COMPOSITION FACTORS OF THE FUNCTOR OF THE COMPLEX CHARACTERS

by

Mehmet Arslan

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ABSTRACT

COMPOSITION FACTORS OF THE FUNCTOR OF THE COMPLEX CHARACTERS

When we consider $\mathbb{C}R_{\mathbb{C}}$ as a map sending any finite group G to the complex vector space $\mathbb{C}R_{\mathbb{C}}(G)$ of complex valued class functions on G, it becomes an A-fibered biset functor for any group $A \leq \mathbb{C}^{\times}$. Its structure is known for tirivial fiber groups $A = 1$ and $A = \mathbb{C}^{\times}$. While it is a direct sum of simple biset functors in the case that $A = 1$, in the other case it is a simple $\mathbb{C}^{\times}$-fibered biset functor. We noticed that as the fiber group grows, some of simple summands of 1-fibered biset functor $\mathbb{C}R_{\mathbb{C}}$ unite and form new fibered simple summands.

In this thesis, we investigate the structure of the functor $\mathbb{C}R_{\mathbb{C}}$ for two intermediate fiber groups. The first one is a group containing all p^n-th roots of unity for any $n \in \mathbb{N}$ and for any prime number p from a fixed set of primes π. The second one is the group of all p^n-th roots of unity for a fixed $n \in \mathbb{N}$. For both cases, we identify its new fibered simple summands by determining uniting summands via defining equivalence relations on them.

ÖZET

KARMAŞIK KARAKTERLER İZLECİNİN BİLEŞKE ÇARPANLARI

Herhangi bir sonlu G grubunu, grubun sınıf fonksiyonlarının karmaşık vektör uzayı olan $\mathbb{C}R_{\mathbb{C}}(G)$ uzayına gönderen bir dönüşüm olarak düşündüğümüzde $\mathbb{C}R_{\mathbb{C}}$, her $A \leq \mathbb{C}^{\times}$ grubu için bir A-fiberli ikili küme izleci olur. Yapısı, $A = 1$ ve $A = \mathbb{C}^{\times}$ aşikar fiber grupları için bilinmektedir. Fiber group $A = 1$ olduğunda basit ikili küme izleçlerinin direkt toplamı iken diğer durumda bir basit $\mathbb{C}^{\times}$-fiberli ikili küme izlecidir. Fiber grup büyüdüğünde direkt toplamdaki basit parçalardan bazılarının birleştiğini ve yeni fiberli basit parçaları oluşturduğunu farkettik.

Bu tezde, $\mathbb{C}R_{\mathbb{C}}$ izlecinin yapısını iki ara fiber grubu için inceledik. Birincisi, asal sayıların sabitlenmiş bir π altkümesinden alınan her asl sayı p ve her $n \in \mathbb{N}$ için birim elemanın p^n. dereceden bütün köklerini içeren bir gruptur. İkincisi, sabit bir $n \in \mathbb{N}$ için birim elemanın p^n. dereceden bütün köklerinin grubudur. İki durumda da, ilk durumdaki parçalardan hangilerinin birleştiğini üzerlerinde denklik bağıntıları tanımlayarak belirledik ve yeni fiberli basit parçaları bulduk.

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LIST OF SYMBOLS

$B^A(G,H)$	Burnside group of A-fibered (G,H)-bisets
$\mathcal{C}$	A-fibered bisets category
$\mathbb{C}$	The field of complex numbers
$\mathbb{C}^\times$	The multiplicative group of non-zero complex numbers
$\mathrm{Cf}(G,\mathbb{C})$	Set of all complex-valued class functions on the group G
C_n	Multiplicative cyclic group of order n
$\mathcal{F}$	Category of A-fibered biset functors
$\lvert G:H\rvert$	Index of $H \leq G$ in G
$\mathrm{GL}(n,\mathbb{C})$	Multiplicative group of invertible $n \times n$ complex matrices
$\mathrm{Irr}(G)$	Set of all irreducible characters of the group G
(m,n)	The greatest common divisor of the numbers $m,n \in \mathbb{N}$
$\mathbb{N}$	The set of natural numbers
n_π	π-part of n
$o(g)$	The order of a group element g
$\mathbb{R}$	The field of real numbers
R-Mod	Category of left R-modules
S_n	Symmetric group of degree n
$\mathrm{tr}(B)$	Trace, sum of all diagonal entries, of the square matrix B
$\mathbb{Z}$	The set of integers

1. INTRODUCTION

Representation theory is the study of algebraic structures in linear algebra by representing elements of structures as linear transformations of vector spaces. This makes abstract structures more concrete by describing the structure in terms of matrices and their algebraic operations as matrix operations. Algebraic structures studied in this way include groups, associative algebras, and Lie algebras. In this thesis, we are interested in representations of finite groups, in which group elements represented by invertible matrices such that the group operation is the matrix multiplication. Representations of finite groups allow us to state group theoretic problems in terms of linear algebra, which reduces the complexity of problems because linear algebra is a well understood branch of mathematics. Just as that the group concept is an abstraction of symmetries of geometric objects, representations of finite groups can be considered as abstractions of groups beause the results obtained via represantation theory can extend beyond the boundaries of the theory itself. For instance, although the fact that every group of order square of a prime number is abelian can be proved in group theory, it is an evident consequence of basic facts of representations. Therefore understanding representations of finite groups can enable us to understand structures of groups. The theory of representations of finite groups starts with representations over a field of characteristic zero. If the field is also algebraically closed, then three different notions, namely representations, modules of group algebras and characters can be studied interchangeably whichever offers the easiest way. In this thesis, prefer characters and consider character rings of finite groups. However, analyzing character rings for each group separately is not an easy task. In the sense of abstraction, Bouc added another link to the chain and introduced biset functors since representation rings turn out to be the evaluations of a biset functor at the finite groups.

The character of a group representation is a function on the group that associates the trace of the matrix of each group element to the corresponding group element. Characters contain all of the essential information of representations in a more compact way. One of the fundamental constructions in the theory is the ring of characters of a

finite group. Based on its importance, the character ring is studied in many different ways since its introduction by Richard Brauer. Artin's and Brauer's induction theorems describe generating sets for the character ring. These two fundamental theorems show that only the induction map provides a convenience for determining characters of finite groups. Therefore, including all module theoretic maps, induction, restriction, isogation, inflation, and deflation, should have given much better results. This idea results in the birth of biset functors. To name a few of those studying the biset structure of the ring of characters, Thevenaz and Webb's theorem describes its Mackey functor structure, whereas Bouc's Theorem describes its biset functor structure. The structure of it as a Green biset functor is investigated by Romero. More recently, Boltje and Coşkun show that this functor is simple as a $\mathbb{C}^\times$-fibered biset functor. Moreover, analyzing the ring of characters as a biset functor enables us with determining the structure of it for any finite group instead of determining it for each group separately. Also, fiber actions provide a look into their structure under extra conditions. In this thesis, we are aiming to study the structure of the functor of complex characters as a fibered biset functor for some nice choices of non-trivial fiber groups $A < \mathbb{C}^\times$. In particular, our aim is to identify simple A-fibered subfunctors of it. Since achieving this aim mainly depends on the decomposition of A-fibered bisets, we analyze the structure of A-fibered bisets and decompose them with our choices of fiber groups. Boltje and Coşkun decompose A-fibered bisets in general provided that A is a divisible group. On the other hand, Coşkun and Yılmaz adapted Boltje and Coşkun's decomposition to abelian groups. We used Coşkun and Yılmaz's decomposition for the first part of our main theorems. In the second part, since we choose a fiber group that is not divisible, we decompose fibered bisets. But, as it is a challenging task we first obtain it for cyclic groups.

The thesis is designed as follows. In Section 2, the basic definitions, examples and theorems about representations and characters of finite groups are introduced. Section 3 is devoted to details of bisets and fibered bisets. Also, Burnside group is introduced in the same section. The definition of fibered biset functors and the structure of simple fibered biset functors constitute the topics of Section 4. Applications of some of previous results to abelian groups is shown in Section 5. Detailed information about

the functor of complex character ring takes place in Section 6. We conclude the thesis with the statements and the proofs of the main theorems in Section 7.

We now give some notations that will be valid throughout the thesis. First, unless otherwise explicitly stated, we assume all groups to be finite and fix a multiplicatively written (not necessarily finite) abelian group A. For any group G, we set

$$G^A := \mathrm{Hom}(G, A) \quad \text{and} \quad G^* := \mathrm{Hom}(G, \mathbb{C})$$

and view them as abelian groups with pointwise multiplication. The order of any element $g \in G$ is denoted by $o(g)$. If π is a set of prime numbers and $n \in N$, then we denote by n_π and $n_{\pi'}$ the π and π'-parts of n, respectively. That is, $n = n_\pi n_{\pi'}$ where $n_{\pi'}$ is the largest factor of n provided that $p \nmid n_{\pi'}$ for any $p \in \pi$. For an abelian group G, we denote by the subgroup of order $|G|_\pi$ by G_π.

2. REPRESENTATIONS AND CHARACTERS

In this chapter, we introduce three different notions of representation theory, namely representations, modules and characters. We give just necessary but sufficient details to establish the close connection among them and to justify the reason why we can use them interchangeably. All results in this chapter and more can be found in [11], [10] and [9]. Throughout this chapter, G is an arbitrary finite group unless otherwise stated.

Definition 2.1. *A* representation *of G over $\mathbb{C}$, or shortly a* $\mathbb{C}$-representation *of G, is a group homomorphism*

$$\rho : G \to \mathrm{GL}(n, \mathbb{C}) \tag{2.1}$$

for some n, where $\mathrm{GL}(n, \mathbb{C})$ denotes the multiplicative group of invertible $n \times n$ matrices with entries in complex numbers. The number n is called the degree $\deg(\rho)$ *of ρ. A representation is said to be* faithful *if it is injective.*

Another definition of a representation can be given as follows. Let V be an n-dimensional $\mathbb{C}$-vector space, and let $\mathrm{GL}(V)$ denote the group of all automorphisms of V, under the composition of maps. That is, $\mathrm{GL}(V)$ is the group of all bijective $\mathbb{C}$-linear transformations on V. A *representation* of G over $\mathbb{C}$ is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Actually, this definition is equivalent to the former one because after fixing a basis of V, every automorphism corresponds to an $n \times n$ invertible matrix with complex number entries and vice versa. The *degree* of a representation is given as $\dim(V)$ in the latter definition since the correspondence verifies $\dim(V) = \deg(\rho) = n$.

Examples 2.1. (i) The map $\rho : G \to \mathrm{GL}(n, \mathbb{C})$, $g \mapsto \rho(g) = I_n$ for all $g \in G$, where I_n is the identity $n \times n$ matrix, is a representation of G. If we take $n = 1$, ρ is called the *trivial representation* of G.

(ii) Let $C_3 = \{1, a, a^2\}$ be the cyclic group of order 3, generated by a. Then,

$$\rho : C_3 \to \mathrm{GL}(2, \mathbb{C}),\ a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{2.2}$$

is a $\mathbb{C}$-representation of C_3 of order 2.

(iii) (*Permutation Representation*) Let $X = \{x_1, \ldots, x_n\}$ be a set on which G acts from the left. Consider the $\mathbb{C}$-vector space

$$\mathbb{C}X = \{c_1 x_1 + \cdots + c_n x_n \mid c_i \in \mathbb{C}, 1 \leq i \leq n\}. \tag{2.3}$$

If $x = c_1 x_1 + \cdots + c_n x_n \in \mathbb{C}X$ is an arbitrary element, the map

$$\rho_g : \mathbb{C}X \to \mathbb{C}X,\ x \mapsto gx := c_1(gx_1) + \cdots + c_n(gx_n) \tag{2.4}$$

gives an automorphism of $\mathbb{C}X$ for each $g \in G$. Then

$$\rho : G \to \mathrm{GL}(\mathbb{C}X),\ g \mapsto \rho_g \tag{2.5}$$

is a $\mathbb{C}$-representation of G of degree n, namely the permutation representation of G on $\mathbb{C}X$.

(iv) (*Regular Representation*) Take $X = G$ in the previous example with the action induced by the group multiplication. In this case, the corresponding representation is called the regular representation of G.

(v) Let G be the symmetric group S_n of degree n. For any $\sigma \in G$, define the action $\sigma \cdot e_i := e_{\sigma(i)}$ on $\mathbb{R}^n$, where $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, $1 \leq i \leq n$ are the canonical basis elements of $\mathbb{R}^n$ such that 1 is the i-th entry. The $\mathbb{R}$-linear extension

$$\sigma \cdot (r_1 e_1 + \cdots + r_n r_n) := r_1 e_{\sigma(1)} + \cdots + r_n e_{\sigma(n)} \tag{2.6}$$

of the action gives an automorphism of $\mathbb{R}^n$ as in Example 3. Arbitrary elements $(r_1, \ldots, r_n) \in \mathbb{R}^n$ are here expressed as $r_1 e_1 + \cdots + r_n e_n$.

Remark 2.1. The first example shows that any group has a representation of order n for any n. The fifth one is actually a special case of the third one. Indeed, if we take $x_i = e_i$ and define the action of $G = S_n$ as the $\mathbb{C}$-linear extension of $\sigma \cdot e_i := e_{\sigma(i)}$, $1 \leq i \leq n$, we obtain the representation in the last example.

Definition 2.2. *Let X be an arbitrary finite set. The set $\mathbb{C}X$ is called the* complex linearization *or* $\mathbb{C}$-linearization *of the set X. The* complex linearization map *is the induced map*

$$\mathrm{lin}_{\mathbb{C}} : X \mapsto \mathbb{C}X. \tag{2.7}$$

Next, we regard the ways of relating representations of groups that are algebraically connected such as subgroups, quotient groups and isomorphic groups.

2.1. New Representations from Old Ones

(i) Let $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ be a representation of G and $T \in \mathrm{GL}(n, \mathbb{C})$. Then, the map $\sigma(g) := T^{-1}\rho(g)T$ is also a representation of G of the same degree n. The representations ρ and σ are said to be *equivalent representations*.

(ii) If ρ is a $\mathbb{C}$-representation of G, and $H \leq G$, then

$$\mathrm{Res}^G_H \rho : H \to \mathrm{GL}(n, \mathbb{C}),\ \mathrm{Res}^G_H \rho(h) := \rho(h) \tag{2.8}$$

is a representation of H, namely the *restriction* of ρ from G to H.

(iii) Let ρ be a $\mathbb{C}$-representation of H of degree n for some $H \leq G$. Choose a left transversal $\{t_1 H, \ldots, t_m H\}$ of H in G, i.e. a set of representatives of left cosets G/H, where $m = |G : H|$ is the index of H in G. We define the homomorphism

$$\mathrm{Ind}^G_H \rho : G \to \mathrm{GL}(nm, \mathbb{C}) \tag{2.9}$$

as $m \times m$ block matrices where each block is an $n \times n$ matrix afforded by ρ. For any $g \in G$, the ij-th block of $\mathrm{Ind}_H^G \rho(g)$ is equal to $\rho(t_i^{-1} g t_j)$ if $t_i^{-1} g t_j \in H$, or to $[0]_{n \times n}$ otherwise. The representation $\mathrm{Ind}_H^G \rho$ is called the *induction* of ρ from H to G. The degree of $\mathrm{Ind}_H^G \rho$ is equal to $|G : H| \deg(\rho)$.

(iv) From any representation ρ of G/N, where $N \trianglelefteq G$, we can obtain a representation of G, that assigns to every $g \in G$ the value $\rho(gN)$. This new representation is called the *inflation* of ρ from G/N to G, and denoted by $\mathrm{Inf}_{G/N}^G \rho$.

(v) Let ρ be a representation of G whose kernel contains $N \trianglelefteq G$. Then, the *deflation* of ρ from G to G/N is the representation defined as $\mathrm{Def}_{G/N}^G \rho(gN) := \rho(g)$ for all $g \in G$.

(vi) If $f : H \to G$ is a group isomorphism for some group H, we may define a representation ${}_H\mathrm{Iso}_G^f \rho$ of H from any representation ρ of G via f. This new representation is called the *isogation* from G to H, and is defined as ${}_H\mathrm{Iso}_G^f \rho(h) := \rho(f(h))$ for any $h \in H$. We sometimes write ${}_H\mathrm{Iso}_G \rho$ if the isomorphism f is clear from the context.

In fact, each of the ways above gives a map between representations of related groups. For instance, Res_H^G sends the representations of G to the representations of the subgroup $H \leq G$.

Remark 2.2. Since $\mathrm{tr}(BC) = \mathrm{tr}(CB)$ for any $n \times n$ matrices, the traces of equivalent representations are the same. In other words, if $\rho, \sigma : G \to \mathrm{GL}(n, \mathbb{C})$ are equivalent representations, then $\mathrm{tr}(\rho(g)) = \mathrm{tr}(\sigma(g))$ for all $g \in G$.

Remark 2.3. Let ρ be a $\mathbb{C}$-representation of G with $\deg(\rho) = n$. Consider the $\mathbb{C}$-vector space $V = \mathbb{C}^n$, the space of $n \times 1$ column matrices with complex entries. We can equip V with the G-action $g \cdot v := \rho(g)v \in \mathbb{C}^n$ for any $g \in G$, $v \in V$. This action satisfies the following conditions for all $g, h \in G$, $v, u \in V$, $c \in \mathbb{C}$.

(i) $g \cdot v \in V$

(ii) $h \cdot (g \cdot v) = (hg) \cdot v$

(iii) $1 \cdot v = v$

(iv) $g \cdot (cv) = c(g \cdot v)$

(v) $g \cdot (v + u) = g \cdot v + g \cdot u$

In fact, any $\mathbb{C}$-vector space V is said to be a left $\mathbb{C}G$-*module* if it is equipped with a left G-action satisfying the conditions (i)-(v). This points to that if G has a representation, we can obtain a $\mathbb{C}G$-module through the representation. On the other hand, if V is a $\mathbb{C}G$-module, the conditions (i),(iv) and (v) ensure that the map sending any element v of V to the element $v \cdot g$ is an automorphism of V for any $g \in G$. Let $\mathcal{B}$ be a basis of V and let $[g]_{\mathcal{B}}$ denote the matrix corresponding to the linear transformation above relative to the basis $\mathcal{B}$. Then, the map $g \mapsto [g]_{\mathcal{B}}$ yields a representation of G, which means that the existence of a $\mathbb{C}G$-module guarantees the existence of a representation of G. Clearly, the representations of the form $[g]_{\mathcal{B}}$ depend on choices of bases. If we choose a basis other than $\mathcal{B}$, we obtain a representation equivalent to $[g]_{\mathcal{B}}$. This is a special case of a result which we state later

The first item in Section 2.1 suggests that from a given representation, we may obtain infinitely many new representations as T varies. We call such representations equivalent because it indeed gives an equivalence relation on representations. From now on, when we say distinct representations, we refer to a set of unequivalent representations.

Definition 2.3. *A non-zero $\mathbb{C}G$-module V is called* irreducible *or* simple *if it has no non-trivial $\mathbb{C}G$-submodule, that is $\{0\}$ and V are the only $\mathbb{C}G$-submodules of V. In connection with this definition, a representation $\rho \to \mathrm{GL}(n, \mathbb{C})$ is said to be* irreducible *if the $\mathbb{C}G$-module $\mathbb{C}^n$ induced by ρ is irreducible.*

Definition 2.4. *Let V_1 and V_2 be $\mathbb{C}G$-modules. A linear transformation $L : V_1 \to V_2$ is said to be a $\mathbb{C}G$-homomorphism if $L(g \cdot v) = g \cdot L(V)$ for all $g \in G$, $v \in V_1$. In the case that L is bijective, it is called a* $\mathbb{C}G$-isomorphism*, and V_1 and V_2 are called* isomorphic $\mathbb{C}G$*-modules.*

Now, we have given all that is needed to state connection between representations of G and $\mathbb{C}G$-modules.

Theorem 2.1. *There is a one-to-one correspondence between isomorphism classes of simple $\mathbb{C}G$-modules, and equivalence classes of irreducible $\mathbb{C}$-representations of G.*

A fundamental result that describes the structure of the algebra $\mathbb{C}G$ is Maschke's Theorem.

Theorem 2.2 (Maschke)**.** *If V is a $\mathbb{C}G$-module and V_1 is a $\mathbb{C}G$-submodule of V, then there exists a $\mathbb{C}G$-submodule V_2 of V such that $V = V_1 \oplus V_2$.*

As a consequence of Maschke's Theorem, any $\mathbb{C}G$-module can be written as a direct sum of its simple submodules. Therefore, we can focus on isomorphism classes of simple $\mathbb{C}G$-modules or on equivalence classes of irreducible representations of G interchangeably to understand all $\mathbb{C}G$-modules and representations of G. But, this raises the questions of how we can identify simple modules and how many non-isomorphic simple modules there are. The answers are hidden in the regular $\mathbb{C}G$-module.

Definition 2.5. *The $\mathbb{C}$-linearization*

$$\mathbb{C}G = \left\{ \sum_{g \in G} c_g g \ : \ c_g \in \mathbb{C} \right\} \tag{2.10}$$

of G is a $\mathbb{C}G$-module with the action $g' \cdot \left(\sum_{g \in G} c_g g \right) := \sum_{g \in G} c_g (g'g)$ for any $g' \in G$. It is called the regular $\mathbb{C}G$-module*.*

Theorem 2.3. *Every simple $\mathbb{C}G$-module is isomorphic to a simple submodule of the regular $\mathbb{C}G$-module $\mathbb{C}G$.*

The last theorem answers the first question but not the second one. It only implies the finiteness of non-isomorphic simple $\mathbb{C}G$-modules. The exact number of them is revealed after introducing characters.

Definition 2.6. *Let ρ be a representation of G. Then the* character χ *of ρ is the map*

$$\chi : G \to \mathbb{C}, \ g \mapsto \chi(g) := \mathrm{tr}(\rho(g)). \tag{2.11}$$

We say that χ is a character of G if it is the character of a representation of G. Similarly, we define the character of a $\mathbb{C}G$-module V using a representation obtained from V.

Definition 2.7. *The* regular character χ_{reg} *of G is the character of the regular $\mathbb{C}G$-module $\mathbb{C}G$.*

Definition 2.8. *A character of G is called* irreducible *if it is the character of a simple $\mathbb{C}G$-module. We denote the set of all irreducible characters of G by* $\text{Irr}(G)$.

Examples 2.2. (i) The character χ of the first representation in Examples 2.1 is $\chi(g) = n$ for any $g \in G$.

(ii) The map sending the elements $1, a, a^2$ to $2, -1, -1$, respectively, is the character of the representation $\rho : C_3 \to \text{GL}(2, \mathbb{C})$ in 2.1.

(iii) The regular character χ_{reg} of G is the character of the regular representation introduced in Examples 2.1. It sends the identity element to the order $|G|$ of G, and sends each non-identity element to 0.

Remark 2.2 implies that the characters of equivalent representations are the same. Therefore, since equivalent representations correspond to isomorphic $\mathbb{C}G$-modules, all isomorphic $\mathbb{C}G$-modules have the same character.

Actually, this is a very surprising and powerful result because to understand representations one needs to deal with n^2 complex numbers, whereas the character theory reduces this to a single complex number. In the character theory, we can try to reach our aims by taking advantage of field properties and convenience of $\mathbb{C}$ in comparison to $\text{GL}(n, \mathbb{C}), n > 1$. To sum up, it is more advantageous to study irreducible characters of G to understand all representations of G. However, the definition of irreducible characters depends on simple modules. Luckily, the inner product below provides us with a way to detect irreducible characters without going into module theory.

Definition 2.9. *A function $f : G \to \mathbb{C}$ is said to be a* class function *if it is constant on conjugacy classes of elements in G, i.e. $f(g) = f(x^{-1}gx)$ for all $g, x \in G$. The set of $\mathbb{C}$-valued class functions on G is denoted by* $\text{Cf}(G, \mathbb{C})$.

The set $\mathrm{Cf}(G, \mathbb{C})$ naturally forms a $\mathbb{C}$-vector space under pointwise addition and multiplication by scalars from $\mathbb{C}$. This space has an inner product defined as

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)} \tag{2.12}$$

where $f_1, f_2 \in \mathrm{Cf}(G, \mathbb{C})$ and $\overline{f_2(g)}$ is the complex conjugation of $f_2(g)$. As characters are class functions, we can talk about the inner product of characters and state the following theorem.

Theorem 2.4. *A character χ of G is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Since every $\mathbb{C}G$-module is a direct sum of simple $\mathbb{C}G$-modules, we need to find out the relation between the characters and the direct sum.

Theorem 2.5. *Let χ and ψ be the characters of $\mathbb{C}G$-modules M and N, respectively. Then, the character of the $\mathbb{C}G$-module $M \oplus N$ is $\chi + \psi$, where*

$$(\chi + \psi)(g) := \chi(g) + \psi(g) \tag{2.13}$$

for all $g \in G$.

As a result of this theorem and Maschke's Theorem, every character χ is of the form $\chi = d_1\chi_1 + \cdots + d_r\chi_r$, where $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$, and $d_1, \ldots, d_r$ are non-negative integers. Moreover, together with Theorem 2.3, we deduce that every irreducible character is a summand of the regular character χ_{reg}, that is

$$\chi_{\mathrm{reg}} = e_1\chi_1 + \cdots + e_r\chi_r \tag{2.14}$$

where $e_1, \ldots, e_r$ are now positive integers. The coefficients $d_1, \ldots, d_r$ of any character χ is calculated by $d_i = \langle \chi, \chi_i \rangle$, $i \in \{1, \ldots, r\}$ due to the next theorem and the linearity of the inner product.

Theorem 2.6. *Irreducible characters of G form an orthonormal set in the vector space* $\mathrm{Cf}(G,\mathbb{C})$*, i.e. if* $\mathrm{Irr}(G)=\{\chi_1,\ldots,\chi_r\}$*, then*

$$\langle\chi_i,\chi_j\rangle=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise.}\end{cases} \tag{2.15}$$

Finally, the question about the number of non-isomorphic simple $\mathbb{C}G$-modules is answered by the next theorem

Theorem 2.7. *The set of irreducible characters* $\mathrm{Irr}(G)$ *is an orthonormal basis for the vector space* $\mathrm{Cf}(G,\mathbb{C})$ *whose dimension is equal to the number of conjugacy classes of* G*.*

Hence, any class function $f\in\mathrm{Cf}(G,\mathbb{C})$ is of the from $f=c_1\chi_1+\cdots+c_r\chi_r$ where $\mathrm{Irr}(G)=\{\chi_1,\ldots,\chi_r\}$ and $c_i\in\mathbb{C}$, $1\leq i\leq r$. However, although all characters are class functions, the converse is not always true. The next theorem helps us decide whether a class function is a character or not.

Theorem 2.8. *Let* $f=c_1\chi_1+\cdots+c_r\chi_r\in\mathrm{Cf}(G,\mathbb{C})$ *where* $\mathrm{Irr}(G)=\{\chi_1,\ldots,\chi_r\}$*. Then, f is a character if and only if $c_1,\ldots,c_r$ are non-negative integers.*

Consequently, the set of characters of G is a semigroup. Then, the $\mathbb{Z}$-linear combinations of characters of G becomes an additive abelian group. The following theorem shows that we can also equip the set of characters with a multiplication in order to get a ring structure.

Theorem 2.9. *Let χ and ψ be the characters of $\mathbb{C}G$-modules M and N, respectively. Then, the character of the $\mathbb{C}G$-module $M\otimes N$ is $\chi\psi$, where*

$$(\chi\psi)(g):=\chi(g)\psi(g) \tag{2.16}$$

for all $g\in G$.

Hence, the product of two characters of G is again a character of G. Now, we can define the complex character ring.

2.2. Character Ring

Let $R_{\mathbb{C}}(G)$ denote the set of $\mathbb{Z}$-linear combinations of all characters of G. Then, $(R_{\mathbb{C}}(G), +, \cdot)$ is a ring, where

$$(\chi + \psi)(g) := \chi(g) + \psi(g) \quad \text{and} \quad (\chi \cdot \psi)(g) := \chi(g)\psi(g)$$

for all $g \in G$.

Since irreducible characters are generators of all characters, the character ring can be given as

$$R_{\mathbb{C}}(G) := \bigoplus_{\chi \in \mathrm{Irr}(G)} \mathbb{Z}\chi. \tag{2.17}$$

The elements of $R_{\mathbb{C}}(G)$ are called *virtual characters* or *generalized characters* of G.

The module theoretical correspondant of the character ring is defined as the Grothendieck group of the category of finitely generated $\mathbb{C}G$-modules. Precisely, $R_{\mathbb{C}}(G)$ is the quotient of the free abelian group on the set of isomorphism classes of finitely generated $\mathbb{C}G$-modules, by the subgroup generated by all the elements of the form

$$[M \oplus N] - [M] - [N] \tag{2.18}$$

where $[M]$ and $[N]$ denote the isomorphism classes of the $\mathbb{C}G$-modules M and N, respectively. In fact, what is done by this quotient group is inducing a semi-group structure on the set of isomorphism classes of finitely generated $\mathbb{C}G$-modules via

$$[M \oplus N] := [M] + [N]. \tag{2.19}$$

Letting integer coefficients yields an additive abelian group structure and the tensor product over $\mathbb{C}$ completes the ring structure. To sum up, the ring

$$R_{\mathbb{C}}(G) := \bigoplus_{i=1}^{r} \mathbb{Z}[M_i] \tag{2.20}$$

is the representation ring of G, where $M_1, \ldots, M_r$ are the complete list of all non-isomorphic simple modules. This ring is commutative because $M \otimes N$ is isomorphic to $N \otimes M$ for all $\mathbb{C}G$-modules M and N.

2.3. Maps Between Character Rings

In 2.1, we have introduced the operations induction, restriction, inflation, deflation and isogation connecting representations of algebraically related groups. Since these maps are essential in representation theory, they are of great importance in the sequel. We now give the exact correspondence of them in terms of modules and characters, separately. For this aim, let $H \leq G$ and $N \trianglelefteq G$ unless otherwise stated. Consider the regular $\mathbb{C}G$-module $\mathbb{C}G$. As its action is induced by the group action, we can equipped it with a two sided $\mathbb{C}G$-action, $\mathbb{C}H$-action, or $\mathbb{C}(G/N)$-action. For example, if we take $\mathbb{C}G$ as a left $\mathbb{C}G$-module and as a right $\mathbb{C}H$-module, we call $\mathbb{C}G$ a $(\mathbb{C}G, \mathbb{C}H)$-bimodule. For a $\mathbb{C}H$-module M, the tensor product $\mathbb{C}G \otimes_{\mathbb{C}H} M$ yields a $\mathbb{C}G$-module when we regard $\mathbb{C}G$ as a $(\mathbb{C}G, \mathbb{C}H)$-bimodule. The module $\mathbb{C}G \otimes_{\mathbb{C}H} M$ is the induced module. The definition in character theory is a bit more complicated. For a character χ of H, the induced character $\mathrm{Ind}_H^G \chi$ of G is given as

$$\mathrm{Ind}_H^G \chi(g) := \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi(x^{-1}gx). \tag{2.21}$$

Hence, the induction map is defined as follows in both sense

$$\begin{aligned} \mathrm{Ind}_H^G : R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G),\ [M] &\mapsto [\mathbb{C}G \otimes_{\mathbb{C}H} M] \\ \chi &\mapsto \mathrm{Ind}_H^G \chi. \end{aligned}$$

The restriction of a character is just evaluating it on the elements of a subgroup. Similarly, the map is given

$$\begin{aligned} \mathrm{Res}^G_H : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H),\ [M] &\mapsto [\mathbb{C}G \otimes_{\mathbb{C}G} M] \\ \chi &\mapsto \chi|_H \end{aligned}$$

where $\mathbb{C}G$ is now a $(\mathbb{C}H, \mathbb{C}G)$-bimodule. The inflation of a $\mathbb{C}(G/N)$-module M from $\mathbb{C}(G/N)$ to $\mathbb{C}G$ is to consider M as a $\mathbb{C}G$-module via the intuitive way. In the case that $\mathbb{C}(G/N)$ is taken as a $(\mathbb{C}G, \mathbb{C}(G/N))$-bimodule, the iflation of M isomorphic to the $\mathbb{C}G$-module $\mathbb{C}(G/N) \otimes_{\mathbb{C}(G/N)} M$. Thus

$$\begin{aligned} \mathrm{Inf}^G_{G/N} : R_{\mathbb{C}}(G/N) \to R_{\mathbb{C}}(G),\ [M] &\mapsto [\mathbb{C}(G/N) \otimes_{\mathbb{C}G} M] \\ \chi &\mapsto \mathrm{Inf}^G_{G/N}\chi \end{aligned}$$

where $\mathrm{Inf}^G_{G/N}\chi(g) := \chi(gN)$ for any $g \in G$. Similarly, if we take $\mathbb{C}(G/N)$ as a $(\mathbb{C}(G/N), \mathbb{C}G)$-bimodule, we define

$$\begin{aligned} \mathrm{Def}^G_{G/N} : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G/N),\ [M] &\mapsto [\mathbb{C}(G/N) \otimes_{\mathbb{C}(G/N)} M] \\ \chi &\mapsto \mathrm{Def}^G_{G/N}\chi \end{aligned}$$

where $\mathrm{Def}^G_{G/N}\chi(gN) := \chi(gN)$ if $N \leq \ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$. If $\ker(\chi)$ does not contain N, then $\mathrm{Def}^G_{G/N}\chi$ gives the zero character directly. Lastly, let $f : H \to G$ be an isomorphism for some group H. When we afford the H-action via the isomorphism f, the $\mathbb{C}G$-module $\mathbb{C}G$ becomes a $(\mathbb{C}H, \mathbb{C}G)$-bimodule. Then

$$\begin{aligned} {}_H\mathrm{Iso}^f_G : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H),\ [M] &\mapsto [\mathbb{C}G \otimes_{\mathbb{C}G} M] \\ \chi &\mapsto {}_H\mathrm{Iso}^f_G\chi \end{aligned}$$

where ${}_H\mathrm{Iso}^f_G\chi(h) := \chi(f(h))$ for any $h \in H$.

As it is generally easier to study small groups, we try to reach to global knowledge from local knowledge, that is deducing results for groups from results for their subgroups. The following main theorems shows practicability of these maps for this aim.

Theorem 2.10 (Brauer)**.** *Let E_p be the set of p-elementary subgroups of G, i.e. subgroups of G isomorphic to direct product of cyclic groups and p-groups. Then*

$$R_{\mathbb{C}}(G) = \sum_{H \in \bigcup_p E_p} \mathrm{Ind}_H^G R_{\mathbb{C}}(H) \tag{2.22}$$

that is every character of G is a $\mathbb{Z}$-linear combination of characters induced from p-elementary subgroups of G to G.

Theorem 2.11 (Artin)**.** *Each character of a group is a $\mathbb{Q}$-linear combination of characters induced from cyclic subgroups of G to G.*

As the induction theorems of Brauer and Artin show, we have powerful and useful results with only the induction map. Therefore, it is natural to expect that one may obtain more powerful results by using all maps. Biset functor notion serves to meet this expectation by unifying the treatment of all these five basic operations.

3. BISETS AND FIBERED BISETS

This chapter is devoted to basic knowledge about bisets and fibered bisets. One may find almost all about bisets in [5]. As for fibered case, we refer the reader to [4], [8] and [1]. We mostly follow the notations of [4]. Throughout this chapter, the letters G, H, K are reserved for finite groups, whereas X, Y, Z denote sets.

Definition 3.1. *A set X is said to be a* left G-set *if there is a left G-action on it, i.e. there exists a map $f : G \times X \to X$, $(g, x) \mapsto g \cdot x$ subject to the following conditions*

(i) $1 \cdot x = x$
(ii) $g \cdot (g' \cdot x) = (gg') \cdot x$

for all $g, g' \in G$, $x \in X$.

Definition 3.2. *A set X is called a* (G, H)-biset *if it has a left G-action and a right H-action such that the actions commute with each other, that is*

$$(g \cdot x) \cdot h = g \cdot (x \cdot h) \tag{3.1}$$

for all $g \in G, h \in H, x \in X$.

Remark 3.1. A (G, H)-biset X can also be considered as a left $(G \times H)$-set via the action $(g, h) \cdot x := g \cdot x \cdot h^{-1}$, which we generally use for convenience.

Examples 3.1. Let $H \leq G$ and $N \trianglelefteq G$ unless otherwise stated.

(i) Every left G-set is a (G, H)-biset with the trivial right H-action. By the same token, any set X is a (G, H)-biset with the trivial actions, that is $g \cdot x \cdot h = x$ for all $g \in G, h \in H, x \in X$.
(ii) The group G itself is a (G, G)-biset with the actions induced by the group multiplication. This biset is called the *identity* (G, G)-biset, and denoted by Id_G.
(iii) The group G is a (G, H)-biset with the group multiplication. This biset is called

the *induction* from H to G, denoted by Ind_H^G.

(iv) Similarly, the *restriction* biset Res_H^G from G to H is the (H,G)-biset G with the group multiplication.

(v) Consider the quotient group G/N. It has an obvious right (G/N)-action, the group multiplication in G/N, and the left G-action $g \cdot (g'N) := gg'N$. The $(G, G/N)$-biset G/N is the *inflation* from G/N to G, which we denote by $\mathrm{Inf}_{G/N}^G$.

(vi) Likewise, the *deflation* biset $\mathrm{Def}_{G/N}^G$ from G/N to G is the $(G/N, G)$-biset G/N.

(vii) Let $f : G \to H$ be a group isomorphism. Then, the group H is an (H,G)-biset with the actions $h \cdot h' \cdot g := hh' f(g)$ for all $g \in G$, $h, h' \in H$. This biset is denoted by ${}_H\mathrm{Iso}_G^f$, and called the *isogation* biset.

Definition 3.3. *Let X be a (G,H)-biset. For any $x \in X$, the set*

$$G \cdot x \cdot H := \{g \cdot x \cdot h \mid (g,h) \in G \times H\} \tag{3.2}$$

is called the (G,H)-orbit *of x. The set of (G,H)-orbits in X is denoted by $G\backslash X/H$. If the set $G\backslash X/H$ is a singleton, that is, if we can obtain whole X by performing the (G,H)-action on any element of X, we call X* transitive.

Because the set $G\backslash X/H$ gives a partition of X, any (G,H)-biset is a disjoint union of its (G,H)-orbits. Also, since every (G,H)-orbit is clearly a transitive (G,H)-biset, every (G,H)-biset is a disjoint union of transitive ones. On the other hand, disjoint union of bisets are again bisets. Therefore, we can focus only on transitive bisets instead of all bisets.

Definition 3.4. *Let X and Y be two (G,H)-bisets. A map $f : X \to Y$ is called a* morphism *of (G,H)-bisets if it is a (G,H)-equivariant map, i.e.*

$$f(g \cdot x \cdot h) = g \cdot f(x) \cdot h \tag{3.3}$$

for all $g \in G, h \in H, x \in X$. A morphism f is an isomorphism *if it is bijective. Class of all (G,H)-bisets and their morphisms form a category denoted by ${}_G\mathsf{set}_H$.*

Lemma 3.1. (i) *For any $U \leq G \times H$, the set $\left(\frac{G\times H}{U}\right)$ of left cosets in $G \times H$ is a transitive (G, H)-biset.*

(ii) *Every transitive (G, H)-biset X is isomorphic to $\left(\frac{G\times H}{U}\right)$ for some $U \leq G \times H$.*

The proof follows from the fact that any transitive G-set is isomorphic to G/K for some $K \leq G$ and that, as we stated in Remark 3.1, any A-fibered (G, H)-biset is a $(G \times H)$-set. The subgroup U above is the stabilizer of some $x \in X$ in $G \times H$, that is

$$U = S_x := \{(g, h) \in G \times H \mid (g, h) \cdot x = x\}. \tag{3.4}$$

Here, different choice of $x \in X$, gives a subgroup conjugate to U in $G \times H$. Consequently, there is bijective correspondence between

(i) isomorphism classes $[X]$ of transitive (G, H)-bisets, and
(ii) conjugacy classes $[U]$ of subgroups of $G \times H$.

Due to this bijection, from now on, we denote transitive (G, H)-bisets and their isomorphism classes by

$$\left(\frac{G \times H}{U}\right) \quad \text{and} \quad \left[\frac{G \times H}{U}\right]$$

respectively, for appropriate subgroups U. Now, let us see new notations of some of our examples given in 3.1. In the examples, we introduce some other notations that one can encounter in the rest of the thesis.

(i) $\mathrm{Id}_G \cong \left(\frac{G\times G}{G\times G}\right)$.
(ii) ${}_GG_H := \mathrm{Ind}_H^G \cong \left(\frac{G\times H}{\Delta(H)}\right)$, where $\Delta(H) := \{(h, h) \mid h \in H\}$.
(iii) ${}_HG_G := \mathrm{Res}_H^G \cong \left(\frac{H\times G}{\Delta(H)}\right)$.
(iv) ${}_G(G/N)_{G/N} := \mathrm{Inf}_{G/N}^G \cong \left(\frac{G\times G/N}{\Delta_\pi(G)}\right)$, where $\Delta_\pi(H) := \{(g, gN) \mid g \in G\}$ and π is the canonical projection of G onto G/N.
(v) ${}_{G/N}(G/N)_G := \mathrm{Def}_{G/N}^G \cong \left(\frac{G/N\times G}{{}_\pi\Delta(G)}\right)$ where ${}_\pi\Delta(H) := \{(gN, g) \mid g \in G\}$.

(vi) ${}_H\mathrm{Iso}_G^f \cong \left(\frac{H\times G}{{}_f\Delta(G)}\right)$, where ${}_f\Delta(G) := \{(f(g), g) \mid g \in G\}$ for a group isomorphism $f : G \to H$.

We prefer to use these new notations because they are more suitable when we consider bisets as maps. We show how bisets become maps after introducing the Mackey product and the Burnside group.

3.1. Mackey Product of Bisets

Let $X \in {}_G\mathsf{set}_H$ and $Y \in {}_H\mathsf{set}_K$. Then, the *Mackey product* $X \times_H Y \in {}_G\mathsf{set}_K$ of X and Y is defined as the set of H-orbits under the action

$$(x, y) \cdot h = (x \cdot h, h^{-1} \cdot y) \tag{3.5}$$

for all $(x, y) \in X \times Y$, $h \in H$. The H-orbit $(x, y) \cdot H \in X \times_H Y$ of (x, y) is denoted by $(x,_H y)$.

For any $X, X' \in {}_G\mathsf{set}_H$ and $Y, Y' \in {}_H\mathsf{set}_K$, we have the distributive laws

$$\begin{aligned} X \times_H (Y \sqcup Y') &\cong (X \times_H Y) \sqcup (X \times_H Y') \\ (X \sqcup X') \times_H Y &\cong (X \times_H Y) \sqcup (X' \times_H Y). \end{aligned} \tag{3.6}$$

3.2. Burnside Group of Bisets

The Burnside group $B(G, H)$ of the (G, H)-bisets is the Grothendieck group of the category ${}_G\mathsf{set}_H$. Definitively, it is the quotient of the free abelian group on the set of isomorphism classes of the (G, H)-bisets, by the subgroup generated by the elements of the form

$$[X \sqcup Y] - [X] - [Y] \tag{3.7}$$

where $X, Y \in {}_G\mathsf{set}_H$, and $[X]$ is the isomorphism class of X. Since every (G, H)-biset is the disjoint union of transitive ones, we have

$$B(G, H) := \sum_{[U] \in s_{G \times H}} \mathbb{Z} \left[\frac{G \times H}{U} \right] \tag{3.8}$$

where $s_{G \times H}$ is a set of representatives of $(G \times H)$-conjugacy classes of subgroups of $G \times H$. Recall Section 2.2 for details of the construction of Grothendieck groups.

Let $X \in {}_G\mathsf{set}_H$ and $Y \in {}_H\mathsf{set}_1 := {}_H\mathsf{set}$. As we stated in Examples 3.1, Y is an H-set, and $X \times_H Y \in {}_G\mathsf{set}_1 := {}_G\mathsf{set}$ is a G-set by the definition of the Mackey product. Therefore, by the distributivity of the Mackey product on the disjoint union, the (G, H)-biset X can be considered as a map as follows

$$X \times_H - : B(H) \to B(G),\ [Y] \mapsto [X \times_H Y] \tag{3.9}$$

where $B(G) := B(G, 1)$ denotes the Burnside group of left G-sets after identifying G with $G \times 1$.

Remark 3.2. We can regard bisets as an abstraction of bimodules because for any (G, H)-biset X, the $\mathbb{C}$-linearization $\mathbb{C}X$ of X naturally becomes a $(\mathbb{C}G, \mathbb{C}H)$-bimodule. By tensor product with this bimodule, we can define the map

$$\mathbb{C}X \otimes_{\mathbb{C}H} - : R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G),\ [M] \mapsto [\mathbb{C}X \otimes_{\mathbb{C}H} M]. \tag{3.10}$$

We need the notations below to calculate the Mackey product of two bisets precisely.

Notations 3.1. Let $U \leq G \times H$ and $V \leq H \times K$. Then, we set

$$\begin{aligned} U * V &= \{(g,k) \in G \times K \mid (g,h) \in U, (h,k) \in V \text{ for some } h \in H\}, \\ p_1(U) &:= \{g \in G \mid (g,h) \in U \text{ for some } h \in H\}, \\ p_2(U) &:= \{h \in H \mid (g,h) \in U \text{ for some } g \in G\}, \\ k_1(U) &:= \{g \in G \mid (g,1) \in U\} \text{ and } k_2(U) := \{h \in H \mid (1,h) \in U\}. \end{aligned}$$

All the sets above are subgroups of the related groups. Furthermore, we have $k_1(U) \trianglelefteq p_1(U)$, $k_2(U) \trianglelefteq p_2(U)$, and

$$p_1(U)/k_1(U) \cong p_2(U)/k_2(U) \cong U/(k_1(U) \times k_2(U)). \tag{3.11}$$

As it is seen, we get a quintuple $(P, K, \eta, L, Q) := (p_1(U), k_1(U), \eta, k_2(U), p_2(U))$, from a given $U \leq G \times H$, where the isomorphism $\eta : Q/L \to P/K$ is determined via U as $\eta(hL) = gK$ if $(g,h) \in U$. Conversely, a given quintuple (P, K, η, L, Q) satisfying $K \trianglelefteq P \leq G$ and $L \trianglelefteq Q \leq H$, and that $\eta : Q/L \to P/K$ is an isomorphism determines a unique subgroup $U = \{(g,h) \in P \times Q \mid \eta(hL) = gK\} \leq G \times H$. This is known as Goursat's Theorem. Via these notations, we can give the Mackey formula, which enables us to identify Mackey product of transitive bisets.

3.3. Mackey Formula

Theorem 3.1. ([5, Lemma 2.3.24]) *Let $U \leq G \times H$ and $V \leq H \times K$. Then*

$$\left(\frac{G \times H}{U}\right) \times_H \left(\frac{H \times K}{V}\right) \cong \bigsqcup_{h \in [p_2(U) \backslash H / p_1(V)]} \left(\frac{G \times K}{U *^{(h,1)} V}\right) \tag{3.12}$$

where $[p_2(U)\backslash H/p_1(V)]$ is a set of representatives of $(p_2(U), p_1(V))$-orbits in H.

Although our biset examples look very simple, they are of great importance. In fact, Bouc proved that any transitive biset is the Mackey product of five of them.

Theorem 3.2 (Bouc)**.** *Let* $(P, K, \eta, L, Q) = (p_1(U), k_1(U), \eta, k_2(U), p_2(U))$ *be the datum associated to some* $U \leq G \times H$*. Then,*

$$\left(\frac{G \times H}{U}\right) \cong \mathrm{Ind}_P^G \times_P \mathrm{Inf}_{P/K}^P \times_{P/K\ P/K} \mathrm{Iso}_{Q/L}^{\eta} \times_{Q/L} \mathrm{Def}_{Q/L}^Q \times_Q \mathrm{Res}_Q^H. \tag{3.13}$$

Now, we let another player into the game, namely the fiber group A. We aim to analyze fibered bisets to obtain similar results as in the ordinary case.

Definition 3.5. *Let* A *be a multiplicatively written (not necessarily finite) abelian group. An* A*-set* X *is said to be an* A-fibered (G, H)-biset *if the following conditions hold.*

(i) A *acts freely on* X*, i.e. if* $a \cdot x = x$ *for all* $x \in X$*, then* $a = 1$*.*
(ii) The set of A*-orbits in* X *is a finite* (G, H)*-biset.*
(iii) All three actions commute with each other.

In contrast to the ordinary bisets, in the fibered case, we allow sets to be infinite but we want the set of A-orbits of them to be finite. The fibered case can be regarded as a generalization of the former one because when we take $A = 1$, the trivial group, we obtain the ordinary bisets.

Examples 3.2. (i) As remarked above all bisets are A-fibered bisets for the fiber group $A = 1$.

(ii) Let $G = S_3$, $H =<(13)>$ and $A =<(123)> \cong C_3$. In this case, G becomes an A-fibered (G, H)-biset.

Definition 3.6. *An* A*-fibered* (G, H)*-biset* X *is* transitive *if the set of* A*-orbits in* X *is a transitive* (G, H)*-biset.*

Recall that if X is a transitive A-fibered (G, H)-biset, the set of A-orbits in X is isomorphic to $\left(\frac{G \times H}{U}\right)$ as a biset, where $U = S_{[x]}$ for some A-orbit $[x]$. Then, if $(g, h) \in U$, we have $(g, h) \cdot [x] = [(g, h) \cdot x] = [x]$ by the commutativity of the actions

on X. Since $[(g,h)\cdot x]=[x]$, then $(g,h)\cdot x = a\cdot x$ for some $a\in A$. The element $a\in A$ is uniquely determined because A-action is free on X. Therefore, the map

$$\phi_x : U \to A,\ (g,h)\mapsto \phi_x(g,h)=a \tag{3.14}$$

is a well-defined group homomorphism.

Definition 3.7. *The pair $(S_{[x]},\phi_x)$ above is called the* stabilizing pair *of x.*

Notation 3.2. We denote by $\mathcal{M}_{G\times H}(A)$ the set of all pairs (U,ϕ), where U is a subgroup of $G\times H$ and $\phi : U\to A$ is a group homomorphism.

The set $\mathcal{M}_{G\times H}(A)$ admits a $(G\times H)$-conjugation via ${}^{(g,h)}(U,\phi) := ({}^{(g,h)}U, {}^{(g,h)}\phi)$, where ${}^{(g,h)}\phi\left({}^{(g,h)}u\right) := \phi(u)$ for all $u\in U$. Moreover, $\mathcal{M}_{G\times H}(A)$ is a partially ordered set (poset) with the ordering given by

$$(U,\phi)\preceq(V,\psi) \text{ if } U\leq V \text{ and } \phi=\psi|_U.$$

Clearly, the poset structure is invariant under conjugation, i.e. if $(U,\phi)\preceq(V,\psi)$, then ${}^{(g,h)}(U,\phi)\preceq {}^{(g,h)}(V,\psi)$ for all $(g,h)\in G\times H$. We denote the conjugacy class of $(U,\phi)\in\mathcal{M}_{G\times H}(A)$ by $[U,\phi]_{G\times H}$.

Lemma 3.2. *For any $(U,\phi)\in\mathcal{M}_{G\times H}(A)$, the set $\left(\frac{G\times H\times A}{U_\phi}\right)$ is a transitive A-fibered (G,H)-biset, where $U_\phi := \{(u,\phi(u^{-1})\mid u\in U\}\leq G\times H\times A$.*

Definition 3.8. *A* morphism *of A-fibered (G,H)-bisets is a morphism of (G,H)-bisets that is also A-equivariant. A bijective morphism is called* isomorphism. *The category whose objects are A-fibered (G,H)-bisets with their morphisms denoted by ${}_G\mathsf{set}^A_H$.*

With this definition, we have all to show the characterization of transitive fibered bisets.

Theorem 3.3. *There exists a bijection between isomorphism classes $[X]$ of transitive A-fibered (G,H)-bisets and conjugacy classes $[U,\phi]_{G\times H}$ of elements in $\mathcal{M}_{G\times H}(A)$.*

In detail, for a given transitive A-fibered (G,H)-biset X, the corresponding element of $\mathcal{M}_{G\times H}(A)$ is $(S_{[x]},\phi_x)$ for some A-orbit $[x]$ in X. The inverse correspondence of the bijection is as shown in Lemma 3.2.

As a result of this bijection, we denote transitive A-fibered (G,H)-bisets and their isomorphism classes by

$$\left(\frac{G\times H}{U,\phi}\right) \quad \text{and} \quad \left[\frac{G\times H}{U,\phi}\right]$$

respectively, for an appropriate pair $(U,\phi)\in\mathcal{M}_{G\times H}(A)$. In the sequel, we show that ordinary bisets constitute a major part of the fibered biset formulae also. But, when we use ordinary (G,H)-bisets in the fibered case, we use the notation $\left(\frac{G\times H}{U,1}\right)$ instead of $\left(\frac{G\times H}{U}\right)$.

Definition 3.9. *Let $X\in {}_G\mathsf{set}^A_H$. The* opposite $X^{\mathrm{op}}\in {}_H\mathsf{set}^A_G$ *of X is the A-fibered (H,G)-biset X with the same A-action, and with the $(H\times G)$-action given via*

$$(h,g)\cdot x := (g^{-1},h^{-1})\cdot x. \tag{3.15}$$

Remark 3.3. If $X\cong\left(\frac{G\times H}{U,\phi}\right)$, then $X^{\mathrm{op}}\cong\left(\frac{H\times G}{U^{\mathrm{op}},\phi^{\mathrm{op}}}\right)$, where

$$U^{\mathrm{op}} := \{(h,g)\mid (g,h)\in U\}\leq H\times G \quad \text{and} \quad \phi^{\mathrm{op}}(h,g) := (\phi(g,h))^{-1}.$$

3.4. Tensor Product of Fibered Bisets

Let $X\in {}_G\mathsf{set}^A_H$ and $Y\in {}_H\mathsf{set}^A_K$. Then, the Mackey product $X\times_{AH}Y\in {}_G\mathsf{set}_K$ of X and Y is defined as the set of orbits in $X\times Y$ under the $(A\times H)$-action

$$(x,y)\cdot(a,h) = (x\cdot(a,h),(a^{-1},h^{-1})\cdot y) \tag{3.16}$$

for all $(x, y) \in X \times Y$, $(a, h) \in A \times H$. The $(A \times H)$-orbit of (x, y) is denoted by $(x,_{AH} y)$. The set $X \times_{AH} Y$ is both an A-set and a (G, H)-biset via the actions

$$\begin{aligned} a \cdot (x,_{AH} y) &:= (a \cdot x,_{AH} y) = (x,_{AH} a \cdot y), \\ (g, k) \cdot (x,_{AH} y) &:= (g \cdot x,_{AH} y \cdot k^{-1}), \end{aligned} \tag{3.17}$$

respectively. We have the following properties on $X \times_{AH} Y$.

(i) For every $a \in A$, we have $a \cdot (x,_{AH} y) = (x \cdot a,_{AH} y) = (x,_{AH} a \cdot y) = (x,_{AH} y \cdot a)$. The equations in the left and right hold because A is abelian and we can switch a left A-action to a right A-action. To verify the middle equation, observe that $(x \cdot a,_{AH} y) = (a, 1) \cdot (x \cdot a,_{AH} y) = ((x \cdot a) \cdot a^{-1},_{AH} a \cdot y) = (x,_{AH} a \cdot y)$ as $(x \cdot a,_{AH} y)$ is an $(A \times H)$-orbit.

(ii) By the same token, $(x \cdot h,_{AH} y) = (x,_{AH} h \cdot y)$, for any $h \in H$.

Despite all these properties, $X \times_{AH} Y$ is not always an A-fibered biset because A-action may not remain free. Therefore, we define the tensor product

$$X \otimes_{AH} Y \in {}_G\mathsf{set}_K^A \tag{3.18}$$

as the union of the elements of $X \times_{AH} Y$ on which A acts freely. Since the construction is the same as the Mackey product of bisets, the distribution on the disjoint union is also valid for the tensor product. We use the notation $x \otimes_{AH} y$ for elements of $X \otimes_{AH} Y$.

We need the following notations for Boltje and Coşkun's formula calculating the tensor product of transitive fibered bisets.

Notations 3.3. If $(U, \phi) \in \mathcal{M}_{G \times H}(A)$ and $(V, \psi) \in \mathcal{M}_{H \times K}(A)$, then the homomorphism $\phi * \psi : U * V \to A$ is defined by

$$(\phi * \psi)(g, k) = \phi(g, h)\psi(h, k) \tag{3.19}$$

for some choice of $h \in H$ such that $(g,h) \in U$ and $(h,k) \in V$. Sometimes, we need only parts of the datum $(p_1(U), k_1(U), \eta, k_2(U), p_2(U))$, which are *left invariants* $l(U,\phi) := (p_1(U), k_1(U), \phi_1)$ and *right invariants* $r(U,\phi) := (p_2(U), k_2(U), \phi_2)$ of (U,ϕ), where the homomorphisms $\phi_i : k_i(U) \to A$, $i = 1,2$ are defined through the equation $\phi|_{(k_1(U)\times k_2(U))} := \phi_1 \times (\phi_2)^{-1}$. We take the inverse of the second homomorphism in order to have formulae in the sequel looked nicer.

Adopting the notations above and those of Section 3.3, the formula for the tensor product of transitive A-fibered bisets is given in [4, Corollary 2.5] as

$$\left(\frac{G\times H}{U,\phi}\right) \otimes_{AH} \left(\frac{H\times K}{V,\psi}\right) \cong \bigsqcup_{\substack{x\in[p_2(U)\backslash H/p_1(V)] \\ \phi_2|_{H_x} = {}^x\psi_1|_{H_x}}} \left(\frac{G\times K}{U *^{(x,1)} V, \phi *^{(x,1)} \psi}\right) \tag{3.20}$$

where $H_x = k_2(U) \cap {}^x k_1(V)$. Due to the condition $\phi_2|_{H_x} = {}^x\psi_1|_{H_x}$, the homomorphism $\phi *^{(x,1)} \psi$ is independent of the choice of $h \in H$.

Let $\left(\frac{G\times H}{U,\phi}\right)$ be any transitive A-fibered (G,H)-biset and (P,K,η,L,Q) be the quintuple $(p_1(U), k_1(U), \eta, k_2(U), p_2(U))$ afforded by U. As we stated in Theorem 3.2, any transitive biset is the Mackey product of five canonical bisets. A similar decomposition for any A-fibered (G,H)-biset is obtained partially in [4]. It is precisely

$$\left(\frac{G\times H}{U,\phi}\right) \cong \mathrm{Ind}_P^G \otimes_{AP} \mathrm{Inf}_{P/\hat{K}}^P \otimes_{A(P/\hat{K})} Y \otimes_{A(Q/\hat{L})} \mathrm{Def}_{Q/\hat{L}}^Q \otimes_{AQ} \mathrm{Res}_Q^H \tag{3.21}$$

where $\hat{K}$ and $\hat{L}$ are kernels of ϕ_1 and ϕ_2, respectively. Let us denote the stabilizing pair of the transitive A-fibered $(P/\hat{K}, Q/\hat{L})$-biset Y by $(\bar{U}, \bar{\phi})$. Here, the first and the second projections of $\bar{U}$ are full, i.e. $p_1(\bar{U}) = P/\hat{K}$ and $p_2(\bar{U}) = Q/\hat{L}$, and the homomorphisms $\bar{\phi}_1$ and $\bar{\phi}_2$ are faithful. Boltje and Coşkun decomposed Y fully with the following additional condition on A. To state the full decomposition, we need many new definitions that we will not use in the sequel, therefore see [4, Section 10] for further details. Since we also impose the condition on A in certain parts of the thesis, we recall it.

Hypothesis 3.1. *There exists a (unique) set π of primes such that for every $n \in \mathbb{N}$, the n-torsion part of A is cyclic of order n_π, where n_π denotes the π-part of n.*

The meaning of the hypothesis is that A is divisible, that is if A contains p-th roots of unity for some prime number $p \in \pi$, then A contains p^n-th roots of unity for every $n \in \mathbb{N}$.

3.5. Burnside Group of Fibered Bisets

The Burnside group $B^A(G,H)$ of the A-fibered (G,H)-bisets is the Grothendieck group of the category ${}_G\mathsf{set}^A_H$. As in the ordinary case, due to the fact that transitive A-fibered (G,H)-bisets form a basis for $B^A(G,H)$ and Theorem 3.3, we have

$$B^A(G,H) := \sum_{[U,\phi]\in[\mathcal{M}_{G\times H}(A)]} \mathbb{Z}\left[\frac{G\times H}{U,\phi}\right] \tag{3.22}$$

where $[\mathcal{M}_{G\times H}(A)]$ is a set of representatives of $(G\times H)$-conjugacy classes of elements in $\mathcal{M}_{G\times H}(A)$. Recall 2.2 for details of the construction of Grothendieck groups. As tensor product is possible in $E_G = \mathrm{End}_{\mathcal{C}}(G) := RB^A(G,G)$, it also has a ring structure and hence, E_G is an R-algebra.

Definition 3.10. *The elements of $B^A(G,H)$ are called* virtual *A-fibered (G,H)-bisets.*

Similar to the ordinary case, fibered bisets can be regarded as maps via the tensor product. That is

$$X\otimes_{AH} - : B^A(H) \to B^A(G),\ [Y] \mapsto [X\otimes_{AH} Y] \tag{3.23}$$

where X is an A-fibered (G,H)-biset.

3.6. Idempotents in E_G

Let $\mathcal{M}_G(A)$ denote the set $\mathcal{M}_{G\times 1}(A)$. Recall that $\mathcal{M}_G(A)$ is a G-set via conjugation. We denote by $\mathcal{M}_G(A)^G$ the set of G-fixed points in $\mathcal{M}_G(A)$, i.e. the set of pairs $(K,\kappa) \in \mathcal{M}_G(A)$ such that $K \trianglelefteq G$, and $\kappa(k^g) = \kappa(k)$ for all $k \in K$, $g \in G$. For $(K,\kappa) \in \mathcal{M}_G(A)^A$, Boltje and Coşkun introduced the A-fibered (G,G)-biset

$$\mathrm{E}_{K,\kappa} := \left(\frac{G\times G}{\Delta_K(G), \phi_\kappa}\right) \tag{3.24}$$

where $\Delta_K(G) := \{(gk, g) \mid g \in G, k \in K\} = (K \times 1)\Delta(G) = (1 \times K)\Delta(G)$, and $\phi_\kappa(gk, g) := \kappa(k)$. They called $\mathrm{E}_{K,\kappa}$ *reduced* if it cannot be factored through a group of smaller order than $|G|$, and found the necessary and sufficient conditions to be reduced when A satisfies Hypothesis 3.1 ([4, Corollary 10.13]). It is an idempotent in E_G. Moreover, if $(K,\kappa) \preceq (L,\lambda)$ for some $(L,\lambda) \in \mathcal{M}_G(A)^A$, then $\mathrm{E}_{K,\kappa} \otimes_{AG} E_{L,\lambda} \cong E_{L,\lambda}$ ([4, Proposition 4.2]). Set $e_{K,\kappa} := [\mathrm{E}_{K,\kappa}] \in E_G = RB^A(G,G)$, and

$$f_{K,\kappa} := \sum_{(K,\kappa)\preceq(L,\lambda)\in\mathcal{M}_G(A)^A} \mu^{\triangleleft}_{(K,\kappa),(L,\lambda)} e_{L,\lambda} \tag{3.25}$$

where $\mu^{\triangleleft}_{(K,\kappa),(L,\lambda)}$ is the Möbius coefficient with respect to the poset $\mathcal{M}_G(A)^A$.

Lemma 3.3. *Following the notation above, we have*

$$\sum_{(K,\kappa)\in\mathcal{M}_G(A)^A} f_{K,\kappa} = e_{1,1} = 1 \in E_G. \tag{3.26}$$

The idempotent $\mathrm{E}_{K,\kappa}$ is crucial for us because we show that it is a multiplier of any transitive A-fibered biset when A does not satisfy Hypothesis 3.1 in our set-up. That is why, we also find when $\mathrm{E}_{K,\kappa}$ is reduced to achieve full decomposition of any transitive fibered biset for cyclic groups.

4. A-FIBERED BISET FUNCTORS

Definition 4.1. *Let R be a commutative ring with unity. The A-fibered biset category $\mathcal{C} := \mathcal{C}_R^A$ of finite groups is the category defined as below*

(i) The objects of $\mathcal{C}$ are finite groups.

(ii) For any two objects G and H, $\mathrm{Hom}_{\mathcal{C}}(G,H) := RB^A(H,G) := R \otimes_{\mathbb{Z}} B^A(H,G)$, i.e. morphisms from G to H are the R-linear extensions of the virtual A-fibered (H,G)-bisets.

(iii) The composition of morphisms in $\mathcal{C}$ is the R-linear extension of the tensor product of fibered bisets.

(iv) For any object G, the identity morphism of G is $R \otimes_{\mathbb{Z}} [\mathrm{Id}_G] = R \otimes_{\mathbb{Z}} \left[\frac{G\times G}{G\times G}\right]$.

Remark 4.1. The category $\mathcal{C}$ is an R-linear category, that is for any objects G and H, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(G,H)$ is an R-module, and the composition of morphisms is R-bilinear.

Definition 4.2. *An A-*fibered biset functor *F over R is an R-linear functor from the category $\mathcal{C}$ to the category R-*Mod*, that is a functor F from $\mathcal{C}$ to R-*Mod* such that the maps that F induces between sets of morphisms are R-linear.*

Examples 4.1. (i) The map RB^A sending any finite group G to the Burnside group $RB^A(G)$ of left A-fibered G-sets, is an A-fibered biset functor over R, called the *A-fibered Burnside functor*. For any virtual biset $\gamma \in RB^A(G,H) = \mathrm{Hom}_{\mathcal{C}}(H,G)$, the map $RB^A(\gamma) : RB^A(H) \to RB^A(G)$ is the R-linear extension of the map $X \otimes_{AH} - : B^A(H) \to B^A(G)$ shown in Section 3.4.

(ii) Consider the map

$$R_{\mathbb{C}} : G \mapsto R_{\mathbb{C}}(G) \tag{4.1}$$

sending any finite group G to its representation ring $R_{\mathbb{C}}(G)$. Then, $R_{\mathbb{C}}$ is an A-fibered biset functor for any $A \leq \mathbb{C}^\times$, where $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ is the multiplicative group of invertible complex numbers. We call it the *functor of complex character*

ring.

(iii) Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$, and let $A \leq \mathbb{F}^\times$. We denote the ring of trivial source $\mathbb{F}G$-modules by $T_{\mathbb{F}}(G)$. Then, the map assigning the ring $T_{\mathbb{F}}(G)$ to each finite group G is an A-fibered biset functor for any $A \leq \mathbb{F}^\times$. See [2] for the definition of trivial source modules.

Together with natural transformations, A-fibered biset functors form a category which we denote by $\mathcal{F} := \mathcal{F}_R^A$. Since the category R-Mod is abelian, the category $\mathcal{F}$ is also abelian with the pointwise evaluation of kernels and cokernels. That is to say, if $f : F_1 \to F_2$ is the natural transformation, then

$$(\ker(f))(G) = \ker(f_G) \quad \text{and} \quad (\operatorname{coker}(f))(G) = \operatorname{coker}(f_G) \tag{4.2}$$

where $f_G : F_1(G) \to F_2(G)$ for any finite group G. This property enables us to define subfunctors, quotient functors, projective functors, simple functors, etc.

Definition 4.3. *A group G is said to be* minimal *for an A-fibered biset functor F if $F(G) \neq 0$ and $F(H) = 0$ for any group H such that $|H| < |G|$.*

Let $(K, \kappa) \in \mathcal{M}_G(A)^G$, a G-fixed pair in $\mathcal{M}_G(A)$. The canonical basis elements $\left[\frac{G\times H}{U,\phi}\right] \in B^A(G, G)$ satisfying $l(U, \phi) = (G, K, \kappa) = r(U, \phi)$ form a group $\Gamma_{G,K,\kappa}$. The identity element of the group is $e_{K,\kappa}$ shown in 3.6, and the inverse of $\left[\frac{G\times H}{U,\phi}\right]$ is the opposite biset $\left[\frac{H\times G}{U^{\mathrm{op}},\phi^{\mathrm{op}}}\right]$.

Definition 4.4. *Let $(K, \kappa) \in \mathcal{M}_G(A)^G$ and $(L, \lambda) \in \mathcal{M}_H(A)^H$. If there exists a pair $(U, \phi) \in \mathcal{M}_{G\times H}(A)$ such that $l(U, \phi) = (G, K, \kappa)$ and $r(U, \phi) = (H, L, \lambda)$, then (G, K, κ) and (H, L, λ) are said to be* linked.

Remark 4.2. Assume the hypothesis of the definition above. The set

$${}_{G,K,\kappa}\Gamma_{H,L,\lambda} := \{\left[\frac{G \times H}{U, \phi}\right] \mid l(U, \phi) = (G, K, \kappa) \text{ and } r(U, \phi) = (H, L, \lambda)\} \tag{4.3}$$

is a $(\Gamma_{G,K,\kappa}, \Gamma_{H,L,\lambda})$-biset. Obviously this set is non-empty if and only if (G, K, κ) and (H, L, λ) are linked. Notice that ${}_{G,K,\kappa}\Gamma_{H,L,\lambda}$ induces a bijection

$$(R \otimes_{\mathbb{Z}\, G,K,\kappa}\Gamma_{H,L,\lambda}) \otimes_{R\Gamma_{H,L,\lambda}} - \ : \ \mathrm{Irr}(R\Gamma_{H,L,\lambda}) \to \mathrm{Irr}(R\Gamma_{G,K,\kappa}) \tag{4.4}$$

between irreducible left modules of the related algebras.

Theorem 4.1. *[4, Theorem 9.2] Any simple A-fibered biset functor S is parametrized by the quadruples $(G, K, \kappa, [V])$, that is S is of the form $S_{G,K,\kappa,[V]}$, where G is a minimal group for S, the pair $(K, \kappa) \in \mathcal{M}_G(A)^G$ such that $\mathrm{E}_{K,\kappa}$ is reduced, and $[V]$ is the isomorphism class of the irreducible $R\Gamma_{G,K,\kappa}$-module V.*

Examples 4.2. (i) If $\mathbb{F}$ is a field, then the the first functor example $\mathbb{F}B^A$ above is a projective A-fibered biset functor and it is an indecomposable object of $\mathcal{F}$. Besides, $\mathbb{F}B^A$ is a projective cover of the simple functor $S_{1,1,1,[\mathbb{F}]}$.

(ii) The functor $\mathbb{C}R_{\mathbb{C}} : G \mapsto \mathbb{C}R_{\mathbb{C}}(G) := \mathbb{C} \otimes_{\mathbb{Z}} R_{\mathbb{C}}(G)$ is the $\mathbb{C}$-linear extension of the second functor in Examples 4.1. This functor is also an A-fibered biset functor for any $A \leq \mathbb{C}^\times$. Besides, it is simple and isomorphic to the simple functor $S_{1,1,1,[\mathbb{C}]}$ when $A = \mathbb{C}^\times$.

5. ABELIAN CASE

The basic theory of fibered bisets and fibered biset functors are almost fully analyzed by Boltje and Coşkun in the case that the fiber group satisfies Hypothesis 3.1. In the present thesis, we free the fiber group from this condition. Because it is a challenging task, we decided to move step by step. As a first step, we aim to obtain our results for cyclic groups. That is why, we need to cover previous results for abelian groups, then reduce them to cyclic groups when it is necessary. From now on, in this chapter all groups are assumed to be abelian groups unless otherwise stated.

Let $G = <g>$ be a cyclic group of order n. Since characters of abelian groups are group homomorphisms, any character $\psi : G \to \mathbb{C}$ is of the form $g^i \mapsto a^i$ for some $i \in \mathbb{N}$, where a is an n-th root of unity. Because any n-th root unity a is of the form $a = \omega^j$, $1 \leq j \leq n$ for a fixed primitive n-th root of unity ω, any character of G is of the form $\psi_j : G \to \mathbb{C}$, $g^i \mapsto \omega^{ij}$ for some $1 \leq j \leq n$. Hence, if we denote the character of G sending g to ω by χ, we obtain that $\psi_j = \chi^j$, that is $\mathrm{Irr}(G) = \{\chi, \chi^2, \ldots, \chi^n = 1\}$. With the multiplication $\chi^i \cdot \chi^j := \chi^{i+j}$, the set $\mathrm{Irr}(G)$ becomes a group. Moreover, $\mathrm{Irr}(G)$ is a cyclic group generated by χ, and isomorphic to G. Note that $\mathrm{Irr}(G) = \mathrm{Hom}(G, \mathbb{C}^\times)$.

In general, let G be an abelian group and let $|G| = {p_1}^{n_1}{p_2}^{n_2} \ldots {p_r}^{n_r}$ be the prime factorization of $|G|$. Since $G \cong C_{{p_1}^{n_1}} \times C_{{p_2}^{n_2}} \times \cdots \times C_{{p_r}^{n_r}}$, any character (homomorphism) $\chi : G \to \mathbb{C}$ is of the form $(\chi_1)^{j_1}(\chi_2)^{j_2} \ldots (\chi_r)^{j_r}$, $1 \leq j_i \leq {p_i}^{n_i}$, where χ_j is the generator of $\mathrm{Irr}(C_{{p_i}^{n_i}})$, $1 \leq i \leq r$.

Recall that we did not give the final decomposition of any transitive A-fibered biset due to its complication and the excessive notations even if A satisfies Hypothesis 3.1. However, in the case that G and H are abelian groups and A satisfies the hypothesis, we can state the full decomposition of a transitive A-fibered (G, H)-biset with a new fibered biset in addition to the five canonical bisets: *Twist*.

Until the end of this chapter, we assume that A satisfies Hypothesis 3.1.

Definition 5.1. *If $\varphi : G \to A$ is a group homomorphism, then the Twist by φ at G is the A-fibered (G,G)-biset*

$$\mathrm{Tw}_G^{\varphi} = \left(\frac{G \times G}{\Delta(G), \Delta(\varphi)}\right) \tag{5.1}$$

where $\Delta(G) := \{(g,g) \mid g \in G\}$ and $\Delta(\varphi)(g,g) := \varphi(g)$ for any $g \in G$.

Let $(U,\phi) \in \mathcal{M}_{G\times H}(A)$ and $(P,K,\eta,L,Q) = (p_1(U), k_1(U), \eta, k_2(U), p_2(U))$ be the invariants determined by (U,ϕ). We use the notation $\tilde{\phi} = \tilde{\phi}_1 \times \tilde{\phi}_2$ for an extension of ϕ to $P \times Q$ which exists since the group $P \times Q$ is abelian and A is divisible by the hypothesis.

Theorem 5.1. ([7, Coşkun-Yılmaz]) *Let G, H, A and (U,ϕ) be as above. Then*

$$\left(\frac{G \times H}{U,\phi}\right) \cong \mathrm{Ind}_P^G \ \mathrm{Tw}_P^{\tilde{\phi}_1} \ \mathrm{Inf}_{P/K \ P/K}^P \mathrm{Iso}_{Q/L}^{\eta} \ \mathrm{Def}_{Q/L}^Q \mathrm{Tw}_Q^{\tilde{\phi}_2} \ \mathrm{Res}_Q^H. \tag{5.2}$$

We need to introduce some notations to classify simple fibered biset functors whose minimal groups are abelian. We denote by $\bar{E}_G$ the subalgebra of the algebra $E_G = RB^A(G,G)$ consisting of the A-fibered (G,G)-bisets which cannot be factored through a group of smaller order. The algebra $\bar{E}_G$ plays an essential role in the classification. Its structure is analyzed generally in [4, Section 8]. We describe its structure when G is abelian. Note that, in this paper, we only need the case that the minimal group is cyclic, but we include a more general case since the same arguments still work in this case.

Let $\left(\frac{G\times G}{U,\phi}\right)$ be a transitive A-fibered (G,G)-biset which does not factor through a group of smaller order. Then, it is decomposed as

$$\left(\frac{G \times G}{U,\phi}\right) \cong \mathrm{Tw}_G^{\tilde{\phi}_1} \otimes_{AG \ G} \mathrm{Iso}_G^{\lambda} \otimes \mathrm{Tw}_G^{\tilde{\phi}_2} \tag{5.3}$$

since we deduce by Theorem 5.1 that

$$P = Q = G, \quad K = L = 1, \quad \text{and} \quad U = \{(g, \lambda(g)) \in G \times G \mid \lambda \in \mathrm{Out}(G)\}.$$

The homomorphism $\tilde{\phi} = \tilde{\phi}_1 \times \tilde{\phi}_2$ above is an extension of ϕ to $G \times G$. Moreover, from the tensor product formula, we easily obtain

$$ {}_G\mathrm{Iso}_G^{\lambda} \otimes_{AG} \mathrm{Tw}_G^{\phi} \cong \mathrm{Tw}_G^{\phi\circ\lambda} \otimes_{AG\ G}\mathrm{Iso}_G^{\lambda}. \tag{5.4}$$

Therefore, the algebra $\bar{E}_G$ is generated by all A-fibered (G, G)-bisets of the form $\mathrm{Tw}_G^{\phi} \otimes_{AG\ G}\mathrm{Iso}_G^{\lambda}$, where $\phi \in G^A = \mathrm{Hom}(G, A)$ and $\lambda \in \mathrm{Out}(G)$. We use the notation

$$[\phi, \lambda]_G := \mathrm{Tw}_G^{\phi} \otimes_{AG\ G}\mathrm{Iso}_G^{\lambda} \tag{5.5}$$

for short. Now, if $\lambda, \mu \in \mathrm{Out}(G)$ and $\phi, \psi \in G^A$, again from the tensor product formula, we obtain the following equations

$$[1, \lambda]_G \cdot [1, \mu]_G = [1, \lambda \cdot \mu]_G, \tag{5.6}$$

$$[1, \lambda]_G \cdot [\phi, 1]_G \cdot [1, \lambda^{-1}] = [\phi \circ \lambda, 1]_G, \tag{5.7}$$

$$[\phi, 1]_G \cdot [\psi, 1]_G = [\phi \circ \psi, 1]_G. \tag{5.8}$$

As a result, by the equations above we construct an algebra isomorphism between $\bar{E}_G$ and $R[G^A \rtimes \mathrm{Out}(G)]$, via the map

$$\bar{E}_G \to R[G^A \rtimes \mathrm{Out}(G)], \quad [\phi, \lambda] \mapsto \phi \cdot \lambda. \tag{5.9}$$

5.1. Simple Functors

Let S be a simple fibered biset functor such that its minimal group is abelian. In this case, it can easily be shown that $S(G)$ is a simple E_G-module.

The evaluation $V := S(G)$ is actually a simple $\bar{E}_G$-module as the minimality of G implies that any A-fibered (G, G)-biset which factors through a group of smaller order annihilates $S(G)$. Thus, V is a simple $R[G^A \rtimes \mathrm{Out}(G)]$-module by Map 5.9. Furthermore, any minimal group for S should be isomorphic with G ([4, Proposition 9.4]).

Indeed, recall that general parametrization of any simple A-fibered biset functor is of the form $S_{G,K,\kappa,V}$ for some quadruple (G, K, κ, V) such that $\mathrm{E}_{K,\kappa}$ is reduced. In the case A satisfies the hypothesis, it is shown that $\mathrm{E}_{K,\kappa}$ is reduced if and only if K is a cyclic π-group such that $K \leq Z(G) \cap G'$, and κ is a faithful character of K ([4, Corollary 10.13]). Since we take G as abelian in our case, there is only one possibility for such (K, κ), which is $(1, 1)$. Moreover, in this case, we have $\Gamma_{G,1,1} \cong G^A \rtimes \mathrm{Out}(G)$.

On the other hand, let H be another minimal group for S. We claim that $H \cong G$. Indeed, by Theorem 9.2 in [4], we have $|H| = |G|$ and by Proposition 9.5 in [4], if $S(H)$ is non-zero then there is a section $H_1 \trianglelefteq H_2 \leq H$ of H and a subgroup L of $H^* = H_2/H_1$ such that $G \cong H^*/L$ and $L \cap (H^*)' = 1$. Now since $|G| = |H|$, we must have $G \cong H$.

As a result, we have proved the following theorem.

Theorem 5.2. *Let F be a field. Then, there is a bijective correspondence between the set of isomorphism classes of simple A-fibered biset functors with an abelian minimal group and the set of pairs (G, V), where G runs over all finite abelian groups, up to isomorphism, and V runs over the isomorphism classes of simple $k[G^A \rtimes \mathrm{Out}(G)]$-modules.*

6. FUNCTOR OF COMPLEX CHARACTER RING

This chapter is devoted to details related to the structure of the functor of complex character ring introduced in Examples 4.1. It is studied as various objects such as a Mackey functor, a Green biset functor, an ordinary biset functor and a $\mathbb{C}^\times$-fibered biset functor. For completeness, we include known results regarding the aspects in which the complex character ring treated. We refer to [13] and [12] for preliminaries on Mackey functors and Green biset functors, respectively. Before recalling the results, we need the following definition.

Definition 6.1. *For any character or equivalently homomorphism $\zeta : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$, the map $\tilde{\zeta} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$ given by*

$$\tilde{\zeta}(x) = \begin{cases} \zeta(x) & \text{if } x \in (\mathbb{Z}/m\mathbb{Z})^\times \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

for any $x \in \mathbb{Z}/m\mathbb{Z}$ is called a Dirichlet character *modulo m. A Dirichlet character $\tilde{\zeta} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$ is called* primitive *if ζ cannot be factored through a proper quotient of $(\mathbb{Z}/m\mathbb{Z})^\times$, that is it is not induced from any character of smaller modulus. We denote by Γ the set of pairs (m, ζ), where $m \in \mathbb{Z}^+$ and $\tilde{\zeta}$ is a primitive Dirichlet character modulo m.*

Theorem 6.1. *Let $\mathbb{F}$ be a field of characteristic zero. Then*

(i) ([13, Thévenaz-Webb]) *For any finite group G, there is an isomorphism*

$$\mathbb{Q}R_{\mathbb{C}}^G \cong \bigoplus_{\substack{(H,V):H \text{ cyclic} \\ H \leq_G G}} n_{H,V} S_{H,V}^G \tag{6.2}$$

of Mackey functors for G over $\mathbb{Q}$. Here, $n_{H,V}$ denotes the multiplicity of the $\mathbb{Q}[N_G(H)/H]$-module V in $\mathbb{Q}(\zeta_{|H|})$, and $S_{H,V}^G$ is the simple Mackey functor for G parameterized by (H, V).

(ii) ([12, Romero]) *The functor $\mathbb{C}R_{\mathbb{C}}$ is simple as a Green biset functor.*

(iii) ([5, Bouc]) *There is an isomorphism*

$$\mathbb{C}R_{\mathbb{C}} \cong \bigoplus_{(m,\zeta)\in\Gamma} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta} \tag{6.3}$$

of biset functors, where $\mathbb{C}_\zeta$ denotes the $\mathbb{C}\mathrm{Out}(\mathbb{Z}/m\mathbb{Z})$-module $\mathbb{C}$ on which the group $\mathrm{Out}(\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ acts via ζ.

(iv) ([4, Boltje-Coşkun]) *There is an isomorphism*

$$\mathbb{F}R_{\mathbb{C}} \cong S_{1,1,1,[\mathbb{F}]} \tag{6.4}$$

of $\mathbb{C}^\times$-fibered biset functors where the right hand side is the simple functor parameterized by the trivial group.

We pointed out in Examples 4.2 that $\mathbb{C}R_{\mathbb{C}}$ is an A-fibered biset functors for any $A \leq \mathbb{C}^\times$. When we take $A = 1$, it becomes an ordinary biset functor. Note that Bouc's decomposition can be thought as a decomposition of 1-fibered biset functors, and hence the above two results cover the two extreme cases where the fiber group is the smallest and the largest. For the rest of the thesis, we want to determine the structure of $\mathbb{C}R_{\mathbb{C}}$ as an A-fibered biset functor for some other nice choices of the fiber group A. We should give some details about the simple summands of $\mathbb{C}R_{\mathbb{C}}$ to reach our aim. More details can be found in [5, Chapter 7].

As in Chapter 4, let B^A denote the A-fibered Burnside functor, sending any group G to the Burnside group $B^A(G)$. The well-known $\mathbb{C}$-linearization map associates to any transitive A-fibered G-set X with the stabilizing pair (U, ϕ) the monomial $\mathbb{C}G$-module $\mathbb{C}X$ with monomial basis X. In other words, $\mathbb{C}X$ is the $\mathbb{C}$-vector space with basis X and the $\mathbb{C}G$-action inherited from the G-action on X. It is easy to show that

$$\mathbb{C}X \cong \mathrm{Ind}_U^G \mathbb{C}_\phi \tag{6.5}$$

as $\mathbb{C}G$-modules, where $\mathbb{C}_\phi$ denotes the 1-dimensional representation of U with the character ϕ. Now, the linearization map is defined as the linear extension of this correspondence. Similarly, if Y is an A-fibered (H, G)-biset, then the linearization of Y can be regarded as a monomial $(\mathbb{C}H, \mathbb{C}G)$-bimodule and hence we obtain a group homomorphism

$$R_{\mathbb{C}}(Y) : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H) \tag{6.6}$$

given by $R_{\mathbb{C}}(Y)([M]) = [\mathbb{C}Y \otimes_{\mathbb{C}G} M]$. For simplicity, we denote $R_{\mathbb{C}}(Y)$ by ${}_RY$. It is shown in [4, Subsection 11B] that with this action of fibered bisets, the functor $R_{\mathbb{C}}$ becomes an A-fibered biset functor.

In this thesis, we want to work with characters instead of $\mathbb{C}G$-modules. The following lemma describes the above group homomorphism in terms of characters.

Lemma 6.1. *Let $\chi \in R_{\mathbb{C}}(G)$ be the character of the $\mathbb{C}G$-module M and $Y = \left(\frac{H\times G}{V,\psi}\right)$ be a transitive A-fibered (H, G)-biset. Then the character ${}_RY(\chi)$ of the module $[\mathbb{C}Y \otimes_{\mathbb{C}G} M]$ is given by*

$${}_RY(\chi)(h) = \frac{1}{|V|} \sum_{\substack{x\in H, g\in G \\ (h^x, g)\in V}} \psi(h^x, g)\chi(g). \tag{6.7}$$

Proof. Since $\mathbb{C}Y$ is a $(\mathbb{C}H, \mathbb{C}G)$-bimodule, then $[\mathbb{C}Y \otimes_{\mathbb{C}G} M] \in R_{\mathbb{C}}(H)$. Therefore, ${}_RY(\chi) \in R_{\mathbb{C}}(H)$ is a character of H. Hence, by the formula of the characters of tensor product of modules found in [5, Lemma 7.1.3], the character ${}_RY(\chi)$ is given by

$${}_RY(\chi)(h) = \frac{1}{|G|} \sum_{g\in G} \theta(h, g)\chi(g) \tag{6.8}$$

where θ is the character of the monomial $(\mathbb{C}H, \mathbb{C}G)$-bimodule $\mathbb{C}Y$. Now, we need to identify the character θ. It is equal to the function sending any $(h, g) \in H \times G$ to the trace of the endomorphism $y \mapsto (h, g)y$ of $\mathbb{C}Y$. Since Y is an A-fibered (H, G)-biset and $A \leq \mathbb{C}^\times$, the basis of $\mathbb{C}Y$ is a set $[Y]/\sim$ of the representatives of the A-orbits $[Y]$

of Y. It is precisely equal to

$$\theta(h,g) = \sum_{\substack{y\in[Y]/\sim \\ (h,g)[y]=[y]}} \psi_y(h,g) \tag{6.9}$$

where $\psi_y(h,g) \in A$ is determined by the equation $(h,g)y = \psi_y(h,g)y$. On the other hand, since Y is transitive, there exists some $(a,b)V \in (H\times G)/V$ satisfying the equation $(a,b)[y] = [y']$ for any $[y],[y'] \in [Y]$. Also, if (h,g) stabilizes $[y]$, then $(h,g)^{(a,b)}$ stabilizes $[y']$. Thus,

$$\begin{aligned} {}_RY(\chi)(h) &= \frac{1}{|G|}\sum_{g\in G}\theta(h,g)\chi(g) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{(a,b)V\in(H\times G)/V \\ (h^a,g^b)\in V}} \psi(h^a,g^b)\chi(g). \end{aligned}$$

Since $\varphi_b : G\to G, g\mapsto g^b$ is an isomorphism and the characters are class functons, we have

$$\begin{aligned} {}_RY(\chi)(h) &= \frac{1}{|G|}\sum_{{}_bg\in G}\sum_{\substack{(a,b)V\in(H\times G)/V \\ (h^a,g)\in V}} \psi(h^a,g)\chi(g). \end{aligned} \tag{6.10}$$

If $(a,b)V \in (H\times G)/V$, then for each $(u,v)\in V$ and for any $[y]\in[Y]$,

$$(h^{au},g^v)[y] = (h^a,g)^v[y] = (h^a,g)[y]. \tag{6.11}$$

Then

$$\begin{aligned}
{}_RY(\chi)(h) &= \frac{1}{|G|}\frac{1}{|V|}\sum_{{}_bg\in G}\sum_{\substack{(a,b)\in(H\times G)\\(h^a,g)\in V}}\psi(h^a,g)\chi(g)\\
&= \frac{1}{|G|}\frac{1}{|V|}\sum_{g\in G}\sum_{\substack{(a,b)\in H\times G\\(h^a,g)\in V}}\psi(h^a,g)\chi(g)\\
&= \frac{1}{|G|}\frac{1}{|V|}\sum_{g\in G}\sum_{a\in H}\sum_{\substack{b\in G:\\(h^a,g)\in V}}\psi(h^a,g)\chi(g)\\
&= \frac{1}{|V|}\sum_{\substack{a\in H,g\in G\\(h^a,g)\in V}}\psi(h^a,g)\chi(g). \qquad (6.12)
\end{aligned}$$

□

We also need the explicit descriptions of the actions of basic fibered bisets. Note that if Y is one of induction, restriction, inflation, deflation or isogation bisets, then the above formula becomes the corresponding well-known maps in Section 2.3 from character theory.

Since we have long expressions containing biset actions on characters in the sequel, we prefer to use the notation $Y\cdot\chi$ or $Y\chi$ instead of ${}_RY(\chi)$.

Proposition 6.1. *Let $\varphi : G \to A$ be a homomorphism for some group G and let $\chi \in \mathrm{Irr}(G)$. Then $\mathrm{Tw}_G^\varphi\chi = \varphi\chi$.*

Proof. The chain of equations

$$\begin{aligned}
(\mathrm{Tw}_G^\varphi\chi)(g) &= \frac{1}{|\Delta(G)|}\sum_{\substack{x\in G,g'\in G\\(g^x,g')\in\Delta(G)}}\Delta(\varphi)((g^x,g'))\chi(g')\\
&= \frac{1}{|G|}\sum_{x\in G}\Delta(\varphi)((g^x,g^x))\chi(g^x)\\
&= \frac{1}{|G|}\sum_{x\in G}\varphi(g)\chi(g) = \varphi(g)\chi(g) \qquad (6.13)
\end{aligned}$$

is a direct result of the formula in Lemma 6.1 and completes the proof. □

Proposition 6.2. *Let G be an abelian group, $\chi \in \mathrm{Irr}(G)$ and $\mathrm{E}_{K,\kappa} = \left(\frac{G\times G}{\Delta_K(G),\phi_\kappa}\right)$ be an idempotent for some pair $(K,\kappa) \in \mathcal{M}_G(A)$. Then*

$$\mathrm{E}_{K,\kappa}\chi = \begin{cases} \chi & \text{if } \chi|_K = \kappa \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

Proof. Keeping in mind that G is abelian, we obtain from Lemma 6.1 that

$$\begin{aligned} (\mathrm{E}_{K,\kappa}\chi)(g) &= \frac{1}{|\Delta_K(G)|} \sum_{\substack{x\in G, g'\in G \\ (g^x,g')\in\Delta_K(G)}} \phi_\kappa((g^x,g'))\chi(g') \\ &= \frac{1}{|G||K|} \sum_{x\in G, k\in K} \phi_\kappa((g,gk))\chi(gk) \\ &= \frac{1}{|K|} \sum_{k\in K} \phi_\kappa((g,gk))\chi(gk) \\ &= \frac{1}{|K|} \sum_{k\in K} \kappa(k^{-1})\chi(gk). \end{aligned}$$

Since any character of an abelian group is a homomorphism, we have

$$\begin{aligned} (\mathrm{E}_{K,\kappa}\chi)(g) &= \frac{1}{|K|} \sum_{k\in K} \kappa(k^{-1})\chi(g)\chi(k) \\ &= \chi(g)\left(\frac{1}{|K|}\sum_{k\in K}\kappa(k^{-1})\chi(k)\right) = \begin{cases} \chi(g) & \text{if } \chi|_K = \kappa \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by the row orthogonality relation of characters. □

Notation 6.1. Let $\mathrm{E}_{K,\kappa}$ be an idempotent in E_G for some finite group G. Abusing the notation, we denote this idempotent by $E_{A,\kappa}$ if $\kappa : K \to A$ is an isomorphism, and we set $e_{A,\kappa} := [E_{A,\kappa}]$.

Lemma 6.2. *Let $f_{A,\kappa}$ be the virtual A-fibered biset defined in Section 3.6. If G is a finite cyclic group, then $f_{A,\kappa} = e_{A,\kappa}$.*

Proof. Note first that $f_{A,\kappa} = f_{K,\kappa}$ for some $K \leq G$ such that $K \cong A$, and

$$f_{K,\kappa} := \sum_{(K,\kappa) \preceq (L,\lambda) \in \mathcal{M}_G(A)^A} \mu^{\triangleleft}_{(K,\kappa),(L,\lambda)} e_{L,\lambda}. \tag{6.15}$$

In the case that G is cyclic, the Möbius coefficient $\mu^{\triangleleft}_{(K,\kappa),(L,\lambda)}$ becomes

$$\mu^{\triangleleft}_{(K,\kappa),(L,\lambda)} = \begin{cases} 0 & \text{if } |L:K| \text{ has repeated prime factors} \\ 1 & \text{if } K = L \\ -1 & \text{if } K \text{ is maximal in } L. \end{cases} \tag{6.16}$$

On the other hand, since $\kappa : K \to A$ is an isomorphism, the only pair satisfying $(K,\kappa) \preceq (L,\lambda)$ is (K,κ) itself due to the condition $\lambda|_K = \kappa$, that is $f_{A,\kappa} = e_{A,\kappa}$. □

Lemma 6.3. *Let G be a finite cyclic group of order m. Then,*

$$\sum_{\alpha} (e_{A,\alpha}\chi) = \chi \tag{6.17}$$

for any irreducible character $\chi \in \mathrm{Irr}(G)$.

Proof. First, notice that $\mathcal{M}_G(A)^A = \mathcal{M}_G(A)$ if G is abelian. Then $f_{K,\kappa}$ takes the form

$$f_{K,\kappa} = \sum_{(K,\kappa) \preceq (L,\lambda) \in \mathcal{M}_G(A)} \mu^{\triangleleft}_{(K,\kappa),(L,\lambda)} e_{L,\lambda} = e_{K,\kappa} - \sum_{\substack{p^2 \nmid |L:K| \\ \lambda|_K = \kappa}} e_{L,\lambda} \tag{6.18}$$

for any $(K,\kappa) \in \mathcal{M}_G(A)$ by Equations 3.25 and 6.16. The number p in the second sum above is any prime number because Möbius coefficient is zero if $|L:K|$ has repeated prime factors. Now, consider $f_{K,\kappa}\chi$ for any $\chi \in \mathrm{Irr}(G)$. It equals zero if the sum $\sum_{\substack{p^2 \nmid |L:K| \\ \lambda|_K = \kappa}} e_{L,\lambda}$ is non-zero, i.e. if there exist pairs (L,λ) such that K is maximal in L and $\lambda|_K = \kappa$. Indeed, if $e_{K,\kappa}\chi = 0$, that is if $\chi|_K \neq \kappa$, then $\sum_{\substack{p^2 \nmid |L:K| \\ \lambda|_K = \kappa}} e_{L,\lambda}\chi = 0$ because $\chi|_L$ cannot be equal to λ due to the equality $\lambda|_K = \kappa$. On the other hand, if $e_{K,\kappa}\chi = \chi$, that is if $\chi|_K = \kappa$, then $f_{K,\kappa}\chi = 0$ again because $\sum_{\substack{p^2 \nmid |L:K| \\ \lambda|_K = \kappa}} e_{L,\lambda}\chi = \chi$. Since

G is cyclic, there exists for certain at least one pair (L, λ) with $\chi|_L = \lambda$, which implies the equalities $\sum_{\substack{p^2 \nmid |L:K| \\ \lambda|_K = \kappa}} e_{L,\lambda}\chi = \chi$ and $f_{K,\kappa}\chi$. We know by Lemma 6.2 that $f_{A,\alpha} = e_{A,\alpha}$ for any isomorphism $\alpha : A \to A$. If we set

$$f_A := \sum_{\alpha} f_{A,\alpha} = \sum_{\alpha} e_{A,\alpha} \tag{6.19}$$

then

$$1 = \sum_{(K,\kappa)\in\mathcal{M}_G(A)} f_{K,\kappa} = f_A + f'_A \quad \text{where} \quad f'_A := \sum_{\substack{(K,\kappa)\in\mathcal{M}_G(A) \\ K\ncong A}} f_{K,\kappa} = 1 - f_A$$

by Lemma 3.3. Notice that if (K, κ) is any pair with $K \ncong A$, then there exists surely a pair (L, λ) such that $\lambda|_K = \kappa$ because G is cyclic and κ is not an isomorphism. Therefore $f_{K,\kappa}\chi = 0$ by the argument above which implies that $f'_A\chi = 0$. Now, if we apply the identity element $1 \in E_G$ to χ, we obtain

$$\chi = 1 \cdot \chi = (f_A + f'_A) \cdot \chi = f_A\chi + f'_A\chi = f_A\chi = \sum_{\alpha} e_{A,\alpha}\chi \tag{6.20}$$

which is what we desire to prove. □

7. MAIN RESULTS

We have stated that $\mathbb{C}R_{\mathbb{C}}$ is a semisimple biset functor and a simple $\mathbb{C}^\times$-fibered biset functor. In other words, while $\mathbb{C}R_{\mathbb{C}}$ is semisimple for the minimal fiber group $A = 1 \leq \mathbb{C}^\times$, it is simple for the maximal fiber group $A = \mathbb{C}^\times$. Our aim is to study the structure of $\mathbb{C}R_{\mathbb{C}}$ for some non-trivial cases, i.e. for some specific groups A such that $1 < A < \mathbb{C}^\times$. We separate our investigation into two parts:

(i) Part I: Large Fiber Group
(ii) Part II: Small Fiber Group

where by a large fiber group, we mean a group satisfying Hypothesis 3.1. Otherwise, we call it a small fiber group.

7.1. Part I: Large Fiber Group

We first concentrate on a fiber group satisfying the hypothesis. For simplicity, we let π be a set of prime numbers and let $A = \pi^\infty$ be an abelian group containing all p^n-th roots of unity for all powers $n \in \mathbb{N}$ and all primes $p \in \pi$. We are aiming to show that $\mathbb{C}R_{\mathbb{C}}$ is still semisimple as a π^∞-fibered biset functor by determining a decomposition of it into simple summands.

Before stating our first main theorem of this part, we introduce some notation. Let Γ be the set of all pairs (m, ζ) as defined in the previous section. We define a relation on Γ as follows. Two pairs $(m, \zeta), (n, \nu) \in \Gamma$ are said to be *π-equivalent*, written $(m, \zeta) \equiv (n, \nu)$ if the π'-parts $m_{\pi'}$ and $n_{\pi'}$ are equal and after identifying the groups $\mathbb{Z}/m_{\pi'}\mathbb{Z} \cong \mathbb{Z}/n_{\pi'}\mathbb{Z}$, we have

$$\mathbb{C}_{\zeta_{\pi'}} \cong \mathbb{C}_{\nu_{\pi'}}. \tag{7.1}$$

Remark 7.1. The last condition means that the π'-parts $\tilde{\zeta}_{\pi'}$ and $\tilde{\nu}_{\pi'}$ of ζ and ν affords the same one dimensional representation of $\mathbb{Z}/m_{\pi'}\mathbb{Z} \cong \mathbb{Z}/n_{\pi'}\mathbb{Z}$, and hence they differ by a non-zero complex scalar.

Clearly, this is an equivalence relation on the set Γ. We denote the equivalence class containing (m, ζ) by $[m, \zeta]$ and write $\Gamma_{\pi\infty}$ for the set of equivalence classes. It is also clear that each equivalence class contains a unique pair (n, ν) where n is a π'-number. Now, we can state our first main result of this part.

Theorem 7.1. *Assume the above notation. Further assume that m is a π'-number and denote by $S^A_{m,\zeta}$ the π^∞-fibered subfunctor of $\mathbb{C}R_{\mathbb{C}}$ generated by the simple biset subfunctor $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta}$. Then, there is an isomorphism*

$$S^A_{m,\zeta} \cong \bigoplus_{(n,\nu)\in[m,\zeta]} S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu} \tag{7.2}$$

of biset functors.

Proof. To simplify the proof of the theorem, we introduce the following temporary notation

$$S_{[m,\zeta]} = \bigoplus_{(n,\nu)\in[m,\zeta]} S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}. \tag{7.3}$$

Clearly, $S_{[m,\zeta]}$ is a biset functor and the first part of the theorem claims that forgetting the fibered structure of the functor $S^A_{m,\zeta}$, we obtain an isomorphism $S_{[m,\zeta]} \cong S^A_{m,\zeta}$ of biset functors. We prove the claim in two steps. First, let $(n, \nu) \in [m, \zeta]$. We show that the simple biset subfunctor $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}$ of $\mathbb{C}R_{\mathbb{C}}$ is contained in $S^A_{m,\zeta}$. Since m is a π'-number, there is a π-number k such that $n = mk$ and without loss of generality, the π'-part $\tilde{\nu}_{\pi'}$ of $\tilde{\nu}$ coincides with $\tilde{\zeta}$. On the other hand, the π-part $\tilde{\nu}_\pi$ of $\tilde{\nu}$ is a virtual character of $\mathbb{Z}/k\mathbb{Z}$, and hence it is a complex linear combination of the irreducible

characters of $\mathbb{Z}/k\mathbb{Z}$, say

$$\tilde{\nu}_\pi = \sum_{\chi \in \mathrm{Irr}(\mathbb{Z}/k\mathbb{Z})} c_\chi \chi \tag{7.4}$$

for some complex numbers c_χ. Moreover, since k is a π-number, $\mathbb{Z}/k\mathbb{Z}$ embeds in A and hence each irreducible character χ of $\mathbb{Z}/k\mathbb{Z}$ induces a twist biset $\mathrm{Tw}^\chi_{\mathbb{Z}/k\mathbb{Z}}$. Thus, putting

$$\mathrm{Tw}_{\tilde{\nu}_\pi} = \sum_\chi c_\chi \mathrm{Tw}^\chi_{\mathbb{Z}/k\mathbb{Z}} \tag{7.5}$$

we obtain

$$\tilde{\nu}_\pi = \mathrm{Tw}_{\tilde{\nu}_\pi} \cdot 1 \tag{7.6}$$

where 1 denotes the trivial character of the group $\mathbb{Z}/k\mathbb{Z}$. Moreover, writing $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, we can regard the A-fibered $(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/k\mathbb{Z})$-biset $\mathrm{Tw}_{\tilde{\nu}_\pi}$ as an A-fibered $(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$-biset by letting $\mathbb{Z}/m\mathbb{Z}$ act trivially on both sides. Therefore, the above equality becomes

$$\tilde{\nu}_\pi \times 1 = \mathrm{Tw}_{\tilde{\nu}_\pi} \cdot 1 \tag{7.7}$$

where 1 on the right hand side is the trivial character of the group $\mathbb{Z}/n\mathbb{Z}$. On the other hand, we clearly have

$$1 \times \tilde{\nu}_{\pi'} = \mathrm{Inf}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \tilde{\zeta}. \tag{7.8}$$

Combining these two equalities, we get

$$\tilde{\nu} = \tilde{\nu}_\pi \times \tilde{\nu}_{\pi'} = \mathrm{Tw}_{\tilde{\nu}_\pi} \cdot \mathrm{Inf}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \cdot \tilde{\zeta}. \tag{7.9}$$

In particular, $\tilde{\nu}$ is contained in the A-fibered subfunctor generated by $\tilde{\zeta}$. We already

know that the simple biset functor $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}$ is generated by $\tilde{\nu}$. Thus we have proved that

$$S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu} \subseteq S^A_{m,\zeta} \tag{7.10}$$

for all $(n,\nu) \in [m,\zeta]$. In particular, we have shown that

$$S_{[m,\zeta]} \subseteq S^A_{m,\zeta} \tag{7.11}$$

as required.

To prove the reverse inclusion, it is sufficient to show that any simple biset subfunctor of $S^A_{m,\zeta}$ is parameterized by a pair equivalent to (m,ζ). Indeed, since $\mathbb{C}R_{\mathbb{C}}$ is semisimple, the subfunctor $S^A_{m,\zeta}$ is also semisimple and hence it is a sum of its simple subfunctors. Therefore, let $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu} \subseteq S^A_{m,\zeta}$. We need to show is that (n,ν) is equivalent to (m,ζ). Since $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}(\mathbb{Z}/n\mathbb{Z}) \subseteq S^A_{m,\zeta}(\mathbb{Z}/n\mathbb{Z})$ and the functor $S^A_{m,\zeta}$ is generated by $\tilde{\zeta}$, we should have

$$\tilde{\nu} = \gamma \cdot \tilde{\zeta} \tag{7.12}$$

for some virtual A-fibered biset $\gamma \in B^A(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$. Recall that π^∞ satisfies Hypothesis 3.1. Hence, by the decomposition given in Theorem 5.1, we deduce that $\tilde{\nu}$ is a $\mathbb{C}$-linear combination of elements of the form

$$\mathrm{Ind}_P^{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{Tw}_P^{\tilde{\phi}_1}\ \mathrm{Inf}_{P/K}^P\ \mathrm{c}^\eta_{P/K,Q/L}\ \mathrm{Def}_{Q/L}^Q\ \mathrm{Tw}_Q^{\tilde{\phi}_2}\ \mathrm{Res}_Q^{\mathbb{Z}/m\mathbb{Z}}\ \tilde{\zeta} \tag{7.13}$$

for appropriate choices of the notation. But $\mathbb{Z}/m\mathbb{Z}$ is a minimal group for the functor $S^A_{m,\zeta}$. Thus the maps factoring through a group of smaller order annihilates $\tilde{\zeta}$ and hence any transitive summand of γ must be of the form

$$\mathrm{Ind}_{\mathbb{Z}/s\mathbb{Z}}^{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{Tw}_{\mathbb{Z}/s\mathbb{Z}}^{\phi}\ \mathrm{Inf}_{\mathbb{Z}/m\mathbb{Z}}^{\mathbb{Z}/s\mathbb{Z}}\ \tilde{\zeta} \tag{7.14}$$

where $\phi : \mathbb{Z}/s\mathbb{Z} \to A$ is a homomorphism. Also, since $\mathbb{Z}/n\mathbb{Z}$ is a minimal group for the biset functor $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}$, the induction maps in the above decomposition must be trivial, that is, s must be equal to n. Thus the above form reduces to

$$\mathrm{Tw}^{\phi}_{\mathbb{Z}/n\mathbb{Z}} \, \mathrm{Inf}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \, \tilde{\zeta}. \tag{7.15}$$

By its transitivity we can divide the inflation map above into two parts as

$$\mathrm{Tw}^{\phi}_{\mathbb{Z}/n\mathbb{Z}} \, \mathrm{Inf}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/n_{\pi'}\mathbb{Z}} \, \mathrm{Inf}^{\mathbb{Z}/n_{\pi'}\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \, \tilde{\zeta} \tag{7.16}$$

where $n_{\pi'}$ denotes the π'-part of n. We need to show that the inflation map on the right side is trivial and hence m is the π'-part of n. Assume it is not trivial. Then we have

$$\mathrm{Tw}^{\phi}_{\mathbb{Z}/n\mathbb{Z}} \, \mathrm{Inf}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/n_{\pi'}\mathbb{Z}} \, \mathrm{Inf}^{\mathbb{Z}/n_{\pi'}\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \, \tilde{\zeta} = \mathrm{Tw}^{\phi}_{\mathbb{Z}/n\mathbb{Z}} \, (\tau \times 1) \tag{7.17}$$

where $\tau = \mathrm{Inf}^{\mathbb{Z}/n_{\pi'}\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \, \tilde{\zeta}$. On the other hand, because of the structure of the group A and the fact that ϕ is a homomorphism whose image is in A, ϕ must be trivial on the π'-part of $\mathbb{Z}/n\mathbb{Z}$, i.e. following the previous notations, ϕ must be of the form $1 \times \phi_\pi$. Hence, the expression above becomes

$$\mathrm{Tw}^{\phi}_{\mathbb{Z}/n\mathbb{Z}} \, (\tau \times 1) = \mathrm{Tw}^{1\times\phi_\pi}_{\mathbb{Z}/n\mathbb{Z}} \, (\tau \times 1) = \tau \times \phi_\pi. \tag{7.18}$$

But the last map $\tau \times \phi_\pi$ can be given as $\mathrm{Inf}^{\mathbb{Z}/n_{\pi'}\mathbb{Z}}_{\mathbb{Z}/m\mathbb{Z}} \, (\tilde{\zeta} \times \phi_\pi)$, which contradicts the primitivity of ν. Therefore, $n_{\pi'} = m$ and all together imply that $(m, \zeta) \equiv (n, \nu)$, which is what we want to show to justify

$$S^A_{m,\zeta} \subseteq S_{[m,\zeta]}. \tag{7.19}$$

□

Theorem 7.2. *The π^∞-fibered biset functor $S^A_{m,\zeta}$ is simple.*

Proof. We have already shown that the π^∞-fibered biset functor $S^A_{m,\zeta}$ is cyclic and generated by $\tilde{\zeta}$. Thus it is sufficient to show that any non-generator must be zero. This is equivalent to show that the intersection of the kernels of all maps

$$S^A_{m,\zeta}(G) \to S^A_{m,\zeta}(\mathbb{Z}/m\mathbb{Z}) \tag{7.20}$$

induced by A-fibered $(\mathbb{Z}/m\mathbb{Z}, G)$-bisets is zero for any group G.

Let $0 \neq \psi \in S^A_{m,\zeta}(G)$. Assume, for a contradiction, that for any A-fibered $(\mathbb{Z}/m\mathbb{Z}, G)$-biset X, the induced map

$$X : S^A_{m,\zeta}(G) \to S^A_{m,\zeta}(\mathbb{Z}/m\mathbb{Z}) \tag{7.21}$$

annihilates ψ, that is, we have $X(\psi) = 0$. By Theorem 7.1, we have

$$S^A_{m,\zeta}(G) = \bigoplus_{(n,\nu)} S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}(G) \tag{7.22}$$

where the sum is over all pairs (n, ν) equivalent to (m, ζ). Therefore, we can write

$$\psi = \sum_{(n,\nu)} \psi_{(n,\nu)} \tag{7.23}$$

where $\psi_{(n,\nu)} \in S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}(G)$. Now let (n_0, ν_0) be a minimal pair equivalent to (m, ζ) subject to the condition that $\psi_{(n_0,\nu_0)} \neq 0$ but $\psi_{(n,\nu)} = 0$ for all $n < n_0$.

Now, by its choice, the element ψ lies in the kernel of any composite map ZY where

$$Y : S^A_{m,\zeta}(G) \to S^A_{m,\zeta}(\mathbb{Z}/n_0\mathbb{Z}) \quad \text{and} \quad Z : S^A_{m,\zeta}(\mathbb{Z}/n_0\mathbb{Z}) \to S^A_{m,\zeta}(\mathbb{Z}/m\mathbb{Z})$$

are induced by A-fibered bisets. In particular, for any π^∞-fibered $(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n_0\mathbb{Z})$-biset V and for any $(\mathbb{Z}/n_0\mathbb{Z}, G)$-biset U, we have

$$(V \otimes_{A(\mathbb{Z}/n_0\mathbb{Z})} U)(\psi) = 0. \tag{7.24}$$

Now, the image of ψ under any such biset U can be evaluated. Indeed, since $\mathbb{Z}/n\mathbb{Z}$ is a minimal group for the simple biset functor $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}$, for each $n > n_0$ and any pair (n, ν) equivalent to (m, ζ), we have

$$U(\psi_{(n,\nu)}) = 0. \tag{7.25}$$

Hence, the ones lying in the summands $S_{\mathbb{Z}/n_0\mathbb{Z},\mathbb{C}_{\nu_i}}(\mathbb{Z}/n_0\mathbb{Z})$, $1 \leq i \leq t$, for some integer $t \geq 1$, are the only non-zero components of $U(\psi)$.

For simplicity, we denote by ψ_i the component $\psi_{(n_0,\nu_i)}$ of ψ. As a subfunctor of $\mathbb{C}R_{\mathbb{C}}$, the evaluation $S_{\mathbb{Z}/n_0\mathbb{Z},\mathbb{C}_{\nu_i}}(\mathbb{Z}/n_0\mathbb{Z})$ is generated by $\tilde{\nu}_i$. This implies the following equality

$$U(\psi) = \sum_{i=1}^{t} U(\psi_i) = \sum_{i=1}^{t} c_i\tilde{\nu}_i \in \bigoplus_{i=1}^{t} S_{\mathbb{Z}/n_0\mathbb{Z},\mathbb{C}_{\nu_i}}(\mathbb{Z}/n_0\mathbb{Z}) \tag{7.26}$$

for some $c_i \in \mathbb{C}$. Also, since each pair (n_0, ν_i) is equivalent to (m, ζ), we have $n_0 = mp$ for some π-number p and $\tilde{\nu}_i = \tilde{\zeta} \times (\tilde{\nu}_i)_\pi$. Now since $(\tilde{\nu}_i)_\pi$ is a character of $\mathbb{Z}/p\mathbb{Z}$ for each i, there are complex numbers $d_{ij} \in \mathbb{C}$ such that

$$(\tilde{\nu}_i)_\pi = \sum_{j=1}^{p} d_{ij}\chi_j. \tag{7.27}$$

Here, χ_j runs over the all irreducible characters of $\mathbb{Z}/p\mathbb{Z}$. Thus, $U(\psi)$ is actually of the form

$$U(\psi) = \sum_{i=1}^{t}\sum_{j=1}^{p} c_i(\tilde{\zeta} \times d_{ij}\chi_j) = \sum_{i=1}^{t}\sum_{j=1}^{p} c_i d_{ij}(\tilde{\zeta} \times \chi_j). \tag{7.28}$$

Note that if we deflate the character $\tilde{\zeta} \times \chi_j$ to the quotient $\mathbb{Z}/m\mathbb{Z}$, we get zero unless χ_j is the trivial character. Hence, deflation annihilates all the terms in the above some except for the trivial character χ_1. Moreover, given any non-trivial character χ_j of $\mathbb{Z}/p\mathbb{Z}$, we can multiply $U(\psi)$ by the twist biset $\mathrm{Tw}_{\mathbb{Z}/n_0\mathbb{Z}}^{1\times(\chi_j)^{-1}}$ to trivialize the π-part of the corresponding summand. Hence given any index j, we have

$$0 = \mathrm{Def}_{\mathbb{Z}/m\mathbb{Z}}^{\mathbb{Z}/n_0\mathbb{Z}} \mathrm{Tw}_{\mathbb{Z}/n_0\mathbb{Z}}^{1\times(\chi_j)^{-1}} U(\psi) = \sum_{i=1}^{t} c_i d_{ij} \tilde{\zeta}. \tag{7.29}$$

Since $\tilde{\zeta}$ is non-zero, the above equality implies $\sum_{i=1}^{t} c_i d_{ij} = 0$. Now we multiply this equality by χ_j and sum over j for $1 \leq j \leq p$ to get

$$0 = \sum_{j=1}^{p} \left(\sum_{i=1}^{t} c_i d_{ij} \chi_j \right) = \sum_{i=1}^{t} c_i \left(\sum_{j=1}^{p} d_{ij} \chi_j \right) = \sum_{i=1}^{t} c_i (\tilde{\nu}_i)_\pi. \tag{7.30}$$

But, $(\tilde{\nu}_i)_\pi$'s are linearly independent as they form a subset of the set of all primitive characters of $\mathbb{Z}/n_0\mathbb{Z}$. Therefore we must have $c_1 = c_2 = ... = c_t = 0$ and hence $U(\psi_i) = 0$ for each i and for all $(\mathbb{Z}/n_0\mathbb{Z}, G)$-bisets U. In particular, we see that the element ψ_i lies in the intersection of kernels of all maps

$$U : S_{\mathbb{Z}/n_0\mathbb{Z}, \mathbb{C}_{\nu_i}}(G) \to S_{\mathbb{Z}/n_0\mathbb{Z}, \mathbb{C}_{\nu_i}}(\mathbb{Z}/n_0\mathbb{Z}). \tag{7.31}$$

induced by $(\mathbb{Z}/n_0\mathbb{Z}, G)$-bisets. Therefore, since $S_{\mathbb{Z}/n_0\mathbb{Z}, \mathbb{C}_{\nu_i}}$ is a simple biset functor, ψ_i must be equal to zero for each i, which contradicts to the minimality of n_0. Therefore ψ must be zero, as required. □

Corollary 7.1. *The π^∞-fibered biset functor $\mathbb{C}R_\mathbb{C}$ is semisimple and there is an isomorphism*

$$\mathbb{C}R_\mathbb{C} \cong \bigoplus_{[m,\zeta] \in \Gamma_{\pi^\infty}} S_{m,\zeta}^A \tag{7.32}$$

of π^∞-fibered biset functors.

Proof. It easily follows from Theorems 7.1 and 7.2. □

7.2. Part II: Small Fiber Group

Recall that Theorem 5.1 states that $\mathrm{E}_{K,\kappa}$ is always decomposable for a large fiber group because it is not one of the multipliers. This is why, twist is the only fibered biset in addition to the canonical bisets as we saw in the decomposition. On the other hand, for a small fiber group, $\mathrm{E}_{K,\kappa}$ may not disappear. Therefore, we need to determine new conditions for $\mathrm{E}_{K,\kappa}$ to be reduced because Hypothesis 3.1 is no longer satisfied. We need some lemmas to find the necessary and sufficient conditions for $\mathrm{E}_{K,\kappa}$ to be reduced.

Lemma 7.1. *If $\mathrm{E}_{K,\kappa}$ is reduced, then κ is faithful and does not extend to G, that is, there is no homomorphism $\varphi : G \to A$ such that $\varphi(k) = \kappa(k)$ for some $1 \neq k \in K$.*

Proof. Let $\mathrm{E}_{K,\kappa}$ be reduced. Then, because of the decomposition (3.21), the homomorphisms $(\phi_\kappa)_1 = (\phi_\kappa)_2 = \kappa$ must be faithful. The irreducibility of $\mathrm{E}_{K,\kappa}$ implies the irreducibility of $\mathrm{E}_{K,\kappa} \otimes_{AG} \mathrm{Tw}_G^\varphi$ for any homomorphism $\varphi : G \to A$. Indeed, if $\mathrm{E}_{K,\kappa} \otimes_{AG} \mathrm{Tw}_G^\varphi$ were reducible, it would be decomposed as

$$\mathrm{E}_{K,\kappa} \otimes_{AG} \mathrm{Tw}_G^\varphi \cong \left(\frac{G \times H}{U, \phi_1}\right) \otimes_{AH} \left(\frac{H \times G}{U, \phi_2}\right) \tag{7.33}$$

for some group H with $|H| < |G|$. Then, consider the homomorphism $\varphi^{-1} : G \to A$ defined as $\varphi^{-1}(g) := (\varphi(g))^{-1}$ for any $g \in G$. If we multiply the right hand-side of Expression 7.33 by $\mathrm{Tw}_G^{\varphi^{-1}}$, we obtain

$$\mathrm{E}_{K,\kappa} \otimes_{AG} \left(\mathrm{Tw}_G^\varphi \otimes_{AG} \mathrm{Tw}_G^{\varphi^{-1}}\right) \cong \mathrm{E}_{K,\kappa} \cong \left(\frac{G \times H}{U, \phi_1}\right) \otimes_{AH} \left(\left(\frac{H \times G}{U, \phi_2}\right) \otimes_{AG} \mathrm{Tw}_G^{\varphi^{-1}}\right)$$

which contradicts the irreducibility of $\mathrm{E}_{K,\kappa}$. Therefore, $\mathrm{E}_{K,\kappa} \otimes_{AG} \mathrm{Tw}_G^\varphi$ is also reduced for any homomorphism $\varphi : G \to A$. Then, $\ker[(\phi_\kappa * \Delta(\varphi))_i]$ must be trivial for $i = 1, 2$.

In other words, for any $1 \neq k \in K$, the evaluation

$$(\phi_\kappa * \Delta(\varphi))(1,k) = \phi_\kappa(1,k)\Delta(\varphi)(k,k) = \kappa(k^{-1})\varphi(k) \tag{7.34}$$

cannot be equal to 1, i.e. $\varphi(k) \neq \kappa(k)$. □

Lemma 7.2. *If G and H are groups of coprime orders, then the map*

$$\begin{aligned} \Theta : RB^A(G,G) \otimes_R RB^A(H,H) &\longrightarrow RB^A(G \times H, G \times H) \\ (X,Y) &\longmapsto X \times Y \end{aligned}$$

is a ring isomorphism.

Proof. The proof is almost the same as that of Proposition 2.5.14 in [5]. □

Remark 7.2. As a result of Lemma 7.2, we have the following isomorphism

$$\mathrm{E}_{K,\kappa} \cong \bigtimes_{\substack{p:prime \\ p||G|}} \mathrm{E}_{K_p,\kappa_p} \tag{7.35}$$

where $\mathrm{E}_{K_p,\kappa_p} := \left(\frac{G_p \times G_p}{\Delta_{K_p}(G_p),\phi_{\kappa_p}}\right)$. This isomorphism enables us to reduce the proof to p-groups because $\mathrm{E}_{K,\kappa}$ is reducible if and only if $\mathrm{E}_{K_p,\kappa_p}$ is reducible for some prime $p \mid |G|$.

Theorem 7.3. *Let $(K,\kappa) \in \mathcal{M}_G(A)$ for a cyclic group G. Then, the A-fibered (G,G)-biset $\mathrm{E}_{K,\kappa}$ is reduced if and only if the homomorphism κ is faithful, and $|A|_p < |G|_p$ for any prime number p dividing $|K|$.*

Proof. Let $\mathrm{E}_{K,\kappa}$ be reduced for a cyclic group $G = <g>$. By Lemma 7.1, we know that κ is faithful. Then, $\kappa_p : K_p \to A$ is also faithful for any prime number p dividing $|K|$. Suppose, for a contradiction, that $|A|_p \geq |G|_p$ for some prime number p. As a result of this assumption, A contains all $|G|_p$-th roots of unity. Therefore, the faithful homomorphism $\kappa_p : K_p \to A$ can be extended to a faithful homomorphism $\lambda : G_p \to A$.

In other words, $\lambda(k) = \kappa_p(k)$ for any $k \in K_p$. Since this contradicts the assertion of Lemma 7.1, following the notation of Remark 7.2, the idempotent $\mathrm{E}_{K_p,\kappa_p}$ is not reduced. Then, neither is $\mathrm{E}_{K,\kappa}$ by Remark 7.2.

Conversely, let κ be a faithful homomorphism and $|A|_p < |G|_p$ for any prime number p dividing $|K|$. Assume that $\mathrm{E}_{K,\kappa}$ is not reduced. Then, we can decompose $\mathrm{E}_{K,\kappa}$ as

$$\left(\frac{G \times G}{\Delta_K(G), \phi_\kappa}\right) \cong \left(\frac{G \times H}{U, \varphi}\right) \otimes_{AH} \left(\frac{H \times G}{V, \psi}\right) \tag{7.36}$$

for some group H such that $|H| < |G|$. Since $(x, x) \in \Delta_K(G)$ for any $x \in G$, then $(g, g) \in \Delta_K(G)$. By the definition of $U*V$, there exists some $h \in H$ such that $(g, h) \in U$ and $(h, g) \in V$. We claim that $(o(g), o(h)) = d > 1$. Indeed, if they were relatively prime, $<g, h>$ would be equal to $<g> \times <h>$. In that case, $U * V = \Delta_K(G)$ would be equal to $G \times G$ since $<g, h> \leq U$ and $<h, g> \leq V$. The equality $\Delta_K(G) = G \times G$ implies $G = K$. As $\kappa : K \to A$ is faithful, $|K|_p = |G|_p$ must be less than or equal to $|A|_p$ for any prime number p dividing $|K|$, which contradicts the condition $|A|_p < |G|_p$. Therefore $d > 1$. Now, consider the group $<g^a, h^a>$, where $a := \frac{o(h)}{(o(g),o(h))} = \frac{o(h)}{d}$. Due to the facts $(o(g), a) = 1$ and $o(h) < o(g)$, we have $<g^a, h^a> \cong G$. Let $\theta : <g^a, h^a> \to A$ be the restriction of φ to $<g^a, h^a>$, i.e. $\theta := \varphi|_{<g^a,h^a>}$, and let

$$L := k_1(<g^a, h^a>) \cap k_1(U) \leq k_1(\Delta_K(G)) = K. \tag{7.37}$$

The group L is non-trivial because of the facts $<g^{ad}, h^{ad}> = <g^{ad}, 1> = <g^d, 1>$ and $<g^d, 1> \leq U$. Altogether, we obtain $\theta|_{L\times 1} = \kappa|_L$ as

$$\kappa(g^d) = \phi_\kappa(g^d, 1) = \varphi(g^d, 1)\psi(1, 1) = \varphi(g^d, 1) = \theta(g^d, 1). \tag{7.38}$$

Since κ is faithful, so is $\theta|_{L\times 1}$. Let p be a prime number dividing $|L| = |L \times 1|$. Then, faithfulness of $\theta|_{L\times 1}$ implies the faithfulnes of θ_p. which yields in turn that $| <g^a, x^a> |_p = |G|_p \leq |A|_p$. This is a contradiction because $L \leq K$, which means p

divides $|K|$ also. □

Lemma 7.3. *Let G and H be cyclic groups and let $U \leq G \times H$ with the corresponding quintuple (G, K, η, L, H). If $|G| \geq |H|$, then $U = U_G U_L$ such that $U_L = 1 \times L$ and $U_G = <g, h> \cong G$ for some generators g and h of G and H, respectively.*

Proof. Let g be a generator of G. By Goursat Lemma, there exists a generator h of H such that $(g, h) \in U$ as $G/K \cong H/L$. Then, $U_G \leq U$, where $U_G := <g, h> \cong G$. Now, take an arbitrary element $(x, y) \in U$. Notice that $(x, y) = (x, z)(1, z^{-1}y)$, where $(x, z) = (g^a, h^a)$ such that $x = g^a$ for some $a \in \mathbb{Z}$. Hence, we have shown that $U \subseteq U_G U_L$, where $U_L := 1 \times L$. The reverse inclusion is straightforward. □

Remark 7.3. If we take $|H| \geq |G|$ in Lemma 7.3, we obtain $U = U_H U_K$ such that $U_K = K \times 1$, and $U_H \cong H$. It can be easily shown by using opposite bisets and following the steps above.

Now, we can find a general formula for the decomposition of any transitive A-fibered (G, H)-biset $\left(\frac{G \times H}{U, \phi}\right)$ into products of canonical A-fibered bisets. In the light of Lemma 7.2, it suffices to obtain a formula for p-groups. Assume the conditions and notations of Lemma 7.3. Not to deal with complicated notations of the general decomposition (3.21), we assume also that $\mathrm{Ker}(\phi_1) = 1 = \mathrm{Ker}(\phi_2)$. Lemma 7.3 claims that each pair $(U, \phi) \in \mathcal{M}_{G \times H}(A)$ is of the form $(U_G \cdot U_L,\ \alpha \cdot \beta)$, where $\alpha : U_G \to A$ and $\beta : U_L \to A$ are homomorphisms. Notice that the condition $\mathrm{Ker}(\phi_1) = 1 = \mathrm{Ker}(\phi_2)$ forces β to be faithful. Let $\bar{\alpha} : G \to A$ be a homomorphism defined as $\bar{\alpha}(g) := \alpha(g, h)$, where (g, h) is a generator of U_G. After all these settings, the decomposition is as follows

$$\begin{aligned}
\left(\frac{G \times H}{U, \phi}\right) &= \left(\frac{G \times H}{U_G \cdot U_L, \alpha \cdot \beta}\right) \cong \mathrm{Tw}_G^{\bar{\alpha}} \otimes_{AG} \left(\frac{G \times H}{U_G \cdot U_L, 1 \cdot \beta}\right) \\
&\cong \mathrm{Tw}_G^{\bar{\alpha}} \otimes_{AG} \mathrm{Inf}_{G/M}^{G} \otimes_{A(G/M)\ G/M} \mathrm{Iso}_H \otimes_{AH} \left(\frac{H \times H}{\Delta(H) \cdot U_L, 1 \cdot \beta}\right) \\
&= \mathrm{Tw}_G^{\bar{\alpha}} \otimes_{AG} \mathrm{Inf}_{G/M}^{G} \otimes_{A(G/M)\ G/M} \mathrm{Iso}_H \otimes_{AH} \mathrm{E}_{L,\beta}^{H}
\end{aligned}$$

where $M \leq G$ such that $G/M \cong H$, and

$$\mathrm{E}^{H}_{L,\beta} := \left(\frac{H \times H}{\Delta_L(H), \phi_\beta}\right) = \left(\frac{H \times H}{\Delta(H) \cdot U_L, 1 \cdot \beta}\right). \tag{7.39}$$

At this point, we separate the decomposition into two cases because it depends on the orders of the groups. In the light of Theorem 7.3, if $|A|_p < |H|$, then $\mathrm{E}^{H}_{L,\beta}$ is reduced, and the decomposition above is in its final form. However, if $|A|_p \geq |H|$, then $\mathrm{E}^{H}_{L,\beta}$ is decomposable by Theorem 7.3. In this case,

$$\begin{aligned} \mathrm{E}^{H}_{L,\beta} &\cong \mathrm{Tw}^{\xi}_{H} \otimes_{AH} \left(\frac{H \times H}{\Delta_L(H), 1}\right) \otimes_{AH} \mathrm{Tw}^{\xi^{-1}}_{H} \\ &\cong \mathrm{Tw}^{\xi}_{H} \otimes_{AH} \mathrm{Inf}^{H}_{H/L} \otimes_{A(H/L)} \mathrm{Def}^{H}_{H/L} \otimes_{AH} \mathrm{Tw}^{\xi^{-1}}_{H} \end{aligned}$$

where $\xi : H \to A$ is a homomorphism satisfying $\xi|_L = \beta$. After combining two decompositions, the final form is as follows.

$$\begin{aligned} \left(\frac{G \times H}{U, \phi}\right) &\cong \mathrm{Tw}^{\bar{\alpha}}_{G} \otimes_{AG} \mathrm{Inf}^{G}_{G/M} \otimes_{A(G/M)\ G/M} \mathrm{Iso}_{H} \otimes_{AH} \mathrm{E}^{H}_{L,\beta} \\ &\cong \mathrm{Tw}^{\bar{\alpha}}_{G}\ \mathrm{Inf}^{G}_{G/M\ G/M}\mathrm{Iso}_{H}\ \mathrm{Tw}^{\xi}_{H}\ \mathrm{Inf}^{H}_{H/L}\ \mathrm{Def}^{H}_{H/L}\ \mathrm{Tw}^{\xi^{-1}}_{H} \end{aligned}$$

with tensor product over appropriate groups between each two of them. Hence, we have proved the following theorem.

Theorem 7.4. *Let G and H be cyclic p-groups such that $|G| \geq |H|$. Then, assuming the notation above,*

$$\left(\frac{G \times H}{U, \phi}\right) \cong \begin{cases} \mathrm{Tw}^{\bar{\alpha}}_{G} \otimes_{AG} \mathrm{Inf}^{G}_{G/M} \otimes_{A(G/M)\ G/M} \mathrm{Iso}_{H} \otimes_{AH} \mathrm{E}^{H}_{L,\beta} & \text{if } |A|_p < |H| \\ \mathrm{Tw}^{\bar{\alpha}}_{G}\ \mathrm{Inf}^{G}_{G/M\ G/M}\mathrm{Iso}_{H}\ \mathrm{Tw}^{\xi}_{H}\ \mathrm{Inf}^{H}_{H/L}\ \mathrm{Def}^{H}_{H/L}\ \mathrm{Tw}^{\xi^{-1}}_{H} & \text{if } |A|_p \geq |H|. \end{cases}$$

Remark 7.4. Assume the notations and the conditions of Remark 7.3. Then, any transitive A-fibered (G,H)-biset is of the form

$$\left(\frac{G\times H}{U,\phi}\right)=\left(\frac{G\times H}{U_K\cdot U_H,\alpha\cdot\beta}\right) \tag{7.40}$$

where $\alpha : U_K \to A$ and $\beta : U_H \to A$ are homomorphisms. Hence, by taking the opposite of the biset and using Theorem 7.4, the theorem below easily follows.

Theorem 7.5. *Assume the hypothesis of the remark above. Then,*

$$\left(\frac{G\times H}{U,\phi}\right)\cong\begin{cases}\mathrm{E}_{K,\alpha}\otimes_{AG}{}_G\mathrm{Iso}_{H/N}\otimes_{A(H/N)}\mathrm{Def}^H_{H/N}\otimes_{AH}\mathrm{Tw}^{\bar\beta}_H & \text{if } |A|_p<|G|\\ \mathrm{Tw}^{\tau}_G\ \mathrm{Inf}^G_{G/K}\ \mathrm{Def}^G_{G/K}\ \mathrm{Tw}^{\tau^{-1}}_G\ {}_G\mathrm{Iso}_{H/N}\ \mathrm{Def}^H_{H/N}\ \mathrm{Tw}^{\bar\beta}_H & \text{if } |A|_p\geq|G|\end{cases}$$

where $N\leq H$ with $G\cong H/N$, $\bar\beta : H\to A$ is defined as $\bar\beta(h):=\beta(h,g)$ for $(h,g)\in U_H$, and $\tau : G\to A$ is a homomorphism satisfying $\tau|_K=\alpha$.

Now, we need to fix our small fiber group and introduce another equivalence relation on Γ in order to state our main theorems of this part. Let p be a fixed prime number and let $A\leq\mathbb{C}^\times$ be the group of all $p^{\mathfrak{n}}$-th roots of unity for a fixed $\mathfrak{n}\in\mathbb{N}$. From now on, we say $p^{\mathfrak{n}}$-fibered instead of A-fibered to point the fixed fiber group. We denote the $p^{\mathfrak{n}}$-fibered biset

$$\left(\frac{\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}}{\Delta_K(\mathbb{Z}/n\mathbb{Z}),\phi_\kappa}\right) \tag{7.41}$$

by $\mathrm{E}^n_{K,\kappa}$. If κ is an isomorphism, i.e. if $K\cong A$, we use $\mathrm{E}^n_{A,\kappa}$ by an abuse of the notation. Notice that when we use $\mathrm{E}^n_{A,\kappa}$, it is implicitly assumed that $n\geq|A|$. We say that the pairs $(m,\zeta),(n,\nu)\in\Gamma$ are $p^{\mathfrak{n}}$-*equivalent*, and write $(m,\zeta)\equiv(n,\nu)$ if the following conditions hold.

(i) $m_{p'}=n_{p'}$ and $\mathbb{C}_{\zeta_{p'}}\cong\mathbb{C}_{\nu_{p'}}$ after identifying the groups $\mathbb{Z}/m_{p'}\mathbb{Z}\cong\mathbb{Z}/n_{p'}\mathbb{Z}$.

(ii) Either (a) $m_p,n_p\leq|A|$, or (b) $m_p=n_p>|A|$ and $\mathrm{E}^n_{A,\alpha}\tilde\zeta=c_\alpha\mathrm{E}^n_{A,\alpha}\tilde\nu$ for any isomorphism $\alpha : A\to A$ and for some $c_\alpha\in\mathbb{C}$ depending on α.

Let $<n,\nu>$ denote the equivalence class of the pair (n,ν), and let Γ_{p^n} be the set of equivalence classes in Γ. Notice that any class of pairs equivalent with respect to the part (a), contains a unique pair (n,ν) such that n is a p'-number. Now, we can state our first main theorem of this part.

Theorem 7.6. *If $S^A_{n,\nu}$ is the p^n-fibered subfunctor of $\mathbb{C}R_{\mathbb{C}}$ generated by the simple biset subfunctor $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}$, then, adopting the notations above, there is an isomorphism*

$$S^A_{n,\nu} \cong \bigoplus_{(m,\zeta)\in<n,\nu>} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta} \tag{7.42}$$

of biset functors.

We need the following lemma to prove the theorem.

Lemma 7.4. *Let $\mathrm{E}^n_{K,\kappa}$ be any reduced idempotent in $B^A(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n\mathbb{Z})$. Then, for any pair $(n,\nu)\in\Gamma$ and for any homomorphism $\varphi:\mathbb{Z}/n\mathbb{Z}\to A$*

$$\mathrm{E}^n_{K,\kappa}\ (\varphi\tilde{\nu}) = c_\varphi \mathrm{E}^n_{K,\kappa}\ \tilde{\nu} \tag{7.43}$$

for some $c_\varphi\in\mathbb{C}$ depending on φ.

Proof. As the Dirichlet characters are also virtual characters of $\mathbb{Z}/n\mathbb{Z}$, we can write $\tilde{\nu} = \sum_{i=1}^n c_i\chi^i$, where $\mathrm{Irr}(\mathbb{Z}/n\mathbb{Z}) = <\chi>$. As we know from the preliminaries, the coefficient c_i is equal to $\langle\tilde{\nu},\chi^i\rangle$ for all $1\le i\le n$. The exact values of the coefficients are obtained from the equation

$$\tilde{\nu} = c\sum_{i=1}^{n}\overline{\tilde{\nu}(i)}\chi^i \tag{7.44}$$

for some $c\in\mathbb{C}$, where $\overline{\tilde{\nu}(i)}$ is the complex conjugation of $\tilde{\nu}(i)$. Since we work in complex vector space of characters, we omit the constant coefficient c for short. These

coefficients have the relation

$$\langle \tilde{\nu}, \chi^i \rangle = \overline{\tilde{\nu}(i)} \langle \tilde{\nu}, \chi \rangle \tag{7.45}$$

for all $1 \leq i \leq n$ ([6, Proposition 2.1.39]). We aim to calculate the coefficients of $\varphi\tilde{\nu}$ for any homomorphism $\varphi : \mathbb{Z}/n\mathbb{Z} \to A$ via Relation 7.45. First, note that any homomorphism $\varphi : \mathbb{Z}/n\mathbb{Z} \to A$ is an irreducible character of $\mathbb{Z}/n\mathbb{Z}$, i.e. $\varphi = \chi^s$ for some $1 \leq s \leq n$. Since $n_p > |A|$ by reducedness, the homomorphism φ is not faithful, which implies $s|n$. Keeping this in mind, consider the following chain of equations

$$\begin{aligned} \langle \varphi\tilde{\nu}, \chi^i \rangle &= \frac{1}{n} \sum_{j=1}^{n} \varphi(j)\tilde{\nu}(j)\overline{\chi^i(j)} \\ &= \frac{1}{n} \sum_{j=1}^{n} \tilde{\nu}(j)\overline{\chi^{i+r}(j)} = \langle \tilde{\nu}, \chi^{i+r} \rangle \end{aligned}$$

where $\varphi = \chi^s = \overline{\chi^r}$ such that $r + s = n$. Hence, Equation 7.45 yields that

$$\langle \varphi\tilde{\nu}, \chi^i \rangle = \langle \tilde{\nu}, \chi^{i+r} \rangle = \overline{\tilde{\nu}(i+r)} \langle \tilde{\nu}, \chi \rangle. \tag{7.46}$$

Again by omitting the constant coefficient, we deduce

$$\varphi\tilde{\nu} = \sum_{i=1}^{n} \overline{\tilde{\nu}(i+r)} \chi^i. \tag{7.47}$$

As $\mathrm{E}^n_{K,\kappa}$ keeps the homomorphisms whose restriction to K is κ as they are, and annihilates the others,

$$\mathrm{E}^n_{K,\kappa}\left(\varphi\tilde{\nu}\right) = \sum_{l=1}^{|G:K|} \overline{\tilde{\nu}(i+r+l|K|)} \chi^{i+l|K|} \tag{7.48}$$

where $i \in \{1, 2, \ldots, |K|\}$ is the smallest number satisfying $\chi^i|_K = \kappa$. Observe that, since $(i, n) = 1$ and $r|n$, $(i + r, n) = 1$. Therefore, there exists a unique $x \in (\mathbb{Z}/n\mathbb{Z})^\times$, up to (mod n), such that $ix \equiv i + r \pmod{n}$. As $|K|$ divides n, $ix \equiv i + r \pmod{|K|}$, and moreover $ix + lx|K| \equiv i + r + l|K| \pmod{|K|}$ for any $l \in \{1, 2, \ldots, |G : K|\}$. So,

x does not depend on l. Taking all into the consideration, we reach the equalities

$$\begin{aligned} \mathrm{E}^n_{K,\kappa}\ (\varphi\tilde{\nu}) &= \sum_{l=1}^{|G:K|} \overline{\tilde{\nu}(i+r+l|K|)}\chi^{i+l|K|} = \sum_{l=1}^{|G:K|} \overline{\tilde{\nu}((i+l|K|)x)}\chi^{i+l|K|} \\ &= \overline{\tilde{\nu}(x)} \sum_{l=1}^{|G:K|} \overline{\tilde{\nu}(i+l|K|)}\chi^{i+l|K|} = \overline{\tilde{\nu}(x)}\mathrm{E}^n_{K,\kappa}\ \tilde{\nu}. \end{aligned}$$

We are done because r is determined by φ uniquely, that is $c_\varphi := \overline{\tilde{\nu}(x)}$ depends only on φ. □

Proof of Theorem 7.6. First, set the following temporary notation for short

$$S_{<n,\nu>} = \bigoplus_{(m,\zeta)\in<n,\nu>} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta}. \tag{7.49}$$

It is obvious that $S_{<n,\nu>}$ is a biset functor. Let $(m,\zeta) \equiv (n,\nu)$. We know for certain that $\tilde{\zeta}_{p'} = c\tilde{\nu}_{p'}$ for some $c \in \mathbb{C}^\times$ by Remark 7.1. If they are equivalent with respect to the part (a), the bisets $\mathrm{Tw}^{\chi}_{\mathbb{Z}/m_p\mathbb{Z}}$ and $\mathrm{Tw}^{\psi}_{\mathbb{Z}/n_p\mathbb{Z}}$ can be defined for all characters $\chi \in \mathrm{Irr}(\mathbb{Z}/m_p\mathbb{Z})$ and $\psi \in \mathrm{Irr}(\mathbb{Z}/n_p\mathbb{Z})$ because $m_p, n_p \leq |A|$. We can express $\tilde{\zeta}_p$ and $\tilde{\nu}_p$ as

$$\tilde{\zeta}_p = \sum_{\chi\in\mathrm{Irr}(\mathbb{Z}/m_p\mathbb{Z})} c_\chi\chi \quad \text{and} \quad \tilde{\nu}_p = \sum_{\psi\in\mathrm{Irr}(\mathbb{Z}/n_p\mathbb{Z})} c_\psi\psi$$

since they are virtual characters of $\mathbb{Z}/m_p\mathbb{Z}$ and $\mathbb{Z}/n_p\mathbb{Z}$, respectively. Setting the notations

$$\mathrm{Tw}_{\tilde{\nu}_p^{-1}\times 1} := \sum_{\psi} {c_\psi}^{-1}\mathrm{Tw}^{\psi^{-1}\times 1}_{\mathbb{Z}/n\mathbb{Z}} \quad \text{and} \quad \mathrm{Tw}_{\tilde{\zeta}_p\times 1} := \sum_{\chi} c_\chi \mathrm{Tw}^{\chi\times 1}_{\mathbb{Z}/m\mathbb{Z}}$$

we obtain

$$\tilde{\nu}_{p'} = \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/n_{p'}\mathbb{Z}}(1\times\tilde{\nu}_{p'}) = \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/n_{p'}\mathbb{Z}}\left(\mathrm{Tw}_{\tilde{\nu}_p^{-1}\times 1}(\tilde{\nu}_p\times\tilde{\nu}_{p'})\right) = \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/n_{p'}\mathbb{Z}}\mathrm{Tw}_{\tilde{\nu}_p^{-1}\times 1}\ \tilde{\nu}.$$

Combining the previous equation and $\tilde{\zeta}_{p'} = c\tilde{\nu}_{p'}$, we get

$$\begin{aligned}\tilde{\zeta} &= \mathrm{Tw}_{\tilde{\zeta}_p \times 1}\, \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/m_{p'}\mathbb{Z}}\, \tilde{\zeta}_{p'} = \mathrm{Tw}_{\tilde{\zeta}_p \times 1}\, \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/m_{p'}\mathbb{Z}}\, (c\tilde{\nu}_{p'}) \\ &= c\left(\mathrm{Tw}_{\tilde{\zeta}_p \times 1}\, \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/m_{p'}\mathbb{Z}}\, \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/n_{p'}\mathbb{Z}}\, \mathrm{Tw}_{\tilde{\nu}_p^{-1} \times 1}\, \tilde{\nu}\right)\end{aligned}$$

which implies $S^A_{n,\nu} = S^A_{m,\zeta}$. Since $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu} \subseteq S^A_{n,\nu}$, then $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta} \subseteq S^A_{n,\nu} = S^A_{m,\zeta}$ for any pair $(m,\zeta) \in < n,\nu >$. This shows that

$$S_{<n,\nu>} \subseteq S^A_{n,\nu}. \tag{7.50}$$

Since the biset functor $\mathbb{C}R_{\mathbb{C}}$ is semisimple, the subfunctor $S^A_{n,\nu}$ is also semisimple, and is a direct sum of $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta}$ for some $(m,\zeta) \in \Gamma$. For the reverse inclusion, let $n \leq |A|$ be a p'-number. We need to show that if $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta}$ is a summand of $S^A_{n,\nu}$, then $(m,\zeta) \equiv (n,\nu)$ with respect to the part (a). Therefore, let $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta} \subseteq S^A_{n,\nu}$. This inclusion requires $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta}(\mathbb{Z}/m\mathbb{Z}) \subseteq S^A_{n,\nu}(\mathbb{Z}/m\mathbb{Z})$. Since $S^A_{n,\nu}$ is generated by $\tilde{\nu}$, we can obtain $\tilde{\zeta}$ through $\tilde{\nu}$, i.e. there exists some virtual $p^{\mathfrak{n}}$-fibered biset $\gamma \in B^A(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ such that

$$\tilde{\zeta} = \gamma \cdot \tilde{\nu}. \tag{7.51}$$

By the structure of the Burnside group of the fibered bisets and Decomposition 3.21, the character $\tilde{\zeta}$ is a $\mathbb{C}$-linear combination of elements of the form

$$\mathrm{Ind}^{\mathbb{Z}/m\mathbb{Z}}_P \otimes_{AP} \mathrm{Inf}^P_{P/\hat{K}} \otimes_{A(P/\hat{K})} Y \otimes_{A(Q/\hat{L})} \mathrm{Def}^Q_{Q/\hat{L}} \otimes_{AQ} \mathrm{Res}^{\mathbb{Z}/n\mathbb{Z}}_Q\, \tilde{\nu} \tag{7.52}$$

for appropriate choices of the notation. As $\mathbb{Z}/n\mathbb{Z}$ is a minimal group for the biset functor $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\nu}$, the biset $\mathrm{Def}^Q_{Q/\hat{L}} \otimes_{AQ} \mathrm{Res}^{\mathbb{Z}/n\mathbb{Z}}_Q$ must be trivial because any biset through a group of smaller order than n annihilates $\tilde{\nu}$. Likewise, $\mathrm{Ind}^{\mathbb{Z}/m\mathbb{Z}}_P \otimes_{AP} \mathrm{Inf}^P_{P/\hat{K}}$ is also trivial due to the minimality of $\mathbb{Z}/m\mathbb{Z}$ for $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta}$. Hence, Expression 7.52 is actually

of the form

$$\left(\frac{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}}{U,\phi}\right)\ \tilde{\nu}. \tag{7.53}$$

Since $\left(\frac{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}}{U,\phi}\right)$ is obtained after applying Decomposition 3.21, the projections of U are full. We assume that $m \geq n$ without losing generality to use Theorem 7.4. We would use Theorem 7.5 otherwise. Then, we deduce that $\tilde{\zeta}$ is a $\mathbb{C}$-linear combination of elements of the form

$$\mathrm{Tw}^{\varphi}_{\mathbb{Z}/m\mathbb{Z}} \otimes_{A(\mathbb{Z}/m\mathbb{Z})} \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}} \otimes_{A(\mathbb{Z}/n\mathbb{Z})} {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}} \otimes_{A(\mathbb{Z}/n\mathbb{Z})} \mathrm{E}^{n}_{K,\kappa}\ \tilde{\nu} \tag{7.54}$$

for some $K \leq \mathbb{Z}/n\mathbb{Z}$, some faithful homomorphism $\kappa : K \to A$ and some isomorphism $\eta : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. By the biset $\mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}$ above, we deduce that $n|m$. Since the biset $\mathrm{E}^{n}_{K,\kappa}$ is decomposable by Theorem 7.3, the elements above become

$$\mathrm{Tw}^{\varphi}_{\mathbb{Z}/m\mathbb{Z}}\ \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\ {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{Tw}^{\xi}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{Inf}^{\mathbb{Z}/n\mathbb{Z}}_{(\mathbb{Z}/n\mathbb{Z})/K}\ \mathrm{Def}^{\mathbb{Z}/n\mathbb{Z}}_{(\mathbb{Z}/n\mathbb{Z})/K}\ \mathrm{Tw}^{\xi^{-1}}_{\mathbb{Z}/n\mathbb{Z}}\ \tilde{\nu} \tag{7.55}$$

by Theorem 7.4, where $\xi : \mathbb{Z}/n\mathbb{Z} \to A$ is a homomorphism satisfying $\xi|_K = \kappa$. Notice that ξ is faithful because κ is, and this implies that $K = 1$ due to the facts that n is a p'-number and A is a p-group. Then the elements above are actually in the form

$$\mathrm{Tw}^{\varphi}_{\mathbb{Z}/m\mathbb{Z}}\ \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\ {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \tilde{\nu} \tag{7.56}$$

By its transitivity we can divide the inflation map above into two parts as

$$\mathrm{Tw}^{\varphi}_{\mathbb{Z}/m\mathbb{Z}}\ \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/m_{p'}\mathbb{Z}}\ \mathrm{Inf}^{\mathbb{Z}/m_{p'}\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\ {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \tilde{\nu}. \tag{7.57}$$

Since $|A|$ is a p-number, the homomorphism $\varphi : \mathbb{Z}/m\mathbb{Z} \to A$ must be trivial on the p'-part of $\mathbb{Z}/m\mathbb{Z}$, that is φ must be of the form $\varphi = \varphi_p \times 1 : (\mathbb{Z}/m_p\mathbb{Z}) \times \mathbb{Z}/m_{p'}\mathbb{Z} \to A$.

Hence, transitive summands of γ must be of the form

$$\mathrm{Tw}^{\varphi_p\times 1}_{\mathbb{Z}/m\mathbb{Z}}\,\mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/m_{p'}\mathbb{Z}}\,\mathrm{Inf}^{\mathbb{Z}/m_{p'}\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\;{}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\;\tilde{\nu} = \mathrm{Tw}^{\varphi_p\times 1}_{\mathbb{Z}/m\mathbb{Z}}(\theta\times 1) = \varphi\theta\times 1 \tag{7.58}$$

where $\theta := \mathrm{Inf}^{\mathbb{Z}/m_{p'}\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\;{}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\;\tilde{\nu}$. But the last map $\varphi\theta\times 1$ can be given as $\mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/m_{p'}\mathbb{Z}}\,\varphi\theta$, which contradicts the primitivity of $\tilde{\zeta}$. Therefore, the inflation map $\mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/m_{p'}\mathbb{Z}}$ must be identity, i.e. $m_{p'} = n$, which yields $(m,\zeta)\equiv(n,\nu)$ because $\tilde{\zeta}_{p'} = \tilde{\nu}$. This justifies the inclusion

$$S^A_{n,\nu} \subseteq S_{<n,\nu>} \tag{7.59}$$

since the pair (m,ζ) is chosen arbitrarily and completes the proof for the part (a).

As for the part (b), let $(m,\zeta)\in<n,\nu>$. Then, we are given that $m=n$ such that $m_p = n_p > |A|$, $\zeta_{p'} = c\nu_{p'}$ for some $c\in\mathbb{C}$, and $\mathrm{E}^n_{A,\alpha}\tilde{\zeta} = c_\alpha \mathrm{E}^n_{A,\alpha}\tilde{\nu}$ for any isomorphism $\alpha: A\to A$ and for some $c_\alpha\in\mathbb{C}$ depending on α. If we sum all $\mathrm{E}_{A,\alpha}\tilde{\zeta}$, we obtain

$$\tilde{\zeta} = \sum_\alpha \mathrm{E}_{A,\alpha}\tilde{\zeta} = \sum_\alpha c_\alpha \mathrm{E}_{A,\alpha}\tilde{\nu} \tag{7.60}$$

by Lemma 6.3. This equality implies that $\tilde{\zeta}$ is generated by $\tilde{\nu}$, that is $\tilde{\zeta}\in S^A_{n,\nu}$. As $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\zeta}$ is generated by $\tilde{\zeta}$, we have $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\zeta}\subseteq S^A_{n,\nu}$. But $(n,\zeta)\in<n,\nu>$ is chosen arbitrarily, therefore

$$S_{<n,\nu>}\subseteq S^A_{n,\nu}. \tag{7.61}$$

The begining of the verification of the reverse inclusion goes identical with the one of the part (a). By skipping this identical part, if we take $S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\zeta}\subseteq S^A_{n,\nu}$, we deduce that $\tilde{\zeta}$ is a $\mathbb{C}$-linear combination of elements of the form

$$\mathrm{Tw}^{\varphi}_{\mathbb{Z}/m\mathbb{Z}}\otimes_{A(\mathbb{Z}/m\mathbb{Z})}\mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\otimes_{A(\mathbb{Z}/n\mathbb{Z})}{}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\otimes_{A(\mathbb{Z}/n\mathbb{Z})}\mathrm{E}^n_{K,\kappa}\;\tilde{\nu} \tag{7.62}$$

for some $K \leq \mathbb{Z}/n\mathbb{Z}$, for some faithful homomorphism $\kappa : K \to A$ and for some isomorphism $\eta : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. By Lemma 7.2, we can take m and n as p-numbers. Since $m \geq n > |A|$, the homomorphism φ cannot be faithful, and it is of the form $\varphi = \mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/|A|\mathbb{Z}}\ \tau$ for some homomorphism $\tau : \mathbb{Z}/|A|\mathbb{Z} \to A$. We can consider φ as $\mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\ \theta$ due to the transitivity of the inflation maps and the fact $n > |A|$, where $\theta = \mathrm{Inf}^{\mathbb{Z}/n\mathbb{Z}}_{\mathbb{Z}/|A|\mathbb{Z}}\ \tau$. On the other hand, ${}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}} \otimes_{A(\mathbb{Z}/n\mathbb{Z})} \mathrm{E}^{n}_{K,\kappa}\ \tilde{\nu}$ is a map from $\mathbb{Z}/n\mathbb{Z}$ to A by the actions of the bisets, and then, it is inflated to $\mathbb{Z}/m\mathbb{Z}$. Altogether, Expression 7.62 can be regarded as

$$\mathrm{Inf}^{\mathbb{Z}/m\mathbb{Z}}_{\mathbb{Z}/n\mathbb{Z}}\ [\theta \left({}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}} \otimes_{A(\mathbb{Z}/n\mathbb{Z})} \mathrm{E}^{n}_{K,\kappa}\ \tilde{\nu} \right)] \tag{7.63}$$

by the action of the twist map. But, $\mathbb{Z}/m\mathbb{Z}$ is minimal for $\tilde{\zeta}$, therefore n must be equal to m. i.e. Expression 7.62 must be of the form

$$\mathrm{Tw}^{\varphi}_{\mathbb{Z}/n\mathbb{Z}} \otimes_{A(\mathbb{Z}/n\mathbb{Z})} {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}} \otimes_{A(\mathbb{Z}/n\mathbb{Z})} \mathrm{E}^{n}_{K,\kappa}\ \tilde{\nu}. \tag{7.64}$$

In general, since we can follow the same steps for any prime number q instead of p, we proved implicitly that $m_{p'} = n_{p'}$, which means $m = n$. The following equations

$$\begin{aligned} \mathrm{Tw}^{\varphi}_{\mathbb{Z}/n\mathbb{Z}}\ {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{E}^{n}_{K,\kappa}\ \tilde{\nu} &= {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{Tw}^{\varphi\circ\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{E}^{n}_{K,\kappa}\ \tilde{\nu} \\ &= {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{E}^{n}_{K,\kappa(\varphi\circ\eta)|_K} \otimes_{A(\mathbb{Z}/n\mathbb{Z})} \mathrm{Tw}^{\varphi}_{\mathbb{Z}/n\mathbb{Z}}\ \tilde{\nu} \end{aligned}$$

can easily be shown by the tensor product formula, where $\varphi \circ \eta$ is the function composition of φ and η. As the homomorphism φ is not faithful, then $(\varphi \circ \eta)|_K$ is not either. Therefore, the homomorphism $\kappa(\varphi \circ \eta)|_K$ is faithful. Indeed, if we suppose $(\kappa(\varphi \circ \eta)|_K)(k) := \kappa(k)\varphi(\eta(k)) = 1$ for some $k \in K$, then $\kappa^{-1}(k) = \varphi(\eta(k))$. The last equation obliges k to be 1 because we work on cyclic p-groups, and equality at an element implies equality on a p-group, which is impossible. Hence, as κ runs over faithful homomorphisms, so does $\kappa((\varphi \circ \eta)|_K)$. Therefore, $\tilde{\zeta}$ is a $\mathbb{C}$-linear combination

of elements

$$\begin{aligned} {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{E}^{n}_{K,\lambda}\ \mathrm{Tw}^{\varphi}_{\mathbb{Z}/n\mathbb{Z}}\ \tilde{\nu} &= {}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{E}^{n}_{K,\lambda}\ (\varphi\tilde{\nu}) \\ &= c_{\varphi}\left({}_{\mathbb{Z}/n\mathbb{Z}}\mathrm{Iso}^{\eta}_{\mathbb{Z}/n\mathbb{Z}}\ \mathrm{E}^{n}_{K,\lambda}\ \tilde{\nu}\right) = c_{\varphi}\left(\mathrm{E}^{n}_{K,\lambda}\ \tilde{\nu}\right) \end{aligned}$$

by Lemma 7.4 and the action of isogation bisets, where $\lambda := \kappa((\varphi \circ \eta)|_K) : K \to A$ is a faithful homomorphism. Recall from Section 3.6 that if we multiply $\mathrm{E}^{n}_{K,\lambda}\ \tilde{\nu}$ from the left by $\mathrm{E}^{n}_{A,\alpha}$, we obtain $\mathrm{E}^{n}_{A,\alpha}\tilde{\nu}$ if $(K,\lambda) \preceq (A,\alpha)$ and zero otherwise. Consequently, we deduce that $\mathrm{E}^{n}_{A,\alpha}\ \tilde{\zeta}$ is a $\mathbb{C}$-linear combination of elements of the form $\mathrm{E}^{n}_{A,\alpha}\ \tilde{\nu}$, that is

$$\mathrm{E}^{n}_{A,\alpha}\ \tilde{\zeta} = c\mathrm{E}^{n}_{A,\alpha}\ \tilde{\nu}, \tag{7.65}$$

where $\alpha : A \to A$ is an arbitrary isomorphism. Therefore, it is valid for all such isomorphism, i.e. $(n,\zeta) \equiv (n,\nu)$, which implies $S^{A}_{n,\nu} \subseteq S_{<n,\nu>}$. Hence, we have proved that

$$S^{A}_{n,\nu} = S_{<n,\nu>} \tag{7.66}$$

which justifies the assertion of the theorem. □

Theorem 7.7. *The $p^{\mathfrak{n}}$-fibered biset functor $S^{A}_{n,\nu}$ is simple.*

Proof. Theorem 7.6 encapsulates the information that $S^{A}_{n,\nu}$ is a cyclic $p^{\mathfrak{n}}$-fibered biset functor generated by $\tilde{\nu}$. Hence, to verify that $S^{A}_{n,\nu}$ is a simple $p^{\mathfrak{n}}$-fibered biset functor, it suffices to show that the intersection of the kernels of all maps

$$S^{A}_{n,\nu}(G) \to S^{A}_{n,\nu}(\mathbb{Z}/n\mathbb{Z}) \tag{7.67}$$

induced by $p^{\mathfrak{n}}$-fibered $(\mathbb{Z}/n\mathbb{Z}, G)$-bisets is zero for any group G. We consider equivalence classes with respect to the part (b) because the proof for the part (a) is identical with that of Theorem 7.2. Let $\psi \in S^{A}_{n,\nu}(G)$ be in the intersection of the kernels of all maps, that is $X(\psi) = 0$ for each $p^{\mathfrak{n}}$-fibered $(\mathbb{Z}/n\mathbb{Z}, G)$-bisets X. We are done if we show that

$\psi = 0$. As ψ is in the kernels of all maps, it is also in the kernels of maps induced by ordinary $(\mathbb{Z}/n\mathbb{Z}, G)$-bisets. In other words, $X(\psi) = 0$ for any map

$$X : S^A_{n,\nu}(G) \to S^A_{n,\nu}(\mathbb{Z}/n\mathbb{Z}) \tag{7.68}$$

induced by an ordinary $(\mathbb{Z}/n\mathbb{Z}, G)$-biset. Note that, we can express ψ as a sum as follows

$$\psi = \sum_{(n,\zeta)\in<n,\nu>} \psi_{(n,\zeta)} \tag{7.69}$$

because we know that

$$S^A_{n,\nu}(G) = \bigoplus_{(n,\zeta)\in<n,\nu>} S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\zeta}(G). \tag{7.70}$$

Since $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\zeta}$ is an ordinary biset functor, it is closed under biset actions. Therefore, $X(\psi_{(n,\zeta)}) = 0$ for any $(n, \zeta) \in< n, \nu >$ as every X is assumed to be induced by bisets. On the other hand, the biset functor $S_{\mathbb{Z}/n\mathbb{Z},\mathbb{C}_\zeta}$ is simple. Therefore, if $X(\psi_{(n,\zeta)}) = 0$ for any $(\mathbb{Z}/n\mathbb{Z}, G)$-biset, then $\psi_{(n,\zeta)} = 0$ for any $(n, \zeta) \in< n, \nu >$. Hence, $\psi = 0$ since all summands of ψ is zero. □

Corollary 7.2. *The $p^{\mathfrak{n}}$-fibered biset functor $\mathbb{C}R_{\mathbb{C}}$ is semisimple and there is an isomorphism*

$$\mathbb{C}R_{\mathbb{C}} \cong \bigoplus_{<n,\nu>\in\Gamma_p} S^A_{n,\nu} \tag{7.71}$$

of $p^{\mathfrak{n}}$-fibered biset functors.

Proof. It is immediate from Theorems 7.6 and 7.7. □

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