VERTICAL DISTRIBUTION PROBLEMS IN ZETA-FUNCTION THEORY

by

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M.S., Mathematics, Boğaziçi University, 2009

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics

Boğaziçi University

2015

ACKNOWLEDGEMENTS

First of all, I would like to express my deepest gratitude to my supervisor Cem Yalçın Yıldırım for his excellent guidance, valuable suggestions, encouragements, and patience. I am glad to have the chance to study with him.

Besides my supervisor, I would like to thank the rest of my thesis committee for careful reading of the text and valuable remarks.

I am grateful to my family and friends for their encouragements and supports.

Finally, I would like to thank TUBITAK for supporting me during the Ph.D. program.

ABSTRACT

VERTICAL DISTRIBUTION PROBLEMS IN ZETA-FUNCTION THEORY

In this thesis, we focus on the vertical distribution problems of the zeros of the Riemann zeta-function and other related functions. In the first half of our study we modify Montgomery's argument [1] in such a way that we can obtain some analogues of the pair correlation of zeta zeros, which provide some gap and multiplicity results. In the second half of our study we estimate the averages studied in [2] over the zeta maximas on the critical line instead of zeros so that we arrive at a result on the number of distinct zeta zeros.

ÖZET

ZETA-FONKSİYONU TEORİSİNDE DİKEY DAĞILIM PROBLEMLERİ

Bu tezde Riemann zeta ve diğer alakalı fonksiyonların sıfırlarının dikey dağılımı problemlerine odaklandık. Çalışmamızın ilk yarısında boşluk ve çok katlılık sonuçları sağlayan zeta sıfırlarının ikili korelasyonu analoglarını elde edebilecek şekilde Montgomery'nin argümanını [1] değiştirdik. Çalışmamızın ikinci yarısında [2]'de çalışılmış olan averajları zeta sıfırları yerine $1/2-$doğrusundaki zeta maksimumları üzerinden hesapladık öyle ki farklı zeta sıfırlarının sayısı üzerine bir netice elde ettik.

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LIST OF SYMBOLS

$e(\theta)$	$= e^{2\pi i\theta}$
$f^{(j)}(s)$	$j^{\text{ th}}$ derivative of $f(s)$, $f^{(0)} = f$.
$L(s,\chi)$	A Dirichlet L-function.
$N(T)$	The number of complex zeros of $\zeta(s)$ lying in the upper-half plane up to the height T.
$N_s(T)$	The number of simple complex zeros of $\zeta(s)$ lying in the upper-half plane up to the height T.
$N_d(T)$	The number of distinct complex zeros of $\zeta(s)$ lying in the upper-half plane up to the height T.
$Z_1(s)$	$= \zeta'(s) - \frac{1}{2}\frac{\chi'}{\chi}(s)\zeta(s)$.
β	The real part of a complex zero of the zeta function.
$\Gamma(s)$	$= \int_0^\infty e^{-x}x^{s-1}dx$ for $\sigma > 0$; called the *Gamma function*.
γ	The imaginary part of a complex zero of the zeta function.
$\zeta(s)$	The Riemann zeta-function.
$\Lambda(n)$	$= \log p$ if $n = p^k$, $= 0$ otherwise; known as the *von Mangoldt Lambda function*.
$\Lambda^{*(j)}(n)$	The $j-$th convolution of Λ; generated by the Dirichlet series $\left(-\frac{\zeta'}{\zeta}(s)\right)^j$, $j \in \mathbb{N}$.
$\mu(n)$	$= (-1)^{\omega(n)}$ for square-free n, $n = 0$ otherwise. Known as the *Möbius mu function*.
ρ	$= \beta + i\gamma$; a complex zero of the zeta function.
τ	$= \|t\| + 4$.
$\tau_k(n)$	The generalized divisor function defined by the relation $\sum_{n\geq 1}\frac{\tau_k(n)}{n^s} = (\zeta(s))^k$, where $k \in \mathbb{N}$.
υ	The imaginary part of the complex zeros of $Z_1(s)$.
$\phi(n)$	The number a, $1 \leq a \leq n$, for which $(a,n) = 1$; knowns as *Euler's Totient function*.
$\chi(n)$, $\psi(n)$	A Dirichlet character.
$\chi(s)$	The function defined by the functional equation for $\zeta(s)$.
$\omega(n)$	The number of distinct primes dividing n.

ϱ	A complex zero of $Z_1(s)$.
$[\![x]\!]$	The unique integer such that $[\![x]\!] \leq x < [\![x]\!] + 1$; called the *integer part* of x.
$\{x\}$	$= x - [\![x]\!]$; called the *fractional part* of x.
$f(x) = O(g(x))$	$\lvert f(x)\rvert \leq C\lvert g(x)\rvert$, where C is an absolute constant.
$f(x) \ll g(x)$	$f(x) = O(g(x))$.
$f(x) = O_{\alpha_1,\alpha_2,\ldots}(g(x))$	$\lvert f(x)\rvert \leq Cg(x)$, where C is a constant which depends on α_1, $\alpha_2, \ldots$.
$f(x) \ll_{\alpha_1,\alpha_2,\ldots} g(x)$	$f(x) = O_{\alpha_1,\alpha_2,\ldots}(g(x))$.
$f(x) \gg g(x)$	$g(x) = O(f(x))$.
$f(x) \gg_{\alpha_1,\alpha_2,\ldots} g(x)$	$g(x) = O_{\alpha_1,\alpha_2,\ldots}(f(x))$.
$f(x) \asymp g(x)$	$f(x) = O(g(x))$ and $g(x) = O(f(x))$.
$f(x) \sim g(x)$	$\lim_{x\to\infty} f(x)/g(x) = 1$.

LIST OF ACRONYMS/ABBREVIATIONS

GRH	The generalized Riemann Hypothesis.
RH	The Riemann Hypothesis.

1. INTRODUCTION AND STATEMENT OF RESULTS

The Riemann zeta-function, defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s > 1,$$

plays a prominent role in Number Theory. The first unsolved problem occuring to our minds pertaining to $\zeta(s)$ is the Riemann Hypothesis (abbreviated by RH) stating that all complex zeros of $\zeta(s)$ lie on the critical line $\Re s = 1/2$. RH says everything about the horizontal distribution of the zeros. However, even if we assume RH, there remains the vertical distribution problem as to how complex zeros are distributed on the critial line. In this thesis we focus on some problems about the vertical distribution of zeta zeros on the critical line.

Let $\rho = \beta + i\gamma$ run through the nontrivial zeros (i.e. complex zeros) of $\zeta(s)$ and $m_\zeta(\rho)$ denote the multiplicity of ρ. The Riemann-von Mangoldt formula states that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{1.1}$$

where $N(T)$ is the number of zeros of $\zeta(s)$ with $0 < \gamma < T$. We also have some other countings:

$$\begin{aligned} N_s(T) &= |\{\rho : 0 < \gamma < T,\, \zeta(\rho) = 0,\, m_\zeta(\rho) = 1\}|, \\ N_d(T) &= |\{\rho : 0 < \gamma < T,\, \zeta(\rho) = 0\}|, \\ N_0(T) &= |\{\rho : 0 < \gamma < T,\, \zeta(\rho) = 0,\, \beta = 1/2\}|. \end{aligned}$$

It readily follows from (1.1) that the average gap between consecutive zeros is $\sim 2\pi/\log T$. In understanding the distribution of the zeros on the critical line, we have two significant quantities:

$$C_S := \liminf_{n\to\infty} \frac{(\gamma_{n+1}-\gamma_n)\log\gamma_n}{2\pi} \quad \text{and} \quad C_L := \limsup_{n\to\infty} \frac{(\gamma_{n+1}-\gamma_n)\log\gamma_n}{2\pi}, \tag{1.2}$$

where γ_n represents the imaginary part of the n-th zero in the upper half plane. Although there is no need to put any restriction on the real parts of the zeros when studying these quantities, RH is assumed in most calculations.

It is conjectured that $C_S = 0$ and $C_L = \infty$, which indicates the existence of arbitrarily small and large gaps between zeros of $\zeta(s)$. In capturing small differences Montgomery [1] introduces the double sum over zeta zeros

$$F_{\zeta,\zeta}(x,T) = \sum_{0<\gamma,\tilde{\gamma}\le T} x^{i(\gamma-\tilde{\gamma})} w(\gamma-\tilde{\gamma}), \quad x>0, \tag{1.3}$$

or the version with the normalizer and the substitution $x = T^\alpha$

$$F_{\zeta,\zeta}(\alpha) = F_{\zeta,\zeta}(\alpha,T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\tilde{\gamma}\le T} T^{i\alpha(\gamma-\tilde{\gamma})} w(\gamma-\tilde{\gamma}), \tag{1.4}$$

where $\tilde{\gamma}$ is the ordinate of a nontrivial zero of $\zeta(s)$, and $w(u) = 4/(4+u^2)$ is a suitable weight function. Concerning $F_{\zeta,\zeta}(\alpha)$ assuming RH he computed

$$F_{\zeta,\zeta}(\alpha) = (1+o(1))T^{-2|\alpha|}\log T + |\alpha| + o(1), \tag{1.5}$$

as $T\to\infty$, uniformly for $|\alpha| \le 1-\epsilon$. By convolving $F_{\zeta,\zeta}(\alpha)$ with appropriate kernels, Montgomery deduced that $C_S \le 0.68$ and

$$N_s(T) \ge (\frac{2}{3}+o(1))\frac{T\log T}{2\pi}. \tag{1.6}$$

The analogous process leading to these conclusions will be seen in §15. However, with a completely different method, Montgomery and Odlyzko improved to $C_S \le 0.5179$, and there are some small improvements following the same method. The current record is

0.5155, due to H. M. Bui, M. B. Milinovich and N. C. Ng. There is a barrier at 0.5.

The small gap problem is itself of interest, however, there is also some extra motivation behind these attempts in breaking the barrier at 0.5. This problem is closely related to the class number problem and Siegel zeros. The existence of consecutive zeros whose distance is less than the half of the average implies not only remarkably improved but also effective lower bounds for the value of Dirichlet-L functions at 1, which clear zeros off the larger line segment near 1, possibly the Siegel zeros. For detailed treatises on the subject we refer the reader to [3] and [4].

The first half of our thesis is devoted to studying some analogues of $F_{\zeta,\zeta}$. In §6-8 Montgomery's argument is modified in such a way that beyond obtaining a different proof of (1.5) we give a new method of correlating the zeros of two (possibly different) functions. These three sections were originally presented in [5].

In the second half of our study we focus on the multiplicity of the Riemann zeros. It is conjectured that all zeros are simple, in other words,

$$N(T) = N_s(T) = N_d(T).$$

We remark that this problem is also associated with the gap problems. Obviously, if there are infinitely many non-simple zeros, then $C_S = 0$. Besides the conditional result (1.6), Montgomery pointed out that in addition to RH assuming the Pair Correlation Conjecture, concerned with the behaviour of $F_{\zeta,\zeta}(\alpha)$ outside $\alpha \in [-1, 1]$, it follows that

$$N_s(T) \sim \frac{T \log T}{2\pi}, \tag{1.7}$$

that's to say that almost all zeros are simple. Conrey, Ghosh and Gonek [2] developed a different approach to this problem. Their starting point is a simple Cauchy-Schwarz inequality application:

$$|\sum_{0<\gamma\le T} B\zeta'(\rho)|^2 \le N_s(T)\sum_{0<\gamma\le T}|B\zeta'(\rho)|^2,$$

where

$$B(s)=\sum_{n\le y}\frac{b(n)}{n^s}, \qquad\qquad b(n)=\mu(n)P\left(\frac{\log\frac{y}{n}}{\log y}\right),$$

$y=T^\theta$, and $P(\cdot)$ is a polynomial with real coefficients which satisfies $P(0)=0$, $P(1)=1$. Assuming the Generalized Lindelöf Hypothesis and RH, they calculated the above two averages for $\theta<1/2$ as

$$\sum_{0<\gamma\le T} B\zeta'(\rho)\sim\left(\frac{1}{2}+\theta\int_0^1 P(x)dx\right)\frac{T(\log T)^2}{2\pi},$$
$$\sum_{0<\gamma\le T}|B\zeta'(\rho)|^2\sim\frac{\mathfrak{s}T(\log T)^3}{2\pi},$$

where

$$\mathfrak{s}=\frac{1}{3}+\left(\theta\int_0^1 P(x)dx\right)^2+\theta\int_0^1 P(x)dx+\frac{1}{2\theta}\int_0^1\left(P'(x)\right)^2dx. \tag{1.8}$$

The calculus of variations gives the optimal choice $P(x)=-\theta x^2+(1+\theta)x$. With this choice,

$$N_s(T)\ge\left(\frac{19}{27}+o(1)\right)\frac{T\log T}{2\pi}. \tag{1.9}$$

They also observed that

$$2N_s(T)\le\sum_{0<\gamma\le T}\frac{(m_\zeta(\rho)-2)(m_\zeta(\rho)-3)}{m_\zeta(\rho)}=\sum_{0<\gamma\le T}m_\zeta(\rho)-5N(T)+6N_d(T).$$

Combining this with (1.9) and the result

$$\sum_{0<\gamma\le T} m_\zeta(\rho) \le (4/3+o(1))N(T),$$

which was proven in [1], they derived that

$$N_d(T) \ge \left(\frac{5}{6}+o(1)\right)\frac{T\log T}{2\pi}. \tag{1.10}$$

Our plan is to estimate the same averages over the complex zeros of $Z_1(s)$, defined by

$$Z_1(s) := \zeta'(s) - \frac{1}{2}\frac{\chi'}{\chi}(s)\zeta(s). \tag{1.11}$$

Let ϱ denote the complex zeros of $Z_1(s)$, $\upsilon = \Im\varrho$, and $m_{Z_1}(\varrho)$ the multiplicity of ϱ. Now the application of the Cauchy-Schwarz inequality produces $N_d(T)$ instead of $N_s(T)$. More precisely,

$$|\sum_{0<\Im\varrho\le T} B\zeta'(\varrho)|^2 \le (N_d(T)+O(1)) \sum_{0<\Im\varrho\le T} |B\zeta'(\varrho)|^2. \tag{1.12}$$

To see this we must show that

$$\sum_{\substack{0<\Im\varrho\le T\\ \zeta'(\varrho)\ne 0}} 1 = N_d(T) + O(1).$$

We first note that if ρ is a common zero of $\zeta(s)$ and $Z_1(s)$, then it is seen directly from the definition of $Z_1(s)$ that $m_{Z_1}(\rho) = m_\zeta(\rho) - 1$. So,

$$\sum_{\substack{0<\Im\varrho\leq T\\ \zeta'(\varrho)\neq 0}} 1 = \sum_{0<\Im\varrho\leq T} 1 - \sum_{\substack{0<\Im\varrho\leq T\\ \zeta'(\varrho)=0}} 1 = \sum_{0<\Im\varrho\leq T} 1 - \sum_{0<\Im\rho\leq T} \frac{m_\zeta(\rho)-1}{m_\zeta(\rho)}$$
$$= \sum_{0<\Im\rho\leq T} \frac{1}{m_\zeta(\rho)} + O(1) = N_d(T) + O(1).$$

Here we've appealed to Hall's Z_1-analogue of (1.1), which will be seen in §3.

In §20 we prove that on GRH (the Generalized Riemann Hypothesis)

$$\sum_{0<\Im\varrho\leq T} B\zeta'(\varrho) = \mathfrak{R}_1(P,\theta)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2 + O\left(T(\log T)^{3/2}(\log\log T)^A\right) \tag{1.13}$$

and

$$\sum_{0<\Im\varrho\leq T} |B\zeta'(\varrho)|^2 = \mathfrak{R}_2(P,\theta)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^3 + O\left(T(\log T)^{5/2}(\log\log T)^A\right), \tag{1.14}$$

where

$$\mathfrak{R}_1(P,\theta) := \frac{3-e^2}{4} + \sum_{i_1=1}^{k} a_{i_1} i_1! \sum_{\kappa'\geq 1} \frac{(-2\theta)^{\kappa'}\left(\frac{F_{1,1}(\kappa'+1;2;2)}{2} - F_{1,1}(\kappa'+1;3;2)\right)}{(i_1+\kappa')!} \tag{1.15}$$

and

$$\mathfrak{R}_2(P,\theta) := \frac{e^2-5}{8\theta}\int_0^1 (P'(t))^2\,dt - \theta\int_0^1 (P(t))^2\,dt + \frac{e^2-5}{4} \tag{1.16}$$
$$-\frac{1}{2}\sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2} i_1! i_2! \sum_{j_5=0,1} \frac{\theta^{-[j_5=0]}}{(i_2-[j_5=0])!(3-j_5)!}$$
$$\sum_{\kappa'\geq 2} \frac{(-2\theta)^{\kappa'}\left((3-j_5)F_{1,1}(\kappa'+1;3-j_5;2) - 4F_{1,1}(\kappa'+1;4-j_5;2)\right)}{(i_1+\kappa'+i_2-[j_5=0])(i_1+\kappa'-1)!}.$$

Here and throughout the work we frequently use the Iverson notation that for a statement S, $[S]=1$ if S is true, and $[S]=0$ if S is false. Here $F_{1,1}$ denotes the confluent hypergeometric series, defined as

$$F_{1,1}(a;b;z) := \sum_{n=0}^{\infty} \frac{a_{(n)}z^n}{b_{(n)}n!},$$

where

$$a_{(n)} = \prod_{j=0}^{n-1}(a+j).$$

Combining (1.12), (1.13) and (1.14), we see that

$$N_d(T) \geq \frac{(\mathfrak{R}_1(P,\theta))^2}{\mathfrak{R}_2(P,\theta)}(1+o(1))\frac{T}{2\pi}\log T. \tag{1.17}$$

In §21 we try to obtain a lower bound as large as possible for $N_d(T)$ by choosing appropriate P. Among the polynomials P with $\deg P = 3$, $P(0)=0$, $P(1)=1$, the optimal choice is

$$P(x) = a_1x + a_2x^2 + a_3x^3,$$

where

$$a_1 = 0.75816\cdots, \; a_2 = 0.267977\cdots, \; a_3 = -0.0261367\cdots.$$

With this choice, (1.17) becomes

$$N_d(T) \geq 0.7734\cdots(1+o(1))\frac{T}{2\pi}\log T. \tag{1.18}$$

To derive a result on $N_s(T)$ from (1.18), first observe that

$$\begin{aligned}N_d(T) &= N_s(T) + \frac{1}{2}\sum_{0<\gamma\le T,\, m_\zeta(\rho)=2} 1 + \frac{1}{3}\sum_{0<\gamma\le T,\, m_\zeta(\rho)=3} 1 + \cdots \\ &\le N_s(T) + \frac{N(T)-N_s(T)}{2}.\end{aligned}$$

Employing (1.18) we get

$$N_s(T) \ge 0.5468\cdots(1+o(1))\frac{T}{2\pi}\log T. \tag{1.19}$$

2. BASIC FACTS and NOTATION

We write $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$. We have the convention $0^0 = 1$. Let $s = \sigma + it$, w denote complex variables, and $\tau = |t| + 4$. Throughout the article ϵ and A denote arbitrarily small positive and sufficiently large positive numbers, respectively. The constant implied by $O-$term may depend on ϵ, A. We use paranthesis to show the dependence of one variable or constant on some others. The constants denoted by the same symbol need not have the same value at each occurrence. If a power series or a partial sum of it is in question, then the index of the sum starts with 0. However, as regards a Dirichlet series or summatory functions of arithmetic functions, we make the index of the sum start with 1. It will be useful to set down certain formulae and estimates.

The functional equation of the Riemann zeta-function can be expressed in the asymmetric form

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{2.1}$$

where

$$\chi(s) := 2^s \pi^{-1+s} \sin\left(\frac{1}{2}s\pi\right)\Gamma(1-s) \tag{2.2}$$

$$= \pi^{s-\frac{1}{2}} \frac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma(\frac{1}{2}s)}. \tag{2.3}$$

Firstly, we state two well-known asymptotic formulas involving Γ function:

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right), \tag{2.4}$$

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right). \tag{2.5}$$

These formulas are valid as $|s| \to \infty$, in the angle $-\pi + \delta < \arg s < \pi - \delta$, for any fixed $\delta > 0$. By using the above formulas and some fundamental properties of $\Gamma-$function, it is easy to show that

$$\frac{\chi'}{\chi}(s) = \log 2\pi - \frac{\Gamma'}{\Gamma}(s) + \frac{\pi}{2}\tan\frac{\pi s}{2} = \log 2\pi - \frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2}\cot\frac{\pi s}{2}, \tag{2.6}$$

$$\chi(s) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\left(-it\log\frac{|t|}{2\pi e} + \frac{i\pi}{4}\operatorname{sgn}(t)\right)\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{2.7}$$

$$\frac{\chi'}{\chi}(s) = -\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right), \tag{2.8}$$

the last two of which holds uniformly in $\alpha \leq \sigma \leq \beta$ and $|t| \geq 1$, for any fixed real numbers α and β, where

$$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if } \ t > 0 \\ 0 & \text{if } \ t = 0 \\ -1 & \text{if } \ t < 0. \end{cases}$$

From (2.6),

$$\frac{d}{ds}\frac{\chi'}{\chi}(s) = -\frac{d}{ds}\frac{\Gamma'}{\Gamma}(s) + \left(\frac{\pi}{2}\sec\frac{\pi s}{2}\right)^2. \tag{2.9}$$

The trigonometric part is $\ll \exp(-\pi|t|)$ if t does not belong to $\epsilon-$neighborhood of odd integers, and since the error term of (2.5) is analytic, by Cauchy's integral, $\frac{d}{ds}\frac{\Gamma'}{\Gamma}(s) \ll |s|^{-1}$ in where the Stirling formula holds, so that

$$\frac{d}{ds}\frac{\chi'}{\chi}(s) \ll |t|^{-1} \tag{2.10}$$

in the region $|s| \gg 1$ and $-\pi + \delta < \arg s < \pi - \delta$, excluding $\epsilon-$neighborhood of odd integers.

From (1.1) it follows that

$$N(T+1)-N(T)\ll \log T, \tag{2.11}$$

which means that for $T \geq T_0$, there are at most $O(\log T)$ non-trivial zeros whose imaginary parts lying in $[T,T+1]$ and among the gaps between the ordinates of these zeros there must be a gap of length $\gg (\log T)^{-1}$.

Consider the standard formula

$$\frac{\zeta'}{\zeta}(\sigma+iT)=\sum_{|T-\gamma|\leq 1}\frac{1}{s-\rho}+O(\log T)\ .$$

This estimate is for large T and uniformly for $-1\leq\sigma\leq 2$ and the sum is limited to those ρ for which $|T-\gamma|\leq 1$. Since there are at most $O(\log T)$ terms in the sum of the above formula and each term is $\ll \log T$ if T is chosen suitably according to the above reasoning, we have the estimate

$$\frac{\zeta'}{\zeta}(\sigma+iT)\ll(\log T)^2\,,\qquad(\text{for }\ -1\leq\sigma\leq 2). \tag{2.12}$$

The Riemann zeta function is a meromorphic function with a simple pole with residue 1 at $s=1$. So, it has a Laurent expansion in the neighborhood of $s=1$

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1}+\gamma_0+\gamma_1\,(s-1)+\gamma_2\,(s-1)^2+\ldots \\ &= \frac{1}{s-1}+O\,(1),\qquad(s\to 1).\end{aligned} \tag{2.13}$$

By differentiation of the above Laurent series, it is easy to see that for $j\in\mathbb{N}$

$$\zeta^{(j)}(s)=\frac{(-1)^j j!}{(s-1)^{j+1}}+O_j\,(1),\qquad(s\to 1), \tag{2.14}$$

and then

$$\frac{\zeta^{(j)}}{\zeta}(s) = \frac{(-1)^j j!}{(s-1)^j} + O_j(1), \qquad (s \to 1). \tag{2.15}$$

The well-known convexity bound for $\zeta(s)$ can be generalized to its derivatives. We have, for $k = 0, 1, 2, ...$ and $|t| \geq 1$

$$\zeta^{(k)}(\sigma + it) \ll_{\epsilon,k} \begin{cases} |t|^{\frac{1}{2}-\sigma+\epsilon} & \text{if} \quad \sigma \leq 0 \\ |t|^{\frac{1}{2}(1-\sigma)+\epsilon} & \text{if} \quad 0 \leq \sigma \leq 1 \\ |t|^{\epsilon} & \text{if} \quad \sigma \geq 1, \end{cases} \tag{2.16}$$

with an arbitrarily fixed $\epsilon > 0$. (See Gonek [6], section 2) Together with this unconditional one, we need some conditional order results. For the followings we consult chapter 13 of [7]:
Let χ be any Dirichlet character modulo q, $q \in \mathbb{Z}^+$, and suppose that $L(s,\chi) \neq 0$ for $\sigma > 1/2$. Then there exists an absolute constant $C > 0$ such that

$$|L(s,\chi)| \leq \exp\left(\frac{C \log q\tau}{\log\log q\tau}\right) \tag{2.17}$$

uniformly for $1/2 \leq \sigma \leq 3/2$; and

$$\frac{L'}{L}(s,\chi) \ll \left(1 + (\log q\tau)^{2-2\sigma}\right) \min\left(\frac{1}{|\sigma - 1|}, \log\log q\tau\right) \tag{2.18}$$

uniformly for $1/2 + 1/\log\log q\tau \leq \sigma \leq 3/2$. See Exercises 4 and 12 of §13.2 in [7] for the second estimate; Exercise 8 and some results throughout the book, for example Exercise 3 of §2.3 and (10.20), for the first one.

We now give an estimate for the inverse of $L(s,\chi)$. Let χ be any Dirichlet character modulo q, $q \in \mathbb{Z}^+$, and suppose that $L(s,\chi) \neq 0$ for $\sigma > 1/2$. It then follows that

$$\log L(s,\chi) = \log L(s_1,\chi) - i\int_{\sigma}^{3/2} \frac{L'}{L}(\alpha+it,\chi)d\alpha, \tag{2.19}$$

where $s_1 = 3/2 + it$, $s = \sigma + it$, and $1/2 + 1/\log\log q\tau \leq \sigma \leq 3/2$. By (2.18), the integral becomes

$$\begin{aligned}
&= \left(\int_{\substack{\sigma\leq\alpha\leq 3/2 \\ |\alpha-1|\leq(\log\log q\tau)^{-1}}} + \int_{\substack{\sigma\leq\alpha\leq 3/2 \\ (\log\log q\tau)^{-1}<|\alpha-1|\leq\epsilon}} + \int_{\substack{\sigma\leq\alpha\leq 3/2 \\ |\alpha-1|>\epsilon}} \right) \frac{L'}{L}(\alpha+it,\chi)d\alpha \\
&\ll (\log q\tau)^{\epsilon} + \int_{\substack{\sigma\leq\alpha\leq 3/2 \\ |\alpha-1|>\epsilon}} (\log q\tau)^{2-2\alpha} d\alpha \\
&\ll (\log q\tau)^{\epsilon} + \frac{(\log q\tau)^{2-2\alpha}}{-2\log\log q\tau}\Big|_{\sigma}^{3/2} \ll \frac{\log q\tau}{\log\log q\tau}.
\end{aligned}$$

Combining this with (2.19), we obtain

$$|\log|L(s,\chi)|| \leq \sqrt{(\log|L(s,\chi)|)^2 + (\arg L(s,\chi))^2} = |\log L(s,\chi)| \leq \frac{A\log q\tau}{\log\log q\tau},$$

so

$$-\frac{A\log q\tau}{\log\log q\tau} \leq \log|L(s,\chi)|.$$

Taking exponential of the both sides we obtain the desired result, under GRH,

$$\frac{1}{L(s,\chi)} \ll \exp\left(\frac{A\log q\tau}{\log\log q\tau}\right) \tag{2.20}$$

uniformly for $1/2 + 1/\log\log q\tau \leq \sigma \leq 3/2$.

Regarding the generalized divisor function we have some notes. We denote the number of ways expressing n as a product of k positive integers by $\tau_k(n)$. In addition, this arithmetic function can be produced by the relation

$$\sum_{n\geq 1} \frac{\tau_k(n)}{n^s} = (\zeta(s))^k$$

for $\sigma > 1$. We have two familiar facts on $\tau_k(n)$:

(i) for bounded values of k, $\tau_k(n) \ll n^\epsilon$,

(ii) $\tau_k(mn) \leq \tau_k(m)\tau_k(n)$.

We will employ these items without referring to, especially in §17 and §18.

3. THE RELATIVE MAXIMA OF $|\zeta(\frac{1}{2}+it)|$

One of the common points of our two main works is the function

$$Z_1(s) := \zeta'(s) - \frac{1}{2}\frac{\chi'}{\chi}(s)\zeta(s) \tag{3.1}$$

introduced by Conrey and Ghosh [8]. In this section we give some fundamental properties of $Z_1(s)$. From (2.1) and (3.1), it follows that

$$Z_1(s) = -\chi(s)Z_1(1-s), \tag{3.2}$$

which is analogous to the functional equation for $\zeta(s)$. On taking logarithmic derivatives we also see that

$$\frac{Z_1'}{Z_1}(s) = \frac{\chi'}{\chi}(s) - \frac{Z_1'}{Z_1}(1-s). \tag{3.3}$$

The well-known Hardy's Z-function is defined by

$$Z(t) = \big(\chi(\frac{1}{2}+it)\big)^{-\frac{1}{2}}\zeta(\frac{1}{2}+it), \tag{3.4}$$

which is real for real t and satisfies $|Z(t)| = |\zeta(\frac{1}{2}+it)|$, since

$$|\chi(\frac{1}{2}+it)|^2 = \chi(\frac{1}{2}+it)\chi(\frac{1}{2}-it) = 1,$$

which can be easily deduced from (2.1). Taking derivatives of both sides of (3.4) with respect to t we obtain

$$\begin{aligned} Z'(t) &= -\frac{i}{2}\big(\chi(\frac{1}{2}+it)\big)^{-\frac{3}{2}}\chi'(\frac{1}{2}+it)\zeta(\frac{1}{2}+it) + i\big(\chi(\frac{1}{2}+it)\big)^{-\frac{1}{2}}\zeta'(\frac{1}{2}+it) \\ &= i\big(\chi(\frac{1}{2}+it)\big)^{-\frac{1}{2}}Z_1(\frac{1}{2}+it), \end{aligned}$$

which gives $|Z'(t)| = |Z_1(\frac{1}{2} + it)|$. So on the critical line $Z_1(s)$ vanishes at the relative maxima of $|\zeta(\frac{1}{2} + it)|$ and at the multiple zeros of $\zeta(s)$ if ever these exist. In the literature we have two results concerning the zeros of $Z_1(s)$. In [8] Conrey and Ghosh showed that

- $N_1(T) := \#\{\varrho : Z_1(\varrho) = 0\,\text{and}\,0 < \upsilon \leq T\}$

$$= N(T) + O(\log T). \tag{3.5}$$

- Assuming RH,

$$\#\left\{\varrho : Z_1(\varrho) = 0,\ 0 < \upsilon \leq T\,\text{and}\,\Re\varrho \neq \frac{1}{2}\right\} \ll \log T.$$

- Assuming RH, if $Z_1(\varrho) = 0$ and υ is sufficiently large, then

$$|\Re\varrho - \frac{1}{2}| \leq \frac{1}{9}.$$

In [9] Hall improved these to

- Assuming RH,

$$N_1(T) = N(T) - \frac{1}{2}\text{sgn}\frac{Z'(T)}{Z(T)} + \frac{3}{2}.$$

- Under the truth of RH, all the non-trivial zeros of $Z_1(s)$ lie on the critical line.

Another point related to the zeros of $Z_1(s)$ is the number of relative maxima between two consecutive zeta zeros on the critical line. We first give a formula for $\frac{Z'}{Z}(t)$. If we take $s = 1/2 + it$ in

$$\frac{\zeta'}{\zeta}(s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s),$$

it is easily seen that $\frac{\chi'}{\chi}(\frac{1}{2}+it)$ is a real number. Combining this with the logarithmic derivative of (3.4) with respect to t, we see that

$$\frac{Z'}{Z}(t) = \operatorname{Im}\frac{\zeta'}{\zeta}(\frac{1}{2}-it). \tag{3.6}$$

Taking imaginary parts in the partial fraction formula (from Chapter 12 of [10])

$$\frac{\zeta'}{\zeta}(s) = C - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}+1) + \sum_{\rho}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big), \tag{3.7}$$

where C is an absolute real constant, we have

$$\operatorname{Im}\frac{\zeta'}{\zeta}(\frac{1}{2}-it) = -\frac{t}{\frac{1}{4}+t^2} - \frac{1}{2}\operatorname{Im}\frac{\Gamma'}{\Gamma}(\frac{5}{4}-\frac{it}{2}) + \operatorname{Im}\sum_{\rho}\frac{1}{\frac{1}{2}-it-\rho}. \tag{3.8}$$

Now we assume RH, and use (3.3), to obtain

$$\frac{Z'}{Z}(t) = \operatorname{Im}\frac{\zeta'}{\zeta}(\frac{1}{2}-it) = \frac{\pi}{4} - O\big(\frac{1}{t}\big) + \sum_{\gamma>0}\frac{2t}{t^2-\gamma^2}. \tag{3.9}$$

This formula reveals that as t crawls up on the critical line from a zero of $\zeta(s)$ to the next zero, there will be only one point where $Z'(t) = 0$. Hence we obtain that on RH, the zeros of $Z'(t)$ are interlaced with the zeros of $Z(t)$, and $|\zeta(\frac{1}{2}+it)|$ has exactly one maximum between consecutive zeta zeros in the upper half-plane (and of course by symmetry also in the lower half-plane).

We now derive two estimates of Z_1'/Z_1 on half-planes with infinitely many holes. By the definition of Z_1 we see that

$$\frac{Z_1'}{Z_1}(s) = \frac{\frac{\zeta'}{\zeta}(s) - \frac{2}{\frac{\chi'}{\chi}(s)}\frac{\zeta''}{\zeta}(s) + \frac{\left(\frac{\chi'}{\chi}(s)\right)'}{\frac{\chi'}{\chi}(s)}}{1 - \frac{2}{\frac{\chi'}{\chi}(s)}\frac{\zeta'}{\zeta}(s)}. \tag{3.10}$$

By (2.5) and (2.6), we have $\frac{\chi'}{\chi}(s) \gg \log|s|$ in where (2.10) holds. Further, if $\sigma \geq 1+\epsilon$, then the Dirichlet series in (3.10) are $\ll 1$ and so $\frac{Z_1'}{Z_1}(s) \ll 1$. Here we should note that if $|s|$ is not large enough, we may lose the lower bound for denominator of (3.10), so we must check the finite region $|s| \ll 1, \sigma \geq 1+\epsilon$ separately. At this point, from the study of Hall [9] we know that the real zeros of $Z_1(s)$ are all simple and located at $s = 1/2$, z_m, $1 - z_m$, where $z_m \in (2m+1, 2m+3)$ for each $m \in \mathbb{Z}^+$, and that $Z_1(s)$ has simple poles at $s = 0, 3, 5...$, and a double pole at $s = 1$. Here one may ask why doesn't the existence of the real zeros of $Z_1(s)$ violate the boundedness result above even though we just exclude $\epsilon-$disc of positive, odd integers? The answer is, as $\sigma \to +\infty$, the zeros fall into these discs! In the end, we conclude that

$$\frac{Z_1'}{Z_1}(s) \ll 1 \tag{3.11}$$

in the region $\sigma \geq 1+\epsilon$ not intersecting with ϵ-neighborhood of any real zero or pole of Z_1. We next study on a negative half-planes. In view of the second formula of χ'/χ in (2.5) and (2.6),

$$\frac{\chi'}{\chi}(s) \ll \log|s| \tag{3.12}$$

as $\sigma \to -\infty$ and s is not in $\epsilon-$neighborhood of any even integer. If $\sigma \leq -\epsilon$ and s does not belong to ϵ-neighborhood of any real zero of Z_1 or any even integer, then by (3.11),

$$\frac{Z_1'}{Z_1}(1-s) \ll 1 \tag{3.13}$$

So, combined (3.12) and (3.13) in (3.3), it follows that

$$\frac{Z_1'}{Z_1}(s) = O(\log|s|), \tag{3.14}$$

provided that $\sigma \leq -\epsilon$ and s does not belong to ϵ-neighborhood of any real zero of Z_1 or any negative, even integer. Similar to the case of positive half-plane, as $\sigma \to -\infty$, the real zeros fall into $\epsilon-$neighborhood of negative, even integers. The similarity is actually consequence of the symmetry with respect to the point $1/2$ arising from (3.2).

Just as the estimate for $\frac{\zeta'}{\zeta}(s)$, we approximate $\frac{Z_1'}{Z_1}(s)$ by a finite sum of partial fractions $\frac{1}{s-\varrho}$ over the zeros of Z_1 near to s:

$$\frac{Z_1'}{Z_1}(s) = \sum_{|s-\varrho|\leq 1} \frac{1}{s-\varrho} + O(\log|t|) \tag{3.15}$$

for $-1 \leq \sigma \leq 2$, $|t| \geq 2$. The proofs of the Riemann zeta function case also holds for this case. The formula can be easily seen from the estimate $Z_1(s) \ll |t|^A$ throughout the region $-1 \leq \sigma \leq 2$, $|t| \geq 2$ and the Lemma due to Landau:

If $f(s)$ is regular, and

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M \quad (M > 1) \tag{3.16}$$

in the circle $|s - s_0| \leq r$, then

$$\left|\frac{f'(s)}{f(s)} - \sum_{\rho} \frac{1}{s-\rho}\right| < \frac{AM}{r} \quad (|s - s_0| \leq r/4), \tag{3.17}$$

where ρ runs through the zeros of $f(s)$ such that $|\rho - s_0| \leq r/2$.

From the formula $N_1(T) = N(T) + O(\log T)$ we deduce that

$$N_1(T+1) - N_1(T) \ll \log T, \tag{3.18}$$

i.e., the number of zeros of $Z_1(s)$ with $|\Im\varrho - T| \leq 1$ is $\ll \log T$. Among the ordinates of these zeros there must be a gap of length $\gg (\log T)^{-1}$. Hence by varying T by a bounded amount, we can ensure that

$$|\Im\varrho - T| \gg (\log T)^{-1} \tag{3.19}$$

for all zeros ϱ. With this choice of T, each term in the sum of the formula (3.15) is $\ll \log T$, and the number of terms is also $\log T$ so that we have

$$\frac{Z_1'}{Z_1}(\sigma + iT) \ll (\log T)^2 \quad (-1 \leq \sigma \leq 2) \tag{3.20}$$

if T is chosen so that the condition (3.19) is satisfied.

Finally, we'll finish this section off by acquiring formulas approximating $Z_1'(s)/Z_1(s)$ by Dirichlet series. The first two approximations will be unconditional. Since $\chi(s) \gg \log|s|$ in the region $\sigma \geq 1+\epsilon$, $|t| \geq A(\epsilon)$, we can expand the denominator $\left(1 + \frac{2}{\frac{\chi'}{\chi}(s)}\frac{\zeta'}{\zeta}(s)\right)^{-1}$ in (3.10) as a power series, and then using (2.10) we obtain

$$\frac{Z_1'}{Z_1}(s) = -\sum_{k \leq \frac{\log T}{\log\log T}} \left(\frac{-2}{\frac{\chi'}{\chi}(s)}\right)^k \sum_{m=1}^{\infty} \frac{\Lambda^{*(k+1)}(m)}{m^s}$$

$$+ \sum_{k \leq \frac{\log T}{\log\log T}} \left(\frac{-2}{\frac{\chi'}{\chi}(s)}\right)^{k+1} \sum_{m=1}^{\infty} \frac{\lambda^{*(k)}(m)}{m^s} \tag{3.21}$$

$$+O\left(\exp\left((A(\epsilon) - \log\log|s|)\frac{\log T}{\log\log T}\right) + \frac{1}{|t|\log|s|}\right),$$

to which we will appeal in deriving an explicit formula for Z_1. Here the coefficients $\Lambda^{*(k)}(m)$ and $\lambda^{*(k)}$ are produced by the relations

$$\sum_{m=1}^{\infty} \frac{\Lambda^{*(k)}(m)}{m^s} := \left(-\frac{\zeta'}{\zeta}(s)\right)^k, \tag{3.22}$$

$$\sum_{m=1}^{\infty} \frac{\lambda^{*(k)}(m)}{m^s} := \left(-\frac{\zeta'}{\zeta}(s)\right)^k \frac{\zeta''}{\zeta}(s) \tag{3.23}$$

for $k \in \mathbb{N}$, where

$$\sum_{m=1}^{\infty} \frac{\Lambda_j(m)}{m^s} := \frac{\zeta^{(j)}}{\zeta}(s),\ j \in \mathbb{N}. \tag{3.24}$$

If we replace χ'/χ by the right-hand side of (2.8) in the above calculations, we have

$$\frac{Z_1'}{Z_1}(s) = -\sum_{k\le\frac{\log|t|}{\log\log|t|}}\left(\frac{2}{\log\frac{|t|}{2\pi}}\right)^k\sum_{m=1}^{\infty}\frac{\Lambda^{*(k+1)}(m)}{m^s}$$
$$+\sum_{k\le\frac{\log|t|}{\log\log|t|}}\left(\frac{2}{\log\frac{|t|}{2\pi}}\right)^{k+1}\sum_{m=1}^{\infty}\frac{\sum_{d|m}\Lambda^{*(k)}(d)\Lambda_2\left(\frac{m}{d}\right)}{m^s}+O\left(\frac{1}{|t|}\exp\left(\frac{A(\epsilon)\log|t|}{\log\log|t|}\right)\right) \tag{3.25}$$

in the region $1+\epsilon\le\sigma\le\sigma_0$, $|t|\gg 1$, where σ_0 is an arbitrarily fixed number $>1+\epsilon$. However, we need a very similar formula in which we are arbitrarily close to $\sigma=1$. If we take $q=1$ in (2.18), we easily derive that

$$\frac{\zeta^{(j)}}{\zeta}(s)\ll_j(\log\log|t|)^j,\ (j\in\mathbb{Z}^+,\ \sigma\ge 1,\ |t|\ge t_0) \tag{3.26}$$

under the truth of RH. Proceeding as in the unconditional cases, with the aid of (2.8) and (3.26), we arrive at

$$\frac{Z_1'}{Z_1}(s) = -\sum_{k\le\frac{\log|t|}{\log\log|t|}}\left(\frac{2}{\log\frac{|t|}{2\pi}}\right)^k\sum_{m=1}^{\infty}\frac{\Lambda^{*(k+1)}(m)}{m^s}$$
$$+\sum_{k\le\frac{\log|t|}{\log\log|t|}}\left(\frac{2}{\log\frac{|t|}{2\pi}}\right)^{k+1}\sum_{m=1}^{\infty}\frac{\lambda^{*(k)}(m)}{m^s}+O\left(|t|^{-1}\exp\left(\frac{A\log|t|\log\log\log|t|}{\log\log|t|}\right)\right) \tag{3.27}$$

in the region $1<\sigma\le\sigma_0$, $|t|\gg 1$.

In fact, expanding the denominator in (3.10) as geometric series does not necessitate the assumption RH. Unconditionally, by the Vinogradov-Korobov theory we have

$$\frac{\zeta'}{\zeta}(s)\ll(\log|t|)^{2/3+\epsilon},$$

where $\sigma \geq 1$, $|t| \gg 1$. This gives the same formula with the error term $O(|t|^{-1/3+\epsilon})$, which leads to a smaller range of α in Theorem 11.1.

4. ON THE ZEROS OF χ'

Here we will show that apart from two exceptional zeros on the critical line all the zeros of $\chi'(s)$ lie on the real axis. It follows from the definition of $\chi(s)$, and the simple properties of the sine and the Gamma functions that all the zeros of $\chi(s)$ are simple and located at $s = -2n$, $n \in \mathbb{N}$. Then by Rolle's theorem, there exists at least one zero of $\chi'(s)$, say $x_n \in (-2n-2, -2n)$. The uniqueness of x_n in $(-2n-2, -2n)$ can be derived from the second formula in (2.6). We know that $\frac{\Gamma'}{\Gamma}(1-s)$ is analytic, and ≥ 0 for $s \leq 0$, which can be seen from the integral representation of Γ, and $\frac{\Gamma'}{\Gamma}(1-s) \sim \log(1-s)$ as $s \to -\infty$. However, $\cot \frac{\pi s}{2} < 0$ in $(-2n-1, -2n)$, $\cot \frac{\pi s}{2} \to -\infty$ as $s \to (-2n)^-$, and $\cot \frac{\pi s}{2} > 0$ in $(-2n-2, -2n-1)$, $\cot \frac{\pi s}{2} \to +\infty$ as $s \to (-2n-2)^+$. So $x_n \in (-2n-2, -2n-1)$ by the intermediate value theorem. Further, comparing the derivatives of the logarithm and the cotangent functions we see the uniqueness of x_n's.

From the functional equation of $\zeta(s)$ it follows that

$$\chi(s)\chi(1-s) = 1,$$

from which by logarithmic differentiation we obtain

$$\frac{\chi'}{\chi}(s) = \frac{\chi'}{\chi}(1-s),$$

which shows the existence and the uniqueness of the zeros on the positive real-axis. If we denote these zeros by the sequence $(y_n)_{n\in\mathbb{N}}$, then $y_n = 1-x_n$ and $y_n \in (2n+1, 2n+3)$ for each $n \in \mathbb{N}$.

We next show that $\chi'(s) \neq 0$ if $\Im s \neq 0$ and $\Re s \neq 1/2$, and locate the two exceptional zeros on the critical line. First observe that

$$\Im\frac{\chi'}{\chi}(u+iv) = \frac{(2u-1)v}{(u^2+v^2)((1-u)^2+v^2)} + \frac{v}{4}\sum_{n\geq 1}\left(\frac{1}{\left(\frac{1-u}{2}+n\right)^2+\left(\frac{v}{2}\right)^2} - \frac{1}{\left(\frac{u}{2}+n\right)^2+\left(\frac{v}{2}\right)^2}\right), \quad (4.1)$$

which can be obtained by combining the formulas that

$$\frac{\Gamma'}{\Gamma}(s) = 1 - \frac{1}{s} - \gamma_e + \sum_{n\geq 1}\left(\frac{1}{n+1} - \frac{1}{s+n}\right) \quad (4.2)$$

and

$$\frac{\chi'}{\chi}(s) = \log\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-\frac{s}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right), \quad (4.3)$$

which is the logarithmic derivative of (2.3). Here γ_e denotes Euler's constant. From (4.1) we see that $\Im\frac{\chi'}{\chi}(u+iv)$ is non-zero unless $u = 1/2$ or $v = 0$. Furthermore if $u > 1/2$, then $\Im\frac{\chi'}{\chi}(u+iv)$ is positive; if $u < 1/2$, then $\Im\frac{\chi'}{\chi}(u+iv)$ is negative. For the case $u = 1/2$, again using (4.2) and (4.3), we see that

$$\Re\frac{\chi'}{\chi}(1/2+iv) = \log\pi - 1 + \gamma_0 + \frac{4}{1+4v^2} - \sum_{n\geq 1}\frac{-\frac{3}{4}(1/4+n)+(v/2)^2}{(n+1)((1/4+n)^2+(v/2)^2)}.$$

Mathematica then works out that the zeros of χ' on the critical line are located at $z_{+1/2} = 1/2 + i6.28984\ldots$ and $z_{-1/2} = 1/2 - i6.28984\ldots$.

5. EXPLICIT FORMULA FOR $Z_1(s)$

We begin with the contour integral

$$\frac{1}{2\pi i}\int_{(c)}\frac{Z_1'}{Z_1}(w+1/2)k(w,s)x^w dw,$$

where (c) denotes the line of points whose real parts are c, and

$$k(w,s) := \frac{2\sigma-1}{(w-(s-1/2))(w-(1/2-\overline{s}))},$$

$x \geq 1$, and s belongs to the region $\sigma_0 \leq \sigma \leq \sigma_1$, where σ_0 and σ_1 are arbitrarily fixed numbers satisfying $\sigma_0 > 1+\epsilon$, excluding $\epsilon-$neighborhood of any real zero of Z_1 or χ' or any pole of Z_1. Here we choose $c = 1/2+\epsilon$ so that on the line (c) there is no singularity of $\frac{Z_1'}{Z_1}(w+1/2)$, and that $\frac{\chi'}{\chi}(w+1/2)$ is non-zero, and that $\sigma - 1/2 > c$. Thus we do not encounter the pole of $k(w,s)$ at $w = s-1/2$ when moving the line of integration to the left.

Here the kernel $k(w,s)$ was introduced by Farmer and Gonek in [11] to derive an explicit formula for $\xi'(s)$. Although it is not very different from Montgomery's derivation of the explicit formula for $\zeta(s)$, especially when the logarithmic derivative of a function for which we want to find an explicit formula has not a Dirichlet series representation, the approach in [11] is more convenient than that of Montgomery.

Let R be the rectangular contour joining the points $c-iV$, $c+iV$, $-U+iV$ and $-U-iV$, where U and V are large positive numbers so that $U \geq V \geq 2(|t|+1)$ and that the estimates (3.14) and (3.20) are valid for $\frac{Z_1'}{Z_1}(w+1/2)$ on the left-vertical and the horizontal sides of R, respectively. We remind that in every interval of length 1 it is possible to find at least one U and V. However, (3.20) remains true when the real part of variable of Z_1'/Z_1 lies on bounded ranges, for example $[-1,2]$ in (3.20). Thus, we divide $[-U,c]$ into three parts the first of which we apply (3.20) to $\frac{Z_1'}{Z_1}(w+1/2)$; in the remaining parts we apply (3.14). Then,

$$\int_{c+iV}^{-U+iV}, \int_{-U-iV}^{c-iV} \frac{Z_1'}{Z_1}(w+1/2)k(w,s)x^w dw \ll \frac{\log^2 V}{V^2}\int_{-3/2}^{c} x^u du$$
$$+ \frac{\log V}{V^2}\int_{-V}^{-3/2} x^u du + \int_{-U}^{-V} \frac{\log(u^2+V^2)}{u^2+V^2} x^u du$$
$$\ll \frac{x^c \log^2 V}{V^2 \log 2x},$$

$$\int_{-U-iV}^{-U+iV} \frac{Z_1'}{Z_1}(w+1/2)k(w,s)x^w dw \ll \frac{V\log(U^2+V^2)}{x^U U^2},$$

both of which tend to 0 as $U \to \infty$, then $V \to \infty$.

Here we assume the Riemann Hypothesis which implies that $\Re\varrho = 1/2$, due to the work of Hall. Inside R we encounter the simple poles of the integrand at $w = 0,\ 1/2,\ -1/2,\ 1/2 - z_1, \ldots, 1/2 - z_\ell,\ iv,\ 1/2 - \overline{s}$, where $|v| < V$ and $1 - z_{\ell+1} < -U + 1/2 < 1 - z_\ell$. Collecting the above results concerning the contributions of the horizontal and the left-vertical sides of R, letting $U \to \infty$, then $V \to \infty$, by the residue theorem we can conclude that

$$\frac{1}{2\pi i}\int_{(c)} \frac{Z_1'}{Z_1}(w+1/2)k(w,s)x^w dw = -k\left(-\frac{1}{2}, s\right)x^{-1/2} - 2k\left(\frac{1}{2}, s\right)x^{1/2} + k(0,s)$$
$$- x^{1/2-\overline{s}}\frac{Z_1'}{Z_1}(1-\overline{s}) + \sum_{\varrho} k(iv,s)x^{iv} + \sum_{m\geq 1} x^{1/2-z_m}k(1/2-z_m, s).$$

Since $1 + 2m < z_m < 3 + 2m$ and s has distance $\geq \epsilon$ from the nearest real zero of Z_1, the above sum over m is

$$\ll x^{-5/2}\left(\left(\sum_{m\ll|t|+1} + \sum_{m\gg|t|+1}\right)\frac{1}{|(1-z_m-s)(z_m-s)|}\right).$$

In the first range, the summand is trivially $\ll \left(\frac{1}{|t|+1}\right)^2$. The second part is $\ll \sum_{m\gg|t|+1}\frac{1}{m^2} \ll \frac{1}{|t|+1}$, by the integral test.

Using the definition of the kernel $k(w,s)$, we obtain

$$\frac{1}{2\pi i}\int_{(c)}\frac{Z_1'}{Z_1}(w+1/2)k(w,s)x^w dw = -\sum_{\varrho}\frac{(2\sigma-1)x^{iv}}{(v-t)^2+(1/2-\sigma)^2} - x^{1/2-\overline{s}}\frac{Z_1'}{Z_1}(1-\overline{s}) + O\left(\frac{x^{1/2}}{(|t|+1)^2}\right) + O\left(\frac{x^{-5/2}}{|t|+1}\right). \quad (5.1)$$

We next write the integral on the left in terms of a version of the discontinuous integral. Let T be a real variable tending to $+\infty$. In (3.21) we have expressed Z_1'/Z_1 as an approximate Dirichlet series. Employing this the integral becomes

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(c)}\frac{Z_1'}{Z_1}(w+1/2)k(w,s)x^w dw = \qquad (5.2)\\
&-\sum_{k\le\frac{\log T}{\log\log T}}(-2)^k\sum_{m=1}^{\infty}\frac{\Lambda^{*(k+1)}(m)}{m^{1/2}}\frac{1}{2\pi i}\int_{(c)}\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w dw\\
&+\sum_{k\le\frac{\log T}{\log\log T}}(-2)^{k+1}\sum_{m=1}^{\infty}\frac{\lambda^{*(k)}(m)}{m^{1/2}}\frac{1}{2\pi i}\int_{(c)}\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^{k+1}}\left(\frac{x}{m}\right)^w dw\\
&+\sum_{k\le\frac{\log T}{\log\log T}}(-2)^k\sum_{m=1}^{\infty}\frac{\Lambda^{*(k+1)}(m)}{m^{1/2}}\frac{1}{2\pi i}\int_{c-iA(\epsilon)}^{c+iA(\epsilon)}\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w dw\\
&-\sum_{k\le\frac{\log T}{\log\log T}}(-2)^{k+1}\sum_{m=1}^{\infty}\frac{\lambda^{*(k)}(m)}{m^{1/2}}\frac{1}{2\pi i}\int_{c-iA(\epsilon)}^{c+iA(\epsilon)}\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^{k+1}}\left(\frac{x}{m}\right)^w dw\\
&+O\left(x^c\int_{-A(\epsilon)}^{A(\epsilon)}\left|\frac{Z_1'}{Z_1}(c+1/2+iv)k(c+iv,s)\right|dv\right)\\
&+O\left(x^c\int_{|v|\ge A(\epsilon)}|k(c+iv,s)|\left(\exp\left(\frac{(A(\epsilon)-\log\log|v|)\log T}{\log\log T}\right)+\frac{1}{|v|\log|v|}\right)dv\right)\\
&=\sum_{1\le i\le 6}C_i,\ \text{say.}
\end{aligned}$$

On the line segment $[c-iA(\epsilon), c+iA(\epsilon)]$, $\frac{Z_1'}{Z_1}(c+1/2+iv)=O(1)$ and $k(c+iv,s)\ll(|t|+1)^{-2}$, so that $C_5\ll\frac{x^c}{(|t|+1)^2}$. For the last O-term, $k(c+iv,s)\ll(\max\{|v|+1,|t|+1\})^{-2}$ if $|v|\le|t|/2$ or $|v|\ge 3|t|/2$; $\ll 1$ if $|t|/2<|v|<3|t|/2$ and $|v-t|\le 1$; $\asymp|v-t|^{-2}$

otherwise, from which we deduce that

$$C_6 \ll x^c \exp\left(\frac{A(\epsilon)\log T}{\log\log T}\right) \max\left\{\exp\left(-\log\log(|t|+3)\frac{\log T}{\log\log T}\right), (|t|+1)^{-1}\right\}.$$

We remark that this upper bound is $\ll x^c \exp\left(\frac{A(\epsilon)\log T}{\log\log T}\right)(|t|+1)^{-1}$ when $|t| \leq T$, which will be used later. By the choice of c, in C_3 and C_4, $\left(\frac{\chi'}{\chi}(w+1/2)\right)^{-1}$ is $O(1)$. In addition to this, using (2.15), it follows that C_3, $C_4 \ll \frac{x^c}{(|t|+1)^2}\exp\left(A(\epsilon)\frac{\log T}{\log\log T}\right)$. Combining the results on the C_i's, we obtain

$$\begin{aligned} C_3 + C_4 + C_5 + C_6 \ll x^c \exp\left(\frac{A(\epsilon)\log T}{\log\log T}\right) \\ \times \max\left\{\exp\left(-\log\log(|t|+3)\frac{\log T}{\log\log T}\right), (|t|+1)^{-1}\right\}. \end{aligned} \tag{5.3}$$

We treat the integrals in the remaining parts C_1 and C_2 in two cases. Assume first $x/m \geq 1$. Let the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ denote the negative and the positive real zeros of χ', respectively, so that $1 - x_n = y_n$, $-2n-2 < x_n < -2n$ and $2n+1 < y_n < 2n+3$ for $n \in \mathbb{N}$. Choose $U_n > 0$ such that $x_{n+1} - 1/2 < -U_n < x_n - 1/2$ for sufficiently large $n \in \mathbb{N}$ and $\frac{\chi'}{\chi}(w+1/2) \gg \log U_n$ on the line $\Re w = -U_n$, which can be seen from the second formula in (2.6). Between the lines (c) and $(-U_n)$ we encounter the simple pole of the integrand at $w = 1/2 - \overline{s}$, coming from the simple pole of $k(w,s)$, the pole at $w = z_{1/2}$ and the simple poles at $w = x_\ell - 1/2$, where $0 \leq \ell \leq n$. Then the residue theorem gives that

$$\frac{1}{2\pi i}\int_{c-iU_n}^{c+iU_n} \frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k}\left(\frac{x}{m}\right)^w dw = \sum_{\mathfrak{d}\in\left\{\frac{1}{2}-\overline{s}, z_{1/2}-1/2, x_0-1/2, \ldots, x_n-1/2\right\}} \mathrm{Res}_{w=\mathfrak{d}}\left\{\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w\right\}$$

$$+\left(\int_{-U_n+iU_n}^{c+iU_n}+\int_{-U_n-iU_n}^{-U_n+iU_n}+\int_{c-iU_n}^{-U_n-iU_n}\right)\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w dw.$$

For sufficiently large n, $k(w,s) \ll U_n^{-2}$ and $\frac{\chi'}{\chi}(w+1/2) \gg \log U_n$ on the three sides above. We then see that as n tend to ∞, the three integrals on the right-hand side tends to 0. The residue of the pole at $w = 1/2-\overline{s}$ is $-\left(\frac{\chi'}{\chi}(1-\overline{s})\right)^{-k}\left(\frac{x}{m}\right)^{1/2-\overline{s}}$. So

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c)}\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w dw &= -\left(\frac{\chi'}{\chi}(1-\overline{s})\right)^{-k}\left(\frac{x}{m}\right)^{1/2-\overline{s}}\\
&+\frac{1}{(k-1)!}\sum_{\ell\in\mathbb{N}}\frac{d^{k-1}}{dw^{k-1}}\left\{k(w,s)\left(\frac{w-(x_\ell-1/2)}{\frac{\chi'}{\chi}(w+1/2)}\right)^k\left(\frac{x}{m}\right)^w\right\}_{w=x_\ell-1/2}\\
&+\frac{1}{(rk-1)!}\frac{d^{rk-1}}{dw^{rk-1}}\left\{k(w,s)\left(\frac{(w-(z_{+1/2}-1/2))^r}{\frac{\chi'}{\chi}(w+1/2)}\right)^k\left(\frac{x}{m}\right)^w\right\}_{w=z_{+1/2}-1/2}\\
&+\frac{1}{(rk-1)!}\frac{d^{rk-1}}{dw^{rk-1}}\left\{k(w,s)\left(\frac{(w-(z_{-1/2}-1/2))^r}{\frac{\chi'}{\chi}(w+1/2)}\right)^k\left(\frac{x}{m}\right)^w\right\}_{w=z_{-1/2}-1/2},\end{aligned}$$

where r is the degree of the zero at $w = z_{\pm 1/2}-1/2$. By Cauchy's integral formula with a circle of an ϵ-radius around $x_\ell - 1/2$ and $z_{\pm1/2}-1/2$, we see that the above derivatives are $\ll (A(\epsilon))^k(k-1)!(x/m)^{x_\ell-1/2+\epsilon}(|x_\ell-s||y_\ell-s|)^{-1}$ and $\ll (A(\epsilon))^k(rk-1)!(x/m)^{\epsilon}(|t|+1)^{-2}$, respectively. In the second case, we have used $r = O(1)$ and $\Re z_{\pm1/2} = 1/2$. It then follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c)}\frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w dw &= -\left(\frac{\chi'}{\chi}(1-\overline{s})\right)^{-k}\left(\frac{x}{m}\right)^{1/2-\overline{s}}\\
&+O\left(\frac{(A(\epsilon))^k(x/m)^\epsilon}{(|t|+1)^2}+(A(\epsilon))^k(x/m)^{x_0-1/2+\epsilon}\sum_{\ell\in\mathbb{N}}\frac{(x/m)^{x_\ell-x_0}}{|x_\ell-s||y_\ell-s|}\right).\end{aligned}\tag{5.4}$$

The sum over ℓ is $(|t|+1)^{-1}$, which can be seen by separating the range of summation into two parts, namely $\ell \ll (|t|+1)$ and $\ell \gg (|t|+1)$ and by transforming the second part into an infinite integral via the integral test.

In the case $x/m < 1$, $(x/m)^w$ tends to 0 as $\Re w \to \infty$ so that we pull the line of integration to the right and sum the residues of the poles at $w = s - 1/2$ and at $w = y_\ell - 1/2$, $\ell \in \mathbb{N}$. Similar to the $x/m \geq 1$ case, we have

$$\frac{1}{2\pi i}\int_{(c)} \frac{k(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w dw = -\left(\frac{\chi'}{\chi}(s)\right)^{-k}\left(\frac{x}{m}\right)^{s-1/2} + O\left(\frac{(A(\epsilon))^k (x/m)^{y_0-1/2-\epsilon}}{|t|+1}\right). \quad (5.5)$$

Returning to C_1 and C_2, we first introduce the following notation that

$$\alpha(m,s) := \sum_{k\leq \frac{\log T}{\log\log T}} (-2)^k \left(\Lambda^{*(k+1)}(m)\left(\frac{\chi'}{\chi}(s)\right)^{-k} + 2\lambda^{*(k)}(m)\left(\frac{\chi'}{\chi}(s)\right)^{-k-1}\right). \quad (5.6)$$

Then using (5.4) and (5.5), we get

$$\begin{aligned} C_1 + C_2 = {} & x^{-1/2}\left(\sum_{m\leq x}\alpha(m,1-\overline{s})\left(\frac{x}{m}\right)^{1-\overline{s}} + \sum_{m>x}\alpha(m,s)\left(\frac{x}{m}\right)^{s}\right) \qquad (5.7)\\ & + O\left(\frac{x^{x_0-1/2+\epsilon}}{|t|+1}\sum_{k\leq \frac{\log T}{\log\log T}}(A(\epsilon))^k \sum_{m\leq x}\frac{\Lambda^{*(k+1)}(m)+\lambda^{*(k)}(m)}{m^{x_0+\epsilon}}\right)\\ & + O\left(\frac{x^{\epsilon}}{(|t|+1)^2}\sum_{k\leq \frac{\log T}{\log\log T}}(A(\epsilon))^k \sum_{m\leq x}\frac{\Lambda^{*(k+1)}(m)+\lambda^{*(k)}(m)}{m^{1/2+\epsilon}}\right)\\ & + O\left(\frac{x^{y_0-1/2-\epsilon}}{|t|+1}\sum_{k\leq \frac{\log T}{\log\log T}}(A(\epsilon))^k \sum_{m> x}\frac{\Lambda^{*(k+1)}(m)+\lambda^{*(k)}(m)}{m^{y_0-\epsilon}}\right). \end{aligned}$$

The evaluation of the sums over m in the error terms are similar, these can be bounded by the same quantity. In the first sum, since $m \leq x$, $m^{-x_0-\epsilon} \leq m^{-1-\epsilon}x^{1-x_0}$. We replace $m^{-x_0-\epsilon}$ by $m^{-1-\epsilon}x^{1-x_0}$, then the resulting Dirichlet polynomial is a part of the Dirichlet series which is $\sim \epsilon^{-k-2}$ by (2.15), so that the first error term can be bounded by the right-hand side of (5.3).

The last step before the derivation of the explicit formula for Z_1 is that we replace $-x^{1/2-\overline{s}}\frac{Z_1'}{Z_1}(1-\overline{s})$ in (5.1) by

$$x^{1/2-\overline{s}}\frac{Z_1'}{Z_1}(\overline{s}) - x^{1/2-\overline{s}}\frac{\chi'}{\chi}(\overline{s}) = x^{1/2-\overline{s}}\log\frac{|t|+2}{2\pi} + O(x^{1/2-\sigma}), \tag{5.8}$$

which follows from (2.8), (3.3) and (3.11).

As a summary of what we have done up to now, we combine (5.1), (5.2), (5.3), (5.7), (5.8) in the following Theorem.

Theorem 5.1. *(On RH) Let $s=\sigma+it$ be a complex number satisfying $1+\epsilon<\sigma\le\sigma_1$, where σ_1 is an arbitrarily fixed number, and that s does not coincide with the zeros of χ' and Z_1, and the poles of Z_1. Then for large T,*

$$\begin{aligned}
&\sum_{\varrho}\frac{(2\sigma-1)x^{iv}}{(v-t)^2+(\sigma-1/2)^2} = \\
&-x^{-1/2}\left(\sum_{m\le x}\alpha(m,1-\overline{s})\left(\frac{x}{m}\right)^{1-\overline{s}} + \sum_{m>x}\alpha(m,s)\left(\frac{x}{m}\right)^{s}\right) + x^{1/2-\overline{s}}\log\frac{|t|+2}{2\pi} \\
&+O_\sigma\left(x^{\frac{1}{2}+\epsilon}\exp\left(\frac{A(\epsilon)\log T}{\log\log T}\right)\max\left\{\exp\left(\frac{-(\log\log(|t|+3))\log T}{\log\log T}\right), \frac{1}{|t|+1}\right\}\right) \\
&+O_\sigma(x^{1/2-\sigma}).
\end{aligned}$$

6. A SKETCH OF MONTGOMERY'S DERIVATION

The explicit formula from Landau's book [12] is that, if $x > 1$ and $x \neq p^k$ (p prime, and k a positive integer), then

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{r=1}^{\infty} \frac{x^{-2r-s}}{2r+s} \tag{6.1}$$

provided $s \neq 1$, $s \neq \rho$, $s \neq -2r$. Here s is a complex variable and the standard notation $s = \sigma + it$ will be employed, ρ runs through the nontrivial zeros of $\zeta(s)$, and the series over ρ is convergent with the interpretation

$$\lim_{U \to \infty} \sum_{|\mathrm{Im}\,\rho| \leq \mathrm{U}} \frac{x^{\rho-s}}{\rho - s}. \tag{6.2}$$

Upon assuming RH and manipulating this equation, Montgomery obtained the explicit formula

$$\begin{aligned}(2\sigma - 1)\sum_{\gamma} \frac{x^{i\gamma}}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} \quad = \quad & -x^{-\frac{1}{2}}\Big(\sum_{n \leq x} \Lambda(n)(\frac{x}{n})^{1-\sigma+it} \sum_{n > x} \Lambda(n)(\frac{x}{n})^{\sigma+it}\Big) \\ & -\frac{\zeta'}{\zeta}(1-\sigma+it)x^{\frac{1}{2}-\sigma+it} \\ & +\frac{(2\sigma-1)x^{\frac{1}{2}}}{(\sigma-1+it)(\sigma-it)} \\ & -x^{-\frac{1}{2}}\sum_{r=1}^{\infty} \frac{(2\sigma-1)x^{-2r}}{(\sigma-1-it-2r)(\sigma+it+2r)}, \end{aligned} \tag{6.3}$$

valid for $\sigma > 1$, and all $x \geq 1$. In this formula Montgomery used the logarithmic derivative of the functional equation of $\zeta(s)$ to replace $\frac{\zeta'}{\zeta}(1-\sigma+it)$ by $-\frac{\zeta'}{\zeta}(\sigma - it) - \log(|t|+2) + O(1) = -\log(|t|+2) + O_\sigma(1)$ (for s in a fixed strip in $\sigma > 1$), and he put upper-bound estimates for the last two terms of (6.3). Montgomery then took $\sigma = \frac{3}{2}$, squared the modulus of both sides, and then integrated both sides over t from 0 to T. From the expression thus arising from the left hand-side of (6.3), he discarded those $\gamma \notin [0, T]$ within an error of $O(\log^3 T)$, and in order to evaluate the integral he extended the range of integration to $\int_{-\infty}^{\infty}$ within an even smaller error. In this way the left-hand side of (6.3) led to the expression (1.3). To carry out the integration of the square of the series involving $\Lambda(n)$ coming from the right-hand side of (6.2), Montgomery had recourse to Parseval's formula for Dirichlet series from [13]:

$$\text{If } \sum_{n=1}^{\infty} n|a_n|^2 \text{ converges, then } \int_0^T \Big|\sum_n a_n n^{-it}\Big|^2 dt = \sum_n |a_n|^2 (T + O(n)),$$

The result of this calculation was (1.5).

7. A MODIFIED APPROACH

With $\sigma = \frac{5}{2}$ in (6.3) reads

$$\begin{aligned}
\sum_{\gamma} \frac{4x^{i\gamma}}{4+(t-\gamma)^2} &= -x^{-\frac{1}{2}}\Big(\sum_{n\leq x} \Lambda(n)(\frac{x}{n})^{-\frac{3}{2}+it} + \sum_{n>x} \Lambda(n)(\frac{x}{n})^{\frac{5}{2}+it}\Big) \\
&\quad -\frac{\zeta'}{\zeta}(-\frac{3}{2}+it)x^{-2+it} + \frac{4x^{\frac{1}{2}}}{(\frac{3}{2}+it)(\frac{5}{2}-it)} \\
&\quad -4x^{-\frac{1}{2}} \sum_{r=1}^{\infty} \frac{x^{-2r}}{(\frac{3}{2}-2r-it)(\frac{5}{2}+2r+it)}.
\end{aligned}$$

For the $\frac{\zeta'}{\zeta}$ term here we use

$$\begin{aligned}
\frac{\zeta'}{\zeta}(1-\sigma+it) &= -\frac{\zeta'}{\zeta}(\sigma-it) + \log\pi + \frac{1}{(\sigma-it)(1-\sigma+it)} \\
&\quad -\frac{1}{2}\Big(\frac{\Gamma'}{\Gamma}(\frac{3-\sigma+it}{2}) + \frac{\Gamma'}{\Gamma}(1+\frac{\sigma-it}{2})\Big), \qquad (7.1)
\end{aligned}$$

which follows from the formulas in pp. 80–82 of [10], with $\sigma = \frac{5}{2}$. Hence we have

$$\begin{aligned}
\sum_{\gamma} \frac{4x^{i(\gamma-t)}}{4+(\gamma-t)^2} &= -x^{-2}\sum_{n\leq x}\Lambda(n)n^{\frac{3}{2}-it} - x^2\sum_{n>x}\Lambda(n)n^{-\frac{5}{2}-it} \\
&\quad +x^{-2}\Big(\frac{\zeta'}{\zeta}(\frac{5}{2}+it) - \log\pi\Big) \\
&\quad +\frac{x^{-2}}{2}\Big(\frac{\Gamma'}{\Gamma}(\frac{1}{4}+\frac{it}{2}) + \frac{\Gamma'}{\Gamma}(\frac{9}{4}-\frac{it}{2})\Big) \\
&\quad +\frac{x^{-2}}{(\frac{5}{2}-it)(\frac{3}{2}-it)} + \frac{4x^{\frac{1}{2}-it}}{(\frac{3}{2}+it)(\frac{5}{2}-it)} \\
&\quad -4x^{-\frac{1}{2}-it}\sum_{r=1}^{\infty}\frac{x^{-2r}}{(\frac{3}{2}-2r-it)(\frac{5}{2}+2r+it)}. \qquad (7.2)
\end{aligned}$$

Recalling (2.5) and doing elementary estimations, we can simplify (7.2) into

$$\sum_{\gamma} \frac{4x^{i(\gamma-t)}}{4+(\gamma-t)^2} = -x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}-it} - x^2\sum_{n>x}\Lambda(n)n^{-\frac{5}{2}-it} + x^{-2}\big(\log(|t|+3)+O(1)\big) + O\Big(\frac{x^{\frac{1}{2}}}{(|t|+1)^2}\Big) + O\Big(\frac{x^{-\frac{5}{2}}}{|t|+1}\Big) \tag{7.3}$$

for $x \ge 1$.

When t runs through a set of values we sum both sides of (7.3) over the relevant t. This will be feasible if one can calculate the sums over t, in particular $\sum_t p^{-iat}$ where p is a prime and a is a natural number. In the following sections several examples will be presented.

8. PAIR CORRELATION OF ZETA ZEROS

We apply our method first to the quantity (1.3) considered by Montgomery. So, letting t run through those ordinates $\tilde{\gamma}$ of the zeros of the Riemann zeta-function which are in the interval $(0, T]$, from (7.3) we have

$$\begin{aligned}\sum_{\gamma} \sum_{0<\tilde{\gamma}\le T} \frac{4x^{i(\gamma-\tilde{\gamma})}}{4+(\gamma-\tilde{\gamma})^2} &= -x^{-2} \sum_{n\le x} \Lambda(n) n^{\frac{3}{2}} \sum_{0<\tilde{\gamma}\le T} n^{-i\tilde{\gamma}} - x^2 \sum_{n>x} \frac{\Lambda(n)}{n^{\frac{5}{2}}} \sum_{0<\tilde{\gamma}\le T} n^{-i\tilde{\gamma}} \\ &+x^{-2} \sum_{0<\tilde{\gamma}\le T} \big(\log\tilde{\gamma} + O(1)\big) + O\Big(x^{\frac{1}{2}} \sum_{0<\tilde{\gamma}\le T} \frac{1}{\tilde{\gamma}^2}\Big) \\ &+O\Big(x^{-\frac{5}{2}} \sum_{0<\tilde{\gamma}\le T} \frac{1}{\tilde{\gamma}}\Big) \qquad (8.1)\end{aligned}$$

for $x \ge 1$. From the count

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \qquad (8.2)$$

of zeta zeros with ordinates in $(0, T]$, we easily estimate the last terms of (8.1) and re-state (8.1) as

$$\begin{aligned}\sum_{\gamma} \sum_{0<\tilde{\gamma}\le T} \frac{4x^{i(\gamma-\tilde{\gamma})}}{4+(\gamma-\tilde{\gamma})^2} &= -x^{-2} \sum_{n\le x} \Lambda(n) n^{\frac{3}{2}} \sum_{0<\tilde{\gamma}\le T} n^{-i\tilde{\gamma}} - x^2 \sum_{n>x} \frac{\Lambda(n)}{n^{\frac{5}{2}}} \sum_{0<\tilde{\gamma}\le T} n^{-i\tilde{\gamma}} \\ &+\frac{T\log^2 T}{2\pi x^2}\big(1 + O(\frac{1}{\log T})\big) + O(x^{\frac{1}{2}}). \qquad (8.3)\end{aligned}$$

In the sum on the left-hand side we can exclude those $\gamma \notin [0, T]$ within an error of $O(\log^3 T)$, by a standard calculation based upon (8.2). This exclusion leads to the error term

$$\ll \sum_{\gamma>T}\sum_{0<\tilde\gamma\le T}\frac{1}{4+(\gamma-\tilde\gamma)^2}+\sum_{\gamma\le 0}\sum_{0<\tilde\gamma\le T}\frac{1}{4+(\gamma-\tilde\gamma)^2} \tag{8.4}$$
$$= \sum_{1\le m\le T}\sum_{m-1<\tilde\gamma\le m}\sum_{n\ge 0}\sum_{T+n+1\ge\gamma>T+n}\frac{1}{4+(\gamma-\tilde\gamma)^2}$$
$$+\sum_{1\le m\le T}\sum_{m-1<\tilde\gamma\le m}\sum_{n\ge 0}\sum_{-n-1<\gamma\le -n}\frac{1}{4+(\gamma-\tilde\gamma)^2}.$$

The first quantity on the right of the equality is, by (8.2),

$$\ll (\log T)^2\sum_{1\le m\le T}\sum_{n\ge 0}\frac{1}{4+(T+n-m)^2}\ll(\log T)^2\sum_{1\le m\le T}\frac{1}{T-m+1}\ll(\log T)^3. \tag{8.5}$$

Similar calculations can be done for the last quantity in (8.4).

We now have recourse to Gonek's [14] unconditional result

$$\sum_{0<\gamma\le T} y^\rho = -\frac{T}{2\pi}\Lambda(y)+O(y\log 2yT\log\log 3y)$$
$$+O\big(\min\big(T,\frac{y}{\langle y\rangle}\big)\log y\big)+O\big(\min\big(T,\frac{1}{\log y}\big)\log 2T\big)\big) \tag{8.6}$$

for $y, T>1$, where ρ denotes a complex zero of $\zeta(s)$ and $\langle y\rangle$ denotes the distance from y to the nearest prime power other than y itself. Assuming RH and using (8.6) for the inner sum occurring in the first two terms of the right-hand side of (8.3) , the contribution from the first term of the right-hand side of (8.6) is

$$\frac{T}{2\pi}\Big(\frac{1}{x^2}\sum_{n\le x}n\Lambda(n)^2+x^2\sum_{n>x}\frac{\Lambda(n)^2}{n^3}\Big) \tag{8.7}$$
$$= \frac{T}{2\pi}\Big(\frac{1}{x^2}\big(\frac{x^2}{2}\log x-\frac{x^2}{4}+O(x^{\frac{3}{2}}\log^3 2x)\big)+x^2\big(\frac{\log x}{2x^2}+\frac{1}{4x^2}+O(x^{-\frac{5}{2}}\log^3 2x)\big)\Big)$$
$$= \frac{T}{2\pi}\log x+O(Tx^{-\frac{1}{2}}\log^3 2x).$$

Here we have calculated the sums from

$$\sum_{n\le x}\Lambda(n) = x + O(x^{\frac{1}{2}}\log^2 2x) \tag{8.8}$$

which is the prime number theorem under RH. The first error term in (8.6) contributes

$$\begin{aligned}&\ll \frac{1}{x^2}\sum_{n\le x} n^2\Lambda(n)\log 2nT\log\log 3n + x^2\sum_{n>x}\frac{\Lambda(n)}{n^2}\log 2nT\log\log 3n\\ &\ll x\log 2xT\log\log 3x.\end{aligned} \tag{8.9}$$

The second error term in (8.6) contributes

$$\ll \frac{1}{x^2}\sum_{n\le x} n\Lambda(n)\log n\min(T,n) + x^2\sum_{n>x}\frac{\Lambda(n)}{n^3}\log n\min(T,n) \tag{8.10}$$

$$\ll \begin{cases} x\log 2x + \frac{x^2}{T}\log T & \text{if} \quad x\le T,\\ T\log x & \text{if} \quad x>T.\end{cases}$$

Finally the last error term of (8.6) contributes

$$\ll \frac{1}{x^2}\sum_{n\le x} n\Lambda(n)\frac{\log 2T}{\log n} + x^2\sum_{n>x}\frac{\Lambda(n)}{n^3}\frac{\log 2T}{\log n} \ll \frac{\log T}{\log 2x}. \tag{8.11}$$

Combining the above we find that

$$\begin{aligned}F_{\zeta,\zeta}(x,T) \quad &= \quad \frac{T\log^2 T}{2\pi x^2}\big(1+O(\frac{1}{\log T})\big) + \frac{T}{2\pi}\log x + O(x\log 2xT\log\log 3x)\\ &\quad +O\big(\frac{x^2\log T}{T}\big) + O\big(\frac{T\log^3 2x}{x^{\frac{1}{2}}}\big)\end{aligned} \tag{8.12}$$

as $T\to\infty$. This gives an asymptotic result for $1\le x = o\big(\frac{T}{\log\log T}\big)$.

9. SOME AVERAGES OF $\Lambda^{*(k)}$ AND $\lambda^{*(k)}$

In the following section we generalize Gonek's result on Λ in [14] to $\Lambda^{*(k)}$ and $\lambda^{*(k)}$. Lemma 9.4 is the extension of Gonek's $k = 1$ case, Lemma 9.5 is the $\lambda^{*(k)}-$ analogue of Lemma 9.4. We will use these in estimating the Landau sums in the next section. Let's start with some basic exercises.

Lemma 9.1. *Let k, $M \in \mathbb{N}$. Then we have the following binomial identities:*

$$\sum_{j=0}^{M} \binom{j+k}{k} = \binom{M+k+1}{k+1}, \tag{9.1}$$

$$\sum_{j=0}^{M} \binom{j+k}{k}(M-j+1) = \binom{M+k+2}{k+2}. \tag{9.2}$$

Proof. We first note that for $m \in \mathbb{N}$

$$\left(\sum_{n=0}^{\infty} z^n\right)^m = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^n \tag{9.3}$$

where for $r, s \in \mathbb{Z}^{\geq -1}$ the binomial coefficient is defined by

$$\binom{r}{s} := \begin{cases} 1 & \text{if} \quad r = s, \\ 0 & \text{if} \quad s = -1,\ r \geq 0, \\ 0 & \text{if} \quad r < s, \\ \frac{r!}{s!(r-s)!} & \text{otherwise.} \end{cases} \tag{9.4}$$

From (9.3) we see that comparing the coefficients of z^M in the power series

$$(1 + z + z^2 + \cdots)\frac{1}{(1-z)^{k+1}} \quad \text{and} \quad \frac{1}{(1-z)^{k+2}}$$

gives (9.1). Similarly, using

$$\frac{1}{(1-z)^{k+1}}\left(\frac{1}{1-z}\right)'=\frac{1}{(1-z)^{k+3}}$$

we can see (9.2).

Lemma 9.2. *Let α, $\ell\in\mathbb{N}$ and p a prime number. Define*

$$\sum_{n=1}^{\infty}\frac{\Lambda^{*(\ell)}(n)}{n^s}:=\left(-\frac{\zeta'}{\zeta}(s)\right)^{\ell}, \tag{9.5}$$

$$\sum_{n=1}^{\infty}\frac{\lambda^{*(\ell)}(n)}{n^s}:=\left(-\frac{\zeta'}{\zeta}(s)\right)^{\ell}\frac{\zeta''}{\zeta}(s). \tag{9.6}$$

Then

$$\begin{aligned}\Lambda^{*(\ell)}\left(p^{\alpha}\right)&=\binom{\alpha-1}{\ell-1}\log^{\ell}p,\\ \lambda^{*(\ell)}\left(p^{\alpha}\right)&=\left(2\binom{\alpha}{\ell+1}-\binom{\alpha-1}{\ell}\right)\log^{\ell+2}p.\end{aligned}$$

Proof. If $\ell=0$, then $\left(\frac{\zeta'}{\zeta}(s)\right)^{0}=1$, so

$$\Lambda^{*(0)}\left(p^{\alpha}\right)=\begin{cases}1 & \text{if} \quad \alpha=0,\\ 0 & \text{if} \quad \alpha>0.\end{cases} \tag{9.7}$$

Assume $\ell\in\mathbb{Z}^{+}$. Then

$$\begin{aligned}\Lambda^{*(\ell)}\left(p^{\alpha}\right)&=\sum_{\substack{\alpha_1+\cdots+\alpha_\ell=\alpha\\ \alpha_1,\ldots,\alpha_\ell\in\mathbb{Z}^{+}}}\Lambda\left(p^{\alpha_1}\right)\ldots\Lambda\left(p^{\alpha_\ell}\right)\\ &=\log^{\ell}p\sum_{\substack{\alpha_1+\cdots+\alpha_\ell=\alpha\\ \alpha_1,\ldots,\alpha_\ell\in\mathbb{Z}^{+}}}1\\ &=\binom{\alpha-1}{\ell-1}\log^{\ell}p.\end{aligned}$$

We now deal with $\lambda^{*(\ell)}(p^\alpha)$. By (9.6) and the first result of the lemma we have

$$\lambda^{*(\ell)}(p^\alpha) = \log^\ell p \sum_{r=0}^{\alpha} \binom{r-1}{\ell-1} \Lambda_2\left(p^{\alpha-r}\right). \tag{9.8}$$

The values of the arithmetic function Λ_2 can be calculated easily by the formula $\Lambda_2(n) = \sum_{d|n} \mu(d) \left(\log \frac{n}{d}\right)^2$, and (9.8) becomes

$$\begin{aligned}
\lambda^{*(\ell)}(p^\alpha) &= \log^{\ell+2} p \sum_{\ell \le r < \alpha} \binom{r-1}{\ell-1} (2\alpha - 2r - 1) \\
&= 2\log^{\ell+2} p \sum_{j=0}^{\alpha-\ell-1} \binom{j+\ell-1}{\ell-1} (\alpha - j - \ell) - \log^{\ell+2} p \sum_{j=0}^{\alpha-\ell-1} \binom{j+\ell-1}{\ell-1},
\end{aligned}$$

by the substitution $j = r - \ell$. Then the proof is completed by the two results of Lemma 9.1.

Lemma 9.3. *For $k \in \mathbb{N}$, $a \in \mathbb{R}$, we have*

$$\sum_{r=0}^{k} \frac{(a)_r}{r!} = \frac{(a+1)_k}{k!},$$

where $(a)_k := \prod_{i=0}^{k-1}(a+i)$, with the convention $(a)_0 := 1$.

Proof. We show this by induction on k. The case $k = 0$ is trivial. Assume the claim holds for $k \in \mathbb{N}^+$, then

$$\begin{aligned}
\sum_{r=0}^{k+1} \frac{(a)_r}{r!} &= \frac{(a)_{k+1}}{(k+1)!} + \sum_{r=0}^{k} \frac{(a)_r}{r!} \\
&= \frac{(a)_{k+1}}{(k+1)!} + \frac{(a+1)_k}{k!} \\
&= \frac{a(a+1)_k}{(k+1)!} + \frac{(a+1)_k}{k!} \\
&= \left(\frac{a}{k+1} + 1\right) \frac{(a+1)_k}{k!} \\
&= \frac{(a+1)_{k+1}}{(k+1)!}.
\end{aligned}$$

By induction the lemma is proved. $\square$

Lemma 9.4. *Let $k \in \mathbb{N}$. We have for $N \geq 1$*

$$\sum_{n \leq N/2} \frac{\Lambda^{*(k)}(N-n)}{n} \ll \log^k 2N \frac{(\log\log 3N + 7)_k}{(k-1)!},$$

and

$$\sum_{n \leq N/2} \frac{\Lambda^{*(k)}(N+n)}{n} \ll \log^k 2N \frac{(\log\log 3N + 7)_k}{(k-1)!},$$

with the convention that $(-1)! = 1$.

Proof. The case $k = 0$ is trivial. Assume $k \geq 1$. It follows from mathematical induction on k and the fact $\sum_{d|n} \Lambda(d) = \log n$ that $\Lambda^{*(k)}(n) \leq \log^k n$ for all $n \in \mathbb{Z}^+$. Using this and applying Stieltjes integration, we obtain

$$\sum_{n \leq N/2} \frac{\Lambda^{*(k)}(N-n)}{n} \leq \log^k 2N \int_{1^-}^{N/2} \frac{dS_k(x,N)}{x}, \tag{9.9}$$

and

$$S_k(x,N) := \left| \left\{ N - x < n \leq N : \Lambda^{*(k)}(n) \neq 0 \right\} \right|. \tag{9.10}$$

To obtain an upper bound for $S_k(x,N)$ we first quote the result of Tudesq in [15]:

$$\sum_{\substack{y < n \leq y+x \\ \omega(n) = k}} 1 \ll \frac{x}{\log 2x} \frac{(\log\log 3x + 6)_{k-1}}{(k-1)!}, \tag{9.11}$$

where $1 \leq x \leq y$, $k \in \mathbb{Z}^+$, and the constant implied by $\ll$ is absolute. Employing this we have

$$S_k(x,N) \le \sum_{i=1}^{k} \sum_{\substack{N-x<n\le N \\ \omega(n)=i}} 1 \ll \frac{x}{\log 2x} \sum_{i=1}^{k} \frac{(\log\log 3x+6)_{i-1}}{(i-1)!}$$
$$= \frac{x}{\log 2x} \frac{(\log\log 3x+7)_{k-1}}{(k-1)!}, \tag{9.12}$$

by Lemma 9.3. We then return to (9.9), and by integration by parts we obtain

$$\sum_{n\le N/2} \frac{\Lambda^{*(k)}(N-n)}{n} \le \log^k 2N \left\{ \frac{S_k(x,N)}{x}\Big|_{1^-}^{N/2} + \int_{1^-}^{N/2} \frac{S_k(x,N)}{x^2} dx \right\}$$

Using the upper bound in (9.12) for $S_k(x,N)$ in the above inequality, we complete the proof of the first claim. The second sum can be treated in the same manner.

Here we note that the above results cannot be deduced from one form of the prime number theorem and partial summation, because, if $u = o(N/\log N)$, the upper bound for the sum $\sum_{n\le u} \Lambda(N-n)$ obtained from the prime number theorem is worse than the trivial bound $u\log N$.

Lemma 9.5. *Let $k \in \mathbb{N}$. We have for $N \ge 1$*

$$\sum_{n\le N/2} \frac{\lambda^{*(k)}(N-n)}{n} \ll \log^{k+2} 2N \frac{(\log\log 3N+7)_{k+2}}{(k+1)!},$$

and

$$\sum_{n\le N/2} \frac{\lambda^{*(k)}(N+n)}{n} \ll \log^{k+2} 2N \frac{(\log\log 3N+7)_{k+2}}{(k+1)!}.$$

Proof. We follow exactly the same lines of the proof of the above lemma. Assume $k \ge 1$, $k=0$ is again a trivial case. From definition of $\lambda^{(k)}(n)$ we see that $\lambda^{*(k)}(n) \le \log^{k+2} n$ for all $n \in \mathbb{Z}^+$, by using the facts that $\Lambda^{*(k)}(n) \le \log^k n$ and $\sum_{d|n} \Lambda_2(d) = \log^2 n$.

Employing this bound, we have

$$\sum_{n\le N/2}\frac{\lambda^{*(k)}(N-n)}{n}\le \log^{k+2}2N\int_{1^-}^{N/2}\frac{dT_k(x,N)}{x}, \tag{9.13}$$

where

$$T_k(x,N):=\Big|\big\{N<n\le N+x:\lambda^{*(k)}(n)\ne 0\big\}\Big|. \tag{9.14}$$

To bound $T_k(x,N)$ first observe that $\lambda^{*(k)}(n)\ne 0$ implies that $1\le\omega(n)\le k+2$. Similar to the estimation of $S(x)$, we have

$$T_k(x,N)\ll\frac{x}{\log 2x}\frac{(\log\log 3x+7)_{k+1}}{(k+1)!}. \tag{9.15}$$

Using this we obtain

$$\int_{1^-}^{N/2}\frac{dT_k(x,N)}{x}=\frac{T_k(x,N)}{x}\Big|_{1^-}^{N/2}+\int_{1^-}^{N/2}\frac{T_k(x,N)}{x^2}dx\ll\frac{(\log\log 3N+7)_{k+2}}{(k+1)!}. \tag{9.16}$$

Combining (9.13) and (9.16) gives the result. The second claim can be similarly handled.

10. SOME GENERALIZATIONS OF LANDAU'S SUM

We shall calculate the Landau sums

$$\sum_{T/2<\upsilon\leq T} n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+a)\right)^{-k} \qquad \text{and} \sum_{0<\gamma(\chi)\leq T} n^{\rho(\chi)},$$

where $k\in\mathbb{N}$, ϱ run through the zeros of $Z_1(s)$, $a=-2$ or 2, and $\rho(\chi)$, $\gamma(\chi)>0$, are the zeros of $L(s,\chi)$, where χ is a Dirichlet character modulo $q\in\mathbb{Z}^+$. Before the estimation of these sums, we give the following lemma which is necessary for the oscillating integrals to be encountered.

Lemma 10.1. *For A large,*

$$\int_A^B w^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^r} = \begin{cases} O\left(\frac{1}{|\log w|(\log A)^r}\right) & \text{if } w\neq 1,\\ \frac{B-A}{\left(\log\frac{A}{2\pi}\right)^r}+O\left(\frac{rA}{(\log A)^{r+1}}\right) & \text{if } w=1,\end{cases}$$

where $A<B\leq 2A$, $w>0$, $r\in\mathbb{N}$, $r=o(\log A)$. The constants of the above $O-$terms are absolute.

Proof. We recall Lemma 4.3 of [16]:

Let $F(x)$, $G(x)$ be real functions, $G(x)/F'(x)$ monotonic, and $F'(x)/G(x)\geq m>0$, or $\leq -m<0$, throughout the interval $[A,B]$. Then

$$\left|\int_A^B G(x)e^{iF(x)}dx\right|\leq\frac{4}{m}.$$

Using this, if $w\neq 1$, we have

$$\left|\int_A^B w^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^r}\right|\leq\frac{4}{|\log w|\left(\log\frac{A}{2\pi}\right)^r}.$$

By Bernoulli's inequality, under the restriction $r = o(\log A)$, we have

$$\frac{1}{\left(\log \frac{A}{2\pi}\right)^r} = \frac{1}{(\log A)^r \left(1 - \frac{\log 2\pi}{\log A}\right)^r} \leq \frac{1}{(\log A)^r \left(1 - r\frac{\log 2\pi}{\log A}\right)} \ll \frac{1}{(\log A)^r}, \tag{10.1}$$

which completes the proof of the case $w \neq 1$.

If $w = 1$, then integrating by parts gives

$$\begin{aligned}\int_A^B \frac{dt}{\left(\log \frac{t}{2\pi}\right)^r} &= \frac{t}{\left(\log \frac{t}{2\pi}\right)^r}\Bigg|_A^B + r\int_A^B \frac{dt}{\left(\log \frac{t}{2\pi}\right)^{r+1}} \\ &= \frac{B}{\left(\log \frac{B}{2\pi}\right)^r} - \frac{A}{\left(\log \frac{A}{2\pi}\right)^r} + O\left(\frac{r(B-A)}{\left(\log \frac{A}{2\pi}\right)^{r+1}}\right). \end{aligned} \tag{10.2}$$

Since $A < B \leq 2A$, by the calculations in (10.1), the above error term is

$$\ll \frac{rA}{(\log A)^{r+1}}. \tag{10.3}$$

By the mean value theorem of Elementary Calculus and (10.1), we see that

$$\frac{1}{\left(\log \frac{B}{2\pi}\right)^r} - \frac{1}{\left(\log \frac{A}{2\pi}\right)^r} = \frac{-(B-A)r}{C\left(\log \frac{C}{2\pi}\right)^{r+1}} = O\left(\frac{r}{(\log A)^{r+1}}\right) \tag{10.4}$$

for some $C \in (A, B)$. Combining (10.2), (10.3) and (10.4) gives the result. $\square$

The calculations and Bernoulli's inequality applications that have been seen in (10.1) and (10.4) will be done repeatedly throughout the paper without any explanation.

Coming back to the main tasks in this section, let's start with the average over the complex zeros of $Z_1(s)$. In light of Lemma 10.1, assume $k = o(\log T)$, where T is a real variable tending to $+\infty$. If $n = 1$, then the sum is almost trivial, since, by (2.8),

$$\frac{\chi'}{\chi}(\varrho + a) = -\log\frac{T}{2\pi} + O(1)$$

for $T/2 < \upsilon \leq T$, from which it follows that

$$\sum_{T/2<\upsilon\leq T}\left(\frac{\chi'}{\chi}(\varrho + a)\right)^{-k} = \frac{(-1)^{k+1}T}{4\pi}\left(\log\frac{T}{2\pi}\right)^{-k+1} + O\left((k+1)T(\log T)^{-k}\right). \quad (10.5)$$

Assume $n > 1$. By the result of Hall on the zeros of $Z_1(s)$ under the Riemann hypothesis, the residue theorem implies that

$$\sum_{T/2<\upsilon\leq T} n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho + a)\right)^{-k} =$$

$$\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s + a)\right)^{-k} n^{-s}\,ds + O(n^{-\frac{1}{2}}(\log T)^{1-k}), \quad (10.6)$$

where $\mathcal{C}$ is the rectangle (oriented counterclockwise) having vertices at $-\delta + it_r$, $1 + \delta + it_r$ $(r = 1, 2)$ with $\delta = \dfrac{1}{\log T}$ and $t_1 = T/2 + O(1)$, $t_2 = T + O(1)$ chosen such that (3.20) is valid on the horizontal sides of $\mathcal{C}$. This choice causes the error term in (10.6), which can be estimated trivially by using (2.8) and (3.5). By (2.8) and (3.20), we see that the integrals along the horizontal sides of $\mathcal{C}$ satisfy

$$\int_{-\delta\pm it_r}^{1+\delta\pm it_r}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s + a)\right)^{-k} n^{-s}\,ds \ll \frac{(\log T)^{2-k}}{\log n}\exp\left(\frac{\log n}{\log T}\right). \quad (10.7)$$

Inserting the right-hand sides of (2.8) and (3.27) instead of $\dfrac{\chi'}{\chi}(s+a)$ and $\dfrac{Z_1'}{Z_1}(s)$, respectively, the integral along the right vertical side of $\mathcal{C}$ becomes

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{1+\delta+it_1}^{1+\delta+it_2} \frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+a)\right)^{-k} n^{-s}ds = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad & \\
-\frac{(-1)^k}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell}\sum_{m=1}^{\infty}\frac{\Lambda^{*(\ell+1)}(m)}{(mn)^{1+\delta}}\int_{t_1}^{t_2}\frac{(mn)^{-it}dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}} & \\
+\frac{(-1)^k}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell+1}\sum_{m=1}^{\infty}\frac{\lambda^{*(\ell)}(m)}{(mn)^{1+\delta}}\int_{t_1}^{t_2}\frac{(mn)^{-it}dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}} & \\
+O\left(\frac{k}{n^{1+\delta}}\int_{t_1}^{t_2}\left|\frac{Z_1'}{Z_1}(1+\delta+it)\right|\left(\log\frac{t}{2\pi}\right)^{-k-1}\frac{dt}{t}\right) & \\
+O\left(\frac{(\log T)^{-k}}{n}\exp\left(\frac{A\log T\log\log\log T}{\log\log T}-\frac{\log n}{\log T}\right)\right). &
\end{aligned}
\tag{10.8}
$$

By (2.8), (2.10), (3.10) and (3.26), $\left|\dfrac{Z_1'}{Z_1}(1+\delta+it)\right| \ll \log\log|t|$ for $|t|\gg 1$, so the first error term is

$$
\ll \frac{k\log\log T}{n(\log T)^k}\exp\left(-\frac{\log n}{\log T}\right). \tag{10.9}
$$

For the above oscillating integrals we first note that $m>1$, otherwise $\Lambda^{*(\ell+1)}(m)$ and $\lambda^{*(\ell)}(m)$ vanish. Applying then Lemma 10.1 to these integrals, and using (2.15), the sums over ℓ can be absorbed in the error term which is

$$
\ll \frac{(\log T)^{1-k}}{n^{1+\delta}\log n}\sum_{\ell\le\frac{\log T}{\log\log T}} A^{\ell} \ll \frac{(\log T)^{-k}}{n\log n}\exp\left(\frac{A\log T}{\log\log T}-\frac{\log n}{\log T}\right). \tag{10.10}
$$

It remains to calculate the integral along the left-vertical side of $\mathcal{C}$. Substituting $s \to 1-s$, and then using (3.3) we obtain

$$\int_{-\delta+it_2}^{-\delta+it_1} \frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+a)\right)^{-k}\frac{ds}{n^s} = -\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+a)\right)^{-k}\frac{ds}{n^{1-s}} + \int_{1+\delta-it_2}^{1+\delta-it_1}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+a)\right)^{-k} n^{s-1}ds.$$

We integrate the first integral on the right by parts and then use the estimates (2.8) and (2.10) to have

$$\begin{aligned}\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+a)\right)^{-k}\frac{ds}{n^{1-s}} &= \frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+a)\right)^{-k}\frac{n^{s-1}}{\log n}\Big|_{1+\delta-it_2}^{1+\delta-it_1}\\ &- \frac{1}{\log n}\int_{1+\delta-it_2}^{1+\delta-it_1}\left(\frac{\chi'}{\chi}\right)'(s)\left(\frac{\chi'}{\chi}(1-s+a)\right)^{-k} n^{s-1}ds\\ &- \frac{1}{\log n}\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{\chi'}{\chi}(s)\frac{d}{ds}\left\{\left(\frac{\chi'}{\chi}(1-s+a)\right)^{-k}\right\} n^{s-1}ds\\ &\ll \frac{(\log T)^{1-k}}{\log n}\exp\left(\frac{\log n}{\log T}\right). \end{aligned} \tag{10.11}$$

Letting $s = 1+\delta-it$, and using (2.8) and (3.27) we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+a)\right)^{-k} n^{s-1}ds = \\ -\frac{(-1)^k n^\delta}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^\ell\sum_{m=1}^\infty\frac{\Lambda^{*(\ell+1)}(m)}{m^{1+\delta}}\int_{t_1}^{t_2}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}\\ +\frac{(-1)^k n^\delta}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell+1}\sum_{m=1}^\infty\frac{\lambda^{*(\ell)}(m)}{m^{1+\delta}}\int_{t_1}^{t_2}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}}\\ +O\left(kn^\delta\int_{t_1}^{t_2}\left|\frac{Z_1'}{Z_1}(1+\delta-it)\right|\left(\log\frac{t}{2\pi}\right)^{-k-1}\frac{dt}{t}\right)\\ +O\left((\log T)^{-k}\exp\left(\frac{A\log T\log\log\log T}{\log\log T}+\frac{\log n}{\log T}\right)\right). \end{aligned} \tag{10.12}$$

It is easy to see that the second error term above dominates the others that occur in (10.7), (10.8), (10.10) (10.11) and the first error term in (10.12), and as a result putting all together,

$$\sum_{T/2<v\le T} n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+a)\right)^{-k} = \tag{10.13}$$
$$-\frac{(-1)^k n^{\delta}}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell}\sum_{m=1}^{\infty}\frac{\Lambda^{*(\ell+1)}(m)}{m^{1+\delta}}\int_{t_1}^{t_2}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}$$
$$+\frac{(-1)^k n^{\delta}}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell+1}\sum_{m=1}^{\infty}\frac{\lambda^{*(\ell)}(m)}{m^{1+\delta}}\int_{t_1}^{t_2}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}}$$
$$+O\left((\log T)^{-k}\exp\left(\frac{A\log T\log\log\log T}{\log\log T}+\frac{\log n}{\log T}\right)\right).$$

We divide the above sums over m into three parts; the term with $m=n$, the terms with $0<|m-n|<n/2$, the terms with $|m-n|\ge n/2$. By Lemma 10.1, the diagonal term $m=n$ gives

$$\frac{(-1)^{k+1}T}{4\pi n}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell}\left(\frac{\Lambda^{*(\ell+1)}(n)}{\left(\log\frac{T}{2\pi}\right)^{k+\ell}}-\frac{2\lambda^{*(\ell)}(n)}{\left(\log\frac{T}{2\pi}\right)^{k+\ell+1}}\right)$$
$$+O\left(\frac{T}{n}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell}(k+\ell+1)\left(\frac{\Lambda^{*(\ell+1)}(n)}{\left(\log\frac{T}{2\pi}\right)^{k+\ell+1}}+\frac{2\lambda^{*(\ell)}(n)}{\left(\log\frac{T}{2\pi}\right)^{k+\ell+2}}\right)\right). \tag{10.14}$$

The contribution from the non-diagonal terms to (10.13) is

$$\ll n^{\delta}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell}\sum_{m\ne n}\frac{\Lambda^{*(\ell+1)}(m)}{m^{1+\delta}}\frac{1}{\left|\log\frac{m}{n}\right|(\log T)^{k+\ell}}$$
$$+n^{\delta}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell+1}\sum_{m\ne n}\frac{\lambda^{*(\ell)}(m)}{m^{1+\delta}}\frac{1}{\left|\log\frac{m}{n}\right|(\log T)^{k+\ell+1}},$$

by Lemma 10.1. For $m\le n/2$ and $m\ge 3n/2$, we have $|\log\frac{n}{m}|\gg 1$, so using this and (2.15) the sum of the terms with $|m-n|\ge n/2$ is bounded by

$$n^{\delta}(\log T)^{1-k}\sum_{\ell\le\frac{\log T}{\log\log T}} A^{\ell}\ll(\log T)^{-k}\exp\left(\frac{A\log T}{\log\log T}+\frac{\log n}{\log T}\right). \tag{10.15}$$

In the remaining contribution to (10.13) we can use $|\log\frac{n}{m}| \gg \frac{|n-m|}{m}$, so the sum of the terms with $0 < |m-n| < n/2$ is

$$\le (\log T)^{-k} \sum_{\ell \le \frac{\log T}{\log\log T}} \left(\frac{2}{\log T}\right)^{\ell} \sum_{0<|m-n|<n/2} \frac{\Lambda^{*(\ell+1)}(m)}{|m-n|} + (\log T)^{-k} \sum_{\ell \le \frac{\log T}{\log\log T}} \left(\frac{2}{\log T}\right)^{\ell+1} \sum_{0<|m-n|<n/2} \frac{\lambda^{*(\ell)}(m)}{|m-n|}.$$

By considering separately the inner sums, $n/2 < m < n$ and $n < m < 3n/2$, from Lemma 9.4 and 9.5, we see that the above is dominated by

$$(\log T)^{-k}(\log 2n) \sum_{\ell \le \frac{\log T}{\log\log T}} \left(\frac{2\log 2n}{\log T}\right)^{\ell} \frac{(\log\log 3n+7)_{\ell+1}}{\ell!} + (\log T)^{-k}(\log 2n) \sum_{\ell \le \frac{\log T}{\log\log T}} \left(\frac{2\log 2n}{\log T}\right)^{\ell+1} \frac{(\log\log 3n+7)_{\ell+2}}{(\ell+1)!}.$$

The first factors in the above summands are trivially $O\left(\exp\left(\frac{A\log T}{\log\log T}\right)\right)$, subject to the constraint $\log n \ll \log T$, so that the above bound is

$$\ll (\log T)^{-k} \exp\left(\frac{A\log T}{\log\log T}\right) \sum_{\ell \le \frac{\log T}{\log\log T}} \frac{(\log\log 3n+7)_{\ell}}{\ell!}$$
$$= (\log T)^{-k} \exp\left(\frac{A\log T}{\log\log T}\right) \frac{(\log\log 3n+8)_{[\![\frac{\log T}{\log\log T}]\!]}}{[\![\frac{\log T}{\log\log T}]\!]!},$$

by Lemma 9.3. Observe that

$$\frac{(\log\log 3n+8)_{[\![\frac{\log T}{\log\log T}]\!]}}{[\![\frac{\log T}{\log\log T}]\!]!} = \exp\left(\sum_{i=[\![\log\log 3n]\!]+8}^{[\![\frac{\log T}{\log\log T}]\!]} \log\left(1+\frac{\log\log 3n+7}{i}\right)\right) \times \prod_{1\le i\le \log\log 3n+7} \left(1+\frac{\log\log 3n+7}{i}\right).$$

The product is trivially

$$\leq (\log\log 3n + 8)^{[\![\log\log 3n]\!]+7}.$$

Using $\log(1+x) < x$, for $|x| < 1$, the exponential part above is

$$\leq \exp\left((\log\log 3n + 7) \sum_{i=[\![\log\log 3n]\!]+8}^{[\![\frac{\log T}{\log\log T}]\!]} \frac{1}{i} \right) \leq \exp\left((\log\log 3n + 7) \log\log T \right).$$

So the contribution of the terms with $0 < |m-n| < n/2$ is

$$\ll (\log T)^{-k} \exp\left(\frac{A \log T}{\log\log T} \right).$$

Combining the above in (10.13) leads to

Theorem 10.1. *(RH) As $T \to +\infty$, for $a = -2$ or $a = 2$, $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ satisfying $\log n \ll \log T$ and $k = o(\log T)$, we have*

$$\sum_{T/2<\upsilon\leq T} n^{-\varrho} \left(\frac{\chi'}{\chi}(\varrho + a) \right)^{-k} =$$

$$\frac{(-1)^{k+1} T}{4\pi n \left(\log \frac{T}{2\pi}\right)^k} \left(-[n=1] \log \frac{T}{2\pi} + \sum_{\ell \leq \frac{\log T}{\log\log T}} 2^\ell \left(\frac{\Lambda^{*(\ell+1)}(n)}{\left(\log \frac{T}{2\pi}\right)^\ell} - \frac{2\lambda^{*(\ell)}(n)}{\left(\log \frac{T}{2\pi}\right)^{\ell+1}} \right) \right)$$

$$+ O\left(\frac{T(k+1) \left([n=1] \log \frac{T}{2\pi} + \sum_{\ell \leq \frac{\log T}{\log\log T}} A^\ell \left(\frac{\Lambda^{*(\ell+1)}(n)}{\left(\log \frac{T}{2\pi}\right)^\ell} + \frac{2\lambda^{*(\ell)}(n)}{\left(\log \frac{T}{2\pi}\right)^{\ell+1}} \right) \right)}{n \left(\log \frac{T}{2\pi}\right)^{k+1}} \right)$$

$$+ O\left((\log T)^{-k} \exp\left(\frac{A \log T \log\log\log T}{\log\log T} \right) \right).$$

We now deal with the special case of Theorem 10.1 that $k = 0$ and n is a prime power. Assume $n = p^r$, $r \in \mathbb{Z}^+$, $\log n \ll \log T$. From Lemma 9.2, we see that $\Lambda^{*(\ell+1)}(n) = 0$ and $\lambda^{*(\ell)}(n) = 0$ for $\ell \geq r$. If $r \leq \frac{\log T}{\log\log T} + 1$, then Theorem 10.1

becomes

$$\sum_{T/2<v\le T} n^{-\varrho} = \frac{T\log p}{4\pi p^r}\left(\left(1+\frac{2\log p}{\log\frac{T}{2\pi}}\right)^r - 2\right)$$
$$+ O\left(\frac{T}{p^r}\left(1+\frac{A\log p}{\log\frac{T}{2\pi}}\right)^r + \exp\left(\frac{A\log T\log\log\log T}{\log\log T}\right)\right).$$

Otherwise, $r > \frac{\log T}{\log\log T}$, adding and substracting the terms with $\frac{\log T}{\log\log T} < \ell \le r$, we again arrive at the main term of the first case, within an error term

$$\ll \frac{T\log p}{p^r}\sum_{\frac{\log T}{\log\log T}<\ell\le r}\left(\binom{r-1}{\ell}\left(\frac{2\log p}{\log\frac{T}{2\pi}}\right)^{\ell} + 2\binom{r}{\ell+1}\left(\frac{2\log p}{\log\frac{T}{2\pi}}\right)^{\ell+1}\right.$$
$$\left. + \binom{r-1}{\ell}\left(\frac{2\log p}{\log\frac{T}{2\pi}}\right)^{\ell+1}\right)$$
$$\ll \frac{T\log p}{p^r}\sum_{\frac{\log T}{\log\log T}<\ell\le r}\binom{r}{\ell+1}\left(\frac{A\log p}{\log T}\right)^{\ell}.$$

Employing the inequalities $\binom{r}{\ell+1} \le \left(\frac{re}{\ell+1}\right)^{\ell+1}$, which is a simple consequence of the Stirling formula, and $\log n = r\log p \ll \log T$, the above can be simplified to

$$\frac{T\log p^r}{p^r}\sum_{\frac{\log T}{\log\log T}<\ell\le r}\left(\frac{Ar\log p}{\ell\log T}\right)^{\ell} \ll \frac{T\log p^r}{p^r}\sum_{\frac{\log T}{\log\log T}<\ell\le r}\left(\frac{A}{\ell}\right)^{\ell}$$
$$\ll \frac{T\log p^r}{p^r}\left(\frac{A\log\log T}{\log T}\right)^{\frac{\log T}{\log\log T}} \ll p^{-r}\exp\left(\frac{A\log T\log\log\log T}{\log T}\right).$$

At the end of these calculations, in the case considered Theorem 10.1 reduces to the following Corollary.

Corollary 10.1. *(RH) Suppose that $n = p^r$ is a prime power, $r \in \mathbb{Z}^+$. As $T \to +\infty$,* *if* $\log n \ll \log T$*, then we have*

$$\sum_{T/2<v\leq T} n^{-\varrho} = \frac{T\log p}{4\pi p^r}\left(\left(1+\frac{2\log p}{\log\frac{T}{2\pi}}\right)^r - 2\right)$$
$$+O\left(\frac{T}{p^r}\left(1+\frac{A\log p}{\log\frac{T}{2\pi}}\right)^r + \exp\left(\frac{A\log T\log\log\log T}{\log\log T}\right)\right).$$

For the remainder of this section we shall be dealing with the Landau sum over the zeros of $L(s,\chi)$. If $n=1$, then

$$\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)} = N_\chi(T) = |\rho(\chi) :\ L(\rho(\chi),\chi)=0,\ 0<\gamma(\chi)\leq T|$$
$$= \frac{T}{2\pi}\log\frac{T}{2\pi} + O\left(T\log 2q\right) \tag{10.16}$$

for any Dirichlet character χ modulo $q\in\mathbb{Z}^+$. Assume $n>1$. If χ is principal, then

$$L(s,\chi) = \zeta(s)\prod_{p|q}\left(1-\frac{\chi(p)}{p^s}\right). \tag{10.17}$$

In addition to the zeros of $\zeta(s)$, $L(s,\chi)$ has extra zeros on the imaginary axis, and the number of these additional zeros up to height T is $O(T\log q)$. So the contribution of these zeros to the sum $\displaystyle\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)}$ is $O(T\log 2q)$, and by Gonek's uniform estimate quoted in (8.6) we have

$$\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)} = -\frac{T}{2\pi}\Lambda(n) + O(n\log 2nT\log\log 3n) + O(T\log 2q)$$
$$+ O\big(\min\big(T, \frac{n}{\langle n\rangle}\big)\log n\big). \tag{10.18}$$

Assume χ is non-principal, so $q\geq 3$. Then there exists $q_1|q$ such that χ is induced by a primitive character χ^* modulo q_1. Since (10.17) holds if we write $L(s,\chi^*)$ instead of

$\zeta(s)$, we can reduced an imprimitive case to a primitive one, and similarly we have

$$\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)} = \sum_{0<\gamma(\chi^*)<T} n^{\rho(\chi^*)} + O(T\log q). \tag{10.19}$$

Suppose $T \geq T_0$ is not the imaginary part of the zeros of $L(s,\chi^*)$ and further that $|T-\gamma(\chi^*)| \gg 1/\log qT$ for any $\gamma(\chi^*)$. This restriction on T is harmless within the error term $O(n\log qT)$. For this chosen T we have

$$\frac{L'}{L}(\sigma+iT) \ll \log^2 qT, \tag{10.20}$$

uniformly on any bounded range of σ. We put $c = 1 + 1/\log 3n$, and consider the integral around the rectangle R joining the points $c+ic_0$, $c+iT$, $1-c+iT$, $1-c+ic_0$, where $c_0$is a sufficiently large constant so that the estimate (10.20) holds along the line segment $[1-c+ic_0, c+ic_0]$. By the residue theorem we have

$$\begin{aligned}\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)} =& \frac{1}{2\pi i}\left(\int_{c+ic_0}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+ic_0} + \int_{1-c+ic_0}^{c+ic_0}\right)\frac{L'}{L}(s,\chi^*)n^s ds \\ &+ O((n+T)\log qT) = I_1+I_2+I_3+I_4+O((n+T)\log qT), \text{ say.}\end{aligned} \tag{10.21}$$

Here we note that the contribution of the terms with $0 < \gamma(\chi) < c_0$ to the above sum is trivially absorbed in the error term $O(n\log q)$. It follows from (10.20) that $I_2 + I_4 \ll n\log^2 qT$. It is possible to sharpen this estimate by means of the asymptotic formula for L'/L with an error term $\log qT$. But this weak result is sufficient for our aims.

We now deal with I_1. Replacing $\frac{L'}{L}(s,\chi^*)$ by $-\sum_{m=2}^{\infty}\Lambda(m)\chi^*(m)m^{-s}$ and integrating term by term, we find that

$$I_1 = -\sum_{m=2}^{\infty}\Lambda(m)\chi^*(m)\left(\frac{n}{m}\right)^c\frac{1}{2\pi}\int_{c_0}^{T}\left(\frac{n}{m}\right)^{it}dt$$
$$= -\frac{T-c_0}{2\pi}\Lambda(n)\chi^*(n) + O\left(\sum_{\substack{m=2\\ m\neq n}}^{\infty}\Lambda(m)\left(\frac{n}{m}\right)^c\min\left(T,\frac{1}{|\log(n/m)|}\right)\right).$$

By Lemma 2 in [14] we obtain

$$I_1 = -\frac{T}{2\pi}\Lambda(n)\chi^*(n) + O\left(n\log 2n\log\log 3n\right) + O\left(\log n\min\left(T,\frac{n}{\langle n\rangle}\right)\right).$$

Finally we treat I_3, and then this case is completed. We first write the logarithmic derivative of the functional equation for $L(s,\chi^*)$:

$$\frac{L'}{L}(s,\chi^*) = -\log\frac{q_1|t|}{2\pi} - \frac{L'}{L}(1-s,\overline{\chi^*}) + O\left(\frac{1}{|t|}\right), \qquad (-1<\sigma<2,\ |t|\geq t_0).$$

Using this, I_3 becomes

$$I_3 = \frac{n^{1-c}}{2\pi}\int_{c_0}^{T}n^{it}\frac{L'}{L}(c-it,\overline{\chi^*})dt + \frac{n^{1-c}}{2\pi}\int_{c_0}^{T}n^{it}\log\frac{q_1 t}{2\pi}dt + O(\log T).$$

Integration by parts the second integral is $\ll \frac{\log q_1 T}{\log 2n}$. Using the Dirichlet series representation of L'/L the first integral becomes

$$= -\frac{n^{1-c}}{2\pi}\sum_{m\geq 2}\Lambda(m)\overline{\chi^*}(m)m^{-c}\int_{c_0}^{T}(nm)^{it}dt \ll \sum_{m\geq 2}\frac{\Lambda(m)}{m^c\log nm} \ll 1.$$

Collecting all these results we obtain the q-analogue of (8.6):

Proposition 10.1. *Suppose that $T\geq T_0$ and $n\in\mathbb{Z}^+$. Let χ be a Dirichlet character* *modulo* $q \in \mathbb{Z}^+$. *Then if* $n = 1$,

$$\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)} = \frac{T}{2\pi} \log \frac{T}{2\pi} + O\left(T \log 2q\right); \tag{10.22}$$

if χ *is principal and* $n > 1$,

$$\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)} = -\frac{T}{2\pi}\Lambda(n) + O(n \log 2nT \log\log 3n) + O(T \log q)$$
$$+ O\big(\min\big(T, \frac{n}{\langle n \rangle}\big) \log n\big);$$

if χ *is non-principal and* $n > 1$,

$$\sum_{0<\gamma(\chi)<T} n^{\rho(\chi)} = -\frac{T}{2\pi}\Lambda(n)\chi^*(n) + O(T \log q) + O(n \log^2 qT)$$
$$+ O\left(n \log 2n \log\log 3n\right) + O\left(\log n \min\left(T, \frac{n}{\langle n \rangle}\right)\right),$$

where χ^* *is a primitive character induces* χ.

11. CORRELATION OF ZETA ZEROS WITH THE RELATIVE MAXIMA OF $|\zeta(\frac{1}{2}+it)|$

We introduce the correlation function

$$G_{\zeta,Z_1}(x,T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\upsilon\le T} x^{i(\gamma-\upsilon)}\omega(\gamma-\upsilon), \tag{11.1}$$

where $x \ge 1$ and $T \to +\infty$. Here $\omega(u)$ is a suitable weighting function, $\omega(u) = 4/(4+u^2)$. From (3.5) and (7.3) we have

$$\begin{aligned} \sum_{\gamma}\sum_{T/2\le\upsilon\le T} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} &= -x^{-2}\sum_{n\le x}\Lambda(n)n^2 \sum_{T/2\le\upsilon\le T} n^{-\varrho} \\ &\quad - x^2\sum_{n>x}\Lambda(n)n^{-2}\sum_{T/2\le\upsilon\le T} n^{-\varrho} + \frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right) \\ &\quad + O\Big(\frac{x^{\frac{1}{2}}\log T}{T}\Big) + O(x^{-\frac{5}{2}}\log T) \\ &= \Sigma_1 + \Sigma_2 + \frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right) + O\Big(\frac{x^{\frac{1}{2}}\log T}{T}\Big) + O(x^{-\frac{5}{2}}\log T) \end{aligned} \tag{11.2}$$

for $x \ge 1$ and $T \ge T_0$. In the same way as for (8.3), those $\gamma \notin [T/2, T]$ can be discarded from the sum on the left-hand side within an error of $O(\log^3 T)$. In view of Corollary 10.1 we impose the constraint $\log x \ll \log T$ and divide the sum over $n > x$ into two parts, $T^M \ge n > x$ and $n > T^M$, where M is large fixed constant, so that Corollary 10.1 is applicable to the first $\upsilon-$sum and the first part of the second $\upsilon-$sum. However, the remainder of the second sum can be estimated trivially. As a result, we have

$$\begin{aligned}\Sigma_1 =&\frac{T}{4\pi x^2}\sum_{p\le x} p(\log p)^2 - \frac{T}{2\pi x^2}\left(\log\frac{T}{2\pi}\right)^{-1}\sum_{p\le x} p(\log p)^3 \\ &+ O\left(\frac{T}{x^2}\sum_{p\le x} p(\log p)\right) + O\left(\frac{T}{x^2}\sum_{\substack{p^r\le x\\ r\in\mathbb{Z}^+\\ r\ge 2}} p^r(\log p)^2\left(1+\frac{A\log p}{\log\frac{T}{2\pi}}\right)^r\right) \\ &+ O\left(\exp\left(\frac{A\log T\log\log\log T}{\log\log T}\right)x^{-2}\sum_{n\le x}\Lambda(n)n^2\right) \\ =&\Sigma_{1,1}-\Sigma_{1,2}+\Sigma_{1,3}+\Sigma_{1,4}+\Sigma_{1,5},\text{ say,}\end{aligned} \tag{11.3}$$

and

$$\begin{aligned}\Sigma_2 =&\frac{Tx^2}{4\pi}\sum_{T^M\ge p>x}\frac{(\log p)^2}{p^3} - \frac{Tx^2}{2\pi}\left(\log\frac{T}{2\pi}\right)^{-1}\sum_{T^M\ge p>x}\frac{(\log p)^3}{p^3} \\ &+ O\left(Tx^2\sum_{T^M\ge p>x}\frac{\log p}{p^3}\right) + O\left(Tx^2\sum_{\substack{T^M\ge p^r>x\\ r\in\mathbb{Z}^+\\ r\ge 2}}\frac{(\log p)^2}{p^{3r}}\left(1+\frac{A\log p}{\log\frac{T}{2\pi}}\right)^r\right) \\ &+ O\left(\exp\left(\frac{A\log T\log\log\log T}{\log\log T}\right)x^2\sum_{T^M\ge n>x}\Lambda(n)n^{-2}\right) \\ &+ O\left(x^2T(\log T)\sum_{n>T^M}\Lambda(n)n^{-5/2}\right) \\ =&\Sigma_{2,1}-\Sigma_{2,2}+\Sigma_{2,3}+\Sigma_{2,4}+\Sigma_{2,5}+\Sigma_{2,6},\text{ say.}\end{aligned} \tag{11.4}$$

In $\Sigma_{1,5}$ we remove the factor n^2 from the sum trivially, then by the prime number theorem,

$$\Sigma_{1,5} \ll x\exp\left(\frac{A\log T\log\log\log T}{\log\log T}\right). \tag{11.5}$$

As simple consequences of the prime number theorem and partial summation, we have

$$\sum_{n>x} \frac{\Lambda(n)}{n^{\ell_1}} = \frac{1}{(\ell_1 - 1)x^{\ell-1}} + O\left(\frac{1}{x^{\ell-1}\log 2x}\right), \tag{11.6}$$

$$\sum_{p\le x} p^{\ell_2} \log^k p = \frac{x^{\ell_2+1}\log^{k-1} x}{\ell_2 + 1} + O\left(x^{\ell_2+1}\log^{k-2} 2x\right), \tag{11.7}$$

$$\sum_{p>x} \frac{\log^k p}{p^{\ell_3}} = \frac{\log^{k-1} x}{(\ell_3 - 1)x^{\ell_3-1}} + O\left(\frac{\log^{k-2} 2x}{x^{\ell_3-1}}\right), \tag{11.8}$$

for ℓ_2, $x \ge 1$, ℓ_1, $\ell_3 \ge 2$ and $k = 1, 2$ or 3. In view of these results, compared the main terms coming from $\Sigma_{1,1}$ and $\Sigma_{1,2}$ with the error term in (11.5), we are forced to require the condition $x \le T^{1-\epsilon}$. So

$$\Sigma_{1,1} - \Sigma_{1,2} = \Sigma_{2,1} - \Sigma_{2,2} = \frac{T\log x}{8\pi}\left(1 - \frac{2\log x}{\log\frac{T}{2\pi}} + O\left(\frac{1}{\log 2x}\right)\right), \tag{11.9}$$

$$\Sigma_{1,3}, \Sigma_{2,3} \ll T \tag{11.10}$$

if $M > 1$. For $\Sigma_{1,4}$ and $\Sigma_{2,4}$ we use the inequality that $(1+u)^v \le \exp(uv)$ for $u, v > 0$, so that

$$\begin{aligned}\Sigma_{1,4} + \Sigma_{2,4} \ll{}& Tx^{-2} \sum_{2\le r\le\frac{\log x}{\log 2}} \sum_{p\le x^{1/r}} p^{r+\frac{Ar}{\log\frac{T}{2\pi}}} \log^2 p \\ &+ Tx^2 \sum_{2\le r\le C\log x} \sum_{p>x^{1/r}} \frac{(\log p)^2}{p^{3r-\frac{Ar}{\log\frac{T}{2\pi}}}} - Tx^2 \sum_{2\le r\le C\log x} \sum_{p>T^{M/r}} \frac{(\log p)^2}{p^{3r-\frac{Ar}{\log\frac{T}{2\pi}}}} \\ &+ Tx^2 \sum_{C\log x<r\le\frac{M\log T}{\log 2}} \sum_{p\le T^{\frac{M}{r}}} \frac{(\log p)^2}{p^{3r-\frac{Ar}{\log\frac{T}{2\pi}}}},\end{aligned} \tag{11.11}$$

where $C > 0$ is so large that $x^{1/r} < 2$ for $r > C\log x$. By (11.7) and (11.8), the first two parts on the right above is $\ll \dfrac{T\log x}{\sqrt{x}}$. The third is, by (11.8),

$$\ll \frac{x^2\log xT}{T^{5M/2-1}} \ll 1 \tag{11.12}$$

if $M > 6/5$. In $\Sigma_{2,5}$ and $\Sigma_{2,6}$ by (11.6) we have

$$\Sigma_{2,5} + \Sigma_{2,6} \ll \left(x + \frac{x^2}{T^M} \right) \exp \left(\frac{A \log T \log\log\log T}{\log\log T} \right) \ll T^{1-\epsilon} \tag{11.13}$$

under the constraints $x \leq T^{1-\epsilon}$ and $M > 6/5$. For the last one in (11.11), we note that the sum over p is trivially $\ll 2^{-3r}$. Then the fourth part is

$$\ll \frac{T}{x^{2C\log 2 - 2}} \ll \frac{T \log x}{\sqrt{x}} \tag{11.14}$$

if C is sufficiently large. Finally, putting all findings together we arrive at

$$\begin{aligned}\sum_{\gamma} \sum_{T/2 < \upsilon \leq T} \frac{4x^{i(\gamma-\upsilon)}}{4 + (\gamma - \upsilon)^2} = & \frac{T \log x}{4\pi} \left(1 - \frac{2 \log x}{\log \frac{T}{2\pi}} + O\left(\frac{1}{\log 2x} \right) \right) \\ & + \frac{T \log^2 T}{4\pi x^2} \left(1 + O\left(\frac{1}{\log T} \right) \right)\end{aligned}$$

for sufficiently large T, say $T > T_0$. We replace T by $T/2$, $T/2^2, \ldots$ and add all the results. This process contains $\ll \log T$ steps. We then have

$$\begin{aligned}\sum_{\gamma} \sum_{0 < \upsilon \leq T} \frac{4x^{i(\gamma-\upsilon)}}{4 + (\gamma - \upsilon)^2} = & \sum_{\gamma} \sum_{0 < \upsilon \leq T_0} \frac{4x^{i(\gamma-\upsilon)}}{4 + (\gamma - \upsilon)^2} \\ & + \frac{T \log x}{4\pi} \left(\sum_{0 \leq k \ll \log T} 2^{-k} \right) \left(1 + O\left(\frac{1}{\log 2x} \right) \right) \\ & - \frac{T \log x}{2\pi} \sum_{0 \leq k \ll \log T} \frac{1}{2^k \left(\log \frac{T}{2\pi} - k \log 2 \right)} \\ & + \frac{T \log^2 T}{4\pi x^2} \left(\sum_{0 \leq k \ll \log T} 2^{-k} \left(1 + O\left(\frac{k}{\log T} \right) \right) \right) \left(1 + O\left(\frac{1}{\log T} \right) \right).\end{aligned} \tag{11.15}$$

For any $d > 1$ and $C > 0$, we have

$$\sum_{0 \leq k \leq C} d^{-k} = \left(1 - \frac{1}{d} \right)^{-1} \left(1 + O\left(\exp\left(-(\log d) [\![C]\!] \right) \right) \right).$$

We examine the second sum over k in (11.15) by separating the range into two parts according to whether $k \ll \log\log T$. The first part on the right of (11.15) is $\ll \log T$, which can be seen by dividing the double sum into two parts, according to $\gamma \in (0, T_0]$ or not and using (2.11). Then, in the same way as for the pair correlation of zeta zeros, we discard the terms with $\gamma \notin (0, T_0]$ within an error term of $O\left((\log T)^3\right)$. As a result of these two steps,

$$\sum_{0<\gamma,\upsilon\leq T} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} = \frac{T\log x}{2\pi}\left(1-\frac{2\log x}{\log T}+O\left(\frac{1}{\log 2x}\right)\right)$$
$$+\frac{T\log^2 T}{2\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right).$$

Putting $x = T^\alpha$ and re-defining $G_{\zeta,Z_1}(x,T)$ as

$$G_{\zeta,Z_1}(\alpha) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\nu\leq T} T^{i\alpha(\gamma-\nu)}\omega(\gamma-\nu), \tag{11.16}$$

we arrive at

Theorem 11.1. *Assume the Riemann Hypothesis. Then $G_{\zeta,Z_1}(\alpha)$ is asymptotically real and even. Further, uniformly for $-1+\epsilon \leq \alpha \leq 1-\epsilon$, we have*

$$G_{\zeta,Z_1}(\alpha) = |\alpha| - 2|\alpha|^2 + (1+o(1))\,T^{-2|\alpha|}\log T + o(1).$$

12. CORRELATION OF THE ZEROS OF TWO DISTINCT DIRICHLET $L-$ FUNCTIONS

Let χ, ψ be two primitive character modulo q, q', respectively. Here q and q' are fixed positive integers. We quote the analogue of Montgomery's explicit formula from [17]:

$$(2\sigma-1)\sum_{\gamma(\chi)}\frac{x^{i\gamma(\chi)}}{(\sigma-\frac{1}{2})^2+(t-\gamma(\chi))^2}=-x^{-\frac{1}{2}}\Big(\sum_{n\leq x}\Lambda(n)\chi(n)(\frac{x}{n})^{1-\sigma+it}$$
$$+\sum_{n>x}\Lambda(n)\chi(n)(\frac{x}{n})^{\sigma+it}\Big)+x^{\frac{1}{2}-\sigma+it}\left(\log\frac{q\tau}{2\pi}+O_\sigma(1)\right)+O\left(\frac{x^{-1/2-\mathfrak{a}}}{\tau}\right),$$

where $\sigma>1$, $x\geq 1$ and

$$\mathfrak{a}=\begin{cases}0 & \text{if} \quad \chi(-1)=1,\\ 1 & \text{if} \quad \chi(-1)=-1.\end{cases}$$

This formula is valid under the truth of GRH for $L(s,\chi)$. Taking $\sigma=5/2$, and letting t run through those ordinates $\gamma(\psi)\in(0,T]$ of the zeros of $L(s,\psi)$, we obtain

$$\sum_{\gamma(\chi)}\sum_{0<\gamma(\psi)<T}\frac{4x^{i(\gamma(\chi)-\gamma(\psi))}}{4+(\gamma(\psi)-\gamma(\chi))^2}=-x^{-2}\sum_{n\leq x}\Lambda(n)\chi(n)n^2\sum_{0<\gamma(\psi)<T}n^{-\varrho(\psi)}$$
$$-x^2\sum_{n>x}\Lambda(n)\chi(n)n^{-2}\sum_{0<\gamma(\psi)<T}n^{-\varrho(\psi)}+x^{-2}\sum_{0<\gamma(\psi)<T}(\log q\gamma(\psi)+O(1))$$
$$+O\left(x^{-1/2}\log^2 T\right).$$

As in the previous two pair correlation cases, on the left-hand side above the contribution of the terms with $\gamma(\chi)\notin(0,T]$ is $O\left(\log^3 T\right)$. By Stieltjes integration and the formula

$$\begin{aligned} N(T,\psi) &= \big| \{\sigma + it : 0 < \sigma < 1,\, 0 < t < T,\, L(\sigma + it, \psi) = 0\} \big| \\ &= \frac{T}{2\pi} \log \frac{q'T}{2\pi} - \frac{T}{2\pi} + O(\log T), \end{aligned}$$

we find that

$$\sum_{0<\gamma(\psi)<T} \log q\gamma(\psi) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^2 \left(1 + O\left(\frac{1}{\log T} \right) \right).$$

It then follows from the above result and Proposition 10.1 that

$$\begin{aligned} \sum_{0<\gamma(\psi),\gamma(\chi)\le T} \frac{4x^{i(\gamma(\chi)-\gamma(\psi))}}{4 + (\gamma(\psi) - \gamma(\chi))^2} &= \frac{T}{2\pi x^2} \sum_{n\le x} \Lambda^2(n)(\overline{\psi}\chi)(n)n \qquad (12.1) \\ &+ \frac{Tx^2}{2\pi} \sum_{n>x} \frac{\Lambda^2(n)(\overline{\psi}\chi)(n)}{n^3} + \frac{T}{2\pi x^2} \left(\log \frac{qT}{2\pi} \right)^2 \left(1 + O\left(\frac{1}{\log T} \right) \right) \\ &+ O\Bigg((\log T)^3 + \frac{T \log\log T}{x^2} \sum_{n\le x} \Lambda(n)n + \frac{(\log T)^2}{x^2} \sum_{n\le x} \Lambda(n)n^2 \qquad (12.2) \\ &+ Tx^2 \log\log T \sum_{n>x} \frac{\Lambda(n)}{n^3} + x^2 (\log T)^2 \sum_{n>x} \frac{\Lambda(n)}{n^2} \Bigg) \end{aligned}$$

under the restriction $1 \le x \le T^{1-\epsilon}$. If $\chi \neq \psi$, then we have

$$\sum_{n\le x} \Lambda^2(n)(\chi\psi)(n) \ll x \exp(-\tilde{c}\sqrt{\log x}) \qquad (12.3)$$

for some $\tilde{c} > 0$. This assertion can be proved easily by first observing that

$$\sum_{n\ge 1} \frac{\Lambda^2(n)(\chi\psi)(n)}{n^s} = -\frac{d}{ds} \sum_{n\ge 1} \frac{\Lambda(n)(\chi\psi)(n)}{n^s},$$

and then modifying the proof of prime number theorem for arithmetic progressions. By (12.3) and partial summation,

$$\sum_{n>x} \frac{\Lambda^2(n)(\overline{\psi}\chi)(n)}{n^3} \ll x^{-2}\exp(-\tilde{c}\sqrt{\log x}). \tag{12.4}$$

We divide the range of integral we encounter in the partial summation application into two parts, $[x, x^2]$ and $[x^2, \infty)$, and in the second range we bound $\sum_{n\le u} \Lambda^2(n)(\overline{\chi}\psi)(n)$ by the amount $u \log u$ with the aid of the prime number theorem. The calculations of the sums over n in (12.1) when $\chi = \psi$ are the same as those in §8. As a result, if $\chi \neq \psi$,

$$\sum_{0<\gamma(\psi),\gamma(\chi)\le T} \frac{4x^{i(\gamma(\chi)-\gamma(\psi))}}{4+(\gamma(\psi)-\gamma(\chi))^2} = \frac{T}{2\pi x^2}\left(\log\frac{T}{2\pi}\right)^2\left(1+O\left(\frac{1}{\log T}\right)\right);$$

if $\chi = \psi$,

$$\begin{aligned}\sum_{0<\gamma(\psi),\gamma(\chi)\le T} \frac{4x^{i(\gamma(\chi)-\gamma(\psi))}}{4+(\gamma(\psi)-\gamma(\chi))^2} = \frac{T\log x}{2\pi}\left(1+O\left(\frac{1}{\log 2x}\right)\right)& \\ + \frac{T}{2\pi x^2}\left(\log\frac{T}{2\pi}\right)^2\left(1+O\left(\frac{1}{\log T}\right)\right)&.\end{aligned}$$

We define

$$F_{\chi,\psi}(\alpha) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma(\chi),\gamma(\psi)\le T} T^{i\alpha(\gamma(\chi)-\gamma(\psi))}\omega(\gamma(\chi)-\gamma(\psi)), \tag{12.5}$$

then we get

Theorem 12.1. *Assume GRH for $L(s,\chi)$ or $L(s,\psi)$. Then $F_{\chi,\psi}(\alpha)$ is asymptotically real and even. Further, uniformly for $-1+\epsilon \le \alpha \le 1-\epsilon$, we have*

$$F_{\chi,\psi}(\alpha) = E(\chi,\psi)|\alpha| + (1+o(1))\,T^{-2|\alpha|}\log T + o(1).$$

where

$$E(\chi, \psi) = \begin{cases} 0 & \textit{if} \quad \chi \neq \psi, \\ 1 & \textit{otherwise.} \end{cases}$$

For any imprimitive character χ mod q, there exist a primitive character $\tilde{\chi}$ mod $\tilde{q}|q$ such that

$$L(s, \chi) = L(s, \tilde{\chi}) \prod_{p|q} \left(1 - \frac{\tilde{\chi}(p)}{p^s}\right),$$

from which it follows that the zeros of $L(s, \chi)$ and $L(s, \tilde{\chi})$ lying on the critical line coincide. So we drop the restriction that χ and ψ are primitive.

In the above pair correlation result if we take an arbitrary non-real character χ and $\psi = \overline{\chi}$, then the first part of the main term vanishes, and for $\alpha \geq \frac{\log\log T}{\log T}$, the secondary main term is dominated by the error term. In light of the functional equation of $L(s, \psi)$ we know that the zeros of $L(s, \psi)$ lying in the upper half-plane coincide precisely with the zeros of $L(s, \chi)$ lying in the lower half-plane. We thus arrive at the interesting conclusion that the zeros of $L(s, \chi)$ with positive imaginary part are not correlated to the zeros of $L(s, \chi)$ with negative imaginary part.

13. SOME SUMS INVOLVING COEFFICIENTS RELATED TO VON MANGOLDT FUNCTION

In this part we deal with the sum

$$\tilde{S}_{k,\ell,\iota_1,\iota_2}(x) := \sum_{n\le x} \Delta(k,\iota_1,n)\Delta(\ell,\iota_2,n),$$

where k, $\ell \in \mathbb{N}$, ι_1, $\iota_2 \in \{0,1\}$, and

$$\sum_{n=1}^{\infty} \frac{\Delta(k,\iota_1;n)}{n^s} := \left(-\frac{\zeta'}{\zeta}(s)\right)^k \left(\frac{\zeta''}{\zeta}(s)\right)^{\iota_1}.$$

This sum will be crucial in estimating the pair correlation of $Z_1(s)$. We also note that Farmer and Gonek treated the case $\iota_1 = \iota_2 = 0$ in [11], but they did not make the k, ℓ-dependence of the error terms explicit in their calculations so that their result is not sufficient for our purpose.

Lemma 13.1. *For prime p and $j \in \mathbb{Z}^+$,*

$$\Delta(j,\iota;pn) = \begin{cases} j(\log p)\Delta(j-1,\iota;n) + (\log p)[\iota=1] & \text{if} \quad (p,n)=1, \\ \times\left(2\Delta(j+1,0;n) + (\log p)\Delta(j,0;n)\right) & \\ O\left(j(\log p)(\log pn)^{j+2\iota-1}\right) & \text{if} \quad p|n. \end{cases}$$

Proof. The case $\iota = 0$ can be found in the proof of Proposition 5.1 of [11]. So assume $\iota = 1$. For $(p,n)=1$, we have

$$\begin{aligned} \Delta(j,\iota;pn) &= \sum_{d|(pn)} \Lambda^{*(j)}(pn/d)\Lambda_2(d) \\ &= \sum_{d|n} \Lambda^{*(j)}(pn/d)\Lambda_2(d) + \sum_{d|n} \Lambda^{*(j)}(n/d)\Lambda_2(pd) \end{aligned}$$

Applying the case $\iota = 0$ to the above j-fold convolution of Λ we obtain

$$\Delta(j,\iota;pn) = j(\log p)\sum_{d|n}\Lambda^{*(j-1)}(n/d)\Lambda_2(d) + 2(\log p)\sum_{\substack{d|n\\ d>1}}\Lambda^{*(j)}(n/d)\Lambda(d) + (\log p)^2\Lambda^{*(j)}(n),$$

which is the desired result. Here we have evaluated the values of Λ_2 by means of the formula

$$\Lambda_2(n) = \Lambda(n)\log n + \sum_{d|n}\Lambda(d)\Lambda(n/d). \tag{13.1}$$

Assume now $(p,n) > 1$. Let $n = p^r n'$, $(n',p) = 1$, $r \in \mathbb{Z}^+$. First observe that

$$\begin{aligned}\Delta(j,\iota;pn) &= \sum_{d|(pn)}\Lambda^{*(j)}(pn/d)\Lambda_2(d)\\ &= \sum_{d|n}\Lambda^{*(j)}(pn/d)\Lambda_2(d) + \sum_{\substack{d'|n'\\ d'>1}}\Lambda^{*(j)}(n'/d')\Lambda_2(d'p^{r+1}) + \Lambda^{*(j)}(n')\Lambda_2(p^{r+1}).\end{aligned}$$

By (13.1), $\Lambda_2(p^{r+1}) = (2r+1)\log^2 p$ and $\Lambda_2(d'p^{r+1}) = 2(\log p)\Lambda(d')$ for $d' > 1$ and $(d',p) = 1$. Employing the bound $\Lambda^{*(j)}(n) \le \log^j n$ and the case $\iota = 0$, we obtain

$$\Delta(j,\iota;pn) \le j(\log p)(\log pn)^{j-1}\sum_{d|n}\Lambda_2(d) + 2(\log p)(\log n')^j\sum_{d'|n'}\Lambda(d') + (2r+1)(\log p)^2(\log n')^j.$$

Then the identity, $\sum_{d|n}\Lambda_o(d) = (\log n)^o$ for $o = 1$ or 2, completes the case $(p,n) > 1$.

Lemma 13.2. *For $k \in \mathbb{N}$ we have*

$$\sum_{n\le x}\frac{\Lambda(n)}{n}\left(\log\frac{x}{n}\right)^k = \frac{(\log x)^{k+1}}{k+1} + O\left((\log x)^k\right).$$

Proof. If $k = 0$, then the assertion is exactly the well-known fact:

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1). \tag{13.2}$$

Assume $k > 0$. By partial summation we convert the sum considered into an integral and then using the above formula, the result follows.

$$\begin{aligned}
\sum_{n \le x} \frac{\Lambda(n)}{n} \left(\log \frac{x}{n}\right)^k &= k \int_1^x (\log u + O(1)) \left(\log \frac{x}{u}\right)^{k-1} \frac{du}{u} \\
&= -(\log u) \left(\log \frac{x}{u}\right)^k \Big|_1^x + \int_1^x \left(\log \frac{x}{u}\right)^k \frac{du}{u} + O\left(k \int_1^x \left(\log \frac{x}{u}\right)^{k-1} \frac{du}{u}\right) \\
&= \frac{(\log x)^{k+1}}{k+1} + O\left((\log x)^k\right).
\end{aligned}$$

We state Lemma 6.3 of [11]:

Lemma 13.3. *If $k \ge 2$ and $\ell \ge 1$ then*

$$\sum_{p \le x} \frac{(\log p)^k}{p} \left(\log \frac{x}{p}\right)^\ell = \frac{(k-1)!\ell!}{(k+\ell)!} (\log x)^{k+\ell} + O\left(\frac{(k-1)!\ell!}{(k+\ell-1)!} (\log x)^{k+\ell-1}\right).$$

It is easy to check that the Lemma also holds for $\ell = 0$. Further, if $k + \ell \ll \log x$, then

$$\sum_{p \le x} \frac{(\log p)^k}{p} \left(\log \frac{x}{p}\right)^\ell \ll \frac{(k-1)!\ell!}{(k+\ell)!} (\log x)^{k+\ell}. \tag{13.3}$$

Lemma 13.4. *Let $\mathfrak{a}$ be 1 or any prime number. Suppose that $j \in \mathbb{N}$, $\iota = 0$ or 1 and $1 \le j + \iota$. Then*

$$\sum_{n \le x/\mathfrak{a}} \Delta(j, \iota; \mathfrak{a}n) \le \begin{cases} \log x & \textit{if } j = 1,\ \iota = 0,\ \mathfrak{a} \ne 1, \\ A(\log x)^2 + \frac{Ax \log x}{\mathfrak{a}} & \textit{if } j = 0,\ \iota = 1,\ \mathfrak{a} \ne 1, \\ \frac{A^j x (\log x)^{j+2\iota-1}}{(j+2\iota-1)!\mathfrak{a}} & \textit{otherwise.} \end{cases}$$

Proof. Assume $\mathfrak{a} \le x$, otherwise there is nothing to prove. We first treat the case $\iota = 0$, and then aim at reducing the remaining to the treated one. Since $\iota + j \ge 1$, $j \ge 1$. Here $j = 1$ is the exceptional case. If $\mathfrak{a} = 1$, then by Chebyshev's estimate, we have $\sum_{n \le x} \Lambda(n) \ll x$. If $\mathfrak{a} \ne 1$, then n must be a power of $\mathfrak{a}$. So

$$\sum_{n \le x/\mathfrak{a}} \Lambda(\mathfrak{a}n) = (\log \mathfrak{a}) \sum_{0 \le r \le \frac{\log \frac{x}{\mathfrak{a}}}{\log \mathfrak{a}}} 1 \le \log x. \tag{13.4}$$

The proof of the rest of the considered case is based on mathematical induction on $j \ge 2$. Observe that

$$\begin{aligned} \sum_{n \le x/\mathfrak{a}} \Delta(j, 0; \mathfrak{a}n) &= \sum_{n \le x/\mathfrak{a}} \sum_{d | (\mathfrak{a}n)} \Delta(j-1, 0; \mathfrak{a}n/d) \Lambda(d) \\ &= \sum_{\substack{d \le x/\mathfrak{a} \\ (d,\mathfrak{a})=1}} \Lambda(d) \sum_{e \le \frac{x}{\mathfrak{a}d}} \Delta(j-1, 0; \mathfrak{a}e) + \sum_{\substack{d \le x \\ (d,\mathfrak{a})>1}} \Lambda(d) \sum_{e \le x/d} \Delta(j-1, 0; e) \end{aligned} \tag{13.5}$$

For $j = 2$, the induction basis, by the exceptional case above, Chebyshev's estimate and Lemma 13.2, the right-hand side of (13.5) becomes

$$\ll x \sum_{d \le x} \frac{\Lambda(d)}{d} \ll x \log x$$

if $\mathfrak{a} = 1$;

$$\ll \sum_{d \le x/\mathfrak{a}} \Lambda(d) \log \frac{x}{d} + \frac{x}{\mathfrak{a}} \sum_{d' \le \frac{x}{\mathfrak{a}}} \frac{\Lambda(\mathfrak{a}d')}{d} \ll \frac{x}{\mathfrak{a}} \log x + \frac{x \log \mathfrak{a}}{\mathfrak{a}} \sum_{r \ge 0} \frac{1}{\mathfrak{a}^r} \ll \frac{x}{\mathfrak{a}} \log x$$

if $\mathfrak{a}$ is prime. Assume $j > 2$ and the related part in our assertion holds for $j - 1$, then using the induction hypothesis in (13.5) gives that

$$\sum_{n \le x/\mathfrak{a}} \Delta(j, 0; \mathfrak{a}n) \le \frac{A^{j-1}x}{(j-2)!} \sum_{d \le \frac{x}{\mathfrak{a}}} \frac{\Lambda(d)}{d} \left(\log \frac{x}{\mathfrak{a}d}\right)^{j-2} + \frac{A^{j-1}x(\log x)^{j-2}}{(j-2)!\mathfrak{a}} \sum_{d' \le \frac{x}{\mathfrak{a}}} \frac{\Lambda(\mathfrak{a}d')}{d}. \tag{13.6}$$

The sum over d', as before, is $\ll \log \mathfrak{a}$. Then, combining the above bound with Lemma 13.2 provides a confirmation for our assertion for $j > 2$ and this finishes the case $\iota = 0$.

Assume $\iota = 1$. From (13.1), it follows that for $j \in \mathbb{N}$

$$\Delta(j,1;n) = \sum_{d|n} \Delta(j,0;n/d)\Lambda(d)\log d + \Delta(j+2,0;n), \tag{13.7}$$

from which it is seen that

$$\begin{aligned}\sum_{n\le x/\mathfrak{a}} \Delta(j,1;\mathfrak{a}n) = & \sum_{\substack{d\le x/\mathfrak{a}\\(d,\mathfrak{a})=1}} \Lambda(d)(\log d) \sum_{e\le \frac{x}{\mathfrak{a}d}} \Delta(j,0;\mathfrak{a}e)\\ &+ \sum_{\substack{d\le x\\(d,\mathfrak{a})>1}} \Lambda(d)(\log d) \sum_{e\le x/d} \Delta(j,0;e) + \sum_{n\le x/\mathfrak{a}} \Delta(j+2,0;\mathfrak{a}n).\end{aligned} \tag{13.8}$$

We examine the rest in five cases:

Case 1: $j = 0$ and $\mathfrak{a} = 1$

By (13.8), the $\iota = 0$-case and Chebyshev's estimate,

$$\sum_{n\le x} \Delta(0,1;n) = \sum_{d\le x} \Lambda(d)(\log d) + \sum_{n\le x} \Delta(2,0;n) \ll x\log x. \tag{13.9}$$

Case 2: $j = 0$ and $\mathfrak{a} \ne 1$

By (13.8) and the $\iota = 0$-case,

$$\begin{aligned}\sum_{n\le x/\mathfrak{a}} \Delta(0,1;\mathfrak{a}n) &= \sum_{\substack{d\le x\\(d,\mathfrak{a})>1}} \Lambda(d)\log d + \sum_{n\le x/\mathfrak{a}} \Delta(2,0;\mathfrak{a}n)\\ &= \sum_{\substack{d\le x\\(d,\mathfrak{a})>1}} \Lambda(d)\log d + O\left(\frac{x\log x}{\mathfrak{a}}\right)\\ &= (\log \mathfrak{a})^2 \sum_{1\le r\le \frac{\log x}{\log \mathfrak{a}}} r + O\left(\frac{x\log x}{\mathfrak{a}}\right) \ll (\log x)^2 + \frac{x\log x}{\mathfrak{a}},\end{aligned}$$

which is the second exceptional case of the Lemma.

Case 3: $j = \mathfrak{a} = 1$

By (13.8), the $\iota = 0$-case, Chebyshev's estimate and Lemma 13.2,

$$\sum_{n \le x} \Delta(1,1;n) = \sum_{d \le x} \Lambda(d)(\log d) \sum_{e \le \frac{x}{d}} \Delta(1,0;e) + \sum_{n \le x} \Delta(3,0;n) \ll x(\log x)^2. \qquad (13.10)$$

Case 4: $j = 1$ and $\mathfrak{a} \ne 1$

By (13.8), the $\iota = 0$-case and Chebyshev's estimate,

$$\sum_{n \le x/\mathfrak{a}} \Delta(1,1;\mathfrak{a}n) \ll \sum_{\substack{d \le x/\mathfrak{a} \\ (d,\mathfrak{a})=1}} \Lambda(d)(\log d)\left(\log \frac{x}{d}\right) + x \sum_{\substack{d \le x \\ (d,\mathfrak{a})>1}} \frac{\Lambda(d)\log d}{d} + \frac{x(\log x)^2}{\mathfrak{a}} \qquad (13.11)$$

$$\ll (\log x)^2 \sum_{d \le x/\mathfrak{a}} \Lambda(d) + \frac{x}{\mathfrak{a}} \sum_{d' \le x/\mathfrak{a}} \frac{\Lambda(\mathfrak{a}d')\log \mathfrak{a}d'}{d'} + \frac{x(\log x)^2}{\mathfrak{a}} \ll \frac{x(\log x)^2}{\mathfrak{a}}.$$

Case 5: $j \ge 2$

By (13.8) and the $\iota = 0$-case,

$$\sum_{n \le x/\mathfrak{a}} \Delta(j,1;\mathfrak{a}n) \le \frac{A^{j-1} x \log x}{(j-1)!\mathfrak{a}} \left\{ \sum_{d \le x} \frac{\Lambda(d)}{d}\left(\log \frac{x}{d}\right)^{j-1} + \sum_{d' \le x/\mathfrak{a}} \frac{\Lambda(\mathfrak{a}d)}{d}\left(\log \frac{x}{\mathfrak{a}d}\right)^{j-1} \right\} + \frac{A^{j+1} x (\log x)^{j+1}}{(j+1)!\mathfrak{a}}. \qquad (13.12)$$

Since $\Lambda(\mathfrak{a}d') \le \Lambda(d')$, Lemma 13.2 is applicable to both of the sums on the right so that the result follows.

Theorem 13.1. *Let ι_1, $\iota_2 = 0$ or 1. For $k, \ell \in \mathbb{N}$ with $k + \iota_1$, $\ell + \iota_2 \geq 1$, we have*

$$\tilde{S}_{k,\ell,\iota_1,\iota_2}(x) = \frac{P(k,\ell,\iota_1,\iota_2)}{(k+\ell+2\iota_1+2\iota_2-1)!} x(\log x)^{k+\ell+2\iota_1+2\iota_2-1} + O\left(\frac{A^{k+\ell} x(\log x)^{k+\ell+2\iota_1+2\iota_2-2}}{(\max\{k,\ell\})!}\right),$$

where

$$P(k,\ell,\iota_1,\iota_2) := \begin{cases} k! & \textit{if} \quad \iota_1 = \iota_2 = 0 \quad \textit{and} \quad k = \ell, \\ 2(\max\{k,\ell\})! & \textit{if} \quad \iota_1 - \iota_2 = \pm 1 \quad \textit{and} \quad k - \ell = \mp 1, \\ (\max\{k,\ell\})! & \textit{if} \quad \iota_1 - \iota_2 = \pm 1 \quad \textit{and} \quad k - \ell = \mp 2, \\ (k+2)! + 4(k+1)! + 2k! & \textit{if} \quad \iota_1 = \iota_2 = 1 \quad \textit{and} \quad k = \ell, \\ 2(\max\{k,\ell\}+1)! & \textit{if} \quad \iota_1 = \iota_2 = 1 \quad \textit{and} \quad k - \ell = \mp 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We divide our theorem into three parts:

(i) $\iota_1 = \iota_2 = 0$, $k, \ell \geq 1$,

(ii) $\iota_1 = 0$, $\iota_2 = 1$, $k \geq 1$, $\ell \geq 0$,

(iii) $\iota_1 = \iota_2 = 1$, $k, \ell \geq 0$.

We start with the first part. We will prove that for $k, \ell \in \mathbb{Z}^+$, $x \geq 1$,

$$\tilde{S}_{k,\ell,0,0}(x) = \frac{P(k,\ell,0,0)}{(k+\ell-1)!} x(\log x)^{k+\ell-1} + O^*\left(H(k,\ell) x(\log x)^{k+\ell-2}\right), \tag{13.13}$$

by induction on $k + \ell \geq 2$. Here * in the O-term indicates that the implicit constant of the error term is 1, and $H(k,\ell)$ will be determined later.

The basis step, $k+\ell=2$, contains only one sum:

$$\sum_{n\le x}\Lambda(n)\Lambda(n),$$

which equals $x\log x+O(x)$. This can be obtained by partial summation and using the prime number theorem, and is agree with the assertion if one choose $H(1,1)$ sufficiently large. Take any pair (k,ℓ) with $k,\ell\in\mathbb{Z}^+$ and $k+\ell>2$. Assume that for all $(\tilde{k},\tilde{\ell})$ satisfying $\tilde{k},\tilde{\ell}\in\mathbb{Z}^+$ and $2\le\tilde{k}+\tilde{\ell}<k+\ell$, (13.13) holds, which is the induction hypothesis. Since $k+\ell\ge 3$, we can assume without loss of generality $2\le\ell$ and $1\le k\le\ell$. By means of Lemma 13.1, we obtain

$$\begin{aligned}
\tilde{S}_{k,\ell,0,0}(x) &= \sum_{p\le x}\log p\sum_{\substack{e\le x/p\\(p,e)=1}}\Delta(k,0;ep)\Delta(\ell-1,0;e)\\
&+\sum_{p\le x}\log p\sum_{\substack{e\le x/p\\(p,e)>1}}\Delta(k,0;ep)\Delta(\ell-1,0;e)\\
&+\sum_{\substack{p^\alpha\le x\\\alpha\ge 2}}\log p\sum_{e\le x/p^\alpha}\Delta(k,0;ep^\alpha)\Delta(\ell-1,0;e)\\
&=k\sum_{p\le x}(\log p)^2\tilde{S}_{k-1,\ell-1,0,0}(x/p)\\
&-k\sum_{p\le x}(\log p)^2\sum_{\substack{e\le x/p\\(e,p)>1}}\Delta(k-1,0;e)\Delta(\ell-1,0;e)\\
&+\sum_{p\le x}\log p\sum_{\substack{e\le x/p\\(p,e)>1}}\Delta(k,0;ep)\Delta(\ell-1,0;e)\\
&+\sum_{\substack{p^\alpha\le x\\\alpha\ge 2}}\log p\sum_{e\le x/p^\alpha}\Delta(k,0;ep^\alpha)\Delta(\ell-1,0;e)\\
&=\square_1+\square_2+\square_3+\square_4,
\end{aligned}$$

say. The first box constitutes the main term. If $k=1$, then e must be 1, otherwise $\Delta(k-1,0;e)$ vanishes, but if this is the case, then $\Delta(\ell-1,0;e)$ vanishes since $\ell\ge 2$. So, in the case $k=1$ and $\ell\ge 2$, the main term disappears, as $P(k,\ell,\iota_1,\iota_2)$ suggests. Assume $k,\ell\ge 2$. This ensures the applicability of the induction hypothesis and we have

$$\square_1 = \frac{kxP(k-1,\ell-1,0,0)}{(k+\ell-3)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k+\ell-3}$$
$$+ O^*\left(kH(k-1,\ell-1)x\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k+\ell-4}\right).$$

The remaining part of the inductive step is examined in two cases. First assume $k+\ell \ll \log x$. It then follows from Lemma 13.3 and (13.3) that

$$\square_1 = x\,(\log x)^{k+\ell-1}\,\frac{kP(k-1,\ell-1,0,0)}{(k+\ell-1)!}$$
$$+ O^*\left(Ax(\log x)^{k+\ell-2}\left(\frac{kH(k-1,\ell-1)}{(k+\ell-2)(k+\ell-3)} + \frac{kP(k-1,\ell-1,0,0)}{(k+\ell-2)!}\right)\right).$$

From the definition of $P(k,\ell,\iota_1,\iota_2)$ we have

$$P(k,\ell,0,0) = kP(k-1,\ell-1,0,0).$$

If we choose temporarily

$$H(k,\ell) = \frac{A^{k+\ell}k!}{(k+\ell-2)!}, \tag{13.14}$$

then the contribution of $\square_1$ becomes

$$\square_1 = \frac{P(k,\ell,0,0)}{(k+\ell-1)!}x\,(\log x)^{k+\ell-1} + O^*\left(H(k,\ell)x\,(\log x)^{k+\ell-2}\right).$$

All the remaining parts, $\square_2$, $\square_3$, $\square_4$, can be estimated in a similar manner; we remove Δ-factor involving the variable k by using Lemma 13.1, and then calculate the remaining sum by Lemma 13.4. We only treat $\square_2$. If $k=1$, then $\square_2=0$. If $k=\ell=2$, then p must be $\le\sqrt{x}$, otherwise the inner sum is void, then by Lemma 13.1 and the first exceptional case of the Lemma, we obtain

$$\square_2 \ll \sum_{p\le\sqrt{x}} (\log p)^3 \sum_{e'\le x/p^2} \Delta(1,0;pe') \ll \sum_{p\le\sqrt{x}} (\log p)^3 \left(\log \frac{x}{p}\right) \ll \sqrt{x}(\log x)^4.$$

If $k \ge 2$ and $\ell > 2$, then together with the substitution $e = e'p$, using Lemma 13.1, we obtain

$$\square_4 \ll k(k-1)\,(\log x)^{k-2} \sum_{p\le x} (\log p)^3 \sum_{e'\le x/p^2} \Delta(\ell-1,0;pe').$$

Applying Lemma 13.4 to the above inner sum, we see that

$$\begin{aligned}\square_4 &\ll \frac{A^\ell k(k-1)x\,(\log x)^{k+\ell-4}}{(\ell-2)!} \sum_{p\le x} \frac{(\log p)^3}{p^2} \\ &\ll \frac{A^\ell x\,(\log x)^{k+\ell-2}}{(\ell-2)!}\end{aligned}$$

provided that $k \ll \log x$. This bound also leads to the new choice of $H(k,\ell)$:

$$H(k,\ell) = \frac{A^{k+\ell}}{\ell!}. \tag{13.15}$$

In the second case, $k+\ell \gg \log x$, (13.13) also holds, since the sum dealt with vanishes and the main term is already absorbed in the error term with the choice in (13.15).

Reducing (2) to (1)

For the second part assume $\iota_1 = 0$, $\iota_2 = 1$, $k \ge 1$, $\ell \ge 0$. By means of (13.7) and

Lemma 13.1 we can relate (2) to (1) so that

$$\begin{aligned}
\tilde{S}_{k,\ell,0,1}(x) &= k\sum_{p\le x}(\log p)^3 \sum_{e\le x/p} \Delta(k-1,0;e)\Delta(\ell,0;e)\\
&\quad - k\sum_{p\le x}(\log p)^3 \sum_{\substack{e\le x/p\\ (e,p)>1}} \Delta(k-1,0;e)\Delta(\ell,0;e)\\
&\quad + \sum_{p\le x}(\log p)^2 \sum_{\substack{e\le x/p\\ (e,p)>1}} \Delta(k,0;ep)\Delta(\ell,0;e)\\
&\quad + \sum_{p^\alpha\le x,\,\alpha\ge 2} \alpha(\log p)^2 \sum_{e\le x/p^\alpha} \Delta(k,0;ep^\alpha)\Delta(\ell,0;e) + \tilde{S}_{k,\ell+2,0,0}(x)\\
&= \circ_1 + \circ_2 + \circ_3 + \circ_4 + \tilde{S}_{k,\ell+2,0,0}(x),
\end{aligned}$$

say. We handle $\circ_2, \circ_3$ and $\circ_4$ by arguments exactly the same as in the estimation of the error terms of (1). If $k = 1(\ell = 0)$, then e must be 1, and then ℓ must be $0(k$ must be 1) and by the prime number theorem and (1),

$$\circ_1 + \tilde{S}_{k,\ell+2,0,0}(x) = \sum_{p\le x}(\log p)^3 + O(x\log x) = x(\log x)^2 + O(x\log x), \tag{13.16}$$

as expected. However, for any (k,ℓ) with $k = 1$ or $\ell = 0$, but not both of them, $\circ_1$ vanishes. Assume $k \ge 2$, $\ell \ge 1$, then (1) is applicable to $\circ_1$ and we have

$$\begin{aligned}
\circ_1 + \tilde{S}_{k,\ell+2,0,0}(x) &= \frac{kxP(k-1,\ell,0,0)}{(k+\ell-2)!}\sum_{p\le x}\frac{(\log p)^3}{p}\left(\log\frac{x}{p}\right)^{k+\ell-2}\\
&\quad + O\left(\frac{A^{k+\ell}x}{(\max\{k-1,\ell\})!}\sum_{p\le x}\frac{(\log p)^3}{p}\left(\log\frac{x}{p}\right)^{k+\ell-3}\right)\\
&\quad + \frac{P(k,\ell+2,0,0)}{(k+\ell+1)!}x(\log x)^{k+\ell+1} + O\left(\frac{A^{k+\ell}x(\log x)^{k+\ell}}{(\max\{k,\ell+2\})!}\right).
\end{aligned}$$

Employing Lemma 13.3 and (13.3), it follows that

$$\circ_1 + \tilde{S}_{k,\ell,0,1}(x) = \frac{2kP(k-1,\ell,0,0) + P(k,\ell+2,0,0)}{(k+\ell+1)!}x(\log x)^{k+\ell+1}$$

$$+O\Bigg(A^{k+\ell}x(\log x)^{k+\ell}\Big(\frac{(k+\ell-3)!}{(k+\ell)!(\max\{k-1,\ell\})!}$$

$$+\frac{1}{(\max\{k,\ell+2\})!}+\frac{P(k-1,\ell,0,0)}{(k+\ell)!}\Big)\Bigg).$$

Paying attention to the definition of $P(k,\ell,\iota_1,\iota_2)$, the right-hand side confirms our Theorem in the case considered.

Reducing (3) to (2)

For the third part assume $\iota_1=1$, $\iota_2=1$, $k,\ell\geq 0$. We first deal with the exceptional case $k=\ell=0$. By (13.7), we see that

$$\tilde{S}_{0,0,1,1}(x)=\sum_{n\leq x}\Lambda_2(n)\Lambda(n)\log n+\tilde{S}_{0,2,1,0}(x).$$

Only prime numbers contribute to the sum on the right and by the prime number theorem,

$$\sum_{n\leq x}\Lambda_2(n)\Lambda(n)\log n=x(\log x)^3+O\left(x(\log x)^2\right).$$

By (2), $\tilde{S}_{0,2,1,0}(x)\sim\frac{1}{3}x(\log x)^3$. Comparing these results with our assertion, we are done. Assume without loss of generality, $k\geq 1$. From (13.7), it follows that

$$\begin{aligned}\tilde{S}_{k,\ell,1,1}(x)=&\sum_{p\leq x}(\log p)^2\sum_{\substack{e\leq x/p\\(e,p)=1}}\Delta(k,1;ep)\Delta(\ell,0;e)\\&+\sum_{p\leq x}(\log p)^2\sum_{\substack{e\leq x/p\\(e,p)>1}}\Delta(k,1;ep)\Delta(\ell,0;e)\\&+\sum_{p^\alpha\leq x,\,\alpha\geq 2}\alpha(\log p)^2\sum_{e\leq x/p^\alpha}\Delta(k,1;ep^\alpha)\Delta(\ell,0;e)+\tilde{S}_{k,\ell+2,1,0}(x).\end{aligned}$$

We only treat the first part, say $\diamond$, which carries a part of the main term. If $\ell=0$, then e must be 1. But then for any $k\geq 1$, $\Delta(k,1;p)=0$. So in the case of $\ell=0$ and $k\geq 1$, $\tilde{S}_{k,0,1,1}(x)\sim\tilde{S}_{k,2,1,0}(x)$, which was already calculated in (2). Assume $\ell\geq 1$.

With the use of Lemma 13.1, it follows that

$$
\begin{aligned}
\diamond &= k\sum_{p\le x}(\log p)^3\tilde{S}_{k-1,\ell,1,0}(x/p) + 2\sum_{p\le x}(\log p)^3\tilde{S}_{k+1,\ell,0,0}(x/p)\\
&+\sum_{p\le x}(\log p)^4\tilde{S}_{k,\ell,0,0}(x/p) - k\sum_{p\le x}(\log p)^3\sum_{\substack{e\le\frac{x}{p}\\(e,p)>1}}\Delta(k-1,1;e)\Delta(\ell,0;e)\\
&-2\sum_{p\le x}\log^3 p\sum_{\substack{e\le\frac{x}{p}\\(e,p)>1}}\Delta(k+1,0;e)\Delta(\ell,0;e) - \sum_{p\le x}\log^4 p\sum_{\substack{e\le\frac{x}{p}\\(e,p)>1}}\Delta(k,0;p)\Delta(\ell,0;e).
\end{aligned}
$$

For the first three parts, which make up the main term, we employ the previous two cases and then Lemma 13.3. The estimation of the remaining parts is very similar to those in (1) and (2).

Corollary 13.1.

$$
\sum_{n\le x}\Delta(k,\iota_1,n)\Delta(\ell,\iota_2,n)n = \frac{P(k,\ell,\iota_1,\iota_2)x^2(\log x)^{k+\ell+2\iota_1+2\iota_2-1}}{2(k+\ell+2\iota_1+2\iota_2-1)!} + O\left(\frac{A^{k+\ell}x^2(\log x)^{k+\ell+2\iota_1+2\iota_2-2}}{(\max\{k,\ell\})!}\right);
$$

if $k+\ell\ll\log x$,

$$
\sum_{n>x}\frac{\Delta(k,\iota_1,n)\Delta(\ell,\iota_2,n)}{n^3} = \frac{P(k,\ell,\iota_1,\iota_2)(\log x)^{k+\ell+2\iota_1+2\iota_2-1}}{2x^2(k+\ell+2\iota_1+2\iota_2-1)!} + O\left(\frac{A^{k+\ell}(\log x)^{k+\ell+2\iota_1+2\iota_2-2}}{x^2(\max\{k,\ell\})!}\right).
$$

We employ Theorem 13.1 in an easy application of partial summation in both cases. However, the second result requires the additional condition, $k+\ell\ll\log x$. So in this case we give some details. By partial summation,

$$\begin{aligned}\sum_{n>x}\frac{\Delta(k,\iota_1,n)\Delta(\ell,\iota_2,n)}{n^3} &= -\frac{P(k,\ell,\iota_1,\iota_2)(\log x)^{k+\ell+2\iota_1+2\iota_2-1}}{x^2(k+\ell+2\iota_1+2\iota_2-1)!}\\ &\quad + O\left(\frac{A^{k+\ell}(\log x)^{k+\ell+2\iota_1+2\iota_2-2}}{x^2(\max\{k,\ell\})!}\right)\\ &\quad + \frac{3P(k,\ell,\iota_1,\iota_2)}{(k+\ell+2\iota_1+2\iota_2-1)!}\int_x^\infty \frac{(\log u)^{k+\ell+2\iota_1+2\iota_2-1}}{u^3}du\\ &\quad + O\left(\frac{A^{k+\ell}}{(\max\{k,\ell\})!}\int_x^\infty \frac{(\log u)^{k+\ell+2\iota_1+2\iota_2-2}}{u^3}du\right).\end{aligned}$$

It is enough to deal with the integral in the main term. By integrating by parts,

$$\begin{aligned}\int_x^\infty \frac{(\log u)^{k+\ell+2\iota_1+2\iota_2-1}}{u^3}du &= -\frac{(\log u)^{k+\ell+2\iota_1+2\iota_2-1}}{2u^2}\Big|_x^\infty\\ &\quad + \frac{k+\ell+2\iota_1+2\iota_2-1}{2}\int_x^\infty \frac{(\log u)^{k+\ell+2\iota_1+2\iota_2-2}}{u^3}du\\ &= \frac{(\log x)^{k+\ell+2\iota_1+2\iota_2-1}}{2x^2} + \frac{k+\ell+2\iota_1+2\iota_2-1}{2}\int_x^\infty \frac{(\log u)^{k+\ell+2\iota_1+2\iota_2-2}}{u^3}du.\end{aligned}$$

The sum of the first quantities on the right of the above two results forms the main term of the assertion. The justification of the error term is provided by the fact that under the condition $m \ll \log x$,

$$\int_x^\infty \frac{(\log u)^m}{u^3}du \ll \frac{(A\log x)^m}{x^2}, \tag{13.17}$$

which can be shown by mathematical induction on m.

14. PAIR CORRELATION OF THE ZEROS OF Z_1

After going to the explicit formula for $Z_1(s)$ in Theorem 5.1, we allow t to be ordinates $\tilde{\upsilon}$ of the zeros of Z_1 in $(T/2, T]$, we sum the both sides of the explicit formula over $\tilde{\upsilon}$. As before, we neglect the terms with $\upsilon > T$ and $\upsilon \leq 0$, which cause an error term bounded by $\log^3 T$, and take $\sigma = 5/2$. As a result of these steps, the explicit formula becomes

$$\sum_{\substack{0<\upsilon\leq T\\ T/2<\tilde{\upsilon}\leq T}} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} = \tag{14.1}$$

$$- x^{-2} \sum_{k\leq \frac{\log T}{\log\log T}} (-2)^k \sum_{m\leq x} \Lambda^{*(k+1)}(m) m^2 \sum_{\substack{T/2<\tilde{\upsilon}\leq T\\ \tilde{\varrho}=1/2+i\tilde{\upsilon}}} m^{-\tilde{\varrho}} \left(\frac{\chi'}{\chi}(-3/2+i\tilde{\upsilon})\right)^{-k}$$

$$+ x^{-2} \sum_{k\leq \frac{\log T}{\log\log T}} (-2)^{k+1} \sum_{m\leq x} \lambda^{*(k)}(m) m^2 \sum_{\substack{T/2<\tilde{\upsilon}\leq T\\ \tilde{\varrho}=1/2+i\tilde{\upsilon}}} m^{-\tilde{\varrho}} \left(\frac{\chi'}{\chi}(-3/2+i\tilde{\upsilon})\right)^{-k-1}$$

$$- x^{2} \sum_{k\leq \frac{\log T}{\log\log T}} (-2)^k \sum_{m>x} \frac{\Lambda^{*(k+1)}(m)}{m^2} \sum_{\substack{T/2<\tilde{\upsilon}\leq T\\ \tilde{\varrho}=1/2+i\tilde{\upsilon}}} m^{-\tilde{\varrho}} \left(\frac{\chi'}{\chi}(5/2+i\tilde{\upsilon})\right)^{-k}$$

$$+ x^{2} \sum_{k\leq \frac{\log T}{\log\log T}} (-2)^{k+1} \sum_{m>x} \frac{\lambda^{*(k)}(m)}{m^2} \sum_{\substack{T/2<\tilde{\upsilon}\leq T\\ \tilde{\varrho}=1/2+i\tilde{\upsilon}}} m^{-\tilde{\varrho}} \left(\frac{\chi'}{\chi}(5/2+i\tilde{\upsilon})\right)^{-k-1}$$

$$+ x^{-2} \sum_{T/2<\tilde{\upsilon}\leq T} \left(\log\frac{\tilde{\upsilon}}{2\pi} + O(1)\right) + O\left(x^{1/2+\epsilon} \exp\left(\frac{A(\epsilon)\log T}{\log\log T}\right) \sum_{T/2<\tilde{\upsilon}\leq T} \frac{1}{\tilde{\upsilon}}\right).$$

It is obvious that $\sum_{T/2<\tilde{\upsilon}\leq T} \frac{1}{\tilde{\upsilon}} \ll \log T$ by (3.5), so that the last error term in (14.1) is

$$\ll x^{1/2+\epsilon} \exp\left(\frac{A(\epsilon)\log T}{\log\log T}\right). \tag{14.2}$$

By (3.5) and the simple estimate that $\log \frac{\tilde{v}}{2\pi} = \log \frac{T}{2\pi} + O(1)$ for $T/2 < \tilde{v} \le T$, we see that

$$\sum_{T/2<\tilde{v}\le T} \left(\log \frac{\tilde{v}}{2\pi} + O(1) \right) = \frac{T(\log T)^2}{4\pi} \left(1 + O\left(\frac{1}{\log T} \right) \right). \tag{14.3}$$

Taking into account the restriction $\log m \ll \log T$ in Theorem 10.1, we impose the condition $x \le T^2$. In the first two parts of the right of (14.1), Theorem 10.1 is applicable. However, the third and the fourth parts involve terms violating $\log m \ll \log T$. Define

$$\mathfrak{I} = -x^2 \sum_{k \le \frac{\log T}{\log\log T}} (-2)^k \sum_{m \ge T^2} \frac{\Lambda^{*(k+1)}(m)}{m^2} \sum_{\substack{T/2<\tilde{v}\le T \\ \tilde{\varrho}=1/2+i\tilde{v}}} m^{-\tilde{\varrho}} \left(\frac{\chi'}{\chi}(5/2+i\tilde{v}) \right)^{-k}$$

$$\mathfrak{J} = x^2 \sum_{k \le \frac{\log T}{\log\log T}} (-2)^{k+1} \sum_{m \ge T^2} \frac{\lambda^{*(k)}(m)}{m^2} \sum_{\substack{T/2<\tilde{v}\le T \\ \tilde{\varrho}=1/2+i\tilde{v}}} m^{-\tilde{\varrho}} \left(\frac{\chi'}{\chi}(5/2+i\tilde{v}) \right)^{-k-1}.$$

Firstly, by (2.8), the contribution of the innermost sum in $\mathfrak{I}$ is an amount $\ll m^{-1/2}T$ $(\log T)^{1-k}$. After employing the bound $\Lambda^{(k+1)}(m) \le (\log m)^{k+1}$, the sum over $m \ge T^2$ in $\mathfrak{I}$ can be transformed to

$$\int_{T^2}^{\infty} \frac{(\log u)^{k+1}}{u^{5/2}} du$$

via the integral test. Under the condition $k \le \log T$,

$$\int_{T^2}^{\infty} \frac{(\log u)^k}{u^{5/2}} du \ll \frac{(2\log T)^k}{T^3},$$

similar to (13.17). Combining these we obtain

$$\mathfrak{I} \ll x^2 T^{-2} (\log T) \sum_{k \le \frac{\log T}{\log\log T}} A^k \ll x^2 T^{-2} \exp\left(\frac{A \log T}{\log\log T} \right), \tag{14.4}$$

which also holds for $\mathfrak{J}$. However, in calculation of the right of (14.1), together with the application of Theorem 10.1, the last error term in Theorem 10.1 produces

$$O\left(x \exp\left(\frac{A(\log T)(\log\log\log T)}{\log\log T}\right)\right),$$

which will be seen in (14.5). This amount dominates (14.4) provided that $x \leq T^2$.

On the other hand, using Theorem 10.1 to calculate the innermost sum in $\mathfrak{I}$ and $\mathfrak{J}$ despite of the restriction $\log m \ll \log T$, and then repeating all other steps in the above discussion, we see that

$$\mathfrak{I}, \mathfrak{J} \ll x^2 T^{-3} \exp\left(\frac{A(\log T)(\log\log\log T)}{\log\log T}\right) \ll x \exp\left(\frac{A(\log T)(\log\log\log T)}{\log\log T}\right),$$

provided that $x \leq T^3$. Comparing the error terms occuring in our three observations, we conclude that the misuse carried out is harmless, and so we ignore the restriction $\log m \ll \log T$.

Together with (14.2), (14.3) and the above remarks, (14.1) takes the following form:

$$\sum_{\substack{0<\upsilon\leq T\\ T/2<\tilde{\upsilon}\leq T}} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} = \frac{T\left(\Theta(0,0;x;2) - 2\Theta(0,1;x;2) + \Theta(1,1;x;2)\right)}{4\pi x^2} \tag{14.5}$$

$$+\frac{Tx^2\left(\tilde{\Theta}(0,0;x;2) - 2\tilde{\Theta}(0,1;x;2) + \tilde{\Theta}(1,1;x;2)\right)}{4\pi} + \frac{T\log^2 T}{4\pi x^2}\left(1 + O\left(\frac{1}{\log T}\right)\right)$$

$$+O\left(\frac{T\left(\Theta(0,0;x;A) + \Theta(0,1;x;A) + \Theta(1,1;x;A)\right)}{x^2\log T}\right)$$
$$+O\left(Tx^2\left(\tilde{\Theta}(0,0;x;A) + \tilde{\Theta}(0,1;x;A) + \tilde{\Theta}(1,1;x;A)\right)\right)$$
$$+O\left(x\exp\left(\frac{A(\log T)(\log\log\log T)}{\log\log T}\right)\right),$$

where ι_1 and ι_2 are 0 or 1, and

$$\Theta(\iota_1,\iota_2;x;\mathfrak{e}) := \sum_{k,\ell\le\frac{\log T}{\log\log T}} \left(\frac{\mathfrak{e}}{\log\frac{T}{2\pi}}\right)^{k+\ell+\iota_1+\iota_2} \sum_{m\le x} \Delta(k+[\iota_1=0],\iota_1;m)\Delta(\ell+[\iota_2=0],\iota_2;m)m,$$

$$\tilde{\Theta}(\iota_1,\iota_2;x;\mathfrak{e}) := \sum_{k,\ell\le\frac{\log T}{\log\log T}} \left(\frac{\mathfrak{e}}{\log\frac{T}{2\pi}}\right)^{k+\ell+\iota_1+\iota_2} \sum_{m>x} \frac{\Delta(k+[\iota_1=0],\iota_1;m)\Delta(\ell+[\iota_2=0],\iota_2;m)}{m^3},$$

$\mathfrak{e}\ll 1$ and $\Delta(\cdot,\cdot;\cdot)$ was introduced at the beginning §13. From Corollary 13.1 it follows that

$$\Theta(\iota_1,\iota_2;x;\mathfrak{e}) = x^2(\log x) \sum_{k,\ell\le\frac{\log T}{\log\log T}} \left(\frac{\mathfrak{e}\log x}{\log\frac{T}{2\pi}}\right)^{k+\ell+\iota_1+\iota_2} \times \frac{P(k+[\iota_1=0],\ell+[\iota_2=0],\iota_1,\iota_2)}{2(k+\ell+\iota_1+\iota_2+1)!}$$

$$+O\left(x^2 \sum_{k,\ell\le\frac{\log T}{\log\log T}} \left(\frac{A\log x}{\log\frac{T}{2\pi}}\right)^{k+\ell+\iota_1+\iota_2} \frac{1}{(\max\{k,\ell\})!}\right).$$

If $\log x \ll \log T$, then the above error term is

$$\ll x^2 \sum_{\ell\le\frac{\log T}{\log\log T}} \frac{A^\ell}{\ell!} \sum_{k\le\ell} A^k \ll x^2 \sum_{\ell\le\frac{\log T}{\log\log T}} \frac{A^\ell}{\ell!} \ll x^2.$$

Similarly,

$$\tilde{\Theta}(\iota_1, \iota_2; x; \mathfrak{e}) = \frac{\log x}{x^2} \sum_{k,\ell \le \frac{\log T}{\log\log T}} \left(\frac{\mathfrak{e} \log x}{\log \frac{T}{2\pi}} \right)^{k+\ell+\iota_1+\iota_2} \times \frac{P(k+[\iota_1=0], \ell+[\iota_2=0], \iota_1, \iota_2)}{2(k+\ell+\iota_1+\iota_2+1)!} + O\left(x^{-2}\right).$$

From the definition of P, it follows that

$$x^4\tilde{\Theta}(0,0;x;\mathfrak{e}) = \Theta(0,0;x;\mathfrak{e}) = \frac{x^2\log x}{2} \sum_{k \le \frac{\log T}{\log\log T}} \left(\frac{\mathfrak{e} \log x}{\log \frac{T}{2\pi}} \right)^{2k} \frac{(k+1)!}{(2k+1)!} + O(x^2),$$

$$x^4\tilde{\Theta}(0,1;x;\mathfrak{e}) = \Theta(0,1;x;\mathfrak{e}) = x^2(\log x) \sum_{k \le \frac{\log T}{\log\log T}} \left(\frac{2 \log x}{\log \frac{T}{2\pi}} \right)^{2k+1} \frac{(k+1)!}{(2k+2)!} + \frac{x^2\log x}{2} \sum_{k \le \frac{\log T}{\log\log T}} \left(\frac{2 \log x}{\log \frac{T}{2\pi}} \right)^{2k+2} \frac{(k+2)!}{(2k+3)!} + O(x^2),$$

$$x^4\tilde{\Theta}(1,1;x;\mathfrak{e}) = \Theta(1,1;x;\mathfrak{e}) = 2x^2(\log x) \sum_{k \le \frac{\log T}{\log\log T}} \left(\frac{\mathfrak{e} \log x}{\log \frac{T}{2\pi}} \right)^{2k+3} \frac{(k+2)!}{(2k+4)!} + \frac{x^2\log x}{2} \sum_{k \le \frac{\log T}{\log\log T}} \left(\frac{\mathfrak{e} \log x}{\log \frac{T}{2\pi}} \right)^{2k+2} \frac{(k+2)!+4(k+1)!+2k!}{(2k+3)!} + O(x^2).$$

From the Taylor series we see that replacing the above partial sums by their power series produces an error term $\ll T^{-1+\epsilon}$, provided that $\log x \ll \log T$. Combining these results with (14.5) gives

$$\sum_{\substack{0<\upsilon\leq T\\ T/2<\tilde{\upsilon}\leq T}} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} = \frac{T\log x}{4\pi}\left(1-\frac{4\log x}{\log\frac{T}{2\pi}}+\sum_{k\geq 0}\left(\frac{2\log x}{\log\frac{T}{2\pi}}\right)^{2k+2}\frac{2k!}{(2k+2)!}\right) \tag{14.6}$$
$$+\frac{T(\log T)^2}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O(T)$$

under the restriction $x \leq T^{1-\epsilon}$. As for the pair correlation of the zeros of $\zeta(s)$ and $Z_1(s)$, we can extend the range $[T/2, T]$ for $\tilde{\upsilon}$ to $(0, T]$ within the above error terms. Take $x = T^\alpha$ and define

$$F_{Z_1,Z_1}(\alpha) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\upsilon,\tilde{\upsilon}\leq T} \frac{4T^{i\alpha(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2}. \tag{14.7}$$

The above sum is symmetric in the variables υ and $\tilde{\upsilon}$, so that $F_{Z_1,Z_1}(\alpha)$ is even. We also note that, as in the Montgomery's case, $F_{Z_1,Z_1}(\alpha)$ has an integral representation, namely

$$\sum_{0<\upsilon,\tilde{\upsilon}\leq T} \frac{4T^{i\alpha(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left|\sum_{0<\upsilon\leq T}\frac{T^{i\alpha\upsilon}}{1+(\upsilon-t)^2}\right| dt, \tag{14.8}$$

which gives the positivity of $F_{Z_1,Z_1}(\alpha)$. As a consequence we obtain

Theorem 14.1. *Assume RH. For real α, $T \geq 2$, let $F_{Z_1,Z_1}(\alpha)$ be defined by (14.7). Then $F_{Z_1,Z_1}(\alpha)$ is positive real, even and*

$$F_{Z_1,Z_1}(\alpha) = (1+o(1))T^{-2|\alpha|}\log T + |\alpha| - 4|\alpha|^2$$
$$+|\alpha|\sum_{k=0}^{\infty}\frac{2k!}{(2k+2)!}(2|\alpha|)^{2k+2}+o(1),$$

as T tends to infinity; this holds uniformly for $|\alpha| \leq 1-\epsilon$.

15. SOME CLASSICAL APPLICATION OF PAIR CORRELATION RESULTS

In [1] Montgomery used the asymptotic formula for the pair correlation function for ζ-function to examine differences between zeta zeros and obtained some important corollaries on simple zeros and small gaps between zeros of ζ-function. Firstly, we apply the Fourier theoretic machinery to $F_{Z_1,Z_1}(\alpha)$. Analogous to (3) in [1],

$$\sum_{0<\upsilon,\tilde{\upsilon}\leq T} r\left(\frac{(\upsilon-\tilde{\upsilon})\log T}{2\pi}\right) w(\upsilon-\tilde{\upsilon}) = \frac{T\log T}{2\pi}\int_{-\infty}^{\infty} \hat{r}(\alpha)F_{Z_1,Z_1}(\alpha)d\alpha, \tag{15.1}$$

where r, $\hat{r} \in L^1(\mathbb{R})$ and

$$\hat{r}(\alpha) = \int_{-\infty}^{\infty} r(u)e(-\alpha u)du, \qquad e(u) = e^{2\pi i u}. \tag{15.2}$$

From (14.7) and (15.2), (15.1) is immediately seen. Using Theorem 14.1 and the Fourier pair

$$r(u) = \left(\frac{\sin \pi\lambda u}{\pi\lambda u}\right)^2, \qquad \hat{r}(\alpha) = \frac{1}{\lambda}\max(1-\frac{|\alpha|}{\lambda}, 0), \qquad \epsilon < \lambda < 1-\epsilon, \tag{15.3}$$

in (15.1) yields that

$$\begin{aligned}\sum_{0<\upsilon\leq T} m_{Z_1}(\varrho) = \sum_{\substack{0<\upsilon,\tilde{\upsilon}\leq T\\ \upsilon=\tilde{\upsilon}}} 1 &\leq \sum_{0<\upsilon,\tilde{\upsilon}\leq T}\left(\frac{\sin\frac{(\upsilon-\tilde{\upsilon})\lambda\log T}{2}}{\frac{(\upsilon-\tilde{\upsilon})\lambda\log T}{2}}\right)^2 w(\upsilon-\tilde{\upsilon})\\ &= \frac{T\log T}{\lambda\pi}(1+o(1))\int_0^{\lambda}\left(1-\frac{\alpha}{\lambda}\right)\left(\frac{\log T}{T^{2\alpha}}+\alpha-4\alpha^2+\sum_{k\geq 0}\frac{k!(2\alpha)^{2k+3}}{(2k+2)!}\right)d\alpha\\ &= \frac{T\log T}{2\pi}\left(\frac{1}{\lambda}+\frac{\lambda}{3}-\frac{2\lambda^2}{3}+2\sum_{k\geq 0}\frac{k!(2\lambda)^{2k+3}}{(2k+4)(2k+5)(2k+2)!}+o(1)\right),\end{aligned}$$

where $m_{Z_1}(\varrho)$ is the multiplicity of ϱ. Letting $\lambda \to 1^-$, we have

$$\sum_{0<\upsilon\leq T} m_{Z_1}(\varrho) \leq (1.14161\cdots + o(1))\frac{T\log T}{2\pi},$$

Hence,

$$\sum_{0<\upsilon\leq T;\, \varrho \text{ is simple}} 1 \geq \sum_{0<\upsilon\leq T} (2 - m_{Z_1}(\varrho)) \geq (0.85838\cdots + o(1))\frac{T\log T}{2\pi}. \tag{15.4}$$

In small gaps problem, differently from Montgomery's case, consider the Fourier pair

$$r(u) = \left(\frac{\sin \pi u}{\pi u}\right)^2 \left(\frac{1}{1-u^2}\right), \qquad \hat{r}(\alpha) = \max(1 - |\alpha| + \frac{\sin 2\pi|\alpha|}{2\pi}, 0). \tag{15.5}$$

Employing the pair $(r(u/\lambda), |\lambda|\hat{r}(\lambda\alpha))$ in (15.1) gives that

$$\begin{aligned}
\sum_{0<\upsilon\leq T} m_{Z_1}(\varrho) + 2\sum_{0<\upsilon-\tilde{\upsilon}\leq\frac{2\pi\lambda}{\log T}} 1 &= \sum_{\substack{0<\upsilon,\tilde{\upsilon}\leq T\\ |\upsilon-\tilde{\upsilon}|\leq\frac{2\pi\lambda}{\log T}}} 1 \\
&\geq \sum_{0<\upsilon,\tilde{\upsilon}\leq T} \left(\frac{\sin\frac{(\upsilon-\tilde{\upsilon})\log T}{2\lambda}}{\frac{(\upsilon-\tilde{\upsilon})\log T}{2\lambda}}\right)^2 \frac{w(\upsilon-\tilde{\upsilon})}{1-\left(\frac{(\upsilon-\tilde{\upsilon})\log T}{2\pi\lambda}\right)^2} \\
&= \frac{|\lambda| T\log T}{2\pi}\int_{-\infty}^{\infty} \max(1-|\lambda\alpha| + \frac{\sin 2\pi|\lambda\alpha|}{2\pi}, 0) F_{Z_1,Z_1}(\alpha)d\alpha.
\end{aligned}$$

Assume $\epsilon < \lambda < 1-\epsilon$. It is easy to check that

$$1 - \lambda\alpha + \frac{\sin 2\pi\lambda\alpha}{2\pi} \geq 0 \tag{15.6}$$

for $\alpha \in [0, 1/\lambda]$. Together with this, taking into consideration the positivity of $F_{Z_1,Z_1}(\alpha)$, we can reduce the range of the integration to $[-1,1]$.(In fact decrease first to $[-1+\delta, 1-\delta]$ in which the asymptotic formula for $F_{Z_1,Z_1}(\alpha)$ is valid, then expand it to $[-1,1]$

by letting $\delta \to 0$.) Thus, by Theorem 14.1,

$$\sum_{0<\upsilon\le T} m_{Z_1}(\varrho) + 2 \sum_{0<\upsilon-\tilde{\upsilon}\le \frac{2\pi\lambda}{\log T}} 1 \ge \frac{\lambda T\log T}{\pi}(1+o(1))$$
$$\times \int_0^1 (1-\lambda\alpha+\frac{\sin 2\pi\lambda\alpha}{2\pi}) \left(\frac{\log T}{T^{2\alpha}} + \alpha - 4\alpha^2 + \sum_{k\ge 0}\frac{k!(2\alpha)^{2k+3}}{(2k+2)!}\right) d\alpha$$
$$= \frac{\lambda T\log T}{\pi}(1+o(1))\left(\frac{1}{2} + \int_0^1 (1-\lambda\alpha+\frac{\sin 2\pi\lambda\alpha}{2\pi})\right.$$
$$\left. \times \left(\alpha - 4\alpha^2 + \sum_{k\ge 0}\frac{k!(2\alpha)^{2k+3}}{(2k+2)!}\right) d\alpha\right).$$

Either we have infinitely many multiple zeros which provides the smallest gaps one can obtain, or we can assume

$$\sum_{0<\upsilon\le T} m_{Z_1}(\varrho) \sim \frac{T\log T}{2\pi}.$$

So

$$\sum_{0<\upsilon-\tilde{\upsilon}\le \frac{2\pi\lambda}{\log T}} 1 \ge \frac{C(\lambda)T\log T}{2\pi}(1+o(1)),$$

where

$$C(\lambda) := \frac{\lambda}{2} + \lambda\int_0^1 (1-\lambda\alpha+\frac{\sin 2\pi\lambda\alpha}{2\pi}) \left(\alpha - 4\alpha^2 + \sum_{k\ge 0}\frac{k!(2\alpha)^{2k+3}}{(2k+2)!}\right) d\alpha - 1/2.$$

We then have $C(0.89661) > 0$. These two results together constitute

Corollary 15.1. *Assuming RH, more than* 85.838% *of the zeros of* $Z_1(s)$ *are simple, and a positive proportion of the gaps between consecutive zeros of* $Z_1(s)$ *are smaller than* 0.89661 *times the average spacing. Thus,*

$$\liminf_{n\to\infty} \frac{(\upsilon_{n+1}-\upsilon_n)\log \upsilon_n}{2\pi} \le 0.89661,$$

where $0 \le \upsilon_1 \le \upsilon_2 \le \cdots$ *denotes the imaginary parts of the zeros of* $Z_1(s)$ *in the upper half plane.*

The same results as Corollary 15.1 were proved for ξ' by Farmer, Gonek and Lee [18] (which is a corrected version of [11]). They also expressed that the results of Corollary 15.1 should hold by way of upon proving these for ξ' they gave an explanation as to why the leading orders of pair correlation functions for the zeros of ξ' and of Z_1 are equal.

We next question whether the application of the same process to $F_{\zeta,Z_1}(\alpha)$, the pair correlation function of the zeta zeros and zeta maximas on the critical line, gives rise to some valuable results. We cannot benefit from Montgomery's Fourier pair in capturing small gaps between successive zeta zeros and maximas on the critical line, due to the loss of positivity of $F_{\zeta,Z_1}(\alpha)$, which is essential in the case of $F_{\zeta,\zeta}(\alpha)$. On the other hand, similar to (15.1), we have

$$r(0) \sum_{\substack{0<\gamma,\upsilon\le T \\ \gamma=\upsilon}} 1 \le \sum_{0<\gamma,\upsilon\le T} r\left(\frac{(\gamma-\upsilon)\log T}{2\pi}\right) w(\gamma-\upsilon) = \frac{T\log T}{2\pi}\int_{-\infty}^{\infty} \hat{r}(\alpha)G(\alpha)d\alpha \tag{15.7}$$

if $r \in L^1(\mathbb{R})$ is chosen to satisfy the conditions $r(u) \ge 0$ and $r(0) > 0$. Observe that $\gamma = \upsilon$ implies that $m(\rho) \ge 2$ and $m(\varrho) = m(\rho) - 1$. Thus, by the definition of the Fourier transform,

$$\sum_{0<\gamma\le T} (m_\rho - 1) \le \frac{T\log T}{2\pi} \frac{\int_{-\infty}^{\infty} \hat{r}(\alpha)G(\alpha)d\alpha}{\int_{-\infty}^{\infty} \hat{r}(\alpha)d\alpha}, \tag{15.8}$$

from which it follows that

$$\sum_{0<\gamma\le T;\, \rho \,\text{is simple}} 1 \ge \sum_{0<\gamma\le T} (2-m_\rho) \ge \frac{T\log T}{2\pi}\left(1 - \frac{\int_{-\infty}^{\infty} \hat{r}(\alpha)G(\alpha)d\alpha}{\int_{-\infty}^{\infty} \hat{r}(\alpha)d\alpha}\right). \tag{15.9}$$

So one can beat the current record on simple zeros of zeta function by finding a Fourier pair satisfying the above positivity criterion and making the above quotient of the integrals as small as possible.

So far we have evaluated several pair correlation functions, some of which can be derived from Montgomery's approach, namely $F_{\zeta,\zeta}(\alpha)$, $F_{Z_1,Z_1}(\alpha)$, while the others cannot be. In the remaining part of this section we apply our new technique to a product of at least two Dirichlet series, for example $\zeta\cdot Z_1$ and $L(s,\chi)\cdot L(s,\psi)$. This sort of applications are again impossible for Montgomery's method.

Consider the pair correlation function

$$F_{\zeta\cdot Z_1,\zeta\cdot Z_1}(\alpha) = \left(\frac{T}{\pi}\log T\right)^{-1} \sum_{0<t_1,t_2\leq T} T^{i\alpha(t_1-t_2)} w(t_1-t_2), \tag{15.10}$$

where t_1 and t_2 denote the ordinates of the non-real zeros of $\zeta\cdot Z_1(s)$. The number of these zeros of $\zeta\cdot Z_1$ up to T is $\sim \frac{T\log T}{\pi}$. On assuming RH, on the upper-critical line it is possible to enumerate these zeros as

$$0<\gamma_1\leq \upsilon_1\leq \gamma_2\leq \upsilon_2\leq \cdots .$$

So the gaps between consecutive zeta zeros and Z_1 zeros fill the critical line. The average spacing is $\sim \frac{\pi}{\log T}$, half the average of zeta(or Z_1) zeros. Observe that

$$F_{\zeta\cdot Z_1,\zeta\cdot Z_1}(\alpha) = \frac{1}{2}\left(F_{\zeta,\zeta}(\alpha) + F_{\zeta,Z_1}(\alpha) + F_{Z_1,\zeta}(\alpha) + F_{Z_1,Z_1}(\alpha)\right). \tag{15.11}$$

From the previous pair correlation theorems we have

$$F_{\zeta\cdot Z_1,\zeta\cdot Z_1}(\alpha) = (2+o(1))T^{-2|\alpha|}\log T + 2|\alpha| - 4|\alpha|^2 + \frac{1}{2}\sum_{k=0}^{\infty}\frac{k!(2|\alpha|)^{2k+3}}{(2k+2)!} + o(1). \tag{15.12}$$

In the left of (14.8) we also take any sequence of points instead of Z_1-zeros, so this brings the positivity of $F_{\zeta\cdot Z_1,\zeta\cdot Z_1}(\alpha)$.

In the beginning of this section we have two applications of small gaps and simple zeros of $Z_1(s)$. We here repeat the same steps with the sole difference that we use $\left(\frac{T}{\pi}\log T\right)^{-1}$ as a normalizer instead of $\left(\frac{T}{2\pi}\log T\right)^{-1}$. Thus,

$$\sum_{0<t_1\leq T} m_{\zeta\cdot Z_1}(1/2+it_1)+2\sum_{0<t_1-t_2\leq\frac{2\pi\lambda}{\log T}} 1\geq \frac{\lambda T\log T}{\pi}\int_{-\infty}^{\infty}\max(1-|\lambda\alpha|+\frac{\sin 2\pi|\lambda\alpha|}{2\pi},0)F_{\zeta\cdot Z_1,\zeta\cdot Z_1}(\alpha)d\alpha, \quad (15.13)$$

where $\epsilon<\lambda<1-\epsilon$ and $m_{\zeta\cdot Z_1}$ is the multiplicity of zeros of $\zeta\cdot Z_1$. Similar to Z_1-case, assume

$$\sum_{0<t_1\leq T} m_{\zeta\cdot Z_1}(1/2+it_1)\sim\frac{T\log T}{\pi}, \quad (15.14)$$

otherwise we easily conclude that

$$\liminf_{n\to\infty}\frac{(\gamma_{n+1}-\gamma_n)\log\gamma_n}{2\pi}=0. \quad (15.15)$$

Following Z_1-case, by (15.12) and (15.14), (15.13) becomes

$$\sum_{0<t_1-t_2\leq\frac{2\pi\lambda}{\log T}} 1\geq\frac{\tilde{C}(\lambda)T\log T}{\pi}(1+o(1)),$$

where

$$\tilde{C}(\lambda):=2\lambda+2\lambda\int_0^1(1-\lambda\alpha+\frac{\sin 2\pi\lambda\alpha}{2\pi})\left(2\alpha-4\alpha^2+\frac{1}{2}\sum_{k\geq 0}\frac{k!(2\alpha)^{2k+3}}{(2k+2)!}\right)d\alpha-1.$$

We then have $\tilde{C}(0.39421)>0$. So a positive proportion of the gaps between consecutive zeros of $\zeta\cdot Z_1(s)$ are smaller than 0.78842 times the average spacing, which implies three possible cases:

- A positive proportion of the gaps between consecutive zeros of ζ are smaller than 0.39421 times the average spacing of the zeta zeros.
- A positive proportion of the gaps between consecutive zeros of Z_1 are smaller than 0.39421 times the average spacing of the Z_1 zeros.
- A positive proportion of the gaps between consecutive zeros of ζ and Z_1 (of the form (γ_n, υ_n) or $(\upsilon_n, \gamma_{n+1})$ for $n \in \mathbb{Z}^+$) are smaller than 0.78842 times the average spacing of the $\zeta \cdot Z_1$ zeros.

Considering the small gap result on Z_1 in this section and the result in [19] that

$$\liminf_{n\to\infty} \frac{(\gamma_{n+1} - \gamma_n)\log \gamma_n}{2\pi} \leq 0.5172, \tag{15.16}$$

although the first two possibilities represent remarkable improvements, the third one is weaker than the result suggested by (15.16).

As for the simplicity problem of $\zeta \cdot Z_1$, keeping the normalizer issue in mind, Z_1-case can be adapted as follows.

$$\sum_{0<t_1\leq T} m_{\zeta\cdot Z_1}(1/2 + it_1) \leq \frac{\tilde{\tilde{C}}(\lambda) T \log T}{\pi}(1 + o(1)), \tag{15.17}$$

where

$$\tilde{\tilde{C}}(\lambda) = \frac{2}{\lambda}\left(1 + \int_0^{\lambda}\left(1 - \frac{\alpha}{\lambda}\right)\left(2\alpha - 4\alpha^2 + \frac{1}{2}\sum_{k\geq 0}\frac{k!(2\alpha)^{2k+3}}{(2k+2)!}\right)d\alpha\right),$$

which tends to $2.2375\cdots$ as $\lambda \to 1^-$. If t_1 is a zero of $\zeta \cdot Z_1$, there are only three possible cases: $1/2 + it_1$ is a simple zero of $\zeta(s)$ or $Z_1(s)$, or a common zero of $\zeta(s)$ and $Z_1(s)$. In the last case we have

$$m_{\zeta\cdot Z_1}(1/2 + it_1) = 2m_\zeta(1/2 + it_1) - 1,$$

which can be seen from the definition of $Z_1(s)$. In view of these, using Corollory 15.1, we see that the left-hand side of (15.17) is

$$= \sum_{\substack{0<\gamma\le T \\ m_\zeta(1/2+i\gamma)=1}} 1 + \sum_{\substack{0<\gamma\le T \\ m_\zeta(1/2+i\gamma)>1}} (2m_\zeta(1/2+it_1)-1) + \sum_{\substack{0<\upsilon\le T \\ m_{Z_1}(1/2+i\upsilon)=1}} 1$$
$$\ge (1.8538+o(1))\frac{T\log T}{2\pi} + 2 \sum_{\substack{0<\gamma\le T \\ m_\zeta(1/2+i\gamma)>1}} 1.$$

Combining this with (15.17), we obtain

$$\sum_{\substack{0<\gamma\le T \\ m_\zeta(1/2+i\gamma)>1}} 1 \le (1.3107+o(1))\frac{T\log T}{2\pi},$$

which is worse than trivial.

Finally, we deal with the product $L(s,\chi)\cdot L(s,\psi)$, where χ and ψ are two primitive characters. Assume GRH for $L(s,\chi)$ and $L(s,\psi)$. By Theorem 12.1,

$$\begin{aligned} F_{L(s,\chi)\cdot L(s,\psi),L(s,\chi)\cdot L(s,\psi)}(\alpha) :&= \left(\frac{T}{\pi}\log T\right)^{-1} \sum_{0<t_1,t_2\le T} T^{i\alpha(t_1-t_2)} w(t_1-t_2) \\ &= \frac{1}{2}\left(F_{\chi,\chi}(\alpha)+F_{\chi,\psi}(\alpha)+F_{\psi,\chi}(\alpha)+F_{\psi,\psi}(\alpha)\right) \\ &= (2+o(1))T^{-2|\alpha|}\log T + |\alpha| + o(1). \end{aligned} \tag{15.18}$$

Employing the pair $(r(u/\lambda), |\lambda|\hat{r}(\lambda\alpha))$, where $(r,\hat{r})$ comes from (15.5), and employing (15.18), we obtain

$$\sum_{0<t_1\le T} m_{L(s,\chi)\cdot L(s,\psi)}(1/2+it_1) + 2 \sum_{0<t_1-t_2\le \frac{2\pi\lambda}{\log T}} 1 \ge$$
$$\frac{\lambda T\log T}{\pi}\int_{-\infty}^{\infty} \max(1-|\lambda\alpha| + \frac{\sin 2\pi|\lambda\alpha|}{2\pi}, 0)((2+o(1))T^{-2|\alpha|}\log T + |\alpha| + o(1))d\alpha, \tag{15.19}$$

$$\geq \frac{2\lambda T \log T}{\pi}(1+o(1))\left(1+\int_0^1 (1-\lambda\alpha+\frac{\sin 2\pi\lambda\alpha}{2\pi})\alpha d\alpha\right).$$

So,

$$\liminf_{n\to\infty} \frac{(t_{n+1}-t_n)\log t_n}{\pi} = 0,$$

where $(t_n)_{n\in\mathbb{N}}$ is an increasing sequence of the imaginary parts of the zeros of $L(s,\chi)\cdot L(s,\psi)$, or

$$\sum_{0<t_1\leq T} m_{L(s,\chi)\cdot L(s,\psi)}(1/2+it_1) \sim \frac{T\log T}{\pi},$$

which gives that

$$\sum_{0<t_1-t_2\leq \frac{2\pi\lambda}{\log T}} 1 \geq \frac{T\log T}{\pi}(1+o(1))\left(\lambda+\lambda\int_0^1 (1-\lambda\alpha+\frac{\sin 2\pi\lambda\alpha}{2\pi})\alpha d\alpha - 1/2\right).$$

A simple Mathematica calculation gives that the smallest λ making the above coefficient positive is ≥ 0.343705, so that

$$\liminf_{n\to\infty} \frac{(t_{n+1}-t_n)\log t_n}{\pi} \leq 0.68741 \tag{15.20}$$

If we take $\chi \equiv 1$, which reduces $L(s,\chi)$ to $\zeta(s)$, and $\psi(n) = \chi_d(n) := \left(\frac{d}{n}\right)_K$, where d is the discriminant of a quadratic number field K and $\chi_d(n)$ is defined by the Kronecker symbol, then $L(s,\chi)\cdot L(s,\psi)$ returns to the Dedekind zeta function $\zeta_K(s)$ of the K. So the formula (15.20) is also valid for the non-trivial zeros of $\zeta_K(s)$.

16. EXTENSIONS OF SOME LEMMAS IN [2] AND [6]

We start with a technical result in [2]:

Lemma 16.1. *Suppose that* $A(s) = \sum_{n\geq 1} a(n)n^{-s}$ *for* $\sigma > 1$, *where*

$$a(n) \ll \tau_{k_1}(n)(\log n)^{\ell_1}$$

for some non-negative integers k_1 *and* ℓ_1. *Let* $B(s) = \sum_{n\leq y} b(n)n^{-s}$, *where*

$$b(n) \ll \tau_{k_2}(n)(\log n)^{\ell_2}$$

for non-negative integers k_2 *and* ℓ_2 *and where*

$$T^{\epsilon} \ll y \ll T$$

for some $\epsilon > 0$. *Also, let*

$$I = \frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)B(1-s)A(s)ds, \tag{16.1}$$

where $c = 1 + 1/\log T$. *Then we have*

$$I = \sum_{n\leq y} \frac{b(n)}{n} \sum_{m\leq \frac{nT}{2\pi}} a(m)e\left(\frac{-m}{n}\right) + O_{\epsilon}\left(T^{1/2}y(\log T)^{B}\right)$$

for some B. *The admissible value for* B *is* $\ell_1 + \ell_2 + k_1 + k_2$.

A similar integral of the kind in (16.1) will occur in our work. Before introducing it we remark upon two aspects of Lemma 16.1. This lemma is used in [2] finitely many times. So the k_1, k_2, ℓ_1, ℓ_2−dependencies of the implicit constant of the error term is negligible. But in our case ℓ_1 will be forced to be $o(\log T)$ and it can go to $+\infty$ as $T \to +\infty$. So these dependencies, at least ℓ_1-dependency, must be made explicit in our calculations. One other concern emerging from this distinction is about how near $\sigma = 1$ the line of integration is. Is $\dfrac{1}{\log T}-$nearness still possible or does it lead some troubles? We here work on the line $\sigma = 1 + \epsilon$.

This section is devoted to estimating of the integral

$$I_1 = \frac{1}{2\pi i} \int_{1+\epsilon+iT/2}^{1+\epsilon+iT} \chi(1-s)B(1-s)A(s) \left(\log \frac{t}{2\pi}\right)^{\omega} ds, \tag{16.2}$$

where $\omega \in \mathbb{Z}$ and $|\omega| = o(\log T)$. I_1 differs mainly from I by the power of $\log \frac{t}{2\pi}$ in its integrand. Since ω can be negative, it is more convenient to work on $[1+\epsilon+iT/2, 1+\epsilon+iT]$ than on $[1+\epsilon+i, 1+\epsilon+iT]$. We follow closely the proof of Lemma 16.1 in [2] with small changes. After the interchange of the integral and the sums in the integrand, together with the use of (2.7), I is reduced to the integral of Lemma 3 in [6]:

Lemma 16.2. *For $m = 0, 1, 2, ...,$ A large, and $A < r \leq B \leq 2A$,*

$$\begin{aligned}\int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m dt =&(2\pi)^{1-a} r^a \, e^{-ir+\frac{\pi i}{4}} \left(\log \frac{r}{2\pi}\right)^m \\ &+ E(r, A, B) \left(\log A\right)^m,\end{aligned}$$

while for $r \leq A$ or $r > B$,

$$\int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m dt = E(r, A, B) \left(\log A\right)^m,$$

where

$$E(r, A, B) = O\left(A^{a-\frac{1}{2}}\right) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r| + A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-r| + B^{\frac{1}{2}}}\right). \tag{16.3}$$

The same step suggests us to extend this lemma to negative values of m. In [6] the $m-$dependence of $E(r,A,B)$ is skipped, since it is presumably an unnecessary detail. Here our extension is $m-$uniform.

Lemma 16.3. *Let a be a fixed number such that $1 < a < 1/2 + 1/\log 2$, A large, and $m \in \mathbb{Z}$ with $|m| = o(\log A)$. We have, for $A < r \le B \le 2A$,*

$$\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt$$
$$= (2\pi)^{1-a} r^a e^{-ir+\frac{\pi i}{4}}\left(\log\frac{r}{2\pi}\right)^m + E(r,A,B)(\log A)^m,$$

while for $r \le A$ or $r > B$,

$$\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt = E(r,A,B)(\log A)^m.$$

Here, $E(r,A,B)$ is as in (16.3), further the constants implied by its O-terms do not depend on m.

[20] includes a slightly different version of this result.

Proof. Case 1: $m < 0$ and $A < r \le B \le 2A$

In the range $A \le t \le B$, we can write

$$\left[\log\frac{t}{2\pi}\right]^{-1} = \left[\left(\log\frac{r}{2\pi}\right)\left(1+\frac{\log\frac{t}{r}}{\log\frac{r}{2\pi}}\right)\right]^{-1} = \left(\log\frac{r}{2\pi}\right)^{-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\log\frac{t}{r}}{\log\frac{r}{2\pi}}\right)^n.$$

The above power series is absolutely and uniformly convergent for $t \in [A,B]$. Then

$$\left[\log\frac{t}{2\pi}\right]^m = \left(\log\frac{r}{2\pi}\right)^m\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{\log\frac{t}{r}}{\log\frac{r}{2\pi}}\right)^n\right]^{-m}$$
$$= \left(\log\frac{r}{2\pi}\right)^m\left\{1+\sum_{n=1}^{\infty}(-1)^n\binom{-m+n-1}{-m-1}\left(\frac{\log\frac{t}{r}}{\log\frac{r}{2\pi}}\right)^n\right\}$$

by (9.3). Using the above expansion, we have

$$\begin{aligned}
&\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt \\
&\qquad = \left(\log\frac{r}{2\pi}\right)^m \int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt \\
&+ \sum_{n=1}^{\infty}(-1)^n\binom{-m+n-1}{-m-1}\left(\log\frac{r}{2\pi}\right)^{m-n}\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{r}\right)^n dt \\
&\qquad = \left(\log\frac{r}{2\pi}\right)^m\left[S_0 + \sum_{n=1}^{\infty}(-1)^n\binom{-m+n-1}{-m-1}\left(\log\frac{r}{2\pi}\right)^{-n} S_n\right], \text{ say.} \qquad (16.4)
\end{aligned}$$

Taking $m = 0$ in Lemma 16.2, we have

$$S_0 = (2\pi)^{1-a} r^a e^{-ir+\frac{\pi i}{4}} + E(r, A, B).$$

For $n = 1$, applying integration by parts with

$$u = \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}, \quad dv = \left(\log\frac{t}{r}\right)\exp\left[it\log\frac{t}{re}\right] dt$$

gives

$$S_1 = -i\exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\Bigg|_A^B + \frac{i\left(a-\frac{1}{2}\right)}{2\pi}\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{3}{2}} dt,$$

and then, trivial estimation gives $S_1 \ll A^{a-\frac{1}{2}}$. Similarly, for $n \geq 2$, we have

$$\begin{aligned}
S_n &= -i\exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{r}\right)^{n-1}\Bigg|_A^B \\
&+ \frac{i}{2\pi}\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{3}{2}}\left(\log\frac{t}{r}\right)^{n-2}\left\{\left(a-\frac{1}{2}\right)\log\frac{t}{r} + n - 1\right\} dt.
\end{aligned}$$

Since $A < r \leq B \leq 2A$ and $t \in [A, B]$, we have $\left|\log\frac{t}{r}\right| \leq \log 2 < 1$. So we can say that $S_n \ll n(\log 2)^n A^{a-1/2} \ll c^n A^{a-1/2}$, for any fixed number $c > \log 2$. Then the last sum in (16.4) is

$$\ll A^{a-\frac{1}{2}} \sum_{n=1}^{\infty} \binom{-m+n-1}{-m-1} \left(\frac{c}{\log \frac{r}{2\pi}}\right)^n$$

$$= A^{a-\frac{1}{2}} \left[\left(\frac{1}{1 - \frac{c}{\log \frac{r}{2\pi}}} \right)^{-m} - 1 \right] \ll \frac{|m| A^{a-\frac{1}{2}}}{\log A},$$

as long as $|m| = o(\log A)$. Combining these results completes the first case.

Case 2: $m \geq 0$ and $A < r \leq B \leq 2A$

This case is already done in Lemma 16.2. Here the m-dependence of the error terms is made explicit. Firstly,

$$\int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m dt$$

$$= \sum_{j \leq m} \binom{m}{j} \left(\log \frac{r}{2\pi}\right)^{m-j} \int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{r}\right)^j dt. \quad (16.5)$$

Returning to the first case, we realize that the integral above matches the already calculated S_j, $0 \leq j \leq m$, and so the result follows.

By the way, (16.5) is included in (16.4), which can be seen by taking into consideration the most general definition of the binomial coefficients; if we let m of the binomial expression in (16.4) be positive, then the infinite series turns to a finite sum. So we could in fact handle $A < r \leq B$ in one case.

Case 3: $A - \sqrt{A} < r \leq A$

We will use the following well-known result ([16], Lemma 4.5):

Let $F(t)$ and $G(t)$ be real functions, $F(t)$ twice differentiable, $G(t)/F'(t)$ monotonic, $|G(t)| \leq M$, and let $F''(t) \geq \delta > 0$, or $F''(t) \leq -\delta < 0$, throughout an interval

$[A, B]$. Then

$$\left| \int_A^B G(t) \exp\{iF(t)\}\, dt \right| \le \frac{8M}{\sqrt{\delta}}. \tag{16.6}$$

We let

$$F(t) = t \log \frac{t}{re} \qquad \text{and} \qquad G(t) = \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m,$$

and see that

$$F''(t) \ge \frac{1}{2A} \qquad \text{and} \qquad |G(t)| \ll A^{a-1/2}(\log A)^m \tag{16.7}$$

for $t \in [A, B]$ and $m = o(\log A)$. Since

$$\left(\frac{G(t)}{F'(t)}\right)' = \frac{\left(\frac{t}{2\pi}\right)^{a-\frac{3}{2}} \left(\log \frac{t}{2\pi}\right)^m}{2\pi \log \frac{t}{r}} \left[a - \frac{1}{2} + \frac{m}{\log \frac{t}{2\pi}} - \frac{1}{\log \frac{t}{r}}\right],$$

we see that, with our restrictions on a and $|m| = o(\log A)$, $G/F'(t)$ is monotone decreasing if A is sufficiently large. Hence applying (16.6), we obtain

$$\int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m dt \ll A^a (\log A)^m,$$

which completes the proof for the cases $A - \sqrt{A} < r \le A$. The case $B \le r < B + \sqrt{B}$ is similarly handled.

Case 4: $r \le A - \sqrt{A}$

Now $\log t/r \neq 0$ for $t \in [A, B]$. Integrating by parts, we have

$$\begin{aligned}\int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m dt = \\ & - i \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \left(\log \frac{t}{2\pi}\right)^m \left(\log \frac{t}{r}\right)^{-1} \Bigg|_A^B \\ & + \frac{i(a-1/2)}{2\pi} \int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{3}{2}} \left(\log \frac{t}{2\pi}\right)^m \left(\log \frac{t}{r}\right)^{-1} dt \\ & + \frac{im}{2\pi} \int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{3}{2}} \left(\log \frac{t}{2\pi}\right)^{m-1} \left(\log \frac{t}{r}\right)^{-1} \\ & - \frac{i}{2\pi} \int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{3}{2}} \left(\log \frac{t}{2\pi}\right)^m \left(\log \frac{t}{r}\right)^{-2} dt.\end{aligned} \tag{16.8}$$

The first three terms of the right-hand side of (16.8) are trivially

$$\ll \frac{A^{a-1/2}(\log A)^m}{|\log \frac{r}{A}|} \ll \frac{A^{a+1/2}\,(\log A)^m}{|A-r|}$$

since $|\log r/A| \gg (A-r)/A$ and $(\log A/2\pi)^{-m} \asymp (\log A)^{-m}$ under $|m| = o(\log A)$. The three real-valued functions in the integrand of the last integral in the right-hand side of (16.8) are monotone, and so applying the second mean value theorem of integral calculus three times gives

$$\begin{aligned}&\int_A^B \exp\left[it \log \frac{t}{re}\right] \left(\frac{t}{2\pi}\right)^{a-\frac{3}{2}} \left(\log \frac{t}{2\pi}\right)^m \left(\log \frac{t}{r}\right)^{-2} dt \\ &\asymp \frac{A^{a-3/2}(\log A)^m}{|\log \frac{r}{A}|^3} \int_{A_1}^{B_1} \exp\left[it \log \frac{t}{re}\right] \left(\log \frac{t}{r}\right) dt \quad (\text{for some } A_1, B_1 \in [A, B]) \\ &\ll \frac{A^{a+1/2}(\log A)^m}{|A-r|}\end{aligned}$$

since $|A - r| \geq \sqrt{A}$ and $|\log r/A| \gg (r - A)/A$.

Combining the cases $A - \sqrt{A} < r \leq A$ and $r \leq A - \sqrt{A}$ gives the definition of $E(r, A, B)$. The remaining $r \geq B + \sqrt{B}$ case can be done similarly. □

With the use of Lemma 16.2 the main term of Lemma 16.1 can be derived. For the error term calculations the result of Shiu on the generalized divisor function is needed. We quote from [21]:

$$\sum_{x-y<n\leq x} \tau_k(n) \ll_{\epsilon,k} y(\log x)^{k-1} \tag{16.9}$$

for $y \gg x^{\epsilon}$. Further, before stating the fundamental result of this section, we continue with the following simple result.

Lemma 16.4. *Suppose $k \in \mathbb{Z}^+$, $\ell \in \mathbb{N}$. We have*

$$\sum_{n=1}^{\infty} \frac{\tau_k(n)(\log n)^{\ell}}{n^s} \leq \frac{A^{k+\ell}\ell!}{|s-1|^{k+\ell}} \tag{16.10}$$

for $|s-1| \ll 1$ and $\Re s > 1$.

Proof. By (2.13), for $|s-1| \ll 1$,

$$(\zeta(s))^k \leq \left(\frac{A}{|s-1|}\right)^k. \tag{16.11}$$

By Cauchy's integral formula we obtain

$$\sum_{n=1}^{\infty} \frac{\tau_k(n)(\log n)^{\ell}}{n^s} = (-1)^{\ell} \frac{d^{\ell}}{ds^{\ell}} (\zeta(s))^k = \frac{(-1)^{\ell}\ell!}{2\pi i} \int \frac{(\zeta(w))^k \, dw}{(w-s)^{\ell+1}},$$

where the integral is over the circle $|w-s| \asymp |s-1|$. Employing (16.11), the result follows trivially. $\square$

Lemma 16.5. *Let $\omega \in \mathbb{Z}$, $k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ satisfying $|\omega|$, $\ell_1 = o(\log T)$ for large T, and $k_1, k_2, \ell_2 = O(1)$. Suppose that $A(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}$ and $B(s) = \sum_{n\leq y} \frac{b(n)}{n^s}$ for $\sigma > 1$, where*

$$a(n) \ll \tau_{k_1}(n)(\log n)^{\ell_1}$$
$$b(n) \ll \tau_{k_2}(n)(\log n)^{\ell_2},$$

and $T^\epsilon \ll y \ll T$. *Then*

$$I_1 = \sum_{n\le y}\frac{b(n)}{n}\sum_{\frac{nT}{4\pi}<m\le\frac{nT}{2\pi}} a(m)e\left(\frac{-m}{n}\right)\left(\log\frac{m}{n}\right)^\omega + O\left(yT^{1/2+\epsilon}(\log T)^{\omega+\ell_1}\right).$$

Proof. The boundedness of k_1, k_2 and ℓ_2 will be repeatedly used. Inserting (2.7) into the integrand of I_1 gives that

$$\begin{aligned} I_1 =& \frac{e^{-i\pi/4}}{2\pi}\sum_{n\le y} b(n)n^\epsilon \sum_{m\ge 1}\frac{a(m)}{m^{1+\epsilon}}\int_{T/2}^{T}\left(\frac{t}{2\pi}\right)^{1/2+\epsilon}\exp\left(it\log\frac{tn}{2\pi me}\right)\left(\log\frac{t}{2\pi}\right)^\omega dt \\ &+ O\left(T^{1/2+\epsilon}(\log T)^\omega\left|\sum_{n\le y} b(n)n^\epsilon\sum_{m\ge 1} a(m)m^{-1-\epsilon}\right|\right). \end{aligned}$$

By the definition of $b(n)$ and the fact that $\tau_k(m) \ll m^\epsilon$ for bounded values of k,

$$\left|\sum_{n\le y} b(n)n^\epsilon\right| \ll y^{1+\epsilon}. \tag{16.12}$$

It then follows from Lemma 16.4 that the above O-term is

$$\ll A^{\ell_1}\ell_1! yT^{1/2+\epsilon}(\log T)^\omega \ll yT^{1/2+\epsilon}(\log T)^{\omega+\ell_1}, \tag{16.13}$$

by the Stirling formula and the condition on ℓ_1. From Lemma 16.3,

$$\begin{aligned} I_1 =& \sum_{n\le y}\frac{b(n)}{n}\sum_{\frac{nT}{4\pi}<m\le\frac{nT}{2\pi}} a(m)e\left(\frac{-m}{n}\right)\left(\log\frac{m}{n}\right)^\omega + O\left(yT^{1/2+\epsilon}(\log T)^{\omega+\ell_1}\right) \\ &+ O\left(\sum_{n\le y}|b(n)|n^\epsilon\sum_{m\ge 1}|a(m)|m^{-1-\epsilon}(\log T)^\omega E\left(2\pi m/n, T/2, T\right)\right). \end{aligned} \tag{16.14}$$

We divide the second error term into three parts, say S_1, S_2 and S_3, corresponding to the three parts forming $E\left(2\pi m/n, T/2, T\right)$ in (16.3). From (16.12) and Lemma 16.4, S_1 is dominated by the bound in (16.13). To deal with S_2 we consider the terms

$m < \dfrac{nT}{8\pi}$, $\dfrac{nT}{8\pi} \le m < \dfrac{3nT}{8\pi}$ and $m \ge \dfrac{3nT}{8\pi}$ separately:

$$S_2 \ll \sum_{n\le y} |b(n)| n^{\epsilon} \left(\sum_{m<\frac{nT}{8\pi}} + \sum_{\frac{nT}{8\pi}\le m<\frac{3nT}{8\pi}} + \sum_{m\ge\frac{3nT}{8\pi}} \right) \frac{|a(m)| m^{-1-\epsilon} T^{3/2+\epsilon} (\log T)^{\omega}}{|T/2 - 2\pi m/n| + \sqrt{T/2}}.$$

If $m < \dfrac{nT}{8\pi}$ or $m \ge \dfrac{3nT}{8\pi}$ then

$$|T/2 - 2\pi m/n| + \sqrt{T/2} \gg T,$$

so the contribution from these terms can be absorbed by the bound in (16.13) similarly to S_1. The remaining terms are the pairs (n, m) with $T/4 \le 2\pi m/n < 3T/4$, and for these $m^{-1-\epsilon} \ll (nT)^{-1-\epsilon}$. We see that the interval $[T/4, 3T/4]$ is covered by the subintervals of the form $[T/2 \pm (2^{\nu} - 1)T^{1/2}, T/2 \pm (2^{\nu+1} - 1)T^{1/2}]$, where $\nu \in \mathbb{N}$ and $\nu \ll \log T$. We have

$$\begin{aligned} &\sum_{n\le y} |b(n)| n^{\epsilon} \sum_{\frac{nT}{8\pi}\le m<\frac{3nT}{8\pi}} \frac{|a(m)| m^{-1-\epsilon} T^{3/2+\epsilon} (\log T)^{\omega}}{|T/2 - 2\pi m/n| + \sqrt{T/2}} \\ &\ll T^{\epsilon} (\log T)^{\omega+\ell_1} \sum_{n\le y} \frac{|b(n)|}{n} \sum_{0\le\nu\ll\log T} \frac{1}{2^{\nu} - 1 + \frac{1}{\sqrt{2}}} \\ &\times \sum_{\frac{T}{2}\pm(2^{\nu}-1)T^{\frac12} \le \frac{2\pi m}{n} < \frac{T}{2}\pm(2^{\nu+1}-1)T^{\frac12}} \tau_{k_1}(m). \end{aligned}$$

By (16.9), the inner-most sum is

$$\ll 2^{\nu} T^{1/2} n (\log T)^{k_1 - 1}.$$

The $\nu-$sum is $\ll 1$ and the $n-$sum is estimated as in (16.12) so that the bound in (16.13) is still dominant. S_3 can be handled similarly and we're done. $\square$

17. A SUM OF DIRICHLET COEFFICIENTS RELATED TO THE VON MANGOLDT FUNCTION AND ITS GENERALIZATIONS

Let $\nu \in \mathbb{N}$, for $\nu_1, \nu_2 \in \{0, 1\}$, we set

$$L_{\nu_1,\nu_2}(s;\nu) := \sum_{m=1}^{\infty} \frac{\alpha(m;\nu,\nu_1,\nu_2)}{m^s}$$
$$= \left(-\frac{\zeta'}{\zeta}(s)\right)^{\nu} \left(\frac{\zeta''}{\zeta}(s)\right)^{\nu_1} (\zeta(s))^{\nu_2+1} (B(s))^{\nu_2},$$
$$L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi) := \sum_{m=1}^{\infty} \frac{\alpha(md;\nu,\nu_1,\nu_2)\psi(m)}{m^s}$$

for $\sigma > 1$, where $d, \eta \in \mathbb{Z}^+$, ψ is any Dirichlet character modulo η, and the Dirichlet polynomial $B(s)$ is defined by

$$B(s) = \sum_{n \le y} \frac{b(n)}{n^s}, \tag{17.1}$$

where $b(n) = \mu(n)P\left(\frac{\log \frac{y}{n}}{\log y}\right)$ and $P(\cdot)$ is a polynomial with real coefficients. We omit the $P-$dependence of the errors in our estimations. In addition to these, assume that d is square-free, $d \le y$, and $(\eta, d) = 1$. Without mentioning we will use these conditions on d and η frequently. The main aim of this section is to estimate the average

$$\sum_{w/2 < m \le w} \alpha(md;\nu,\nu_1,\nu_2)\psi(m), \quad w \ge 1. \tag{17.2}$$

To do this we first examine the analytic properties of $L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)$. We start this part by determining the size of the Dirichlet coefficients $\alpha(m;\nu,\nu_1,\nu_2)$.

Lemma 17.1. $|\alpha(m;\nu,\nu_1,\nu_2)| \le \tau_{2\nu_2+2}(m)(\log m)^{\nu+2\nu_1}$.

Proof. Disregarding the restrictions on ν_1 and ν_2, we define $\tilde{\alpha}(m;\nu,\nu_1,\nu_2,\nu_3)$ by the relation

$$\sum_{m=1}^{\infty} \frac{\tilde{\alpha}(m;\nu,\nu_1,\nu_2,\nu_3)}{m^s} = \left(-\frac{\zeta'}{\zeta}(s)\right)^{\nu} \left(\frac{\zeta''}{\zeta}(s)\right)^{\nu_1} (\zeta(s))^{\nu_2} (B(s))^{\nu_3},$$

where $\nu, \nu_1, \nu_2, \nu_3 \in \mathbb{N}$. We first prove that

$$\tilde{\alpha}(m;\nu,\nu_1,\nu_2,0) \leq \tau_{\nu_2+1}(m)(\log m)^{\nu+2\nu_1}, \tag{17.3}$$

by induction on $\nu+\nu_1+\nu_2$. The case $\nu+\nu_1+\nu_2 = 0$ is trivial. It is enough to show that if the assertion is true for a triple (ν,ν_1,ν_2), then it is also true for $(\nu+1,\nu_1,\nu_2)$, (ν,ν_1+1,ν_2) and (ν,ν_1,ν_2+1). Consider the first case. We see that

$$\begin{aligned}\tilde{\alpha}(m;\nu+1,\nu_1,\nu_2,0) &= \sum_{d|m} \tilde{\alpha}(m/d;\nu,\nu_1,\nu_2,0)\Lambda(d) \\ &\leq \tau_{\nu_2+1}(m)(\log m)^{\nu+2\nu_1} \sum_{d|m} \Lambda(d) = \tau_{\nu_2+1}(m)(\log m)^{\nu+1+2\nu_1}.\end{aligned}$$

The last equality is a direct consequence of the product $-\dfrac{\zeta'}{\zeta}(s)\zeta(s) = -\zeta'(s)$. Similarly, the case (ν,ν_1+1,ν_2) follows from the identity $\sum_{d|n} \Lambda_2(d) = \log^2 m$. We derive the final case by observing that

$$\begin{aligned}\tilde{\alpha}(m;\nu,\nu_1,\nu_2+1,0) &= \sum_{d|m} \tilde{\alpha}(d;\nu,\nu_1,\nu_2,0) \\ &\leq (\log m)^{\nu+2\nu_1} \sum_{d|m} \tau_{\nu_2+1}(d) = \tau_{\nu_2+2}(m)(\log m)^{\nu+2\nu_1},\end{aligned}$$

which finishes the proof of (17.3). Here the identity we have employed in the last line can be derived from $(\zeta(s))^{\nu_2+1}\zeta(s) = (\zeta(s))^{\nu_2+2}$. It is clear that the $m-$th coefficient of $(B(s))^{\nu_3}$ is $\le A^{\nu_3}\tau_{\nu_3}(m)$. So by (17.3),

$$\tilde{\alpha}(m;\nu,\nu_1,\nu_2,\nu_3) \le A^{\nu_3}(\log m)^{\nu+2\nu_1}\sum_{d|m}\tau_{\nu_2+1}(d)\tau_{\nu_3}\left(\frac{m}{d}\right)$$
$$= A^{\nu_3}\tau_{\nu_2+\nu_3+1}(m)(\log m)^{\nu+2\nu_1},$$

which contains the assertion of the lemma as a special case.

17.1. The left of the line $\sigma = 1$

We next show that $L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)$ possesses an analytic continuation outside the region $\sigma > 1$, where the series converges. By means of Lemma 3 in [2] $L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)$ can be represented in terms of Dirichlet L-functions and some elementary parts. We have

$$L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi) = \tag{17.4}$$
$$\sum_{efgh=d}\left(\sum_{m\ge 1}\frac{\psi(m)\Lambda_2^{*(\nu_1)}(me)}{m^s}\right)\left(\sum_{d_1\ldots d_\nu=g}\prod_{j=1}^{\nu}\sum_{\substack{m\ge 1\\(m,ed_1\ldots d_{j-1})=1}}\frac{\psi(m)\Lambda(md_j)}{m^s}\right)$$
$$\times B_{\nu_2}(s,\psi,h,eg)\left(\sum_{\substack{m\ge 1\\(m,heg)=1}}\frac{\tau_{\nu_2+1}(fm)\psi(m)}{m^s}\right)$$

where

$$B_{\nu_2}(s,\psi,h,eg) := \sum_{\substack{m\le y/h\\(m,eg)=1}}\frac{\psi(m)I_{\nu_2}(hm)b(hm)}{m^s}, \tag{17.5}$$

and

$$I_u(\ell) := \begin{cases} 1 & \text{if}\, u = 1, \\ 1 & \text{if}\, u = 0\,\text{and}\,\ell = 1, \\ 0 & \text{if}\, u = 0\,\text{and}\,\ell > 1. \end{cases} \tag{17.6}$$

Clearly, I_u is totally multiplicative. To avoid any vagueness we emphasize that the sum over $\nu-$tuples comprised of positive divisors of g, in the case of $\nu = 0$, is 1 if $g = 1$; and is 0 for any value of $g > 1$. Observe that

$$\sum_{\substack{m\geq 1\\(m,ed_1\dots d_{j-1})=1}} \frac{\psi(m)\Lambda(md_j)}{m^s} = 0 \tag{17.7}$$

if $\omega(d_j) \geq 2$, and we can write in general

$$\sum_{\substack{m\geq 1\\(m,ed_1\dots d_{j-1})=1}} \frac{\psi(m)\Lambda(md_j)}{m^s} = -I(d_j)\frac{L'}{L}(s, \psi\psi_{0,ed_1\dots d_{j-1}}) + \Lambda(d_j)\left(1 - \frac{\psi(d_j)}{d_j^s}\right)^{-1}, \tag{17.8}$$

where $\psi_{0,q}$ is the principal character modulo q, and

$$I(n) := \begin{cases} 1 & \text{if} \qquad n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{17.9}$$

Here $\psi\psi_{0,ed_1\dots d_{j-1}}$ is a character modulo $\eta ed_1 \dots d_{j-1}$. We now try to find a formula for the first sum over m in (17.4). We see that the sum is 0 when $\omega(e) > 2$; 1 when $\nu_1 = 0$ and $e = 1$; 0 when $\nu_1 = 0$ and $e > 1$. Assume $\nu_1 = 1$. First recall the identity

$$\Lambda_2(m) = \Lambda(m)\log m + \sum_{d|m} \Lambda(d)\Lambda(m/d). \tag{17.10}$$

If $\omega(e) = 2$, say $e = pq$ for some primes p, q, then by (17.10),

$$\sum_{m\geq 1} \frac{\psi(m)\Lambda_2(me)}{m^s} = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} \frac{\psi(p^\alpha q^\beta)\Lambda_2(p^{\alpha+1}q^{\beta+1})}{p^{\alpha s}q^{\beta s}} = \Lambda_2(e)\prod_{p|e}\left(1-\frac{\psi(p)}{p^s}\right)^{-1}.$$

If $\omega(e) = 1$, then e is prime and

$$\sum_{m\geq 1} \frac{\psi(m)\Lambda_2(me)}{m^s} = \sum_{\alpha=0}^{\infty} \frac{\psi(e^\alpha)\Lambda_2(e^{\alpha+1})}{e^{\alpha s}} + \sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{\infty}\sum_{\substack{q:prime\\ q\neq e}} \frac{\psi(e^\alpha q^\beta)\Lambda_2(e^{\alpha+1}q^\beta)}{e^{\alpha s}q^{\beta s}}.$$

From (17.10), we have $\Lambda_2(e^{\alpha+1}) = (2\alpha+1)(\log e)^2$ and $\Lambda_2(e^{\alpha+1}q^\beta) = 2(\log e)(\log q)$. It then follows that

$$\begin{aligned}
\sum_{m\geq 1} \frac{\psi(m)\Lambda_2(me)}{m^s} &= 2(\log e)^2 \sum_{\alpha=0}^{\infty} \alpha\left(\frac{\psi(e)}{e^s}\right)^\alpha + (\log e)^2 \sum_{\alpha=0}^{\infty}\left(\frac{\psi(e)}{e^s}\right)^\alpha \\
&\quad + 2(\log e)\sum_{\alpha=0}^{\infty}\left(\frac{\psi(e)}{e^s}\right)^\alpha \sum_{\substack{q:prime\\ q\neq e}} (\log q) \sum_{\beta=1}^{\infty}\left(\frac{\psi(q)}{q^s}\right)^\beta \\
&= -2(\log e)\frac{d}{ds}\sum_{\alpha=0}^{\infty}\left(\frac{\psi(e)}{e^s}\right)^\alpha + (\log e)^2\left(1-\frac{\psi(e)}{e^s}\right)^{-1} \\
&\quad + 2(\log e)\left(1-\frac{\psi(e)}{e^s}\right)^{-1}\sum_{\substack{q:prime\\ q\neq e}} \frac{\psi(q)\log q}{q^s-\psi(q)} \\
&= (\log e)\left(1-\frac{\psi(e)}{e^s}\right)^{-1}\left(\log e - 2\frac{L'}{L}(s,\psi)\right).
\end{aligned}$$

If $\omega(e) = 0$, then clearly $\sum_{m\geq 1} \frac{\psi(m)\Lambda_2(me)}{m^s} = \frac{L''}{L}(s,\psi)$. If we try to put all these into one form, we find that

$$\sum_{m\geq 1} \frac{\psi(m)\Lambda_2^{*(\nu_1)}(me)}{m^s} =$$
$$I_{\nu_1}(e)\prod_{p|e}\left(1-\frac{\psi(p)}{p^s}\right)^{-1}\sum_{\mathfrak{a}=0}^{2}\binom{2}{\mathfrak{a}}\left((-1)^{\mathfrak{a}}\frac{L^{(\mathfrak{a})}}{L}(s,\psi)\right)^{\nu_1}\Lambda_{2-\mathfrak{a}}(e). \quad (17.11)$$

For the last Dirichlet series of the right-hand side of (17.4), consider the Euler product representation

$$\sum_{\substack{m\geq 1\\(m,heg)=1}} \frac{\tau_{\nu_2+1}(fm)\psi(m)}{m^s} =$$
$$\left(\prod_{p\nmid efgh}\sum_{m=0}^{\infty}\frac{\tau_{\nu_2+1}(p^m)\psi(p^m)}{p^{ms}}\right)\left(\prod_{p|f}\sum_{m=0}^{\infty}\frac{\tau_{\nu_2+1}(p^{m+1})\psi(p^m)}{p^{ms}}\right).$$

Using the identity

$$\left(\sum_{m=0}^{\infty} z^m\right)^{\nu_2+1} = \sum_{m=0}^{\infty}\binom{\nu_2+m}{m} z^m \tag{17.12}$$

and the fact that $\tau_{\nu_2+1}(p^m) = \binom{\nu_2+m}{m}$, we obtain

$$\sum_{\substack{m\geq 1\\(m,heg)=1}} \frac{\tau_{\nu_2+1}(fm)\psi(m)}{m^s} = L^{\nu_2+1}(s,\psi)\prod_{p|heg}\left(1-\frac{\psi(p)}{p^s}\right)^{\nu_2+1}$$
$$\prod_{p|f}\frac{p^s}{\psi(p)}\left(1-\left(1-\frac{\psi(p)}{p^s}\right)^{\nu_2+1}\right), \tag{17.13}$$

which is the same as the result (1.4.2) in [16] when $\nu_2 = 1$ and $\eta heg = 1$. Here ν_2 could be taken as an arbitrary positive integer. However, we need it for $\nu_2 = 0, 1$, and in these cases, the last product in (17.13) is

$$= \prod_{p|f}\left(2-\frac{\psi(p)}{p^s}\right)^{\nu_2}. \tag{17.14}$$

In view of (17.8), (17.11), (17.13) and (17.14) we can organize (17.4) as follows.

$$L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi) = L^{\nu_2+1}(s,\psi)\sum_{\mathfrak{a}=0}^{2}\binom{2}{\mathfrak{a}}\left((-1)^{\mathfrak{a}}\frac{L^{(\mathfrak{a})}}{L}(s,\psi)\right)^{\nu_1}\sum_{efgh=d}\Lambda_{2-\mathfrak{a}}(e) \tag{17.15}$$
$$I_{\nu_1}(e)\prod_{p|g}\Lambda(p)\prod_{p|eg}\left(1-\frac{\psi(p)}{p^s}\right)^{\nu_2}\prod_{p|f}\left(2-\frac{\psi(p)}{p^s}\right)^{\nu_2}\prod_{p|h}\left(1-\frac{\psi(p)}{p^s}\right)^{\nu_2+1}$$
$$B_{\nu_2}(s,\psi,h,eg)\sum_{d_1\dots d_\nu=g}{}^{'}\prod_{\substack{1\le j\le\nu\\ d_j=1}}\left(-\frac{L'}{L}(s,\psi\psi_{0,ed_1\dots d_{j-1}})\right).$$

Here $\sum'$ is over $\nu-$tuples $(d_1,...,d_\nu)$ satisfying $\omega(d_i)\le 1$ for all $i=1,...,\nu$. This restrictions follows from (17.7) and (17.8). The right-hand side of (17.15) defines an analytic function in the half-plane $\sigma\le 1$, except possibly at the zeros of L-functions above and $s=1$. If ψ is the principal character modulo η, then $L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)$ has a pole at $s=1$ of order $\nu+2\nu_1+\nu_2+1$, otherwise, it's analytic at $s=1$.

17.2. Order Estimates

This part is devoted to obtaining an upper bound for $L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)$. Here we restrict ourselves to primitive characters ψ, and accept that the character modulo 1 is primitive.

Assume GRH. From (2.18), we know that

$$\frac{L'}{L}(s,\psi)\ll\log\eta\tau \tag{17.16}$$

for $\sigma\ge\frac{1}{2}+\epsilon$, $|s-1|\gg\frac{1}{\log\eta\tau}$. To estimate $\frac{L''}{L}(s,\psi)$ first consider

$$\frac{L''}{L}(s,\psi)=\frac{d}{ds}\frac{L'}{L}(s,\psi)+\left(\frac{L'}{L}(s,\psi)\right)^2. \tag{17.17}$$

Via Cauchy's integral along a disc with center s and radius ϵ and (2.18), we can deal with the above derivative, and then

$$\frac{L''}{L}(s,\psi) \ll (\log \eta\tau)^2 \tag{17.18}$$

for $\sigma \geq \frac{1}{2}+\epsilon$, $|s-1| \gg \frac{1}{\log \eta\tau}$. The second of the conditions shaping the region on which the above estimates hold is required only when $\eta = 1$. To extend these results to $\frac{L'}{L}(s, \psi\psi_{0,ed_1\ldots d_{j-1}})$, we firstly note that

$$\frac{L'}{L}(s,\psi\psi_{0,ed_1\ldots d_{j-1}}) = \frac{L'}{L}(s,\psi) + \sum_{p|ed_1\ldots d_{j-1}} \frac{\psi(p)\log p}{p^s - \psi(p)}. \tag{17.19}$$

We treat the divisor sum on the right by following almost the same lines of the proof of Lemma 7 in [2]. For $\sigma \geq 1/2+\epsilon$,

$$\begin{aligned}\sum_{p|ed_1\ldots d_{j-1}} \frac{\psi(p)\log p}{p^s-\psi(p)} &\ll \sum_{\substack{p\leq \log(ed_1\ldots d_{j-1})\\ p|ed_1\ldots d_{j-1}}} \frac{\log p}{p^{1/2+\epsilon}} + \sum_{\substack{p> \log(ed_1\ldots d_{j-1})\\ p|ed_1\ldots d_{j-1}}} \frac{\log p}{p^{1/2+\epsilon}} \\ &\ll (\log(ed_1\ldots d_{j-1}))^{1/2-\epsilon}, \end{aligned} \tag{17.20}$$

where we have used $\sum_{p|n}\log p = \log n$ for squarefree n and Chebyshev's estimates. Combining (17.16), (17.19) and (17.20), we obtain

$$\frac{L'}{L}(s,\psi\psi_{0,ed_1\ldots d_{j-1}}) \ll \log \eta e d_1\ldots d_{j-1}\tau \tag{17.21}$$

for $\sigma \geq \frac{1}{2}+\epsilon$, $|s-1| \gg \frac{1}{\log \eta\tau}$. Trivially,

$$\prod_{p|e}\left(1-\frac{\psi(p)}{p^s}\right)^{\nu_2}\prod_{p|f}\left(2-\frac{\psi(p)}{p^s}\right)^{\nu_2}\prod_{p|g}\left(1-\frac{\psi(p)}{p^s}\right)^{-1} \ll \tau_3(d) \ll d^\epsilon \tag{17.22}$$

for $\sigma \geq \frac{1}{2}+\epsilon$.

We next show that

$$B_{\nu_2}(s,\psi,h,eg) = I_{\nu_2}(h)\mu(h)\sum_{m\le y/h}\frac{I_{\nu_2}(m)\psi\psi_{0,egh}(m)\mu(m)P\left(\frac{\log\frac{y}{hm}}{\log y}\right)}{m^s} \ll y^{\nu_2\epsilon},$$

which is clear when $\nu_2 = 0$. In order to arrive at this, in the case of $\nu_2 = 1$, we apply Perron's formula (See [22], A.3):

Let $A(s) = \sum_{m\ge 1} a_m m^{-s}$ converge absolutely for $\sigma > 1$ and $|a_m| < C\Phi(m)$, where $C > 0$ and for $x \ge x_0$, $\Phi(x)$ is monotonically increasing. Let further

$$\sum_{m\ge 1}|a_m|m^{-\sigma} \ll (\sigma-1)^{-\alpha}$$

as $\sigma \to 1^+$ for some $\alpha > 0$. If $w = u+iv$ (u, v real) is arbitrary, $b > 0$, $T > 0$, $u+b > 1$, then

$$\sum_{m\le x} a_m m^{-w} = (2\pi i)^{-1}\int_{b-iT}^{b+iT} A(s+w)x^s s^{-1}ds + O\left(x^b T^{-1}(u+b-1)^{-\alpha}\right)$$
$$+ O\left(T^{-1}\Phi(2x)x^{1-u}\log 2x\right) + O\left(\Phi(2x)x^{-u}\right), \quad (17.23)$$

and the estimate is uniform in x, T, b and u provided that b and u are bounded. Instead of $(u+b-1)^{-\alpha}$ in the first error term, it is possible to write $\sum_{m\ge 1}|a_m|m^{-u-b}$, which is more convenient in some cases.

If we choose $a_m = \mu(m)\psi\psi_{0,egh}(m)$, $b = 1+\epsilon-\sigma$ and $w = s$, then for $x \ge 1$

$$\sum_{m\le x}\frac{\mu(m)\psi\psi_{0,egh}(m)}{m^s} = (2\pi i)^{-1}\int_{1+\epsilon-\sigma-iT}^{1+\epsilon-\sigma+iT}\frac{x^z dz}{L(s+z,\psi\psi_{0,egh})z}$$
$$+ O\left(\frac{x^{1+\epsilon-\sigma}}{T} + \frac{x^{1-\sigma}\log 2x}{T} + \frac{1}{x^\sigma}\right). \quad (17.24)$$

Consider the region determined by the line segments $[1+\epsilon-\sigma-iT, 1+\epsilon-\sigma+iT]$, $[1/2+\epsilon-\sigma\pm iT, 1+\epsilon-\sigma\pm iT]$, $[1/2+\epsilon-\sigma\pm i\epsilon/2, 1/2+\epsilon-\sigma\pm iT]$, and the semi-circle

D, $1/2+\epsilon-\sigma+\epsilon e^{i\theta}/2$, $-\pi/2 \le \theta \le \pi/2$. Under GRH the region contains only one pole, which is at $z=0$, so by the residue theorem, we obtain

$$\sum_{m\le x}\frac{\mu(m)\psi\psi_{0,egh}(m)}{m^s}=\frac{1}{L(s,\psi\psi_{0,egh})}+O\left(\frac{x^{1+\epsilon-\sigma}}{T}+\frac{x^{1-\sigma}\log 2x}{T}+\frac{1}{x^{\sigma}}\right)$$
$$-\frac{1}{2\pi i}\left(\int_{1+\epsilon-\sigma+iT}^{\frac{1}{2}+\epsilon-\sigma+iT}+\int_{\frac{1}{2}+\epsilon-\sigma+iT}^{\frac{1}{2}+\epsilon-\sigma+i\epsilon/2}+\int_D+\int_{\frac{1}{2}+\epsilon-\sigma-i\epsilon/2}^{\frac{1}{2}+\epsilon-\sigma-iT}+\int_{\frac{1}{2}+\epsilon-\sigma-iT}^{1+\epsilon-\sigma-iT}\right) \quad (17.25)$$

$$\times\frac{x^z dz}{L(s+z,\psi\psi_{0,egh})z}.$$

We then employ (2.20) and choose $T=\sqrt{x}$ so that under GRH we have

$$\sum_{m\le x}\frac{\mu(m)\psi\psi_{0,egh}(m)}{m^s}\ll x^{\epsilon}\exp\left(\frac{A\log\eta eghx}{\log\log\eta eghx}\right) \quad (17.26)$$

for $\sigma\ge 1/2+\epsilon$. Together with this result, via partial summation it follows that

$$\begin{aligned}B_1(s,\psi,h,eg)&=O(y^{\epsilon})+\frac{\mu(h)}{\log y}\int_1^{y/h}\sum_{m\le u}\frac{\psi\psi_{0,egh}(m)\mu(m)}{m^s}P'\left(\frac{\log\frac{y}{hu}}{\log y}\right)\frac{du}{u}\\&=O(y^{\epsilon})\end{aligned}$$

for $\sigma\ge 1/2+\epsilon$ and $\log\eta eg\ll\log 2y$.

Finally, together with our all findings, we conclude that if we assume the truth of GRH and have $\log\eta d\ll\log 2y$, then in the region $\sigma\ge 1/2+\epsilon$ and $|s-1|\gg\frac{1}{\log\eta\tau}$, if necessary, $L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)$ has no pole and

$$L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)\ll A^{\nu}\exp\left(\frac{A\log\eta\tau}{\log\log\eta\tau}\right)y^{\epsilon}\sum_{g|d}\frac{\nu!\,(\log\eta d\tau)^{\nu-\omega(g)}}{(\nu-\omega(g))!}\prod_{p|g}\Lambda(p). \quad (17.27)$$

Grouping the terms having the same number of distinct prime factors, then putting

$j = \omega(g)$, $0 \le j \le \omega(d)$, the divisor sum becomes

$$= \sum_{0 \le j \le \omega(d)} \binom{\nu}{j} (\log \eta d\tau)^{\nu - j} \sum_{\substack{g|d \\ \omega(g)=j}} \Lambda^{*(j)}(g).$$

Since the number of divisors of d formed by $j-$distinct primes is $\binom{\omega(d)}{j}$ and $\Lambda^{*(j)}(g) \le (\log d)^j$, the above quantity is

$$\le (\log \eta d\tau)^{\nu} \sum_{0 \le j \le \omega(d)} \binom{\nu}{j} \binom{\omega(d)}{j}.$$

Here, by Vandermonde's convolution, the $j-$sum is $\binom{v+\omega(d)}{\omega(d)}$. As a result of Stirling's formula and the inequality $(1+u)^v \le \exp(uv)$ for $u, v \ge 0$,

$$\binom{v+\omega(d)}{\omega(d)} \le \left(1 + \frac{\nu}{\omega(d)}\right)^{\omega(d)} e^{\omega(d)} \le e^{\nu + \omega(d)}.$$

Coming back to (17.27), if we keep the restrictions valid, we arrive at

$$L_{\nu_1,\nu_2}(s; d; \nu; \eta, \psi) \ll A^{\nu} \exp\left(\frac{A \log \eta\tau}{\log\log \eta\tau}\right) y^{\epsilon} \left(\log \eta d\tau\right)^{\nu}. \tag{17.28}$$

17.3. The Calculation of the Average

We are now in a position to estimate (17.2). Assume $w \ge 1$. Applying (17.23) to $\alpha(md; \nu, \nu_1, \nu_2)\psi(m)$, using Lemma 17.1, if $\log d \ll \log 2w$, we obtain

$$\sum_{m \le w} \alpha(md; \nu, \nu_1, \nu_2)\psi(m) = (2\pi i)^{-1} \int_{1+\epsilon-iT}^{1+\epsilon+iT} L_{\nu_1,\nu_2}(s; d; \nu; \eta, \psi) \frac{w^s}{s} ds$$
$$+ O\left(\frac{w^{1+\epsilon}}{T} \sum_{m \ge 1} \frac{|\alpha(md; \nu, \nu_1, \nu_2)|}{m^{1+\epsilon}} + \frac{A^{\nu} w^{1+\epsilon}}{T} (\log 2w)^{\nu} + w^{\epsilon}(A \log 2w)^{\nu}\right). \tag{17.29}$$

To estimate the sum in the error term, we again use Lemma 17.1, then expand $(\log md)^{\nu+2\nu_1}$ by the binomial theorem, and then from Lemma 16.4, it follows that

$$\begin{aligned}
\sum_{m\geq 1} \frac{|\alpha(md;\nu,\nu_1,\nu_2)|}{m^{1+\epsilon}} &\ll \sum_{m\geq 1} \frac{\tau_{2\nu_2+2}(md)(\log md)^{\nu+2\nu_1}}{m^{1+\epsilon}} \\
&\ll d^\epsilon \sum_{r\leq \nu+2\nu_1} \binom{\nu+2\nu_1}{r} (\log d)^{\nu-r} \sum_{m\geq 1} \frac{\tau_{2\nu_2+2}(m)(\log m)^r}{m^{1+\epsilon}} \\
&\ll A^\nu d^\epsilon (\log 2w)^\nu \sum_{r\leq \nu+2\nu_1} \frac{(\nu+2\nu_1)\cdots(\nu+2\nu_1-r+1)}{(\log 2w)^r} \\
&\ll A^\nu d^\epsilon (\log 2w)^\nu
\end{aligned}$$

provided that $\nu \ll \log 2w$. If we continue with these conditions, we have

$$\begin{aligned}
\sum_{m\leq w} \alpha(md;\nu,\nu_1,\nu_2)\psi(m) = (2\pi i)^{-1} \int_{1+\epsilon-iT}^{1+\epsilon+iT} L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi) \frac{w^s}{s} ds \\
+ O\left(\frac{w^{1+\epsilon}(A\log 2w)^\nu}{T} + w^\epsilon (A\log 2w)^\nu \right). \quad (17.30)
\end{aligned}$$

Under the truth of GRH, inside and on the rectangular contour Γ connecting the points $1+\epsilon+iT$, $1/2+\epsilon+iT$, $1/2+\epsilon-iT$ and $1+\epsilon-iT$, there may be at most one pole, which is at $s=1$ and of order $\nu+2\nu_1+\nu_2+1$, depending on whether $\eta=1$, so that the residue theorem implies that

$$\begin{aligned}
&\sum_{m\leq w} \alpha(md;\nu,\nu_1,\nu_2)\psi(m) = \\
&\quad \frac{[\eta=1]}{(\nu+2\nu_1+\nu_2)!} \frac{d^{\nu+2\nu_1+\nu_2}}{ds^{\nu+2\nu_1+\nu_2}} \left\{ (s-1)^{\nu+2\nu_1+\nu_2+1} L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi) \frac{w^s}{s} \right\}_{s=1} \\
&\quad - \left(\int_{1+\epsilon+iT}^{1/2+\epsilon+iT} + \int_{1/2+\epsilon+iT}^{1/2+\epsilon-iT} + \int_{1/2+\epsilon-iT}^{1+\epsilon-iT} \right) L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi) \frac{w^s ds}{s} \\
&\quad + O\left(\frac{w^{1+\epsilon}(A\log 2w)^\nu}{T} + w^\epsilon (A\log 2w)^\nu \right).
\end{aligned}$$

In addition to $\log \eta d \ll \log 2w$ and $\nu \ll \log 2w$, we impose the condition $w^\epsilon \ll y \ll w$, and choose $T = w^{1/2}$, then by (17.28), we have

$$\int_{1+\epsilon+iT}^{1/2+\epsilon+iT}, \int_{1/2+\epsilon-iT}^{1+\epsilon-iT} \cdots \ll \frac{A^\nu \exp\left(\frac{A\log\eta T}{\log\log\eta T}\right) y^\epsilon w^{1+\epsilon}(\log\eta dT)^\nu}{T}$$
$$\ll A^\nu w^{1/2+\epsilon}(\log 2w)^\nu$$

and

$$\int_{1/2+\epsilon+iT}^{1/2+\epsilon-iT} \cdots \ll A^\nu \exp\left(\frac{A\log\eta\ T}{\log\log\eta T}\right) y^\epsilon w^{1/2+\epsilon}(\log\eta dT)^\nu \int_{1/2+\epsilon+iT}^{1/2+\epsilon-iT} \frac{|ds|}{|s|}$$
$$\ll A^\nu w^{1/2+\epsilon}(\log 2w)^\nu,$$

so that

$$\sum_{m\le w}\alpha(md;\nu,\nu_1,\nu_2)\psi(m) = \tag{17.31}$$
$$\frac{[\eta=1]}{(\nu+2\nu_1+\nu_2)!}\frac{d^{\nu+2\nu_1+\nu_2}}{ds^{\nu+2\nu_1+\nu_2}}\left\{(s-1)^{\nu+2\nu_1+\nu_2+1}L_{\nu_1,\nu_2}(s;d;\nu;\eta,\psi)\frac{w^s}{s}\right\}_{s=1}$$
$$+ O\left(A^\nu w^{1/2+\epsilon}(\log 2w)^\nu\right),$$

under the assumption of GRH. By means of $w^\epsilon \ll y \ll w$, the first restriction, $\log\eta d \ll \log 2w$, implies that of (17.28).

18. SUMS INVOLVING μ–FUNCTION AND GENERALIZED Λ–FUNCTION

We begin with the sum

$$\sum_{n\le y} \frac{\mu(n)\Lambda^{*(j)}(n)F_1(n)}{n},$$

where $j \in \mathbb{N}$ and $F_1(n) = \prod_{p|n}(1 + f(p)/p)$ with $f(p) \ll 1$ for all p. We'll prove that

$$\sum_{n\le y} \frac{\mu(n)\Lambda^{*(j)}(n)F_1(n)}{n} = \frac{(-1)^j(\log y)^j}{j!} + O^*\left(\sum_{1\le k\le j} \alpha_{j,k}(\log y)^{j-k}(\log\log 3y)^k\right), \tag{18.1}$$

where $y \ge 1$ and $\alpha_{j,k}$ will be suitably chosen later. Here $*$ notation has the same meaning as in Theorem 13.1 of §13. For $j = 0$, there is nothing to prove. The case of $j = 1$ also holds by Mertens' formula. We now show that if (18.1) is true for $j \ge 1$, then it also holds for $j + 1$. Assume the validity of the case $j \ge 1$. Writing $\Lambda^{*(j+1)}$ as a product of $\Lambda^{*(j)}$ and Λ, we have

$$\sum_{n\le y} \frac{\mu(n)\Lambda^{*(j+1)}(n)F_1(n)}{n} = -\sum_{n\le y} \frac{\mu(n)\Lambda^{*(j)}(n)F_1(n)}{n} \sum_{\substack{p\le y/n \\ (p,n)=1}} \frac{F_1(p)\log p}{p}. \tag{18.2}$$

We quote Lemma 3.9 of [23]: for large square-free j,

$$\sum_{p|j} \frac{\log p}{p} = O(\log\log j),$$

from which, together with one of Mertens' theorem, it follows that

$$\sum_{\substack{p\le y/n\\(p,n)=1}} \frac{F_1(p)\log p}{p} = \sum_{p\le y/n}\frac{\log p}{p} + O\left(\sum_{p|n}\frac{\log p}{p}+1\right) \tag{18.3}$$
$$= \log\frac{y}{n} + O^*\left(A_1\log\log 3n\right)$$

for some absolute constant $A_1>0$. Inserting this into (18.2), we have

$$\sum_{n\le y}\frac{\mu(n)\Lambda^{*(j+1)}(n)F_1(n)}{n} = -\sum_{n\le y}\frac{\mu(n)\Lambda^{*(j)}(n)F_1(n)}{n}\log\frac{y}{n}$$
$$+O^*\left(A_1(\log\log 3y)\Big|\sum_{n\le y}\frac{\mu(n)\Lambda^{*(j)}(n)|F_1(n)|}{n}\Big|\right) = \mathcal{P}_1+\mathcal{P}_2,$$

say. From (18.1), it follows immediately,

$$\mathcal{P}_2 \le A_1(\log\log 3y)\frac{(\log y)^j}{j!} + A_1\sum_{1\le k\le j}\alpha_{j,k}(\log y)^{j-k}(\log\log 3y)^{k+1}.$$

By (18.1) and partial summation, we see that

$$\mathcal{P}_1 = -\int_1^y \sum_{n\le u}\frac{\mu(n)\Lambda^{*(j)}(n)F_1(n)}{n}\frac{du}{u} = \frac{(-1)^{j+1}}{j!}\int_1^y\frac{(\log u)^j du}{u}$$
$$+O^*\left(\sum_{1\le k\le j}\alpha_{j,k}(\log\log 3y)^k\int_1^y\frac{(\log u)^{j-k}du}{u}\right)$$

$$=\frac{(-\log y)^{j+1}}{(j+1)!}+O^*\left(\sum_{1\le k\le j}\frac{\alpha_{j,k}}{j-k+1}(\log\log 3y)^k(\log y)^{j-k+1}\right)$$

Combining the results of $\mathcal{P}_1$ and $\mathcal{P}_2$, it is not hard to see that

$$\sum_{n\le y}\frac{\mu(n)\Lambda^{*(j+1)}(n)F_1(n)}{n}=\frac{(-\log y)^{j+1}}{(j+1)!}+O^*\Bigg(\left(\frac{A_1}{j!}+\frac{\alpha_{j,1}}{j}\right)(\log y)^j\log\log 3y$$
$$+\sum_{k=2}^{j}\left(A_1\alpha_{j,k-1}+\frac{\alpha_{j,k}}{j-k+1}\right)(\log\log 3y)^k(\log y)^{j-k+1}+A_1\alpha_{j,j}(\log\log 3y)^{j+1}\Bigg).$$

Choosing $\alpha_{j,k}:=\dfrac{A^j}{(j-k)!}$ for $1\le k\le j$, the whole error term becomes

$$\le\sum_{1\le k\le j+1}\alpha_{j+1,k}(\log\log 3y)^k(\log y)^{j+1-k}.$$

So, by mathematical induction, we complete the proof of (18.1).

Proposition 18.1. *For $j\in\mathbb{N}$, $y\ge 1$ and F_1 as defined in the beginning of the section, we have*

$$\sum_{n\le y}\frac{\mu(n)\Lambda^{*(j)}(n)F_1(n)}{n}=\frac{(-1)^j(\log y)^j}{j!}\left(1+O\left(\frac{A^j\log\log 3y}{\log 2y}\right)\right).$$

If $j\gg\log y/\log\log 3y$ then the sum is void, because the number of distinct prime divisors of n is $\ll\log n/\log\log 3n$. In the remaining cases of j, namely $j\ll\log y/\log\log 3y$, the error term of (18.1) is

$$=\frac{A^j(\log y)^{j-1}\log\log 3y}{(j-1)!}\sum_{1\le k\le j}\frac{(j-1)!}{(j-k)!}\left(\frac{\log\log 3y}{\log y}\right)^{k-1}$$
$$=\frac{A^j(\log y)^{j-1}\log\log 3y}{(j-1)!}\sum_{1\le k\le j}\prod_{\ell=1}^{k-1}(j-1-\ell)\left(\frac{\log\log 3y}{\log y}\right)^{k-1}.$$

Since $j-1-\ell\ll\frac{\log y}{\log\log 3y}$ for each $\ell=1,\ldots,k-1$, the above summand is $\le A^{k-1}$, and so the sum is bounded by A^j. Finally, the whole error term above becomes

$$\leq \frac{A^j(\log y)^{j-1}\log\log 3y}{j!}.$$

We complete this section by the following Proposition.

Proposition 18.2. *For $k \in \mathbb{Z}^+$, $\iota = 1$ or 2, $y \geq 1$ and F_1 as defined in the previous Proposition, we have*

$$\sum_{\substack{n\leq y\\(n,k)=1}} \frac{\mu^\iota(n)\Lambda(n)F_1(n)}{n} = (-1)^\iota \log y + O\left(\log\log 3k\right),$$
$$\sum_{\substack{n\leq y\\(n,k)=1}} \frac{\mu^\iota(n)\Lambda_2(n)F_1(n)}{n} = [\iota = 2](\log y)^2 + O\left((\log\log 3yk)(\log y + \log\log 3k)\right).$$

Clearly,

$$\sum_{\substack{n\leq y\\(n,k)=1}} \frac{\mu^\iota(n)\Lambda(n)F_1(n)}{n} = (-1)^\iota \sum_{\substack{p\leq y\\(p,k)=1}} \frac{F_1(p)\log p}{p},$$

which is almost the same as (18.3), and hence we are done. In the second sum to be estimated we recall the formula $\Lambda_2(n) = \Lambda(n)\log n + \sum_{d|n}\Lambda(d)\Lambda(n/d)$ to write

$$\sum_{\substack{n\leq y\\(n,k)=1}} \frac{\mu^\iota(n)\Lambda_2(n)F_1(n)}{n} = \sum_{\substack{n\leq y\\(n,k)=1}} \frac{\mu^\iota(n)\Lambda(n)F_1(n)\log n}{n}$$
$$+ \sum_{\substack{d\leq y\\(d,k)=1}} \frac{\mu^\iota(d)\Lambda(d)F_1(d)}{d} \sum_{\substack{e\leq y/d\\(e,dk)=1}} \frac{\mu^\iota(e)\Lambda(e)F_1(e)}{e}$$

$$= (-1)^\iota \sum_{\substack{p\leq y\\(p,k)=1}} \frac{F_1(p)(\log p)^2}{p} + \sum_{\substack{p_1\leq y\\(p_1,k)=1}} \frac{F_1(p_1)\log p_1}{p_1} \sum_{\substack{p_2\leq y/p_1\\(p_2,p_1k)=1}} \frac{F_1(p_2)\log p_2}{p_2}.$$

Here p_1 and p_2 run through prime numbers. Based on (18.3), by partial summation, we obtain

$$\sum_{\substack{p\le y\\(p,k)=1}} \frac{F_1(p)(\log p)^2}{p} = \frac{(\log y)^2}{2} + O\left((\log y)(\log\log 3k)\right)$$

and

$$\sum_{\substack{p_1\le y\\(p_1,k)=1}} \frac{F_1(p_1)\log p_1}{p_1} \sum_{\substack{p_2\le y/p_1\\(p_2,p_1k)=1}} \frac{F_1(p_2)\log p_2}{p_2} = \sum_{\substack{p_1\le y\\(p_1,k)=1}} \frac{F_1(p_1)\log p_1}{p_1}\log\frac{y}{p_1} + O\left((\log\log 3yk)\sum_{\substack{p_1\le y\\(p_1,k)=1}} \frac{F_1(p_1)\log p_1}{p_1}\right).$$

$$= \frac{(\log y)^2}{2} + O\left((\log\log 3yk)(\log y + \log\log 3k)\right).$$

Putting the last three results into one form gives the second part of the proposition.

19. SOME GENERAL LEMMAS

Lemma 19.1. *Let $k, \ell \in \mathbb{N}$, then for $y \geq 1$*

$$I = \int_1^y (\log u)^\ell \left(\log \frac{y}{u}\right)^k \frac{du}{u} = B(\ell+1, k+1)(\log y)^{\ell+k+1},$$

where $B(\cdot\,,\,\cdot)$ denotes the Beta-function, which is a special function defined by

$$B(z,w) := \int_0^1 t^{z-1}(1-t)^{w-1} dt \tag{19.1}$$

for $\Re z$, $\Re w > 0$.

However, the Beta-function can be expressed in various ways. By the formula

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{19.2}$$

it can be represented in terms of the Gamma function. If z, $w \in \mathbb{Z}^+$, then

$$\Gamma(z) = (z-1)!. \tag{19.3}$$

Proof. Substituting $u = e^v$ gives that

$$I = \int_0^{\log y} v^\ell (\log y - v)^k dv.$$

It follows from the second change of variable $v = s \log y$ and the integral representation of the Beta function that

$$I = (\log y)^{k+\ell+1} \int_0^1 s^\ell (1-s)^k dv = B(\ell+1, k+1)(\log y)^{k+\ell+1}.$$

$\square$

Lemma 19.2. *Let $k, \ell \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$. Assume*

$$A(y) := \sum_{n\le y} a_n = \alpha(\log y)^m + E(y), \quad E(y) \ll f(y)(\log y)^{m-1}, \tag{19.4}$$

where $f(y)$ is positive and increasing on $(0,\infty]$. Then for $x, y \ge 1$,

$$\sum_{n\le y} a_n \left(\log \frac{y}{n}\right)^k (\log nx)^\ell = \alpha m \sum_{i=0}^{\ell} \binom{\ell}{i} (\log x)^{\ell-i} (\log y)^{i+m+k} B(i+m, k+1)$$
$$+ O\left((\ell+k+m) f(y) \sum_{i=0}^{\ell} \binom{\ell}{i} (\log x)^{\ell-i} (\log y)^{i+m+k-1} B(i+m, k+1)\right).$$

Proof. We write the sum as a Stieltjes integral:

$$\begin{aligned}
\sum_{n\le y} a_n \left(\log \frac{y}{n}\right)^k (\log nx)^\ell &= \sum_{i=0}^{\ell} \binom{\ell}{i} (\log x)^{\ell-i} \sum_{n\le y} a_n \left(\log \frac{y}{n}\right)^k (\log n)^i \\
&= \sum_{i=0}^{\ell} \binom{\ell}{i} (\log x)^{\ell-i} \int_{1^-}^{y} (\log u)^i \left(\log \frac{y}{u}\right)^k dA(u) \\
&= \sum_{i=0}^{\ell} \binom{\ell}{i} (\log x)^{\ell-i} I_{i,k}(y),
\end{aligned} \tag{19.5}$$

where $\displaystyle\int_{1^-}^{y} = \lim_{\substack{a\to 1\\ a<1}} \int_a^y$. Using (19.4) and some standart properties of Stieltjes integration, we obtain

$$\begin{aligned}
I_{i,k}(y) = \alpha(\log u)^{m+i} \left(\log \frac{y}{u}\right)^k \Big|_{1^-}^{y} + E(u)(\log u)^i \left(\log \frac{y}{u}\right)^k \Big|_{1^-}^{y} \\
- \int_{1^-}^{y} A(u) d\Big[(\log u)^i \left(\log \frac{y}{u}\right)^k\Big] = I'_{i,k}(y) + I''_{i,k}(y) - I'''_{i,k}(y),
\end{aligned}$$

say. Firstly,

$$I'_{i,k}(y) = \begin{cases} \alpha(\log y)^{m+i} & \text{if} \quad k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It immediately follows from the definition of $E(y)$ that $E(1^-) = -\alpha(\log 1^-)^m$. So

$I''_{i,k}(y) = 0$ for $k \geq 1$. If $k = 0$, then trivially

$$I''_{i,0}(y) \ll f(y)(\log y)^{i+m-1}.$$

For the last part, first assume $i, k \in \mathbb{Z}^+$. Then

$$I'''_{i,k}(y) = \alpha \int_1^y \frac{\left(i \log \frac{y}{u} - k \log u\right) (\log u)^{i+m-1} \left(\log \frac{y}{u}\right)^{k-1} du}{u} \\ + O\left(f(y) \int_1^y \frac{\left(i \log \frac{y}{u} + k \log u\right) (\log u)^{i+m-2} \left(\log \frac{y}{u}\right)^{k-1} du}{u} \right).$$

By Lemma 19.1, (19.2) and (19.3) we see that

$$I'''_{i,k}(y) = -\alpha m B(i+m, k+1)(\log y)^{i+m+k} \\ + O\left(f(y)(i+m+k)B(i+m, k+1)(\log y)^{i+m+k-1}\right). \quad (19.6)$$

Similarly, we have

$$I'''_{i,0}(y) = \frac{\alpha i (\log y)^{i+m}}{i+m} + O\left(f(y)(i+m)B(i+m, 1)(\log y)^{i+m-1}\right) \quad (19.7)$$

for $i > 0$, and

$$I'''_{0,k}(y) = -\alpha m B(m, k+1)(\log y)^{m+k} + O\left(f(y)(m+k)B(m, k+1)(\log y)^{k+m-1}\right) \quad (19.8)$$

for $k > 0$. Then it is easy to check that for $i, k \in \mathbb{N}$

$$I_{i,k}(y) = \alpha m B(i+m, k+1)(\log y)^{i+m+k} \\ + O\left(f(y)(i+m+k)B(i+m, k+1)(\log y)^{i+m+k-1}\right). \quad (19.9)$$

Combining this with (19.5), the result is apparent.

20. THE CALCULATIONS OF $\sum_{0<\Im\varrho\le T} B\zeta'(\varrho)$ AND $\sum_{0<\Im\varrho\le T} |B\zeta'(\varrho)|^2$

Consider the sums

$$\Delta_1 := \sum_{T/2<\Im\varrho\le T} B\zeta'(\varrho) \qquad \text{and} \qquad \Delta_2 := \sum_{T/2<\Im\varrho\le T} |B\zeta'(\varrho)|^2.$$

ϱ, $\Im\varrho > 0$, run through the zeros of $Z_1(s)$. The Dirichlet polynomial $B(s)$ is defined by

$$B(s) = \sum_{n\le y} \frac{b(n)}{n^s}, \tag{20.1}$$

where $b(n) = \mu(n)P\left(\frac{\log\frac{y}{n}}{\log y}\right)$. Here $P(\cdot)$ is a polynomial with real coefficients which satisfies $P(0) = 0$ and $P(1) = 1$, $y = (T/(2\pi))^\theta$, $\epsilon \le \theta \le 1$ for an arbitrarily small number $\epsilon > 0$. The implicit constants of the error terms may depend on P and this will not stated explicitly.

Assume RH. By the work of Hall, mentioned in §3, we know that all non-real zeros of $Z_1(s)$ lie on the critical line so that

$$|B\zeta'(\varrho)|^2 = B\zeta'(\varrho)B\zeta'(1-\varrho),$$

which is necessary to transform the second sum into the contour integral for an application of Cauchy's residue theorem. Before the residue theorem application and estimating the contour integrals, we note the following estimate: For $-2 \le \sigma \le 2$,

$$B(s) \ll \begin{cases} \frac{\max\{y^{1-\sigma},1\}}{|1-\sigma|} & \text{if } |\sigma-1| \ge (\log y)^{-1}, \\ \log y & \text{if } |\sigma-1| \le (\log y)^{-1}. \end{cases} \tag{20.2}$$

It follows from the residue theorem that

$$\Delta_j = \frac{1}{2\pi i}\int f_j(s)\frac{Z_1'}{Z_1}(s)ds + O\left(yT^{1/2+\epsilon}\right), \quad (j=1,2) \tag{20.3}$$

where the integral is taken over the positively oriented rectangle with vertices $1+\epsilon+iT_1$, $1+\epsilon+iT_2$, $-\epsilon+iT_2$ and $-\epsilon+iT_1$,

$$f_1(s) := B(s)\zeta'(s) \qquad \text{and} \qquad f_2(s) := B(s)\zeta'(s)B(1-s)\zeta'(1-s). \tag{20.4}$$

Here T_1 and T_2 are obtained by varying $T/2$ and T by a bounded amount, i.e.,

$$T_1 = T/2 + O(1), \qquad \text{and} \quad T_2 = T + O(1);$$

so that the condition (3.19) is satisfied. These changes on $T/2$ and T mean adding or deleting $O(\log T)$ terms each of size $\ll yT^{1/2+\epsilon}$. Hence this variation leads to the error term in (20.3).

We examine the contour integrals in (20.3) by dividing them into four parts:

$$\begin{aligned}
\Delta_j^{(1)} &= \frac{1}{2\pi i}\int_{1+\epsilon+iT_1}^{1+\epsilon+iT_2} f_j(s)\frac{Z_1'}{Z_1}(s)ds, & \Delta_j^{(2)} &= \frac{1}{2\pi i}\int_{1+\epsilon+iT_2}^{-\epsilon+iT_2} f_j(s)\frac{Z_1'}{Z_1}(s)ds,\\
\Delta_j^{(3)} &= \frac{1}{2\pi i}\int_{-\epsilon+iT_2}^{-\epsilon+iT_1} f_j(s)\frac{Z_1'}{Z_1}(s)ds, & \Delta_j^{(4)} &= \frac{1}{2\pi i}\int_{-\epsilon+iT_1}^{1+\epsilon+iT_1} f_j(s)\frac{Z_1'}{Z_1}(s)ds.
\end{aligned}$$

By the choices of T_1 and T_2, (3.20) holds on the horizontal sides of the contour. Employing (2.16), (3.20) and (20.2), we estimate the integrals along horizontal sides the contour trivially, and we have

$$\Delta_1^{(2)},\, \Delta_1^{(4)},\, \Delta_2^{(2)},\, \Delta_2^{(4)} \ll yT^{1/2+\epsilon}. \tag{20.5}$$

We next show that

$$\Delta_1^{(1)} \ll \exp\left(\frac{A(\log T)(\log\log\log T)}{\log\log T}\right) \tag{20.6}$$

as follows. Using (3.27), $\Delta_1^{(1)}$ takes the following form:

$$\begin{aligned}\Delta_1^{(1)} &= \frac{1}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell}\sum_{k\le y}\frac{b(k)}{k^{1+\epsilon}}\sum_{m,n\ge 1}\frac{\Lambda^{(\ell+1)}(m)(\log n)}{(mn)^{1+\epsilon}}\int_{T_1}^{T_2}\frac{dt}{(nmk)^{it}\left(\log\frac{t}{2\pi}\right)^{\ell}}\\ &-\frac{1}{2\pi}\sum_{\ell\le\frac{\log T}{\log\log T}} 2^{\ell+1}\sum_{k\le y}\frac{b(k)}{k^{1+\epsilon}}\sum_{m,n\ge 1}\frac{(\log n)\sum_{d|m}\Lambda_2\left(\frac{m}{d}\right)\Lambda^{*(\ell)}(d)}{(mn)^{1+\epsilon}}\\ &\times\int_{T_1}^{T_2}\frac{dt}{(nmk)^{it}\left(\log\frac{t}{2\pi}\right)^{\ell+1}} + O\left(\exp\left(\frac{A(\log T)(\log\log\log T)}{\log\log T}\right)\right).\end{aligned}$$

Applying the second mean-value theorem gives that

$$\int_{T_1}^{T_2}\frac{dt}{(nmk)^{it}\left(\log\frac{t}{2\pi}\right)^{\ell}} \ll \left(\log\frac{T_1}{2\pi}\right)^{-\ell}(\log nmk)^{-1}.$$

Together with this, since $B(1+\epsilon+it)$, $\zeta'(1+\epsilon+it)$, $\frac{\zeta'}{\zeta}(1+\epsilon+it)$, $\frac{\zeta''}{\zeta}(1+\epsilon+it) \ll 1$, we have

$$\Delta_1^{(1)} \ll \sum_{\ell\le\frac{\log T}{\log\log T}}\left(\frac{A}{\log T}\right)^{\ell} + O\left(\exp\left(\frac{A(\log T)(\log\log\log T)}{\log\log T}\right)\right).$$

The sum can be trivially absorbed in the error term, so this completes the case.

We are now come to the integrals $\Delta_1^{(3)}$, $\Delta_2^{(1)}$ and $\Delta_2^{(3)}$ that constitute the main terms of Δ_1 and Δ_2. Contrary to $\Delta_1^{(1)}$, $\Delta_2^{(1)}$ produces a part of the main term of the asymptotic formula for Δ_2. By the change of variable $s \to 1-s$ and then taking the complex conjugates of the integrals we have

$$\overline{\Delta_j^{(3)}} = -\frac{1}{2\pi i}\int_{1+\epsilon+iT_1}^{1+\epsilon+iT_2} f_j(1-s)\frac{Z_1'}{Z_1}(1-s)ds \tag{20.7}$$

for $j = 1, 2$. Using (2.8) and (3.3) in (20.7) gives that

$$\overline{\Delta_1^{(3)}} = \frac{1}{2\pi i} \int_{1+\epsilon+iT_1}^{1+\epsilon+iT_2} B(1-s)\zeta'(1-s) \left(\frac{Z_1'}{Z_1}(s) + \log \frac{t}{2\pi} \right) ds \qquad (20.8)$$
$$+ O\left(\left| \int_{1+\epsilon+iT_1}^{1+\epsilon+iT_2} B(1-s)\zeta'(1-s)\frac{ds}{t} \right| \right),$$

and

$$\overline{\Delta_2^{(3)}} = \Delta_2^{(1)} + \frac{1}{2\pi i} \int_{1+\epsilon+iT_1}^{1+\epsilon+iT_2} B(1-s)\zeta'(1-s)B(s)\zeta'(s) \log \frac{t}{2\pi} ds$$
$$+ O\left(\left| \int_{1+\epsilon+iT_1}^{1+\epsilon+iT_2} B(1-s)\zeta'(1-s)B(s)\zeta'(s)\frac{ds}{t} \right| \right), \qquad (20.9)$$

since $f_2(s) = f_2(1-s)$. In the above integrals replacing $\zeta'(1-s)$ by the right-hand side of the formula

$$\zeta'(1-s) = -\chi(1-s) \left(\zeta(s) \log \frac{|t|}{2\pi} + \zeta'(s) \right) + O\left(\frac{|\zeta(1-s)|}{|t|} \right) \qquad (|t| \geq t_0)\,, \qquad (20.10)$$

which can be easily derived from the first derivative of the functional equation (2.1) of $\zeta(s)$ and (2.8), the integrals in (20.8) and (20.9) take a form which is more appropriate for an application of Lemma 16.5. The error terms in (20.8) and (20.9), and the error term stemming from the change in the range of integrals from $[1+\epsilon+iT_1, 1+\epsilon+iT_2]$ to $[1+\epsilon+iT/2, 1+\epsilon+iT]$ are dominated by $O\left(yT^{1/2+\epsilon}\right)$, which follows trivially from (2.16), (3.11) and (20.2). For the last estimate ϵ should be chosen so that the $1+\epsilon$ line does not pass through any real zero or pole of Z_1. As a summary of all our findings we write

$$\begin{aligned}
\overline{\Delta_1} = &- \frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)\zeta(s)\frac{Z_1'}{Z_1}(s)\log\frac{t}{2\pi}ds \\
&- \frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)\zeta'(s)\frac{Z_1'}{Z_1}(s)ds \\
&- \frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)\zeta(s)\left(\log\frac{t}{2\pi}\right)^2 ds \\
&- \frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)\zeta'(s)\log\frac{t}{2\pi}ds \\
&+ O\left(\int_{\frac{T}{2}}^{T} |B(-\epsilon-it)\zeta(-\epsilon-it)|\left(\left|\frac{Z_1'}{Z_1}(1+\epsilon+it)\right| + \log\frac{t}{2\pi}\right)\frac{dt}{t}\right) \\
&+ O\left(yT^{1/2+\epsilon}\right)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Delta_2} = &- 2\Re\left\{\frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)B(s)\zeta'(s)\zeta(s)\frac{Z_1'}{Z_1}(s)\log\frac{t}{2\pi}ds\right\} \\
&- 2\Re\left\{\frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)B(s)\left(\zeta'(s)\right)^2\frac{Z_1'}{Z_1}(s)ds\right\} \\
&- \frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)B(s)\zeta'(s)\zeta(s)\left(\log\frac{t}{2\pi}\right)^2 ds \\
&- \frac{1}{2\pi i}\int_{1+\epsilon+i\frac{T}{2}}^{1+\epsilon+iT} \chi(1-s)B(1-s)B(s)\left(\zeta'(s)\right)^2\log\frac{t}{2\pi}ds \\
&+ O\left(\int_{\frac{T}{2}}^{T} |B\zeta(-\epsilon-it)B\zeta'(1+\epsilon+it)|\left(\left|\frac{Z_1'}{Z_1}(1+\epsilon+it)\right| + \log\frac{t}{2\pi}\right)\frac{dt}{t}\right) \\
&+ O\left(yT^{1/2+\epsilon}\right)
\end{aligned}$$

Using (2.7), (2.16), (3.11), (3.27) and (20.2), the integrals in $O-$terms above and the error term occuring when we replace Z_1'/Z_1 by its Dirichlet series approximation in

(3.27) are bounded by $O\left(yT^{1/2+\epsilon}\right)$. So

$$\overline{\Delta_1} = \sum_{\nu \le \frac{\log T}{\log\log T}} 2^{\nu}\left(F_{0,0}(\nu+1) - 2F_{1,0}(\nu) - F_{0,0}(\nu+2) + 2F_{1,0}(\nu+1)\right) \tag{20.11}$$
$$- F_{0,0}(0) + F_{0,0}(1) + O\left(yT^{1/2+\epsilon}\right)$$

and

$$\overline{\Delta_2} = F_{0,1}(1) - F_{0,1}(2) + O\left(yT^{1/2+\epsilon}\right) \tag{20.12}$$
$$- \Re\left\{\sum_{\nu \le \frac{\log T}{\log\log T}} 2^{\nu+2}\left(\frac{F_{0,1}(\nu+2)}{2} - F_{1,1}(\nu+1) - \frac{F_{0,1}(\nu+3)}{2} + F_{1,1}(\nu+2)\right)\right\}$$

where

$$F_{\nu_1,\nu_2}(\nu) := \frac{1}{2\pi i}\int_{1+\epsilon+iT/2}^{1+\epsilon+iT} \chi(1-s)B(1-s)L_{\nu_1,\nu_2}(s;\nu)\left(\log\frac{t}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2} ds, \tag{20.13}$$

and $\nu_1, \nu_2 = 0$ or 1, $\nu \le \dfrac{\log T}{\log\log T} + 3$.

Now our calculations are reduced to one general form. From Lemma 17.1 we see that an application of Lemma 16.5 causes an error term bounded by $O\left(yT^{1/2+\epsilon}\right)$. So we have

$$F_{\nu_1,\nu_2}(\nu) = \sum_{n_1 \le y} \frac{b(n_1)}{n_1} \sum_{\frac{n_1 T}{4\pi} < m \le \frac{n_1 T}{2\pi}} \alpha(m;\nu,\nu_1,\nu_2)e\left(-\frac{m}{n_1}\right)\left(\log\frac{m}{n_1}\right)^{-\nu-2\nu_1+\nu_2+2} \tag{20.14}$$
$$+O\left(yT^{1/2+\epsilon}\right),$$

For $Y \in (X/2, X]$ and $|r| = o(\log X)$,

$$(\log Y)^r = (\log X)^r \left(1 + O\left(\frac{r}{\log X}\right)\right),$$

which can be easily deduced from the mean-value theorem. Employing this we remove the above logarithm factor from the inner-most sum over m, so that

$$F_{\nu_1,\nu_2}(\nu) = \left(\log \frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2} \left(1 + O\left(\frac{\nu+1}{\log T}\right)\right) \sum_{n_1 \le y} \frac{b(n_1)}{n_1}$$
$$\sum_{\frac{n_1 T}{4\pi} < m \le \frac{n_1 T}{2\pi}} \alpha(m; \nu, \nu_1, \nu_2) e\left(-\frac{m}{n_1}\right) + O\left(yT^{1/2+\epsilon}\right). \quad (20.15)$$

To convert the additive character $e(\cdot)$ above into a character sum we use the formula (5.11) in [2]:

$$e\left(-\frac{m}{k}\right) = \sum_{q|k} \sum_{\psi}{}^{*} \tau\left(\overline{\psi}\right) \sum_{\substack{d|m \\ d|k}} \psi\left(\frac{m}{d}\right) \delta(q, k, d, \psi),$$

where the $*$ indicates that the sum is over all primitive characters mod q. (The character mod 1 which induces all other principal characters will be included as a primitive character.) Here for a character ψ mod q,

$$\delta(q, k, d, \psi) = \sum_{\substack{e|d \\ e|k/q}} \frac{\mu(d/e)}{\phi(k/e)} \overline{\psi}\left(\frac{-k}{eq}\right) \psi\left(\frac{d}{e}\right) \mu\left(\frac{k}{eq}\right), \quad (20.16)$$

which is (5.12) of [2]. Together with this, some simple changes of variables give that

$$F_{\nu_1,\nu_2}(\nu) = \left(\log \frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2} \left(1 + O\left(\frac{\nu+1}{\log T}\right)\right) \sum_{n_1 \le y} \frac{1}{n_1} \sum_{\eta \le y/n_1} \frac{b(\eta n_1)}{\eta}$$
$$\sum_{\psi \bmod \eta}{}^{*} \tau\left(\overline{\psi}\right) \sum_{d|n_1\eta} \delta(\eta, n_1\eta, d, \psi) \sum_{\frac{\eta n_1 T}{4\pi d} < m \le \frac{\eta n_1 T}{2\pi d}} \alpha\left(dm; \nu, \nu_1, \nu_2\right) \psi(m) + O\left(yT^{\frac{1}{2}+\epsilon}\right).$$

While the part comprised of the terms with $\eta = 1$ will give the main term, the rest will be a part of the error term, as in [2]. However, we differ from [24] and [2] in estimating the contribution of the terms with $\eta > 1$. Unfortunately we are able to adapt neither the techniques, such as large sieve inequalities, higher moments of Dirichlet-L functions, in various proofs of Bombieri-Vinogradov theorem, which can be found in [2], [25] and [26], nor the powerful method seen in [24] to deal with the terms $\eta \geq (\log T)^A$. Without seperating the terms with $\eta > 1$ into two parts, by assuming the much stronger assumption GRH, we handle these terms constituting the error term.

20.1. Error-term calculations

In $F_{\nu_1,\nu_2}(\nu)$, since $d|n_1\eta$, $y \gg T^\epsilon$ and $n_1\eta \leq y \leq T$, we have $\frac{n_1\eta T}{2\pi d} \gg T$ and so $\log \eta d \ll \log \frac{n_1\eta T}{2\pi d}$, $\nu \leq \frac{\log T}{\log\log T} \ll \log \frac{n_1\eta T}{2\pi d}$ and $\left(\frac{n_1\eta T}{2\pi d}\right)^\epsilon \ll y \ll \frac{n_1\eta T}{2\pi d}$, so that we can apply (17.31) to the inner-most sum to get

$$F_{\nu_1,\nu_2}(\nu) = \frac{\left(\log \frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}}{(\nu+2\nu_1+\nu_2)!}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right)\sum_{n_1\leq y}\frac{b(n_1)}{n_1}\sum_{d|n_1}\delta(1,n_1,d,\psi_{0,1})$$
$$\times \frac{d^{\nu+2\nu_1+\nu_2}}{ds^{\nu+2\nu_1+\nu_2}}\left\{(s-1)^{\nu+2\nu_1+\nu_2+1}L_{\nu_1,\nu_2}(s;d;\nu;1,\psi_{0,1})\frac{\left(\frac{n_1T}{2\pi d}\right)^s(1-2^{-s})}{s}\right\}_{s=1}$$
$$+\,O\Big(A^\nu T^{\frac{1}{2}+\epsilon}\sum_{n_1\eta\leq y}\frac{|\mu(\eta n_1)|}{(n_1\eta)^{\frac{1}{2}}}\sum_{\psi \bmod \eta}^{*}|\tau\left(\overline{\psi}\right)|\sum_{d|n_1\eta}\frac{|\delta(\eta,n_1\eta,d,\psi)|}{d^{\frac{1}{2}}}+yT^{\frac{1}{2}+\epsilon}\Big). \tag{20.17}$$

Since $n_1\eta$ must be square-free, otherwise $\mu(\eta n_1) = 0$, by (5.13) of [2],

$$|\delta(\eta,n_1\eta,d,\psi)| \leq \frac{(d,n_1)}{\phi(n_1\eta)}.$$

Also using the facts that $|\tau\left(\overline{\psi}\right)| = \sqrt{\eta}$ and that the number of primitive characters modulo η is $\le \phi(\eta)$, the error term becomes

$$\ll A^\nu T^{1/2+\epsilon} \sum_{n_1\eta\le y} \frac{|\mu(\eta n_1)|\phi(\eta)}{n_1^{1/2}\phi(\eta n_1)} \sum_{d|n_1\eta} \frac{(d,n_1)}{d^{1/2}} + yT^{1/2+\epsilon}. \tag{20.18}$$

Since $(\eta, n_1) = 1$, due to the factor $|\mu(\eta n_1)|$, we can write any divisor of ηn_1 as a product of a divisor of n_1 and a divisor of η uniquely so that the innermost sum is

$$= \sum_{e|n_1} e^{1/2} \sum_{f|\eta} \frac{1}{f^{1/2}} \ll n_1^{1/2+\epsilon}\eta^\epsilon.$$

Then the bound in (20.18) is $\ll A^\nu T^{1-\epsilon}$, provided that $y = (T/(2\pi))^\theta \le T^{1/2-\epsilon}$.

20.2. Main-term Calculations

Continuing from (20.17), by the last result of the preceding subsection and (20.16), we have

$$\begin{aligned} F_{\nu_1,\nu_2}(\nu) = {} & \frac{\left(\log\frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}}{(\nu+2\nu_1+\nu_2)!} \left(1 + O\left(\frac{\nu+1}{\log T}\right)\right) \sum_{n_1\le y} \frac{b(n_1)}{n_1} \sum_{\substack{d|n_1\\ e|d}} \frac{\mu\left(\frac{d}{e}\right)\mu\left(\frac{n_1}{e}\right)}{\phi\left(\frac{n_1}{e}\right)} \\ & \times \frac{d^{\nu+2\nu_1+\nu_2}}{ds^{\nu+2\nu_1+\nu_2}} \left\{ \frac{(s-1)^{\nu+2\nu_1+\nu_2+1} L_{\nu_1,\nu_2}(s;d;\nu;1,\psi_{0,1}) \left(\frac{n_1 T}{2\pi d}\right)^s \left(1-\frac{1}{2^s}\right)}{s} \right\}_{s=1} \\ & + O\left(A^\nu T^{1-\epsilon}\right). \end{aligned} \tag{20.19}$$

We first substitute $dn_1' = n_1$ and then relabel n_1' by n_1 so that

$$F_{\nu_1,\nu_2}(\nu) = \frac{\left(\log \frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right)}{(\nu+2\nu_1+\nu_2)!} \sum_{n_1 \le y} \frac{\mu^2(n_1)}{n_1\phi(n_1)} \sum_{\substack{d \le \frac{y}{n_1} \\ (d,n_1)=1}} \frac{\mu(d)F_1(d)}{d}$$

$$P\left(\frac{\log \frac{y}{dn_1}}{\log y}\right) \frac{d^{\nu+2\nu_1+\nu_2}}{ds^{\nu+2\nu_1+\nu_2}} \left\{ \frac{(s-1)^{\nu+2\nu_1+\nu_2+1} L_{\nu_1,\nu_2}(s;d;\nu;1,\psi_{0,1}) \left(\frac{n_1 T}{2\pi}\right)^s}{s\left(1-\frac{1}{2^s}\right)^{-1}} \right\}_{s=1}$$

$$+ O\Big(A^\nu T^{1-\epsilon}\Big), \tag{20.20}$$

where

$$F_1(d) = \prod_{p|d}(1+f_1(p)) \qquad \text{and} \qquad f_1(p) = 1/(p-1). \tag{20.21}$$

In view of (17.5) and (17.15), we make the change of variable $d = efgh$ and re-organize (20.20) as follows.

$$F_{\nu_1,\nu_2}(\nu) = \frac{\left(\log \frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right)}{(\nu+2\nu_1+\nu_2)!} \sum_{\mathfrak{a}=0}^{2} \binom{2}{\mathfrak{a}} \sum_{g \le y} \frac{\mu(g)F_1(g)}{g} \prod_{p|g} \Lambda(p)$$

$$\sum_{\substack{e \le \frac{y}{g} \\ (e,g)=1}} \frac{\Lambda_{2-\mathfrak{a}}(e) I_{\nu_1}(e)\mu(e)F_1(e)}{e} \sum_{\substack{n_1 \le \frac{y}{eg} \\ (n_1,eg)=1}} \frac{\mu^2(n_1)}{n_1\phi(n_1)} \frac{d^{\nu+2\nu_1+\nu_2}}{ds^{\nu+2\nu_1+\nu_2}} \Bigg\{ G_{\nu_1,\nu_2}(s;e,g;\nu;\mathfrak{a})$$

$$\left(\frac{n_1 T}{2\pi}\right)^s \sum_{\substack{f \le \frac{y}{n_1 eg} \\ (f,n_1 eg)=1}} \frac{\mu(f)F_1(f)}{f} \prod_{p|f}\left(2-\frac{1}{p^s}\right)^{\nu_2} \sum_{\substack{h \le \frac{y}{fn_1 eg} \\ (h,fn_1 eg)=1}} \frac{\mu^2(h) I_{\nu_2}(h) F_1(h)}{h}$$

$$\prod_{p|h}\left(1-\frac{1}{p^s}\right)^{\nu_2+1} P\left(\frac{\log \frac{y}{hfn_1 eg}}{\log y}\right) \sum_{\substack{n_2 \le \frac{y}{h} \\ (n_2,heg)=1}} \frac{\mu(n_2) I_{\nu_2}(n_2)}{n_2^s} P\left(\frac{\log \frac{y}{n_2 h}}{\log y}\right) \Bigg\}_{s=1}$$

$$+ O\Big(A^\nu T^{1-\epsilon}\Big), \tag{20.22}$$

where

$$G_{\nu_1,\nu_2}(s;e,g;\nu;\mathfrak{a}) = (s-1)^{\nu+2\nu_1+\nu_2+1}\frac{\zeta^{\nu_2+1}(s)}{s}\left((-1)^{\mathfrak{a}}\frac{\zeta^{(\mathfrak{a})}}{\zeta}(s)\right)^{\nu_1}\left(1-\frac{1}{2^s}\right) \tag{20.23}$$
$$\prod_{p|eg}\left(1-\frac{1}{p^s}\right)^{\nu_2}\sum_{d_1\dots d_\nu=g}{}' \prod_{\substack{1\le j\le\nu\\ d_j=1}}\left(-\frac{L'}{L}(s,\psi_{0,ed_1\dots d_{j-1}})\right).$$

Let $P(x)=a_1x+\cdots+a_kx^k$, where $a_1,\dots,a_k\in\mathbb{R}$ and $a_1+\cdots+a_k=P(1)=1$. We evaluate the above derivative by the generalized Leibniz rule. Then $F_{\nu_1,\nu_2}(\nu)$ becomes

$$F_{\nu_1,\nu_2}(\nu) = \frac{\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}}{(\nu+2\nu_1+\nu_2)!}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right)\sum_{i_1,i_2=1}^{k}\frac{a_{i_1}a_{i_2}}{(\log y)^{i_1+i_2}}\sum_{\mathfrak{a}=0}^{2}\binom{2}{\mathfrak{a}} \tag{20.24}$$
$$\sum_{\substack{j_1+j_2+j_3+j_4+j_5=\nu+2\nu_1+\nu_2\\ j_1,j_2,j_3,j_4,j_5\in\mathbb{N}}}\binom{\nu+2\nu_1+\nu_2}{j_1,j_2,j_3,j_4,j_5}\sum_{g\le y}\frac{\mu(g)F_1(g)}{g}\prod_{p|g}\Lambda(p)$$
$$\sum_{\substack{e\le\frac{y}{g}\\ (e,g)=1}}\frac{\Lambda_{2-\mathfrak{a}}(e)I_{\nu_1}(e)\mu(e)F_1(e)}{e}\sum_{\substack{n_1\le\frac{y}{eg}\\ (n_1,eg)=1}}\frac{\mu^2(n_1)F_1(n_1)}{n_1}\left(\log\frac{n_1T}{2\pi}\right)^{j_2}$$
$$G^{(j_1)}_{\nu_1,\nu_2}(1;e,g;\nu;\mathfrak{a})\sum_{\substack{f\le\frac{y}{n_1eg}\\ (f,n_1eg)=1}}\frac{\mu(f)F_1(f)}{f}\frac{d^{j_3}}{ds^{j_3}}\left\{\prod_{p|f}\left(2-\frac{1}{p^s}\right)^{\nu_2}\right\}_{s=1}$$
$$\sum_{\substack{h\le\frac{y}{fn_1eg}\\ (h,fn_1eg)=1}}\frac{\mu^2(h)I_{\nu_2}(h)F_1(h)}{h}\frac{d^{j_4}}{ds^{j_4}}\left\{\prod_{p|h}\left(1-\frac{1}{p^s}\right)^{\nu_2+1}\right\}_{s=1}\left(\log\frac{y}{hfn_1eg}\right)^{i_1}$$
$$\sum_{\substack{n_2\le\frac{y}{h}\\ (n_2,heg)=1}}\frac{\mu(n_2)I_{\nu_2}(n_2)(-\log n_2)^{j_5}}{n_2}\left(\log\frac{y}{n_2h}\right)^{i_2}+O\Big(A^\nu T^{1-\epsilon}\Big).$$

Considering (2.13), (2.15) and (17.19), we see the order of the zero of $G_{\nu_1,\nu_2}(s;e,g;\nu;\mathfrak{a})$ at $s=1$ is $(2-\mathfrak{a})\nu_1+\omega(g)$, and so

$$G^{(j_1)}_{\nu_1,\nu_2}(1;e,g;\nu;\mathfrak{a}) = 0 \tag{20.25}$$

for $j_1 < (2-\mathfrak{a})\nu_1 + \omega(g)$; moreover,

$$G_{\nu_1,\nu_2}^{((2-\mathfrak{a})\nu_1+\omega(g))}(1;e,g;\nu;\mathfrak{a}) = ((2-\mathfrak{a})\nu_1+\omega(g))!\frac{(\mathfrak{a}!)^{\nu_1}}{2} \prod_{p|eg}\left(1-\frac{1}{p}\right)^{\nu_2}\binom{\nu}{\omega(g)}\omega(g)!. \quad (20.26)$$

We pay attention to the case $\nu = 0$. Due to the binomial coefficient $\binom{\nu}{\omega(g)}$, $\omega(g)$ must be 0, otherwise $\binom{\nu}{\omega(g)} = 0$, so, in the case $\nu = 0$ the only non-zero contribution can come from $g = 1$, which reminds us of the notice coming after (17.6).

In the case of $j_1 \geq (2-\mathfrak{a})\nu_1 + \omega(g)$ we estimate the j_1-th derivative of G_{ν_1,ν_2} at $s = 1$ by means of Cauchy's integral formula. To estimate G_{ν_1,ν_2} around $\asymp (\log\log y)^{-1}$ neighborhood of $s = 1$ we deal with the product over prime divisors appearing in (20.23) in a general setting. We'll prove that for any square-free positive integer Q,

$$(\log\log 3Q)^{-A} \ll \prod_{p|Q}\left(1-\mathfrak{r}p^{-s}\right) \ll (\log\log 3Q)^{A} \quad (20.27)$$

uniformly for $|\sigma - 1| \ll (\log\log 3Q)^{-1}$ and $|\mathfrak{r}| \ll 1$. For the first inequality, we need the assumption that

$$\prod_{p\ll 1}\left(1+\frac{\mathfrak{r}}{p^s-\mathfrak{r}}\right) \ll 1 \text{ for } |\sigma-1| \ll (\log\log 3Q)^{-1}.$$

We first deal with the upper bound. If $Q \ll 1$, the product over primes $p|Q$ is finite. Assume $Q \gg 1$. Observe that

$$\prod_{p|Q}\left(1-\mathfrak{r}p^{-s}\right) \ll \exp\left(\sum_{p|Q}\log\left(1+|\mathfrak{r}|p^{-\Re s}\right)\right) \ll \exp\left(|\mathfrak{r}|\sum_{p|Q}p^{-\Re s}\right)$$
$$\ll \exp\left(A\sum_{1\ll p\leq(\log 2Q)^2}p^{-1} + \frac{A}{\log 2Q}\sum_{p|Q,\, p\geq(\log 2Q)^2}1\right).$$

From Mertens' theorem and the fact that the number of distinct prime divisors of Q

is trivially $\ll \log 2Q$, we reach the one side of (20.27). The lower bound side reduces to the already proven part by seeing

$$\prod_{p|Q}\left(1-\mathfrak{r}p^{-s}\right)^{-1}=\prod_{p|Q}\left(1+\frac{\mathfrak{r}}{p^s-\mathfrak{r}}\right)\ll\prod_{p|Q,\,p\gg 1}\left(1+\frac{O(1)}{p^\sigma}\right).$$

Similarly, we obtain for any square-free positive integer Q

$$\sum_{p|Q}\frac{\log p}{p^s}\ll\log\log Q \tag{20.28}$$

uniformly for $|\sigma-1|\ll(\log\log 3Q)^{-1}$. When $s=1$, the result reduces to Lemma 3.9 in [23]. In addition to the last two estimates we use (2.13), (2.15) and (17.19) to get

$$G_{\nu_1,\nu_2}(s;e,g;\nu;\mathfrak{a})\ll A^\nu\binom{\nu}{\omega(g)}\omega(g)!(\log\log y)^{A-\omega(g)},\quad |s-1|\ll(\log\log y)^{-1},$$

and then

$$\begin{aligned}G^{(j_1)}_{\nu_1,\nu_2}(1;e,g;\nu;\mathfrak{a})&=\frac{j_1!}{2\pi i}\int_{|s-1|\asymp(\log\log y)^{-1}}\frac{G_{\nu_1,\nu_2}(s;e,g;\nu;\mathfrak{a})ds}{(s-1)^{j_1+1}}\\&\ll A^\nu j_1!\binom{\nu}{\omega(g)}\omega(g)!(\log\log y)^{j_1-\omega(g)+A}\end{aligned} \tag{20.29}$$

for $j_1\geq(2-\mathfrak{a})\nu_1+\omega(g)$.

We now have some arrangements on the $g-$sum in (20.24). As indicated above, for having a nonzero value of the derivatives of G_{ν_1,ν_2}, $\omega(g)\leq j_1-(2-\mathfrak{a})\nu_1$. We split the sum into pieces by collecting the terms according to number of their distinct prime divisors. Also, with the formula that for square-free $g\in\mathbb{Z}^+$

$$\prod_{p|g}\Lambda(p)=\frac{\Lambda^{*(\omega(g))}(g)}{\omega(g)!},$$

we have

$$F_{\nu_1,\nu_2}(\nu) = \frac{\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}}{(\nu+2\nu_1+\nu_2)!}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right)\sum_{i_1,i_2=1}^{k}\frac{a_{i_1}a_{i_2}}{(\log y)^{i_1+i_2}}\sum_{\mathfrak{a}=0}^{2}\binom{2}{\mathfrak{a}} \tag{20.30}$$

$$\sum_{\substack{j_1+j_2+j_3+j_4+j_5=\nu+2\nu_1+\nu_2\\ j_1,j_2,j_3,j_4,j_5\in\mathbb{N}}}\binom{\nu+2\nu_1+\nu_2}{j_1,j_2,j_3,j_4,j_5}\sum_{0\le\ell\le j_1-(2-\mathfrak{a})\nu_1}\frac{1}{\ell!}\sum_{g\le y}\frac{\mu(g)F_1(g)\Lambda^{*(\ell)}(g)}{g}$$

$$\sum_{\substack{e\le\frac{y}{g}\\(e,g)=1}}\frac{\Lambda_{2-\mathfrak{a}}(e)I_{\nu_1}(e)\mu(e)F_1(e)}{e}\sum_{\substack{n_1\le\frac{y}{eg}\\(n_1,eg)=1}}\frac{\mu^2(n_1)F_1(n_1)}{n_1}\left(\log\frac{n_1T}{2\pi}\right)^{j_2}$$

$$G^{(j_1)}_{\nu_1,\nu_2}(1;e,g;\nu;\mathfrak{a})\sum_{\substack{f\le\frac{y}{n_1eg}\\(f,n_1eg)=1}}\frac{\mu(f)F_1(f)}{f}\frac{d^{j_3}}{ds^{j_3}}\left\{\prod_{p|f}\left(2-\frac{1}{p^s}\right)^{\nu_2}\right\}_{s=1}$$

$$\sum_{\substack{h\le\frac{y}{fn_1eg}\\(h,fn_1eg)=1}}\frac{\mu^2(h)I_{\nu_2}(h)F_1(h)}{h}\frac{d^{j_4}}{ds^{j_4}}\left\{\prod_{p|h}\left(1-\frac{1}{p^s}\right)^{\nu_2+1}\right\}_{s=1}\left(\log\frac{y}{hfn_1eg}\right)^{i_1}$$

$$\sum_{\substack{n_2\le\frac{y}{h}\\(n_2,heg)=1}}\frac{\mu(n_2)I_{\nu_2}(n_2)(-\log n_2)^{j_5}}{n_2}\left(\log\frac{y}{n_2h}\right)^{i_2}+O\Big(A^{\nu}T^{1-\epsilon}\Big).$$

To avoid some possible difficulties arising when $\nu\gg\log\log T$, we separately handle this part in the estimation of $F_{\nu_1,\nu_2}(\nu)$. By (20.27) and plain Cauchy integral formula applications, we see that

$$\frac{d^{j_3}}{ds^{j_3}}\left\{\prod_{p|f}\left(2-\frac{1}{p^s}\right)^{\nu_2}\right\}_{s=1}\ll j_3!\tau_{1+\nu_2}(f)(\log\log y)^{\nu_2(j_3+A)},$$
$$\frac{d^{j_4}}{ds^{j_4}}\left\{\prod_{p|h}\left(1-\frac{1}{p^s}\right)^{\nu_2+1}\right\}_{s=1}\ll j_4!(\log\log y)^{j_4+A}.$$

Together with these, we employ the fact that $\Lambda_{2-\mathfrak{a}}(e)\ll(\log e)^{2-\mathfrak{a}}$, (20.27) and (20.29) to get

$$F_{\nu_1,\nu_2}(\nu) \ll A^\nu T(\log T)^{-\nu-2\nu_1+\nu_2+2} \sum_{\mathfrak{a}=0}^{2} \sum_{\substack{j_1+\cdots+j_5=\nu+2\nu_1+\nu_2 \\ j_1,j_2,j_3,j_4,j_5\in\mathbb{N}}} \frac{(\log T)^{j_2+j_5+2-\mathfrak{a}}}{j_2!j_5!}$$
$$\sum_{0\le\ell\le j_1-(2-\mathfrak{a})\nu_1} \binom{\nu}{\ell} (\log\log y)^{j_1-\ell+j_3+j_4+A} \sum_{g\le y} \frac{|\mu(g)|\Lambda^{*(\ell)}(g)}{g} \sum_{e\le\frac{y}{g}} \frac{1}{e} \sum_{n_1\le\frac{y}{eg}} \frac{1}{n_1}$$
$$\sum_{f\le\frac{y}{n_1eg}} \frac{\tau_{1+\nu_2}(f)}{f} \sum_{h\le\frac{y}{fn_1eg}} \frac{1}{h} \sum_{n_2\le\frac{y}{h}} \frac{1}{n_2} + O\Big(A^\nu T^{1-\epsilon}\Big).$$

The e, n_1, h, n_2-sums are all bounded by $O(\log y)$, while in the case of the $f-$sum, by the well-known result

$$\sum_{f\le y} \tau_2(f) = y\log y + O(y),$$

we have the bound $O\left((\log y)^{\nu_2+1}\right)$. Applying Proposition 18.1, the $g-$sum is

$$\ll \frac{A^\ell(\log y)^\ell}{\ell!}.$$

From all these results, it follows that

$$F_{\nu_1,\nu_2}(\nu) \ll A^\nu T(\log T)^{-\nu+A} \sum_{\substack{j_1+j_2+j_3+j_4+j_5=\nu+2\nu_1+\nu_2 \\ j_1,j_2,j_3,j_4,j_5\in\mathbb{N}}} \frac{(\log T)^{j_2+j_5}}{j_2!j_5!}$$
$$\sum_{0\le\ell\le j_1} \binom{\nu}{\ell} \frac{(\log y)^\ell(\log\log y)^{j_1-\ell+j_3+j_4+A}}{\ell!} + O\Big(A^\nu T^{1-\epsilon}\Big).$$

Observe that

$$\sum_{0\le\ell\le j_1} \binom{\nu}{\ell} \frac{(\log y)^\ell(\log\log y)^{j_1-\ell}}{\ell!}$$
$$= \frac{(\log y)^{j_1}}{j_1!} \sum_{0\le\ell\le j_1} \binom{\nu}{\ell} \left(\frac{\log\log y}{\log y}\right)^{j_1-\ell} (\ell+1)_{j_1-\ell} \ll \frac{A^\nu(\log T)^{j_1}}{j_1!} \quad (20.31)$$

since $j_1 \ll \log T/\log\log T$, whence

$$F_{\nu_1,\nu_2}(\nu) \ll A^\nu T(\log T)^{-\nu+A} \sum_{\substack{j_1+j_2+j_3+j_4+j_5=\nu+2\nu_1+\nu_2 \\ j_1,j_2,j_3,j_4,j_5\in\mathbb{N}}} \frac{(\log T)^{j_1+j_2+j_5}(\log\log y)^{j_3+j_4+A}}{j_1!j_2!j_5!} + O\Big(A^\nu T^{1-\epsilon}\Big)$$

$$\ll \frac{A^\nu T((\log T)(\log\log T))^A}{(\nu+2\nu_1+\nu_2)!} \sum_{\substack{j_1+j_2+j_3+j_4+j_5=\nu+2\nu_1+\nu_2 \\ j_1,j_2,j_3,j_4,j_5\in\mathbb{N}}} \binom{\nu+2\nu_1+\nu_2}{j_1,j_2,j_3,j_4,j_5} j_3!j_4! \times \left(\frac{\log\log T}{\log T}\right)^{j_3+j_4} + O\Big(A^\nu T^{1-\epsilon}\Big)$$

$$\ll \frac{A^\nu T((\log T)(\log\log T))^A}{(\nu+2\nu_1+\nu_2)!} \tag{20.32}$$

since $j_3, j_4 \ll \log T/\log\log T$, which signifies that $F_{\nu_1,\nu_2}(\nu)$ with $\nu \gg \log\log T$ does not contribute to the main term of Δ_1 and Δ_2 as will be explicitly seen later.

Assume henceforth $\nu \ll \log\log T$ and we continue from (20.30). We call $S_{\nu_2}(x)$ the finite sequence formed by the above successive sums starting with the one whose variable is x and extending to the last sum, including the n_2-sum. We now express the three sums in $S_{\nu_2}(f)$ as contour integrals in order. Let (c) denote the contour $s = c+it$, $-\infty < t < \infty$. Applying the well-known inversion formula

$$\frac{1}{2\pi i}\int_{(c)} \frac{x^z dz}{z^{\varkappa+1}} = \begin{cases} 0 & \text{if}\, 0 < x \le 1, \\ \frac{(\log x)^\varkappa}{\varkappa!} & \text{if}\, x \ge 1, \end{cases} \tag{20.33}$$

for $c > 0$ and $\varkappa \in \mathbb{Z}^+$, we obtain

$$S_{\nu_2}(n_2) = \frac{d^{j_5}}{dv^{j_5}} \left\{ \sum_{\substack{n_2 \le y/h \\ (n_2, heg)=1}} \frac{\mu(n_2) I_{\nu_2}(n_2)}{n_2^v} \left(\log \frac{y}{n_2 h} \right)^{i_2} \right\}_{v=1} \tag{20.34}$$
$$= \frac{i_2!}{2\pi i} \frac{d^{j_5}}{dv^{j_5}} \left\{ \int_{(1)} \mathcal{Z}_1^{\nu_2}(v+w, eg) \prod_{p|h} \left(1 - \frac{1}{p^{v+w}}\right)^{-\nu_2} \frac{\left(\frac{y}{h}\right)^w}{w^{i_2+1}} dw \right\}_{v=1},$$

where

$$\mathcal{Z}_1(u, d) = \frac{1}{\zeta(u)} \prod_{p|d} \left(1 - \frac{1}{p^u}\right)^{-1}. \tag{20.35}$$

Inserting the last result into $S_{\nu_2}(f)$ and exchanging the order of the finite sums, the derivatives and the integral suitably, we obtain

$$S_{\nu_2}(f) = \frac{i_2!}{2\pi i} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5}}{dv^{j_5}} \left\{ \int_{(1)} \mathcal{Z}_1^{\nu_2}(v+w, eg) \frac{y^w}{w^{i_2+1}} \sum_{\substack{f \le \frac{y}{n_1 eg} \\ (f, n_1 eg)=1}} \frac{\mu(f) F_1(f)}{f} \right. \tag{20.36}$$
$$\prod_{p|f} \left(2 - \frac{1}{p^s}\right)^{\nu_2} \sum_{\substack{h \le \frac{y}{f n_1 eg} \\ (h, f n_1 eg)=1}} \frac{\mu^2(h) I_{\nu_2}(h) F_1(h)}{h^{1+w}} \prod_{p|h} \left(1 - \frac{1}{p^u}\right)^{\nu_2+1} \left(1 - \frac{1}{p^{v+w}}\right)^{-\nu_2}$$
$$\left. \left(\log \frac{y}{h f n_1 eg} \right)^{i_1} dw \right\}_{s,u,v=1}.$$

Substituting $d = fh$, we have

$$S_{\nu_2}(f) = \frac{i_2!}{2\pi i} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5}}{dv^{j_5}} \left\{ \int_{(1)} \mathcal{Z}_1^{\nu_2}(v+w, eg) \sum_{\substack{d \le \frac{y}{n_1 eg} \\ (d, n_1 eg)=1}} \frac{\mu(d) F_1(d) \left(\log \frac{y}{d n_1 eg} \right)^{i_1}}{d} \right. \tag{20.37}$$

$$\sum_{h|d} \frac{\mu(h) I_{\nu_2}(h)}{h^w} \prod_{p|h} \left(1 - \frac{1}{p^u}\right)^{\nu_2+1} \left(1 - \frac{1}{p^{v+w}}\right)^{-\nu_2} \prod_{p|\frac{d}{h}} \left(2 - \frac{1}{p^s}\right)^{\nu_2} \frac{y^w dw}{w^{i_2+1}} \Bigg\}_{s,u,v=1}.$$

We notice that the above summand is composed of multiplicative functions apart from the power of the logarithm so that we can translate its generating Dirichlet series into Euler's product form. Again consulting (20.33), we derive that

$$S_{\nu_2}(f) = \frac{i_1! i_2!}{(2\pi i)^2} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5}}{dv^{j_5}} \Bigg\{ \int_{(1)} \mathcal{Z}_1^{\nu_2}(v+w, eg) \frac{y^w}{w^{i_2+1}}$$
$$\int_{(1)} \mathcal{Z}_2(z, w, u, s, v; n_1 eg; \nu_2) \left(\frac{y}{n_1 eg}\right)^z \frac{dz dw}{z^{i_1+1}} \Bigg\}_{s,u,v=1}, \quad (20.38)$$

where

$$\mathcal{Z}_2(z, w, u, s, v; \ell; \nu_2) = \sum_{\substack{d \geq 1 \\ (d,\ell)=1}} \frac{\mu(d) F_1(d)}{d^{1+z}}$$
$$\sum_{h|d} \frac{\mu(h) I_{\nu_2}(h)}{h^w} \prod_{p|h} \left(1 - \frac{1}{p^u}\right)^{\nu_2+1} \left(1 - \frac{1}{p^{v+w}}\right)^{-\nu_2} \prod_{p|\frac{d}{h}} \left(2 - \frac{1}{p^s}\right)^{\nu_2} \quad (20.39)$$

$$= \prod_{p \nmid \ell} \left(1 - \frac{1 + \nu_2 \left(1 - p^{-s} - p^{-w} \left(1 - p^{-u}\right)^2 \left(1 - p^{-v-w}\right)^{-1}\right)}{p^z (p-1)}\right).$$

When z and w lie on the line (1), and $|s-1|, |v-1|, |u-1| \leq \epsilon$, we have

$$\mathcal{Z}_1^{\nu_2}(v+w, eg) \mathcal{Z}_2(z, w, u, s, v; n_1 eg; \nu_2) \ll 1. \quad (20.40)$$

Since $i_1, i_2 \geq 1$, both of the contour integrals in (20.38) are uniformly convergent so that we can exchange the order of derivatives and the integrals:

$$S_{\nu_2}(f) = \frac{i_1!i_2!}{(2\pi i)^2} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5}}{dv^{j_5}} \Bigg\{ \int_{(1)} \left(\frac{y}{n_1 eg}\right)^z \frac{1}{z^{i_1+1}}$$
$$\int_{(1)} \mathcal{Z}_1^{\nu_2}(v+w, eg)\mathcal{Z}_2(z, w, u, s, v; n_1eg; \nu_2) \frac{y^w dwdz}{w^{i_2+1}} \Bigg\}_{s,u,v=1}. \quad (20.41)$$

The infinite contours we will encounter in the rest of the paper are uniformly convergent. Although (20.40) may vary on these contours, that $i_1, i_2 \geq 1$ ensures the uniform convergence, and we do not need to truncate these infinite contours.

At this point we recall some standard information of the Riemann zeta-function concerning the horizontal distribution of its complex zeros. The classical zero-free region theorems state that there exists a constant $c_1 > 0$ such that

$$\zeta(\sigma + it) \neq 0 \qquad \text{for} \qquad \sigma \geq 1 - \frac{c_1}{\log(|t| + 4)}, \qquad -\infty < t < \infty. \quad (20.42)$$

Further, in the same region, we know that

$$\zeta(\sigma + it) - \frac{1}{\sigma + it - 1}, \ \frac{1}{\zeta(\sigma + it)} \ll \log(|t| + 4). \quad (20.43)$$

Returning to the $w-$integral, to avoid the complex zeros of ζ, we must have

$$\Re(v + w) \geq 1 - \frac{c_1}{\log(|\Im(v + w)| + 4)} \quad (20.44)$$

in the case of $\nu_2 = 1$. Let $\mathcal{L}(v)$ be the contour described by

$$1 - \frac{c_1}{3\log(|\Im w| + 4)} + i\Im w - v, \qquad -\infty < \Im w < \infty. \quad (20.45)$$

Given that $\Re z = 1$, $|s-1|, |u-1|, |v-1| \leq \epsilon$, from the product representation in (20.39), it follows that $\mathcal{Z}_2(z, w, u, s, v; n_1eg; \nu_2)$ is an analytic function of w on $\mathcal{L}(v)$ and to the right of $\mathcal{L}(v)$. Here ϵ is sufficiently small so that 0 in the $w-$plane lies in the region whose borders are (1) and $\mathcal{L}(v)$. Further, the factor of the product in (20.39) reads

$$\left(1-\frac{1+\nu_2\left(1-p^{-s}-\left(1-p^{-u}\right)^2\left(1-p^{-v}\right)^{-1}\right)}{p^z(p-1)}\right)$$

at $w=0$, which tends to $1-\frac{1}{p^z(p-1)}\neq 0$ as $s,u,v\to 1$. So $\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)$ does not vanish at $w=0$. According to (20.42), $\mathcal{Z}_1(v+w,eg)$ does not produce any singularity between (1) and $\mathcal{L}(v)$. Assume $v\neq 1$ for a while. With the aid of (2.13), we conclude that the order of the pole at $w=0$ of the $w-$integral in (20.41) is i_2+1. Thus, the residue theorem implies that

$$\begin{aligned}\frac{i_2!}{2\pi i}\int_{(1)}&\mathcal{Z}_1^{\nu_2}(v+w,eg)\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\frac{y^w dw}{w^{i_2+1}}\\ &=i_2!\mathrm{Res}_{w=0}\left\{\mathcal{Z}_1^{\nu_2}(v+w,eg)\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\frac{y^w}{w^{i_2+1}}\right\}\end{aligned}$$

$$+\frac{i_2!}{2\pi i}\int_{\mathcal{L}(v)}\mathcal{Z}_1^{\nu_2}(v+w,eg)\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\frac{y^w dw}{w^{i_2+1}}$$

$$=\sum_{\substack{r_1+r_2+r_3=i_2\\ \nu_2=0\Rightarrow r_1,r_2=0}}\binom{i_2}{r_1,r_2,r_3}\left(\mathcal{Z}_1^{(r_1)}(v,eg)\right)^{\nu_2}\frac{d^{r_2}}{dw^{r_2}}\left\{\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\right\}_{w=0}\quad(20.46)$$

$$\times(\log y)^{r_3}+\frac{i_2!}{2\pi i}\int_{\mathcal{L}(0)}\mathcal{Z}_1^{\nu_2}(w,eg)\mathcal{Z}_2(z,w-v,u,s,v;n_1eg;\nu_2)\frac{y^{w-v}dw}{(w-v)^{i_2+1}}.$$

The last change is due to the substitution $w\to w-v$. Now the three factors in the summand above are continuous, actually differentiable, function of v in a neighborhood of $v=1$, including $v=1$, so we no more need the exclusion $v\neq 1$. If $\nu_2=0$ then the $w-$integral does not depend on v, s and u, so its j_5-th order derivative at $v=1$ is 0 unless $j_5 = 0$. Taking derivatives with respect to v gives that

$$\frac{i_2!}{2\pi i}\frac{d^{j_5}}{dv^{j_5}}\left\{\int_{(1)} \mathcal{Z}_1^{\nu_2}(v+w,eg)\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\frac{y^w dw}{w^{i_2+1}}\right\}_{v=1} =$$

$$[\nu_2 = 0 \Rightarrow j_5 = 0] \sum_{\substack{r_1+r_2+r_3=i_2\\ 0\leq r_4\leq j_5\\ \nu_2=0\Rightarrow r_1,r_2=0}} \binom{j_5}{r_4}\binom{i_2}{r_1,r_2,r_3}\left(\mathcal{Z}_1^{(r_1+r_4)}(1,eg)\right)^{\nu_2}$$

$$\times\frac{d^{j_5-r_4}}{dv^{j_5-r_4}}\frac{d^{r_2}}{dw^{r_2}}\left\{\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\right\}_{\substack{w=0\\ v=1}}(\log y)^{r_3}$$

$$+\frac{[\nu_2 = 0 \Rightarrow j_5 = 0]}{2\pi i}\sum_{r_5+r_6+r_7=j_5}\binom{j_5}{r_5,r_6,r_7}y^{-1}(-\log y)^{r_6}(i_2+r_7)!$$
$$\times\int_{\mathcal{L}(0)}\mathcal{Z}_1^{\nu_2}(w,eg)\frac{d^{r_5}}{dv^{r_5}}\left\{\mathcal{Z}_2(z,w-v,u,s,v;n_1eg;\nu_2)\right\}_{v=1}\frac{y^w dw}{(w-1)^{i_2+r_7+1}}.$$

Here we note that if $\nu_2 = 1$, then $r_1 + r_4$ must be ≥ 1, otherwise $\mathcal{Z}_1^{(r_1+r_4)}(1,eg) = 0$, which is seen from (2.13). Returning to (20.41), the last result gives rise to

$$S_{\nu_2}(f) = \mathcal{T}_1 + \mathcal{T}_2, \tag{20.47}$$

where

$$\mathcal{T}_1 = \frac{i_1!\,[\nu_2 = 0 \Rightarrow j_3, j_4, j_5 = 0]}{2\pi i}\sum_{\substack{r_1+r_2+r_3=i_2\\ 0\leq r_4\leq j_5\\ \nu_2=1\Rightarrow r_1+r_4\geq 1\\ \nu_2=0\Rightarrow r_1,r_2=0}}\binom{j_5}{r_4}\binom{i_2}{r_1,r_2,r_3}\left(\mathcal{Z}_1^{(r_1+r_4)}(1,eg)\right)^{\nu_2} \tag{20.48}$$

$$(\log y)^{r_3}\int_{(1)}\frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{j_5-r_4}}{dv^{j_5-r_4}}\frac{d^{r_2}}{dw^{r_2}}\left\{\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\right\}_{\substack{w=0\\ s,u,v=1}}\frac{\left(\frac{y}{n_1eg}\right)^z dz}{z^{i_1+1}}$$

and

$$\mathcal{T}_2 = \frac{i_1!\,[\nu_2 = 0 \Rightarrow j_3, j_4, j_5 = 0]}{(2\pi i)^2} \sum_{r_5+r_6+r_7=j_5} \binom{j_5}{r_5, r_6, r_7} y^{-1}(-\log y)^{r_6}(i_2+r_7)!$$
$$\times \int_{\mathcal{L}(0)} \mathcal{Z}_1^{\nu_2}(w, eg) \int_{(1)} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{\mathcal{Z}_2(z, w-v, u, s, v; n_1eg; \nu_2)\right\}_{s,u,v=1} \quad (20.49)$$

$$\times \left(\frac{y}{n_1eg}\right)^z \frac{y^w dzdw}{(w-1)^{i_2+r_7+1} z^{i_1+1}}.$$

Before handling $\mathcal{T}_1$ and $\mathcal{T}_2$, we examine $\mathcal{Z}_2(z, w, u, s, v; \ell; \nu_2)$ in detail. Assume $\Re z, \Re w \geq \epsilon$ and $|s-1|, |u-1|, |v-1| \leq \epsilon$, which are enough for $\mathcal{Z}_2(z, w, u, s, v; \ell; \nu_2)$ to be well-defined. However, we want to find out an analytic continuation $\mathcal{Z}_2(z, w, u, s, v; \ell; \nu_2)$ whose domain of analyticity extends the w and z variables to larger regions, while we keep the ranges of s, u and v the same. Continuing from (20.39), we have

$$\mathcal{Z}_2(z, w, u, s, v; \ell; \nu_2) = \prod_p \left(1 - \frac{1 + \nu_2\left(1 - \frac{1}{p^s} - \frac{\left(1-\frac{1}{p^u}\right)^2}{p^w}\left(1 - \frac{1}{p^{v+w}}\right)^{-1}\right)}{p^z(p-1)}\right)$$
$$\prod_{p|\ell} \left(1 - \frac{1 + \nu_2\left(1 - p^{-s} - p^{-w}\left(1-p^{-u}\right)^2\left(1-p^{-v-w}\right)^{-1}\right)}{p^z(p-1)}\right)^{-1} \quad (20.50)$$

$$= \prod_p \left(1 - \frac{1+\nu_2 - \frac{\nu_2}{p^w}}{p^{1+z}}\right)\left(1 - \frac{1+\nu_2 - \frac{\nu_2}{p^w} - \nu_2 p^{1-s} - \nu_2 p^{1-w} f(u,v,w,p)}{\left(p^{1+z} - 1 - \nu_2 + \nu_2 p^{-w}\right)(p-1)}\right)$$
$$\prod_{p|\ell}\left(1 - \frac{1 + \nu_2 - \nu_2 p^{-s} - \nu_2 p^{-w} - \nu_2 p^{-w} f(u,v,w,p)}{p^z(p-1)}\right)^{-1}$$

$$= \frac{\zeta(1+z+w)\prod_p\left(1-\frac{1+\nu_2-\nu_2p^{-w}-\nu_2p^{1-s}-\nu_2p^{1-w}f(u,v,w,p)}{(p^{1+z}-1-\nu_2+\nu_2p^{-w})(p-1)}\right)}{\zeta((1+\nu_2)(1+z+w))\zeta^{1+\nu_2}(1+z)}$$
$$\prod_p\left(1-\frac{1+2p^{1+z}g(p,z,w)}{(p^{1+z}-1)^2}\right)^{\nu_2}\prod_{p|\ell}\left(1-\frac{1+\nu_2-\frac{\nu_2}{p^s}-\frac{\nu_2}{p^w}-\frac{\nu_2 f(u,v,w,p)}{p^w}}{p^z(p-1)}\right)^{-1},$$

where

$$f(u,v,w,p)=\left(1-p^{-u}\right)^2\left(1-p^{-v-w}\right)^{-1}-1 \tag{20.51}$$

$$g(p,z,w)=\left(1+p^{-1-z-w}\right)^{-1}-1. \tag{20.52}$$

It is easy to see that for $\Re w$, $\Re z \geq -\epsilon$,

$$f(u,v,w,p),\ g(p,z,w) \ll p^{-1+\epsilon}, \tag{20.53}$$

and both of the infinite products above are bounded and analytic. We should determine whether the finite product

$$\prod_{p|\ell}\left(1-\frac{1+\nu_2-\frac{\nu_2}{p^s}-\frac{\nu_2}{p^w}-\frac{\nu_2 f(u,v,w,p)}{p^w}}{p^z(p-1)}\right)^{-1} \tag{20.54}$$

has any singularity or not. Observe that

$$\begin{aligned} p^z(p-1) &= 1+\nu_2-\nu_2p^{-s}-\nu_2p^{-w}-\nu_2p^{-w}f(u,v,w,p) \\ \Leftrightarrow p^{1+z}(p-1) &= p+\nu_2p-\nu_2p^{1-s}-\nu_2p^{1-w}-\nu_2p^{1-w}f(u,v,w,p) \\ \Leftrightarrow \left(p^{1+z}-1-\nu_2+\frac{\nu_2}{p^w}\right)(p-1) &= 1+\nu_2-\frac{\nu_2}{p^w}-\nu_2p^{1-s}-\nu_2p^{1-w}f(u,v,w,p), \end{aligned}$$

which means that there occur cancellations between the zeros coming from

$$\prod_p\left(1-\frac{1+\nu_2-\nu_2p^{-w}-\nu_2p^{1-s}-\nu_2p^{1-w}f(u,v,w,p)}{\left(p^{1+z}-1-\nu_2+\nu_2p^{-w}\right)(p-1)}\right) \tag{20.55}$$

and the poles from (20.54). To see what remains after cancellations, we work out the following limit. For any $(p, w, z, u, s, v, \ell, \nu_2)$ satisfying $p|\ell$ and the above equivalent cases,

$$\lim_{a\to z} \frac{\left(1 - \frac{1+\nu_2 - \frac{\nu_2}{p^w} - \nu_2 p^{1-s} - \nu_2 p^{1-w} f(u,v,w,p)}{(p^{1+a}-1-\nu_2+\nu_2 p^{-w})(p-1)}\right)\left(1 - \frac{1+2p^{1+a} g(p,a,w)}{(p^{1+a}-1)^2}\right)^{\nu_2}}{\left(1 - \frac{1+\nu_2 - \frac{\nu_2}{p^s} - \frac{\nu_2}{p^w} - \frac{\nu_2 f(u,v,w,p)}{p^w}}{p^a(p-1)}\right)}$$
$$= \left(1 - \frac{1}{p^{1+z}}\right)^{-1-\nu_2}\left(1 + \frac{1}{p^{1+z+w}}\right)^{-\nu_2},$$

which is neither 0 nor ∞ in the new extended half-plane for z and w. In view of (2.13) and (20.42), we impose the restriction $\Re z \geq -\frac{c_1}{\log(|\Im z|+4)}$ on z so that the only possible singularity of $\mathcal{Z}_2(z, w, u, s, v; \ell; \nu_2)$ is at $w + z = 0$.

With these findings, we return to the estimations of $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $L(\tilde{z}, \tilde{c})$ be the contour consisting of five parts:

$$\begin{aligned}
&L_1(\tilde{z},\tilde{c}) : 1 + it - \tilde{z}, && \log y \leq t < \infty,\\
&L_2(\tilde{z},\tilde{c}) : 1 + \sigma + i\log y - \tilde{z}, && \frac{-\tilde{c}}{\log\log y} \leq \sigma \leq 0,\\
&L_3(\tilde{z},\tilde{c}) : 1 - \frac{\tilde{c}}{\log\log y} + it - \tilde{z}, && -\log y \leq t \leq \log y,\\
&L_4(\tilde{z},\tilde{c}) : 1 + \sigma - i\log y - \tilde{z}, && \frac{-\tilde{c}}{\log\log y} \leq \sigma \leq 0,\\
&L_5(\tilde{z},\tilde{c}) : 1 + it - \tilde{z}, && -\infty < t \leq -\log y,
\end{aligned}$$

where $\tilde{c} > 0$ and $\tilde{z} \in \mathbb{C}$. Considering the integral in $\mathcal{T}_1$, between the contours (1) and $L(1, c_1/3)$ we encounter just one pole, which is located at $z = 0$. So, by the residue theorem,

$$\frac{1}{2\pi i}\int_{(1)} \frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{j_5-r_4}}{dv^{j_5-r_4}}\frac{d^{r_2}}{dw^{r_2}}\left\{\mathcal{Z}_2(z, w, u, s, v; n_1eg; \nu_2)\right\}_{\substack{w=0\\ s,u,v=1}}\left(\frac{y}{n_1eg}\right)^z \frac{dz}{z^{i_1+1}}$$

$$= \mathrm{Res}_{z=0} \left\{ \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5-r_4}}{dv^{j_5-r_4}} \frac{d^{r_2}}{dw^{r_2}} \left\{ \mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2) \right\}_{\substack{w=0 \\ s,u,v=1}} \frac{\left(\frac{y}{n_1eg}\right)^z}{z^{i_1+1}} \right\}$$

$$+ \frac{1}{2\pi i} \int_{L(1,\frac{c_1}{3})} \frac{\frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5-r_4}}{dv^{j_5-r_4}} \frac{d^{r_2}}{dw^{r_2}} \left\{ \mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2) \right\}_{\substack{w=0 \\ s,u,v=1}} \left(\frac{y}{n_1eg}\right)^z dz}{z^{i_1+1}}$$

$$= \mathcal{T}_{1,1} + \mathcal{T}_{1,2}, \text{ say.} \tag{20.56}$$

We treat $\mathcal{T}_{1,2}$ in the following lemma. If $\nu_2 = 0$, then $r_2 = j_3 = j_4 = j_5 = 0$, and by (20.27), (20.43) and (20.50),

$$\frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5-r_4}}{dv^{j_5-r_4}} \frac{d^{r_2}}{dw^{r_2}} \left\{ \mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2) \right\}_{\substack{w=0 \\ s,u,v=1}} \ll (\log(|\Im z|+4))(\log\log y)^A$$

for $z \in L(1, c_1/3)$. In the case of $\nu_2 = 1$ it follows from (20.27), (20.43), (20.50) and the successive applications of Cauchy's integral formula that for $z \in L(1, c_1/3)$,

$$\frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5-r_4}}{dv^{j_5-r_4}} \frac{d^{r_2}}{dw^{r_2}} \left\{ \mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2) \right\}_{\substack{w=0 \\ s,u,v=1}} = \frac{j_3!j_4!(j_5-r_4)!r_2!}{(2\pi i)^4}$$
$$\times \iiiint_{\substack{|w| \asymp (\log(|\Im z|+4)+\log\log y)^{-1} \\ |s-1|=|u-1|=|v-1|=\epsilon}} \frac{\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\,dw\,du\,ds\,dv}{(s-1)^{j_3+1}(u-1)^{j_4+1}(v-1)^{j_5-r_4+1}w^{r_2+1}}$$

$$\ll A^{j_3+j_4+j_5} j_3!j_4!(j_5-r_4)!(\log(|\Im z|+4))^{r_2+3}(\log\log y)^A.$$

Here we've chosen the circle around $w = 0$ so that we can avoid the pole of $\zeta(1+z+w)$ at $z + w = 0$, and that we can make use of (20.43). With these estimates, the integral in $\mathcal{T}_{1,2}$ can be reduced to

Lemma 20.1. *For $x \geq 1$, $k, m \in \mathbb{N}$, $k \geq 2$, $c > 0$, $m \ll \log\log 3y$,*

$$\int_{L(1,c)}, \int_{L_z(1,c)} \frac{(\log(|w|+4))^m \, x^w dw}{|w|^k} \ll (A \log\log y)^{k+m} \left(x^{-\frac{c}{\log\log y}} + \frac{1}{(\log y)^{k-1}} \right), \quad (20.57)$$

where $L_z(1,c)$, $z \in \mathbb{C}$, is described as follows. If $z \notin L_1(1,c)$ and $z \notin L_5(1,c)$, then $L_z(1,c) = L(1,c)$. Otherwise, we paste a semicircle around z of radius $\asymp (\log(|\Im z| + 4) + \log\log y)^{-1}$ to the part on which z is and then delete the line segment between the intersection points of the semicircle and $L(1,c)$, so that z lies to the right of the new contour.

The difference between two cases is negligible. We only deal with $L(1,c)$. Similar to (13.17), by integration by parts,

$$\int_{L_1(1,c)} \cdots, \int_{L_5(1,c)} \cdots \ll \int_{\log y}^{\infty} \frac{(\log u)^m}{u^k} du \ll \frac{(\log\log y)^m}{(\log y)^{k-1}},$$

which, trivially, dominates the contribution of the ranges $L_2(1,c)$ and $L_4(1,c)$. For the last domain, $L_3(1,c)$, we divide into two parts: $[-\frac{c}{\log\log y} - i, -\frac{c}{\log\log y} + i]$ and the remainder. On the first part, $|w|^{-k} \leq \left(\frac{\log\log y}{c}\right)^k$. So

$$\begin{aligned} \int_{L_3(1,c)} \cdots &\ll x^{-\frac{c}{\log\log y}} \left(A^{k+m} (\log\log y)^k + \int_1^{\log y} \frac{(\log u)^m du}{u^k} \right) \\ &\ll x^{-\frac{c}{\log\log y}} (A \log\log y)^{k+m}. \end{aligned}$$

Combining the contributions of the five parts, we arrive at the assertion.

Employing Lemma 20.1 in $\mathcal{T}_{1,2}$, we easily deduce that

$$\mathcal{T}_{1,2} \ll A^{j_3+j_4+j_5} j_3! j_4! (j_5 - r_4)! (\log\log y)^A \left(\left(\frac{y}{n_1 e g} \right)^{-\frac{c_1}{3 \log\log y}} + \frac{1}{(\log y)^{i_1}} \right). \quad (20.58)$$

We now work out $\mathcal{T}_{1,1}$ as much as our final goals require. In view of (2.13) and (20.50), the r_2-th derivative with respect to w of $\zeta(1+z+w)$ at $w=0$ increases the order of the pole at $z=0$ by r_2 in the case $\nu_2=1$, but this factor is cancelled out in the case $\nu_2=0$. However, $\zeta^{-1-\nu_2}(1+z)$ reduces the order of the pole by $1+\nu_2$. If we denote the order of the pole at $z=0$ in (20.56) by K, then we can say that $K \le (i_1+1)+\nu_2(r_2+1)-(1+\nu_2) = i_1+\nu_2 r_2$ in general. Hence,

$$\mathcal{T}_{1,1} = \frac{1}{(K-1)!}\sum_{\ell_1 \le K-1}\binom{K-1}{\ell_1}\left(\log\frac{y}{n_1 eg}\right)^{K-1-\ell_1}$$
$$\frac{d^{\ell_1}}{dz^{\ell_1}}\left\{ z^{K-i_1-1}\frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{j_5-r_4}}{dv^{j_5-r_4}}\frac{d^{r_2}}{dw^{r_2}}\left\{\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\right\}_{\substack{w=0\\ s,u,v=1}}\right\}_{z=0}.$$

We represent the above sequence of derivatives by consecutive Cauchy integrals, and then employing (2.14), (20.27) and (20.50) gives that

$$\frac{d^{\ell_1}}{dz^{\ell_1}}\left\{ z^{K-i_1-1}\frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{j_5-r_4}}{dv^{j_5-r_4}}\frac{d^{r_2}}{dw^{r_2}}\left\{\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\right\}_{\substack{w=0\\ s,u,v=1}}\right\}_{z=0}$$

$$= \frac{\ell_1!}{2\pi i}\int_{|z|\asymp(\log\log y)^{-1}} \frac{\frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{j_5-r_4}}{dv^{j_5-r_4}}\frac{d^{r_2}}{dw^{r_2}}\left\{\mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)\right\}_{\substack{w=0\\ s,u,v=1}}dz}{z^{\ell_1+i_1-K+2}}$$

$$= \frac{\ell_1!j_3!j_4!(j_5-r_4)!r_2!}{(2\pi i)^5}\int\cdots\int_{\substack{|z|\asymp(\log\log y)^{-1}\\ |s-1|=|u-1|=|v-1|=\epsilon\\ |w|=|z|/2}} \mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2)$$
$$\times\frac{dw\,ds\,du\,dv\,dz}{z^{\ell_1+i_1-K+2}(s-1)^{j_3+1}(u-1)^{j_4+1}(v-1)^{j_5-r_4+1}w^{r_2+1}}$$

$$\ll A^{j_3+j_4+j_5}j_3!j_4!(j_5-r_4)!(\log\log y)^A, \tag{20.59}$$

from which $\mathcal{T}_{1,1}$ simplifies substantially to

$$\mathcal{T}_{1,1} = \frac{\left\{ z^{K-i_1-1} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{j_5-r_4}}{dv^{j_5-r_4}} \frac{d^{r_2}}{dw^{r_2}} \left\{ \mathcal{Z}_2(z,w,u,s,v;n_1eg;\nu_2) \right\}_{\substack{w=0 \\ s,u,v=1}} \right\}_{z=0}}{(K-1)!} \times \left(\log \frac{y}{n_1eg} \right)^{K-1} \tag{20.60}$$

$$+O\left([K \geq 2] A^{j_3+j_4+j_5} j_3! j_4! (j_5-r_4)! \left(\log \frac{y}{n_1eg} \right)^{K-2} (\log\log y)^A \right).$$

If $j_3 = j_4 = j_5 - r_4 = 0$, then we must check whether there are factors in (20.50) reducing the order of the pole at $z = 0$ to determine the exact value of K. Observe that

$$\prod_p \left(1 - \frac{1 + 2p^{1+z} g(p,z,w)}{(p^{1+z}-1)^2} \right)^{\nu_2} \Big|_{z,w=0} = \prod_p \left(1 + \frac{1}{p^2-1} \right)^{\nu_2} \neq 0$$

and

$$\prod_p \left(1 - \frac{1+\nu_2-\nu_2p^{-w}-\nu_2p^{1-s}-\nu_2p^{1-w}f(u,v,w,p)}{(p^{1+z}-1-\nu_2+\nu_2p^{-w})(p-1)} \right) \Big|_{\substack{s,u,v=1 \\ z,w=0}} = \prod_p \left(1 - \frac{1}{(p-1)^2} \right) = 0.$$

In the second observation the zero at $z, w = 0$ is due to the factor $p = 2$. Recalling the cancellations between the poles of (20.54) and the zeros of (20.55), we must have $2|n_1eg$ to avoid any loss in the order of the pole at $z = 0$. As a result, $K = i_1 + \nu_2 r_2 - [2 \nmid n_1eg]$. With the aid of (2.14) and (20.50) we can calculate the coefficient of the highest order term in (20.60). Hence,

$$\mathcal{T}_{1,1} = \frac{2(-1)^{r_2\nu_2}(r_2!)^{\nu_2}[2|n_1eg]}{(i_1+\nu_2r_2-1)!} \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \prod_{\substack{p|n_1eg \\ p>2}}\left(1+\frac{1}{p-2}\right) \times \left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1}$$

$$+O\left([i_1+\nu_2r_2\geq 2]\left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-2}(\log\log y)^A\right). \tag{20.61}$$

If $j_3+j_4+j_5-r_4\geq 1$, then ν_2 must be 1, because, otherwise $j_3=j_4=j_5=0$ as was indicated in (20.48). However, we do not need to know the exact order of the pole and to specify the coefficient of the main term in this case. Based on the facts $K\leq i_1+\nu_2r_2$ and (20.59), (20.60) becomes

$$\mathcal{T}_{1,1} \ll A^{j_3+j_4+j_5} j_3!j_4!(j_5-r_4)! \left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1}(\log\log y)^A. \tag{20.62}$$

We remark that this upper bound also holds when $j_3+j_4+j_5-r_4=0$.

Together with the results on $\mathcal{T}_{1,1}$ and $\mathcal{T}_{1,2}$, we return to $\mathcal{T}_1$ and naturally divide into two parts:

$$\mathcal{T}_1 = \mathcal{T}\mathcal{M}_1 + \mathcal{T}\mathcal{E}_1, \tag{20.63}$$

where the first new component is produced by $\mathcal{T}_{1,1}$ while the second one is related to

$\mathcal{T}_{1,2}$. It then follows from (20.48), (20.56) and (20.58) that

$$\mathcal{TE}_1 \ll A^{j_3+j_4+j_5} j_3!j_4! \left(\left(\frac{y}{n_1 eg} \right)^{-\frac{c_1}{3 \log\log y}} + \frac{1}{(\log y)^{i_1}} \right) (\log\log y)^A$$
$$[\nu_2 = 0 \Rightarrow j_3, j_4, j_5 = 0] \sum_{\substack{r_1+r_2+r_3=i_2 \\ 0 \le r_4 \le j_5 \\ \nu_2=1 \Rightarrow r_1+r_4 \ge 1 \\ \nu_2=0 \Rightarrow r_1, r_2=0}} \binom{j_5}{r_4} (j_5 - r_4)! |\mathcal{Z}_1^{(r_1+r_4)}(1, eg)|^{\nu_2} (\log y)^{r_3}. \quad (20.64)$$

By Cauchy's integral formula along the circle around 1 of radius $(\log\log y)^{-1}$,

$$\mathcal{Z}_1^{(r_1+r_4)}(1, eg) \ll (r_1 + r_4)!(\log\log y)^A \ll A^{r_4} r_4!(\log\log y)^A. \quad (20.65)$$

Inserting this result into (20.64) we get

$$\mathcal{TE}_1 \ll A^{j_3+j_4+j_5} j_3!j_4!j_5! \left(\left(\frac{y}{n_1 eg} \right)^{-\frac{c_1}{3 \log\log y}} + \frac{1}{(\log y)^{i_1}} \right)$$
$$\times (\log y)^{i_2 - \nu_2 [j_5=0]} (\log\log y)^A. \quad (20.66)$$

We examine $\mathcal{TM}_1$ in two cases:

Case 1. $j_3 = j_4 = 0$

Corresponding to the case-analysis in the estimation of $\mathcal{T}_{1,1}$, we separate $\mathcal{TM}_1$ into two parts according to $r_4 = j_5$ or not when $\nu_2 = 1$ and $j_5 = 1$. This separation is not in question for the case $\nu_2 = 0$ or $j_5 \ge 2$. Applying (20.61) and (20.62) to the relevant parts, we have

$$\mathcal{TM}_1 = [j_5 \le \nu_2] \sum_{\substack{r_1+r_2+r_3=i_2 \\ \nu_2=1 \Rightarrow r_1+j_5 \ge 1 \\ \nu_2=0 \Rightarrow r_1, r_2=0}} \binom{i_2}{r_1, r_2, r_3} \left(\mathcal{Z}_1^{(r_1+j_5)}(1, eg) \right)^{\nu_2} (\log y)^{r_3} \quad (20.67)$$

$$\left(\frac{2(-1)^{r_2\nu_2}(r_2!)^{\nu_2}i_1![2|n_1eg]}{(i_1+\nu_2r_2-1)!}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p|n_1eg\\p>2}}\left(1+\frac{1}{p-2}\right)\right.$$
$$\left.\times\left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1}+O\left((\log y)^{i_1+\nu_2r_2-2}(\log\log y)^A\right)\right)$$

$$+O\Bigg([\nu_2=1\text{ and }j_5=1]A^{j_5}(\log\log y)^A\sum_{\substack{r_1+r_2+r_3=i_2\\r_1\geq 1}}\binom{i_2}{r_1,r_2,r_3}$$
$$|\mathcal{Z}_1^{(r_1)}(1,eg)|(\log y)^{r_3}\left(\log\frac{y}{n_1eg}\right)^{i_1+r_2-1}\Bigg)$$

$$+O\Bigg([\nu_2=1\text{ and }j_5\geq 2]A^{j_5}(\log\log y)^A\sum_{\substack{r_1+r_2+r_3=i_2\\0\leq r_4\leq j_5\\r_1+r_4\geq 1}}\binom{j_5}{r_4}\binom{i_2}{r_1,r_2,r_3}$$
$$(j_5-r_4)!|\mathcal{Z}_1^{(r_1+r_4)}(1,eg)|(\log y)^{r_3}\left(\log\frac{y}{n_1eg}\right)^{i_1+r_2-1}\Bigg).$$

The sum above is a one-term sum if $\nu_2=0$, while in the case of $\nu_2=1$, whose highest order terms are those having the smallest possible value for the index r_1. By (2.13) and (20.35), we see that

$$\mathcal{Z}_1^{(1)}(1,eg)=\prod_{p|eg}\left(1-\frac{1}{p}\right)^{-1}. \tag{20.68}$$

With (20.65) and (20.68), (20.67) takes a simpler form:

$$\mathcal{TM}_1=2[2|n_1eg]i_1!(i_2!)^{\nu_2}[j_5\leq\nu_2]\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|eg}\left(1-\frac{1}{p}\right)^{-\nu_2}$$
$$\prod_{\substack{p|n_1eg\\p>2}}\left(1+\frac{1}{p-2}\right)\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\\nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}(\log y)^{r_3}\left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1}}{(i_1+\nu_2r_2-1)!(r_3!)^{\nu_2}}$$

$$+O\left(A^{j_5}j_5!(\log y)^{i_1+i_2-2-\nu_2[j_5=0]+\nu_2[j_5\geq 2]}(\log\log y)^A\right).$$

Case 2. $j_3+j_4\geq 1$

Since $\nu_2=0\Rightarrow j_3,j_4,j_5=0$, ν_2 must be 1. Applying (20.62) we similarly obtain

$$\mathcal{TM}_1\ll A^{j_3+j_4+j_5}j_3!j_4!j_5!(\log y)^{i_1+i_2-1-[j_5=0]}(\log\log y)^A. \tag{20.69}$$

Unifying the cases, we arrive at

$$\begin{aligned}\mathcal{TM}_1 =&2i_1!(i_2!)^{\nu_2}[2|n_1eg\,\text{and}\,j_3,j_4=0\,\text{and}\,j_5\leq\nu_2] \\ &\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|eg}\left(1-\frac{1}{p}\right)^{-\nu_2}\prod_{\substack{p|n_1eg\\p>2}}\left(1+\frac{1}{p-2}\right) \\ &\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\\nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2r_2-1)!(r_3!)^{\nu_2}}\left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1} \\ &+O\left(A^{j_3+j_4+j_5}j_3!j_4!j_5!(\log y)^{i_1+i_2-2-\nu_2[j_5=0]+\nu_2[j_5\geq 2\,\text{or}\,j_3+j_4\geq 1]}(\log\log y)^A\right).\end{aligned} \tag{20.70}$$

Comparing the error terms in (20.66) and (20.70), we conclude that

$$\begin{aligned}\mathcal{T}_1 =&2i_1!(i_2!)^{\nu_2}[2|n_1eg\,\text{and}\,j_3,j_4=0\,\text{and}\,j_5\leq\nu_2] \\ &\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|eg}\left(1-\frac{1}{p}\right)^{-\nu_2}\prod_{\substack{p|n_1eg\\p>2}}\left(1+\frac{1}{p-2}\right) \\ &\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\\nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2r_2-1)!(r_3!)^{\nu_2}}\left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1}\end{aligned} \tag{20.71}$$

$$+ O\Bigg(A^{j_3+j_4+j_5} j_3! j_4! j_5! (\log\log y)^A (\log y)^{i_2-\nu_2[j_5=0]}$$

$$\left((\log y)^{i_1-2+\nu_2[j_5\geq 2 \text{ or } j_3+j_4\geq 1]} + \left(\frac{y}{n_1 eg}\right)^{-c_1/\log\log 3y} \right) \Bigg).$$

We finally end this part, the estimation of the $S_{\nu_2}(f)$, by dealing with $\mathcal{T}_2$. Firstly, consider the $z-$integral in (20.49), which possesses at most 2 singularities located at $z = 0$ and $z = 1 - w$ in the region determined by (1) and $L(1, c_1/3)$. We don't bother to determine the exact orders of the possible poles. We call $\tilde{K}_1$ and $\tilde{K}_2$ the order of the poles at $z = 0$ and $z = 1 - w$, respectively. For the first pole we have $\tilde{K}_1 \leq i_1 - \nu_2$. The second one exists if $\nu_2 = 1$ and $\tilde{K}_2 \leq r_5 + 1$. Then by the residue theorem,

$$\frac{i_1!}{2\pi i} \int_{(1)} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; \nu_2) \right\}_{s,u,v=1} \left(\frac{y}{n_1 eg}\right)^z \frac{dz}{z^{i_1+1}}$$

$$= \frac{i_1!}{(\tilde{K}_1 - 1)!} \sum_{r_8 \leq \tilde{K}_1 - 1} \binom{\tilde{K}_1 - 1}{r_8} \left(\log \frac{y}{n_1 eg}\right)^{\tilde{K}_1 - 1 - r_8}$$

$$\frac{d^{r_8}}{dz^{r_8}} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ z^{\tilde{K}_1 - i_1 - 1} \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; \nu_2) \right\}_{\substack{s,u,v=1\\ z=0}}$$

$$- \frac{(-1)^{i_1}[\nu_2 = 1] \left(\frac{y}{n_1 eg}\right)^{1-w}}{(\tilde{K}_2 - 1)!} \sum_{r_9 + r_{10} + r_{11} = \tilde{K}_2 - 1} \frac{\binom{\tilde{K}_2 - 1}{r_9, r_{10}, r_{11}} (i_1 + r_{11})! \left(\log \frac{y}{n_1 eg}\right)^{r_{10}}}{(w-1)^{i_1 + r_{11} + 1}}$$

$$\frac{d^{r_9}}{dz^{r_9}} \left\{ (z + w - 1)^{\tilde{K}_2} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; \nu_2) \right\}_{s,u,v=1} \right\}_{z=1-w}$$

$$+ \frac{i_1!}{2\pi i} \int_{L(1,c_1/3)} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; \nu_2) \right\}_{s,u,v=1} \frac{\left(\frac{y}{n_1 eg}\right)^z dz}{z^{i_1+1}}.$$

Inserting this into (20.49), we have

$$\mathcal{T}_2 = \frac{i_1!j_5![\nu_2 = 0 \Rightarrow j_3, j_4, j_5 = 0]}{2\pi i y} \sum_{\substack{r_5+r_6+r_7=j_5 \\ r_8 \le \tilde{K}_1 - 1}} \frac{(i_2+r_7)!(-\log y)^{r_6}}{r_5!r_6!r_7!r_8!(\tilde{K}_1 - 1 - r_8)!}$$

$$\times \left(\log \frac{y}{n_1 eg}\right)^{\tilde{K}_1 - 1 - r_8} \int_{L(0,c_2)} \frac{\mathcal{Z}_1^{\nu_2}(w, eg) y^w}{(w-1)^{i_2+r_7+1}}$$

$$\times \frac{d^{r_8}}{dz^{r_8}} \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ z^{\tilde{K}_1 - i_1 - 1} \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; \nu_2) \right\}_{\substack{s,u,v=1 \\ z=0}} dw$$

$$- \frac{(-1)^{i_1}[\nu_2 = 1]}{2\pi i} \sum_{\substack{r_5+r_6+r_7=j_5 \\ r_9+r_{10}+r_{11}=\tilde{K}_2-1}} \frac{j_5!(i_1+r_{11})!(i_2+r_7)!(-\log y)^{r_6}}{r_5!r_6!r_7!r_9!r_{10}!r_{11}!} \left(\log \frac{y}{n_1 eg}\right)^{r_{10}}$$

$$\times \int_{L(0,c_2)} \frac{\mathcal{Z}_1^{\nu_2}(w, eg)(n_1 eg)^{w-1}}{(w-1)^{i_1+i_2+r_7+r_{11}+2}} \frac{d^{r_9}}{dz^{r_9}} \Big\{ (z+w-1)^{\tilde{K}_2}$$
$$\times \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; 1) \right\}_{s,u,v=1} \Big\}_{z=1-w} dw$$

$$+ \frac{i_1!j_5![\nu_2 = 0 \Rightarrow j_3, j_4, j_5 = 0]}{(2\pi i)^2 y} \sum_{r_5+r_6+r_7=j_5} \frac{(-\log y)^{r_6}(i_2+r_7)!}{r_5!r_6!r_7!}$$
$$\times \int_{L(1,c_1/3)} \int_{L_{1-z}(0,c_2)} \frac{\mathcal{Z}_1^{\nu_2}(w, eg) \left(\frac{y}{n_1 eg}\right)^z y^w}{(w-1)^{i_2+r_7+1} z^{i_1+1}}$$

$$\times \frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; \nu_2) \right\}_{s,u,v=1} dwdz.$$

For sufficiently small $0 < c_2 < c_1/3$, $L(0, c_2)$ and $L_{1-z}(0, c_2)$ lie in the right of $\mathcal{L}(0)$, via

Cauchy's theorem, we've replaced the last contour by the first and the second one. By (20.27), (20.35) and (20.43),

$$\mathcal{Z}_1(w, eg) \ll (\log\log y)^A \log(|\Im w| + 4) \tag{20.72}$$

for w on $L(0, c_2)$ or $L_{1-z}(0, c_2)$. Together with this, we use the following Cauchy integral applications,

$$\frac{d^{r_8}}{dz^{r_8}}\frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{r_5}}{dv^{r_5}}\left\{z^{\tilde{K}_1-i_1-1}\mathcal{Z}_2(z, w-v, u, s, v; n_1eg; 0)\right\}_{\substack{s,u,v=1\\ z=0}}$$
$$= \frac{r_8![j_3, j_4, r_5 = 0]}{2\pi i}\int_{|z|\asymp(\log\log y)^{-1}} \frac{\mathcal{Z}_2(z, w-1, 1, 1, 1; n_1eg; 0)dz}{z^{r_8+i_1+2-\tilde{K}_1}}$$

$$\ll [j_3, j_4, j_5 = 0](\log\log y)^A,$$

$$\frac{d^{r_8}}{dz^{r_8}}\frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{r_5}}{dv^{r_5}}\left\{z^{\tilde{K}_1-i_1-1}\mathcal{Z}_2(z, w-v, u, s, v; n_1eg; 1)\right\}_{\substack{s,u,v=1\\ z=0}} = \frac{r_8!j_3!j_4!r_5!}{(2\pi i)^4}$$
$$\times \iiiint_{\substack{|v-1|,|z|\asymp\frac{1}{\log(|\Im w|+4)+\log\log y}\\ |s-1|=|u-1|=\epsilon}} \frac{\mathcal{Z}_2(z, w-v, u, s, v; n_1eg; 1)dvdzduds}{z^{r_8+i_1+2-\tilde{K}_1}(s-1)^{j_3+1}(u-1)^{j_4+1}(v-1)^{r_5+1}}$$

$$\ll A^{j_3+j_4+r_5}j_3!j_4!r_5!(\log\log y)^A(\log(|\Im w|+4)+\log\log y)^{r_5+r_8+i_1-\tilde{K}_1},$$

$$\frac{d^{r_9}}{dz^{r_9}}\left\{(z+w-1)^{\tilde{K}_2}\frac{d^{j_3}}{ds^{j_3}}\frac{d^{j_4}}{du^{j_4}}\frac{d^{r_5}}{dv^{r_5}}\left\{\mathcal{Z}_2(z, w-v, u, s, v; n_1eg; 1)\right\}_{s,u,v=1}\right\}_{z=1-w}$$
$$= \frac{r_9!j_3!j_4!r_5!}{(2\pi i)^4}\iiiint_{\substack{|z+w-1|\asymp(\log(|\Im w|+4)+\log\log y)^{-1}\\ |v-1|=|s-1|=|u-1|=\epsilon}} \mathcal{Z}_2(z, w-v, u, s, v; n_1eg; 1)$$

$$\times\frac{dvdzduds}{(z+w-1)^{r_9+1-\tilde{K}_2}(s-1)^{j_3+1}(u-1)^{j_4+1}(v-1)^{r_5+1}},$$

$$\ll A^{j_3+j_4+j_5} r_9! j_3! j_4! r_5! (\log(|\Im w|+4) + \log\log y)^{r_9+3-\tilde{K}_2} (\log\log y)^A,$$

$$\frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; 0) \right\}_{s,u,v=1}$$
$$\ll [j_3, j_4, j_5 = 0] \log(|\Im z|+4)(\log\log y)^A,$$

$$\frac{d^{j_3}}{ds^{j_3}} \frac{d^{j_4}}{du^{j_4}} \frac{d^{r_5}}{dv^{r_5}} \left\{ \mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; 1) \right\}_{s,u,v=1} = \frac{j_3! j_4! r_5!}{(2\pi i)^3}$$
$$\times \iiint_{\substack{|v-1| \asymp (\log(|\Im w|+|\Im z|+4)+\log\log y)^{-1} \\ |s-1|=|u-1|=\epsilon}} \frac{\mathcal{Z}_2(z, w-v, u, s, v; n_1 eg; 1)\, du ds dv}{(s-1)^{j_3+1}(u-1)^{j_4+1}(v-1)^{r_5+1}}$$

$$\ll A^{j_3+j_4+r_5} j_3! j_4! r_5! (\log(|\Im z|+|\Im w|+4) + \log\log y)^{r_5+3} (\log\log y)^A,$$

where $z \in L(1, c_1/3)$ and $w \in L(0, c_2)$ in all applications except the last one in which $w \in L_{1-z}(0, c_2)$, we have appealed to (2.13), (20.27), (20.43) and (20.50), so that the first two integrals immediately and the double integral after the use of the trivial inequality, $\log(|\Im z|+|\Im w|+4) \leq (\log(|\Im z|+4))(\log(|\Im w|+4))$, take the general form in Lemma 20.1. As a result,

$$\mathcal{T}_2 \ll A^{j_3+j_4+j_5} j_3! j_4! j_5! \Bigg[\sum_{\substack{r_5+r_6+r_7=j_5 \\ r_8 \leq \tilde{K}_1 - 1}} \frac{(\log y)^{r_6+\tilde{K}_1-1-r_8} (\log\log y)^{\nu_2 r_5 + r_7 + A}}{r_6!}$$
$$\times \left(y^{-c_2/\log\log y} + \frac{1}{(\log y)^{i_2+r_7}} \right)$$

$$+ \sum_{\substack{r_5+r_6+r_7=j_5 \\ r_9+r_{10}+r_{11}=\tilde{K}_2-1}} \frac{(\log y)^{r_6+r_{10}} (\log\log y)^{r_7+A}}{r_6! r_{10}!} \Big((n_1 eg)^{\frac{-c_2}{\log\log y}}$$

$$+\frac{1}{(\log y)^{i_1+i_2+r_7+r_{11}+1}}\Big)+\sum_{r_5+r_6+r_7=j_5}\frac{(\log y)^{r_6}(\log\log y)^{2\nu_2 r_5+r_7+A}}{r_6!}$$
$$\times\left(y^{-c_2/\log\log y}+\frac{1}{(\log y)^{i_2+r_7}}\right)\left((y/(n_1eg))^{-c_1/(3\log\log y)}+\frac{1}{(\log y)^{i_1}}\right)\Bigg].$$

In the above, we ignore the quantity $y^{-c_2/\log\log y}$ since it decays more rapidly than $(\log y)^{-i_2-r_7}$. The first sum in the bound for $\mathcal{T}_2$ can be considered as the product of two independent sums since $\tilde{K}_1\leq i_1-\nu_2$, while the second sum depends on r_5 because $\tilde{K}_2\leq r_5+1$. With the same way seen in (20.31) we treat the above sums, for example,

$$\sum_{r_5+r_6+r_7=j_5}\frac{(\log y)^{r_6}(\log\log y)^{r_5+r_7}}{r_6!}\ll\frac{A^{j_5}(\log y)^{j_5}}{j_5!},$$

$$\sum_{\substack{r_5+r_6+r_7=j_5\\ r_9+r_{10}+r_{11}=\tilde{K}_2-1}}\frac{(\log y)^{r_6+r_{10}}(\log\log y)^{r_7}}{r_6!r_{10}!}=\sum_{r_5+r_6+r_7=j_5}\frac{(\log y)^{r_6}(\log\log y)^{r_7}}{r_6!}$$
$$\times\sum_{r_9+r_{10}+r_{11}=\tilde{K}_2-1}\frac{(\log y)^{r_{10}}}{r_{10}!}\ll\frac{A^{j_5}(\log y)^{j_5}}{j_5!},$$

$\mathcal{T}_2$ then becomes

$$\mathcal{T}_2\ll A^{j_3+j_4+j_5}j_3!j_4!(\log\log y)^A\Bigg((\log y)^{j_5+i_1-\nu_2-2}$$
$$+(n_1eg)^{\frac{-c_2}{\log\log y}}(\log y)^{j_5}+(y/(n_1eg))^{-c_1/(3\log\log y)}(\log y)^{j_5-1}\Bigg).\quad(20.73)$$

Combining the above error term with that of (20.71), we deduce that

$$S_{\nu_2}(f)=2i_1!(i_2!)^{\nu_2}[2|n_1eg\text{ and }j_3,j_4=0\text{ and }j_5\leq\nu_2]\quad(20.74)$$
$$\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|eg}\left(1-\frac{1}{p}\right)^{-\nu_2}\prod_{\substack{p|n_1eg\\ p>2}}\left(1+\frac{1}{p-2}\right)$$
$$\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\ \nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2r_2-1)!(r_3!)^{\nu_2}}\left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1}$$

$$+ O\Bigg(A^{j_3+j_4+j_5} j_3! j_4! (\log\log y)^A \Big(j_5! (\log y)^{i_1+i_2-\nu_2[j_5=0]+\nu_2[j_5\geq 2 \text{ or } j_3+j_4\geq 1]-2}$$

$$+ (\log y)^{j_5+i_1-\nu_2-2} + (j_5! (\log y)^{i_2-\nu_2[j_5=0]} + (\log y)^{j_5-1}) \left(\frac{y}{n_1 eg}\right)^{-c_1/(3\log\log 3y)}$$

$$+ (n_1 eg)^{\frac{-c_2}{\log\log y}} (\log y)^{j_5} \Big)\Bigg).$$

After this lengthy calculation, we return to our main task $F_{\nu_1,\nu_2}(\nu)$ and continue with $S_{\nu_2}(n_1)$. Firstly, the components comprising $S_{\nu_2}(f)$ naturally divides $S_{\nu_2}(n_1)$ into four parts. Further, $[2|n_1 eg]$ suggests dividing some n_1-sums into two parts according to $2|n_1$ or not. So,

$$S_{\nu_2}(n_1) = S_{\nu_2}^{(1)}(n_1) + S_{\nu_2}^{(2)}(n_1) + S_{\nu_2}^{(3)}(n_1) + S_{\nu_2}^{(4)}(n_1) + S_{\nu_2}^{(5)}(n_1),$$

where

$$S_{\nu_2}^{(1)}(n_1) = 2 i_1! (i_2!)^{\nu_2} [j_3, j_4 = 0 \text{ and } j_5 \leq \nu_2 \text{ and } 2 \nmid eg]$$
$$\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) F_2(eg;\nu_2) \sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0] \\ \nu_2=0 \Rightarrow r_2=0}} \frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2 r_2 - 1)!(r_3!)^{\nu_2}} \qquad (20.75)$$

$$\sum_{\substack{n_1 \leq \frac{y}{2eg} \\ (n_1, 2eg)=1}} \frac{\mu^2(n_1) F_2(n_1;1)}{n_1} \left(\log \frac{n_1 T}{\pi}\right)^{j_2} \left(\log \frac{y}{2n_1 eg}\right)^{i_1+\nu_2 r_2 - 1},$$

$$S_{\nu_2}^{(2)}(n_1) = 2^{1+\nu_2} i_1! (i_2!)^{\nu_2} [j_3, j_4 = 0 \text{ and } j_5 \leq \nu_2 \text{ and } 2|eg]$$
$$\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) F_2(eg;\nu_2) \sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0] \\ \nu_2=0 \Rightarrow r_2=0}} \frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2 r_2 - 1)!(r_3!)^{\nu_2}} \qquad (20.76)$$

$$\sum_{\substack{n_1\le \frac{y}{eg}\\(n_1,2eg)=1}} \frac{\mu^2(n_1)F_2(n_1;1)}{n_1}\left(\log\frac{n_1T}{2\pi}\right)^{j_2}\left(\log\frac{y}{n_1eg}\right)^{i_1+\nu_2r_2-1},$$

$$S^{(3)}_{\nu_2}(n_1) \ll A^{j_2+j_3+j_4+j_5} j_3!j_4!\Big(j_5!(\log T)^{j_2+i_1+i_2-\nu_2[j_5=0]+\nu_2[j_5\ge 2 \text{ or } j_3+j_4\ge 1]-2} + (\log T)^{j_2+j_5+i_1-\nu_2-2}\Big)(\log\log T)^A \sum_{\substack{n_1\le \frac{y}{eg}\\(n_1,eg)=1}} \frac{\mu^2(n_1)}{n_1},$$

$$S^{(4)}_{\nu_2}(n_1) \ll A^{j_2+j_3+j_4+j_5} j_3!j_4!\left(j_5!(\log T)^{j_2+i_2-\nu_2[j_5=0]} + (\log T)^{j_2+j_5-1}\right) (\log\log T)^A \sum_{\substack{n_1\le \frac{y}{eg}\\(n_1,eg)=1}} \frac{\mu^2(n_1)}{n_1}\left(\frac{y}{n_1eg}\right)^{-c_1/(3\log\log 3y)},$$

$$S^{(5)}_{\nu_2}(n_1) \ll A^{j_2+j_3+j_4+j_5} j_3!j_4!(\log T)^{j_2+j_5}(\log\log T)^A \times \sum_{\substack{n_1\le \frac{y}{eg}\\(n_1,eg)=1}} \frac{\mu^2(n_1)}{n_1}(n_1eg)^{\frac{-c_2}{\log\log 3y}},$$

where $F_2(n;\nu_2) = \prod_{p|n,\, p>2}\left(1+\frac{1+\nu_2}{p-2}\right)$ and we've used (20.27) in the removal of the F_1-function from some n_1-sums. Employing the well-known estimate

$$\sum_{n_1\le y} \frac{\mu^2(n_1)}{n_1} = \frac{6\log y}{\pi^2} + O(1), \tag{20.77}$$

we have

$$S^{(3)}_{\nu_2}(n_1) \ll A^{j_2+j_3+j_4+j_5} j_3!j_4!\Big(j_5!(\log T)^{j_2+i_1+i_2-\nu_2[j_5=0]+\nu_2[j_5\ge 2 \text{ or } j_3+j_4\ge 1]-1} + (\log T)^{j_2+j_5+i_1-\nu_2-1}\Big)(\log\log T)^A.$$

$S^{(4)}_{\nu_2}(n_1)$ and $S^{(5)}_{\nu_2}(n_1)$ are of the same type. We only deal with $S^{(4)}_{\nu_2}(n_1)$. Splitting the range of n_1 into $O(\log y)$ intervals, $[y/(2^{v+1}eg),\, y/(2^v eg)]$, where $0 \le v \ll \log y$, in which it holds that $2^v \le y/(n_1 eg) \le 2^{v+1}$ and then $(y/(n_1 eg))^{\frac{-c_1}{3\log\log y}} \le \exp\left(-c_1' v/\log\log y\right)$ for some $c_1' > 0$. Thus, by (20.77)

$$S^{(4)}_{\nu_2}(n_1) \ll A^{j_2+j_3+j_4+j_5} j_3! j_4! \left(j_5!(\log T)^{j_2+i_2-\nu_2[j_5=0]} + (\log T)^{j_2+j_5-1}\right)$$
$$(\log\log T)^A \sum_{0\le v\ll \log y} \exp\left(\frac{-c_1' v}{\log\log y}\right) \sum_{\frac{y}{2^{v+1}eg}\le n_1 \le \frac{y}{2^v eg}} \frac{\mu^2(n_1)}{n_1}$$

$$\ll A^{j_2+j_3+j_4+j_5} j_3! j_4! \left(j_5!(\log T)^{j_2+i_2-\nu_2[j_5=0]} + (\log T)^{j_2+j_5-1}\right)$$
$$(\log\log T)^A \sum_{0\le v\ll \log y} \exp\left(\frac{-c_1' v}{\log\log y}\right).$$

If we consider the $v-$sum as a geometric series, then we see that

$$\sum_{0\le v\ll \log y} \exp\left(\frac{-c_1' v}{\log\log y}\right) \le \left(1 - \exp\left(\frac{c_1'}{\log\log y}\right)\right)^{-1} \ll \log\log y.$$

As a result,

$$S^{(4)}_{\nu_2}(n_1) \ll A^{j_2+j_3+j_4+j_5} j_3! j_4!$$
$$\left(j_5!(\log T)^{j_2+i_2-\nu_2[j_5=0]} + (\log T)^{j_2+j_5-1}\right)(\log\log T)^A. \quad (20.78)$$

Performing the similar dyadic argument introduced above, we obtain

$$S^{(5)}_{\nu_2}(n_1) \ll A^{j_2+j_3+j_4+j_5} j_3! j_4! (\log T)^{j_2+j_5} (\log\log T)^A.$$

From [27] we use the results we need in the estimations of $S^{(1)}_{\nu_2}(n_1)$ and $S^{(2)}_{\nu_2}(n_1)$:

for $\iota = 0, 1$,

$$\sum_{\substack{n_1 \le \frac{y}{2^\iota eg} \\ (n_1, 2eg)=1}} \frac{\mu^2(n_1) F_2(n_1; 1)}{n_1} = \frac{1}{\mathfrak{G}_2(2eg)} \left(\log \frac{y}{2^\iota eg} + O\left(\log\log 3eg\right) \right)$$
$$+ O\left(\exp\left(\frac{c_3 \sqrt{\log 2eg}}{\log\log 3eg} \right) \left(\frac{y}{eg} \right)^{-1/2} \right), \quad (20.79)$$

which follows from Lemma 2.2, (2.16), (2.17) and (2.19) in [27]. Here

$$\mathfrak{G}_2(n) = \begin{cases} 2\prod_{p>2}\left(1-(p-1)^{-2}\right)\prod_{p|n,\, p>2}\frac{p-1}{p-2} & \text{if } 2|n,\ n \ne 0, \\ 0 & \text{if } n \text{ is odd.} \end{cases} \quad (20.80)$$

If $y/(eg) \ll \exp\left(\frac{2c_3 \sqrt{\log 2eg}}{\log\log 3eg} \right)$, then by (20.27) and (20.77),

$$\sum_{\substack{n_1 \le \frac{y}{2^\iota eg} \\ (n_1, 2eg)=1}} \frac{\mu^2(n_1) F_2(n_1; 1)}{n_1} \ll (\log\log y)^A \sqrt{\log 2eg},$$

from which (20.79) simplifies to

$$\sum_{\substack{n_1 \le \frac{y}{2^\iota eg} \\ (n_1, 2eg)=1}} \frac{\mu^2(n_1) F_2(n_1; 1)}{n_1} = \frac{1}{\mathfrak{G}_2(2eg)} \log \frac{y}{2^\iota eg} + O\left((\log\log y)^A \sqrt{\log 2eg} \right). \quad (20.81)$$

Based on this result, with the help of Lemma 19.2, we settle the n_1-sums in $S^{(1)}_{\nu_2}(n_1)$ and $S^{(2)}_{\nu_2}(n_1)$;

$$\sum_{\substack{n_1 \le \frac{y}{2eg} \\ (n_1, 2eg)=1}} \frac{\mu^2(n_1) F_2(n_1; 1)}{n_1} \left(\log \frac{n_1 T}{\pi} \right)^{j_2} \left(\log \frac{y}{2 n_1 eg} \right)^{i_1 + \nu_2 r_2 - 1}$$
$$(20.82)$$

$$= \frac{1}{\mathfrak{S}_2(2eg)} \sum_{\kappa \le j_2} \binom{j_2}{\kappa} \left(\log \frac{T}{\pi}\right)^{j_2-\kappa} \left(\log \frac{y}{2eg}\right)^{i_1+\nu_2 r_2+\kappa} B(\kappa+1, i_1+\nu_2 r_2)$$
$$\left(1 + O\left(\frac{(\log\log y)^A \sqrt{\log 2eg}}{\log \frac{y}{2eg}}\right)\right)$$

and

$$\sum_{\substack{n_1 \le \frac{y}{eg} \\ (n_1, 2eg)=1}} \frac{\mu^2(n_1) F_2(n_1;1)}{n_1} \left(\log \frac{n_1 T}{2\pi}\right)^{j_2} \left(\log \frac{y}{n_1 eg}\right)^{i_1+\nu_2 r_2-1} \tag{20.83}$$

$$= \frac{1}{\mathfrak{S}_2(2eg)} \sum_{\kappa \le j_2} \binom{j_2}{\kappa} \left(\log \frac{T}{2\pi}\right)^{j_2-\kappa} \left(\log \frac{y}{eg}\right)^{i_1+\nu_2 r_2+\kappa} B(\kappa+1, i_1+\nu_2 r_2)$$
$$\left(1 + O\left(\frac{(\log\log y)^A \sqrt{\log 2eg}}{\log \frac{y}{eg}}\right)\right).$$

Here we ignore the factor $j_2 + i_1 + \nu_2 r_2 + 1$ in the error term coming from Lemma 19.2 since $j_2 + i_1 + \nu_2 r_2 + 1 \ll j_2 \le \nu + 2\nu_1 + \nu_2 \ll \log\log T$. From the mean value theorem of elementary calculus we see that

$$\left(\log \frac{y}{2eg}\right)^{i_1+\nu_2 r_2-1} = \left(\log \frac{y}{eg}\right)^{i_1+\nu_2 r_2-1} \left(1 + O\left(\frac{1}{\log \frac{y}{eg}}\right)\right) \tag{20.84}$$

and

$$\left(\log \frac{T}{\pi}\right)^{j_2-\kappa} = \left(\log \frac{T}{2\pi}\right)^{j_2-\kappa} \left(1 + O\left(\frac{\log\log T}{\log T}\right)\right). \tag{20.85}$$

Combining (20.75), (20.76), (20.82), (20.83), (20.84) and (20.85), we have

$$S^{(1)}_{\nu_2}(n_1) + S^{(2)}_{\nu_2}(n_1) = i_1!(i_2!)^{\nu_2} \left(F_1(eg)\right)^{\nu_2} [j_3, j_4 = 0 \,\text{and}\, j_5 \le \nu_2]$$
$$\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0] \\ \nu_2=0 \Rightarrow r_2=0}} \frac{(-1)^{r_2\nu_2} (\log y)^{r_3}}{(i_1+\nu_2 r_2-1)!(r_3!)^{\nu_2}}$$

$$\sum_{\kappa\le j_2}\binom{j_2}{\kappa}\left(\log\frac{T}{2\pi}\right)^{j_2-\kappa}\left(\log\frac{y}{eg}\right)^{i_1+\nu_2r_2+\kappa}B(\kappa+1,i_1+\nu_2r_2)$$
$$\left(1+O\left(\frac{(\log\log y)^A\sqrt{\log 2eg}}{\log\frac{y}{eg}}\right)\right).$$

Summing up the all results regarding the n_1-sum, we deduce that

$$S_{\nu_2}(n_1)=i_1!(i_2!)^{\nu_2}\left(F_1(eg)\right)^{\nu_2}[j_3,j_4=0\,\text{and}\,j_5\le\nu_2]$$
$$\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\ \nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2r_2-1)!(r_3!)^{\nu_2}}$$

$$\sum_{\kappa\le j_2}\binom{j_2}{\kappa}\left(\log\frac{T}{2\pi}\right)^{j_2-\kappa}\left(\log\frac{y}{eg}\right)^{i_1+\nu_2r_2+\kappa}B(\kappa+1,i_1+\nu_2r_2)$$
$$\left(1+O\left(\frac{(\log\log y)^A\sqrt{\log 2eg}}{\log\frac{y}{eg}}\right)\right)$$

$$+O\Bigg(A^{j_2+j_3+j_4+j_5}j_3!j_4!\Big(j_5!(\log T)^{j_2+i_1+i_2-\nu_2[j_5=0]+\nu_2[j_5\ge 2\,\text{or}\,j_3+j_4\ge 1]-1}$$
$$+(\log T)^{j_2+j_5+i_1-\nu_2+\nu_2[i_1=1]-1}\Big)(\log\log T)^A\Bigg).$$

Together with this result on $S_{\nu_2}(n_1)$, we come back to (20.30) and then

$$F_{\nu_1,\nu_2}(\nu)=\Xi_1+\Xi_2, \tag{20.86}$$

where

$$\Xi_1 = \frac{\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}}{(\nu+2\nu_1+\nu_2)!}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right)\sum_{i_1,i_2=1}^{k}\frac{a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}}{(\log y)^{i_1+i_2}}\sum_{\mathfrak{a}=0}^{2}\binom{2}{\mathfrak{a}} \tag{20.87}$$
$$\sum_{\substack{j_1+j_2+j_5=\nu+2\nu_1+\nu_2\\ j_5\le\nu_2}}\binom{\nu+2\nu_1+\nu_2}{j_1,j_2,j_5}\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\ \nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2r_2-1)!(r_3!)^{\nu_2}}$$
$$\sum_{\kappa\le j_2}\binom{j_2}{\kappa}\left(\log\frac{T}{2\pi}\right)^{j_2-\kappa}B(\kappa+1,i_1+\nu_2r_2)\sum_{\ell\le j_1-(2-\mathfrak{a})\nu_1}\frac{1}{\ell!}\sum_{g\le y}\frac{\mu(g)F_1(g)\Lambda^{*(\ell)}(g)}{g}$$
$$\sum_{\substack{e\le\frac{y}{g}\\(e,g)=1}}\frac{\Lambda_{2-\mathfrak{a}}(e)I_{\nu_1}(e)\mu(e)}{e}G^{(j_1)}_{\nu_1,\nu_2}(1;e,g;\nu;\mathfrak{a})F_1(e)\left(F_1(eg)\right)^{\nu_2}$$
$$\left(\log\frac{y}{eg}\right)^{i_1+\nu_2r_2+\kappa}\left(1+O\left(\frac{(\log\log y)^A\sqrt{\log 2eg}}{\log\frac{y}{eg}}\right)\right)$$

and

$$\Xi_2 \ll A^\nu T^{1-\epsilon}+A^\nu T(\log T)^{-\nu-2\nu_1+\nu_2+1}(\log\log T)^A\sum_{i_1,i_2=1}^{k}\frac{1}{(\log T)^{i_2}}\sum_{\mathfrak{a}=0}^{2}$$
$$\sum_{j_1+j_2+j_3+j_4+j_5=\nu+2\nu_1+\nu_2}\Big(\frac{(\log T)^{j_2+i_2-\nu_2[j_5=0]+\nu_2[j_5\ge2\text{ or }j_3+j_4\ge1]}}{j_1!j_2!}$$
$$+\frac{(\log T)^{j_2+j_5-\nu_2+\nu_2[i_1=1]}}{j_1!j_2!j_5!}\Big)\sum_{\ell\le j_1-(2-\mathfrak{a})\nu_1}\frac{1}{\ell!}\sum_{g\le y}\frac{|\mu(g)|\Lambda^{*(\ell)}(g)}{g}$$
$$\sum_{\substack{e\le\frac{y}{g}\\(e,g)=1}}\frac{\Lambda_{2-\mathfrak{a}}(e)I_{\nu_1}(e)|\mu(e)|}{e}|G^{(j_1)}_{\nu_1,\nu_2}(1;e,g;\nu;\mathfrak{a})|.$$

We separate Ξ_1 into two parts according to ℓ values: $\ell=j_1-(2-\mathfrak{a})\nu_1$ and $\ell<j_1-(2-\mathfrak{a})\nu_1$. While in the first part we appeal to (20.26) for $G^{(j_1)}_{\nu_1,\nu_2}(1;e,g;\nu;\mathfrak{a})$, in the remaining case we employ (20.29). Thus,

$$\Xi_1=\Xi_1'+\Xi_1'', \tag{20.88}$$

where

$$\Xi_1' = \frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right)\sum_{i_1,i_2=1}^{k}\frac{a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}}{(\log y)^{i_1+i_2}}\sum_{\mathfrak{a}=0}^{2}\binom{2}{\mathfrak{a}}$$

$$(\mathfrak{a}!)^{\nu_1}\sum_{j_5\le\nu_2}\sum_{\substack{j_1+j_2=\nu+2\nu_1+\nu_2-j_5\\ j_1\ge(2-\mathfrak{a})\nu_1}}\frac{\binom{\nu}{j_1-(2-\mathfrak{a})\nu_1}}{j_2!}\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\ \nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}(\log y)^{r_3}}{(i_1+\nu_2r_2-1)!(r_3!)^{\nu_2}}$$

$$\sum_{\kappa\le j_2}\binom{j_2}{\kappa}\left(\log\frac{T}{2\pi}\right)^{j_2-\kappa}B(\kappa+1,i_1+\nu_2r_2)\sum_{g\le y}\frac{\mu(g)F_1(g)\Lambda^{*(j_1-(2-\mathfrak{a})\nu_1)}(g)}{g}$$

$$\sum_{\substack{e\le\frac{y}{g}\\ (e,g)=1}}\frac{\Lambda_{2-\mathfrak{a}}(e)I_{\nu_1}(e)\mu(e)F_1(e)}{e}\left(\log\frac{y}{eg}\right)^{i_1+\nu_2r_2+\kappa}$$

$$\left(1+O\left(\frac{(\log\log y)^A\sqrt{\log 2eg}}{\log\frac{y}{eg}}\right)\right)$$

and

$$\Xi_1'' \ll A^{\nu}T(\log T)^{-\nu-2\nu_1+\nu_2+2}\sum_{i_1,i_2=1}^{k}\sum_{\mathfrak{a}=0}^{2}\sum_{j_5\le\nu_2}\sum_{j_1+j_2=\nu+2\nu_1+\nu_2-j_5}\frac{(\log T)^{j_2-\nu_2[j_5=0]}}{j_2!}$$

$$\sum_{0\le\ell<j_1-(2-\mathfrak{a})\nu_1}\binom{\nu}{\ell}(\log\log T)^{j_1-\ell+A}\sum_{g\le y}\frac{|\mu(g)|\Lambda^{*(\ell)}(g)}{g}$$

$$\sum_{\substack{e\le\frac{y}{g}\\ (e,g)=1}}\frac{\Lambda_{2-\mathfrak{a}}(e)I_{\nu_1}(e)|\mu(e)|}{e}\left(1+O\left(\frac{(\log\log y)^A\sqrt{\log 2eg}}{\log\frac{y}{eg}}\right)\right).$$

Ξ_1'' and Ξ_2 are similarly handled, but the contribution of Ξ_1'' dominates that of Ξ_2. We only treat Ξ_1''. The last error term is trivially $\ll(\log\log T)^A\sqrt{\log T}$. Employing Propositions 18.1 and 18.2 we obtain

$$\Xi_1'' \ll A^{\nu}T(\log T)^{-\nu-2\nu_1+\nu_2+5/2}\sum_{\mathfrak{a}=0}^{2}\sum_{j_5\le\nu_2}\sum_{j_1+j_2=\nu+2\nu_1+\nu_2-j_5}\frac{(\log T)^{j_2-\nu_2[j_5=0]}}{j_2!}$$

$$\sum_{0\le\ell<j_1-(2-\mathfrak{a})\nu_1}\binom{\nu}{\ell}\frac{(\log\log T)^{j_1-\ell+A}(\log T)^{(2-\mathfrak{a})\nu_1+\ell}}{\ell!}.$$

The maximum value of the power of $\log T$ in $\ell-$sum is j_1-1. Proceeding very similarly to (20.31) we see that the amount $A^\nu(\log T)^{j_1-1}(\log\log T)^A/j_1!$ absorbs the $\ell-$sum. Hence,

$$\Xi_1'' \ll A^\nu T\left(\log T\right)^{-\nu-2\nu_1+\nu_2+3/2}\left(\log\log T\right)^A \sum_{j_5\le\nu_2}\sum_{j_1+j_2=\nu+2\nu_1+\nu_2-j_5}\frac{(\log T)^{j_1+j_2-\nu_2[j_5=0]}}{j_1!j_2!}$$

$$= A^\nu T\left(\log T\right)^{-\nu-2\nu_1+\nu_2+3/2}\left(\log\log T\right)^A\sum_{j_5\le\nu_2}\frac{(\log T)^{\nu+2\nu_1}}{(\nu+2\nu_1+\nu_2-j_5)!} \times \sum_{j_1+j_2=\nu+2\nu_1+\nu_2-j_5}\binom{\nu+2\nu_1+\nu_2-j_5}{j_1,j_2} \ll \frac{A^\nu T(\log T)^{\nu_2+3/2}(\log\log T)^A}{\nu!}.$$

We're come to the foremost part Ξ_1' of $F_{\nu_1,\nu_2}(\nu)$ which carries the main term. If we pay attention to how the changes in the values of ν_1 and $\mathfrak{a}$ affect the contributions of the e and g sums in view of Propositions 18.1 and 18.2, we realize that $\nu_1 = 0$ implies that $\mathfrak{a} = 2$, and in the case of $\nu_1 = 1$, considering Ξ_1' as a union of three parts corresponding to the three possible values of $\mathfrak{a}$, the sector $\mathfrak{a} = 0$ cannot reach the order of magnitude the other two sectors produce. If $\nu_1 = 1$ and $\mathfrak{a} = 0$, then by partial summation and the second assertion of Proposition 18.2,

$$\sum_{\substack{e\le\frac{y}{g}\\(e,g)=1}}\frac{\Lambda_2(e)\mu(e)F_1(e)}{e}\left(\log\frac{y}{eg}\right)^{i_1+\nu_2r_2+\kappa}\left(1+O\left(\frac{(\log\log y)^A\sqrt{\log 2eg}}{\log\frac{y}{eg}}\right)\right) \ll (\log y)^{i_1+\nu_2r_2+\kappa+3/2}(\log\log y)^A.$$

Apart from the exceptional case in which $\nu_1 = 1$ and $\mathfrak{a} = 0$, by Proposition 18.2 and Lemma 19.2, we have

$$\sum_{\substack{e\le\frac{y}{g}\\(e,g)=1}}\frac{\Lambda_{2-\mathfrak{a}}(e)I_{\nu_1}(e)\mu(e)F_1(e)\left(\log\frac{y}{eg}\right)^{i_1+\nu_2r_2+\kappa}}{e}\left(1+O\left(\frac{(\log\log y)^A\sqrt{\log 2eg}}{\log\frac{y}{eg}}\right)\right)$$

$$= \frac{[\nu_1 = 0 \Rightarrow \mathfrak{a} = 2] \left(\log \frac{y}{g}\right)^{i_1+\nu_2 r_2+\kappa+(2-\mathfrak{a})\nu_1}}{(-(i_1+\nu_2 r_2+\kappa+1))^{(2-\mathfrak{a})\nu_1}} \left(1 + O\left(\frac{(\log\log y)^A \sqrt{\log y}}{\log \frac{y}{g}}\right)\right).$$

Relying on this distinction, we separate Ξ_1' into two parts and then apply the results on the $e-$sum to the relevant parts so that

$$\Xi_1' = \frac{T}{2^{2-\nu_1}\pi} \left(\log \frac{T}{2\pi}\right)^{-\nu-2\nu_1+\nu_2+2} \left(1 + O\left(\frac{\nu+1}{\log T}\right)\right) \sum_{i_1,i_2=1}^{k} \frac{a_{i_1} a_{i_2} i_1! (i_2!)^{\nu_2}}{(\log y)^{i_1+i_2}}$$

$$\sum_{\substack{2-\nu_1 \le \mathfrak{a} \le 2 \\ j_5 \le \nu_2}} \sum_{\substack{j_1+j_2=\nu+2\nu_1+\nu_2-j_5 \\ j_1 \ge (2-\mathfrak{a})\nu_1}} \frac{\binom{\nu}{j_1-(2-\mathfrak{a})\nu_1}}{j_2!} \sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0] \\ \nu_2=0 \Rightarrow r_2=0}} \frac{(-1)^{r_2\nu_2} (\log y)^{r_3}}{(i_1+\nu_2 r_2 - 1)!(r_3!)^{\nu_2}}$$

$$\sum_{\kappa \le j_2} \binom{j_2}{\kappa} \left(\log \frac{T}{2\pi}\right)^{j_2-\kappa} B(\kappa+1, i_1+\nu_2 r_2) \sum_{g \le y} \frac{\mu(g) F_1(g) \Lambda^{*(j_1-(2-\mathfrak{a})\nu_1)}(g)}{g}$$

$$\frac{\left(\log \frac{y}{g}\right)^{i_1+\nu_2 r_2+\kappa+(2-\mathfrak{a})\nu_1}}{(-(i_1+\nu_2 r_2+\kappa+1))^{(2-\mathfrak{a})\nu_1}} \left(1 + O\left(\frac{(\log\log y)^A \sqrt{\log y}}{\log \frac{y}{g}}\right)\right)$$

$$+ O\Bigg([\nu_1 = 1] A^\nu T (\log T)^{-\nu-2\nu_1+\nu_2+7/2} (\log\log T)^A \sum_{j_5 \le \nu_2}$$

$$\sum_{\substack{j_1+j_2=\nu+2\nu_1+\nu_2-j_5 \\ j_1 \ge 2}} \frac{(\log T)^{j_2-\nu_2[j_5=0]}}{j_2!} \binom{\nu}{j_1-2} \sum_{g \le y} \frac{|\mu(g)| \Lambda^{*(j_1-2)}(g)}{g}\Bigg)$$

$$= \Xi_{1,1}' + \Xi_{1,2}', \qquad \text{say.}$$

It follows from Proposition 18.1 that

$$\Xi_{1,2}' \ll [\nu_1 = 1] A^\nu T (\log T)^{-\nu-2\nu_1+\nu_2+3/2} (\log\log T)^A$$

$$\sum_{j_5 \le \nu_2} \sum_{\substack{j_1+j_2=\nu+2\nu_1+\nu_2-j_5 \\ j_1 \ge 2}} \frac{(\log T)^{j_1+j_2-\nu_2[j_5=0]}}{(j_1-2)! j_2!} \binom{\nu}{j_1-2}.$$

Here $j_1 + j_2 - \nu_2[j_5 = 0]$ cannot exceed $\nu + 2\nu_1$. So,

$$\Xi'_{1,2} \ll \frac{A^\nu T(\log T)^{\nu_2+3/2}(\log\log T)^A}{\nu!} \sum_{j_5 \le \nu_2} \sum_{\substack{j_1+j_2=\nu+2\nu_1+\nu_2-j_5 \\ j_1 \ge 2}} \binom{\nu}{j_1-2}\binom{\nu+2\nu_1+\nu_2-j_5-2}{j_1-2}.$$

By (5.23) of [28], the inner-most sum is

$$= \binom{2\nu+2\nu_1+\nu_2-j_5-2}{\nu} \le A^\nu,$$

so that

$$\Xi'_{1,2} \ll \frac{A^\nu T(\log T)^{\nu_2+3/2}(\log\log T)^A}{\nu!}, \tag{20.89}$$

which is the same as the bound for Ξ''_1.

By Propositions 18.1 and Lemma 19.2 we have

$$\sum_{g \le y} \frac{\mu(g)F_1(g)\Lambda^{*(j_1-(2-\mathfrak{a})\nu_1)}(g)}{g}\left(\log\frac{y}{g}\right)^{i_1+\nu_2r_2+\kappa+(2-\mathfrak{a})\nu_1} \left(1+O\left(\frac{(\log\log y)^A\sqrt{\log y}}{\log\frac{y}{g}}\right)\right)$$

$$= \frac{(-1)^{j_1-(2-\mathfrak{a})\nu_1}(i_1+\nu_2r_2+\kappa+(2-\mathfrak{a})\nu_1)!}{(j_1+i_1+\nu_2r_2+\kappa)!}(\log y)^{j_1+i_1+\nu_2r_2+\kappa} \left(1+O\left(\frac{A^{j_1}(\log\log y)^A}{\sqrt{\log y}}\right)\right).$$

Here the bound in (20.89) also works for the part of $\Xi'_{1,1}$ produced by the error term in the formula for the $g-$sum and occurring when inserting the above into $\Xi'_{1,1}$.

Summing up all the results above, with the replacement of y by $\left(\frac{T}{2\pi}\right)^\theta$ and some plain simplifications of expressions of factorial, we have

$$F_{\nu_1,\nu_2}(\nu) = \mathcal{A}(\nu,\nu_1,\nu_2,P,\theta)\frac{T}{2^{2-\nu_1}\pi}\left(\log\frac{T}{2\pi}\right)^{\nu_2+2}\left(1+O\left(\frac{\nu+1}{\log T}\right)\right) + O\left(\frac{A^\nu T(\log T)^{\nu_2+3/2}(\log\log T)^A}{\nu!}\right) \tag{20.90}$$

where

$$\mathcal{A}(\nu,\nu_1,\nu_2,P,\theta) = \sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{2-\nu_1\leq\mathfrak{a}\leq 2}\sum_{j_5\leq\nu_2}\theta^{-\nu_2[j_5=0]} \sum_{\substack{j_1+j_2=\nu+2\nu_1+\nu_2-j_5\\ j_1\geq(2-\mathfrak{a})\nu_1}}\binom{\nu}{j_1-(2-\mathfrak{a})\nu_1}(-\theta)^{j_1}\sum_{\kappa\leq j_2}\frac{\theta^\kappa}{(j_2-\kappa)!} \sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\ \nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}}{(r_3!)^{\nu_2}(j_1+i_1+\nu_2r_2+\kappa)!}.$$

By (5.16) in [28] we see that

$$\sum_{\substack{r_2+r_3=i_2-\nu_2[j_5=0]\\ \nu_2=0\Rightarrow r_2=0}}\frac{(-1)^{r_2\nu_2}}{(r_3!)^{\nu_2}(j_1+i_1+\nu_2r_2+\kappa)!} = \frac{1}{(j_1+i_1+\kappa+\nu_2(i_2-[j_5=0]))!}\binom{j_1+i_1+\kappa+\nu_2(i_2-[j_5=0])-1}{\nu_2(i_2-[j_5=0])},$$

from which, together with the substitution $\kappa' = j_1+\kappa$, $\mathcal{A}(\nu,\nu_1,\nu_2,P,\theta)$ becomes

$$\mathcal{A}(\nu,\nu_1,\nu_2,P,\theta) = \sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{2-\nu_1\leq\mathfrak{a}\leq 2}\sum_{j_5\leq\nu_2}\theta^{-\nu_2[j_5=0]} \sum_{\substack{\kappa'\leq\nu+2\nu_1+\nu_2-j_5\\ \kappa'\geq(2-\mathfrak{a})\nu_1}}\frac{\binom{i_1+\kappa'+\nu_2(i_2-[j_5=0])-1}{\nu_2(i_2-[j_5=0])}\theta^{\kappa'}}{(\nu+2\nu_1+\nu_2-j_5-\kappa')!(i_1+\kappa'+\nu_2(i_2-[j_5=0]))!} \sum_{\substack{j_1\leq\kappa'\\ j_1\geq(2-\mathfrak{a})\nu_1}}\binom{\nu}{j_1-(2-\mathfrak{a})\nu_1}(-1)^{j_1}.$$

Again consulting (5.16) in [28] we have

$$\sum_{\substack{j_1\le\kappa'\\ j_1\ge(2-\mathfrak{a})\nu_1}}\binom{\nu}{j_1-(2-\mathfrak{a})\nu_1}(-1)^{j_1}=(-1)^{(2-\mathfrak{a})\nu_1}\sum_{j_1'\le\kappa'-(2-\mathfrak{a})\nu_1}\binom{\nu}{j_1'}(-1)^{j_1'}$$
$$=(-1)^{\kappa'}\binom{\nu-1}{\kappa'-(2-\mathfrak{a})\nu_1},$$

from which it follows that

$$\mathcal{A}(\nu,\nu_1,\nu_2,P,\theta)=\sum_{i_1,i_2=1}^{k}a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{2-\nu_1\le\mathfrak{a}\le2}\sum_{j_5\le\nu_2}\theta^{-\nu_2[j_5=0]}$$
$$\sum_{\substack{\kappa'\le\nu+2\nu_1+\nu_2-j_5\\ \kappa'\ge(2-\mathfrak{a})\nu_1}}\frac{\binom{\nu-1}{\kappa'-(2-\mathfrak{a})\nu_1}\binom{i_1+\kappa'+\nu_2(i_2-[j_5=0])-1}{\nu_2(i_2-[j_5=0])}(-\theta)^{\kappa'}}{(\nu+2\nu_1+\nu_2-j_5-\kappa')!(i_1+\kappa'+\nu_2(i_2-[j_5=0]))!}.$$

We can drop the constraint $\kappa'\ge(2-\mathfrak{a})\nu_1$ because, otherwise, the first binomial coefficient in the numerator is 0. After this notice without hesitation we make the sum over $\mathfrak{a}$ the inner-most and then use

$$\binom{\mathfrak{r}}{\mathfrak{s}}=\binom{\mathfrak{r}-1}{\mathfrak{s}}+\binom{\mathfrak{r}-1}{\mathfrak{s}-1},\qquad(\mathfrak{r},\,\mathfrak{s}\in\mathbb{Z}),$$

so that

$$\mathcal{A}(\nu,\nu_1,\nu_2,P,\theta)=\sum_{i_1,i_2=1}^{k}a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{j_5\le\nu_2}\theta^{-\nu_2[j_5=0]}\tag{20.91}$$
$$\sum_{\kappa'\le\nu+2\nu_1+\nu_2-j_5}\frac{\binom{\nu-1+\nu_1}{\kappa'}\binom{i_1+\kappa'+\nu_2(i_2-[j_5=0])-1}{\nu_2(i_2-[j_5=0])}(-\theta)^{\kappa'}}{(\nu+2\nu_1+\nu_2-j_5-\kappa')!(i_1+\kappa'+\nu_2(i_2-[j_5=0]))!},$$

for which we need an upper bound. Since

$$\binom{i_1+\kappa'+\nu_2(i_2-[j_5=0])-1}{\nu_2(i_2-[j_5=0])}\le\left(\frac{(i_1+\kappa'+\nu_2(i_2-[j_5=0])-1)e}{i_1+\kappa'-1}\right)^{\nu_2(i_1+\kappa'-1)}$$
$$\ll A^{\nu_2\kappa'},$$

$$\mathcal{A}(\nu,\nu_1,\nu_2,P,\theta) \ll \frac{A^\nu}{(\nu+2\nu_1+\nu_2-j_5+i_1+\nu_2(i_2-[j_5=0]))!}$$
$$\sum_{\kappa'\le \nu+2\nu_1+\nu_2-j_5} \binom{\nu+2\nu_1+\nu_2-j_5+i_1+\nu_2(i_2-[j_5=0])}{i_1+\kappa'+\nu_2(i_2-[j_5=0])}\binom{\nu-1+\nu_1}{\kappa'}.$$

By (5.23) of [28] we see that

$$\mathcal{A}(\nu,\nu_1,\nu_2,P,\theta) \ll \frac{A^\nu}{\nu!}\binom{2\nu+3\nu_1+\nu_2-j_5+i_1+\nu_2(i_2-[j_5=0])-1}{\nu+\nu_1+i_1+\nu_2(i_2-[j_5=0])-1} \tag{20.92}$$
$$\ll \frac{A^\nu}{\nu!}\left(\frac{e(2\nu+3\nu_1+\nu_2-j_5+i_1+\nu_2(i_2-[j_5=0])-1)}{\nu+\nu_1+i_1+\nu_2(i_2-[j_5=0])-1}\right)^{\nu+\nu_1+i_1+\nu_2(i_2-[j_5=0])-1}$$
$$\ll \frac{A^\nu}{\nu!},$$

which holds for any $\nu \ge 0$. Using this bound, (20.90) becomes

$$F_{\nu_1,\nu_2}(\nu) = \mathcal{A}(\nu,\nu_1,\nu_2,P,\theta)\frac{T}{2^{2-\nu_1}\pi}\left(\log\frac{T}{2\pi}\right)^{\nu_2+2}$$
$$+O\left(\frac{A^\nu T(\log T)^{\nu_2+3/2}(\log\log T)^A}{\nu!}\right) \tag{20.93}$$

for $\nu \ll \log\log T$.

Returning to (20.11) and (20.12), we apply (20.93) when $\nu \le \log\log T$ and (20.32) when $\log\log T \le \nu \le \frac{\log T}{\log\log T}$, so that

$$\Delta_1 = \Bigg(\sum_{\nu\le\log\log T} 2^\nu\Big(\mathcal{A}(\nu+1,0,0,P,\theta)-4\mathcal{A}(\nu,1,0,P,\theta)-\mathcal{A}(\nu+2,0,0,P,\theta)$$
$$+4\mathcal{A}(\nu+1,1,0,P,\theta)\Big)-\mathcal{A}(0,0,0,P,\theta)+\mathcal{A}(1,0,0,P,\theta)\Bigg)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2$$
$$+O\left(T(\log T)^{3/2}(\log\log T)^A\sum_{\nu\le\log\log T}\frac{A^\nu}{\nu!}\right)+O\left(yT^{1/2+\epsilon}\right)$$
$$+O\left(T\left((\log T)(\log\log T)\right)^A\sum_{\log\log T<\nu\le\frac{\log T}{\log\log T}}\frac{A^\nu}{\nu!}\right)$$

and

$$\Delta_2 = \sum_{\nu \le \log\log T} 2^{\nu+1}\Big(- \mathcal{A}(\nu+2,0,1,P,\theta) + 4\mathcal{A}(\nu+1,1,1,P,\theta)$$
$$+ \mathcal{A}(\nu+3,0,1,P,\theta) - 4\mathcal{A}(\nu+2,1,1,P,\theta)\Big)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^3$$
$$+ \Big(\mathcal{A}(1,0,1,P,\theta) - \mathcal{A}(2,0,1,P,\theta)\Big)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^3$$
$$+ O\left(T(\log T)^{5/2}(\log\log T)^A \sum_{\nu \le \log\log T}\frac{A^\nu}{\nu!}\right) + O\left(yT^{1/2+\epsilon}\right)$$
$$+ O\left(T\left((\log T)(\log\log T)\right)^A \sum_{\log\log T < \nu \le \frac{\log T}{\log\log T}} \frac{A^\nu}{\nu!}\right).$$

We recall $y = (T/(2\pi))^\theta$ and $\theta \le 1/2 - \epsilon$ for the second error terms in Δ_1 and Δ_2. The $\nu-$sums in the first error terms are $\le e^A$. By Stirling's formula,

$$\sum_{\log\log T < \nu \le \frac{\log T}{\log\log T}} \frac{A^\nu}{\nu!} \ll \frac{A^{\lfloor \log\log T\rfloor}}{\lfloor \log\log T\rfloor !} \ll (\log T)^{-\log\log\log T + A}.$$

The next step is to extend the range of $\nu-$sums in the main terms to ∞ within an error term

$$\ll T(\log T)^{\nu_2+2}\sum_{\nu > \log\log T}\frac{A^\nu}{\nu!} \ll T(\log T)^{A-\log\log\log T}$$

by (20.92). So,

$$\Delta_1 = \Big(\sum_{\nu \le \log\log T} 2^\nu\Big(\mathcal{A}(\nu+1,0,0,P,\theta) - 4\mathcal{A}(\nu,1,0,P,\theta) - \mathcal{A}(\nu+2,0,0,P,\theta)$$
$$+ 4\mathcal{A}(\nu+1,1,0,P,\theta)\Big) - \mathcal{A}(0,0,0,P,\theta) + \mathcal{A}(1,0,0,P,\theta)\Big)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2$$
$$+ O\left(T(\log T)^{3/2}(\log\log T)^A\right)$$

and

$$\Delta_2 = \sum_{\nu \leq \log\log T} 2^{\nu+1}\Big(- \mathcal{A}(\nu+2,0,1,P,\theta) + 4\mathcal{A}(\nu+1,1,1,P,\theta)$$
$$+ \mathcal{A}(\nu+3,0,1,P,\theta) - 4\mathcal{A}(\nu+2,1,1,P,\theta)\Big)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^3$$
$$+ \Big(\mathcal{A}(1,0,1,P,\theta) - \mathcal{A}(2,0,1,P,\theta)\Big)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^3$$
$$+ O\left(T(\log T)^{5/2}(\log\log T)^A\right).$$

In some $\nu-$sums making the change of variable $\nu \to \nu+1$ we obtain

$$\Delta_1 = \Big(\sum_{\nu\geq 0} 2^{\nu}\left(\mathcal{A}(\nu+2,0,0,P,\theta) - 4\mathcal{A}(\nu+1,1,0,P,\theta)\right) + 2\mathcal{A}(1,0,0,P,\theta) \tag{20.94}$$
$$- \mathcal{A}(0,0,0,P,\theta) - 4\mathcal{A}(0,1,0,P,\theta)\Big)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2 + O\left(T(\log T)^{3/2}(\log\log T)^A\right)$$

and

$$\Delta_2 = \Big(\sum_{\nu\geq 0} 2^{\nu+1}\left(-\mathcal{A}(\nu+3,0,1,P,\theta) + 4\mathcal{A}(\nu+2,1,1,P,\theta)\right) - 3\mathcal{A}(2,0,1,P,\theta) \tag{20.95}$$
$$+ 8\mathcal{A}(1,1,1,P,\theta) + \mathcal{A}(1,0,1,P,\theta)\Big)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^3 + O\left(T(\log T)^{5/2}(\log\log T)^A\right).$$

It's easy to see that the general form,

$$\mathfrak{U}_{\nu_1,\nu_2} = \sum_{\nu\geq 0} 2^{\nu+2\nu_1+\nu_2}\mathcal{A}(\nu+2-\nu_1+\nu_2,\nu_1,\nu_2,P,\theta), \tag{20.96}$$

covers the four infinite sums in Δ_1 and Δ_2. Adapting (20.91) to our case, we have

$$\mathcal{A}(\nu+2-\nu_1+\nu_2,\nu_1,\nu_2,P,\theta) = \sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{j_5\le\nu_2}\theta^{-\nu_2[j_5=0]}$$
$$\sum_{\kappa'\le\nu+2+\nu_1+2\nu_2-j_5}\frac{\binom{\nu+1+\nu_2}{\kappa'}\binom{i_1+\kappa'+\nu_2(i_2-[j_5=0])-1}{\nu_2(i_2-[j_5=0])}(-\theta)^{\kappa'}}{(\nu+2+\nu_1+2\nu_2-j_5-\kappa')!(i_1+\kappa'+\nu_2(i_2-[j_5=0]))!}.$$

Inserting this into $\mathfrak{U}_{\nu_1,\nu_2}$ and exchanging the order of the $\nu-$ and $\kappa'-$sums, we have

$$\mathfrak{U}_{\nu_1,\nu_2} = \sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{j_5\le\nu_2}\theta^{-\nu_2[j_5=0]}\sum_{\kappa'\ge 0}\frac{\binom{i_1+\kappa'+\nu_2(i_2-[j_5=0])-1}{\nu_2(i_2-[j_5=0])}(-\theta)^{\kappa'}}{\kappa'!(i_1+\kappa'+\nu_2(i_2-[j_5=0]))!}$$
$$\sum_{\substack{\nu\ge\kappa'-1-\nu_2\\ \nu\ge 0}}\frac{(\nu+1+\nu_2)!2^{\nu+2\nu_1+\nu_2}}{(\nu+1+\nu_2-\kappa')!(\nu+2+\nu_1+2\nu_2-j_5-\kappa')!}.$$

To eliminate the second condition on the ν-index we split the $\kappa'-$sum into two parts so that

$$\mathfrak{U}_{\nu_1,\nu_2} = \mathfrak{U}'_{\nu_1,\nu_2} + \mathfrak{U}''_{\nu_1,\nu_2}, \tag{20.97}$$

where

$$\mathfrak{U}'_{\nu_1,\nu_2} = 2^{2\nu_1+\nu_2}\sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{j_5\le\nu_2}\frac{(-1)^{\nu_2[j_5=0]}}{(\nu_2(i_2-[j_5=0]))!}$$
$$\sum_{0\le\kappa'<1+\nu_2}\frac{(-\theta)^{\kappa'-\nu_2[j_5=0]}}{(i_1+\kappa'+\nu_2(i_2-[j_5=0]))(i_1+\kappa'-1)!}$$
$$\sum_{\nu\ge 0}\frac{(\nu+1+\nu_2)!2^{\nu}}{(\nu+1+\nu_2-\kappa')!(\nu+2+\nu_1+2\nu_2-j_5-\kappa')!}$$

and

$$\mathfrak{U}''_{\nu_1,\nu_2} = 2^{2\nu_1+\nu_2} \sum_{i_1,i_2=1}^{k} a_{i_1} a_{i_2} i_1! (i_2!)^{\nu_2} \sum_{j_5 \le \nu_2} \frac{(-1)^{\nu_2[j_5=0]}}{(\nu_2(i_2-[j_5=0]))!}$$
$$\sum_{\kappa' \ge 1+\nu_2} \frac{(-\theta)^{\kappa'-\nu_2[j_5=0]}}{\kappa'!(i_1+\kappa'+\nu_2(i_2-[j_5=0]))(i_1+\kappa'-1)!}$$
$$\sum_{\nu \ge \kappa'-1-\nu_2} \frac{(\nu+1+\nu_2)!2^{\nu}}{(\nu+1+\nu_2-\kappa')!(\nu+2+\nu_1+2\nu_2-j_5-\kappa')!}.$$

We deal with $\mathfrak{U}'_{\nu_1,\nu_2}$ in two cases.

Case 1. $\nu_2 = 0$

$\mathfrak{U}'_{\nu_1,0}$ simplifies to

$$\mathfrak{U}'_{\nu_1,0} = \sum_{\nu \ge 0} \frac{2^{\nu+2\nu_1}}{(\nu+2+\nu_1)!}.$$

From the MacLaurin expansion of the exponential function, we see that

$$\mathfrak{U}'_{\nu_1,0} = \frac{1}{2^{2-\nu_1}} \sum_{\nu \ge 0} \frac{2^{\nu+2+\nu_1}}{(\nu+2+\nu_1)!} = \frac{e^2-3-2\nu_1}{2^{2-\nu_1}}. \tag{20.98}$$

Case 2. $\nu_2 = 1$

It is more appropriate to write the sums with 2 terms explicitly. Then,

$$\begin{aligned}\mathfrak{U}'_{\nu_1,1} = 2^{2\nu_1+1}\Bigg(& \sum_{i_1,i_2=1}^{k} \frac{a_{i_1}a_{i_2}i_1i_2}{\theta(i_1+i_2-1)} \sum_{\nu\ge 0} \frac{2^{\nu}}{(\nu+4+\nu_1)!} \\ & - \sum_{i_1,i_2=1}^{k} \frac{a_{i_1}a_{i_2}i_2}{i_1+i_2} \sum_{\nu\ge 0} \frac{2^{\nu}(\nu+2)}{(\nu+3+\nu_1)!} + \sum_{i_1,i_2=1}^{k} \frac{a_{i_1}a_{i_2}i_1}{i_1+i_2} \sum_{\nu\ge 0} \frac{2^{\nu}}{(\nu+3+\nu_1)!} \\ & - \sum_{i_1,i_2=1}^{k} \frac{\theta a_{i_1}a_{i_2}}{i_1+i_2+1} \sum_{\nu\ge 0} \frac{2^{\nu}(\nu+2)}{(\nu+2+\nu_1)!}\Bigg).\end{aligned} \tag{20.99}$$

It is straightforward to verify that

$$\int_0^1 (P(t))^2\,dt = \sum_{i_1,i_2=1}^{k} \frac{a_{i_1}a_{i_2}}{i_1+i_2+1} \tag{20.100}$$

$$\int_0^1 (P'(t))^2\,dt = \sum_{i_1,i_2=1}^{k} \frac{a_{i_1}a_{i_2}i_1i_2}{i_1+i_2-1} \tag{20.101}$$

$$\frac{1}{2} = \int_0^1 P(t)P'(t)dt = \sum_{i_1,i_2=1}^{k} \frac{a_{i_1}a_{i_2}i_1}{i_1+i_2} = \sum_{i_1,i_2=1}^{k} \frac{a_{i_1}a_{i_2}i_2}{i_1+i_2} \tag{20.102}$$

The first equality in the third identity follows from the condition $P(1) = 1$ and integration by parts. Similar to (20.98),

$$\sum_{\nu\geq 0} \frac{2^\nu}{(\nu+4+\nu_1)!} = \frac{1}{2^{4+\nu_1}} \sum_{\nu\geq 0} \frac{2^{\nu+4+\nu_1}}{(\nu+4+\nu_1)!} = \frac{e^2 - \frac{19+2\nu_1}{3}}{2^{4+\nu_1}}$$

$$\begin{aligned}\sum_{\nu\geq 0} \frac{2^\nu(\nu+2)}{(\nu+3+\nu_1)!} &= \frac{1}{2^{2+\nu_1}} \sum_{\nu\geq 0} \frac{2^{\nu+2+\nu_1}}{(\nu+2+\nu_1)!} - \frac{1+\nu_1}{2^{3+\nu_1}} \sum_{\nu\geq 0} \frac{2^{\nu+3+\nu_1}}{(\nu+3+\nu_1)!} \\ &= \frac{e^2-3-2\nu_1}{2^{2+\nu_1}} - \frac{(1+\nu_1)\left(e^2-5-4\nu_1/3\right)}{2^{3+\nu_1}} \\ &= 6^{-\nu_1}\left(\frac{e^2-1}{2^3}\right)^{1-\nu_1},\end{aligned}$$

$$\sum_{\nu\geq 0} \frac{2^\nu}{(\nu+3+\nu_1)!} = \frac{1}{2^{3+\nu_1}} \sum_{\nu\geq 0} \frac{2^{\nu+3+\nu_1}}{(\nu+3+\nu_1)!} = \frac{e^2-5-4\nu_1/3}{2^{3+\nu_1}},$$

$$\begin{aligned}\sum_{\nu\geq 0} \frac{2^\nu(\nu+2)}{(\nu+2+\nu_1)!} &= \frac{1}{2^{1+\nu_1}} \sum_{\nu\geq 0} \frac{2^{\nu+1+\nu_1}}{(\nu+1+\nu_1)!} - \frac{\nu_1}{2^{2+\nu_1}} \sum_{\nu\geq 0} \frac{2^{\nu+2+\nu_1}}{(\nu+2+\nu_1)!} \\ &= \frac{e^2-1-2\nu_1}{2^{1+\nu_1}} - \frac{\nu_1\left(e^2-3-2\nu_1\right)}{2^{2+\nu_1}} \\ &= \frac{e^2-1}{2^{1+2\nu_1}}.\end{aligned}$$

Inserting all of these into (20.99) we have

$$\mathfrak{U}'_{\nu_1,1} = -\theta(e^2-1)\int_0^1 (P(t))^2\,dt + \frac{3e^2-19-2\nu_1}{3\theta 2^{3-\nu_1}}\int_0^1 (P'(t))^2\,dt - \frac{(9-e^2)^{\nu_1}}{2^{1+\nu_1}}. \tag{20.103}$$

As regards $\mathfrak{U}''_{\nu_1,\nu_2}$, we substitute $\nu' = \nu - \kappa' + 1 + \nu_2$ and then write the inner-most sum in hypergeometric notation so that

$$\mathfrak{U}''_{\nu_1,\nu_2} = 2^{2\nu_1-1}\sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!(i_2!)^{\nu_2}\sum_{j_5\le\nu_2}\frac{(-2)^{\nu_2[j_5=0]}}{(\nu_2(i_2-[j_5=0]))!(1+\nu_1+\nu_2-j_5)!} \tag{20.104}$$
$$\sum_{\kappa'\ge 1+\nu_2}\frac{(-2\theta)^{\kappa'-\nu_2[j_5=0]}F_{1,1}(\kappa'+1;2+\nu_1+\nu_2-j_5;2)}{(i_1+\kappa'+\nu_2(i_2-[j_5=0]))(i_1+\kappa'-1)!}.$$

As a result of (20.97), (20.98), (20.103) and (20.104), we can list the following four results:

$$\mathfrak{U}_{0,0} = \frac{e^2-3}{4} + \frac{1}{2}\sum_{i_1=1}^{k} a_{i_1}i_1!\sum_{\kappa'\ge 1}\frac{(-2\theta)^{\kappa'}F_{1,1}(\kappa'+1;2;2)}{(i_1+\kappa')!}, \tag{20.105}$$

$$\mathfrak{U}_{1,0} = \frac{e^2-5}{2} + \sum_{i_1=1}^{k} a_{i_1}i_1!\sum_{\kappa'\ge 1}\frac{(-2\theta)^{\kappa'}F_{1,1}(\kappa'+1;3;2)}{(i_1+\kappa')!}, \tag{20.106}$$

$$\begin{aligned}\mathfrak{U}_{0,1} =& -\theta(e^2-1)\int_0^1 (P(t))^2\,dt + \frac{3e^2-19}{24\theta}\int_0^1 (P'(t))^2\,dt - \frac{1}{2}\\ &- \sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!i_2!\sum_{j_5\le 1}\frac{(-2)^{[j_5=0]-1}}{(i_2-[j_5=0])!(2-j_5)!}\\ &\sum_{\kappa'\ge 2}\frac{(-2\theta)^{\kappa'-[j_5=0]}}{(i_1+\kappa'+i_2-[j_5=0])(i_1+\kappa'-1)!}F_{1,1}(\kappa'+1;3-j_5;2),\end{aligned} \tag{20.107}$$

$$
\begin{aligned}
\mathfrak{U}_{1,1} =& -\theta(e^2-1)\int_0^1 (P(t))^2\,dt + \frac{e^2-7}{4\theta}\int_0^1 (P'(t))^2\,dt + \frac{e^2-9}{4} \\
& -\sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}i_1!i_2!\sum_{j_5\le 1}\frac{(-2)^{[j_5=0]+1}}{(i_2-[j_5=0])!(3-j_5)!} \\
& \sum_{\kappa'\ge 2}\frac{(-2\theta)^{\kappa'-[j_5=0]}}{(i_1+\kappa'+i_2-[j_5=0])(i_1+\kappa'-1)!}F_{1,1}(\kappa'+1;4-j_5;2).
\end{aligned}
\tag{20.108}
$$

From (20.91), (20.100), (20.101) and (20.102) it follows that

$$
\mathcal{A}(1,0,0,P,\theta)=\mathcal{A}(0,0,0,P,\theta)=1, \qquad \mathcal{A}(0,1,0,P,\theta)=1/2, \tag{20.109}
$$

$$
\begin{aligned}
\mathcal{A}(1,0,1,P,\theta) &= \sum_{i_1,i_2=1}^{k} a_{i_1}a_{i_2}\left(\frac{i_1i_2}{2\theta(i_1+i_2-1)}+\frac{i_1}{i_1+i_2}\right) \\
&= \frac{1}{2\theta}\int_0^1 (P'(t))^2\,dt + \int_0^1 P(t)P'(t)dt = \frac{1}{2}+\frac{1}{2\theta}\int_0^1 (P'(t))^2\,dt,
\end{aligned}
\tag{20.110}
$$

$$
\begin{aligned}
\mathcal{A}(1,1,1,P,\theta) &= \sum_{i_1,i_2=1}^{k}\left(\frac{a_{i_1}a_{i_2}i_1i_2}{24\theta(i_1+i_2-1)}+\frac{a_{i_1}a_{i_2}(i_1-i_2)}{6(i_1+i_2)}-\frac{\theta a_{i_1}a_{i_2}}{2(i_1+i_2+1)}\right) \\
&= \frac{1}{24\theta}\int_0^1 (P'(t))^2\,dt - \frac{\theta}{2}\int_0^1 (P(t))^2\,dt,
\end{aligned}
\tag{20.111}
$$

$$
\begin{aligned}
\mathcal{A}(2,0,1,P,\theta) &= \sum_{i_1,i_2=1}^{k}\left(\frac{a_{i_1}a_{i_2}i_1i_2}{6\theta(i_1+i_2-1)}-\frac{a_{i_1}a_{i_2}i_2}{2(i_1+i_2)}+\frac{a_{i_1}a_{i_2}i_1}{2(i_1+i_2)}-\frac{\theta a_{i_1}a_{i_2}}{(i_1+i_2+1)}\right) \\
&= \frac{1}{6\theta}\int_0^1 (P'(t))^2\,dt - \theta\int_0^1 (P(t))^2\,dt.
\end{aligned}
\tag{20.112}
$$

Together with (20.94), (20.95), (20.96), (20.105), (20.106), (20.107), (20.108), (20.109), (20.110), (20.111) and (20.112), Δ_1 and Δ_2 becomes

$$\Delta_1 = \mathfrak{R}_1(P,\theta)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2 + O\left(T(\log T)^{3/2}(\log\log T)^A\right) \tag{20.113}$$

and

$$\Delta_2 = \mathfrak{R}_2(P,\theta)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^3 + O\left(T(\log T)^{5/2}(\log\log T)^A\right), \tag{20.114}$$

where $\mathfrak{R}_1(P,\theta)$ and $\mathfrak{R}_2(P,\theta)$ are as defined in (1.15) and (1.16).

In Δ_1 and Δ_2 the averages calculated are over ϱ with $T/2 < \upsilon \le T$. We now extend these ranges to $(0,T]$. We only deal with Δ_1. Note that (20.113) holds for sufficiently large T and the contribution of the zeros with $0 < \upsilon \ll 1$ to Δ_1 is obviously bounded. Writing (20.113) for $T/2$, $T/4, \dots,$ and adding these up we have

$$\begin{aligned}\sum_{0<\upsilon\le T} B\zeta'(1/2+i\upsilon) &= \mathfrak{R}_1(P,\theta)\frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2 \sum_{0\le\kappa\ll\log T}\frac{1}{2^\kappa}\left(1-\frac{\kappa\log 2}{\log\frac{T}{2\pi}}\right)^2 \\ &\quad + O\left(T(\log T)^{3/2}(\log\log T)^A\sum_{0\le\kappa\ll\log T}\frac{1}{2^\kappa}\left(1-\frac{\kappa\log 2}{\log T}\right)^{3/2}\right).\end{aligned} \tag{20.115}$$

The second $\kappa-$sum is

$$\ll \sum_{0\le\kappa\ll\log T}\frac{1}{2^\kappa} \ll \sum_{\kappa\ge 0}\frac{1}{2^\kappa} = 2.$$

Since $\kappa \ll \log T$,

$$\left(1-\frac{\kappa\log 2}{\log\frac{T}{2\pi}}\right)^2 = 1 + O\left(\frac{\kappa}{\log T}\right).$$

Then the first $\kappa-$sum becomes

$$
\begin{aligned}
&= \sum_{0\le\kappa\ll\log T} \frac{1}{2^\kappa} + O\left((\log T)^{-1} \sum_{\kappa\ll\log T} \frac{1}{(2-\epsilon)^\kappa}\right) \\
&= 2 + O\left(\sum_{k\gg\log T} \frac{1}{2^\kappa} + (\log T)^{-1}\right).
\end{aligned}
$$

By the integral test the above tail is $\ll \int_{\log T}^{\infty} 2^{-u} du \ll 2^{-\log T}$. Collecting these error estimates in (20.115) we have the conclusion (1.13).

21. NUMERICAL CALCULATIONS ON $N_d(T)$

Re-define

$$\mathfrak{R}_1(P,\theta) := \frac{3-e^2}{4} + \sum_{i_1=1}^{k} a_{i_1} g_1(i_1,\theta) \tag{21.1}$$

and

$$\mathfrak{R}_2(P,\theta) := \frac{e^2-5}{4} + \sum_{i_1,i_2=1}^{k} a_{i_1} a_{i_2} g_2(i_1,i_2,\theta),$$

where

$$g_1(i_1,\theta) := i_1! \sum_{\kappa'\geq 1} \frac{(-2\theta)^{\kappa'}\left(\frac{F_{1,1}(\kappa'+1;2;2)}{2} - F_{1,1}(\kappa'+1;3;2)\right)}{(i_1+\kappa')!} \tag{21.2}$$

and

$$\begin{aligned} g_2(i_1,i_2,\theta) :=& \frac{(e^2-5)i_1 i_2}{8\theta(i_1+i_2-1)} - \frac{\theta}{i_1+i_2+1} \\ &- \frac{1}{2} i_1! i_2! \sum_{j_5=0,1} \frac{\theta^{-[j_5=0]}}{(i_2-[j_5=0])!(3-j_5)!} \\ &\times \sum_{\kappa'\geq 2} \frac{(-2\theta)^{\kappa'}\left((3-j_5)F_{1,1}(\kappa'+1;3-j_5;2) - 4F_{1,1}(\kappa'+1;4-j_5;2)\right)}{(i_1+\kappa'+i_2-[j_5=0])(i_1+\kappa'-1)!}. \end{aligned} \tag{21.3}$$

We try to find an optimal P maximizing the quantity

$$\frac{(\mathfrak{R}_1(P,\theta))^2}{\mathfrak{R}_2(P,\theta)}$$

subject to the constraints $P(0)=0$, $P(1)=1$ and $\epsilon \leq \theta \leq 1/2-\epsilon$. Take $\theta = 0.499999$, $k=3$, and we abbreviate $g_1(i_1,\theta)$ and $g_2(i_1,i_2,\theta)$ by $g_1(i_1)$ and $g_2(i_1,i_2)$. So, $P(x) =$

$a_1x + a_2x^2 + a_3x^3$ and $a_1 + a_2 + a_3 = 1$. Using this estimate we eliminate a_3 so that

$$\mathfrak{R}_1(P,\theta) = \mathfrak{R}_1(a_1,a_2) := L_0 + L_1a_1 + L_2a_2$$

and

$$\mathfrak{R}_2(P,\theta) = \mathfrak{R}_2(a_1,a_2) := L_3 + L_4a_1^2 + L_5a_2^2 + L_6a_1 + L_7a_1a_2 + L_8a_2.$$

where

$$\begin{aligned}
&L_0 := \frac{3-e^2}{4} + g_1(3), \qquad L_1 := g_1(1) - g_1(3), \qquad L_2 := g_1(2) - g_1(3),\\
&L_3 := \frac{e^2-5}{4} + g_2(3,3), \quad L_4 := g_2(1,1) - g_2(1,3) - g_2(3,1) + g_2(3,3),\\
&L_5 := g_2(2,2) - g_2(2,3) - g_2(3,2) + g_2(3,3),\ L_6 := g_2(1,3) + g_2(3,1) - 2g_2(3,3),\\
&L_7 := g_2(1,2) + g_2(2,1) - g_2(1,3) - g_2(3,1) - g_2(2,3) - g_2(3,2) + 2g_2(3,3),\\
&L_8 := g_2(2,3) + g_2(3,2) - 2g_2(3,3).
\end{aligned}$$

Let $r \in \mathbb{R}$. We first search for the extreme values of $\mathfrak{R}_2(a_1,a_2)$ subject to the constraint $\mathfrak{R}_1(a_1,a_2) = r$. By the method of Lagrange multiplier, the extremum points we're looking for must satisfy the following 3 linear equations:

$$\begin{aligned}
&2L_4a_1 + L_7a_2 + L_6 = \lambda L_1\\
&2L_5a_2 + L_7a_1 + L_8 = \lambda L_2\\
&L_0 + L_1a_1 + L_2a_2 = r
\end{aligned}$$

for some $\lambda \in \mathbb{R}$. We have 3 linear equations and 3 variables a_1, a_2, λ, so the unique solution:

$$a_1 = \frac{L_2(L_1L_8 - L_2L_6) + (L_0 - r)(L_2L_7 - 2L_1L_5)}{2(L_2^2L_4 - L_1L_2L_7 + L_1^2L_5)} \tag{21.4}$$

$$a_2 = \frac{(2L_2L_4 - L_1L_7)(L_0 - r) + L_1(L_1L_8 - L_2L_6)}{2(L_1L_2L_7 - L_1^2L_5 - L_2^2L_4)} \tag{21.5}$$

$$\lambda = \frac{(L_0 - r)(L_7^2 - 4L_4L_5) + 2L_1L_5L_6 - L_2L_6L_7 + 2L_2L_4L_8 - L_1L_7L_8}{2(L_2^2L_4 - L_1L_2L_7 + L_1^2L_5)}, \tag{21.6}$$

provided that the denominators are non-zero, which will be seen later. To decide $\mathfrak{R}_2$ attains its maximum or minimum at (a_1, a_2), we must check the bordered Hessian determinant for $\mathfrak{R}_3 := \mathfrak{R}_2 - \lambda\mathfrak{R}_1$ at (a_1, a_2):

$$|\overline{\mathfrak{H}}| = \begin{vmatrix} 0 & -\frac{\partial\mathfrak{R}_1}{\partial a_1} & -\frac{\partial\mathfrak{R}_1}{\partial a_2} \\ -\frac{\partial\mathfrak{R}_1}{\partial a_1} & \frac{\partial^2\mathfrak{R}_3}{\partial a_1^2} & \frac{\partial^2\mathfrak{R}_3}{\partial a_1\partial a_2} \\ -\frac{\partial\mathfrak{R}_1}{\partial a_2} & \frac{\partial^2\mathfrak{R}_3}{\partial a_1\partial a_2} & \frac{\partial^2\mathfrak{R}_3}{\partial a_2^2} \end{vmatrix} = \begin{vmatrix} 0 & -L_1 & -L_2 \\ -L_1 & 2L_4 & L_7 \\ -L_2 & L_7 & 2L_5 \end{vmatrix}$$
$$= 2(L_1L_2L_7 - L_1^2L_5 - L_2^2L_4) = -0.000267532\cdots < 0, \tag{21.7}$$

the negativity of which implies that $\mathfrak{R}_2$ attains its minimum at (a_1, a_2). So at the same point $\mathfrak{R}_1^2/\mathfrak{R}_2$ attains its maximum value,

$$f(r) := \frac{r^2}{\tilde{L}_0 + \tilde{L}_1(L_0 - r) + \tilde{L}_2(L_0 - r)^2},$$

where

$$\tilde{L}_0 = L_3 - \frac{(L_1L_8 - L_2L_6)^2}{4(L_2^2L_4 + L_1^2L_5 - L_1L_2L_7)},$$
$$\tilde{L}_1 = \frac{L_2(L_6L_7 - 2L_4L_8) - L_1(2L_5L_6 - L_7L_8)}{2(L_2^2L_4 + L_1^2L_5 - L_1L_2L_7)},$$
$$\tilde{L}_2 = \frac{4L_4L_5 - L_7^2}{4(L_2^2L_4 + L_1^2L_5 - L_1L_2L_7)}.$$

In the final step we determine r making $f(r)$ maximum. The first derivative of $f(r)$,

$$f'(r) = \frac{r(2\tilde{L}_0 + 2L_0^2\tilde{L}_2 + 2L_0\tilde{L}_1 - r(\tilde{L}_1 + 2L_0\tilde{L}_2))}{(\tilde{L}_0 + \tilde{L}_1(L_0 - r) + \tilde{L}_2(L_0 - r)^2)^2},$$

has zeros at

$$r = 0 \qquad \text{and} \qquad r = \frac{2\tilde{L}_0 + 2L_0^2\tilde{L}_2 + 2L_0\tilde{L}_1}{\tilde{L}_1 + 2L_0\tilde{L}_2}.$$

Since

$$f''(0) = \frac{2}{\tilde{L}_0 + \tilde{L}_1 L_0 + \tilde{L}_2 L_0^2} = 0.061215\cdots > 0$$

and

$$f''\left(\frac{2\tilde{L}_0 + 2L_0^2\tilde{L}_2 + 2L_0\tilde{L}_1}{\tilde{L}_1 + 2L_0\tilde{L}_2}\right) = \frac{-2\left(\tilde{L}_1 + 2\tilde{L}_2 L_0\right)^4}{\left(\tilde{L}_1^2 - 4\tilde{L}_0\tilde{L}_2\right)^2\left(\tilde{L}_0 + \tilde{L}_1 L_0 + \tilde{L}_2 L_0^2\right)} = -61.8010\cdots < 0,$$

by the second derivative test for local extremas, f attains its maximum at

$$r = \frac{2\tilde{L}_0 + 2L_0^2\tilde{L}_2 + 2L_0\tilde{L}_1}{\tilde{L}_1 + 2L_0\tilde{L}_2}. \tag{21.8}$$

This maximum value is

$$= f\left(\frac{2\tilde{L}_0 + 2L_0^2\tilde{L}_2 + 2L_0\tilde{L}_1}{\tilde{L}_1 + 2L_0\tilde{L}_2}\right) = -\frac{4(\tilde{L}_0 + L_0\tilde{L}_1 + L_0^2\tilde{L}_2)}{\tilde{L}_1^2 - 4\tilde{L}_0\tilde{L}_2} = 0.77345\cdots.$$

Combining (21.4), (21.5) and (21.8), we calculate

$$a_1 = 0.75816\cdots \qquad \text{and} \qquad a_2 = 0.267977\cdot\,.$$

The last three numerical values constitute our main results on $N_d(T)$.

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