z}}(S)$, this image uniquely determines the natural transformation η. Conversely let $(x_P)_{P\leq S} \in \varprojlim_{\mathcal{F}} M$, we define a natural transformation $\eta \in \mathrm{Hom}_{\mathrm{mod}_{\mathcal{F}}}(\underline{\mathbb{Z}}, M)$ by $\eta_S(1) = x_S \in M(S)$. These isomorphisms of abelian groups are inverse to each other. With this definition, if a short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} N' \longrightarrow 0\,,$$

of $\mathcal{F}$-modules is given, we may apply the covariant functor $\mathrm{Hom}_{\mathrm{mod}_{\mathcal{F}}}(\underline{\mathbb{Z}}, -)$ to obtain a long exact sequence

$$0 \longrightarrow N(\mathcal{F}) \xrightarrow{\iota} M(\mathcal{F}) \xrightarrow{\pi^*} N'(\mathcal{F}) \longrightarrow \mathrm{Ext}^1_{\mathrm{mod}_{\mathcal{F}}}(\underline{\mathbb{Z}}, N) \longrightarrow \ldots$$

Groups of $\mathcal{F}$-stable elements are considered in special cases by many authors. In [17], $\mathcal{F}$-stable elements in the Dade group are considered together with the gluing problem. In [1], Reeh considered $\mathcal{F}$-stable S-sets and showed that there is a free basis for the monoid of $\mathcal{F}$-stable S-sets and hence constructed a Burnside ring for $\mathcal{F}$. In the following chapter we will follow his line of thought in detail for fibered $S-$sets. In [4], $\mathcal{F}$-stable elements in cohomology groups are considered by Broto, Levi and Oliver. They showed that the cohomology group $H^*(X; \mathbb{F}_p)$ of a p-complete space X is isomorphic to a certain group of '*stable elements*$'$ in $H^*(BS; \mathbb{F}_p)$ where BS is a Sylow subgroup of X. In [18], Reeh and Yalci n considered $\mathcal{F}$-stable elements in representation rings, ghost rings in relation with the group of Borel-Smith functions, which are constant on $\mathcal{F}$ -conjugacy classes of subgroups of S.

3.5. $\mathcal{F}$-stable elements in fibered Burnside groups

Let S be a finite p-group and $\mathcal{F}$ a saturated fusion system over S. In this section we describe $\mathcal{F}$-stable elements in $B^A(S)$ and $\mathfrak{B}^A(S)$ and determine a free basis for the monoid of $\mathcal{F}$-stable A-fibered S-sets. We follow Reeh's methods from [1].

We begin with the description of an $\mathcal{F}$-module structure on $B^A(G)$. We denote by B^A the $\mathcal{F}$-module where for any $P \leq S$, we let $B^A(P)$ be the A-fibered Burnside group of P, and for each $\phi : Q \to P$, we define

$$\mathrm{Res}_\phi : B^A(P) \to B^A(Q)$$

by

$$\mathrm{Res}_\phi([R,\lambda]_P) = \sum_{x\in[\phi(Q)\backslash P/R]} [\phi^{-1}({}^xR), {}^x\lambda \circ \phi\mid_{\phi^{-1}({}^xR)}]_Q$$

for any $[R,\lambda]_P \in B^A(P)$ (cf. [19]). In the special case, when ϕ is an inclusion, the above formula becomes the usual restriction, and if ϕ is an isomorphism, it reduces to the following formula.

$$\mathrm{Iso}^{\phi}_{Q,P}([R,\lambda]_P) = [\phi^{-1}(R), \lambda\circ\phi|_{\phi^{-1}(R)}]_Q$$

Definition 3.19. *The $\mathcal{F}$-stable elements $B^A(\mathcal{F})$ in $B^A(S)$ is called the A-fibered Burnside group of $\mathcal{F}$.*

Following Reeh [1], it is possible to construct another candidate for the A-fibered Burnside group of $\mathcal{F}$. Recall that the Grothendieck group $B^A(S)$ is constructed as the group completion of the monoid of isomorphism classes of A-fibered S-sets. Hence we can first consider the $\mathcal{F}$-stable elements in this monoid and then take the group completion. We denote this group by $\widetilde{B}^A(\mathcal{F})$. It is shown in [1] that for the case of Burnside groups, these two constructions coincide for saturated fusion systems. Our aim is to generalize this result to fibered Burnside groups by constructing a free basis to the monoid of $\mathcal{F}$-stable A-fibered S-sets. As in the case of $\mathcal{F}$-stable S-sets, we need to consider the ghost ring and mark morphism for $\mathcal{F}$-stable A-fibered S-sets.

Note that the assignment $P \mapsto \mathfrak{B}^A(P)$ becomes an $\mathcal{F}$-module together with the usual definition of restriction of functions through a group homomorphism. To be more precise, if $\phi: Q\to P$ and $f\in\mathfrak{B}^A(P)$ are given, we write $\mathrm{Res}_\phi f$ instead of $\mathfrak{B}^A(\phi)(f)$ and for each $(R,\nu)\in\mathcal{M}^A_Q$, define $(\mathrm{Res}_\phi f)(R,\nu) = f(\phi(R), \nu\circ\phi^{-1}|_{\phi(R)})$.

To find a basis for the ghost ring, we extend the $\mathcal{F}$-conjugacy to the pairs (P,λ)

as follows. Given a morphism $\phi : P \to Q$ in $\mathcal{F}$, we put

$$^{\phi}(P, \lambda) = (\phi(P), \lambda \circ (\phi^{-1}|_{\phi(P)})).$$

With this definition two pairs $(P, \lambda), (Q, \kappa) \in \mathcal{M}_S^A$ are said to be $\mathcal{F}$*-conjugate*, written $(P, \lambda) \sim_{\mathcal{F}} (Q, \kappa)$, if there is an isomorphism $\phi : P \to Q$ in $\mathcal{F}$ such that $^{\phi}(P, \lambda) = (Q, \kappa)$. In this case we write $\phi : (P, \lambda) \to (Q, \kappa)$ and denote the $\mathcal{F}$-conjugacy class of (P, λ) by $[P, \lambda]_{\mathcal{F}}$. In general this class is a union of S-conjugacy classes of pairs $\mathcal{F}$-conjugate to (P, λ). With these definitions, the following result is immediate.

Proposition 3.20. *Let $\mathcal{F}$ be a fusion system over S and $\mathfrak{B}^A(S)$ the ghost group of the fibered Burnside group for S. Then a function $f \in \mathfrak{B}^A(S)$ is $\mathcal{F}$-stable if and only if it is constant on $\mathcal{F}$-conjugacy classes.*

Proof. Let $f \in \mathfrak{B}^A(S)$ be $\mathcal{F}-$stable. Then by definition for any $\phi : P \to S$ in $\mathcal{F}$, we have $\mathrm{Res}_{\phi} f = \mathrm{Res}_P^S f$. Equivalently, for any $(Q, \kappa) \in \mathcal{M}_P^A$, we have $(\mathrm{Res}_{\phi} f)(Q, \kappa) = (\mathrm{Res}_P^S f)(Q, \kappa)$.

In particular, if $\phi : Q \to P$ is an isomorphism in $\mathcal{F}$ with $^{\phi}(Q, \kappa) = (P, \lambda)$, then we get

$$f_{(P,\lambda)} = f_{^{\phi}(Q,\kappa)} = (\mathrm{Res}_{\phi} f)(Q, \kappa) = (\mathrm{Res}_Q^S f)(Q, \kappa) = f_{(Q,\kappa)}.$$

Hence f is constant on $\mathcal{F}-$conjugacy classes in $\mathcal{M}_S^A$.

Conversely let's suppose that $f \in \mathfrak{B}^A(S)$ is constant on $\mathcal{F}-$conjugacy classes in $\mathcal{M}_S^A$ and let $\phi : P \to S$ be a morphism in $\mathcal{F}$. We need to prove that it is $\mathcal{F}-$stable, or $\mathrm{Res}_{\phi} f = \mathrm{Res}_P^S f$. Let $(Q, \kappa) \in \mathcal{M}_P^A$. Then

$$(\mathrm{Res}_P^S f)(Q, \kappa) = f_{(Q,\kappa)} = f_{^{\phi}(Q,\kappa)} = (\mathrm{Res}_{\phi} f)(Q, \kappa).$$

$\square$

Definition 3.21. *The subgroup $\mathfrak{B}^A(\mathcal{F})$ of $\mathfrak{B}^A(S)$ consisting of functions constant on $\mathcal{F}$-conjugacy classes in $\mathcal{M}_S^A$ is called the A-fibered ghost group of $\mathcal{F}$.*

We note that for any $x \in B^A(P)$ and for any injective group homomorphism $\phi : Q \to P$ in $\mathcal{F}$ we can write $\mathrm{Res}_\phi(x) = \mathrm{Iso}^{\phi}_{Q,\phi(Q)}\mathrm{Res}^P_{\phi Q}x$. Hence using the generalized version of the mark morphism

$$\beta_P(x) = (\pi_K \mathrm{Res}^P_K x)_{K \leq P}$$

we introduced at Section 2.2 we get

$$\beta_Q(\mathrm{Res}_\phi(x)) = (\pi_R \mathrm{Res}^Q_R \mathrm{Iso}^{\phi}_{Q,\phi(Q)}\mathrm{Res}^P_{\phi(Q)}x)_{R \leq Q} =$$

$$(\mathrm{Iso}^{\phi|_R}_{R,\phi(R)}\pi_{\phi(R)}\mathrm{Res}^{\phi(Q)}_{\phi(R)}\mathrm{Res}^P_{\phi(Q)}x)_{R\leq Q} = (\mathrm{Iso}^{\phi|_R}_{R,\phi(R)}\pi_{\phi(R)}\mathrm{Res}^P_{\phi(R)}x)_{R\leq Q} =$$

$$\mathrm{Res}_\phi(\pi_K \mathrm{Res}^P_K x)_{K\leq P} = \mathrm{Res}_\phi \beta_P(x).$$

In other words, the mark morphism commutes with generalized restrictions Res_ϕ for any injective group homomorphism $\phi : Q \to P$. Now if $x \in B^A(\mathcal{F})$ and $\phi : P \to S$ is in $\mathcal{F}$ we get

$$\mathrm{Res}_\phi \beta_S(x) = \beta_{\phi(P)}(\mathrm{Res}_\phi x) = \beta_{\phi(P)}(\mathrm{Iso}^{\phi}_{(P,\phi(P))}\mathrm{Res}^S_P x) =$$

$$\beta_{\phi(P)}(\mathrm{Res}^S_{\phi(P)}x) = \mathrm{Res}^S_{\phi(P)}\beta_S(x);$$

i.e. $\beta_S(x) \in \mathfrak{B}^A(\mathcal{F})$. Thus the mark homomorphism β_S restricts to a homomorphism $\beta_{\mathcal{F}} : B^A(\mathcal{F}) \to \mathfrak{B}^A(\mathcal{F})$, still called the mark homomorphism. The following result gives a criterion for an element in $B^A(S)$ to be $\mathcal{F}$-stable. A version of this proposition is proved for S-sets by Reeh [1, Lemma 4.1].

Proposition 3.22. *Let $\mathcal{F}$ be a fusion system over S. Given an element $x \in B^A(S)$, then x is $\mathcal{F}$-stable if and only if $\beta_S^A(x)$ is $\mathcal{F}$-stable.*

Proof. We already know by the remark above that the criterion is necessary; i.e. the mark homomorphism has image in $\mathfrak{B}^A(\mathcal{F})$. Conversely let us suppose that $\beta_S(x)$ is $\mathcal{F}-$stable. Then $\mathrm{Res}_\phi \beta_S(x) = \mathrm{Res}_P^S \beta_S(x)$ for any $\phi : P \to S$ in $\mathcal{F}$. Since β_S commutes with restriction maps we get $\beta_P(\mathrm{Res}_\phi(x)) = \beta_P(\mathrm{Res}_P^S x)$, or since β_P is injective $\mathrm{Res}_\phi(x) = \mathrm{Res}_P^S x$; i.e. x is $\mathcal{F}-$stable. $\square$

4. MAIN RESULTS

Let's fix a prime number p, a $p-$group S and and an abelian group A. Let $\mathcal{F}$ be a saturated fusion system over S and $B^A(\mathcal{F})$ the A-fibered Burnside ring of $\mathcal{F}$ as in the previous chapter. In this final chapter we will prove that $B^A(\mathcal{F})$ is free on a basis of irreducible fibered sets and we will write these sets inductively following J.S. Reeh's original proof for the Burnside ring [1]. We will have to do some adaptations for the non-trivial rings of sub-characters $\lambda : P \to A$, the first two sections will be these preliminaries besides some basic results to introduce $\mathrm{Obs}^A(\mathcal{F})$, the *A-fibered obstruction group of* $\mathcal{F}$. The obstruction group $\mathrm{Obs}^A(\mathcal{F})$ is an $\mathcal{F}-$module, in the final section we will conclude that it is the cokernel of the fibered mark homomorphism by calculating a basis for $\mathcal{F}-$stable fibered sets inductively, following the original proof of Reeh [1] and the definitions below for fibered sets. Finally we should note that after embedding the Burnside ring into the ghost ring, which has the easier componentwise product, one could continue to calculate the idempotents of the fibered Burnside ring of $\mathcal{F}$ by studying the characteristic functions on the $\mathcal{F}-$conjugacy classes which are in the image of the fibered Burnside ring.

4.1. Preliminaries

In the rest of the thesis, a version of fully normalized subgroups for the pairs (P, λ) is necessary. The next definition and the following results introduce this notion together with basic properties.

Definition 4.1. *Let $\mathcal{F}$ be a fusion system over S and $(P, \lambda) \in \mathcal{M}_S^A$. We call (P, λ) fully stabilized in $\mathcal{F}$ or fully $\mathcal{F}$-stabilized if for any $(Q, \kappa) \sim_{\mathcal{F}} (P, \lambda)$, we have*

$$|N_S(P, \lambda)| \geq |N_S(Q, \kappa)|.$$

Clearly if λ is the trivial homomorphism, then $(P, 1)$ is fully $\mathcal{F}$-stabilized if and only if P is fully $\mathcal{F}$-normalized. In general, P need not be fully $\mathcal{F}$-normalized for (P, λ) to be fully $\mathcal{F}$-stabilized. One of the most important properties of fully normalized subgroups is the extension property, Lemma 3.10 given in the previous section.

To prove a similar result for fully stabilized pairs, we give another characterization. Note that for any finite group G, the group $\mathrm{Aut}(G)$ of automorphisms of G acts on the group $\hat{G}$ of homomorphisms from G to A via pre-composition, that is ${}^{\phi}\lambda(x) = \lambda(\phi(x))$ for $x \in G$, $\lambda \in \hat{G}$ and $\phi \in \mathrm{Aut}(G)$

Proposition 4.2. *Let $\mathcal{F}$ be a fusion system over S and $(P, \lambda) \in \mathcal{M}_S^A$. Let $K = \mathrm{Stab}_{\mathrm{Aut}(P)}(\lambda)$ be the stabilizer of λ in $\mathrm{Aut}(P)$. Then the following are equivalent.*

(i) The pair (P, λ) is fully stabilized in $\mathcal{F}$.
(ii) The subgroup P is fully K-normalized in $\mathcal{F}$.

We introduce further notation before proving this proposition. Let K be as in the proposition. Then we write $\mathrm{Aut}_{\mathcal{F}}(P, \lambda) = \mathrm{Aut}_{\mathcal{F}}^K(P), \mathrm{Aut}_S^K(P, \lambda) = \mathrm{Aut}_S^K(P)$. Note that we also have the equality $N_S(P, \lambda) = N_S^K(P)$. Indeed, $N_S(P, \lambda) = \{s \in S \mid {}^s(P, \lambda) = (P, \lambda)\} = \{s \in S \mid c_s \in \mathrm{Aut}_S(P) \cap K\} = \{s \in N_S(P) \mid c_s \in \mathrm{Aut}_S^K(P)\} = N_S^K(P)$.

Proof. Let $(P, \lambda) \in \mathcal{M}_S^A$. Then (P, λ) is fully stabilized if and only if $|N_S^K(P)| \geq |N_S(Q, {}^{\phi}\lambda)|$ for isomorphisms $\phi : Q \to P$ in $\mathcal{F}$. On the other hand

$$\mathrm{Stab}_{\mathrm{Aut}(Q)}({}^{\phi}\lambda)) = \{\theta \in \mathrm{Aut}(Q) \mid {}^{\phi^{-1}\theta\phi}\lambda = \lambda\} = {}^{\phi}K.$$

Therefore, $N_S(Q, {}^{\phi}\lambda) = N_S^{{}^{\phi}K}(Q)$ and P is fully K- normalized in $\mathcal{F}$. $\square$

As a corollary we obtain the following characterizations and the extension property for fully stabilized pairs when $\mathcal{F}$ is saturated.

Corollary 4.3. *Let $\mathcal{F}$ be a saturated fusion system over S and $(P, \lambda) \in \mathcal{M}_S^A$. Then the following are equivalent.*

(i) (P, λ) *is fully stabilized in* $\mathcal{F}$.

(ii) P *is fully centralized in* $\mathcal{F}$ *and* $\mathrm{Aut}_S(P, \lambda)$ *is a Sylow* p*-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(P, \lambda)$.

(iii) *For each* $Q \leq S$ *and for each isomorphism* $\phi : Q \to P$ *in* $\mathcal{F}$*, there are homomorphisms* $\chi \in \mathrm{Aut}_{\mathcal{F}}(P, \lambda)$ *and* $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q, \lambda \circ \phi), S)$ *such that* $\bar{\phi}|_Q = \chi \circ \phi$.

Proof. Let us suppose that (P, λ) is fully stabilized in $\mathcal{F}$. By Proposition 4.2 this is if and only if P is fully $K-$normalized where $K = \mathrm{Stab}_{\mathrm{Aut}(P)}(\lambda)$. Or by Proposition 3.9, equivalently we know that P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P, \lambda)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P, \lambda)$. This shows the equivalence $(i) \Leftrightarrow (ii)$.

To show the equivalence $(ii) \Leftrightarrow (iii)$ we assume that P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P, \lambda)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P, \lambda)$. By Proposition 4.2 we get that for each $Q \leq S$ and for each isomorphism $\phi : Q \to P$ in $\mathcal{F}$, there are homomorphisms $\chi \in \mathrm{Aut}_{\mathcal{F}}^K(P)$ and $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(Q \cdot N_S^{K^\phi}(Q), S) = \mathrm{Hom}_{\mathcal{F}}(N_S^{K^\phi}(Q), S)$ such that $\bar{\phi}|_Q = \chi \circ \phi$ where $K = \mathrm{Stab}_{\mathrm{Aut}(P)}(\lambda)$. But we already know that $N_S^{K^\phi}(Q) = N_S(Q, \lambda \circ \phi)$ and the proof is complete. $\square$

The following lemma is the version of extension property, Lemma 3.10.

Corollary 4.4. *Let* $\mathcal{F}$ *be a saturated fusion system over* S *and* $(Q, \lambda) \in \mathcal{M}_S^A$ *be fully stabilized in* $\mathcal{F}$*. Also let* $(P, \kappa) \in \mathcal{M}_S^A$ *be such that* $(P, \kappa) \sim_{\mathcal{F}} (Q, \lambda)$*. Then there is a morphism* $\phi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P, \kappa), N_S(Q, \lambda))$ *such that* ${}^{\phi}(P, \kappa) = (Q, \lambda)$.

Proof. Let (P, κ) and (Q, λ) be as in the hypothesis. By Corollary 4.3 ((i) implies (iii)) we know that there is an extension $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P, \kappa), S)$ of ϕ. Since any morphism in $\mathcal{F}$ is a composition of an isomorphism in $\mathcal{F}$ followed an inclusion, we have $\psi : \iota \circ \bar{\phi}$ for the inclusion $\iota : P \to S$ and some $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(P, \kappa), N_S(Q, \lambda))$ extending ϕ. $\square$

4.2. Stabilization and a free basis

In this section we show that the monoid of $\mathcal{F}$-stable A-fibered S-sets is free. For this aim the first step is to proof a fibered version of Reeh's stabilization lemma [1, Lemma 4.6]. The result allows us to construct a free basis by induction. To begin with we call a subset $\mathcal{H}$ of $\mathcal{M}_S^A$ a *collection* if it is closed under $\mathcal{F}$-subconjugation. In other words, $\mathcal{H}$ is called a collection if for any $(P,\lambda) \in \mathcal{H}$ and any $(Q,\kappa) \in \mathcal{M}_S^A$ such that there is a homomorphism $\phi : (Q,\kappa) \to (P,\lambda)$, we have $(Q,\kappa) \in \mathcal{H}$.

Lemma 4.5. *Let $\mathcal{F}$ be a saturated fusion system over S and $\mathcal{H}$ a collection in $\mathcal{M}_S^A$. Let X be an A-fibered S-set such that*

(i) $\beta_{P,\lambda}(X) = \beta_{P',\lambda'}(X)$ *for all* $(P,\lambda) \sim_{\mathcal{F}} (P',\lambda')$ *with* $(P,\lambda),(P',\lambda') \notin \mathcal{H}$.
(ii) $c_{P,\lambda}(X) = 0$ *for all* $(P,\lambda) \in \mathcal{H}$.

Then there is an $\mathcal{F}$-stable A-fibered S-set X' such that

(i) $\beta_{P,\lambda}(X') = \beta_{P,\lambda}(X)$ *and* $c_{P,\lambda}(X') = c_{P,\lambda}(X)$ *for all* $(P,\lambda) \notin \mathcal{H}$.
(ii) $c_{P,\lambda}(X') = c_{P,\lambda}(X)$ *for all* (P,λ) *which is fully stabilized in* $\mathcal{F}$. *In particular if* $(P,\lambda) \in \mathcal{H}$ *is fully stabilized in* $\mathcal{F}$, *then* $c_{P,\lambda}(X') = 0$.

Proof. Following J.S. Reeh [1, Lemma 4.6] we proceed by induction on the size of $\mathcal{H}$. If $\mathcal{H} = \emptyset$, then X is $\mathcal{F}$-stable and $X' := X$. Otherwise let $(P,\lambda) \in \mathcal{H}$ be maximal under $\mathcal{F}$-conjugation and fully $\mathcal{F}$-stabilized.

Let $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$, there is a homomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P',\lambda'), N_S(P,\lambda))$ with ${}^{\phi}(P,\lambda) = (P',\lambda')$ by Corollary 4.4 since $\mathcal{F}$ is saturated.

By Proposition 2.2 we get

$$\sum_{(P,\lambda)\leq(Q,\kappa)\in\mathcal{M}^A_{\phi N_S(P',\lambda')}} \mu_{P,Q}\beta_{Q,\kappa}(X) \equiv 0 \qquad \mathrm{mod}\ |\phi(N_S(P',\lambda'))/P|$$

and similarly

$$\sum_{(P',\lambda')\leq(Q',\kappa')\in\mathcal{M}^A_{N_S(P',\lambda')}} \mu_{P',Q'}\beta_{Q',\kappa'}(X) \equiv 0 \qquad \mod |N_S(P',\lambda')/P'|.$$

By the hypothesis of induction

$\beta_{Q,\kappa}(X) = \beta_{Q',\kappa'}(X)$ for all $(Q,\kappa) \sim (Q',\kappa')$ with $(P,\lambda) <_{\mathcal{F}} (Q,\kappa)$.

Hence

$$\beta_{P,\lambda}(X) \ - \beta_{P',\lambda'}(X) \ = \sum_{(P,\lambda)\leq(Q,\kappa)\in\mathcal{M}^A_{\phi_{N_S(P',\lambda')}}} \mu_{P,Q}\beta_{Q,\kappa}(X) \ -$$

$$\sum_{(P',\lambda')\leq(Q',\kappa')\in\mathcal{M}^A_{N_S(P',\lambda')}} \mu_{P',Q'}\beta_{Q',\kappa'}(X) \ = \ 0-0 \qquad \mod |N_S(P',\lambda')/P'|.$$

It follows that $a_{(P',\lambda')} := (\beta_{P,\lambda}(X) \ - \beta_{P',\lambda'}(X))|P|/|N_S(P',\lambda')| \in \mathbb{Z}$ and thanks to Lemma 4.6 below we construct a new A-fibered S-set

$$\tilde{X} := X + \sum_{[P',\lambda']_S\subseteq[P,\lambda]_{\mathcal{F}}} a_{(P',\lambda')}[P',\lambda']_S.$$

We have $c_{Q,\kappa}(\tilde{X}) = c_{Q,\kappa}(X)$ for $(Q,\kappa) \nsim_{\mathcal{F}} (P,\lambda)$. And if $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$ is fully stabilized then we will show below at Lemma 4.6 that $a_{(P',\lambda')} = 0$ and thus $c_{P',\lambda'}(\tilde{X}) = c_{P,\lambda}(X)$.

Since $\beta_{Q,\kappa}([P',\lambda']_S) = 0$ unless $(Q,\kappa) \leq_S (P',\lambda')$ we have $\beta_{Q,\kappa}(\tilde{X}) = \beta_{Q,\kappa}(X)$ for $(Q,\kappa) \notin \mathcal{H}$. It also follows that for each $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$,

$$\beta_{P',\lambda'}(\tilde{X}) = \beta_{P',\lambda'}(X) + \sum_{[\tilde{P},\tilde{\lambda}]_S\subset[P,\lambda]_{\mathcal{F}}} a_{(\tilde{P},\tilde{\lambda})}\beta_{P',\lambda'}[\tilde{P},\tilde{\lambda}]_S =$$

$$\beta_{P',\lambda'}(X) + \lambda_{(P',\lambda')}\beta_{P',\lambda'}[P',\lambda']_S = \beta_{P',\lambda'}(X) + a_{(P',\lambda')}\frac{|N_S(P',\lambda')|}{|P'|} = \beta_{P,\lambda}(X).$$

Now we define $\mathcal{H}' := \mathcal{H}\backslash[P,\lambda]_{\mathcal{F}}$ which is again a collection. We showed above that $\beta_{Q,\kappa}(\tilde{X}) = \beta_{Q',\kappa'}(\tilde{X})$ for $(Q',\kappa') \sim_{\mathcal{F}} (Q,\kappa) \notin \mathcal{H}$ and moreover that $\beta_{P,\lambda}(\tilde{X}) = \beta_{P',\lambda'}(\tilde{X})$ for $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$; i.e. we know that $\beta_{Q,\kappa}(\tilde{X}) = \beta_{Q',\kappa'}(\tilde{X})$ for $(Q',\kappa') \sim_{\mathcal{F}} (Q,\kappa) \notin \mathcal{H}'$. On the other hand, we also saw that $c_{Q,\kappa}(\tilde{X}) = c_{Q,\kappa}(X)$ for $(Q,\kappa) \nsim (P,\lambda)$ or in particular for $(Q,\kappa) \in \mathcal{H}'$ we have $c_{Q,\kappa}(\tilde{X}) = c_{Q,\kappa}(X) = 0$. Therefore $\tilde{X}$ satisfies the hypothesis (i) and (ii) of the lemma. Hence applying the lemma to $\tilde{X}$ by induction we get an A-fibered $S-$set $X' \in B^A(\mathcal{F})$ such that $\beta_{Q,\kappa}(X') = \beta_{Q,\kappa}(\tilde{X})$ and $c_{Q,\kappa}(X') = c_{Q,\kappa}(\tilde{X})$ for all $Q \notin \mathcal{H}'$ and $c_{Q,\kappa}(X') = 0$ if $Q \in \mathcal{H}'$ is fully stabilized.

Finally it follows that $\beta_{Q,\kappa}(X') = \beta_{Q,\kappa}(\tilde{X}) = \beta_{Q,\kappa}(X)$ and $c_{Q,\kappa}(X') = c_{Q,\kappa}(\tilde{X}) = c_{Q,\kappa}(X)$ for all $(Q,\kappa) \notin \mathcal{H}$ and we also have $c_{Q,\kappa}(X') = 0$ if $(Q,\kappa) \in \mathcal{H}$ is fully stabilized. $\square$

The following lemma is necessary to ensure that the element $\tilde{X} \in B^A(S)$ is actually an A-fibered S-set.

Lemma 4.6. *Let $\mathcal{F}$ be a saturated fusion system over S and $(P,\lambda) \in \mathcal{M}_S^A$ be fully stabilized in $\mathcal{F}$. Also let X be an A-fibered S-set such that $c_{P',\lambda'}(X) = 0$ for any $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$. Assume further that $\beta_{Q,\kappa}(X) = \beta_{Q',\kappa'}(X)$ for all $(Q',\kappa') \sim_{\mathcal{F}} (Q,\kappa) \in \mathcal{M}_S^A$ such that $(P,\lambda) \leq_{\mathcal{F}} (Q,\kappa)$. Then $\beta_{P,\lambda}(X) \geq \beta_{P',\lambda'}(X)$ for any $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$.*

Proof. We'll follow J.S Reeh [1, Lemma 4.7]. Let $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$ be given. Since (P,λ) is fully stabilized , there exists by Corollary 4.4, a homomorphism $\phi : N_S(P',\lambda') \to N_S(P,\lambda)$ in $\mathcal{F}$ with $\phi(P',\lambda') = (P,\lambda)$.

Let $(A_1,\lambda_1'),\ldots,(A_k,\lambda_k')$ be the subcharacters of $N_S(P',\lambda')$ that are strictly larger than (P',λ'); i.e. $(P',\lambda') < (A_i,\lambda_i')$ with $A_i \leq N_S(P',\lambda')$. We put $(B_i, {}^{\phi}\lambda_i') := \phi(A_i,\lambda_i')$. We moreover let $(C_1,\lambda_1),\ldots,(C_l,\lambda_l)$ be the subcharacters of $N_S(P,\lambda)$ that are strictly larger than (P,λ) and not in the form $(B_i, {}^{\phi}\lambda'_i)$s ; i.e. $(P,\lambda) < (C_i,\lambda_i)$ with $C_i \leq N_S(P,\lambda)$. We denote the indices $\{1,\dots k\}$ by I and $J := \{1,\dots l\}$.

Because $c_{P,\lambda}(X) = c_{P',\lambda'}(X) = 0$ by assumption no orbit of X is isomorphic to $[P',\lambda']_S$. Hence no fiber Ax in $X^{P',\lambda'}$ has the stabilizer (P',λ'). Let $(P',\lambda') < (K,\kappa)$ be the stabilizing pair of x then as K is a p-group we have $(P',\lambda') < (L,\mu) := (K \cap N_S(P',\lambda'), \mathrm{Res}^K_{K\cap N_S(P',\lambda')}\kappa)$ with $P' \lhd L$ such that $Ax \subset [L,\mu]$. We conclude that $X^{P',\lambda'} = \bigcup_{i\in I} X^{A_i,\lambda'_i}$ or with a similar reasoning $X^{P,\lambda} = \bigcup X^{B_i,{}^{\phi}\lambda'_i} \cup \bigcup X^{C_l,\lambda_l}$. The proof is then completed like Reeh by showing

$$|X^{P,\lambda}| = |\bigcup X^{B_i,{}^{\phi}\lambda'_i} \cup \bigcup X^{C_l,\lambda_l}| \geq |\bigcup X^{B_i,{}^{\phi}\lambda'_i}| = |\bigcup_{i\in I} X^{A_i,\lambda'_i}| = |X^{P',\lambda'}|.$$

We only need to prove the second identity from the last. For $(P,\lambda) <_{\mathcal{F}} (B_i, {}^{\phi}\lambda'_i)$ we have $|X^{A_i,\lambda_i}| = |X^{B_i,{}^{\phi}\lambda'_i}|$ by assumption. Then by the inclusion-exclusion principle

$$|\bigcup X^{B_i,{}^{\phi}\lambda'_i}| = \sum_{\emptyset\neq\Lambda\subset I} (-1)^{|\Lambda|+1} |\bigcap X^{B_i,{}^{\phi}\lambda'_i}| = \sum_{\emptyset\neq\Lambda\subset I} (-1)^{|\Lambda|+1} |X^{\langle B_i,{}^{\phi}\lambda'_i\rangle_\Lambda}| =$$

$$\sum_{\emptyset\neq\Lambda\subset I} (-1)^{|\Lambda|+1} |X^{\langle A_i,\lambda'_i\rangle_\Lambda}| = \dots = |\bigcup X^{A_i,\lambda'_i}|$$

where $\langle B_i,\lambda_i\rangle$ denotes the transitive fibered set corresponding to the subcharacter $\lambda(b) = \prod_{i\in\Lambda} \lambda_i(b_i) \in A$ for $b = \prod_{i\in\Lambda} b_i$ with $b_i \in B_i$. $\square$

4.3. Stable obstructions

With this result we see that for $\mathcal{F}$-stable elements, the obstructions for being in the image of $\beta_{\mathcal{F}}$ is the same as obstructions for being in the image of β_S. However being $\mathcal{F}$-stable puts some restrictions on the obstructions also and it turns out that the group of these stable obstructions is smaller. In the next section we describe this group, in other words, the cokernel of $\beta_{\mathcal{F}}$.

Proposition 4.7. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. For each $\mathcal{F}$-conjugacy class of fibered sets $[P,\lambda]_{\mathcal{F}}$, there is an $\mathcal{F}$-stable fibered set $\alpha_{P.\lambda} \in B^A(\mathcal{F})$ such that*

(i) $\beta_{Q,\kappa}(\alpha_{P,\lambda}) = 0$ *unless* $(Q,\kappa) \leq_{\mathcal{F}} (P,\lambda)$.

(ii) $c_{P',\lambda'}(\alpha_{P,\lambda}) = 1$ *and* $\beta_{P',\lambda'} = |N_S(P',\lambda')|/|P|$ *when* (P',λ') *is fully stabilized and* $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$.

(iii) $c_{Q,\kappa}(\alpha_{P,\lambda}) = 0$ *when* (Q,κ) *is fully stabilized and* $(Q,\kappa) \nsim_{\mathcal{F}} (P,\lambda)$.

Proof. Let $(P,\lambda) \in \mathcal{M}_S^A$ be fully stabilized. We let $X \in B^A(S)$ be the fibered set

$$X = \sum_{[P',\lambda']_S \subset [P,\lambda]_{\mathcal{F}}} \frac{|N_S(P,\lambda)|}{|N_S(P',\lambda')|}.[P',\lambda']_S \in B^A(S).$$

X then satisfies that $\beta_{Q,\kappa}(X) = 0$ unless $(Q,\kappa) \leq_{\mathcal{F}} (P,\lambda)$. Moreover, $\beta_{P'',\lambda''}([P',\lambda']_S) = 0$ unless $[P'',\lambda'']_S = [P',\lambda']_S$. Consequently

$$\beta_{P',\lambda'}(X) = \frac{|N_S(P,\lambda)|}{|N_S(P',\lambda')|}\beta_{P',\lambda'}([P',\lambda']_S) = \frac{|N_S(P,\lambda)|}{|P|}$$

independent of $(P',\lambda') \sim_{\mathcal{F}} (P,\lambda)$.

We let $\mathcal{H}$ be the collection of $(Q,\kappa) <_{\mathcal{F}} (P,\lambda)$ so that $\beta_{Q,\kappa}(X) = \beta_{Q'\kappa'}(X)$ for all $(Q,\kappa) \sim_{\mathcal{F}} (Q',\kappa') \notin \mathcal{H}$. Now by Lemma 4.5 we get a set $\alpha_{P,\lambda}$ with the desired properties. □

Corollary 4.8. *The $\alpha_{P,\lambda}$ in the proposition 4.7 are linearly independent.*

Proof. This is immediate since every $\alpha_{P,\lambda}$ contains $[P,\lambda]_S$ but no other $\alpha_{Q,\kappa}$ does. □

Now as in the group case proved by [1, Reeh, S.] we will define the *obstruction group* $\mathrm{Obs}(\mathcal{F})$ and we will show that it is the cokernel of the mark homomorphism $\beta_{\mathcal{F}}$.

Let $\mathcal{M}_S^A/\mathcal{F}$ denote the $\mathcal{F}-$conjugacy classes $[P,\lambda]_{\mathcal{F}}$ of $(P,\lambda) \in \mathcal{M}_S^A$. Following Reeh we define the A-fibered obstruction group of $\mathcal{F}$ as the direct product :

$$\mathrm{Obs}(\mathcal{F}) = \prod_{\substack{[P,\lambda]_{\mathcal{F}} \in \mathcal{M}_S^A/\mathcal{F}, \\ (P,\lambda) \text{is fully stabilized}}} \mathbb{Z}/[N_S(P,\lambda) : P]\mathbb{Z}.$$

Theorem 4.9. *Let $\mathcal{F}$ be a saturated fusion system over a $p-$group S, and let $B^A(\mathcal{F})$ be the Burnside ring of $\mathcal{F}$, i.e. the subring of $\mathcal{F}-$stable elements in $B^A(S)$. We then have a short-exact sequence*

$$0 \longrightarrow B^A(\mathcal{F}) \xrightarrow{\beta_{\mathcal{F}}^A} \mathfrak{B}^A(\mathcal{F}) \xrightarrow{\Psi} \mathrm{Obs}(\mathcal{F}) \longrightarrow 0$$

where $\beta_{\mathcal{F}}$ is the homomorphism of marks, and $\Psi = \Psi_{\mathcal{F}} : \mathfrak{B}^A(\mathcal{F}) \to \mathrm{Obs}(\mathcal{F})$ is a group homomorphism given by the coordinate functions on $[P,\lambda]_S$:

$$\Psi_{P,\lambda}(f) = \sum_{(P,\lambda) \leq (Q,\kappa) \in \mathcal{M}^A_{N_S(P,\lambda)}} \mu_{P,Q} f_{Q,\kappa} \qquad \mathrm{mod}\ [N_S(P,\lambda) : P]$$

when $(P,\lambda) \in \mathcal{M}_S^A$ is a fully stabilized representative of the conjugacy class $[P,\lambda]_{\mathcal{F}}$.

Proof. We will follow Reeh's line of thought. We take the conjugacy classes in $\mathcal{M}_S^A/\mathcal{F}$ in the reverse order; i.e $(P,\lambda) \leq' (Q,\kappa)$ if $(Q,\kappa) \leq (P,\lambda)$ so that the group homomorphism Ψ can be represented by a lower triangular matrix with 1s on the diagonal, hence it is surjective. Moreover we already know that the mark homomorphism is injective. So we only need to show that $\mathrm{im}(\beta_{\mathcal{F}}) = \ker(\Psi)$.

If $f \in \mathrm{im}(\beta_{\mathcal{F}}) \subset \mathrm{im}(\beta_S)$ then $\Psi_S(f) = 0$ by Proposition 2.6. Hence $(\Psi_{\mathcal{F}})_{P,\lambda}(f) = (\Psi_S)_{P,\lambda}(f) = 0$ for fully stabilized coordinates $(P,\lambda) \in \mathcal{M}_S^A$ and $\Psi(f) = 0$.

Now by computing the cokernel of the submodule $\mathcal{H} = \mathrm{span}\{\alpha_{P,\lambda}\}$ generated by the elements $\alpha_{P,\lambda}$ we defined in Proposition 4.7 we will show the converse to finish the proof. First of all since $\mathcal{H} \leq B^A(\mathcal{F})$ we know that $|\mathrm{coker}(\beta_{\mathcal{F}}|_{\mathcal{H}})| \geq |\mathrm{coker}(\beta_{\mathcal{F}})|$.

On the other hand with respect to the ordered bases of $\mathcal{H}$ and $\mathfrak{B}^A(\mathcal{F})$ the restriction $\beta_{\mathcal{F}}|_{\mathcal{H}}$ of $\beta_{\mathcal{F}}$ can be represented by a lower triangular matrix M since $M_{[Q,\kappa]_S,[P,\lambda]_S} = 0$ unless $(Q,\kappa) \sim_{\mathcal{F}} (P,\lambda)$ or $(Q,\kappa) \leq (P,\lambda)$. The diagonal entries $M_{P,\lambda} = \beta_{P,\lambda}(\alpha_{P,\lambda}) = |N_S(P,\lambda)|/|P|$ are nonzero, so the cokernel of $\beta_{\mathcal{F}}|_{\mathcal{H}}$ is

$$|\mathrm{coker}\beta_{\mathcal{F}}|_{\mathcal{H}}| = \prod_{[P,\lambda]_{\mathcal{F}}} M_{P,\lambda} = \prod_{\substack{[P,\lambda]_{\mathcal{F}} \in \mathcal{M}_S^A/\mathcal{F},\\ (P,\lambda) \text{is fully stabilized}}} \frac{|N_S(P,\lambda)|}{|P|}.$$

Moreover since $\Psi \circ \beta_{\mathcal{F}} = 0$ we have

$$|\mathrm{coker}(\beta_{\mathcal{F}}|_{\mathcal{H}})| = \prod_{\substack{[P,\lambda]_{\mathcal{F}} \in \mathcal{M}_S^A/\mathcal{F},\\ (P,\lambda) \text{is fully stabilized}}} \frac{|N_S(P,\lambda)|}{|P|} = |\mathrm{Obs}^A(\mathcal{F})| \leq |\mathrm{coker}(\beta_{\mathcal{F}})|.$$

Therefore we conlude that $\mathrm{Obs}^A(\mathcal{F}) = \mathrm{coker}(\beta_{\mathcal{F}})$ and the proof is complete. □

At last we will show how to write a basis for the $\mathbb{C}-$fibered Burnside ring of $\mathcal{F}_{D_8}(S_4)$:

Example 4.10 ($\mathbb{C}$-fibered Burnside ring of $\mathcal{F}_{D_8}(S_4)$)**.** *Using the proofs of Lemma 4.5 and Proposition 4.7 we can inductively construct a basis for the Burnside ring $B^{\mathbb{C}}(\mathcal{F}_{D_8}(S_4))$. We continue with the notations introduced at Examples 2.1, 3.2, 3.5.*

To begin with let's determine the $\mathcal{F}-$conjugacy classes and fully stabilized pairs in them: $[1,1]_{\mathcal{F}} = \{(1,1)\}$, $(1,1)$ *is fully stabilized;* $[Z,1]_{\mathcal{F}} = \{(Z,1),(C_2^1,1),(C_2^2,1)\}$, $(Z,1)$ *is fully stabilized;* $[Z,\chi_1|_Z]_{\mathcal{F}} = \{(Z,\chi_1|_Z),(C_2^1,\psi_2|_{C_2^1}),(C_2^2,\psi_2|_{C_2^2})\}$, $(Z,\chi_1|_Z)$ *is fully stabilized;* $[C_2^3,1]_{\mathcal{F}} = \{(C_2^3,1),(C_2^4,1)\}$, *both are fully stabilized;* $[C_2^3,\psi_2|_{C_2^3}]_{\mathcal{F}} = \{(C_2^3,\psi_2|_{C_2^3}),(C_2^4,\psi_2|_{C_2^4})\}$, *both are fully stabilized;* $[C_4,1]_{\mathcal{F}} = \{(C_4,1)\}$, $[C_4,\chi_1]_{\mathcal{F}} = \{(C_4,\chi_1),(C_4,\chi_3)\}$, *both fully stabilized;* $[C_4,\chi_2]_{\mathcal{F}} = \{(C_4,\chi_2)\}$*;* $[V_1,1]_{\mathcal{F}} = \{(V_1,1)\}$, $[V_1,\psi_6]_{\mathcal{F}} = \{(V_1,\psi_6),(V_1,\psi_7),(V_1,\psi_8)\}$*;* $[V_2,1]_{\mathcal{F}} = \{(V_2,1)\}$*;* $[V_2,\psi_{10}]_{\mathcal{F}} = \{(V_2,\psi_{10})\}$, $[V_2,\psi_{11}]_{\mathcal{F}} = \{(V_2,\psi_{11}),(V_2,\psi_{12})\}$ *both fully stabilized. Finally,* $[D_8,1]_{\mathcal{F}}$, $[D_8,\psi_2]_{\mathcal{F}}$, $[D_8,\psi_3]_{\mathcal{F}}$, $[D_8,\psi_4]_{\mathcal{F}}$ *are other conjugacy classes, they consist of single elements that are fully stabilized.*

Now we should write an $\mathcal{F}$-stable element for each conjugacy class as in Proposition 4.7. For $[1,1]_{\mathcal{F}}$ we take $\alpha_{1,1} = [1,1]_{D_8}$ which is $\mathcal{F}$-stable.

For $[Z,1]_{\mathcal{F}}$, as in the proof of Proposition 4.7 we first take

$$X = \frac{|N_{D_8}(Z,1)|}{|N_{D_8}(C_2^1,1)|}[C_2^1,1]_{D_8} + [Z,1]_{D_8} = 2[C_2^1,1]_{D_8} + [Z,1]_{D_8}$$

and let $\mathcal{H} = \{(P,\lambda) \in \mathcal{M}^{\mathbb{C}}_{D_8} : (P,\lambda) <_{\mathcal{F}} (Z,1)\} = \{(1,1)\}$. By Lemma 4.5 we conclude that $\alpha_{Z,1} = X = 2[C_2^1,1]_{D_8} + [Z,1]_{D_8}$ is $\mathcal{F}$-stable.

Similarly, for $[Z,\chi_1|_Z]_{\mathcal{F}}$, $[C_2^3,1]_{\mathcal{F}}$, $[C_2^3,\psi_2|_{C_2^3}]_{\mathcal{F}}$ we get the $\mathcal{F}$-stable sets: $\alpha_{Z,\chi_1|_Z} = 2[C_2^1,\psi_2|_{C_2^1}]_{D_8} + [Z,\chi_1|_Z]_{D_8}$, $\alpha_{C_2^3,1} = [C_2^3,1]_{D_8}$, $\alpha_{C_2^3,\psi_2|_{C_2^3}} = [C_2^3,\psi_2|_{C_2^3}]_{D_8}$.

For $[C_4,1]_{\mathcal{F}}$, we first take $X = [C_4,1]_S$ and $\mathcal{H} = \{(1,1),(Z,1),(C_2^1,1),(C_2^2,1)\}$. Then we define inductively $\tilde{X} = X + a_{(C_2^1,1)}[C_2^1,1]_{D_8}$ and $\mathcal{H}' = \{(1,1)\}$ where as in Lemma 4.5,

$$a_{(C_2^1,1)} = (\beta_{Z,1}(X) - \beta_{C_2^1,1}(X))\frac{|Z|}{|N_{D_8}(C_2^1,1)|} = (2-0)\frac{|Z|}{|V_1|} = 1.$$

Thus, since $(1,1)$ is already fully stabilized, we get the $\mathcal{F}$-stable set $\alpha_{C^4,1} = \tilde{X} = [C_4,1]_{D_8} + [C_2^1,1]_{D_8}$.
Similarly, we get $\alpha_{C_4,\chi_1} = [C_4,\chi_1]_{D_8} + [C_2^1,\psi_2|_{C_2^1}]_{D_8}$ and $\alpha_{C_4,\chi_2} = [C_4,\chi_2]_{D_8} + [C_2^1,1]_{D_8}$.

Further calculations are tedious but similar, hence we do not mention them here. In the end, we should get an $\mathcal{F}$-stable element $\alpha_{P,\lambda}$ for each $\mathcal{F}-$conjugacy class of $\mathcal{M}^{\mathbb{C}}_{D_8}$ where (P,λ) is a fully stabilized representative of its class. We know by Theorem 4.9 that these elements will form a basis of the fibered Burnside ring $B^{\mathbb{C}}(\mathcal{F}_{D_8}(S_4))$.

5. CONCLUSION

In this thesis we extended S.P. Reeh's results (2015) for G-sets to fibered G-sets over an abelian group A. We showed that his techniques work as well for writing a free basis for the $\mathcal{F}-$stable elements of the fibered Burnside ring with some tools we adopted from the theory of fusion systems and represenation theory.

REFERENCES

1. S. Reeh, *The abelian monoid of fusion-stable finite sets is free*, Algebra and Number Theory 9 (2015), no. 10, 2303?2324.

2. L. Barker, *Fibred permutation sets and the idempotents and units of monomial Burnside rings*, J. Algebra 281 (2004), 535-566.

3. R. Boltje, *Integrality conditions for elements in ghost rings of generalized Burnside rings*, J. Algebra to appear.

4. C. Broto, R. Levi, B. Oliver, *The homotopy theory of fusion systems* J. Amer. Math. Soc. 16 (2003), 779-856

5. M. G. Aschbacher, *Finite Group Theory* Finite Group Theory (Cambridge Studies in Advanced Mathematics) 2nd Ed.

6. L. Puig, *Frobenius categories*. Journal of Algebra - J ALGEBRA. 303. 309-357. 10.1016/j.jalgebra.2006.01.023.

7. A. Dress *The Ring of Monomial Representations* JOURNAL OFALGEBRA 18, 137-157(1971)]

8. R. Boltje, *A general theory of canonical induction formulae*, J. Algebra 206 (1998), 293-343.

9. R. Boltje and O. Coşkun, *Fibered biset functors*, Adv. Math., 339 (2018), 540-598.

10. J.P. Serre, *Linear Representations of Finite Groups*, (Graduate texts in mathematics; 42) Translation of Representations lineaires des groupes finis, 2. ed.

11. M. Aschbacher , R. Kessar, B. Oliver *Fusion Systems in Algebra and Topology*

London Mathematical Society Lecture Note Series Book 391 1st Edition

12. D. A. Craven *The Theory of Fusion Systems: an algebraic approach* Cambridge studies in advanced mathematics 2011

13. A. Ruiz and A. Viruel *Exotic Fusion Systems* Math. Z. 248, No. 1, 45-65 (2004; Zbl 1062.20022)

14. P. Webb, *A Guide to Mackey Functors*, pages 805-836 in Handbook of Algebra vol 2 (ed M. Hazewinkel), Elsevier 2000.

15. S. Park, *Mackey functors and sharpness for fusion systems*, Homology, Homotopy Appl. 17 (2015), no. 1, 147?164 (joint with Antonio Diaz)

16. W. Luck, *Transformation Groups and Algebraic K-Theory*, Lecture Notes in Mathematics - Mathematica Gottingensis -Springer (1989)

17. M. Linckelmann and N. Mazza *The Dade group of a fusion system* J. Group Theory 12 , 55-74 (2009)

18. S. Reeh, E. Yalçın *Representation rings for fusion systems and dimension functions*, Math. Z. 288 (2018), Issue 1-2, 509-530.

19. R. Boltje, *Representation rings of finite groups, their species and idempotent formulae*, J. Algebra to appear.

20. S. Bouc and E. Yalçın, *Borel-Smith functions and the Dade group*, J. Algebra **311** (2007), 821-839.

21. Peter Webb, *An Introduction to the Representations and Cohomology of Categories*, pp. 149-173 in: M. Geck, D. Testerman and J. Thevenaz (eds.), Group Representation Theory, EPFL Press (Lausanne) 2007.