ON INFINITE DIMENSIONAL SPHERICAL ANALYSIS

by

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ABSTRACT

ON INFINITE DIMENSIONAL SPHERICAL ANALYSIS

This thesis is concerned with the spherical analysis of two different Olshanski pairs, one of which is related to Heisenberg groups, and the other to the automorphism groups of homogeneous trees. The spherical functions of positive type on the infinite dimensional Heisenberg group $H(\infty)$ which are invariant under the natural action of the infinite dimensional unitary group $U(\infty)$ are determined. On the other hand, we consider an Olshanski pair which is constructed from the stabilizers of the horicycles of homogeneous trees of finite degree, where the horicycles form a partition of the set of vertices of the tree, and then we find all spherical functions of this pair. Finally, we give realizations of the corresponding irreducible unitary representations.

ÖZET

SONSUZ BOYUTLU KÜRESEL ANALİZ ÜZERİNE

Bu tezde biri Heisenberg grupları, diğeri ise homojen ağaçların otomorfizma grupları ile bağlantılı iki farklı Olshanski çiftinin küresel analizi ile ilgilenilmiştir. $H(\infty)$ sonsuz boyutlu Heisenberg grubu üzerinde tanımlı, $U(\infty)$ sonsuz boyutlu üniter grubunun doğal etkisi altında değişmez tüm pozitif tanımlı küresel fonksiyonlar belirlenmiştir. Diğer bir taraftan, sonlu dereceli homojen ağaçların, noktalar kümesinin belirli bir parçalanışını sabitleyen otomorfizma grupları kullanılarak kurulan bir Olshanski çifti ele alınmış ve bu çiftin tüm küresel fonksiyonları bulunmuştur. Son olarak, bu küresel fonksiyonlara karşılık gelen tüm indirgenemez üniter temsiller realize edilmiştir.

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LIST OF SYMBOLS

$B(H)$	The algebra of bounded operators on the Hilbert space H
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^*$	The set of non-zero complex numbers
dim	Dimension
$\overline{f}$	The conjugate of the function f
$\mathcal{F}_\lambda^n$	Fock space
$\widehat{G}$	The unitary dual of G
H_n	Heisenberg group
$H(\infty)$	Infinite dimensional Heisenberg group
H_n^k	Horicyle of a $(k+1)$-homogeneous tree
H_n^∞	Horicyle of a homogeneous tree of countably-infinite degree
$\mathrm{Hom}_G(\pi, \pi^{'})$	The collection of all G-intertwinning operators from π to $\pi^{'}$
Im	Imaginary part
$\mathrm{Ind}_H^G(\pi)$	The representation of G induced from the representation π of H
$L^1(H_n)^{K_n}$	The algebra of integrable, K_n-invariant functions on H_n
$L^2(X, \mu)$	The space of square integrable functions on (X, μ)
$\mathrm{mult}(\pi, \psi)$	The multiplicity of π in ψ
$\mathbb{N}$	{1,2,3,...}
$\mathcal{O}(\mathbb{C}^n)$	The space of complex-valued holomorphic functions on $\mathbb{C}^n$
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^*$	The set of non-zero real numbers
$\mathbb{R}^{>0}$	The set of strictly positive real numbers
Re	Real part
S^1	The circle group
$sgn(\lambda)$	The sign of the real number λ
$span$	The linear span
$U(n)$	The unitary group of $n \times n$ complex matrices
$U(\infty)$	The infinite dimensional unitary group

$U(V)$	The group of unitary operators on the Hilbert space V
V_π	The representation space corresponding to π
$V^\perp$	The orthogonal complement of V
$\overline{z}$	The conjugate of the complex number z
$\rho\restriction_N$	The restriction of ρ to N
$\mathbb{1}_K$	One-dimensional trivial representation of K
1_E	The characteristic function of the set E
$\cdot\setminus\cdot$	Difference of sets
$\lvert\cdot\rvert$	The usual absolute value of a complex number
$\lVert\cdot\rVert$	The usual norm on $\mathbb{C}^n$
$\lVert\cdot\rVert_\lambda$	The norm on the Fock space $\mathcal{F}_\lambda^n$
$\langle\cdot,\cdot\rangle$	An inner product on a Euclidean space or the usual inner product on $\mathbb{C}^n$
$\langle\cdot,\cdot\rangle_\lambda$	The inner product on the Fock space $\mathcal{F}_\lambda^n$
$\ltimes$	Semidirect product
$\otimes$	Tensor product
$\oplus$	Direct sum
$\widehat{\bigoplus}$	Hilbert space direct sum

LIST OF ABBREVIATIONS

GNS	Gelfand-Naimark-Segal

1. INTRODUCTION

In representation theory, studying not only a single group G, but a pair (G, K) for a subgroup K of G, is an important idea to define a reasonable family of representations of the group G. The theory of Gelfand pairs, spherical functions and spherical representations is a well-known example of this idea of studying group pairs. Let G be a locally compact group and K be a compact subgroup of G. Then G possesses a Haar measure so that we have the convolution algebra $L^1(K\backslash G/K)$ of K-bi-invariant, integrable functions with respect to a Haar measure on G. The pair (G, K) is said to be a Gelfand pair if this algebra $L^1(K\backslash G/K)$ is commutative. In the language of representation theory, this definition amounts to say that the multiplicity of the trivial representation of K in each unitary representation of K restricted from an irreducible unitary representation of G is at most one. A K-bi-invariant, non-zero, continuous complex function φ on G is said to be a spherical function for the Gelfand pair (G, K) if

$$\varphi(x)\varphi(y) = \int_K \varphi(xky)\ dk \tag{1.1}$$

for every $x, y \in G$ where dk is the normalized Haar measure on K. An irreducible unitary representation (π, V) of G is called spherical for the Gelfand pair (G, K) if the space V^K of K-invariant vectors in V is non-zero. In this case there is essentially a unique $v \in V^K$ which has norm 1. Then $\varphi(g) = \langle v, \pi(g)(v)\rangle$ is a positive definite spherical function on G. Conversely, there is a construction, the so-called Gelfand-Neimark-Segal construction, which gives a spherical representation of (G, K) corresponding to every positive definite spherical function for (G, K). Therefore there is essentially a one-to-one correspondence between positive definite spherical functions and spherical representations of the Gelfand pair (G, K). It is hence a natural problem to find spherical functions and then to find the realizations of the corresponding spherical representations of a Gelfand pair (G, K) given by the GNS-construction.

A general theory for harmonic analysis on the inductive limit G of some locally compact groups G_n and understanding the irreducible unitary representations of the inductive limit group G in terms of the irreducible unitary representations of G_n are first studied by Olshanski in [20]. With this inductive limit approach, in particular, Olshanski studied the inductive limits of Gelfand pairs, which are now called as Olshanski spherical pairs (Olshanski pairs in short), and he generalized the notion of spherical functions from Gelfand pairs to Olshanski pairs so that the one-to-one correspondence between spherical representations and positive definite spherical functions given by the GNS-construction still holds. From then on, it has also been a natural programme to find all positive definite spherical functions and to make realizations of the corresponding spherical representations for an Olshanski pair (G, K). Several examples of infinite dimensional groups and pairs arising from classical matrix groups and also from the symmetric group were investigated in a large number of different papers such as [13], [20], [22], [23] and [27].

The positive definite spherical functions for a Gelfand pair (G, K) are uniquely determined by the characters of the commutative convolution algebra $L^1(K\backslash G/K)$. In the case of an Olshanski pair (G, K), the group G under discussion is not locally compact in general. Hence we do not have a Haar measure, a convolution and an algebra structure anymore. The problem that initiated this thesis was to construct a structure, an algebra structure if possible, which corresponds to the spherical functions of a general Olshanski pair as there is one in the case of Gelfand pairs. On the way of this algebraic aspect on the abstract theory of Olshanski pairs, we considered the harmonic analysis of two different Olshanski pairs, one is related to the Heisenberg group and the other is related to the automorphism group of a countable-degree tree. We determined the spherical dual, i.e. the positive definite spherical functions for the one related to the Heisenberg group and we constructed both the spherical functions and the spherical representations for the one related to the automorphism group of a homogeneous tree of countable degree. It still remains open whether there exists an algebra structure whose characters determine the positive definite spherical functions for an Olshanski pair or not.

For an n-dimensional complex Euclidean vector space V_n, any closed subgroup K_n of the unitary group $U(V_n)$ acts by automorphisms on the (2n+1)-dimensional Heisenberg group $H_n = V_n \times \mathbb{R}$ where the multiplication on H_n is given by

$$(z,t)(z^{'},t^{'}) = (z+z^{'}, t+t^{'} + \mathrm{Im}\langle z, z^{'}\rangle). \tag{1.2}$$

This action gives a yield to the locally compact group $G_n = K_n \ltimes H_n$. For some special choices of the vector space V_n and the subgroup K_n of the unitary group $U(V_n)$, it was observed that the pair (G_n, K_n) forms a Gelfand pair and the spherical functions for these pairs were determined. The question of determining all closed subgroups K_n of the unitary group $U(V_n)$ such that (G_n, K_n) is a Gelfand pair was answered by Carcano in [5] with a representation-theoretic criteria. $K_n \leq U(V_n)$ was required to act multiplicity free on the polynomial ring $P(V_n)$. There is a description of the bounded spherical functions for such Gelfand pairs, due to Benson, Jenkins and Ratcliff [3]. Recently, Faraut [14] presented a work on the spherical analysis for some special cases of inductive limits of Gelfand pairs associated to Heisenberg groups.

When it comes to the homogeneous trees of finite degree, the so-called Bruhat-Tits trees, they appeared as a special type of Bruhat-Tits buildings of rank one and the automorphism groups of Bruhat-Tits trees gave an attractive family of locally compact, totally disconnected, seperable, metrizable groups. The study of irreducible unitary representations of the automorphism group G of a Bruhat-Tits tree was started by Cartier in [6] and [7]. If K is a maximal compact subgroup of G stabilizing one vertex, then (G, K) is a Gelfand pair and the spherical functions for this pair were computed in [17]. Later in the mid-seventies, all irreducible representations of G were constructed by Olshanski in [19]. The study of trees and the groups acting on trees was also stimulated by the course notes [25] of J-P. Serre where he clarified the connections between trees, amalgams and the p-adic SL_2. On the other hand, the automorphism groups of homogeneous trees of infinite degree first occured in the work of Olshanski [21] where all irreducible unitary representations of these groups were found.

In Chapter 2, we present the introductory and fundamental materials of the thesis. We introduce Olshanski pairs, their spherical functions and spherical representations. In the theory of Gelfand pairs, it is a well-known fact that the positive definite spherical functions and the unitary equivalence classes of spherical representations are in one-to-one corrrespondence. The main objective of this chapter is to reach the result of Olshanski saying that we can carry this fact from the theory of Gelfand pairs to the theory of Olshanski pairs.

Chapter 3 is devoted to harmonic analysis of the Gelfand pairs of the form $(K_n \ltimes H_n, K_n)$ where $H_n = \mathbb{C}^n \times \mathbb{R}$ and K_n is a closed subgroup of the unitary group $U(n) = U(\mathbb{C}^n)$ and to determination of the spherical dual of the Olshanski spherical pair $(U(\infty) \ltimes H(\infty), U(\infty))$ which is the inductive limit of the Gelfand pairs $(K_n \ltimes H_n, K_n)$ where $H_n = \mathbb{C}^n \times \mathbb{R}$ and $K_n = U(n)$. Spherical analysis on the Gelfand pairs of the form $(K_n \ltimes H_n, K_n)$ already exists in the literature, but the works on the spherical representations and the works on the spherical functions are found separately only in some references like [3], [14] and [28]. In this chapter, we bring these works together in a nearly self-contained form by giving the correspondences between the positive definite spherical functions and the spherical representations. For $\lambda \in \mathbb{R}^*$, we introduce the Fock representations $(T_\lambda, \mathcal{F}_\lambda^n)$ of the Heisenberg group H_n and see that they form a class of non-equivalent irreducible unitary representations of H_n by making use of the fact that the Fock spaces $\mathcal{F}_\lambda^n$ are reproducing kernel Hilbert spaces. Following the approach of Wolf in [28], we determine the unitary dual $\widehat{H_n}$ of the Heisenberg group H_n by using Mackey' s machinary that constructs the unitary dual of a locally compact, Type I group by inducing representations from certain closed subgroups. The unitary dual $\widehat{H_n}$ of H_n consists only of the unitary characters and the Fock representations. Applying Mackey machine once more, we determine the unitary dual $\widehat{K_n \ltimes H_n}$ and then the spherical representations for $(K_n \ltimes H_n, K_n)$ (whenever it forms a Gelfand pair) by some multiplicity computations of the trivial representation of K_n in some certain representations. There are two types of spherical representations for a Gelfand pair of the form $(K_n \ltimes H_n, K_n)$: the ones that derive from unitary characters of H_n and the ones that derive from infinite dimensional Fock representations of H_n. In both cases applied, the steps of Mackey machine are performed in great detail.

We then turn our attention to the determination of the positive definite spherical functions for the Gelfand pair $(U(n) \ltimes H(n), U(n))$. In this case, the spherical functions can be considered as functions on H_n which are $U(n)$-invariant and every character of the commutative convolution algebra $L^1(H_n)^{U(n)}$ of integrable $U(n)$-invariant functions on H_n gives rise to a unique bounded spherical function. In [14], Faraut describes a family of characters of the algebra $L^1(H_n)^{U(n)}$. We find explicitly the bounded spherical functions corresponding to these characters. We observe that they are indeed positive definite and they correspond to the spherical representations that derive from infinite-dimensional Fock representations of H_n. We also find explicitly the positive definite spherical functions corresponding to the spherical representations that derive from one-dimensional representations of H_n in order to complete the determination of the spherical dual of $(U(n) \ltimes H(n), U(n))$.

The main results of the thesis are contained in Section 3.3 of Chapter 3 and in Chapter 4. Our main result in Chapter 3 is Theorem 3.3.14 where we determine the spherical dual of the Olshanski spherical pair $(U(\infty) \ltimes H(\infty), U(\infty))$. To simplify the asymptotic functional equation satisfied by the spherical functions for $(U(\infty) \ltimes H(\infty), U(\infty))$, we use two lemmas from analysis on the unitary group $U(n)$, due to Faraut. We also prove a result on positive definite functions on the Heisenberg group H_n which has the key role on positive definiteness arguments in the proof of Theorem 3.3.14.

In Chapter 4, the same problems of harmonic analysis are considered for a different Olshanski pair related to the automorphism groups of homogeneous trees of countably infinite degree. If X is a homogeneous tree of finite degree, one can fix a point ω on the boundary of X and consider the group of stabilizers of the corresponding horicycles which gives rise to a Gelfand pair. In [18] Nebbia found all spherical functions of this pair and described the corresponding spherical representations. In [1] Axelgaard studied an embedding of the k-homogeneous tree into the $(k+1)$-homogeneous tree and the embedding of the corresponding automorphism groups $G_k \subset G_{k+1}$. This way he gets an Olshanski pair. He describes all spherical functions and the corresponding spherical representations. He mentions that such an embedding is also possible for the Gelfand pairs studied by Nebbia in [18] and states in [1] the description of spherical functions and representations of that pair as an open problem. We complete this picture by describing all spherical functions and the realizations of the corresponding spherical representations in this case.

2. OLSHANSKI SPHERICAL PAIRS

In this thesis, we study harmonic analysis on two different Olshanski spherical pairs. In order to explain what it means to study harmonic analysis on an Olshanski spherical pair by formulating the natural problems of the subject, we devote this chapter to the presentation of some basic results on the abstract theory of Olshanski spherical pairs which are the generalizations of some well-known results on the abstract theory of Gelfand pairs. All results and proofs of this introductory chapter are based on the lecture notes [11] of Jacques Faraut on finite and infinite dimensional spherical analysis which form one of the few materials on the abstract theory of Olshanski spherical pairs.

2.1. Olshanski Spherical Pairs

Let G be a topological group and K be a closed subgroup of G. Given a unitary representation (π, V) of G on a Hilbert space V, we always assume that it is continuous in the sense that $\pi : G \to U(V)$ is a continuous group homomorphism with respect to the strong operator topology on the group of unitary operators $U(V)$ on V. Let V^K be the space of vectors in V which are invariant under the action of K, i.e.

$$V^K = \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}.$$

Note that V^K is a closed subspace of V. The following proposition gives a relation between the irreducibility of V and how small V^K is.

Proposition 2.1.1. *Let G be a topological group and K be a closed subgroup of G. Let (π, V) be a unitary representation of G. Then if V has a non-zero, K-invariant, cyclic vector v and* $\dim(V^K) = 1$, *then V is irreducible.*

Proof. Let W be a closed invariant subspace of V. We will show either $W = \{0\}$ or $W = V$. Let P_W be the orthogonal projection on W. Note that $P_W \in \mathrm{Hom}_G(\pi, \pi)$.

Hence $\pi(k)P_W(v) = P_W(\pi(k)v) = P_W(v)$ for all $k \in K$ so that $P_W(v) \in V^K$. Then $\dim(V^K) = 1$ together with $v \in V^K$ implies that $P_W(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $P_W(v) = 0$ so that $v \in W^\perp$. Since W is invariant, then $\langle w, \pi(x)v\rangle = \langle \pi(x^{-1})w, v\rangle = 0$ for all $w \in W$ and $x \in G$. Since v is cyclic, it follows that $W = \{0\}$. If $\lambda \neq 0$, then $v \in W$ and since v is cyclic and W is closed invariant, we get $W = V$. $\square$

Conversely, for an irreducible unitary representation V of a topological group G, every non-zero vector in V is cyclic, but if $V^K \neq \{0\}$, then V^K need not to be one-dimensional. In case K consists only of the identity element, then $V^K = V$ which is not one-dimensional in general. Hence it makes sense to make the following definition of a spherical pair. We say that the pair (G, K) is a spherical pair if for every irreducible unitary representation (π, V) of G, we have $\dim(V^K) \leq 1$.

A function $f : G \to \mathbb{C}$ is called K-left-invariant if $f(kx) = f(x)$ for all $x \in G$ and $k \in K$, K-right-invariant if $f(xk) = f(x)$ for all $x \in G$ and $k \in K$ and K-bi-invariant if it is both K-left-invariant and K-right-invariant.

If G is locally compact, we have a Haar measure μ_G on G. Then the space $L^1(G)$ of complex-valued, integrable functions on G becomes an involutive Banach algebra under convolution where the convolution $f * g$ of $f, g \in L^1(G)$ and the involution f^* of f are defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\ d\mu_G(y), \qquad f^*(x) = \overline{f(x^{-1})}.$$

The space $M^b(G)$ of bounded, complex Borel measures on G is also a Banach algebra with involution where the product of two measures $\mu, \nu \in M^b(G)$ is given by their convolution $\mu * \nu$ defined by

$$\int_G f(x)\ d\mu * \nu(x) = \int_G \int_G f(yz)\ d\mu(y)\ d\nu(z) \tag{2.1}$$

for every measurable function $f : G \to \mathbb{C}$.

An involution is given by μ^* where $\mu^*(f) = \overline{\mu(f^*)}$ for all continuous compactly supported functions $f : G \to \mathbb{C}$. The norm $\|\mu\|$ of a measure μ is defined by $\|\mu\| = |\mu|(G)$ where $|\mu|$ is the total variation of μ. Taking $f = 1_A$ for a Borel subset A of G in the equation (2.1), we get

$$\mu * \nu(A) = \int_G \mu(Ay^{-1})\ d\nu(y). \tag{2.2}$$

For a unitary representation (π, V) of G and a bounded complex measure $\mu \in M^b(G)$, the continuous linear operator $\pi(\mu) \in B(V)$ is defined by

$$\pi(\mu)(v) = \int_G \pi(x)v\ d\mu(x)$$

for all $v \in V$. Then, $\pi : M^b(G) \to B(V)$ gives a $*$-representation of the algebra $M^b(G)$ on the Hilbert space V, i.e $\pi : M^b(G) \to B(V)$ is a linear map satisfying the equations

$$\pi(\mu * \nu) = \pi(\mu)\pi(\nu) \ \text{ and } \ \pi(\mu^*) = \pi(\mu)^* \tag{2.3}$$

for all $\mu, \nu \in M^b(G)$.

The algebra $L^1(G)$ can be regarded as a dense $*$-Banach subalgebra of $M^b(G)$ via the map $f \mapsto f\mu_G$.

Let $L^1(K\backslash G/K)$ be the $*$-Banach subalgebra of $L^1(G)$ consisting of K-bi-invariant, integrable functions on G and $M^b(K\backslash G/K)$ be the $*$-Banach subalgebra of $M^b(G)$ consisting of K-bi-invariant, bounded, complex Borel measures on G. Note that $L^1(K\backslash G/K)$ is a dense Banach $*$-subalgebra of $M^b(K\backslash G/K)$ as well.

If G is locally compact, K is compact and the algebra $L^1(K\backslash G/K)$ is commutative, we say that the pair (G, K) is a Gelfand pair.

Let $((G_n, K_n))_{n\in\mathbb{N}}$ be an increasing sequence of Gelfand pairs (in the sense that $G_n \subseteq G_{n+1}$ and $K_n \subseteq K_{n+1}$ for each $n \in \mathbb{N}$) satisfying the following properties:

G_n is Hausdorff , G_n is a closed subgroup of G_{n+1}, the topology of G_n is the topology induced from the topology of G_{n+1} and $K_n = K_{n+1} \cap G_n$. Define

$$G = \cup_{n=1}^{\infty} G_n, \qquad K = \cup_{n=1}^{\infty} K_n.$$

Then the pair (G, K) is called an Olshanski spherical pair (or Olshanski pair in short). We put the inductive limit topology and the natural multiplication on G. Then G is a Hausdorff topological group which is generally not locally compact and K is a closed subgroup of G which is generally not compact. Also each K_n is a compact, hence a closed subgroup of G.

Now let (G, K) be an Olshanski spherical pair and (π, V) be a unitary representation of G. Let P_n and P be the orthogonal projections on the closed subspaces V^{K_n} and V^K of V respectively. Note that

$$P_n(v) = \int_{K_n} \pi(k_n)v \; d\mu_n(k_n)$$

for every $v \in V$ where μ_n is the normalized Haar measure on K_n. Since $K_n \subseteq K_{n+1}$ for each $n \in \mathbb{N}$, we have $V^{K_{n+1}} \subseteq V^{K_n}$ and since $K = \cup_{n=1}^{\infty} K_n$, we have $V^K = \cap_{n=1}^{\infty} V^{K_n}$. Hence the projections P_n converge to the projection P in the strong operator topology.

Proposition 2.1.2. *If (G, K) is an Olshanski spherical pair, then it is a spherical pair.*

Proof. Let (π, V) be an irreducible unitary representation of G such that $V^K \neq \{0\}$. As (G_n, K_n) is a Gelfand pair, the algebra $L^1(K_n \backslash G_n / K_n)$ is commutative. Then the measure algebra $M^b(K_n \backslash G_n / K_n)$ is also commutative. Let μ_n be the normalized Haar measure on K_n. We extend the measure μ_n to a compactly supported measure ν_n on the Borel σ-algebra of G_n by $\nu_n(E) = \mu_n(E \cap K_n)$. Then $\nu_n \in M^b(K_n \backslash G_n / K_n)$ and $P_n(v) = \int_{G_n} \pi(x)v \; d\nu_n(x)$ for all $v \in V$. For $x \in G_n$, let δ_x be the Dirac measure at x.

Then $\pi(x)v = \int_{G_n} \pi(y)v \, d\delta_x(y)$ for all $v \in V$. Hence

$$P_n = \pi(\nu_n) \text{ and } \pi(x) = \pi(\delta_x). \tag{2.4}$$

By using K_n-bi-invariance of ν_n and Equation 2.2, we get $\nu_n * \delta_x * \nu_n \in M^b(K_n \backslash G_n / K_n)$. Then since $M^b(K_n \backslash G_n / K_n)$ is commutative, for any $x, y \in G_n$

$$\nu_n * \delta_x * \nu_n * \nu_n * \delta_y * \nu_n = \nu_n * \delta_y * \nu_n * \nu_n * \delta_x * \nu_n. \tag{2.5}$$

Now applying π to both sides of the equation (2.5) and using the equations (2.3) and (2.4), we get

$$P_n \pi(x) P_n \pi(y) P_n = P_n \pi(y) P_n \pi(x) P_n$$

for every $x, y \in G_n$. Since $V^{K_{n+m}} \subseteq V^{K_n}$ for all $m \in \mathbb{N}$, $P_{n+m} = P_n P_{n+m} = P_{n+m} P_n$. Hence for all $n, m, m' \in \mathbb{N}$ and $x, y \in G_n$,

$$P_{n+m'} \pi(x) P_n \pi(y) P_{n+m} = P_{n+m'} \pi(y) P_n \pi(x) P_{n+m}.$$

When we take limits as m, m' and then n converges to infinity, we get

$$P \pi(x) P \pi(y) P = P \pi(y) P \pi(x) P$$

for all $x, y \in G$ as P_n converges to P in the strong operator topology. Hence the operator norm closed algebra A generated by the operators $P\pi(x)P$, $x \in G$ is commutative. Since irreducible representations of commutative Banach algebras are at most 1 dimensional and V^K is invariant under the action of A, it sufficies to show that V^K is irreducible as a representation of A.

So assume that $V^K = V_1 \oplus V_2$ where V_1, V_2 are A-invariant orthogonal subspaces of V^K and $V_1 \neq \{0\}$. Let $v_1 \in V_1$ and $v_1 \neq 0$. Then for any $x \in G$ and $v_2 \in V_2$, $\langle P\pi(x)Pv_1, v_2 \rangle = 0$ so that $\langle \pi(x)v_1, v_2 \rangle = 0$. Since (π, V) is an irreducible representation of G, v_1 is a cyclic vector. Hence $V_2 = \{0\}$ as desired. $\square$

2.2. Positive Definite Functions on Groups and GNS-construction

Let G be a group and $\varphi : G \to \mathbb{C}$ be a function. The function φ is called positive definite if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(x_i^{-1} x_j) \geq 0$$

for any finite subset $\{x_1, x_2, ..., x_n\}$ in G and for all systems $\{c_1, c_2, ..., c_n\}$ of complex numbers. Given a unitary representation (π, V) of G and a vector $v \in V$, the function φ defined by

$$\varphi(x) = \langle v, \pi(x)v \rangle$$

is positive definite since

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(x_i^{-1} x_j) = \sum_{i,j=1}^{n} c_i \overline{c_j} \langle \pi(x_i)v, \pi(x_j)v \rangle = \| \sum_{i=1}^{n} c_i \pi(x_i) v \|^2 \geq 0.$$

Remark 2.2.1. If $\varphi : G \to \mathbb{C}$ is positive definite, then for every $x \in G$, $\varphi(x^{-1}) = \overline{\varphi(x)}$.

Assume G is a topological group and K is a closed subgroup of G and denote by $P_1(K \backslash G/K)$ the convex set consisting of continuous, K-bi-invariant, positive definite functions φ on G such that $\varphi(e) = 1$. Let (π, V) be a unitary representation of G with a K-invariant, unit vector $v \in V$. Then for $\varphi(x) = \langle v, \pi(x)v \rangle$, we have $\varphi \in P_1(K \backslash G/K)$. Conversely, any $\varphi \in P_1(K \backslash G/K)$ can be obtained in this way and this is exactly what the so-called Gelfand-Naimark-Segal construction says.

Proposition 2.2.2. *[11] (GNS-construction) Let G be a topological group and K be a closed subgroup of G. Given $\varphi \in P_1(K \backslash G/K)$, there exists a unitary representation*

(π, V) of G with a K-invariant, unit and cyclic vector $v \in V$ such that for all $x \in G$,

$$\varphi(x) = \langle v, \pi(x)v \rangle.$$

The triple (π, V, v) is unique up to isomorphism in the following sense: If (π', V', v') is another triple such that (π', V') is a unitary representation of G with a K-invariant, unit and cyclic vector $v' \in V'$ and $\varphi(x) = \langle v', \pi'(x)v' \rangle$ for all $x \in G$, then there exists an isometric isomorphism $T : V \to V'$ such that $T(v) = v'$ and T is an intertwinning operator between the representations (π, V) and (π', V'), i.e. $T\pi(x) = \pi'(x)T$ for all $x \in G$.

An irreducible unitary representation of G with a non-zero, K-invariant vector is called a spherical representation of the pair (G, K). If (π, V) is a spherical representation of (G, K) with a K-invariant, unit vector $v \in V$, then for $\varphi(x) = \langle v, \pi(x)v \rangle$, we have $\varphi \in \mathrm{Ext}[P_1(K \backslash G/K)]$ where $\mathrm{Ext}[P_1(K \backslash G/K)]$ is the set of extremal points in the convex set $P_1(K \backslash G/K)$. Conversely, for each function φ in $\mathrm{Ext}[P_1(K \backslash G/K)]$ the representation associated with φ by the GNS-construction is a spherical representation of (G, K) ([Proposition 1.4, [11]]).

Remark 2.2.3. Let G be a topological group and K be a closed subgroup of G. From the GNS-construction, it follows that if $\varphi \in P_1(K \backslash G/K)$, then $|\varphi(x)| \leq 1$ for all $x \in G$, so φ is bounded.

2.3. Positive Definite Spherical Functions and Spherical Representations

In this section, let (G, K) be an Olshanski spherical pair which is the inductive limit of the increasing sequence $((G_n, K_n))_{n \in \mathbb{N}}$ of Gelfand pairs.

Definition 2.3.1. A non-zero, continuous, K-bi-invariant function $\varphi : G \to \mathbb{C}$ is said to be a spherical function for the Olshanski spherical pair (G, K) if the functional equation

$$\lim_{n \to \infty} \int_{K_n} \varphi(xky) \; d\mu_n(k) = \varphi(x)\varphi(y)$$

is satisfied for all $x, y \in G$ where μ_n is the normalized Haar measure on K_n.

As an immediate property following the previous definition, note that for a spherical function $\varphi : G \to \mathbb{C}$ for an Olshanski spherical pair (G, K), we have $\varphi(e) = 1$ (here by e we denote the identity element of G) as

$$\varphi(x)\varphi(e) = \lim_{n\to\infty} \int_{K_n} \varphi(xk)\ d\mu_n(k) = \lim_{n\to\infty} \int_{K_n} \varphi(x)\ d\mu_n(k) = \varphi(x)$$

for all $x \in G$ and φ is non-zero.

Olshanski' s definition of a spherical function for an Olshanski spherical pair which is given via Definition 2.3.1 generalizes a famous result for Gelfand pairs as follows:

Proposition 2.3.2. *Let $\varphi \in P_1(K\backslash G/K)$. Then the unitary representation corresponding to φ by the GNS-construction is spherical if and only if φ is spherical.*

Proof. Let (π, V, v) be the triple corresponding to φ by the GNS-construction. Let P be the projection operator onto V^K and P_n be the projection operator onto V^{K_n}. Recall that P_n converges to P strongly. For all $x, y \in G$

$$\begin{aligned}\lim_{n\to\infty} \int_{K_n} \varphi(xky)\ d\mu_n(k) &= \lim_{n\to\infty} \int_{K_n} \langle \pi(x^{-1})v, \pi(k)\pi(y)v\rangle\ d\mu_n(k) \\ &= \lim_{n\to\infty} \langle \pi(x^{-1})v, P_n(\pi(y)v)\rangle \\ &= \langle \pi(x^{-1})v, P\pi(y)v\rangle\end{aligned}$$

and

$$\varphi(x)\varphi(y) = \langle v, \pi(x)v\rangle\overline{\langle \pi(y)v, v\rangle} = \langle v, \pi(x)\langle \pi(y)v, v\rangle v\rangle = \langle \pi(x^{-1})v, \langle \pi(y)v, v\rangle v\rangle.$$

Then since v is cyclic, φ is spherical if and only if $P\pi(y)v = \langle \pi(y)v, v\rangle v$ for all $y \in G$.

Assume first that π is spherical, hence irreducible.

By Proposition 2.1.2, $\dim(V^K) = 1$ so that $V^K = \mathbb{C}v$ and the projection operator P onto V^K is given by $P(w) = \langle w, v\rangle v$ for all $w \in V$. Hence $P\pi(y)v = \langle \pi(y)v, v\rangle v$ for all $y \in G$ and φ is spherical.

Conversely assume that φ is spherical, hence $P\pi(y)v = \langle \pi(y)v, v\rangle v$ for all $y \in G$. Then since v is cyclic, the image V^K of the operator P is just $\mathbb{C}v$ so that $\dim(V^K) = 1$. Then by Proposition 2.1.1 the representation π is irreducible, hence spherical. $\square$

Note that two unitary representations (π, V) and (π', V') of a topological group G are said to be equivalent if there is a continuous vector space isomorphism $T : V \to V'$ such that T is also an intertwinning operator between (π, V) and (π', V'). Moreover, if T is a unitary operator, then (π, V) and (π', V') are called unitarily equivalent. Given an Olshanski spherical pair (G, K), by the GNS-construction we obtain a one-to-one correspondence between the positive definite spherical functions for (G, K) and the unitary equivalence classes of the spherical representations of (G, K). Indeed, a positive definite spherical function φ for (G, K) is matched with the unitary equivalence class of the unitary representation (π, V) associated to φ by the GNS-construction. By Proposition 2.3.2, this unitary representation (π, V) associated to φ by the GNS-construction is spherical. Conversely, the unitary equivalence class $[(\pi, V)]$ of a spherical representation (π, V) of (G, K) is matched with the positive definite function $\varphi : G \to \mathbb{C}$ defined by $\varphi(x) = \langle v, \pi(x)v\rangle$ for all $x \in G$, where v is an arbitrary unit, K-invariant vector in V. By Proposition 2.1.2, $\dim(V^K) = 1$ so that the function φ is independent both from the choice of a unit, K-invariant vector in V and the choice of a spherical representation from the unitary equivalence class $[(\pi, V)]$. Also by Proposition 2.3.2, this positive definite function φ is spherical. Hence both matches are well-defined.

By harmonic analysis for an Olshanski spherical pair (G, K), we mean determining its spherical dual $\Omega(G, K)$ consisting of all positive definite spherical functions for (G, K) and giving realizations of the spherical representations corresponding to the positive definite spherical functions by the GNS-construction. In the following chapters, we study harmonic analysis for two different Olshanski spherical pairs.

3. SPHERICAL DUAL OF $(U(\infty) \ltimes H(\infty), U(\infty))$

Let V_n be a finite dimensional complex Euclidean vector space and $H_n = V_n \times \mathbb{R}$ be the corresponding Heisenberg group where the group operation on H_n is given by

$$(z,t)(z^{'},t^{'}) = (z+z^{'}, t+t^{'} + \operatorname{Im}\langle z, z^{'}\rangle).$$

Let K_n be a closed subgroup of the group $U(V_n)$ of all unitary operators on V_n. Then K_n acts on H_n by automorphisms by $k(z,t) = (kz,t)$.
Hence we have the semidirect product $G_n = K_n \ltimes H_n$ where the multiplication is given by

$$(k,z,t)(k^{'},z^{'},t^{'}) = (kk^{'}, z+kz^{'}, t+t^{'} + \operatorname{Im}\langle z, kz^{'}\rangle).$$

Theorem 3.0.1. *[5] The pair (G_n, K_n) defined as above is a Gelfand pair if and only if K_n acts on the polynomial ring $P(V_n)$ multiplicity free.*

Now we assume that $(V_n)_n$ is an increasing sequence of finite dimensional complex vector spaces, each K_n acts multiplicity free on $P(V_n)$ and that $K_n = \{k \in K_{n+1} : k(V_n) = V_n\}$. We define

$$V = \cup_{n=1}^{\infty} V_n, \qquad H = \cup_{n=1}^{\infty} H_n, \qquad K = \cup_{n=1}^{\infty} K_n, \qquad G = \cup_{n=1}^{\infty} G_n.$$

Then K acts on H by automorphisms by $k(z,t) = (kz,t)$. Furthermore, $G = K \ltimes H$ and the pair (G,K) is an Olshanski spherical pair.

Remark 3.0.2. Let μ_n be the normalized Haar measure on K_n. For a continuous, K_n-bi-invariant function $\varphi : G_n \to \mathbb{C}$, define $\widetilde{\varphi} : H_n \to \mathbb{C}$ by $\widetilde{\varphi}(z,t) = \varphi(1,z,t)$. Then $\widetilde{\varphi}$ is a continuous, K_n-invariant function on H_n, i.e. $\widetilde{\varphi}(kz,t) = \widetilde{\varphi}(z,t)$ for all $k \in K_n$ and $(z,t) \in H_n$. The map $\varphi \mapsto \widetilde{\varphi}$ gives a one-to-one correspondence between the continuous, K_n-bi-invariant functions on G_n and the continuous, K_n-invariant functions on H_n. Moreover, the map $\varphi \mapsto \widetilde{\varphi}$ gives an isomorphism between $L^1(K_n \backslash G_n / K_n)$ and $L^1(H_n)^{K_n}$ as convolution algebras where $L^1(H_n)^{K_n}$ denotes the convolution algebra consisting of all Lebesgue-integrable, K_n-invariant functions on H_n.

We call a non-zero, continuous, K_n-invariant, complex-valued function φ on H_n to be spherical if

$$\int_{K_n} \varphi((z,t)k(z^{'},t^{'}))\ d\mu_n(k) = \varphi(z,t)\varphi(z^{'},t^{'}) \tag{3.1}$$

for all $(z,t), (z^{'},t^{'}) \in H_n$. Then, the map $\varphi \mapsto \widetilde{\varphi}$ gives a one-to-one correspondence between spherical functions for the Gelfand pair (G_n, K_n) and the spherical functions on H_n. Throughout the text, we will identify φ with $\widetilde{\varphi}$.

Similarly if $\varphi : G \to \mathbb{C}$ is a continuous, K-bi-invariant function, then φ can be considered as a continuous, K-invariant function on H where H has the inductive limit topology. A spherical function φ for (G, K) can be seen as a spherical function on H which is defined to be a non-zero, continuous, K-invariant, complex-valued function on H satisfying

$$\lim_{n\to\infty} \int_{K_n} \varphi((z,t)k(z^{'},t^{'}))\ d\mu_n(k) = \varphi(z,t)\varphi(z^{'},t^{'}) \tag{3.2}$$

for all $(z,t), (z^{'},t^{'}) \in H$. Moreover, a positive definite function on G corresponds to a positive definite function on H.

In this chapter, we will take $V_n = \mathbb{C}^n$ and $K_n = U(n)$. Then $H_n = \mathbb{C}^n \times \mathbb{R}$ and $G_n = U(n) \ltimes H_n$. Once we prove the following proposition, we will have (G_n, K_n) is a Gelfand pair by Theorem 3.0.1.

Proposition 3.0.3. *The unitary group $U(n)$ acts multiplicity free on the polynomial ring $P(\mathbb{C}^n)$.*

Proof. Denote by π the representation of $U(n)$ on $P(\mathbb{C}^n)$. It is enough to prove that the commutant $\mathcal{A} := \mathrm{Hom}_{U(n)}(\pi, \pi)$ of the representation π is commutative.

Let $A \in \mathcal{A}$. Given $a = (a_1, a_2, ..., a_n) \in \mathbb{N}^n$, let $p_a(z)$ be the monomial defined by $p_a(z) = z_1^{a_1}.z_2^{a_2}...z_n^{a_n}$ where $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$. Now we fix $a \in \mathbb{N}^n$. For an arbitrary $b = (b_1, b_2, ..., b_n) \in \mathbb{N}^n$, we write $A(p_a(z)) = A_b p_b(z) + q_{\widehat{b}}(z)$ for some $A_b \in \mathbb{C}$ and $q_{\widehat{b}}(z) \in P(\mathbb{C}^n)$ not containing the monomial $p_b(z)$ as a summand. For $\theta \in [0, 2\pi]$ and $l = 1, 2, ..., n$, by $k_{l,\theta} \in U(n)$, let us denote the diagonal matrix, with the lth diagonal entry is $e^{-i\theta}$ and all other diagonal entries are 1. Then,

$$\begin{aligned} e^{ia_l\theta} A_b p_b(z) + e^{ia_l\theta} q_{\widehat{b}}(z) &= A\pi(k_{l,\theta})(p_a(z)) \\ &= \pi(k_{l,\theta})A(p_a(z)) = e^{ib_l\theta} A_b p_b(z) + \pi(k_{l,\theta}) q_{\widehat{b}}(z). \end{aligned}$$

Since $\pi(k_{l,\theta})$ multiples each monomial with a constant, the polynomial $\pi(k_{l,\theta})q_{\widehat{b}}(z)$ does not contain the monomial $p_b(z)$. So, $e^{ia_l\theta} A_b p_b(z) = e^{ib_l\theta} A_b p_b(z)$ for all l and θ and since $a_l, b_l \in \mathbb{N}$, this implies either $A_b = 0$ or $a_l = b_l$ for all l. So, for each monomial $p_a(z) \in P(\mathbb{C}^n)$, there exists a constant $A_a \in \mathbb{C}$ such that $A(p_a(z)) = A_a p_a(z)$. Hence, for any $B \in \mathcal{A}$ and for all monomials $p_a(z) \in P(\mathbb{C}^n)$, we have $AB(p_a(z)) = A_a B_a p_a(z) = B_a A_a p_a(z) = BA(p_a(z))$ which shows by linearity of A and B that A commutes with B. So, we are done. $\square$

Therefore the choices $V_n = \mathbb{C}^n$ and $K_n = U(n)$ give rise to an Olshanski pair $(K \ltimes H, K)$ constructed as previously described when we embed $V_n = \mathbb{C}^n$ into $V_{n+1} = \mathbb{C}^{n+1}$ by $z \mapsto (z, 0)$ and also $K_n = U(n)$ into $K_{n+1} = U(n+1)$ by

$$\begin{pmatrix} k \end{pmatrix} \mapsto \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case, $K = U(\infty)$ where $U(\infty)$ is the infinite dimensional unitary group. Note that an infinite matrix $k = (k_{ij})_{i,j\geq 1}$ is an element of $U(\infty)$ if and only if there is an $N \in \mathbb{N}$ such that $(k_{ij})_{i,j=1}^{N} \in U(N)$ and $k_{ij} = \delta_{ij}$ for $i > N$ or $j > N$. When it is considered as a subgroup of $U(\infty)$, the group $U(N)$ consists of all infinite matrices $k = (k_{ij})_{i,j\geq 1}$ such that $(k_{ij})_{i,j=1}^{N} \in U(N)$ and $k_{ij} = \delta_{ij}$ for $i > N$ or $j > N$. Also, $V = \mathbb{C}^{(\infty)}$ where $\mathbb{C}^{(\infty)}$ consists of all infinite sequences of complex numbers with all but finitely many terms are zero. Given elements $z = (z_i)_{i\in\mathbb{N}} = (z_1, z_2, ..., 0, 0, ...) \in V$ and $w = (w_i)_{i\in\mathbb{N}} = (w_1, w_2, ..., 0, 0, ...) \in V$ we can define the norm of z by $\|z\| = \sqrt{z_1^2 + z_2^2 + \cdots}$ and the inner product of z with w by $\langle z, w\rangle = \sum_{i\in\mathbb{N}} z_i w_i$. The group $K = U(\infty)$ acts on $V = \mathbb{C}^{(\infty)}$ by natural automorphisms and this action preserves the norm. Let $H(\infty) = \mathbb{C}^{(\infty)} \times \mathbb{R}$ be the infinite dimensional Heisenberg group with multiplication defined in just the same way as in the finite dimensional case. The group $K = U(\infty)$ also acts on $H(\infty)$ by $k(z,t) = (kz,t)$ for all $k \in K$, $z \in \mathbb{C}^{(\infty)}$ and $t \in \mathbb{R}$. Then, $H = H(\infty)$ and $G = U(\infty) \ltimes H(\infty)$.

Our main goal in this chapter is to determine the positive definite spherical functions of the Olshanski spherical pair $(U(\infty) \ltimes H(\infty), U(\infty))$.

3.1. Spherical Representations of the Gelfand pair $(U(n) \ltimes H_n, U(n))$

In this section, we will determine the unitary dual of the Heisenberg group H_n and then the unitary dual of $U(n) \ltimes H_n$. For both, we will use the Mackey Little Group Theorem which provides a machinery to construct the irreducible unitary representations of a certain class of locally compact groups by inducing representations of their certain subgroups. Finally, we will find the $(U(n) \ltimes H_n, U(n))$-spherical ones among all irreducible unitary representations of $U(n) \ltimes H_n$.

3.1.1. Fock representations of H_n

Let $\mathcal{O}(\mathbb{C}^n)$ be the space of complex-valued holomorphic functions on $\mathbb{C}^n$. The space $\mathcal{O}(\mathbb{C}^n)$ is a Fréchet space when equipped with the topology of uniform convergence on compact subsets. For $\lambda \in \mathbb{R}^*$, define the Fock space $\mathcal{F}^n_\lambda$ by

$$\mathcal{F}^n_\lambda = \{f \in \mathcal{O}(\mathbb{C}^n) : \|f\|^2_\lambda := (\frac{|\lambda|}{\pi})^n \int_{\mathbb{C}^n} e^{-|\lambda|\|z\|^2}|f(z)|^2 \ \ dz < \infty\}$$

where dz is the Lebesgue measure on $\mathbb{C}^n$.

The norm $\|\cdot\|_\lambda$ on $\mathcal{F}^n_\lambda$ is induced by the inner product defined by

$$\langle f, g\rangle_\lambda = (\frac{|\lambda|}{\pi})^n \int_{\mathbb{C}^n} e^{-|\lambda|\|z\|^2} f(z)\overline{g(z)} \ \ dz.$$

Let γ_n be the probability measure on $\mathbb{C}^n$ given by $d\gamma_n(z) = (\frac{|\lambda|}{\pi})^n e^{-|\lambda|\|z\|^2} dz$. The Fock space $\mathcal{F}^n_\lambda$ is a subspace of the Hilbert space $L^2(\mathbb{C}^n, \gamma_n)$ of all square integrable, complex-valued functions on $\mathbb{C}^n$ with respect to the measure γ_n.

Proposition 3.1.1. *Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{F}^n_\lambda$ and $f \in L^2(\mathbb{C}^n, \gamma_n)$. Assume $(f_m)_{m\in\mathbb{N}}$ converges to f in $L^2(\mathbb{C}^n, \gamma_n)$. Then $(f_m)_{m\in\mathbb{N}}$ converges to f uniformly on compact subsets.*

Proof. Take a bounded domain U in $\mathbb{C}^n$. Then there exists $A > 0$ such that $\|z\|^2 < A$ for all $z \in U$ so that $e^{-|\lambda|\|z\|^2} > e^{-|\lambda|A}$ for all $z \in U$. Hence,

$$(\frac{|\lambda|}{\pi})^n \int_U e^{-|\lambda|A}|f_m(z) - f(z)|^2 \ \ dz < \|f_m - f\|^2_\lambda$$

so that

$$\int_U |f_m(z) - f(z)|^2 \ dz < (\frac{|\lambda|}{\pi})^{-n} e^{|\lambda|A}\|f_m - f\|^2_\lambda. \tag{3.3}$$

Since U is bounded, the restrictions $(f_m)_{\restriction U}$ and also $f_{\restriction U}$ are contained in the space $L^2(U)$ of all square integrable, complex valued functions on U with respect to the Lebesgue measure. Then the equation (3.3) together with the convergence of $(f_m)_{m\in\mathbb{N}}$ to f in $L^2(\mathbb{C}^n, \gamma_n)$ indicates the convergence of $((f_m)_{\restriction U})_{m\in\mathbb{N}}$ to $f_{\restriction U}$ in $L^2(U)$. But each $(f_m)_{\restriction U}$ is holomorphic on U and the convergence in $L^2(U)$ of holomorphic functions implies uniform convergence on compact subsets contained in U. Since every compact set is contained in a bounded domain in $\mathbb{C}^n$, the result follows. $\square$

Corollary 3.1.2. *The Fock space $(\mathcal{F}_\lambda^n, \|\cdot\|_\lambda)$ is a Hilbert space.*

Proof. It is enough to show that $\mathcal{F}_\lambda^n$ is a closed subspace of $L^2(\mathbb{C}^n, \gamma_n)$. Let $f_m \to f$ in $L^2(\mathbb{C}^n, \gamma_n)$ where each $f_m \in \mathcal{F}_\lambda^n$. By Proposition 3.1.1, $f_m \to f$ uniformly on compact subsets as well. Then since each f_m holomorphic, so is f. Hence $f \in \mathcal{F}_\lambda^n$ and we are done. $\square$

For $\nu = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{N}^n$, let $|\nu| := \nu_1 + \nu_2 + ... + \nu_n$ and $\nu! := \nu_1!\nu_2!...\nu_n!$. Also given $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$, let $z^\nu := z_1^{\nu_1} z_2^{\nu_2} ... z_n^{\nu_n}$. Then, $\|z^\nu\|_\lambda^2 = \nu!|\lambda|^{-|\nu|}$ so that $z^\nu \in \mathcal{F}_\lambda^n$ for all $\nu \in \mathbb{N}^n$. Let F_m denote the space of all homogeneous polynomials of degree m in $P(\mathbb{C}^n)$ for $m \in \mathbb{N}$. Then F_m is contained in $\mathcal{F}_\lambda^n$ and since F_m is finite-dimensional, F_m is a Hilbert subspace of $\mathcal{F}_\lambda^n$ for all $\lambda \in \mathbb{R}^*$ and $m \in \mathbb{N}$. Moreover, for $e_\nu = \sqrt{\frac{|\lambda|^{|\nu|}}{\nu!}} z^\nu$, the sequence $\{e_\nu\}_{\nu \in \mathbb{N}^n}$ forms an orthonormal basis for $\mathcal{F}_\lambda^n$. Hence $\mathcal{F}_\lambda^n$ is a seperable Hilbert space and

$$\mathcal{F}_\lambda^n = \widehat{\bigoplus_{m \in \mathbb{N}}} F_m.$$

Proposition 3.1.3. *Let $f \in \mathcal{O}(\mathbb{C}^n)$ and $f(z) = \sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ where the series converges to f in $\mathcal{O}(\mathbb{C}^n)$. Then,*

$$\|f\|_\lambda^2 = \sum_{\nu \in \mathbb{N}^n} \nu!|\lambda|^{-|\nu|}|a_\nu|^2. \tag{3.4}$$

Hence $f \in \mathcal{F}_\lambda^n$ if and only if the series $\sum_{\nu \in \mathbb{N}^n} \nu!|\lambda|^{-|\nu|}|a_\nu|^2$ converges.

Proof. Assume $\sum_{\nu \in \mathbb{N}^n} \nu!|\lambda|^{-|\nu|}|a_\nu|^2 < \infty$. Let

$$f_k(z) = \sum_{m=0}^{k} \sum_{|\nu|=m} a_\nu z^\nu \in \mathcal{F}_\lambda^n.$$

Then, the sequence $(f_k)_k$ forms a Cauchy sequence in the Fock space $\mathcal{F}_\lambda^n$, because given $N > L$

$$\begin{aligned}\|\sum_{m=0}^{N}\sum_{|\nu|=m} a_\nu z^\nu - \sum_{m=0}^{L}\sum_{|\nu|=m} a_\nu z^\nu\|_\lambda^2 = \|\sum_{m=L+1}^{N}\sum_{|\nu|=m} a_\nu z^\nu\|_\lambda^2 &= \sum_{m=L+1}^{N}\sum_{|\nu|=m}\|a_\nu z^\nu\|_\lambda^2 \\ &= \sum_{m=L+1}^{N}\sum_{|\nu|=m}\nu!|\lambda|^{-|\nu|}|a_\nu|^2\end{aligned}$$

can be made arbitrarily small for L large enough by our assumption. Hence, the sequence $(f_k)_k$ has a limit in the Fock space $\mathcal{F}_\lambda^n$ and by Proposition 3.1.1, this limit is identical with its limit in $\mathcal{O}(\mathbb{C}^n)$ which is f. Hence, $f \in \mathcal{F}_\lambda^n$.

Now, if $f \notin \mathcal{F}_\lambda^n$, by the argument above, both $\|f\|_\lambda^2$ and $\sum_{\nu\in\mathbb{N}^n}\nu!|\lambda|^{-|\nu|}|a_\nu|^2$ diverge to infinity so that the equation (3.4) is satisfied. If $f \in \mathcal{F}_\lambda^n$, then since $\{e_\nu\}_{\nu\in\mathbb{N}^n}$ forms an orthonormal basis for $\mathcal{F}_\lambda^n$, we get

$$\begin{aligned}\|f\|_\lambda^2 = \sum_{\nu\in\mathbb{N}^n}|\langle f, \sqrt{\frac{|\lambda|^{|\nu|}}{\nu!}}z^\nu\rangle_\lambda|^2 = \sum_{\nu\in\mathbb{N}^n}\frac{|\lambda|^{|\nu|}}{\nu!}|\langle f, z^\nu\rangle_\lambda|^2 &= \sum_{\nu\in\mathbb{N}^n}\frac{|\lambda|^{|\nu|}}{\nu!}|a_\nu|^2(\nu!|\lambda|^{-|\nu|})^2 \\ &= \sum_{\nu\in\mathbb{N}^n}\nu!|\lambda|^{-|\nu|}|a_\nu|^2\end{aligned}$$

as desired. $\square$

By Proposition 3.1.1, the inclusions from $\mathcal{F}_\lambda^n$ into $\mathcal{O}(\mathbb{C}^n)$ are continuous. Hence, for any $z \in \mathbb{C}^n$ the evaluation map $ev_z : \mathcal{F}_\lambda^n \to \mathbb{C}$ given by $ev_z(f) = f(z)$ is a bounded linear functional. Then, by the Riesz Representation Theorem, for any $z \in \mathbb{C}^n$ we have a unique function $K_z \in \mathcal{F}_\lambda^n$ such that $f(z) = ev_z(f) = \langle f, K_z\rangle_\lambda$ for all $f \in \mathcal{F}_\lambda^n$. That is to say, $\mathcal{F}_\lambda^n$ is a reproducing kernel Hilbert space and the map $\mathcal{K}_\lambda : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by $\mathcal{K}_\lambda(z, w) = K_w(z)$ is the reproducing kernel function of $\mathcal{F}_\lambda^n$. Since $\mathcal{F}_\lambda^n$ is a seperable functional Hilbert space, we can find the kernel function $\mathcal{K}_\lambda$ of $\mathcal{F}_\lambda^n$ explicitly in terms of the orthonormal basis $\{e_\nu\}_{\nu\in\mathbb{N}^n}$ by

$$\mathcal{K}_\lambda(z,w) = \sum_{\nu\in\mathbb{N}^n} e_\nu(z)\overline{e_\nu(w)} = \sum_{\nu\in\mathbb{N}^n} |\lambda|^{|\nu|}\frac{1}{\nu!}z^\nu\overline{w}^\nu = e^{|\lambda|\langle z,w\rangle}.$$

Since F_m is a Hilbert subspace of $\mathcal{F}_\lambda^n$ for all $\lambda \in \mathbb{R}^*$ and $m \in \mathbb{N}$, from Proposition 3.1.1 it follows that for all $z \in \mathbb{C}^n$, there exists a unique function $(K_{\lambda,m})_z \in F_m$ such that $f(z) = \langle f, (K_{\lambda,m})_z\rangle_\lambda$ for all $f \in F_m$. Hence, $(F_m, \langle\cdot,\cdot\rangle_\lambda)$ has a reproducing kernel $K_{\lambda,m}$ defined by $K_{\lambda,m}(z,w) = (K_{\lambda,m})_w(z)$, which can be computed via the orthonormal basis $\{e_\nu\}_{|\nu|=m}$ of $(F_m, \langle\cdot,\cdot\rangle_\lambda)$ as

$$K_{\lambda,m}(z,w) = \sum_{|\nu|=m} e_\nu(z)\overline{e_\nu(w)} = \sum_{|\nu|=m} \frac{|\lambda|^m}{\nu!}z^\nu\overline{w}^\nu = \frac{|\lambda|^m}{m!}\sum_{|\nu|=m}\frac{m!}{\nu!}z^\nu\overline{w}^\nu = \frac{|\lambda|^m}{m!}\langle z,w\rangle^m. \tag{3.5}$$

Note that for all $z \in \mathbb{C}^n$,

$$K_z = \sum_m (K_{\lambda,m})_z$$

where the convergence is in $\mathcal{F}_\lambda^n$, hence uniform on compact subsets.

For $\lambda > 0$, the Fock representation T_λ of H_n on $\mathcal{F}_\lambda^n$ is given by

$$[T_\lambda(z,t)f](w) = e^{\lambda(it-\frac{1}{2}\|z\|^2-\langle w,z\rangle)}f(w+z)$$

for any $(z,t) \in H_n$ and $f \in \mathcal{F}_\lambda^n$. For $\lambda < 0$, we define $T_\lambda(z,t) = T_{-\lambda}(\overline{z},-t)$. For each $\lambda \in \mathbb{R}^*$, the Fock representation $(T_\lambda, \mathcal{F}_\lambda^n)$ defines a unitary group representation of the Heisenberg group H_n. Moreover, it is irreducible which we shall now show.

Proposition 3.1.4. *The Fock representation $(T_\lambda, \mathcal{F}_\lambda^n)$ of the Heisenberg group H_n is irreducible for all $\lambda \in \mathbb{R}^*$.*

Proof. It is enough to prove the assertion for $\lambda > 0$. So let $\lambda > 0$ and W be a non-zero, closed subspace of $\mathcal{F}_\lambda^n$ that is invariant under $T_\lambda(z,t)$ for all $(z,t) \in H_n$. Let P be the orthogonal projection on W. As an orthogonal projection to a subrepresentation of a unitary representation, P commutes with the action of H_n. For the constant 1 function in $\mathcal{F}_\lambda^n$, we have $T_\lambda(-z,t)1 = C_{z,t}K_z$ for all $(z,t) \in H_n$ where $C_{z,t} = e^{\lambda(it-\frac{1}{2}\|z\|^2)}$. It follows that for every $z \in \mathbb{C}^n$,

$$\begin{aligned} P(1)(z) = \langle P(1), K_z\rangle_\lambda &= \langle P(1), (C_{z,t})^{-1}T_\lambda(-z,t)1\rangle_\lambda \\ &= \overline{(C_{z,t})^{-1}}\langle T_\lambda(z,-t)P(1), 1\rangle_\lambda \\ &= \overline{(C_{z,t})^{-1}}\langle P(T_\lambda(z,-t)1), 1\rangle_\lambda \\ &= \overline{(C_{z,t})^{-1}}\langle T_\lambda(z,-t)1, P(1)\rangle_\lambda \\ &= \overline{(C_{z,t})^{-1}}C_{-z,-t}\overline{\langle P(1), K_{-z}\rangle_\lambda} = \overline{P(1)(-z)}. \end{aligned}$$

Hence, $\overline{P(1)}(z) = P(1)(-z)$ for all $z \in \mathbb{C}^n$ so that both $P(1)$ and $\overline{P(1)}$ are holomorphic. This implies that $P(1) \in W$ is a constant function, so $1 \in W$. Then $K_z = (C_{z,0})^{-1}T_\lambda(-z,0)1 \in W$ for all $z \in \mathbb{C}^n$.

Now let $f \in W^\perp$. Since $K_z \in W$ for all $z \in \mathbb{C}^n$, we have $f(z) = \langle f, K_z\rangle_\lambda = 0$ for all $z \in \mathbb{C}^n$. Hence $W = \mathcal{F}_\lambda^n$. □

3.1.2. Mackey Machine and Its Application to H_n

In order to find the unitary dual of the Heisenberg group H_n and the group $U(n) \ltimes H_n$, we need two technical results from Harmonic Analysis.

Let G be a locally compact group and N be a normal closed subgroup of G. Since $N \trianglelefteq G$, G acts on the unitary dual $\widehat{N}$ of N. Indeed, given $g \in G$ and (π, V) an irreducible unitary representation of N, we define $g.(\pi, V) = (\pi^g, V)$ where $\pi^g(n) := \pi(g^{-1}ng)$ for all $n \in N$. Then (π^g, V) is also an irreducible unitary representation of N. If π and ρ are two equivalent irreducible unitary representations of N, then π^g and ρ^g are also equivalent. So given $g \in G$ and $[\pi] \in \widehat{N}$, we define $g.[\pi] = [\pi^g]$ and get an action of G on $\widehat{N}$. Let $G_{[\pi]}$ be the stabilizer of $[\pi] \in \widehat{N}$ in G, i.e.

$$G_{[\pi]} = \{g \in G : [\pi^g] = [\pi]\}.$$

If $g \in N$, then $\pi(g) \in Hom_N(\pi^g, \pi)$ so that $[\pi^g] = [\pi]$. Hence $N \trianglelefteq G_{[\pi]}$. Let $E([\pi])$ be the set of extensions of $[\pi] \in \widehat{N}$ to $\widehat{G_{[\pi]}}$ defined by

$$E([\pi]) = \{[\gamma] \in \widehat{G_{[\pi]}} : \gamma_{\restriction N} \text{ is equivalent to a multiple of } \pi\}.$$

Before stating the next theorem, we need some more terminology. In [[16], Realization II], a useful formulation of producing unitary representations of a locally compact group G by inducing representations of a closed subgroup H is given. We summarize this realization in the following definition.

Definition 3.1.5. Let G be a locally compact group and H be a closed subgroup of G. Let B_G be the Borel σ-algebra of G. The Borel σ-algebra $B_{G/H}$ of the quotient space G/H is defined by $B_{G/H} := \{E \subseteq G/H : p^{-1}(E) \in B_G\}$ where $p : G \to G/H$ is the natural quotient map. Fix a quasi-invariant measure μ on $B_{G/H}$, i.e. the action of G on G/H preserves null sets.
Let Δ_H and Δ_G be the modular functions of H and G. A rho-function for the pair (G, H) is a continuous function $\rho : G \to (0, \infty)$ such that for all $x \in G$ and $h \in H$

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x).$$

Let ρ be a rho-function for (G, H) (the existence of a rho-function for (G, H) is given by [Proposition 2.54, [15]]. Note that if H is a closed normal subgroup, then G/H has the structure of a locally compact group. So μ can be chosen as a Haar measure on G/H and hence ρ can be chosen as the constant 1 function on G by [Theorem 1.5.2, [9]]
.
Let (π, V_π) be a unitary representation of H. Let $V_{\mathrm{Ind}_H^G(\pi)}$ be the Hilbert space consisting of all measurable functions $f : G \to V_\pi$ such that $f(xh) = \pi(h^{-1})f(x)$ for almost all $x \in G$ and for all $h \in H$, and satisfying $\int_{G/H} \|f(x)\|^2 \, d\mu(xH) < \infty$ where the inner

product of $f, g \in V_{\mathrm{Ind}_H^G(\pi)}$ is given by

$$\langle f, g\rangle = \int_{G/H} \langle f(x), g(x)\rangle \; d\mu(xH).$$

Then, the representation $(\mathrm{Ind}_H^G(\pi), V_{\mathrm{Ind}_H^G(\pi)})$ of G induced from the representation π of H, or briefly the induced representation, is given by

$$(\mathrm{Ind}_H^G(\pi)(x)f)(y) = \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} f(x^{-1}y)$$

for all $f \in V_{\mathrm{Ind}_H^G(\pi)}$, $x \in G$ and almost all $y \in G$. $(\mathrm{Ind}_H^G(\pi), V_{\mathrm{Ind}_H^G(\pi)})$ defines a unitary representation of G.

Theorem 3.1.6. *[28] (Mackey Little Group Theorem) Let G be a locally compact group of Type* I *and N be a closed normal subgroup of G which is also of Type* I. *Assume that $\widehat{N}$ has a Borel measurable section under the action of G. Then,*

$$\widehat{G} = \{[\mathrm{Ind}_{G_{[\pi]}}^G(\gamma)] : [\pi] \in \widehat{N} \text{ and } [\gamma] \in E([\pi])\}.$$

Moreover, for $[\pi], [\pi^{'}] \in \widehat{N}$, $[\gamma] \in E([\pi])$ and $[\gamma^{'}] \in E([\pi^{'}])$,

$$[\mathrm{Ind}_{G_{[\pi]}}^G(\gamma)] = [\mathrm{Ind}_{G_{[\pi^{'}]}}^G(\gamma^{'})] \text{ if and only if } [\pi^{'}] = [\pi^g] \text{ and } [\gamma^{'}] = [\gamma^g] \text{ for some } g \in G.$$

The pair (G, N) is said to have the extension property if for every irreducible unitary representation π of N, there exists an irreducible unitary representation $\tilde{\pi}$ of $\widehat{G_{[\pi]}}$ such that $\tilde{\pi}_{\restriction_N} = \pi$.

Corollary 3.1.7. *Let G be a locally compact group and N be a closed normal subgroup of G. Assume G and N are of Type* I *and that the pair (G, N) has the extension property. Given $[\pi] \in \widehat{N}$, let $[\tilde{\pi}] \in \widehat{G_{[\pi]}}$ be such that $\tilde{\pi}_{\restriction_N} = \pi$. Assume also that $\widehat{N}$ has a Borel measurable section under the action of G. Then,*

$$E([\pi]) = \{[\tilde{\pi} \otimes \widehat{\mu}] : [\mu] \in \widehat{G_{[\pi]}/N} \text{ is lifted to } [\widehat{\mu}] \in \widehat{G_{[\pi]}}\}$$

so that

$$\widehat{G} = \{[\mathrm{Ind}_{G_{[\pi]}}^{G}(\tilde{\pi} \otimes \widehat{\mu})] : [\pi] \in \widehat{N} \text{ and } [\mu] \in \widehat{G_{[\pi]}/N} \textit{ is lifted to } [\widehat{\mu}] \in \widehat{G_{[\pi]}}\}.$$

Type I condition is a technical one related to the operator algebraic aspects of the theory of group representations. We do not want to go into this. But we will see that the groups which we study satisfy this condition.

The commutator subgroup $[H_n, H_n]$ of the Heisenberg group equals to its center $Z(H_n) = \{0\} \times \mathbb{R}$. Hence H_n is a nilpotent group of nilpotency class 2. Let $N := \mathbb{R}^n \times \mathbb{R}$. To be more precise, N consists of those elements $((z_1, z_2, ..., z_n), t) \in H_n$ such that $\mathrm{Im}(z_i) = 0$ for all $i = 1, ..., n$. In [10], Dixmier showed that every connected nilpotent Lie group is of Type I. Both the Heisenberg group H_n and its abelian subgroup N are connected nilpotent Lie groups, hence both are of Type I. So we can apply Theorem 3.1.6 to $G = H_n$ and its closed normal subgroup $N := \mathbb{R}^n \times \mathbb{R}$. N is isomorphic to the abelian additive topological group $(\mathbb{R}^{n+1}, +)$. Hence $\widehat{N}$ consists of unitary characters $\Psi_{(r_1, r_2, ..., r_{n+1})} : N \to S^1$ where S^1 is the circle group, $(r_1, r_2, ..., r_{n+1})$ runs over $\mathbb{R}^{n+1}$ and

$$\Psi_{(r_1, r_2, ..., r_{n+1})}((z_1, z_2, ..., z_n), t) := e^{i(r_{n+1}t + \sum_{i=1}^{n} r_i z_i)}$$

for all $((z_1, z_2, ..., z_n), t) \in N$. Given $(z, t) = ((z_1, z_2, ..., z_n), t) \in H_n$, we have

$$(\Psi_{(r_1, r_2, ..., r_{n+1})})^{(z,t)} = \Psi_{(r_1 + 2r_{n+1}\mathrm{Im}(z_1), r_2 + 2r_{n+1}\mathrm{Im}(z_2), ..., r_n + 2r_{n+1}\mathrm{Im}(z_n), r_{n+1})}.$$

So,

$$(H_n)_{[\Psi_{(r_1, r_2, ..., r_{n+1})}]} = \begin{cases} H_n & r_{n+1} = 0 \\ N & r_{n+1} \neq 0. \end{cases}$$

In case $r_{n+1} = 0$, given an irreducible unitary representation $\Psi_{(r_1, r_2, \ldots, r_n, 0)}$ of N, we can extend it to an irreducible unitary representation $\widetilde{\Psi}_{(r_1, r_2, \ldots, r_n, 0)}$ of the stabilizer H_n of $[\Psi_{(r_1, r_2, \ldots, r_n, 0)}]$ by defining

$$\widetilde{\Psi}_{(r_1, r_2, \ldots, r_n, 0)}(z, t) := e^{i(\sum_{i=1}^{n} r_i \operatorname{Re}(z_i))}$$

for all $(z, t) = ((z_1, z_2, \ldots, z_n), t) \in H_n$ in the sense that $(\widetilde{\Psi}_{(r_1, r_2, \ldots, r_n, 0)})_{\restriction N} = \Psi_{(r_1, r_2, \ldots, r_n, 0)}$. Thus, the pair (H_n, N) has the extension property and by Corollary 3.1.7

$$E([\Psi_{(r_1, r_2, \ldots, r_n, 0)}]) = \{[\widetilde{\Psi}_{(r_1, r_2, \ldots, r_n, 0)} \otimes \widehat{\mu}] \in \widehat{H_n} : [\mu] \in \widehat{\mathbb{R}^n} \text{ is lifted to } [\widehat{\mu}] \in \widehat{H_n}\}$$

so that

$$\bigcup_{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n} E([\Psi_{(r_1, r_2, \ldots, r_n, 0)}]) = \{\Phi_w : H_n \to \mathbb{C} : \Phi_w(z, t) = e^{i \operatorname{Re}\langle z, w\rangle},\ w \in \mathbb{C}^n\}.$$

By the family $\{\Phi_w\}_{w \in \mathbb{C}^n}$, we obtain all unitary characters of H_n. By Theorem 3.1.6, we get one more type of representations of H_n that are obtained by inducing the characters $\Psi_{(r_1, r_2, \ldots, r_n, r_{n+1})}$ where $r_{n+1} \neq 0$ from N to H_n. Again by Theorem 3.1.6, it follows that two induced representations $\operatorname{Ind}_N^{H_n}(\Psi_{(r_1, r_2, \ldots, r_n, r_{n+1})})$ and $\operatorname{Ind}_N^{H_n}(\Psi_{(s_1, s_2, \ldots, s_n, s_{n+1})})$ with $r_{n+1}, s_{n+1} \in \mathbb{R}^*$ are equivalent if and only if there exists $(z, t) \in H_n$ such that

$$\begin{aligned} \Psi_{(s_1, s_2, \ldots, s_n, s_{n+1})} &= (\Psi_{(r_1, r_2, \ldots, r_n, r_{n+1})})^{(z,t)} \\ &= \Psi_{(r_1 + 2r_{n+1} \operatorname{Im}(z_1), r_2 + 2r_{n+1} \operatorname{Im}(z_2), \ldots, r_n + 2r_{n+1} \operatorname{Im}(z_n), r_{n+1})}. \end{aligned} \tag{3.6}$$

For non-zero r_{n+1} and s_{n+1}, such $(z, t) \in H_n$ satifying the equation (3.6) exists if and only if $r_{n+1} = s_{n+1}$. Hence

$$\widehat{H_n} = \{\Phi_w :\ w \in \mathbb{C}^n\} \cup \{[\operatorname{Ind}_N^{H_n}(\Psi_{(0,0,\ldots,0,\lambda)})] : \lambda \in \mathbb{R}^*\}.$$

Now, we compute the central characters of the induced representation $\operatorname{Ind}_N^{H_n}(e^{i\lambda t})$ and the Fock representation T_λ of the Heisenberg group H_n where $\lambda \neq 0$. The center

$Z(H_n) = \{0\} \times \mathbb{R}$ of the Heisenberg group H_n is completely contained in N. We denote by $V_{\mathrm{Ind}_N^{H_n}(e^{i\lambda t})}$ the representation space corresponding to the induced representation $\mathrm{Ind}_N^{H_n}(e^{i\lambda t})$ of H_n. Then given $f \in V_{\mathrm{Ind}_N^{H_n}(e^{i\lambda t})}$ and $(0, t_0) \in Z(H_n)$,

$$(\mathrm{Ind}_N^{H_n}(e^{i\lambda t})(0,t_0)f)(z,t) = f((0,-t_0)(z,t)) = f((z,t)(0,-t_0)) = e^{i\lambda t_0} f(z,t)$$

for all $(z,t) \in H_n$.

When we consider the Fock space representation $(T_\lambda, \mathcal{F}_\lambda^n)$ of H_n, given $f \in \mathcal{F}_\lambda^n$ and $(0, t_0) \in Z(H_n)$, for all $w \in \mathbb{C}^n$ we have $(T_\lambda(0,t_0)f)(w) = e^{i\lambda t_0} f(w)$. Hence for each non-zero real number λ, the homomorphism

$$\begin{aligned} Z(H_n) &\to \mathbb{C}^* \\ (0,t) &\mapsto e^{i\lambda t} \end{aligned}$$

is the central character of both $\mathrm{Ind}_N^{H_n}(e^{i\lambda t})$ and T_λ, which are irreducible unitary representations of H_n. Then, it follows that

$$[\mathrm{Ind}_N^{H_n}(e^{i\lambda t})] = [T_\lambda]$$

and we have the following lemma.

Lemma 3.1.8. *The unitary dual $\widehat{H_n}$ of the Heisenberg group H_n consists of two types of representations up to unitary equivalence. First, there are unitary characters Φ_w defined by $\Phi_w(z,t) = e^{i\,\mathrm{Re}\langle z,w\rangle}$ where $w \in \mathbb{C}^n$ and $(z,t) \in H_n$. Second, there are infinite dimensional Fock representations $(T_\lambda, \mathcal{F}_\lambda^n)$ for $\lambda \in \mathbb{R}^*$.*

3.1.3. Spherical Representations of $(U(n) \ltimes H_n, U(n))$

We will now classify the irreducible unitary representations of the locally compact group $G_n = K_n \ltimes H_n$ where K_n is a closed subgroup of the unitary group $U(n)$. For this goal, we will once more follow the steps of Mackey Little Group Theorem and then among all irreducible unitary representations we will uncover the ones with K_n-invariant non-zero vectors. Finally, for the case when $K_n = U(n)$, we will write explicitly the spherical representations of the Gelfand pair $(U(n) \ltimes H_n, U(n))$.

The Heisenberg group H_n can be regarded as a closed normal subgroup of G_n. Let $(k_0, z_0, t_0) \in G_n$. The action of (k_0, z_0, t_0) on a representation π of H_n is given by $\pi^{(k_0,z_0,t_0)}(z,t) = \pi((k_0)^{-1}z, t - 2\operatorname{Im}\langle z, z_0\rangle)$ for all $(z,t) \in H_n$. Hence, for the unitary character Φ_w of H_n where $w \in \mathbb{C}^n$, we have

$$(\Phi_w)^{(k_0,z_0,t_0)}(z,t) = e^{i\operatorname{Re}\langle (k_0)^{-1}z, w\rangle} = e^{i\operatorname{Re}\langle z, k_0 w\rangle} = \Phi_{k_0 w}(z,t)$$

for all $(z,t) \in H_n$ so that $(\Phi_w)^{(k_0,z_0,t_0)} = \Phi_{k_0 w}$. So, the stabilizer $(G_n)_{[\Phi_w]} = (K_n)_w \ltimes H_n$ where $(K_n)_w = \{k \in K_n : kw = w\}$. For the Fock representation T_λ of H_n,

$$((T_\lambda)^{(k_0,z_0,t_0)}(z,t)f)(w) = e^{\lambda(i(t-2\operatorname{Im}\langle z,z_0\rangle)-\frac{1}{2}\|z\|^2-\langle w,(k_0)^{-1}z\rangle)}f(w+(k_0)^{-1}z) \text{ if } \lambda > 0, \text{ and}$$

$$((T_\lambda)^{(k_0,z_0,t_0)}(z,t)f)(w) = e^{\lambda(i(t-2\operatorname{Im}\langle \bar{z},z_0\rangle)+\frac{1}{2}\|z\|^2+\langle w,(k_0)^{-1}\bar{z}\rangle)}f(w+(k_0)^{-1}\bar{z}) \text{ if } \lambda < 0$$

for all $(z,t) \in H_n$, $f \in \mathcal{F}_\lambda^n$ and $w \in \mathbb{C}^n$. Then for each $\lambda \in \mathbb{R}^*$, $(T_\lambda)^{(k_0,z_0,t_0)}(0,t)f = e^{i\lambda t}f$ for all $(0,t) \in H_n$ and $f \in \mathcal{F}_\lambda^n$ so that the central character of $(T_\lambda)^{(k_0,z_0,t_0)}$ is the map $(0,t) \mapsto e^{i\lambda t}$ from $Z(H_n)$ to $\mathbb{C}^*$ which is exactly the central character of T_λ. Hence, $(T_\lambda)^{(k_0,z_0,t_0)}$ is unitarily equivalent to T_λ and the stabilizer $(G_n)_{[T_\lambda]} = G_n$.

In the following two propositions we will determine the set of extensions of irreducible unitary representations of H_n to their stabilizers that we have just found.

Proposition 3.1.9. *[28] Given $w \in \mathbb{C}^n$, let Φ_w be the unitary character of H_n given by $\Phi_w(z,t) = e^{i\operatorname{Re}\langle z,w\rangle}$ for all $(z,t) \in H_n$. Then Φ_w can be extended to a unitary character $\widetilde{\Phi_w}$ of its stabilizer $(K_n)_w \ltimes H_n$ in G_n defined by $\widetilde{\Phi_w}(k,z,t) = \Phi_w(z,t)$. Hence $E([\Phi_w]) = \{[\widetilde{\Phi_w} \otimes \widehat{\mu}] : [\mu] \in \widehat{(K_n)_w}$ is lifted to $[\widehat{\mu}] \in (K_n)_w \widehat{\ltimes} H_n\}$.*

Proof. It is enough to show $\widetilde{\Phi_w} : (K_n)_w \ltimes H_n \to S^1$ is a group homomorphism. So we take two elements (k_0, z_0, t_0) and (k_1, z_1, t_1) from $(K_n)_w \ltimes H_n$. Then since

$$(k_0, z_0, t_0)(k_1, z_1, t_1) = (k_0k_1, z_0 + k_0z_1, t_0 + t_1 + \operatorname{Im}\langle z_0, k_0z_1\rangle)$$

we get

$$\begin{aligned}&\widetilde{\Phi_w}((k_0, z_0, t_0)(k_1, z_1, t_1)) = e^{i \operatorname{Re}\langle z_0+k_0z_1,w\rangle} = e^{i \operatorname{Re}\langle z_0,w\rangle} e^{i \operatorname{Re}\langle k_0z_1,w\rangle}\\ &= e^{i \operatorname{Re}\langle z_0,w\rangle} e^{i \operatorname{Re}\langle z_1,(k_0)^{-1}w\rangle} = e^{i \operatorname{Re}\langle z_0,w\rangle} e^{i \operatorname{Re}\langle z_1,w\rangle} = \widetilde{\Phi_w}(k_0, z_0, t_0)\widetilde{\Phi_w}(k_1, z_1, t_1)\end{aligned}$$

as desired. □

Given $\lambda \in \mathbb{R}^*$, the group K_n has a natural action on the Fock space $\mathcal{F}^n_\lambda$ via the homomorphism $\pi_\lambda : K_n \to U(\mathcal{F}^n_\lambda)$ given by

$$(\pi_\lambda(k)f)(z) = f(k^{-1}z). \tag{3.7}$$

For $\lambda > 0$,

$$\begin{aligned}(T_\lambda(z, t)\pi_\lambda(k^{-1})f)(w) &= e^{\lambda(it-\frac{1}{2}\|z\|^2-\langle w,z\rangle)}(\pi_\lambda(k^{-1})f)(w + z)\\ &= e^{\lambda(it-\frac{1}{2}\|z\|^2-\langle w,z\rangle)}f(kw + kz)\end{aligned}$$

so that

$$\begin{aligned}(\pi_\lambda(k)T_\lambda(z, t)\pi_\lambda(k^{-1})f)(w) &= e^{\lambda(it-\frac{1}{2}\|z\|^2-\langle k^{-1}w,z\rangle)}f(w + kz)\\ &= e^{\lambda(it-\frac{1}{2}\|kz\|^2-\langle w,kz\rangle)}f(w + kz) = (T_\lambda(kz, t)f)(w)\end{aligned} \tag{3.8}$$

for all $k \in K_n$, $(z, t) \in H_n$, $f \in \mathcal{F}^n_\lambda$ and $w \in \mathbb{C}^n$. The equation (3.8) also holds for all $\lambda < 0$. Hence for all $k \in K_n$ and $(z, t) \in H_n$,

$$\pi_\lambda(k)T_\lambda(z,t)\pi_\lambda(k^{-1}) = T_\lambda(kz,t). \tag{3.9}$$

Proposition 3.1.10. *[28] For $\lambda \in \mathbb{R}^*$, the Fock representation T_λ of H_n can be extended to an irreducible unitary representation $\widetilde{T_\lambda}$ of G_n on the Fock space $\mathcal{F}_\lambda^n$ defined by $\widetilde{T_\lambda}(k,z,t) = T_\lambda(z,t)\pi_\lambda(k)$. Hence $E([T_\lambda]) = \{[\widetilde{T_\lambda} \otimes \widehat{\gamma}] : [\gamma] \in \widehat{K_n}$ is lifted to $[\widehat{\gamma}] \in \widehat{G_n}\}$.*

Proof. The fact that $\widetilde{T_\lambda} : G_n \to U(\mathcal{F}_\lambda^n)$ is a group homomorphism follows from Equation 3.9 and irreducibility of $\widetilde{T_\lambda}$ follows from the irreducibility of T_λ which is already proved in Proposition 3.1.4. □

Now Proposition 3.1.9 and Proposition 3.1.10 together with Theorem 3.1.6 gives the unitary dual of the group $G_n = K_n \ltimes H_n$ as stated in the following theorem.

Theorem 3.1.11. *[28] Let K_n be a closed subgroup of the unitary group $U(n)$. Then the unitary dual $\widehat{K_n \ltimes H_n}$ consists of two types of representations:*

First, there are classes of representations of the form $[\mathrm{Ind}_{(K_n)_w \ltimes H_n}^{K_n \ltimes H_n}(\widetilde{\Phi_w} \otimes \widehat{\mu})]$ where $w \in \mathbb{C}^n$, $(K_n)_w = \{k \in K_n : kw = w\}$, $\widetilde{\Phi_w}$ is the character of $(K_n)_w \ltimes H_n$ defined by $\widetilde{\Phi_w}(k,z,t) = \Phi_w(z,t)$ and $[\mu] \in \widehat{(K_n)_w}$ is lifted to $[\widehat{\mu}] \in \widehat{(K_n)_w \ltimes H_n}$.

Second, there are classes of representations of the form $[\widetilde{T_\lambda} \otimes \widehat{\gamma}]$ where $(\widetilde{T_\lambda}, \mathcal{F}_\lambda^n)$ is the irreducible unitary representation of $K_n \ltimes H_n$ given by $\widetilde{T_\lambda}(k,z,t) = T_\lambda(z,t)\pi_\lambda(k)$ and $[\gamma] \in \widehat{K_n}$ is lifted to $[\widehat{\gamma}] \in \widehat{K_n \ltimes H_n}$.

Among all irreducible unitary representations of $K_n \ltimes H_n$ given by Theorem 3.1.11, we will determine the ones with non-zero K_n-invariant vectors. Hence if K_n is a closed subgroup of $U(n)$ acting multiplicity free on the polynomial ring $P(\mathbb{C}^n)$, we will have determined the spherical representations of the Gelfand pair $(K_n \ltimes H_n, K_n)$.

Remark 3.1.12. If G is a locally compact group with a compact subgroup K and (π, V) is a unitary representations of G, then the restriction $(\pi_{\restriction K}, V)$ is a unitary representation of the compact group K. The representation space V has a non-zero vector v such that $\pi(k)v = v$ for all $k \in K$ if and only if $(\pi_{\restriction_K}, \mathbb{C}v)$ is a subrepresentation of $(\pi_{\restriction_K}, V)$ where $\mathbb{C}v$ is the one-dimensional $\mathbb{C}$-linear subspace of V generated by v. But $(\pi_{\restriction_K}, \mathbb{C}v)$ is equivalent to the one-dimensional trivial representation $(\mathbb{1}_K, \mathbb{C})$ of K. Hence V has a non-zero vector v invariant under $\pi(k)$ for all $k \in K$ if and only if $\text{mult}(\mathbb{1}_K, \pi_{\restriction_K}) \geq 1$ where $\text{mult}(\mathbb{1}_K, \pi_{\restriction_K})$ denotes the multiplicity of the trivial representation $\mathbb{1}_K$ of K in $\pi_{\restriction_K}$.

In the light of the previous remark, for the determination of all irreducible unitary representations of the locally compact group $K_n \ltimes H_n$ with non-zero K_n-invariant vectors, we will compute the multiplicities $\text{mult}(\mathbb{1}_{K_n}, (\text{Ind}_{(K_n)_w \ltimes H_n}^{K_n \ltimes H_n}(\widetilde{\Phi_w} \otimes \widehat{\mu}))_{\restriction_{K_n}})$ and $\text{mult}(\mathbb{1}_{K_n}, (\widetilde{T_\lambda} \otimes \widehat{\gamma})_{\restriction_{K_n}})$ consecutively.

Proposition 3.1.13. *For all* $[\mu] \in \widehat{(K_n)_w}$,

$$[(\text{Ind}_{(K_n)_w \ltimes H_n}^{K_n \ltimes H_n}(\widetilde{\Phi_w} \otimes \widehat{\mu}))_{\restriction_{K_n}}] = [\text{Ind}_{(K_n)_w}^{K_n}(\mu)] \tag{3.10}$$

where $\widehat{\mu}$ *is the lifting of* μ *to* $(K_n)_w \ltimes H_n$.

Proof. Since $\widetilde{\Phi_w}$ is a one-dimensional representation of $(K_n)_w \ltimes H_n$, we have $V_{\widetilde{\Phi_w}} = \mathbb{C}$. Thus, $V_{\widetilde{\Phi_w}} \otimes V_{\widehat{\mu}} = V_{\widehat{\mu}} = V_\mu$ and $(\widetilde{\Phi_w} \otimes \widehat{\mu})(k, z, t) = \Phi_w(z, t)\mu(k) = e^{i\,\text{Re}\langle z, w\rangle}\mu(k)$ for all $(k, z, t) \in (K_n)_w \ltimes H_n$.
Now we define a bounded linear operator $A : V_{\text{Ind}_{(K_n)_w}^{K_n}(\mu)} \to V_{(\text{Ind}_{(K_n)_w \ltimes H_n}^{K_n \ltimes H_n}(\widetilde{\Phi_w} \otimes \widehat{\mu}))_{\restriction_{K_n}}}$ by $(Af)(k, z, t) = e^{-i\,\text{Re}\langle z, kw\rangle}f(k)$ for all $f \in V_{\text{Ind}_{(K_n)_w}^{K_n}(\mu)}$ and $(k, z, t) \in K_n \ltimes H_n$. We show A gives an equivalence between the representations of K_n given in the equation (3.10) as follows:

Given $g \in V_{(\text{Ind}_{(K_n)_w \ltimes H_n}^{K_n \ltimes H_n}(\widetilde{\Phi_w} \otimes \widehat{\mu}))_{\restriction_{K_n}}}$, for all $(k, z, t) \in K_n \ltimes H_n$ we have

$$g(k, z, t) = g((k, 0, 0)(1, k^{-1}z, t)) = \widetilde{\Phi_w} \otimes \widehat{\mu}(1, -k^{-1}z, -t)g(k, 0, 0) = e^{-i\,\text{Re}\langle z, kw\rangle}g(k, 0, 0).$$

Then, for $f : K_n \to V_\mu$ given by $f(k) = g(k,0,0)$, we have $Af = g$ and that $f \in V_{\mathrm{Ind}_{(K_n)_w}^{K_n}(\mu)}$. Thus A is onto. Moreover, $Af = 0$ if and only if $e^{-i\,\mathrm{Re}\langle z,kw\rangle} f(k) = 0$ for all $(k,z,t) \in K_n \ltimes H_n$. But this is only possible when $f = 0$. Hence A is also one-to-one.

Let ρ be a rho-function for the pair $(K_n, (K_n)_w)$. Then $\rho_0 : K_n \ltimes H_n \to (0,\infty)$ defined by $\rho_0(k,z,t) = \rho(k)$ is a rho-function for the pair $(K_n \ltimes H_n, (K_n)_w \ltimes H_n)$. Then for all $k_0 \in K_n$, $(k,z,t) \in K_n \ltimes H_n$ and $f \in V_{\mathrm{Ind}_{(K_n)_w}^{K_n}(\mu)}$,

$$\begin{aligned} A(\mathrm{Ind}_{(K_n)_w}^{K_n}(\mu)(k_0)f)(k,z,t) &= e^{-i\,\mathrm{Re}\langle z,kw\rangle}\,\mathrm{Ind}_{(K_n)_w}^{K_n}(\mu)(k_0)f(k) \\ &= e^{-i\,\mathrm{Re}\langle z,kw\rangle}\sqrt{\frac{\rho(k_0^{-1}k)}{\rho(k)}}\,f(k_0^{-1}k) \end{aligned}$$

where

$$\begin{aligned} (\mathrm{Ind}_{(K_n)_w\ltimes H_n}^{K_n\ltimes H_n}(\widetilde{\Phi_w}\otimes\widehat{\mu}))_{\restriction K_n}(k_0)(Af)(k,z,t) &= \sqrt{\frac{\rho_0((k_0^{-1},0,0)(k,z,t))}{\rho_0(k,z,t)}}\,Af(k_0^{-1}k, k_0^{-1}z, t) \\ &= \sqrt{\frac{\rho(k_0^{-1}k)}{\rho(k)}}\,e^{-i\,\mathrm{Re}\langle z,kw\rangle} f(k_0^{-1}k) \end{aligned}$$

as well. Thus, A is an intertwining operator. $\square$

Now Proposition 3.1.13 together with the Frobenious Reciprocity Theorem for compact groups gives that

$$\mathrm{mult}(\mathbb{1}_{K_n}, (\mathrm{Ind}_{(K_n)_w\ltimes H_n}^{K_n\ltimes H_n}(\widetilde{\Phi_w}\otimes\widehat{\mu}))_{\restriction K_n}) = \mathrm{mult}(\mathbb{1}_{K_n}, \mathrm{Ind}_{(K_n)_w}^{K_n}(\mu)) = \mathrm{mult}(\mu, \mathbb{1}_{(K_n)_w}).$$

But, $\mathrm{mult}(\mu, \mathbb{1}_{(K_n)_w}) = 1$ if $\mu = \mathbb{1}_{(K_n)_w}$ and $\mathrm{mult}(\mu, \mathbb{1}_{(K_n)_w}) = 0$ otherwise. Hence, in case $(K_n \ltimes H_n, K_n)$ is a Gelfand pair, a representation of $K_n \ltimes H_n$ of the form $\mathrm{Ind}_{(K_n)_w\ltimes H_n}^{K_n\ltimes H_n}(\widetilde{\Phi_w}\otimes\widehat{\mu})$ is spherical for $(K_n \ltimes H_n, K_n)$ if and only if $\mu = \mathbb{1}_{(K_n)_w}$.

Proposition 3.1.14. *[28] Let K_n be a closed subgroup of $U(n)$. Let $[\gamma] \in \widehat{K_n}$ and $[\widehat{\gamma}] \in \widehat{K_n \ltimes H_n}$ where $\widehat{\gamma}$ is the lifting of γ from K_n to $K_n \ltimes H_n$. Then for all $\lambda \in \mathbb{R}^*$, $\mathrm{mult}(\mathbb{1}_{K_n}, (\widetilde{T_\lambda}\otimes\widehat{\gamma})_{\restriction K_n}) = \mathrm{mult}(\gamma^*, \pi)$ where γ^* is the contragredient of γ and π is the* *natural representation of K_n, as a subgroup of $U(n)$, on the ring of polynomials $P(\mathbb{C}^n)$.*

Proof. Since $\widetilde{T_\lambda}_{\restriction_{K_n}} = \pi_\lambda$ and $\widehat{\gamma}_{\restriction_{K_n}} = \gamma$, we have

$$[(\widetilde{T_\lambda} \otimes \widehat{\gamma})_{\restriction_{K_n}}] = [\pi_\lambda \otimes \gamma].$$

The Fock representation π_λ of K_n is equivalent to the natural representation π of K_n on the ring of polynomials $P(\mathbb{C}^n)$. The representation π decomposes as $\pi = \sum_j \pi_j$ where π_j' s are irreducible representations of K_n. Hence, $\pi \otimes \gamma = \sum_j \pi_j \otimes \gamma$ and

$$\mathrm{mult}(\mathbb{1}_{K_n}, \pi \otimes \gamma) = \sum_j \mathrm{mult}(\mathbb{1}_{K_n}, \pi_j \otimes \gamma).$$

But, $\mathrm{mult}(\mathbb{1}_{K_n}, \pi_j \otimes \gamma) = 1$ if π_j is equivalent to γ^* and $\mathrm{mult}(\mathbb{1}_{K_n}, \pi_j \otimes \gamma) = 0$ otherwise. Hence the result follows. $\square$

We combine Theorem 3.1.11 with the above multiplicity calculations in the next theorem and right after we give the spherical representations of the Gelfand pair $(U(n) \ltimes H_n, U(n))$ as its corollary which is the main result of this section.

Theorem 3.1.15. *[28] Let K_n be a closed subgroup of the unitary group $U(n)$ acting multiplicity free on the polynomial ring $P(\mathbb{C}^n)$. Then there are two types of spherical representations of the Gelfand pair $(K_n \ltimes H_n, K_n)$ given as follows:*

First, there are classes of representations of the form $[\mathrm{Ind}_{(K_n)_w \ltimes H_n}^{K_n \ltimes H_n}(\widetilde{\Phi_w})]$ where $w \in \mathbb{C}^n$, $(K_n)_w = \{k \in K_n : kw = w\}$, $\widetilde{\Phi_w}$ is the character of $(K_n)_w \ltimes H_n$ defined by $\widetilde{\Phi_w}(k, z, t) = \Phi_w(z, t) = e^{i\,\mathrm{Re}\langle z, w\rangle}$.

Second, there are classes of representations of the form $[\widetilde{T_\lambda} \otimes \widehat{\gamma}]$ where $\widehat{\gamma}$ is the lifting of $[\gamma] \in \widehat{K_n}$ to $K_n \ltimes H_n$ such that $\mathrm{mult}(\gamma^*, \pi) = 1$ where γ^* is the contragredient of γ and π is the natural representation of K_n, as a subgroup of $U(n)$, on the ring of polynomials $P(\mathbb{C}^n)$.

Corollary 3.1.16. *There are two types of spherical representations of the Gelfand pair $(U(n) \ltimes H_n, U(n))$ given as follows:*

First, there are classes of representations of the form $[\mathrm{Ind}_{(U(n))_w \ltimes H_n}^{U(n)\ltimes H_n}(\widetilde{\Phi_w})]$ where $w \in \mathbb{C}^n$, $(U(n))_w = \{k \in U(n) : kw = w\}$, $\widetilde{\Phi_w}$ is the character of $(U(n))_w \ltimes H_n$ defined by $\widetilde{\Phi_w}(k, z, t) = \Phi_w(z, t) = e^{i\,\mathrm{Re}\langle z,w\rangle}$.

Second, there are classes of representations of the form $[\widetilde{T_\lambda} \otimes \widehat{\pi_m}]$ where $\widehat{\pi_m}$ is the lifting to $U(n) \ltimes H_n$ of the representation π_m of $U(n)$ on the polynomials of degree m on $\mathbb{C}^n$.

3.2. Spherical Functions of $(U(n) \ltimes H_n, U(n))$

Throughout this section, let $G_n = U(n) \ltimes H_n$ and $K_n = U(n)$. As we have seen in Remark 3.0.2, we may identify the spherical functions for the Gelfand pair (G_n, K_n) by the spherical functions on H_n. Given $h \in L^1(H_n)^{K_n}$ and $\lambda > 0$, consider the operator $T_\lambda(h) \in B(\mathcal{F}_\lambda^n)$. From the equation (3.9), it follows that $T_\lambda(h)$ commutes with the natural action of K_n on F_m. Hence by Schur' s Lemma the restriction of $T_\lambda(h)$ to F_m is a scalar multiple of identity, i.e. for all $f \in F_m$,

$$T_\lambda(h)f = \widehat{h}(\lambda, m)f \tag{3.11}$$

where $\widehat{h}(\lambda, m)$ is the spherical Fourier transform of h given by

$$\widehat{h}(\lambda, m) = \int_{H_n} h(z,t)\varphi_{\lambda,m}(z,t)\; dzdt. \tag{3.12}$$

We want to write the bounded spherical functions $\varphi_{\lambda,m}$ explicitly. For all $f \in F_m$, $h \in L^1(H_n)^{K_n}$ and $w \in \mathbb{C}^n$, we have

$$\widehat{h}(\lambda, m)f(w) = T_\lambda(h)f(w) = \langle T_\lambda(h)f, K_w\rangle_\lambda = \int_{H_n} h(z,t)\ \langle T_\lambda(z,t)f, K_w\rangle_\lambda\ dzdt$$
$$= \int_{H_n} h(z,t)\ (T_\lambda(z,t)f)(w)\ dzdt. \quad (3.13)$$

When we apply the equation (3.13) for a fixed $w \in \mathbb{C}^n$ and $f \in F_m$ such that $f(w) = 1$, then we get

$$\begin{aligned}
\widehat{h}(\lambda, m) &= \int_{H_n} h(z,t)\ (T_\lambda(z,t)f)(w)\ dzdt \\
&= \int_{H_n} h(z,t)\ e^{\lambda(it - \frac{1}{2}\|z\|^2 - \langle w,z\rangle)} f(w+z)\ dzdt \\
&= \int_{K_n}\int_{H_n} h(kz,t)\ e^{\lambda(it - \frac{1}{2}\|kz\|^2 - \langle w,kz\rangle)} f(w+kz)\ dzdtdk \\
&= \int_{K_n}\int_{H_n} h(z,t)\ e^{\lambda(it - \frac{1}{2}\|z\|^2 - \langle w,kz\rangle)} f(w+kz)\ dzdtdk \\
&= \int_{H_n}\int_{K_n} h(z,t)\ e^{\lambda(it - \frac{1}{2}\|z\|^2 - \langle w,kz\rangle)} f(w+kz)\ dkdzdt \\
&= \int_{H_n} h(z,t)\ e^{i\lambda t} e^{-\frac{1}{2}\lambda\|z\|^2} \int_{K_n} e^{-\lambda\langle w,kz\rangle} f(w+kz)\ dk\ dzdt.
\end{aligned}$$

So the function $\varphi(z,t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda\|z\|^2} \int_{K_n} e^{-\lambda\langle w,kz\rangle} f(w+kz)\ dk$ is a continuous K_n-invariant function such that the map $h \mapsto \int_{H_n} h(z,t)\varphi(z,t)\ dzdt$ is a character of the commutative algebra $L^1(H_n)^{K_n}$. Hence φ defines a spherical function for the Gelfand pair (G_n, K_n). Note that φ is bounded and the characters of the commutative algebra $L^1(H_n)^{K_n}$ are uniquely determined by the bounded spherical functions for the Gelfand pair (G_n, K_n) (see [Lemma 6.1.7, [26]]). Therefore, we get

$$\varphi_{\lambda,m}(z,t) = \varphi(z,t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda\|z\|^2} \int_{K_n} e^{-\lambda\langle w,kz\rangle} f(w+kz)\ dk$$

where $w \in \mathbb{C}^n$ is fixed and $f \in F_m$ is chosen such that $f(w) = 1$. To move one more step forward, let us take $w = (0,0,...,0,1) \in \mathbb{C}^n$ and $f(z) = z_n^m$ for $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$.

Then, since $f \in F_m$ and $f(w) = 1$,

$$\varphi_{\lambda,m}(z,t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda\|z\|^2} \int_{K_n} e^{-\lambda\overline{(kz)_n}} (1 + (kz)_n)^m \, dk. \tag{3.14}$$

The integral in the above equation is an integral of some special form which is given and computed in the next proposition.

Proposition 3.2.1. *[14] Let $\lambda \in \mathbb{R}^*$. For $f_1, f_2 \in \mathcal{F}_\lambda^n$, we write $f_1 = \sum_{m=0}^{\infty} f_{1,m}$ and $f_2 = \sum_{m=0}^{\infty} f_{2,m}$ uniquely where $f_{1,m}, f_{2,m} \in F_m$ for all m and both series converges in $\mathcal{F}_\lambda^n$. Then,*

$$\int_{K_n} f_1(kz)\overline{f_2(kz)} \, dk = \sum_{m=0}^{\infty} \frac{1}{d_m} K_{\lambda,m}(z,z) \langle f_{1,m}, f_{2,m} \rangle_\lambda$$

where $K_{\lambda,m}$ is the reproducing kernel of $(F_m, \langle \cdot, \cdot \rangle_\lambda)$ and $d_m = \dim(F_m) = \binom{n+m-1}{m}$.

Proof. Let π_λ be the natural action of K_n on the Fock space $\mathcal{F}_\lambda^n$ given as in the equation (3.7). Then F_m's are pairwise orthogonal, non-equivalent, irreducible unitary subrepresentations of the representation $(\pi_\lambda, \mathcal{F}_\lambda^n)$ of K_n. Let $k \in K_n$ and $z \in \mathbb{C}^n$. Note that $(K_{\lambda,m})_z \in F_m$ for each m. Then for $j = 1, 2$ we have

$$\begin{aligned} f_j(kz) = (\pi_\lambda(k^{-1}) f_j)(z) = \langle \pi_\lambda(k^{-1}) f_j, K_z \rangle_\lambda &= \sum_m \langle \pi_\lambda(k^{-1}) f_j, (K_{\lambda,m})_z \rangle_\lambda \\ &= \sum_m \langle f_j, \pi_\lambda(k)(K_{\lambda,m})_z \rangle_\lambda \\ &= \sum_m \langle f_{j,m}, \pi_\lambda(k)(K_{\lambda,m})_z \rangle_\lambda. \end{aligned}$$

Then by the Schur' s Orthogonality Relations for the matrix coefficients of the irreducible unitary representations (π_λ, F_m) of the compact group K_n, we get

$$\begin{aligned}\int_{K_n} f_1(kz)\overline{f_2(kz)}\,dk &= \sum_{m,m'}\int_{K_n}\langle f_{1,m},\pi_\lambda(k)(K_{\lambda,m})_z\rangle_\lambda\overline{\langle f_{2,m'},\pi_\lambda(k)(K_{\lambda,m'})_z\rangle_\lambda}\,dk\\ &= \sum_{m}\int_{K_n}\langle f_{1,m},\pi_\lambda(k)(K_{\lambda,m})_z\rangle_\lambda\overline{\langle f_{2,m},\pi_\lambda(k)(K_{\lambda,m})_z\rangle_\lambda}\,dk\\ &= \sum_{m=0}^{\infty}\frac{1}{d_m}\langle (K_{\lambda,m})_z,(K_{\lambda,m})_z\rangle_\lambda\langle f_{1,m},f_{2,m}\rangle_\lambda\\ &= \sum_{m=0}^{\infty}\frac{1}{d_m}K_{\lambda,m}(z,z)\langle f_{1,m},f_{2,m}\rangle_\lambda\end{aligned}$$

as desired. □

Corollary 3.2.2. *For $\lambda > 0$, the spherical function $\varphi_{\lambda,m}$ of the Gelfand pair (G_n, K_n) given in the equation (3.14) can be expressed explicitly as*

$$\varphi_{\lambda,m}(z,t) = e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2}L_m^{n-1}(\lambda\|z\|^2)$$

where $L_m^n(x)$ is the generalized Laguerre polynomial of order n and of degree m,

$$L_m^n(x) = n!\sum_{i=0}^{m}\binom{m}{i}\frac{(-x)^i}{(n+i)!}. \tag{3.15}$$

Proof. For $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$, let $f_1(z) = (1+z_n)^m$ and $f_2(z) = e^{-\lambda z_n}$. Since $f_1(z) = \sum_{i=0}^{m}\binom{m}{i}z_n^i$ is a finite sum of monomials in $\mathcal{F}_\lambda^n$, we have $f_1 \in \mathcal{F}_\lambda^n$. Also, $f_2(z) = \sum_{i=0}^{\infty}\frac{(-\lambda)^i}{i!}z_n^i$ and since $\sum_{i=0}^{\infty} i!\lambda^{-i}|\frac{(-\lambda)^i}{i!}|^2 = e^\lambda$, by Proposition 3.1.3 we have $f_2 \in \mathcal{F}_\lambda^n$ as well. Let $f_{1,i}(z) = \binom{m}{i}z_n^i \in F_i$ for $0 \le i \le m$ and $f_{1,i}(z) = 0$ for $i > m$. Let $f_{2,i}(z) = \frac{(-\lambda)^i}{i!}z_n^i \in F_i$ for all i. If we denote the product $n(n+1)...(n+i-1)$ by the Pochhammer symbol $(n)_i$, then $d_i = \dim(F_i) = \binom{n+i-1}{i} = \frac{(n)_i}{i!}$. Then by Proposition 3.2.1,

$$\begin{aligned}
\int_{K_n} f_1(kz)\overline{f_2(kz)}\;dk &= \sum_{i=0}^{\infty} \frac{1}{d_i} K_{\lambda,i}(z,z)\langle f_{1,i}, f_{2,i}\rangle_\lambda \\
&= \sum_{i=0}^{m} \frac{i!}{(n)_i}\frac{\lambda^i}{i!}\|z\|^{2i}\binom{m}{i}\frac{(-\lambda)^i}{i!}\|z_n^i\|_\lambda^2 \\
&= \sum_{i=0}^{m} \frac{i!}{(n)_i}\frac{\lambda^i}{i!}\|z\|^{2i}\binom{m}{i}\frac{(-\lambda)^i}{i!}\frac{i!}{\lambda^i} \\
&= \sum_{i=0}^{m} \frac{1}{(n)_i}\binom{m}{i}(-\lambda)^i\|z\|^{2i} \\
&= (n-1)!\sum_{i=0}^{m}\binom{m}{i}\frac{(-\lambda\|z\|^2)^i}{(n+i-1)!} = L_m^{n-1}(\lambda\|z\|^2).
\end{aligned}$$

Hence, $\varphi_{\lambda,m}(z,t) = e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2}\int_{K_n} f_1(kz)\overline{f_2(kz)}\;dk = e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2}L_m^{n-1}(\lambda\|z\|^2)$ as desired. □

In the following proposition, we give another expression of the spherical function $\varphi_{\lambda,m}$ for $\lambda > 0$, which directly indicates that it is indeed positive definite.

Proposition 3.2.3. *Let $\lambda > 0$. Take any unit vector $f_0 \in F_m$. Then,*

$$\varphi_{\lambda,m}(z,t) = \langle T_\lambda(z,t)f_0, f_0\rangle_\lambda.$$

Hence, $\varphi_{\lambda,m}$ is positive definite.

Proof. Since $\|f_0\|_\lambda^2 = 1$, from the equations (3.11) and (3.12), it follows that for every $h \in L^1(H_n)^{K_n}$,

$$\begin{aligned}
&\int_{H_n} h(z,t)\;\langle T_\lambda(z,t)f_0, f_0\rangle_\lambda\;dzdt \\
&= \langle T_\lambda(h)f_0, f_0\rangle_\lambda = \langle \widehat{h}(\lambda,m)f_0, f_0\rangle_\lambda = \widehat{h}(\lambda,m) = \int_{H_n} h(z,t)\varphi_{\lambda,m}(z,t)\;dzdt.
\end{aligned}$$

Moreover, the function $(z,t) \mapsto \langle T_\lambda(z,t)f_0, f_0\rangle_\lambda$ is a bounded, continuous function on H_n, which is also K_n-invariant by the equation (3.9). Then, the result follows from the uniqueness of a bounded, spherical function of H_n corresponding to the character $h \mapsto \widehat{h}(\lambda, m)$ of the algebra $L^1(H_n)^{K_n}$. $\square$

Theorem 3.2.4. *(i) The positive definite spherical function for the Gelfand pair $(U(n) \ltimes H_n, U(n))$ corresponding to the equivalence class of the spherical representation $\mathrm{Ind}_{(U(n))_w \ltimes H_n}^{U(n) \ltimes H_n}(\widetilde{\Phi_w})$ by the GNS-construction is*

$$\psi_w(k,z,t) = \begin{cases} 1, & \text{if } w = 0 \\ \frac{2^{n-1}(n-1)!}{(\|w\|\|z\|)^{n-1}} J_{n-1}(\|w\|\|z\|), & \text{if } w \neq 0 \end{cases} \tag{3.16}$$

where $J_n(x) = 1 + \sum_{i=0}^{\infty} \frac{1}{i!} \frac{1}{n(n+2)\ldots(n+2m-2)} (\frac{-x^2}{2})^i$ is the Bessel function of order n.

(ii) The positive definite spherical function for the Gelfand pair $(U(n) \ltimes H_n, U(n))$ corresponding to the equivalence class of the spherical representation $\widetilde{T_\lambda} \otimes \widehat{\pi_m}$ by the GNS-construction is $\overline{\varphi_{\lambda,m}}$ if $\lambda > 0$ and it is $\varphi_{-\lambda,m}$ if $\lambda < 0$ where

$$\varphi_{\lambda,m}(k,z,t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda\|z\|^2} L_m^{n-1}(\lambda\|z\|^2) \tag{3.17}$$

and $L_m^n(x) = n! \sum_{i=0}^{m} \binom{m}{i} \frac{(-x)^i}{(n+i)!}$ is the generalized Laguerre polynomial of order n and of degree m.

Proof. (i) Since $(U(n) \ltimes H_n, U(n))$ forms a Gelfand pair, the group $U(n) \ltimes H_n$ is unimodular by [Proposition 6.1.2, [26]]. Then as a closed normal subgroup of $U(n) \ltimes H_n$, the Heisenberg group H_n is unimodular as well. The unimodularity of H_n and the compact group $(U(n))_w$ implies the unimodularity of $(U(n))_w \ltimes H_n$ by [Proposition 3.3.10, [28]]. Hence the constant 1 function is a rho-function for the pair $(U(n) \ltimes H_n, U(n)_w \ltimes H_n)$.

For each $w \in \mathbb{C}^n$, define $f_w : U(n) \ltimes H_n \to \mathbb{C}$ by $f(k,z,t) = e^{-i\,\mathrm{Re}\langle z, kw\rangle}$. It can be easily checked that f_w is a $U(n)$-invariant, unit element in $V_{\mathrm{Ind}_{(U(n))_w \ltimes H_n}^{U(n) \ltimes H_n}(\widetilde{\Phi_w})}$. Hence the positive-spherical function ψ_w corresponding to the spherical representation $\mathrm{Ind}_{(U(n))_w \ltimes H_n}^{U(n) \ltimes H_n}(\widetilde{\Phi_w})$ is given by

$$\begin{aligned}\psi_w(k,z,t) &= \langle f_w, \mathrm{Ind}_{(U(n))_w\ltimes H_n}^{U(n)\ltimes H_n}(\widetilde{\Phi_w})(k,z,t)f_w\rangle \\ &= \int_{U(n)} e^{-i\,\mathrm{Re}\langle z,k'w\rangle}\; d\mu_{U(n)}(k')\end{aligned} \tag{3.18}$$

where $\mu_{U(n)}$ is the normalized Haar measure on $U(n)$. Then for $w=0$, $\psi_w(k,z,t)=1$. By the left $U(n)$-invariance of the normalized Haar measure on $U(n)$, the integral in (3.18) depends only on two parameters: w and the usual norm of z in $\mathbb{C}^n$. Hence ψ_w is a function of $\|z\|$. For $w\neq 0$, it follows from (6.4) in [3] that the integral in (3.18), hence ψ_w, can be expressed in terms of a Bessel function as it is stated in (3.16).

(ii) We start with the case $\lambda>0$. Let π_m be the restriction on F_m of the natural action π_λ of $U(n)$ on $\mathcal{F}_\lambda^n$. Let $\overline{F_m}=\{\overline{f}: f\in F_m\}$ and $\overline{\pi_m}$ be the action of $U(n)$ on $\overline{F_m}$ defined by $\overline{\pi_m}(k)\overline{f}=\overline{\pi_m(k)f}$ for $f\in F_m$ and $k\in U(n)$. Clearly, $[\overline{\pi_m}]=[\pi_m]$ in $\widehat{U(n)}$. Now, consider the reproducing kernel $K_{\lambda,m}$ of $(F_m,\langle\cdot,\cdot\rangle_\lambda)$. Then,

$$K_{\lambda,m}(z,w)=\sum_{|\nu|=m} e_\nu(z)\overline{e_\nu(w)}=\sum_{|\nu|=m}(e_\nu\otimes\overline{e_\nu})(z,w)\in\mathcal{F}_\lambda^n\otimes\overline{F_m}$$

where $\nu\in\mathbb{N}^n$ and $e_\nu(z)=\sqrt{\frac{|\lambda|^{|\nu|}}{\nu!}}z^\nu$ is a unit vector in $\mathcal{F}_\lambda^n$. Since

$$K_{\lambda,m}(kz,kw)=\frac{|\lambda|^m}{m!}\langle kz,kw\rangle^m=\frac{|\lambda|^m}{m!}\langle z,w\rangle^m=K_{\lambda,m}(z,w)$$

for all $k\in U(n)$, the reproducing kernel $K_{\lambda,m}(z,w)$ is a $U(n)$-invariant vector in $\mathcal{F}_\lambda^n\otimes\overline{F_m}$ under the action of $U(n)\ltimes H_n$ via the representation $\widetilde{T_\lambda}\otimes\widehat{\overline{\pi_m}}$. Here, $\widehat{\overline{\pi_m}}$ is the lifting of $\overline{\pi_m}$ from $U(n)$ to $U(n)\ltimes H_n$. So, given $(k,z,t)\in U(n)\ltimes H_n$, we have

$$\begin{aligned}
&\langle \widetilde{T_\lambda} \otimes \widehat{\overline{\pi_m}}(k,z,t) K_{\lambda,m}, K_{\lambda,m}\rangle \\
&= \langle \widetilde{T_\lambda} \otimes \widehat{\overline{\pi_m}}(k,z,t) \sum_{|\nu|=m} (e_\nu \otimes \overline{e_\nu}), \widetilde{T_\lambda} \otimes \widehat{\overline{\pi_m}}(k,0,0) \sum_{|\nu|=m} (e_\nu \otimes \overline{e_\nu})\rangle \\
&= \langle \sum_{|\nu|=m} \widetilde{T_\lambda}(k,z,t) e_\nu \otimes \widehat{\overline{\pi_m}}(k,z,t)\overline{e_\nu}, \sum_{|\nu|=m} \widetilde{T_\lambda}(k,0,0) e_\nu \otimes \widehat{\overline{\pi_m}}(k,0,0)\overline{e_\nu}\rangle \\
&= \sum_{|\nu|=m} \sum_{|\nu'|=m} \langle T_\lambda(z,t)\pi_\lambda(k)e_\nu, \pi_\lambda(k)e_{\nu'}\rangle_\lambda \overline{\langle \pi_m(k)e_\nu, \pi_m(k)e_{\nu'}\rangle_\lambda} \\
&= \sum_{|\nu|=m} \langle T_\lambda(z,t)\pi_\lambda(k)e_\nu, \pi_\lambda(k)e_\nu\rangle_\lambda. \qquad (3.19)
\end{aligned}$$

Since $\pi_\lambda(k)e_\nu$ is a unit vector in F_m, by Proposition 3.2.3,

$$\sum_{|\nu|=m} \langle T_\lambda(z,t)\pi_\lambda(k)e_\nu, \pi_\lambda(k)e_\nu\rangle_\lambda = \sum_{|\nu|=m} \varphi_{\lambda,m}(z,t) = \varphi_{\lambda,m}(z,t) \sum_{|\nu|=m} 1 = \varphi_{\lambda,m}(z,t) \dim F_m.$$

Therefore,

$$\overline{\varphi_{\lambda,m}}(z,t) = (\dim F_m)^{-1} \langle K_{\lambda,m}, \widetilde{T_\lambda} \otimes \widehat{\overline{\pi_m}}(k,z,t) K_{\lambda,m}\rangle$$

and since $[\widetilde{T_\lambda} \otimes \widehat{\overline{\pi_m}}] = [\widetilde{T_\lambda} \otimes \widehat{\pi_m}]$ in $\widehat{U(n) \ltimes H_n}$, the result follows.

To continue with the case $\lambda < 0$, take $(k,z,t) \in U(n) \ltimes H_n$ and choose $k_0 \in U(n)$ such that $k_0 z = \bar{z}$. Then,

$$\begin{aligned}
\langle T_\lambda(z,t)\pi_\lambda(k)e_\nu, \pi_\lambda(k)e_\nu\rangle_\lambda &= \langle T_{-\lambda}(\bar{z},-t)\pi_\lambda(k)e_\nu, \pi_\lambda(k)e_\nu\rangle_\lambda \\
&= \langle T_{-\lambda}(k_0 z,-t)\pi_\lambda(k)e_\nu, \pi_\lambda(k)e_\nu\rangle_\lambda \\
&= \langle \pi_\lambda(k)e_\nu, T_{-\lambda}(-k_0 z,t)\pi_\lambda(k)e_\nu\rangle_\lambda \\
&= \langle \pi_\lambda(k)e_\nu, \pi_\lambda(-k_0) T_{-\lambda}(z,t)\pi_\lambda(-k_0^{-1})\pi_\lambda(k)e_\nu\rangle_\lambda \\
&= \langle \pi_\lambda(-k_0^{-1}k)e_\nu, T_{-\lambda}(z,t)\pi_\lambda(-k_0^{-1}k)e_\nu\rangle_\lambda = \overline{\varphi_{-\lambda,m}}(z,t).
\end{aligned}$$

Hence, following the equation (3.19),

$$\langle \widetilde{T_\lambda} \otimes \widehat{\overline{\pi_m}}(k,z,t)K_{\lambda,m}, K_{\lambda,m}\rangle = \sum_{|\nu|=m} \langle T_\lambda(z,t)\pi_\lambda(k)e_\nu, \pi_\lambda(k)e_\nu\rangle_\lambda = \sum_{|\nu|=m} \overline{\varphi_{-\lambda,m}}(z,t)$$
$$= \overline{\varphi_{-\lambda,m}}(z,t)\dim F_m \tag{3.20}$$

so that

$$\varphi_{-\lambda,m}(z,t) = (\dim F_m)^{-1}\langle K_{\lambda,m}, \widetilde{T_\lambda} \otimes \widehat{\overline{\pi_m}}(k,z,t)K_{\lambda,m}\rangle$$

as desired. □

3.3. Spherical Dual of $(U(\infty) \ltimes H(\infty), U(\infty))$

In the asymptotic functional equation satisfied by the spherical functions for the pair $(U(\infty) \ltimes H(\infty), U(\infty))$, we will confront with integrals over the unitary group of functions which depend only on the first entry. The next lemma will enable us to turn such integrals over the unitary group into integrals over the closed unit ball in $\mathbb{C}$. It is indeed a special case of Lemma 5.1 in [14], but we shall give a simple proof that is specific to this case.

Lemma 3.3.1. *Let $n \geq 2$ and D be the closed unit ball in $\mathbb{C}$. Consider the projection $\Lambda : U(n) \to D$ defined by $\Lambda((u_{ij})_{i,j=1}^n) = u_{11}$. If f is a continuous function on D, then*

$$\int_{U(n)} (f \circ \Lambda)(U) \quad d\mu_{U(n)}(U) = \frac{n-1}{\pi} \int_D f(w)(1-|w|^2)^{n-2} \quad d\mu(w) \tag{3.21}$$

where μ is the Lebesgue measure on D and $\mu_{U(n)}$ is the normalized Haar measure on $U(n)$.

Proof. Let

$$S^{2n-1} = \{(z_1, ..., z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 = 1\}$$

be the unit sphere in $\mathbb{C}^n$ and let $e_1 = (1, 0, ..., 0) \in \mathbb{C}^n$. Let $F : U(n) \to S^{2n-1}$ be the function defined by $F(U) = U(e_1)$, i.e. $F((u_{ij})_{i,j=1}^n) = (u_{11}, u_{21}, ..., u_{n1})$. Since the Haar measure on $U(n)$ is left translation invariant, the pushforward $F_*(\mu_{U(n)})$ of the normalized Haar measure $\mu_{U(n)}$ is a rotation invariant probability measure on the sphere S^{2n-1}. Indeed, given $f : S^{2n-1} \to \mathbb{C}$ and $g \in U(n)$,

$$\begin{aligned}
\int_{S^{2n-1}} (f \circ g)(x)\ dF_*(\mu_{U(n)})(x) &= \int_{U(n)} (f \circ g \circ F)(U)\ d\mu_{U(n)}(U) \\
&= \int_{U(n)} (f \circ F)(gU)\ d\mu_{U(n)}(U) \\
&= \int_{U(n)} (f \circ F)(U)\ d\mu_{U(n)}(U) \\
&= \int_{S^{2n-1}} f(x)\ dF_*(\mu_{U(n)})(x).
\end{aligned}$$

Then since the uniform measure σ^{2n-1} is the unique rotation invariant probability measure on the sphere S^{2n-1}, we get $F_*(\mu_{U(n)}) = \sigma^{2n-1}$. For the function $\Theta : S^{2n-1} \to D$ defined by $\Theta(z_1, ..., z_n) = z_1$, we have $\Theta \circ F = \Lambda$. Then since $F_*(\mu_{U(n)}) = \sigma^{2n-1}$, we get

$$\int_{U(n)} (f \circ \Lambda)(U)\ \ d\mu_{U(n)}(U) = \int_{S^{2n-1}} (f \circ \Theta)\ d\sigma^{2n-1}.$$

Let B_2 denote the open unit ball in $\mathbb{C}$. By Theorem A.4. in [2], for some certain constant $C \in \mathbb{C}$ we have

$$
\begin{aligned}
&\int_{S^{2n-1}} (f \circ \Theta)\ d\sigma^{2n-1} \\
&= C. \int_{B_2} (1-|w|^2)^{n-2} \int_{S^{2n-3}} (f \circ \Theta)(w, \sqrt{1-|w|^2}\zeta)\ d\sigma^{2n-3}(\zeta)\ d\mu(w) \\
&= C. \int_{B_2} (1-|w|^2)^{n-2} \int_{S^{2n-3}} f(w)\ d\sigma^{2n-3}(\zeta)\ d\mu(w) \\
&= C. \int_{B_2} f(w)(1-|w|^2)^{n-2}\ d\mu(w).
\end{aligned}
$$

Hence, for a certain constant $C \in \mathbb{C}$,

$$
\int_{U(n)} (f \circ \Lambda)(U)\ \ d\mu_{U(n)}(U) = C. \int_D f(w)(1-|w|^2)^{n-2}\ d\mu(w).
$$

But, $C^{-1} = \int_D (1-|w|^2)^{n-2}\ d\mu(w) = \int_0^{2\pi} \int_0^1 (1-r^2)^{n-2} r\ dr\ d\theta = \frac{\pi}{n-1}$. So, the equation (3.21) follows. $\square$

The following lemma will play a crucial role in the computation of the limit appearing in the asymptotic functional equation satisfied by the spherical functions for the pair $(U(\infty) \ltimes H(\infty), U(\infty))$.

Lemma 3.3.2. *[12] Let X be a compact space and μ be a measure such that $\mu(U) > 0$ for all nonempty open subset U of X. Let $\delta \geq 0$ be a continuous function on X which attains its maximum at only one point x_0. Then for a continuous function f on X,*

$$
\lim_{n\to\infty} \frac{1}{\int_X \delta(x)^n\ d\mu(x)} \int_X f(x)\delta(x)^n\ d\mu(x) = f(x_0).
$$

Proof. Define $a_n = (\int_X \delta(x)^n\ d\mu(x))^{-1}$. Let $M = \delta(x_0)$. Given $r > 0$, also define $B_r = \{x \in X : \delta(x) > M - r\}$. Note that

$$
(a_n)^{-1} = \int_X \delta(x)^n\ d\mu(x) \geq \int_{B_r} \delta(x)^n\ d\mu(x) \geq \int_{B_r} (M-r)^n\ d\mu(x) = (M-r)^n \mu(B_r)
$$

for all $r > 0$.

Given an open neighbourhood U_0 of x_0, there exists an $r_0 > 0$ such that $\delta(x) \leq M - r_0$ for all $x \in X \backslash U_0$. Hence $B_{r_0} \subseteq U_0$. Then,

$$\begin{aligned} a_n \int_{X\backslash U_0} \delta(x)^n \ d\mu(x) &\leq a_n \int_{X\backslash U_0} (M-r_0)^n \ d\mu(x) \\ &\leq a_n \int_{X\backslash B_{r_0}} (M-r_0)^n \ d\mu(x) \leq \frac{(M-r_0)^n \mu(X\backslash B_{r_0})}{(M-\frac{r_0}{2})^n \mu(B_{\frac{r_0}{2}})} \end{aligned}$$

so that $\lim_{n\to\infty} a_n \int_{X\backslash U_0} \delta(x)^n \ d\mu(x) = 0$.

Since X is compact, every continuous function g on X is bounded so that

$$\lim_{n\to\infty} a_n \int_{X\backslash U_0} g(x)\delta(x)^n \ d\mu(x) = 0$$

as well.

Now let f be a continuous function on X and let $\epsilon > 0$ be given. Choose an open neighbourhood U of x_0 such that for all $x \in U$, $|f(x) - f(x_0)| < \frac{\epsilon}{2}$ is satisfied. By the last equation above, we have $\lim_{n\to\infty} a_n \int_{X\backslash U} (f(x) - f(x_0))\delta(x)^n \ d\mu(x) = 0$. Hence there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n \int_{X\backslash U} (f(x) - f(x_0))\delta(x)^n \ d\mu(x)| < \frac{\epsilon}{2}$. Then, for all $n > N$

$$\begin{aligned} &|a_n \int_X f(x)\delta(x)^n \ d\mu(x) - f(x_0)| \\ &= |a_n \int_X f(x)\delta(x)^n \ d\mu(x) - a_n \int_X f(x_0)\delta(x)^n \ d\mu(x)| \\ &= |a_n \int_X (f(x) - f(x_0))\delta(x)^n \ d\mu(x)| \\ &\leq |a_n \int_U (f(x) - f(x_0))\delta(x)^n \ d\mu(x)| + |a_n \int_{X\backslash U} (f(x) - f(x_0))\delta(x)^n \ d\mu(x)| \\ &< \frac{\epsilon}{2} . \frac{\int_U \delta(x)^n \ d\mu(x)}{\int_X \delta(x)^n \ d\mu(x)} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

$\square$

We will give a type of positive definite functions on the Heisenberg group H_n.

Priorly, we list some useful notions, remarks and theorems related to positive definite kernels, hence to positive definite functions on groups.

Definition 3.3.3. Let X be a non-empty set. A function $K : X \times X \to \mathbb{C}$ is called a positive definite kernel if and only if $\sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i, x_j) \geq 0$ for any finite subset $\{x_1, x_2, ..., x_n\}$ in X and for all systems $\{c_1, c_2, ..., c_n\}$ of complex numbers. That is to say, the matrix $(K(x_i, x_j))_{i,j=1}^{n}$ is positive definite for all $n \in \mathbb{N}$ and for any finite subset $\{x_1, x_2, ..., x_n\}$ in X.

Remark 3.3.4. Let G be a group and $\varphi : G \to \mathbb{C}$ be a function. Define $K_\varphi : G \times G \to \mathbb{C}$ by $K_\varphi(x, y) = \varphi(x^{-1}y)$. Then φ is positive definite if and only if K_φ is a positive definite kernel.

Remark 3.3.5. Let $X \neq \emptyset$ and $f : X \to \mathbb{C}$ be an arbitrary function. Then $K(x, y) = f(x)\overline{f(y)}$ is a positive definite kernel, because $\sum_{i,j=1}^{n} c_i \overline{c_j} f(x_i) \overline{f(x_j)} = \| \sum_{i=1}^{n} c_i f(x_i) \|^2 \geq 0$.

The following theorem, due to Schur, shows that the convex cone of positive definite kernels is closed under pointwise multiplication.

Theorem 3.3.6. *[4] Let X be a non-empty set. If $K_1, K_2 : X \times X \to \mathbb{C}$ are positive definite kernels, then their pointwise product $K_1.K_2 : X \times X \to \mathbb{C}$ is also a positive definite kernel.*

Corollary 3.3.7. *If $K : X \times X \to \mathbb{C}$ is a positive definite kernel, then so is $exp(K)$.*

Proof. By Theorem 3.3.6, for each $i \in \mathbb{N}$ the kernel K^i is positive definite so that the finite sum $\sum_{i=1}^{n} \frac{1}{i!} K^i$ is also positive definite for each $n \in \mathbb{N}$. Then since the pointwise limits of positive definite kernels are again positive definite, we get $exp(K) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i!} K^i$ is positive definite. $\square$

Definition 3.3.8. Let X be a non-empty set. A function $K : X \times X \to \mathbb{C}$ is called a negative definite kernel if and only if K is Hermitian, i.e. $K(x, y) = \overline{K(y, x)}$ for all $x, y \in X$ and $\sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i, x_j) \leq 0$ for any finite subset $\{x_1, x_2, ..., x_n\}$ in X and for all systems $\{c_1, c_2, ..., c_n\}$ of complex numbers with $\sum_{i=1}^{n} c_i = 0$ and $n \geq 2$.

Remark 3.3.9. A real-valued kernel $K : X \times X \to \mathbb{R}$ is negative definite if and only if K is symmetric, i.e. $K(x,y) = K(y,x)$ for all $x, y \in X$ and $\sum_{i,j=1}^{n} c_i c_j K(x_i, x_j) \leq 0$ for any finite subset $\{x_1, x_2, ..., x_n\}$ in X and for all systems $\{c_1, c_2, ..., c_n\}$ of real numbers with $\sum_{i=1}^{n} c_i = 0$ and $n \geq 2$.

There is a beautiful relation between positive definite and negative definite kernels given by Schoenberg in [24] as follows.

Theorem 3.3.10. *Let X be a non-empty set. A function $K : X \times X \to \mathbb{C}$ is a negative definite kernel if and only if $exp(-tK)$ is a positive definite kernel for all $t \in \mathbb{R}^{>0}$.*

For the proof of Theorem 3.3.10, one can see Theorem 3.2.2 in [4]. We are now ready to give some results on positive definite functions on $\mathbb{C}^n$ and H_n.

Proposition 3.3.11. *Let Q be a positive definite quadratic form on $\mathbb{C}^n$. Then, $e^{-Q(z)}$ is a positive definite function on $\mathbb{C}^n$.*

Proof. Let B be the positive definite Hermitian form corresponding to Q. By Remark 3.3.4, we need to show that $K(z,w) = e^{-Q(-z+w)} = e^{-B(-z+w,-z+w)}$ is a positive definite kernel. By Theorem 3.3.10, it is enough to show that $T(z,w) = B(-z+w, -z+w)$ is a negative definite kernel. T is clearly symmetric. Now let $c_1, c_2, ..., c_N \in \mathbb{R}$ with $\sum_{i=1}^{N} c_i = 0$ and $z_1, z_2, ..., z_N \in \mathbb{C}^n$. Then,

$$\begin{aligned}
\sum_{i,j=1}^{N} c_i c_j T(z_i, z_j) &= \sum_{i,j=1}^{N} c_i c_j B(-z_i + z_j, -z_i + z_j) \\
&= -(\sum_{i,j=1}^{N} c_i c_j B(z_i, z_j) + \sum_{i,j=1}^{N} c_i c_j B(z_j, z_i)) \\
&= -2.B(\sum_{i=1}^{N} c_i z_i, \sum_{i=1}^{N} c_i z_i) \leq 0.
\end{aligned}$$

Hence by Remark 3.3.9, T is a negative definite kernel. □

Proposition 3.3.12. *For each $\lambda \in \mathbb{R}$ the map $\psi(z,t) = e^{i\lambda t - \frac{1}{2}|\lambda| \|z\|^2}$ is positive definite on the Heisenberg group H_n.*

Proof. We need to show that $K((z,t),(w,s)) = \psi((z,t)^{-1}(w,s))$ is a positive definite kernel on H_n. Let $z = (z_1, z_2, ..., z_n)$ and $w = (w_1, w_2, ..., w_n)$ for $z_j, w_j \in \mathbb{C}$ for all $j = 1, 2, ..., n$. Then,

$$\begin{aligned}
K((z,t),(w,s)) &= \psi((-z,-t)(w,s)) \\
&= e^{i\lambda(-t+s)} e^{-i\lambda \operatorname{Im}\langle z,w\rangle} e^{-\frac{1}{2}|\lambda|\|-z+w\|^2} \\
&= e^{i\lambda(-t+s)} \prod_{j=1}^{n} e^{-i\lambda \operatorname{Im}(z_j \overline{w_j})} \prod_{j=1}^{n} e^{-\frac{1}{2}|\lambda||-z_j+w_j|^2} \\
&= e^{i\lambda(-t+s)} \prod_{j=1}^{n} e^{-i\lambda \operatorname{Im}(z_j \overline{w_j})} \prod_{j=1}^{n} e^{|\lambda| \operatorname{Re}(z_j \overline{w_j})} \prod_{j=1}^{n} e^{-\frac{1}{2}|\lambda|(|z_j|^2+|w_j|^2)} \\
&= e^{i\lambda(-t+s)} \prod_{j=1}^{n} e^{|\lambda|(\operatorname{Re}(z_j \overline{w_j}) - i.sgn(\lambda) \operatorname{Im}(z_j \overline{w_j}))} \prod_{j=1}^{n} e^{-\frac{1}{2}|\lambda|(|z_j|^2+|w_j|^2)}. \qquad (3.22)
\end{aligned}$$

Now, define $A((z,t),(w,s)) = e^{i\lambda(-t+s)}$ and given $j \in \{1, 2, ..., n\}$, let $B_j((z,t),(w,s)) = z_j \overline{w_j}$ and $C_j((z,t),(w,s)) = e^{-\frac{1}{2}|\lambda|(|z_j|^2+|w_j|^2)}$. Since $A((z,t),(w,s)) = f(z,t)\overline{f(w,s)}$ where $f(z,t) = e^{-i\lambda t}$ for $(z,t) \in H_n$, by Remark 3.3.5 we get that A is a positive definite kernel. $B_j((z,t),(w,s)) = f_j(z,t)\overline{f_j(w,s)}$ where $f_j(z,t) = z_j$ for $(z,t) \in H_n$. Hence B_j is a positive definite kernel by Remark 3.3.5. Then $|\lambda| B_j$ and by Corollary 3.3.7 also $e^{|\lambda| B_j}$ are positive definite kernels. Moreover, since $C_j((z,t),(w,s)) = g_j(z,t)\overline{g_j(w,s)}$ where g_j is the function defined on H_n by $g_j(z,t) = e^{-\frac{1}{2}|\lambda||z_j|^2}$, we have C_j is a positive definite kernel by Remark 3.3.5 as well. The equation (3.22) shows that if $\lambda < 0$, then $K = A \prod_{j=1}^{n} e^{|\lambda| B_j} \prod_{j=1}^{n} C_j$ and if $\lambda \geq 0$, then $K = A \prod_{j=1}^{n} e^{|\lambda| \overline{B_j}} \prod_{j=1}^{n} C_j$. In both cases, as a product of positive definite kernels, K is a positive definite kernel by Theorem 3.3.6. □

Lemma 3.3.13. *Let $\lambda \in \mathbb{R}$ and assume Q is a positive definite quadratic form on $\mathbb{C}^n$. Consider $\varphi : H_n \to \mathbb{C}$ defined by*

$$\varphi(z,t) = e^{i\lambda t} e^{-Q(z)}.$$

Then the function φ is positive definite if and only if $Q(z) \geq \frac{1}{2}|\lambda|\|z\|^2$.

Proof. In case $\lambda = 0$, it follows from Proposition 3.3.11 that $\varphi(z,t) = e^{-Q(z)}$ is positive definite on H_n for any positive definite quadratic form Q on $\mathbb{C}^n$. Now let $\lambda \in \mathbb{R}^*$. Assume $\varphi : H_n \to \mathbb{C}$ given by $\varphi(z,t) = e^{i\lambda t}e^{-Q(z)}$ is positive definite on H_n. Fix $u \in \mathbb{C}^n$ with $\|u\| = 1$. We define the map $\varphi_u : H_1 \to \mathbb{C}$ by

$$\varphi_u(z,t) = \varphi(zu,t) = e^{i\lambda t}e^{-Q(u)|z|^2}$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$. Since φ is positive definite on H_n and $\|u\| = 1$, we get φ_u is positive definite on H_1.

Given $\mu \in \mathbb{R}^{>0}$, let $\psi_{\lambda,\mu} : H_1 \to \mathbb{C}$ defined by $\psi_{\lambda,\mu}(z,t) = e^{i\lambda t}e^{-\mu|z|^2}$. By [p.269, [8]], the ordinary generating function of the sequence $\{L_m(x)\}_{m=0}^{\infty}$ of the Laguerre polynomials is given by

$$\sum_{m=0}^{\infty} L_m(x)r^m = \frac{1}{1-r}e^{-\frac{r}{1-r}x} \tag{3.23}$$

if $-1 < r < 1$. Given $z \in \mathbb{C}$, taking $x = |\lambda||z|^2$ and $r = \frac{2\mu-|\lambda|}{2\mu+|\lambda|}$ in the equation (3.23), we get

$$\sum_{m=0}^{\infty} L_m(|\lambda||z|^2)(\frac{2\mu-|\lambda|}{2\mu+|\lambda|})^m = \frac{2\mu+|\lambda|}{2\lambda}e^{(-\mu+\frac{|\lambda|}{2})|z|^2}. \tag{3.24}$$

Then we multiply both sides of the equation (3.24) with $e^{i\lambda t}e^{-\frac{1}{2}|\lambda||z|^2}$ and we get

$$\psi_{\lambda,\mu}(z,t) = \frac{2|\lambda|}{2\mu+|\lambda|}\sum_{m=0}^{\infty}(\frac{2\mu-|\lambda|}{2\mu+|\lambda|})^m\varphi_{\lambda,m}(z,t) \tag{3.25}$$

where $\varphi_{\lambda,m}(z,t) = e^{i\lambda t}e^{-\frac{1}{2}|\lambda||z|^2}L_m(|\lambda||z|^2)$ is positive definite on H_1 for each $m \in \mathbb{N}$ as it follows from Proposition 3.2.3. Hence $\psi_{\lambda,\mu}$ is positive definite on H_1 if and only if $2\mu \geq |\lambda|$. Since φ_u is positive definite on H_1 and $\varphi_u = \psi_{\lambda,Q(u)}$ for all $u \in \mathbb{C}^n$ with $\|u\| = 1$, we then get $2Q(\frac{z}{|z|}) \geq |\lambda|$ for all non-zero $z \in \mathbb{C}^n$ so that $Q(z) \geq \frac{1}{2}|\lambda|\|z\|^2$ for all $z \in \mathbb{C}^n$.

Conversely, assume $Q(z) \geq \frac{1}{2}|\lambda|\|z\|^2$. Let $\psi(z,t) = e^{i\lambda t - \frac{1}{2}|\lambda|\|z\|^2}$ and $\chi(z,t) = e^{\frac{1}{2}|\lambda|\|z\|^2 - Q(z)}$. By Proposition 3.3.12, ψ is positive definite on H_n. For $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$, let $C(z,w) = B(z,w) - \frac{1}{2}|\lambda|\langle z,w\rangle$ where B is the positive definite Hermitian form corresponding to Q. Then C is also a positive definite Hermitian form on $\mathbb{C}^n$ and by Proposition 3.3.11, $e^{-C(z,z)}$ is a positive definite function on $\mathbb{C}^n$ so that $\chi(z,t) = e^{-C(z,z)}$ is positive definite on H_n. Therefore, $\varphi(z,t) = \psi(z,t)\chi(z,t)$ is positive definite by Theorem 3.3.6. □

Theorem 3.3.14. *Let $\varphi : U(\infty) \ltimes H(\infty) \to \mathbb{C}$ be a continuous and $U(\infty)$-invariant function. Then, φ is a positive definite spherical function for the Olshanski pair $(U(\infty) \ltimes H(\infty), U(\infty))$ if and only if*

$$\varphi(k,z,t) = e^{i\lambda t}e^{-\mu\|z\|^2}$$

for some $\lambda, \mu \in \mathbb{R}$ such that $\mu \geq \frac{1}{2}|\lambda|$.

Proof. Let $\varphi : U(\infty) \ltimes H(\infty) \to \mathbb{C}$ be a continuous and $U(\infty)$-invariant function. Then as already mentioned in Remark 3.0.2, φ does not depend on the parameter k and we can consider φ as a continuous, $U(\infty)$-invariant function on $H(\infty)$. Given $z, w \in \mathbb{C}^{(\infty)}$, there exists $k \in U(\infty)$ such that $kz = w$ if and only if $\|z\| = \|w\|$. Then by $U(\infty)$-invariance of φ, we get $\varphi(z,t) = \varphi((\|z\|, 0, 0, ...), t)$ for all $(z,t) \in H(\infty)$. That is to say, there exists a unique continuous function $F_\varphi : [0,\infty) \times \mathbb{R} \to \mathbb{C}$ such that $\varphi(z,t) = F_\varphi(\|z\|^2, t)$ for all $(z,t) \in H(\infty)$.

Given (z_1,t_1) and (z_2,t_2) in $H(\infty)$, for N sufficently large, we can choose $k_1, k_2 \in U(N)$ such that $z_1 = k_1(\|z_1\|, 0, 0, ...)$ and $z_2 = k_2(\|z_2\|, 0, 0, ...)$. Then by $U(\infty)$-left-invariance of φ and the unimodularity of the Haar measure on the compact group $U(n)$, for all $n > N$ we have

$$\begin{aligned}
&\int_{U(n)} \varphi((z_1,t_1)U(z_2,t_2))\ d\mu_{U(n)}(U)\\
&= \int_{U(n)} \varphi(k_1((\|z_1\|,0,0,...),t_1)Uk_2((\|z_2\|,0,0,...),t_2))\ d\mu_{U(n)}(U)\\
&= \int_{U(n)} \varphi(((\|z_1\|,0,0,...),t_1)U((\|z_2\|,0,0,...),t_2))\ d\mu_{U(n)}(U).
\end{aligned}$$

Hence, φ is $(U(\infty) \ltimes H(\infty), U(\infty))$-spherical if and only if

$$\lim_{n\to\infty} \int_{U(n)} \varphi(xUy)\ d\mu_{U(n)}(U) = \varphi(x)\varphi(y)$$

for all $x = ((r,0,0,...),t) \in H(\infty)$ and $y = ((r^{'},0,0,...),t^{'}) \in H(\infty)$ where $r, r^{'} \in \mathbb{R}^{>0}$.

Now for $r, r^{'} \in \mathbb{R}^{>0}$, let $x = ((r,0,0,...),t) \in H(\infty)$ and $y = ((r^{'},0,0,...),t^{'}) \in H(\infty)$. Given $U = (u_{ij})_{i,j\geq 1} \in U(n)$,

$$\begin{aligned}
\varphi(xUy) &= \varphi((r+u_{11}r^{'}, u_{21}r^{'}, ..., u_{n1}r^{'}, 0, 0, ...), t+t^{'}+rr^{'}\operatorname{Im}(\overline{u_{11}}))\\
&= F_\varphi(r^2+(r^{'})^2+2rr^{'}\operatorname{Re}(u_{11}), t+t^{'}-rr^{'}\operatorname{Im}(u_{11})).
\end{aligned}$$

Then by Lemma 3.3.1,

$$\begin{aligned}
&\int_{U(n)} \varphi(xUy)\ d\mu_{U(n)}(U)\\
&= \frac{n-1}{\pi}\int_D F_\varphi(r^2+(r^{'})^2+2rr^{'}\operatorname{Re}(w), t+t^{'}-rr^{'}\operatorname{Im}(w))(1-|w|^2)^{n-2}\ \ d\mu(w).
\end{aligned}$$

By taking limits as n goes to ∞ of the above equation and applying Lemma 3.3.2,

$$\lim_{n\to\infty} \int_{U(n)} \varphi(xUy)\ d\mu_{U(n)}(U) = F_\varphi(r^2+(r^{'})^2, t+t^{'}).$$

Therefore, it turns out that φ is $(U(\infty) \ltimes H(\infty), U(\infty))$-spherical if and only if F_φ satisfies the multiplicative property given by the equation

$$F_\varphi(r^2, t)F_\varphi((r')^2, t') = F_\varphi(r^2 + (r')^2, t + t') \tag{3.26}$$

for all $r, r', t, t' \in \mathbb{R}$.

Now, to prove the 'only if' part of the statement of the theorem, assume φ is positive definite and spherical for the Olshanski pair $(U(\infty) \ltimes H(\infty), U(\infty))$. Then we have the equation (3.26) which implies $F_\varphi(r^2, t) = F_\varphi(r^2 + 0, 0 + t) = F_\varphi(r^2, 0)F_\varphi(0, t)$ for all $r, t \in \mathbb{R}$. Now let $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R}^{>0} \to \mathbb{C}$ be the continuous functions defined by $f(t) = F_\varphi(0, t)$ and $g(r) = F_\varphi(r, 0)$. Since $F_\varphi(r^2, t) = F_\varphi(0 + r^2, 0 + t) = F_\varphi(0, 0)F_\varphi(r^2, t) = f(0)F_\varphi(r^2, t)$ for all $r, t \in \mathbb{R}$ and φ is non-zero, we have $f(0) = 1$. Since φ is positive definite, φ satisfies the Hermitian symmetry so that for any $t \in \mathbb{R}$, $\|f(t)\|^2 = f(t)\overline{f(t)} = \varphi((0, 0, ...), t)\varphi((0, 0, ...), -t) = F_\varphi(0, t)F_\varphi(0, -t) = F_\varphi(0, 0) = f(0) = 1$. Hence, $f : \mathbb{R} \to S^1$ and since $f(t + s) = F_\varphi(0, t + s) = F_\varphi(0, t)F_\varphi(0, s) = f(t)f(s)$ for all $t, s \in \mathbb{R}$, it follows that $f : \mathbb{R} \to S^1$ is a continuous group homomorphism. Therefore, $f(t) = e^{i\lambda t}$ for some $\lambda \in \mathbb{R}$. When it comes to g, we have $g(r^2 + s^2) = F_\varphi(r^2 + s^2, 0) = F_\varphi(r^2, 0)F_\varphi(s^2, 0) = g(r^2)g(s^2)$ for all $r, s \in \mathbb{R}$ and together with the continuity of g, we get $g(r) = e^{cr}$ for some $c = \alpha + i\beta \in \mathbb{C}$. By positive definiteness of φ, we have $\overline{\varphi(z, 0)} = \varphi(-z, 0)$ for all $(z, 0) \in H(\infty)$ and since $\varphi(z, 0) = g(\|z\|^2)$, we get $e^{\alpha\|z\|^2}e^{i\beta\|z\|^2} = e^{\alpha\|z\|^2}e^{-i\beta\|z\|^2}$ for all $z \in \mathbb{C}^{(\infty)}$. Hence $\beta = 0$. Since φ is bounded, g is also bounded so that $\alpha \leq 0$. So, for all $r \in \mathbb{R}$, $g(r^2) = e^{-\mu r^2}$ for some $\mu \geq 0$. Thus,

$$\varphi(z, t) = F_\varphi(0, t)F_\varphi(\|z\|^2, 0) = f(t)g(\|z\|^2) = e^{i\lambda t}e^{-\mu\|z\|^2}$$

for some $\lambda \in \mathbb{R}$ and $\mu \geq 0$. Since φ is positive definite, the restriction $\varphi_{\restriction H(n)}$ is positive definite on H_n for all n. Then, by Lemma 3.3.13, we get $\mu \geq \frac{1}{2}|\lambda|$.

To prove the 'if part', let $\varphi(z, t) = e^{i\lambda t}e^{-\mu\|z\|^2}$ where $\lambda, \mu \in \mathbb{R}$, $\mu \geq \frac{1}{2}|\lambda|$ and $(z, t) \in H(\infty)$. Since $\varphi_{\restriction H(n)}$ is continuous for each n, φ is continuous on $H(\infty)$ with the inductive limit topology. The function φ is obviously $U(\infty)$-invariant. Moreover, the corresponding function F_φ satisfies the equation (3.26) so that φ defines a spherical function on $H(\infty)$. Since $\mu \geq \frac{1}{2}|\lambda|$, by Lemma 3.3.13, $\varphi_{\restriction H(n)}$ is positive definite on $H(n)$ for each n. Hence, φ is positive definite on $H(\infty)$ as well. $\square$

4. HARMONIC ANALYSIS FOR AN OLSHANSKI PAIR CONSISTING OF STABILIZERS OF HORICYCLES OF HOMOGENEOUS TREES

4.1. The Olshanski Pair (B_ω, B_n)

In this section we give the necessary definitions and notations to construct the second Olshanski pair on which we study.

Let (X, E) be a homogeneous tree of countably infinite degree where X is the set of vertices and E is the set of edges. Let d denote the natural distance on X which counts the number of edges between two points. Fix two distinct elements ω and ω' of the boundary of the tree (X, E). There is a unique doubly infinite chain on the tree (X, E) connecting the boundary points ω and ω'. Denote it by (ω', ω). Enumerate the vertices on (ω', ω) by a sequence $(x_n)_{n\in\mathbb{Z}}$ such that x_n and x_{n+1} are neighbours (i.e. $d(x_n, x_{n+1})$=1), the infinite chain $(x_n)_{n=0}^{\infty}$ corresponds to the boundary point ω and the infinite chain $(x_n)_{n=0}^{-\infty}$ corresponds to the boundary point ω' (for the definitions of the notions used up to now, one can see [1]).

For each vertex x in X, we enumerate its neighbours: If $x \neq x_n$ for any $n \in \mathbb{Z}$, then we choose a bijection $\tau_x : \mathbb{N} \to \{y \in X \mid d(x, y) = 1\}$ such that $\tau_x(1)$ is the neighbour of x with minimal distance to the chain (ω', ω). If $x = x_n$ for some $n \in \mathbb{Z}$, then we choose a bijection $\tau_x : \mathbb{N} \to \{y \in X \mid d(x, y) = 1\}$ such that $\tau_x(1) = x_{n-1}$ and $\tau_x(2) = x_{n+1}$. We fix these bijections $\{\tau_x\}_{x\in X}$.

For each integer $k \geq 2$, we define a subtree (X^k, E^k) of the tree (X, E) as follows: If $x = x_n$ for some $n \in \mathbb{Z}$, we let $x \in X^k$. If $x \neq x_n$ for any $n \in \mathbb{Z}$, then there exists a unique vertex x_{n_0} on the chain (ω', ω) with minimal distance to x. Suppose y_0, y_1, ..., y_m are the vertices on the path from x_{n_0} to x with $y_0 = x_{n_0}$ and $y_m = x$ and that y_{j+1} is a neighbour of y_j. Then we say that $x \in X^k$ if $y_{j+1} \in \tau_{y_j}(\{2, 3, ..., k + 1\})$ for all $j \in \{0, 1, ..., m-1\}$. Let $E^k = \{\{x, y\} \in E \mid x \in X^k,\, y \in X^k\}$. Note that (X^k, E^k) is a locally finite, homogeneous subtree of degree $k+1 \geq 3$ of the tree (X, E).

We now define the horicycle H_n^∞ of the tree (X, E) associated to the boundary point ω, for $n \in \mathbb{Z}$: For $x \in X$, let x_{n_0} be the vertex on the chain $(\omega^{'}, \omega)$ with minimal distance to x. Then we say that x belongs to the horicycle H_n^∞ if the equation $d(x, x_{n_0}) = d(x_n, x_{n_0})$ holds. Given $n \in \mathbb{Z}$ and an integer $k \geq 2$, set the horicycle H_n^k of the subtree (X^k, E^k) by $H_n^k = H_n^\infty \cap X^k$. Note that the families $\{H_n^k\}_{n\in\mathbb{Z}}$ and $\{H_n^\infty\}_{n\in\mathbb{Z}}$ give partitions of X^k and X respectively.

Given an integer $k \geq 2$, let $Aut(X^k)$ be the group of all automorphisms of the tree (X^k, E^k), i.e. the group of all bijections from X^k onto itself which preserve the edges. The group $Aut(X^k)$ is a metrizable Hausdorff topological group with the topology of compact convergence. The collection of sets $U_F(g) = \{h \in Aut(X^k) : h(x) = g(x)$ for all $x \in F\}$ forms a base for this topology where $g \in Aut(X^k)$ and $F \subset X^k$ is finite. Since X^k is a homogeneous tree of finite degree, the group $Aut(X^k)$ is locally compact. One can also see a fundamental system of compact open neighbourhoods of the identity in [19]. Given a vertex $x \in X$, let K_x^k be the subgroup of $Aut(X^k)$ consisting of all automorphisms fixing the vertex x. Then K_x^k is an open compact subgroup of $Aut(X^k)$ and the pair $(Aut(X^k), K_x^k)$ is a Gelfand pair (see [Proposition 2.3.2, [1]]). In particular, the pair $(Aut(X^k), K_{x_n}^k)$ is a Gelfand pair for all $n \in \mathbb{Z}$. Let

$$\mathrm{Stab}(H_n^k) = \{\, g \in Aut(X^k) \mid g(H_n^k) = H_n^k\} \text{ and } B_\omega^k = \cap_{n\in\mathbb{N}}\mathrm{Stab}(H_n^k) = \cap_{n\in\mathbb{Z}}\mathrm{Stab}(H_n^k).$$

It can be observed that

$$B_\omega^k = \{\, g \in Aut(X^k) \mid g(x_n) = x_n \text{ for n sufficiently large}\,\}.$$

Indeed, if $g \in B_\omega^k$ and $g(x_0) = x_0$, then $g(x_n) = x_n$ for all $n \geq 0$. If $g \in B_\omega^k$ and $g(x_0) = x$ for some $x \in H_n^k \backslash \{x_0\}$ and if x_m is the vertex on $(\omega^{'}, \omega)$ with minimal distance to x, then $g(x_n) = x_n$ for all $n \geq m$.

On the group B^k_ω we consider the topology induced from that of $Aut(X^k)$. Let $\{g_m\}_m$ be a sequence in $\mathrm{Stab}(H^k_n)$ converging to $g \in Aut(X^k)$. Let $x \in H^k_n$. Since $g_m \to g$, there exists $M \in \mathbb{N}$ such that $g_m \in U_{\{x\}}(g)$ for all $m \geq M$. In particular $g(x) = g_M(x) \in H^k_n$. So $g \in \mathrm{Stab}(H^k_n)$. This argument shows that $\mathrm{Stab}(H^k_n)$ is a closed subgroup of $Aut(X^k)$ for each $n \in \mathbb{Z}$. Then the group $B^k_\omega = \cap_{n\in\mathbb{Z}}\mathrm{Stab}(H^k_n)$ is also a closed subgroup of $Aut(X^k)$, hence B^k_ω is itself a locally compact topological group.

For all $n \in \mathbb{Z}$, set $B^k_n = B^k_\omega \cap K^k_{x_n}$. Then B^k_n is an open compact subgroup of B^k_ω for which we have the following proposition.

Proposition 4.1.1. *[18] The pair (B^k_ω, B^k_n) is a Gelfand pair for all integers $k \geq 2$ and for all $n \in \mathbb{Z}$.*

Proof. Let $k, n \in \mathbb{Z}$ be such that $k \geq 2$. Let $g \in B^k_\omega$. Suppose that $g \notin B^k_n$. Then since $B^k_\omega = \cup_{n\in\mathbb{Z}} B^k_n$ and $B^k_n \subset B^k_{n+1}$ for all $n \in \mathbb{Z}$, there exists $m > n$ such that $g \in B^k_m \backslash B^k_{m-1}$. Let $m_0 \in \mathbb{Z}$ be such that $m_0 < n < m$. Since $g(x_{m_0}) \neq x_{m_0}$ and

$$\begin{aligned} d(g(x_{m_0}), x_m) &= d(g(x_{m_0}), g(x_m)) \\ &= d(x_{m_0}, x_m) = d(g^{-1}(x_{m_0}), g^{-1}(x_m)) = d(g^{-1}(x_{m_0}), x_m), \end{aligned} \tag{4.1}$$

there exists $h \in B^k_{m_0}$ such that $hg(x_{m_0}) = g^{-1}(x_{m_0})$. This implies that $g^{-1} \in B^k_{m_0} g B^k_{m_0}$. Since $B^k_{m_0} \subset B^k_n$, we get $g^{-1} \in B^k_n g B^k_n$. If $g \in B^k_n$, obviously $g^{-1} \in B^k_n g B^k_n$. Hence for all $g \in B^k_\omega$, we have $g^{-1} \in B^k_n g B^k_n$. Then by Gelfand' s trick [Theorem 6.1.3, [26]], (B^k_ω, B^k_n) is a Gelfand pair. □

By using these Gelfand pairs (B^k_ω, B^k_n), we construct an Olshanski pair as follows:

Let $Aut(X)$ be the group of all automorphisms of the tree (X, E) and let

$$B^\infty_\omega = \{\, g \in Aut(X) \mid g(H^\infty_n) = H^\infty_n \text{ for all } n \in \mathbb{Z} \,\}.$$

The group $Aut(X)$ is a topological group with the compact-open topology. B_ω^∞ is also a topological group with the topology induced from that of $Aut(X)$.

We can embed $Aut(X^k)$ in $Aut(X)$ for each integer $k \geq 2$ as follows: Let $g \in Aut(X^k)$. We will extend g to an automorphism $\tilde{g}$ in $Aut(X)$ (which is called the natural extension of g), so we just need to define $\tilde{g}(x)$ for the vertices $x \in X \backslash X^k$ so that $\tilde{g} \in Aut(X)$. So let $x \in X \backslash X^k$ and let y be the vertex of X^k with minimal distance to x. Assume that $y_0, y_1, ..., y_m$ are the vertices on the path from y to x with $y_0 = y$, $y_m = x$ and $d(y_i, y_{i+1}) = 1$. Since $y_1, ..., y_m$ are not in X^k, for each $i \in \{0, 1, ..., m-1\}$ there exists an integer $k_i > k+1$ such that $y_{i+1} = \tau_{y_i}(k_i)$. We can now define $\tilde{g}(x) = \tilde{g}(y_m)$ where $\tilde{g}(y_m)$ is defined inductively by $\tilde{g}(y_0) = g(y)$ and $\tilde{g}(y_{i+1}) = \tau_{\tilde{g}(y_i)}(k_i)$ for all $i \in \{0, 1, ..., m-1\}$. Then,

Lemma 4.1.2. *[1] The map $\varphi_k : Aut(X^k) \to Aut(X)$ defined by $\varphi_k(g) = \tilde{g}$ is an injective group homomorphism, which is also a homeomorphism onto its image.*

For $g \in B_\omega^k$, we have $\tilde{g} \in B_\omega^\infty$. Hence if we let ψ_k to be the restriction of the map φ_k to B_ω^k, then as a corollary of Lemma 4.1.2, we have

Corollary 4.1.3. *The map $\psi_k : B_\omega^k \to B_\omega^\infty$ defined by $\psi_k(g) = \tilde{g}$ is an injective group homomorphism, which is a homeomorphism onto its image.*

Given $n \in \mathbb{Z}$, we identify B_ω^k and B_n^k with their images $\psi_k(B_\omega^k)$ and $\psi_k(B_n^k)$ in B_ω^∞ respectively. We define

$$B_\omega = \cup_{k=2}^\infty B_\omega^k \quad \text{and} \quad B_n = \cup_{k=2}^\infty B_n^k.$$

We endow B_ω with the inductive limit topology. Then,

Proposition 4.1.4. *The pair (B_ω, B_n) is an Olshanski spherical pair for all $n \in \mathbb{Z}$.*

Proof. By Proposition 4.1.1 and Corollary 4.1.3, we have a sequence of Gelfand pairs $(B_\omega^k, B_n^k)_{k \geq 2}$. Given $g \in B_\omega^k$, the automorphism g is the natural extension of its restriction onto the finite degree subtree X^m for each $m \geq k$. Hence $B_\omega^k \subset B_\omega^{k+1}$, $B_n^k \subset B_n^{k+1}$ and $B_n^k = B_n^{k+1} \cap B_\omega^k$ for all $k \geq 2$.

What is rest is to show that B_ω^k is a closed subgroup of B_ω^{k+1}. For this, we identify each $Aut(X^k)$ with its image in $Aut(X)$ under φ_k. By [Proposition 3.1.2, [1]], $Aut(X^k)$ is a closed subgroup of $Aut(X^{k+1})$. Both of the topologies on B_ω^{k+1} and $Aut(X^{k+1})$ are induced from the same topology of $Aut(X)$ and $B_\omega^{k+1} \subset Aut(X^{k+1})$, so the topology of B_ω^{k+1} is the one induced from the topology of $Aut(X^{k+1})$.
Hence $B_\omega^{k+1} \backslash B_\omega^k = (Aut(X^{k+1}) \backslash Aut(X^k)) \cap B_\omega^{k+1}$ is open in B_ω^{k+1} as desired. □

Within the rest of the text, we fix $n \in \mathbb{Z}$ and consider the Olshanski pair (B_ω, B_n). It is a natural programme to find all positive definite spherical functions and the corresponding spherical representations of this Olshanski pair.

4.2. Spherical Functions for (B_ω, B_n)

The spherical functions for the Gelfand pair $\left(B_\omega^k, B_n^k\right)$ are given by the following result of Nebbia.

Proposition 4.2.1. *[18] The non-trivial spherical functions for the Gelfand pair* $\left(B_\omega^k, B_n^k\right)$ *are the following:*

$$\varphi_{m,k} = 1_{B_m^k} + \frac{1}{1-k} 1_{B_{m+1}^k \backslash B_m^k} \tag{4.2}$$

for every $m \geq n$. *(*1_E *is the characteristic function of the set* E.*)*

Proof. Let φ be a non-trivial spherical function for the pair $\left(B_\omega^k, B_n^k\right)$. If $f, g \in B_m^k \backslash B_{m-1}^k$ for some $m > n$, then $d(f(x_{m-1}), x_n) = d(g(x_{m-1}), x_n)$, therefore $f(x_{m-1})$ can be sent to $g(x_{m-1})$ via an automorphism in B_n^k and so, $g \in B_n^k f B_n^k$. Hence complex-valued B_n^k-bi-invariant functions on B_ω^k are constant on $B_m^k \backslash B_{m-1}^k$ for all $m > n$ and also on B_n^k. In particular, φ is a linear combination of the characteristic functions $1_{B_n^k}$ and $1_{B_m^k \backslash B_{m-1}^k}$ for $m > n$. Let $\varphi(m)$ denote the value of φ on $B_m^k \backslash B_{m-1}^k$ for $m \geq n$.

Let μ be a Haar measure on B^k_ω such that $\mu(B^k_n) = 1$. Then since for each $i \in \mathbb{N}$, B^k_{n+i-1} is an index k subgroup of B^k_{n+i}, we have $\mu(B^k_{n+i}) = k^i$ so that $\mu(B^k_{n+i} \backslash B^k_{n+i-1}) = (k-1)k^{i-1}$. Let $\chi_i = (\lambda_i)^{-1} 1_{B^k_{n+i} \backslash B^k_{n+i-1}}$ where $\lambda_i = (k-1)k^{i-1}$ and $i \in \mathbb{N}$. Direct computation gives that for $j > i$, $\chi_i * \chi_j = \chi_j$. Then as the spherical functions are characters of the commutative convolution algebra $C_c(B^k_n \backslash B^k_\omega / B^k_n)$ of continuous compactly supported B^k_n-bi-invariant functions on B^k_ω, for $j > i$ we have

$$\begin{aligned}
\varphi(n+j) = \int_{B^k_\omega} \chi_j(f)\varphi(f)\ d\mu(f) &= \int_{B^k_\omega} \chi_i * \chi_j(f)\varphi(f)\ d\mu(f) \\
&= \int_{B^k_\omega} \chi_i(f)\varphi(f)\ d\mu(f) \int_{B^k_\omega} \chi_j(f)\varphi(f)\ d\mu(f) \\
&= \varphi(n+i)\varphi(n+j).
\end{aligned}$$

It can be immediately derived from the above equation that if $\varphi(n+i) = 0$ for some $i \in \mathbb{N}$, then $\varphi(n+j) = 0$ for all $j > i$ and moreover that if $\varphi(n+j) \neq 0$ for some non-zero $j \in \mathbb{N}$, then $\varphi(n+i) = 1$ for all $i < j$. Then since φ is non-trivial, we get

$$\varphi = 1_{B^k_{n+i}} + \alpha 1_{B^k_{n+i+1} \backslash B^k_{n+i}}$$

for some $\alpha \in \mathbb{C}$ and $i \in \mathbb{N}$. By [Proposition 6.1.6, [26]], for all $j > i+1$ there is a constant $c_j \in \mathbb{C}$ such that $\varphi * \chi_j = c_j \varphi$. But, for all $j > i+1$, $\varphi * \chi_j = [\alpha.(k-1)+1]k^i \chi_j$ as well. Then we get $[\alpha.(k-1)+1]k^i\chi_j = c_j\varphi$ which is possible for some $j > i+1$ if and only if $c_j = 0 = [\alpha.(k-1)+1]$. Hence $\alpha = \frac{1}{1-k}$ as desired.

Conversely, let $\varphi = 1_{B^k_{n+i}} + \frac{1}{1-k} 1_{B^k_{n+i+1} \backslash B^k_{n+i}}$ for some $i \in \mathbb{N}$. Then φ is B^k_n-bi-invariant. Since for each $m \in \mathbb{N}$ the group B^k_m is compact open, every characteristic function of the form $1_{B^k_m}$ and $1_{B^k_m \backslash B^k_{m-1}}$ is continuous, so is φ. Note that every function in $C_c(B^k_n \backslash B^k_\omega / B^k_n)$ is a finite linear combination of the characteristic functions $1_{B^k_n}$ and $1_{B^k_m \backslash B^k_{m-1}}$ for $m > n$. It can be verified directly that for all $j \in \mathbb{N}$, $\varphi * \chi_j = \chi_j * \varphi = c_j \varphi$ for some $c_j \in \{0, 1\}$ and that $\varphi * 1_{B^k_n} = 1_{B^k_n} * \varphi = \varphi$. Then by [Proposition 6.1.6, [26]], we get φ is spherical for $\left(B^k_\omega, B^k_n\right)$. $\square$

As we see in Proposition 4.2.1, the support of a spherical function for the Gelfand pair $\left(B_\omega^k, B_n^k\right)$ is one of the compact sets B_m^k for some $m > n$. Then by Proposition 4.2.2 which we next prove, all spherical functions for $\left(B_\omega^k, B_n^k\right)$ are positive definite.

Proposition 4.2.2. *[18] If φ is a compactly supported spherical function for a Gelfand pair (G, K), then φ is positive definite.*

Proof. Since φ is a continuous compactly supported K-bi-invariant function on G, so is φ^*. Hence both φ and φ^* are in $L^2(G, \mu)$ where μ is a Haar measure on G. Denote by L_G the left regular representation of G on $L^2(G, \mu)$. Then for all $x \in G$,

$$\begin{aligned}\varphi * \varphi^*(x) = \int_G \varphi(y)\varphi^*(y^{-1}x)\, d\mu(y) &= \int_G \varphi(y)\overline{\varphi(x^{-1}y)}\, d\mu(y)\\ &= \int_G \overline{\overline{\varphi}(y)} L_G(x)\overline{\varphi}(y)\, d\mu(y) = \langle L_G(x)\overline{\varphi}, \overline{\varphi}\rangle.\end{aligned}$$

Hence $\varphi * \varphi^*$ is a function of positive type. By [Proposition 6.1.6, [26]] we have $\varphi(e) = 1$ (here e is the identity element of G) and $\varphi * \varphi^* = \varphi^* * \varphi = \lambda\varphi$ for some complex number λ. Then since $\varphi \neq 0$,

$$\lambda = \varphi * \varphi^*(e) = \int_G \varphi(y)\varphi^*(y^{-1}e)\, d\mu(y) = \int_G \varphi(y)\overline{\varphi(y)}\, d\mu(y) = \|\varphi\|_2^2 > 0$$

where $\|\varphi\|_2$ is the L^2-norm of φ in $L^2(G, \mu)$. Thus $\varphi = (\lambda)^{-1}(\varphi * \varphi^*)$ is positive definite. $\square$

According to Theorem 22.10 in [20], every positive definite spherical function for an Olshanski spherical pair is the uniform limit on compact sets of positive definite spherical functions of the underlying Gelfand pairs. Hence, for each $m \geq n$ we consider the following pointwise limits

$$\lim_{k\to\infty} \varphi_{m,k} = 1_{B_m}$$

as candidates of positive definite spherical functions of the Olshanski pair (B_ω, B_n).

Next we observe that the set $\{1_{B_m} : \ m \geq n\}$ consists of all non-trivial spherical functions for (B_ω, B_n).

Theorem 4.2.3. *The non-trivial spherical functions for the pair (B_ω, B_n) are the characteristic functions 1_{B_m} with $m \geq n$. The spherical functions are all positive definite.*

Proof. Let $\varphi : B_\omega \to \mathbb{C}$ be a spherical function. If $g, h \in B_m \backslash B_{m-1}$ for some integer $m > n$, then the equality $d(x_n, g(x_n)) = d(x_n, h(x_n))$ holds. Hence, there exists $k \in B_n$ satisfying $k(g(x_n)) = h(x_n)$. This shows that $h^{-1}kg \in B_n$, i.e. $g \in B_n h B_n$ and that $\varphi(g) = \varphi(h)$ by the B_n-bi-invariance of φ. So, φ is constant on B_n and $B_m \backslash B_{m-1}$ for all $m > n$. Let us denote by $\varphi(n)$ the value of φ on B_n and by $\varphi(m)$ the value of φ on $B_m \backslash B_{m-1}$ given $m > n$.

Now, fix two integers m and p such that $m > p \geq n$. Let $g \in B_p \backslash B_{p-1}$ and $h \in B_m \backslash B_{m-1}$. Choose L such that $g, h \in B_m^L$. If $k > L$ and $l \in B_n^k$, then $glh(x_m) = gl(x_m) = g(x_m) = x_m$, whereas $glh(x_{m-1}) \neq x_{m-1}$ because otherwise $h(x_{m-1}) = l^{-1}g^{-1}(x_{m-1}) = l^{-1}(x_{m-1}) = x_{m-1}$ which is impossible as $h \in B_m \backslash B_{m-1}$. Hence, $glh \in B_m \backslash B_{m-1}$ and $\varphi(glh) = \varphi(m)$ for all $l \in B_n^k$. Since φ is spherical, we then get

$$\varphi(p)\varphi(m) = \varphi(g)\varphi(h) = \lim_{k \to \infty} \int_{B_n^k} \varphi(glh)\ dl = \lim_{k \to \infty} \int_{B_n^k} \varphi(m)\ dl = \varphi(m).$$

where dl is the normalized Haar measure on B_n^k.

By the equality above, we conclude that if $\varphi(p) = 0$ for $p \geq n$, then $\varphi(m) = 0$ for all $m > p$ and that if $\varphi(m) \neq 0$ for some $m > n$, then $\varphi(p) = 1$ for all $p < m$. So, if $\varphi \neq 0$ and $\varphi \neq 1$, then either $\varphi = \alpha 1_{B_n}$ for some $\alpha \in \mathbb{C}^*$ or $\varphi = 1_{B_{m-1}} + \alpha 1_{B_m \backslash B_{m-1}}$ for some $\alpha \in \mathbb{C}^*$ and $m > n$.

To compute α in case $\varphi = \alpha 1_{B_n}$, take $g \in B_n$. Since φ is spherical,

$$\alpha^2 = \varphi(g)^2 = \lim_{k \to \infty} \int_{B_n^k} \varphi(glg)\ dl = \lim_{k \to \infty} \int_{B_n^k} \varphi(n)\ dl = \varphi(n) = \alpha$$

so that $\alpha = 1$ and $\varphi = 1_{B_n}$.

In case $\varphi = 1_{B_{m-1}} + \alpha 1_{B_m \backslash B_{m-1}}$, take $g \in B_m \backslash B_{m-1}$. There exists L such that $g \in B^L_m \backslash B^L_{m-1}$. Given $k > L$, let $y_1, y_2, ..., y_{k-1}$ be the neighbours of x_m in X^k except from x_{m-1} and x_{m+1}. We may assume that $y_1 = g^{-1}(x_{m-1})$. For $l \in B^k_n$, $l(g(x_{m-1}))$ may take one of the $k-1$ values $y_1, y_2, ..., y_{k-1}$. Therefore, B^k_n is the disjoint union of the sets $A_1, A_2, ..., A_{k-1}$ where $A_i = \{l \in B^k_n : l(g(x_{m-1})) = y_i\}$. Choosing an automorphism $k_{i,j}$ in B^k_n such that $k_{i,j}(y_i) = y_j$, we observe that $k_{i,j}(A_i) = A_j$. It follows that each A_i has the same measure $\frac{1}{k-1}$ with respect to the normalized Haar measure on B^k_n. Then,

$$\begin{aligned}
\int_{B^k_n} \varphi(glg)\ dl &= \int_{A_1} \varphi(glg)\ dl + \int_{B^k_n \backslash A_1} \varphi(glg)\ dl \\
&= \int_{A_1} 1\ dl + \int_{B^k_n \backslash A_1} \varphi(m)\ dl \\
&= \frac{1}{k-1} + \frac{k-2}{k-1}\alpha
\end{aligned}$$

and together with the fact that φ is spherical we get

$$\alpha^2 = (\varphi(m))^2 = (\varphi(g))^2 = \lim_{k\to\infty} \int_{B^k_n} \varphi(glg)\ dl = \lim_{k\to\infty} (\frac{1}{k-1} + \frac{k-2}{k-1}\alpha) = \alpha.$$

So, $\alpha = 1$ and $\varphi = 1_{B_{m-1}} + 1_{B_m \backslash B_{m-1}} = 1_{B_m}$.

Conversely, let $\varphi = 1_{B_m}$ for $m \geq n$. Since B_m is an open subgroup of B_ω and $B_n \subseteq B_m$, the function 1_{B_m} is continuous and B_n-bi-invariant.

Now, let $g, h \in B_\omega$ and $\epsilon > 0$. There exists L such that $g, h \in B^L_\omega$. Since for all m and k, the function $\varphi_{m,k}$ given in the equation (4.2) is spherical for the Gelfand pair (B^k_ω, B^k_n), for all $k > L$ we get

$$\begin{aligned}
&\left| \int_{B^k_n} 1_{B_m}(glh) - 1_{B_m}(g)1_{B_m}(h)\ dl \right| \\
&\leq \int_{B^k_n} |1_{B_m}(glh) - \varphi_{m,k}(glh)|\ dl + |\varphi_{m,k}(g)\varphi_{m,k}(h) - 1_{B_m}(g)1_{B_m}(h)|
\end{aligned} \tag{4.3}$$

But, $|1_{B_m}(glh) - \varphi_{m,k}(glh)|$ is either 0 or $\frac{1}{k-1}$ for any $l \in B_\omega$ and we can choose $N_1 > L$ so that for all $k > N_1$ we have $\frac{1}{k-1} < \frac{\epsilon}{2}$. Since $\varphi_{m,k} \to 1_{B_m}$ as $k \to \infty$, we can choose $N_2 > N_1$ so that for all $k > N_2$, both $|\varphi_{m,k}(g) - 1_{B_m}(g)| < \frac{\epsilon}{4}$ and $|\varphi_{m,k}(h) - 1_{B_m}(h)| < \frac{\epsilon}{4}$. Hence by (4.3) for all $k > N_2$, $\left|\int_{B_n^k} 1_{B_m}(glh) - 1_{B_m}(g)1_{B_m}(h)\, dl\right| < \epsilon$ and this shows 1_{B_m} is spherical for the Olshanski pair (B_ω, B_n).

Every spherical function 1_{B_m} is positive definite because given an arbitrary group G and a subgroup H of G, the characteristic function 1_H of H is positive definite. $\square$

Remark 4.2.4. The restriction of the (B_ω, B_n)-spherical function 1_{B_m} with $m \geq n$ to B_ω^k is the characteristic function $1_{B_m^k}$ which is a non-spherical function for $\left(B_\omega^k, B_n^k\right)$ by Proposition 4.2.1. Hence, the restriction of a spherical function for an Olshanski pair to an underlying locally compact group need not to be spherical for the corresponding Gelfand pair.

4.3. Spherical Representations for (B_ω, B_n)

Now, we will make concrete realizations of the spherical representations of the Olshanski pair (B_ω, B_n) which correspond to the positive definite spherical functions for the pair (B_ω, B_n) by the Gelfand-Naimark-Segal construction.

Given $m \in \mathbb{Z}$, consider the horicycle H_m^∞ containing the vertex x_m. The group B_ω acts transitively on the horicycle H_m^∞. Indeed, if $x, y \in H_m^\infty$, let $m_1 \geq m$ be such that x_{m_1} is the unique vertex on (ω', ω) with minimal distance to x and $m_2 \geq m$ be such that x_{m_2} is the unique vertex on (ω', ω) with minimal distance to y. Assume $m_1 \geq m_2$ and $x, y \in X^k$ for $k \in \mathbb{Z}$. Then we can find an automorphism $g \in Aut(X^k)$ such that $g(x) = y$ and $g(x_n) = x_n$ for all $n \geq m_1$. So, the natural extension $\tilde{g}$ of g is an automorphism in B_w which sends x to y.

The spherical function $\varphi = 1$ corresponds to the one-dimensional trivial representation.

For the spherical function $\varphi_m = 1_{B_m}$ with $m \geq n$, consider the Hilbert space $l^2(H_m^\infty) = L^2(H_m^\infty, \lambda_m)$ where λ_m is the counting measure on H_m^∞. By the action of B_ω on the horicycle H_m^∞, we get a representation π_m of B_ω on the Hilbert space $l^2(H_m^\infty)$ if we define $(\pi_m(g)f)(x) = f(g^{-1}(x))$ where $g \in B_\omega$, $f \in l^2(H_m^\infty)$ and $x \in H_m^\infty$. Indeed, π_m is a group homomorphism from B_ω to the group of unitary operators on the Hilbert space $l^2(H_m^\infty)$. π_m is moreover a continuous representation of B_ω on $l^2(H_m^\infty)$ where we put the strong operator topology on $U(l^2(H_m^\infty))$: Since $\left\{1_{\{x\}} : x \in H_m^\infty\right\}$ form an orthonormal basis for $l^2(H_m^\infty)$, it suffices to prove that the map $g \longmapsto \pi_m(g)1_{\{x\}}$ is continuous for each $x \in H_m^\infty$. Given $x \in H_m^\infty$ and $g_0 \in B_\omega$, the set $U_0 = \{g \in B_\omega : g(x) = g_0(x)\}$ is open in the topology induced from $Aut(X)$. The inductive limit topology on B_ω is stronger than the topology induced from $Aut(X)$. So, the set U_0 is also open in the inductive limit topology. The map $g \longmapsto \pi_m(g)1_{\{x\}}$ is constant on the open set U_0. So, the map $g \longmapsto \pi_m(g)1_{\{x\}}$ is locally constant, hence continuous as desired. Therefore, π_m defines a unitary representation of B_ω.

$1_{\{x_m\}}$ is a B_n-bi-invariant unit vector in $l^2(H_m^\infty)$. By the transitive action of B_ω on $l^2(H_m^\infty)$, we have

$$\begin{aligned}
\overline{span\left\{\pi_m(g)1_{\{x_m\}} : g \in B_\omega\right\}} &= \overline{span\left\{1_{\{g(x_m)\}} : g \in B_\omega\right\}} \\
&= \overline{span\left\{1_{\{y\}} : y \in H_m^\infty\right\}} \\
&= l^2(H_m^\infty)
\end{aligned}$$

so that $1_{\{x_m\}}$ is also a cyclic vector.

Moreover,

$$\begin{aligned}
< 1_{\{x_m\}}, \pi_m(g)1_{\{x_m\}} > &= \sum_{y \in H_m^\infty} 1_{\{x_m\}}(y).\overline{\pi_m(g)1_{\{x_m\}}(y)} \\
&= \sum_{y \in H_m^\infty} 1_{\{x_m\}}(y).1_{\{x_m\}}(g^{-1}y) \\
&= 1_{\{x_m\}}(g^{-1}x_m) \\
&= 1_{B_m}(g).
\end{aligned}$$

We summarize our observations in the following theorem.

Theorem 4.3.1. *The spherical representation of the Olshanski pair (B_ω, B_n) corresponding to the spherical function $\varphi = 1$ by the Gelfand-Naimark-Segal construction is the one-dimensional trivial representation.*

The spherical representation of the Olshanski pair (B_ω, B_n) corresponding to the spherical function $\varphi_m = 1_{B_m}$, $m \geq n$ by the Gelfand-Naimark-Segal construction is the left regular representation π_m on $l^2(H_m^\infty)$.

5. CONCLUSION

The problem which initiated this thesis and which has always been in mind during the study is the following: "Is it possible to construct an algebra structure whose characters correspond to the bounded spherical functions for a general Olshanski spherical pair?". Two facts drived us to seek for such a result. The first one is the existence of a positive answer in the case of Gelfand pairs as we have mentioned and used in several places in the thesis. Secondly, Gelfand pairs are the building blocks of Olshanski pairs. This abstract problem created the need to study spherical harmonic analysis on some concrete examples of infinite dimensional spherical pairs. We first studied an Olshanski pair related to the group of isometries of homogeneous trees of infinite degree and we found all spherical functions and spherical representations of that pair. Then we considered the Olshanski pair $(U(\infty) \ltimes H(\infty), U(\infty))$ related to Heisenberg groups. Inspired by the ideas and works of J. Faraut, we calculated all positive definite spherical functions for this pair. On the way to make realizations of the corresponding irreducible unitary representations, we have developed a couple of approaches to define the analogue of the Fock space $\mathcal{F}_\lambda^\infty$ which works in the case of infinitely many complex variables. They both carry meaningful representations of $H(\infty)$. The question of whether these representations are irreducible and they can be extended to representations of $U(\infty) \ltimes H(\infty)$ which correspond to the positive definite spherical functions we have is still waiting to be worked out. It also seems that to construct the algebraic counterpart of harmonic analysis on Olshanski pairs will be challenging.

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