ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES IN THREE DIMENSIONS

by

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ABSTRACT

ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES IN THREE DIMENSIONS

In this thesis, we reviewed several aspects of asymptotically anti-de Sitter ($AAdS$) spacetimes in three dimensional Einstein gravity by following some important historical work. Starting with a brief introduction to anti-de Sitter (AdS) spacetimes where also the BTZ black hole solution is given we defined Noether-Wald charges using Noether theorems. Next, we compared different definitions of $AAdS$ spacetimes. Here, we adopted the Fefferman-Graham coordinates and solved Einstein equations order by order to prove that the Fefferman-Graham expansion of $AAdS$ spacetimes terminates at second order in three dimensions, as first shown by Skenderis and Solodukhin. Lastly, we considered two sets of boundary conditions and presented their asymptotic symmetry algebras and charge algebras. Imposing Brown-Henneaux boundary conditions we arrived at Bañados metric, which is the most general metric for $AAdS$ spacetimes under these conditions. Then we showed that the asymptotic symmetry algebra is two copies of the Virasoro algebra. Under the Compère-Song-Strominger boundary conditions, we calculated the most general metric and showed the charge algebra is a semidirect sum of Virasoro and Kac-Moody algebras. We concluded with some comments and future research directions.

ÖZET

ÜÇ BOYUTLU ASİMPTOTİK ANTİ-DE SİTTER UZAYZAMANLAR

Bu tezde, tarihe geçmiş bazı önemli çalışmaların izinden giderek üç boyutlu Einstein kütleçekiminde kullanılan asimptotik anti-de Sitter ($AAdS$) uzayzamanların çeşitli özellikleri gözden geçirildi. Anti-de Sitter (AdS) uzayzamanlarına giriş ve BTZ kara delik çözümünden sonra Noether teoremleri kullanılarak Noether-Wald yükleri tanımlandı. Ardından farklı $AAdS$ uzayzaman tanımları karşılaştırıldı. $AAdS$ geometrisini incelemek amacıyla Fefferman-Graham koordinatları benimsenerek Skenderis ile Solodukhin tarafından gösterilmiş olduğu üzere, Einstein alan denklemlerini sırayla çözüldü ve üç boyut için Fefferman-Graham açılımının ikinci mertebede bittiği gösterildi. Tezin son kısmında iki sınır koşulu kümesi için asimptotik simetri cebirleri ile yük cebirleri sunuldu. Brown-Henneaux sınır koşulları kullanılarak, bu koşullar altındaki $AAdS$ uzayzamanlar için en genel metrik olarak Bañados metriğine ulaşıldığı hesaplandı. Sonrasında asimptotik simetri cebirinin Virasoro cebirinin iki kopyasından oluştuğu gösterildi. Ayrıca Compère-Song-Strominger sınır koşulları incelenerek, onlar için en genel metrik bulunup yük cebirinin Virasoro ve Kac-Moody cebirlerinin yarı doğrudan toplamı olduğunu gösterildi. Yorumlar ve gelecek araştırmalar üzerine fikirler ile sonlandırıldı.

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LIST OF SYMBOLS

$:=$	Equality by definition
$\approx$	Equal on shell, i.e. when equations of motion hold
$[\cdot,\cdot]$	Lie bracket
$\{\cdot,\cdot\}$	Poisson bracket
∂	The partial derivative operator, or the boundary operator
AdS_n	The n-dimensional anti-de Sitter spacetime
$F[p,q\,;s]$	F is a function of p and q about the point s
G	Newtonian constant of gravitation
H^n	The n-dimensional hyperbolic space
i_ξ	Interior multiplication by ξ
$\mathcal{L}_\xi$	Lie derivative along the vector field ξ
$\mathbb{R}^{p,q}$	The flat space with a metric of signature (p,q)
S^n	The n-dimensional sphere
$SO(p,q)$	The special orthogonal group of $\mathbb{R}^{p,q}$
$\widetilde{SO}(p,q)$	A universal covering of the special orthogonal group of $\mathbb{R}^{p,q}$
$\mathfrak{so}(p,q)$	The special orthogonal group of $\mathbb{R}^{p,q}$
$Tr(M)$	Trace of the matrix M
δ_a	A variation generated by a
$\epsilon_{\mu_1\ldots\mu_n}$	Levi-Civita tensor in n dimensions
Λ	Cosmological constant

LIST OF ACRONYMS/ABBREVIATIONS

AdS	Anti-de Sitter
AAdS	Asymptotically anti-de Sitter
AlAdS	Asymptotically locally anti-de Sitter
BMS	Bondi-Metzner-Sachs
BTZ	Bañados-Teitelboim-Zanelli
CFT	Conformal field theory
CSS	Compère-Song-Strominger
dS	de Sitter

1. INTRODUCTION

The anti-de Sitter spacetime (AdS) is a maximally symmetric smooth manifold endowed with a Lorentz signature metric whose scalar curvature is constant and negative. The maximally symmetric spaces are the most basic objects that give insights about more complicated geometries, and therefore have their own value. As in the Riemannian case, the maximally symmetric spacetimes have constant scalar curvature, and they are classified by the sign of their curvature: The AdS spacetime has negative curvature, the de-Sitter (dS) spacetime is positively curved and the flat spacetime has zero curvature. They are the Lorentzian analogues of the hyperbolic space, the sphere and the Euclidean space, respectively, and they have similar roles in the Lorentzian geometry. The dS spacetime was found in 1917 by Willem de Sitter as a cosmological solution for vacuum Einstein gravity with a positive cosmological constant, and similarly AdS spacetime solves them for negative cosmological constant. Since dS spacetime is an interesting solution that matches our observations about the universe, it attracted attention along with the Minkowski spacetimes.

The interest in physics community to anti-de Sitter (AdS) spacetime increased in eighties when they were shown to be supersymmetric solutions of supergravity theories. In those years, both physicist [1,2] and mathematicians [3] studied AdS spacetime with different motivations. They became popular again after the AdS/CFT correspondence was proposed by Maldacena [4] and it is still widely studied. According to the AdS/CFT duality, string theory in $d+1$ dimensions is equivalent to conformal field theory on its conformal boundary. The focus on AdS_3 spacetimes is due to the fact that their duals give CFT_2 theories which are well known by physicists. Studying the three dimensional case is easier since there is less freedom, but we also hope to find some insight on difficult problems encountered in four dimensional gravity.

We aim to understand and compare two approaches in the literature used to define asymptotically (locally) AdS ($A(l)AdS$) spacetimes. We show that the Fefferman-

Graham expansion stops at the second order under a simplifying assumption and study the charge and symmetry algebras of $AAdS$ spacetimes. That was first studied by Henneaux et al. in the eighties [1, 2] influenced by relevant work for asymptotically flat spacetimes [5, 6]. Similar to the discovery of BMS algebra, which extended the Poincaré algebra, the algebra of three dimensional $AAdS$ spacetimes was surprisingly larger than what they expected [1]. This discovery opened the way to the AdS/CFT correspondence, as this algebra contained two copies of Virasoro algebra; therefore, it was closely related to the conformal field theories in two dimensions. We want to study the charge algebra of this seminal work [1] and a recent work [7] published in 2013, to see how the algebra changes when different initial conditions are chosen.

In this thesis, Einstein gravity is reviewed very briefly and it is discussed how it fully determines the local properties of spacetime in three dimensions. After an introduction to anti-de Sitter spacetimes, we focus on the three dimensional case. In relation to the Noether theorems, which relate symmetries and charges of spacetimes, the conserved charges for gravity theories are defined. In the last chapter, we compare two approaches used in the literature to define asymptotically AdS spacetimes. After introducing the Fefferman-Graham expansion [3] we show that it takes a special form in three dimensions. In the end, we follow two important works in this area [1, 7], and present their charge algebras.

We assume the reader is familiar with the Riemannian geometry [8, 9] and has basic knowledge of Einstein gravity [10–12]. Our conventions are given in the appendix.

2. EINSTEIN GRAVITY

2.1. Einstein Field Equations

General relativity in n dimensions is described by the *Einstein-Hilbert action* (up to a boundary term)

$$S_{EH}[g] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^n x \sqrt{-g}(R - 2\Lambda), \tag{2.1}$$

where G stands for the Newtonian constant of gravitation and the cosmological constant is denoted by $\Lambda \in \mathbb{R}$. The integral is taken over an n-dimensional smooth manifold $\mathcal{M}$ endowed with a metric $g_{\mu\nu}$ of Lorentz signature $(- + \ldots +)$ and R is the corresponding scalar curvature.

By varying the action with respect to the metric gives the vacuum Einstein field equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{2.2}$$

where $G_{\mu\nu}$ is called the *Einstein tensor*. These equations can be contracted with the inverse metric $g^{\mu\nu}$ to get

$$R = \frac{2n}{n-2} \Lambda \tag{2.3}$$

by assuming $n \geq 3$. Combining this result with the field equations (2.2) we see the Ricci tensor $R_{\mu\nu}$ is pointwise proportional to the metric $g_{\mu\nu}$

$$R_{\mu\nu} = \frac{2}{n-2} \Lambda \, g_{\mu\nu}. \tag{2.4}$$

A metric $g_{\mu\nu}$ obeying (2.4) is called an *Einstein metric* and a manifold $(\mathcal{M}, g)$ endowed with an Einstein metric is called an *Einstein manifold*. The following proposition follows directly from (2.3).

Proposition 2.1. *Any Einstein metric has constant scalar curvature.*

2.2. Weyl Tensor

For $n \geq 3$, the Riemann tensor can be decomposed as a sum [10]

$$\begin{aligned} R_{\mu\nu\rho\sigma} = & C_{\mu\nu\rho\sigma} + \frac{2}{n-2} g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma} \\ & - \frac{2}{(n-1)(n-2)} R (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \end{aligned} \tag{2.5}$$

which is called the *Ricci decomposition* of the Riemann curvature tensor. Here, $C_{\mu\nu\rho\sigma}$ is called the *Weyl tensor* and it denotes the "trace free part" of $R_{\mu\nu\rho\sigma}$. It has the same symmetry properties with the Riemann tensor and plays an important role when investigating the conformal flatness of spacetimes for $n \geq 4$, which is explained below.

A *conformal transformation* is a map from a manifold $(\mathcal{M}, g)$ to a manifold $(\mathcal{N}, \tilde{g})$ such that

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{2.6}$$

where Ω^2 is a positive function, called the *conformal factor* [10, 13, 14]. If the metric of a spacetime can be mapped to a metric of flat spacetime via a conformal map, it is called *conformally flat*.

The Weyl tensor is also called the "conformal tensor" since it has nice properties under conformal maps:

Theorem 2.2. *Under any conformal transformation, the Weyl tensor with one index* *raised remains invariant [10],*

$$C^{\sigma}_{\mu\nu\rho} = \tilde{C}^{\sigma}_{\mu\nu\rho}. \tag{2.7}$$

Since the Weyl tensor vanishes for flat metric the next theorem follows directly from Theorem 2.2.

Theorem 2.3. *For $n \geq 4$, the Weyl tensor vanishes if and only if the spacetime is conformally flat.*

In n dimensions, the Riemann tensor has n^4 components, but only $\frac{1}{12}n^2(n^2-1)$ are independent due to its symmetries and the first Bianchi identity [15]. The Ricci tensor, on the other hand, is a symmetric tensor and therefore has $\frac{1}{2}n(n+1)$ independent components. When $n = 3$, both numbers coincide and do not leave any degree of freedom for the Weyl tensor to carry [16]. Therefore the Riemann tensor is encoded only by the Ricci tensor in three dimensions.

Theorem 2.4. *For $n = 3$, the Weyl tensor always vanishes.*

The conformally flat spacetimes are characterized by the vanishing of the *Cotton tensor* in three dimensions.

2.3. Einstein Gravity in Three Dimensions

Since the Weyl tensor vanishes identically when $n = 3$, the equation (2.5) becomes

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} - \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{2.8}$$

which can be rewritten by using (2.4) and (2.3)

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{2.9}$$

A manifold $(\mathcal{M}, g)$ satisfying (2.9) is said to be of *constant curvature* [16], i.e. it has constant sectional curvature.

Proposition 2.5. *Any manifold of constant curvature is an Einstein manifold.*

It follows directly from (2.9). The converse is not true, in general, however for $n = 3$ it was shown above that for any Einstein metric (2.9) holds.

Proposition 2.6. *Three dimensional Einstein manifolds are of constant curvature.*

Constant curvature spacetimes have the following useful property. In any dimension, there are three special spacetimes of constant curvature which are complete, simply connected and *maximally symmetric*, i.e. they admit the maximum number of Killing vectors, which is $\frac{1}{2}n(n+1)$ in n dimensions. These are de Sitter (dS), flat and anti-de Sitter (AdS) spacetimes, with positive, zero and negative constant curvature, respectively. They are the Lorentzian analogues of S^n, $\mathbb{R}^n$ and H^n. As in Riemannian geometry, it is possible to find a local isometry between constant curvature spacetimes and the maximally symmetric ones. We present the Lorentzian analogue of a theorem from Riemannian geometry [8].

Corollary 2.7. *Any spacetime with constant curvature is locally isometric to a de Sitter, flat or anti-de Sitter spacetime, if it has positive, zero or negative constant curvature, respectively.*

From now on, we will focus on the spacetimes of negative curvature. These are the solutions of Einstein field equations (2.2) with $\Lambda < 0$, therefore they are locally isometric to an AdS spacetime. Nevertheless, one can obtain interesting spacetimes that differ in topological and asymptotic properties from AdS. A three dimensional example, the BTZ black hole, will be explored in Section 3.6.

3. AdS_{d+1} SPACETIMES

In this chapter we define the anti-de Sitter spacetimes and study their properties, show that they are exact solutions of vacuum Einstein field equations (2.2) with negative cosmological constant and that they are maximally symmetric spacetimes of negative constant curvature. In the end, after reviewing some special properties of AdS_3 such as its symmetry algebra we present the BTZ black hole.

3.1. The Hyperboloid and AdS_{d+1}

To give a definition for AdS_{d+1} spacetimes one needs to study a closely related object. Let (T^1, T^2, X^i), $i = 1, \ldots, d$ be the standard coordinates of $\mathbb{R}^{2,d}$ and consider the following hyperboloid embedded in $\mathbb{R}^{2,d}$

$$\sum_{i=1}^{d} (X^i)^2 - (T^1)^2 - (T^2)^2 = -l^2 \tag{3.1}$$

with $l \neq 0$. Without loss of generality, take $l > 0$. The equation reveals that this set of points is sent to itself under the rotations and reflections about the origin in $\mathbb{R}^{2,d}$, i.e. by the group $O(2,d)$, hence it admits the maximum number of Killing vectors in $d+1$ dimensions. We will elaborate on that in Section 3.3. This is also a space of negative constant curvature, which is to be shown in Section 3.2, but it lacks the property of being a spacetime or being simply connected.

The hyperboloid given by (3.1) does not admit a causal structure since it allows closed timelike curves to exist. To get rid of them, one chooses to work with a universal cover of the hyperboloid instead, obtained by unrolling the hyperboloid so that the closed timelike circles, $(T^1)^2 + (T^2)^2 = const.$ and $X^i = const.$, are unwrapped to straight lines. This covering space is called $d+1$-*dimensional anti-de Sitter spacetime,* AdS_{d+1}, and l in (3.1) is called the AdS radius. Note that AdS_{d+1} is described as an immersion in $\mathbb{R}^{2,d}$, not as an embedding.

3.1.1. Global Coordinate Systems

3.1.1.1. Global coordinates. A set of coordinates compatible with (3.1) is

$$\begin{aligned} T^1 &= l\cosh\rho\cos\tau, \\ T^2 &= l\cosh\rho\sin\tau, \\ X^i &= l\sinh\rho\,\Omega^i \quad \text{for } i = 1,\dots,d, \end{aligned} \tag{3.2}$$

where $\rho > 0$ is the radial coordinate, $\tau \in [0, 2\pi)$ and Ω_i is the spherical coordinates of S^{d-1} in $\mathbb{R}^d$ satisfying $\sum_i \Omega_i^2 = 1$. The coordinate τ parametrizes the timelike circles we want to unwrap. Therefore, one needs unidentify the points $\tau \not\sim \tau + 2\pi$ and expand the range of τ from $[0, 2\pi)$ to $\mathbb{R}$ to get the coordinates for AdS_{d+1}. These coordinates are called the *global coordinates* and the induced metric becomes

$$ds^2 = l^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho\sum_{i=1}^{d}(d\Omega^i)^2). \tag{3.3}$$

3.1.1.2. Conformal coordinates. A similar coordinate chart can be obtained by replacing ρ with conformal radial coordinate $\theta \in [0, \frac{\pi}{2})$ such that $\tan\theta = \sinh\rho$. These coordinates are related to the embedding coordinates as follows

$$\begin{aligned} T^1 &= l\sec\theta\cos\tau, \\ T^2 &= l\sec\theta\sin\tau, \\ X^i &= l\tan\theta\,\Omega^i \quad \text{for } i = 1,\dots,d, \end{aligned} \tag{3.4}$$

so that the induced metric is

$$ds^2 = \frac{l^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta\sum_{i=1}^{d}(d\Omega^i)^2). \tag{3.5}$$

These coordinates are referred as the *conformal coordinates*.

3.1.1.3. Static coordinates. Another set of global coordinates for AdS_{d+1} can be created using timelike coordinate $t = l\tau$, and the luminosity distance $r = l\sinh\rho$. The embedding coordinates are written in (r, t, Ω^i)

$$
\begin{aligned}
T^1 &= \sqrt{r^2 + l^2}\,\cos\left(\frac{t}{l}\right), \\
T^2 &= \sqrt{r^2 + l^2}\,\sin\left(\frac{t}{l}\right), \\
X^i &= l\,r\,\Omega^i \quad \text{for } i = 1, \ldots, d,
\end{aligned}
\tag{3.6}
$$

and the metric becomes

$$
ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 \sum_{i=1}^{d} (d\Omega^i)^2. \tag{3.7}
$$

These are the coordinates used by Brown and Henneaux in their famous paper on Brown-Henneaux boundary conditions [1].

3.1.2. Poincaré Patches

Let us first introduce light cone coordinates using T^1 and X^d

$$
u = \frac{T^1 - X^d}{l^2} \qquad \text{and} \qquad v = \frac{T^1 + X^d}{l^2}, \tag{3.8}
$$

and define

$$
t = \frac{T^2}{lu} \qquad \text{and} \qquad x^i = \frac{X^i}{lu} \quad \text{for } i = 1, \ldots, d-1, \tag{3.9}
$$

with $u \neq 0$. The coordinate v can be expressed in terms of other coordinates by using (3.1) as

$$
v = \frac{1}{l^2 u}(1 - u^2 t^2 + u^2 \vec{x}^{\,2}), \tag{3.10}
$$

then one half of the hyperboloid is parametrized by

$$
\begin{aligned}
T^1 &= \frac{1}{2u}(1 + u^2(l^2 + \vec{x}^{\,2} - t^2)), \\
T^2 &= lut, \\
X^i &= lux^i \quad \text{for } i = 1, \ldots, d-1, \\
X^d &= \frac{1}{2u}(1 + u^2(-l^2 + \vec{x}^{\,2} - t^2)),
\end{aligned} \tag{3.11}
$$

where $\vec{x}^{\,2} = \sum_{i=1}^{d-1}(x^i)^2$. The induced metric becomes

$$
ds^2 = l^2 \left(\frac{1}{u^2}\, du^2 + u^2(-dt^2 + d\vec{x}^{\,2}) \right). \tag{3.12}
$$

Here we assumed that $u \neq 0$ for well-defined coordinates, hence the points having $T^1 = X^d$ are not covered by this parametrization. This hyperplane is called the *Poincaré Killing horizon*, and corresponds to the hyperplane $\cos\tau = \Omega^d \sin\theta$ in global coordinates. $u < 0$ gives $\cos\tau < \Omega^d \sin\theta$ and vice versa.

One can not use Poincaré patches to parametrize the entire hyperboloid, and thus AdS_{d+1}, while global coordinates can be used in this fashion, giving them the name. Note that $u = 0$ region is left out in either patch; the hyperboloid is cut into two disconnected pieces and each half is parametrized by one of the patches. To move along closed timelike circles, $(T^1)^2 + (T^2)^2 = const.$ with $X^i = const.$, one must keep passing from one patch to the other, which makes it impossible to modify the Poincaré charts to parametrize AdS_{d+1}, as we did in global coordinates. Nevertheless, Poincaré coordinates can be used to investigate local properties of AdS_{d+1}.

Sometimes the coordinate $r = u/l^2$ is used instead. Then the embedding coordinates are given by

$$
\begin{aligned}
T^1 &= \frac{l^2}{2r}(1+\frac{r^2}{l^4}(l^2+\vec{x}^{\,2}-t^2)),\\
T^2 &= \frac{rt}{l},\\
X^i &= \frac{rx^i}{l} \quad \text{for } i=1,\dots,d-1,\\
X^d &= \frac{l^2}{2r}(1+\frac{r^2}{l^4}(-l^2+\vec{x}^{\,2}-t^2)),
\end{aligned} \tag{3.13}
$$

so that the intrinsic metric is written as

$$
ds^2 = \frac{l^2}{r^2}\,dr^2 + \frac{r^2}{l^2}(-dt^2+d\vec{x}^2). \tag{3.14}
$$

Using the transformation $z = 1/u$ one can obtain another set of Poincaré patches

$$
\begin{aligned}
T^1 &= \frac{1}{2z}(z^2+l^2+\vec{x}^{\,2}-t^2),\\
T^2 &= \frac{lt}{z},\\
X^i &= \frac{lx^i}{z} \quad \text{for } i=1,\dots,d-1,\\
X^d &= \frac{1}{2z}(z^2-l^2+\vec{x}^{\,2}-t^2),
\end{aligned} \tag{3.15}
$$

and the metric

$$
ds^2 = \frac{l^2}{z^2}(dz^2-dt^2+d\vec{x}^2). \tag{3.16}
$$

Before moving on to the properties of AdS_{d+1} let us make some comments on the boundary of AdS_{d+1}. AdS_{d+1} does not have a boundary, but its asymptotic boundary corresponds to the limits $\rho \to \infty$, $\theta \to \frac{\pi}{2}$ and $r \to \infty$ in global coordinates, $u \to \infty$ or $u \to -\infty$, $r \to \infty$ or $r \to -\infty$, and $z \to 0$ in Poincaré coordinates. In order to analyze the behavior of the given metrics near this asymptotic limit, one needs conformal structures. For a review on conformal properties of AdS_{d+1} we refer the reader to Section 3.4.

3.2. Local Properties

Using the global coordinates (ρ, τ, Ω^i) for AdS_{d+1} the nonzero Christoffel symbols of metric (3.2) are listed as

$$
\begin{aligned}
\Gamma^{\tau}_{\tau\rho} &= \Gamma^{\tau}_{\rho\tau} = \tanh\rho \\
\Gamma^{\rho}_{\tau\tau} &= \sinh\rho\cosh\rho \\
\Gamma^{\rho}_{kk} &= -\sinh\rho\cosh\rho\ \hat{g}_{kk} \\
\Gamma^{k}_{k\rho} &= \Gamma^{k}_{\rho k} = \coth\rho \\
\Gamma^{k}_{ij} &= \hat{\Gamma}^{k}_{ij}
\end{aligned}
\tag{3.17}
$$

where the hatted symbols denote the corresponding quantities for S^{d-1}. The contributing components of the Riemann curvature tensor are then

$$
\begin{aligned}
R^{\tau}_{\rho\tau\rho} &= R^{k}_{\rho k\rho} = -1 \\
R^{\rho}_{\tau\rho\tau} &= R^{k}_{\tau k\tau} = \cosh^2\rho \\
R^{\tau}_{k\tau k} &= R^{\rho}_{k\rho k} = R^{n}_{knk} = -\sinh^2\rho\hat{g}_{kk}
\end{aligned}
\tag{3.18}
$$

implying the Ricci tensor to be proportional to the metric with

$$R_{\mu\nu} = -\frac{d}{l^2}g_{\mu\nu} \tag{3.19}$$

and the Ricci scalar to be a negative constant

$$R = -\frac{d(d+1)}{l^2}. \tag{3.20}$$

From (2.3) it is seen that AdS_{d+1} solves the vacuum Einstein field equations (2.2) when AdS radius is taken as $l = \sqrt{-\frac{d(d-1)}{2\Lambda}}$.

3.3. Global Properties

The hyperboloid has the topology of a cylinder, $S^1 \times \mathbb{R}^d$, but due to the modification in τ and t coordinates it is seen that AdS_{d+1} is topologically equivalent to $\mathbb{R}^{d+1}$. Also note that the map that sends (ρ, τ, Ω^i) to (X^i, T^i) is not injective globally but locally, hence AdS_{d+1} can be represented by an immersion in $\mathbb{R}^{2,d}$. This map wraps AdS_{d+1} around the hyperboloid countably many times in τ direction.

Any Killing field of the embedded hyperboloid lifts naturally to its universal covering AdS_{d+1}, therefore the symmetries of AdS_{d+1} give a covering of $O(2,d)$. Considering the continuous symmetries, i.e. after removing reflections, this symmetry group reduces to $\widetilde{SO}(2,2)$, a covering of $SO(2,d)$. The reason lies behind the modification of timelike global coordinates τ and t: In $SO(2,d)$, i.e. considering the hyperboloid, a rotation in the direction of τ with $2\pi k$, $k \in \mathbb{Z}$, reproduces the identity transformation while on AdS_{d+1} it gives a translation in τ direction.

$SO(2,d)$ and its covering $\widetilde{SO}(2,d)$ differ only globally, hence infinitesimally they look the same and have the same Lie algebra $\mathfrak{so}(2,d)$ whose generators are the infinitesimal transformations in the form

$$J_{\mu\nu} = X_\nu \partial_\mu - X_\mu \partial_\nu, \tag{3.21}$$

where $X^\mu = (T^i, X^j)$, $i = 1,2$, $j = 1,\ldots,d$, for convenience.

It is worth noting that the number of generators in $\mathfrak{so}(2,d)$, or equivalently $\widetilde{SO}(2,d)$, coincides with the maximum number of symmetries a $(d+1)$-dimensional space can have, which is equal to $\frac{1}{2}(d+1)(d+2)$. This shows us that the AdS_{d+1} is maximally symmetric, hence the Riemann curvature tensor of AdS_{d+1} can be written as [16]

$$R_{\mu\nu\sigma\lambda} = -\frac{1}{l^2}(g_{\mu\sigma}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\sigma}). \tag{3.22}$$

3.4. Conformal Properties

The asymptotic structure of a spacetime is an important ingredient to construct conserved charges. Here we will investigate the structure at the boundaries of AdS as an example, however, infinity itself stands as an obstacle. To talk about the asymptotic region at infinity one needs to bring that region to a finite distance, and that is achieved by conformal compactification.

3.4.1. Conformal Compactification and Conformal Structure

Conformal compactification of a spacetime is used to bring infinite distances to a finite distance, namely to represent the spacetime at hand on a bounded set and then adding a boundary to it so that it is possible to work on or near the boundary of the spacetime, without losing the causal structure. To preserve the causal structure one needs to use a conformal transformation to send the original spacetime to a bounded one. The next proposition follows directly from the definition of the conformal map we gave in Section 2.2.

Proposition 3.1. *The causal structure is then preserved under any conformal map, i.e. all spacelike/timelike/null vectors are sent to a vector with the same property.*

The conformal compactification was introduced by Penrose in [17] as a tool to study the asymptotic regions of spacetimes and described in [18] in detail. Let us introduce the definition following [13, 14, 18, 19].

Let $(\mathcal{M}, g)$ be a smooth manifold which we call "physical spacetime". We wish to extend it to an "unphysical spacetime", a smooth manifold $\overline{\mathcal{M}}$ with boundary $\mathcal{B}$

and the interior $\mathcal{M}$. The boundary $\mathcal{B}$ and the physical spacetime $\mathcal{M}$ are related by a smooth function Ω defined on $\mathcal{M}$ which can be extended smoothly to $\overline{\mathcal{M}}$ satisfying

- $\Omega > 0$ in $\mathcal{M}$
- $\Omega = 0$, $d\Omega \neq 0$ on $\mathcal{B}$
- the metric $\tilde{g} = \Omega^2 g$ extends smoothly on $\overline{\mathcal{M}}$ and is non-degenerate.

The function Ω is called the *boundary defining function*. If such a function exists the metric g is said to be *conformally compact* and the manifold $(\overline{\mathcal{M}}, \tilde{g})$ is called a *conformal compactification* of $(\mathcal{M}, g)$. Note that if g is conformally compact it must have a second order pole at the boundary so that it becomes well-defined on $\mathcal{B}$ after being multiplied by Ω^2.

The word "compactification" may be misleading as the manifold $(\overline{\mathcal{M}}, \tilde{g})$ is not necessarily compact, rather it is contained in a compact set and therefore called a compactification.

Many boundary defining functions can be derived from an existing one. If Ω is a boundary defining function then so is $\tilde{\Omega} = \Omega e^{\omega}$, where ω is a function with no zeroes or poles on $\mathcal{B}$. Each boundary defining function Ω gives a different metric on the boundary $\mathcal{B}$ which are related to the each other by a smooth positive factor. Such metrics define an equivalence class with the relation $g \sim e^{\omega} g$, and such equivalence classes of metrics are called *conformal structures* [3]. Hence if a conformal compactification $(\overline{\mathcal{M}}, \tilde{g})$ defines a conformal structure $[\tilde{g}]$ on $\mathcal{B}$.

3.4.2. Conformal Boundary of AdS_{d+1}

Among the metrics given in Section 3.1 there are two obvious candidates to use for a conformal compactification of AdS_{d+1}. The first one is already called "conformal coordinates" because the factor $\frac{l^2}{\cos^2\theta}$ in (3.5) can be eliminated by a conformal transformation. This would allow θ to take the value $\frac{\pi}{2}$ where the conformal boundary $\mathcal{B}$ of

AdS_{d+1} lies in these coordinates. The metric $\tilde{g}$ is then given by

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta \sum_{i=1}^{d} (d\Omega^i)^2. \tag{3.23}$$

The conformal boundary at $\theta = \frac{\pi}{2}$ has the topology $\mathbb{R} \times S^{d-1}$.

The other metric we would choose to consider is (3.16) of Poincaré patches. This time the conformal boundary is at $z = 0$ and the metric after the transformation is then given by

$$ds^2 = dz^2 - dt^2 + d\vec{x}^2 \tag{3.24}$$

as the $d + 1$-dimensional Minkowski metric. Because each patch has $z > 0$ or $z < 0$, this transformation maps the AdS to one half of the Minkowski spacetime. This also shows that AdS is conformally flat.

3.5. Symmetries of AdS_3

In the last chapter, we will focus on the 3 dimensional case. AdS_3 is given by the universal covering space of the embedded hyperboloid

$$x^2 + y^2 - u^2 - v^2 = -l^2. \tag{3.25}$$

whose continuous symmetries give a covering of the $SO(2, 2)$ group. (A detailed explanation can be found later in this section and in Section 3.3.) These symmetries are generated by 6 infinitesimal transformations given in (3.21) with $X^\mu = (u, v, x, y)$.

They satisfy $J_{\mu\nu} = -J_{\nu\mu}$ and written in detail as

$$
\begin{aligned}
J_{uv} &= -v\partial_u + u\partial_v & J_{vx} &= x\partial_v + v\partial_x \\
J_{ux} &= x\partial_u + u\partial_x & J_{vy} &= y\partial_v + v\partial_y \\
J_{uy} &= y\partial_u + u\partial_y & J_{xy} &= y\partial_x - x\partial_y.
\end{aligned} \tag{3.26}
$$

The intrinsic picture is much more illuminating when dealing with symmetries, let us pass to the global coordinates (ρ, τ, ϕ) in (3.2)

$$
\begin{aligned}
u &= l\cosh\rho\cos\tau \\
v &= l\cosh\rho\sin\tau \\
x &= l\sinh\rho\cos\phi \\
y &= l\sinh\rho\sin\phi
\end{aligned} \tag{3.27}
$$

and the intrinsic metric given by (3.3) in three dimensions as

$$ds^2 = l^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\phi^2) \tag{3.28}$$

where $\rho > 0$, $\tau \in \mathbb{R}$ and $\phi \in [0, 2\pi)$. The pullbacks of the generators (3.21) can be calculated using

$$J^\mu = g^{\mu\nu}\frac{\partial x^k}{\partial x^\nu}\eta_{kl}J^l \tag{3.29}$$

where latin indices denote the flat metric in $\mathbb{R}^{2,2}$ and greek indices are used for global coordinates (ρ, τ, ϕ). These generators are then given by

$$J_{uv} = \partial_\tau \tag{3.30a}$$

$$J_{ux} = -\tanh\rho \sin\tau \cos\phi \partial_\tau + \cos\tau \cos\phi \partial_\rho - \coth\rho \cos\tau \sin\phi \partial_\phi \tag{3.30b}$$

$$J_{uy} = -\tanh\rho \sin\tau \sin\phi \partial_\tau + \cos\tau \sin\phi \partial_\rho + \coth\rho \cos\tau \cos\phi \partial_\phi \tag{3.30c}$$

$$J_{vx} = \tanh\rho \sin\tau \cos\phi \partial_\tau + \sin\tau \cos\phi \partial_\rho - \coth\rho \sin\tau \sin\phi \partial_\phi \tag{3.30d}$$

$$J_{vy} = \tanh\rho \sin\tau \sin\phi \partial_\tau + \sin\tau \sin\phi \partial_\rho + \coth\rho \sin\tau \cos\phi \partial_\phi \tag{3.30e}$$

$$J_{xy} = -\partial_\phi. \tag{3.30f}$$

These vector fields satisfy the Killing equation $\mathcal{L}_\xi = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. It is easily seen using the Levi-Civita connection for the metric (3.28) whose contributing components can be found from (3.17) as

$$\begin{aligned} \Gamma^\tau_{\tau\rho} &= \Gamma^\tau_{\rho\tau} = \tanh\rho & \Gamma^\rho_{\tau\tau} &= \sinh\rho\cosh\rho \\ \Gamma^\phi_{\phi\rho} &= \Gamma^k_{\rho k} = \coth\rho & \Gamma^\rho_{\phi\phi} &= -\sinh\rho\cosh\rho. \end{aligned} \tag{3.31}$$

A generic Killing vector of AdS_3 can be written as $\frac{1}{2}\omega^{jk}J_{jk}$ using an antisymmetric tensor ω^{jk} in $\mathbb{R}^{2,2}$. We deduce that AdS_3 is stationary because ∂_τ is given in (3.30a) as one of the Killing vector fields. This vector field is orthogonal to ∂_ρ and ∂_ϕ, and spacelike hypersurfaces with constant τ, hence AdS_3 is also static.

As mentioned in Section 3.3, the continuous symmetries of AdS_3 give $\widetilde{SO}(2,2)$, a covering of $SO(2,2)$, and its Lie algebra is $\mathfrak{so}(2,2)$ generated by the infinitesimal transformations in (3.30). To construct another basis for $\mathfrak{so}(2,2)$ one can use the following combinations

$$
\begin{aligned}
L_+ &= \frac{1}{2}(J_{vu} + J_{yx} + J_{vx} + J_{yu}), &\qquad \bar{L}_+ &= \frac{1}{2}(J_{vu} + J_{xy} + J_{vx} + J_{uy}),\\
L_- &= \frac{1}{2}(J_{vu} + J_{yx} + J_{xv} + J_{uy}), &\qquad \bar{L}_- &= \frac{1}{2}(J_{vu} + J_{xy} + J_{xv} + J_{yu}),\\
L_0 &= \frac{1}{2}(J_{ux} + J_{vy}), &\qquad \bar{L}_0 &= \frac{1}{2}(J_{ux} + J_{yv}).
\end{aligned} \tag{3.32}
$$

The barred and unbarred elements commute

$$[L_i, \bar{L}_j] = 0 \quad \text{for } i, j = -1, 0, 1, \tag{3.33}$$

and both generate $\mathfrak{sl}(2, \mathbb{R})$ algebra obeying

$$[L_\pm, L_0] = \pm L_\pm, \qquad\qquad [L_+, L_-] = 2L_0. \tag{3.34}$$

Hence we may use the decomposition $\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$.

$\mathfrak{so}(2,2)$ can also be represented as the direct sum of two $\mathfrak{so}(2,1)$ algebras by rearranging the set of generators as

$$
\begin{aligned}
J_0^+ &= \frac{1}{2}(J_{yx} + J_{vu}) &\qquad J_1^+ &= \frac{1}{2}(J_{yu} + J_{vx}) &\qquad J_2^+ &= \frac{1}{2}(J_{xu} + J_{yv})\\
J_0^- &= \frac{1}{2}(J_{yx} + J_{uv}) &\qquad J_1^- &= \frac{1}{2}(J_{yu} + J_{xv}) &\qquad J_2^- &= \frac{1}{2}(J_{xu} + J_{vy})
\end{aligned} \tag{3.35}
$$

with the following commutation relations

$$[J_a^\pm, J_b^\pm] = \epsilon_{abc} J^{\pm c} \qquad\qquad [J_a^+, J_b^-] = 0. \tag{3.36}$$

3.6. BTZ Black Hole

In three dimensions, all solutions of the Einstein field equations are of constant curvature, therefore locally dS, AdS or flat, i.e. locally isometric to dS, AdS or flat spacetimes, however their global properties may differ. One of the most important exact solutions of the Einstein field equations (2.2) is the BTZ black hole, found by Bañados, Teitelboim and Zanelli in 1992 [20]. It exhibits properties similar to a black hole, such as having event horizons, however, it is a locally AdS_3 spacetime and has no curvature singularity.

The metric of the BTZ black hole is given by

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 \tag{3.37}$$

with $t, \phi \in \mathbb{R}$, $\phi \in [0, 2\pi)$ and $r \geq 0$. It solves the Einstein field equations (2.2) with $\Lambda = -1/l^2$. Here, N and N^ϕ are functions of r given by

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \tag{3.38}$$

$$N^\phi = -\frac{J}{2r^2} \tag{3.39}$$

where we took $8G = 1$. The constants $M, J \in \mathbb{R}$ are linked r_- and r_+ in (3.46) with

$$r_\pm = l \left[\frac{M}{2} \left(1 \pm \sqrt{1 - \left(\frac{J}{Ml} \right)^2} \right) \right]^{1/2} . \tag{3.40}$$

as $r_\pm$ are defined as the positive roots of the function $N^2 = 0$. Then

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad J = \frac{2 r_+ r_-}{l}. \tag{3.41}$$

$r_\pm$ are real if and only if

$$|J| \leq Ml \qquad \text{and} \qquad M > 0. \tag{3.42}$$

r_+ and r_- correspond to the radii of the inner and outer horizon of the black hole. When $r_+ = r_-$ the black hole is said to be *extremal*. In that case, $|J| = Ml$ [20–22].

As all components of the metric depend only on r we see ∂_t and ∂_ϕ are Killing vectors of BTZ black hole. One may find the proof that these are the only two Killing vectors of BTZ in the Section 3.2.6 of [22]. M and J are interpreted as conserved charges associated with ∂_t and ∂_ϕ, respectively, in Chapter 4. To understand why this solution has black hole like properties we will study it as an identification of AdS.

3.6.1. Identification Subgroup and the Quotient Space

It is possible to express the BTZ black hole as an identification of AdS_3. As pointed out at the end of [20], "such a spacetime must arise from identifications of points in anti-de Sitter space through a discrete subgroup of its symmetry group $O(2, 2)$" [20]. The geometry of this quotient space was investigated in a follow-up paper [22]. Here, the non-extremal case is presented; the extremal case can be considered as the limit $r_- \to r_+$ and the details can be found in [22].

While making identifications it is important to preserve the continuity and the smoothness of the metric, therefore we consider isometries of AdS_3, identify the points that are mapped to each other under this map and create a quotient space.

Consider the one-parameter subgroup generated by some Killing vector ξ

$$\gamma : \mathbb{R} \to \widetilde{SO}(2, 2) \qquad\qquad \gamma(t) = e^{t\xi}. \tag{3.43}$$

The curve $\gamma(t)$ living in $\widetilde{SO}(2, 2)$ contains all isometries of AdS_3 generated by ξ and its scalar multiples. This subgroup of isometries acts on the points in AdS_3

$$\Theta : \mathbb{R} \times AdS_3 \to AdS_3 \qquad\qquad \Theta(t,p) = e^{t\xi}p. \tag{3.44}$$

Using any element of $\gamma(t)$ we can define an identification of AdS_3. Let us proceed with $\gamma(2\pi) = e^{2\pi\xi}$. The factor 2π of our choice is conventional and it is used to emphasize the cyclic shape after the identification.

A point p would then be identified with $e^{2\pi k\xi}p$, $k \in \mathbb{Z}$, namely to all points which p is sent to under the discrete subgroup Γ

$$\Gamma = < e^{2\pi\xi} > = \{e^{2\pi k\xi}, k \in \mathbb{Z}\}, \tag{3.45}$$

hence it is called the identification subgroup associated with the Killing vector $2\pi\xi$. As long as the action of Γ is properly discontinuous on AdS_3 we get a quotient manifold AdS_3/Γ given by the equivalence relation $p \sim \exp(2\pi k\xi)p$, $k \in \mathbb{Z}$. More information on identifications of AdS may be found in [23].

For BTZ black hole this discrete subgroup is generated by the Killing vector field [22]

$$\xi = \frac{r_+}{l} J_{yx} - \frac{r_-}{l} J_{vu} - J_{uy} + J_{yx} \tag{3.46}$$

where $0 \leq r_- \leq r_+$.

The inherited metric is well-defined and smooth on this quotient manifold because it remains unchanged along the orbits of Θ and these orbits are closed after the identification. Therefore the quotient space is locally isometric to AdS_3, i.e. locally AdS_3. Since the Riemann curvature tensor is same with AdS_3, it also solves the Einstein field equations. Nevertheless, the causality of this new space is to be questioned before we can call it a spacetime.

3.6.2. Causality

Gluing the points of AdS_3 creates closed curves along the orbits of the map Θ in (3.44). This will cause closed timelike or null curves to appear where ξ is timelike or null, respectively. In order to have a well-defined causal structure one needs to discard such curves in the quotient space. Therefore the regions of AdS_3 where $\xi^\mu \xi_\mu \leq 0$ must be removed before the identification. This, in general, does not guarantee that we end up with a space that admits a causal structure, there may still be closed timelike or null curves left, but at least the ones along the the orbits of Θ are eliminated this way. In the case of BTZ black hole it will be enough as shown in the Section 3.2.5 of [22].

At this point one may ask if we can take a quotient after cutting out some regions from AdS_3. For a well-defined identification we need to check that for each point p in the remaining part of AdS_3 the points $e^{2\pi k \xi} p$, $k \in \mathbb{Z}$, are not cut out. It is easy to see that this is indeed true. Since the norm of the vector ξ does not change along the orbits of Θ, and p and $e^{2\pi k \xi}$ belong to the same orbit for each $p \in AdS_3$, they are either in the remaining region or in the one we cut out. Thus the identification is still well-defined.

3.6.3. The Quotient Space and its Singularities

The further properties of the BTZ black hole are explained thoroughly in [22], but for our purposes it will be enough to state the metric of the BTZ black hole. By choosing a parametrization (t, r, ϕ) on the remaining parts of AdS as given in Section 3.2.3 of [22], the AdS metric becomes the metric we gave in (3.37) with $\phi \in \mathbb{R}$. Then the identification is made through identifying $\phi \sim \phi + 2\pi$.

Since the quotient space has same local properties with AdS_3, there are no curvature singularities in BTZ black hole, even when $r = 0$. The region $r = 0$ corresponds to a surface, not a point, and the Killing vector ∂_ϕ becomes timelike in the regions of AdS_3 which correspond to $r < 0$. This shows that $r = 0$ is only a causal singularity.

There are many papers written about the properties of the BTZ black hole. In addition to the ones we mentioned in this chapter, Carlip's works [24–28] can be used for further study.

4. CONSERVED GLOBAL CHARGES AND THEIR ALGEBRA

Symmetries are related to conserved quantities which in general are called *charges*. In physics, charges play important role such as energy, mass and angular momentum. Historically, it was Emmy Noether who established the relation between the symmetries of a spacetime and the charges that are conserved as time changes in 1915. In this thesis, we are interested in conserved charges at spatial infinity. We will first define a structure called the variational bicomplex and introduce notations and conventions used in the so-called covariant phase space formalism [29–32]. Here, we will follow the lecture notes by Compère and Fiorucci [21]. The Noether's Theorems will come next. Then we will define the Noether-Wald surface charge density and calculate it for Einstein gravity. At the end of this chapter, we will apply this formalism to the BTZ solution (3.37) and show that the constant M that appears in the definition corresponds to a conserved charge.

4.1. Variational Bicomplex

The Lagrangian theories we will consider are the ones where the Lagrangian density L depends on the derivatives of the fields involved to all orders. We aim to generalize the notion of cotangent space so that we can additionally include the variations of the fields as differential forms (so that later we can integrate between two fields). To this end, we are going to define a structure called the variational bicomplex which combines the spacetime manifold and the field space. The construction of the variational bicomplex and more details on its properties are thoroughly given in [33,34], here we will present the information necessary for our use in this thesis.

The spacetime is given by an n-dimensional manifold $\mathcal{M}$, we will denote its coordinates by $\{x^\mu,\, \mu = 1, \ldots, n\}$. Let $p \in \mathcal{M}$. The *cotangent space*, the dual of the tangent space, at the point p is the set of all 1-forms with basis $\{dx^\mu,\, \mu = 1, \ldots, n\}$.

The *interior product* of a 1-form ω by an element of the tangent space ξ is defined as $\xi^\mu \partial_\mu \omega = \xi^\mu \frac{\partial}{\partial\, dx^\mu} \omega$. We may generalize the definition of the interior product by ξ as an operator that sends k-forms to $(k-1)$-forms and denote it by i_ξ. Its inverse operation, the *exterior derivative* of forms, is given by the d operator, $d = dx^\mu \partial_\mu$, as usual.

Now let us define the field space. The *field space*, or *jet space*, is the space of smooth fields and their derivatives of all orders with respect to the spacetime coordinates. The fields and their derivatives, however, are considered in an abstract manner independently of their values on the spacetime manifold. They are used as a coordinate system to locate fields in the field space. To have a well-defined coordinate system, one needs to identify some of the derivatives with each other, for example, $\partial_\mu \partial_\nu \Phi = \partial_\nu \partial_\mu \Phi$ but they should be used as a single coordinate variable. Therefore we define the symmetrized derivatives $\Phi_{\mu\nu}$ as follows

$$\frac{\partial}{\partial \Phi^j_{\mu\nu}} \Phi^i_{\alpha\beta} = \delta^{(\mu}_\alpha \delta^{\nu)}_\beta \delta^i_j. \tag{4.1}$$

Using this construction, a point $(\Phi, \Phi_\mu, \Phi_{\mu\nu}, \dots)$ in the field space corresponds to a specific field and the differential forms at that point can be written using the basis $\{\delta\Phi, \delta\Phi_\mu, \delta\Phi_{\mu\nu}, \dots\}$ where the operator δ is defined as $\delta = \delta\Phi^i_I \frac{\partial}{\partial \Phi^i_I}$ where I is a multi-index that denotes any symmetrized derivative and Einstein summation runs over all i and I. Noting the similarity between the cotangent space of the field space and the spacetime manifold, we follow the conventional choice in [21] by defining all basis elements $\{\delta\Phi, \delta\Phi_\mu, \delta\Phi_{\mu\nu}, \dots\}$ as Grassmann odd variables, i.e. they anticommute with each other, as in the usual exterior algebras. An interior product can also be defined via

$$i_{\delta_a} := \delta_a \Phi^i_I \frac{\partial}{\partial \delta \Phi^i_I}, \tag{4.2}$$

for the transformation generated by an element "a" in some group of transformations, under which the variations in the field space are given by $\delta_a \Phi^i_I$. The variations $\delta_a \Phi^i_I$ are not basis elements anymore, hence they are Grassmann even, i.e. they commute with everything. Note that this inner product maps an arbitrary variation to a variation under a specific transformation denoted by a.

The *jet bundle* or the *variational bicomplex* is a combination of both spaces mentioned above, with coordinates (x^μ, Φ^i_I). This space contains the field space as a fiber at each point of the spacetime manifold $\mathcal{M}$. Now we should adapt the derivative operators d and δ to the variational bicomplex. We will not change the way the exterior derivative $d = dx^\mu \partial_\mu$ is defined, but the partial derivative ∂_μ will be calculated like a total derivative with respect to x^μ where all fields are taken as dependent variables

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \Phi^i_\mu \frac{\partial}{\partial \Phi^i} + \Phi^i_{\mu\nu} \frac{\partial}{\partial \Phi^i_\nu} + \ldots, \tag{4.3}$$

which we will call the *horizontal derivative*. The definition of the *vertical derivative* δ stays the same. These differential operators anticommute and with them involved we define this space as a *variational bicomplex*. A differential form on variational bicomplex is an element of cotangent spaces of the spacetime manifold and the field space, hence uses $\{dx^\mu, \delta\Phi^i_I\}$ as its basis. A form which has p many dx^μ factors and q many $\delta\Phi^i_I$ terms is called a *(p, q)-form*.

The Lagrangian density L and the Lagrangian form $\mathbf{L} = L d^n x$ are natural objects that live on this structure. They depend on the fields and their higher derivatives. We can now observe that arbitrary variations are also naturally defined on variational bicomplex. For example, $\mathbf{L}$ is an $(n, 0)$-form and its variation $\delta\mathbf{L}$ becomes an $(n, 1)$-form in this formalism.

4.2. Noether Theorems

Let us consider a theory described by some Lagrangian density $L[\Phi^i, \Phi^i_I]$ which depends on the fields Φ^i and their derivatives Φ^i_I, and introduce the new notations while showing that its equations of motion are given by the Euler-Lagrange derivative.

Theorem 4.1. *The equations of motion of this theory is given by the* Euler-Lagrange equations

$$\frac{\delta L}{\delta \Phi^i} := \frac{\partial L}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial L}{\partial \partial_\mu \Phi^i} \right) + \partial_\mu \partial_\nu \left(\frac{\partial L}{\partial \partial_\mu \partial_\nu \Phi^i} \right) - \cdots = 0, \tag{4.4}$$

for all i, and the metric $g_{\mu\nu}$ may be included in the set of fields $\{\Phi^i\}$.

Proof. Consider an arbitrary variation of the Lagrangian density L

$$\begin{aligned} \delta L &= \delta \Phi^i \frac{\partial L}{\partial \Phi^i} + \delta \partial_\mu \Phi^i \frac{\partial L}{\partial \partial_\mu \Phi^i} + \ldots \\ &= \delta \Phi^i \frac{\partial L}{\partial \Phi^i} - \partial_\mu \left(\delta \Phi^i \frac{\partial L}{\partial \partial_\mu \Phi^i} \right) - \delta \Phi^i \partial_\mu \frac{\partial L}{\partial \partial_\mu \Phi^i} + \ldots \\ &= \delta \Phi^i \frac{\delta L}{\delta \Phi^i} - \partial_\mu \Theta^\mu [\delta \Phi^i; \Phi]. \end{aligned} \tag{4.5}$$

where the inverse Leibniz rule was applied with a minus sign because we conventionally defined δ and partial derivatives as anticommutative operators. The last term, the divergence of the vector field Θ^μ, contains the total derivative terms coming from the inverse Leibniz operations. Θ^μ is called the *presymplectic potential.* The same equation can be written using differential forms as

$$\delta \mathbf{L} = \delta \Phi^i \frac{\delta \mathbf{L}}{\delta \Phi^i} - d\boldsymbol{\Theta}[\delta \Phi^i; \Phi]. \tag{4.6}$$

One should note that when the above expression is contracted by some interior product δ_a it gives

$$\delta_a \mathbf{L} = \delta_a \Phi^i \frac{\delta \mathbf{L}}{\delta \Phi^i} + d\boldsymbol{\Theta}[\delta_a \Phi^i; \Phi]. \tag{4.7}$$

where the minus sign is changed because δ and d anticommute.

By applying the principle of the stationary action one gets

$$\begin{aligned} 0 &= \delta S && (4.8) \\ &= \int \delta L d^n x && (4.9) \\ &= \int \delta\Phi^i \frac{\delta L}{\delta \Phi^i} - \partial_\mu \Theta^\mu[\delta\Phi^i;\Phi] d^n x && (4.10) \\ &= \int \delta\Phi^i \frac{\delta L}{\delta \Phi^i} d^n x. && (4.11) \end{aligned}$$

Since this equality must be true for any variation $\delta\Phi^i$ we conclude $\dfrac{\delta L}{\delta \Phi^i} = 0$. □

Now assume there is a group G of transformations of the fields Φ^i such that the action

$$S = \int L d^n x \tag{4.12}$$

is invariant under the elements in G. Such a transformation $a \in G$ is called a *global symmetry* of L and it necessarily preserves the Lagrangian form $\mathbf{L}$ up to a total derivative term, i.e. $\delta_a \mathbf{L} = d\boldsymbol{\alpha}[\delta_a\Phi;\Phi]$ for some $(n-1,0)$-form $\boldsymbol{\alpha}$. Then the $(n-1,0)$-form

$$\mathbf{J}[a] = \boldsymbol{\Theta}[\delta_a\Phi;\Phi] - \boldsymbol{\alpha}[\delta_a\Phi;\Phi] \tag{4.13}$$

is closed, $d\mathbf{J}[a] = 0$, when the equations of motion are satisfied. This $(n-1,0)$-form is the Hodge dual of a current J^μ which is called a *conserved current*, i.e. its divergence vanishes when the equations of motion hold. That is the *Noether current* associated with the symmetry generated by $a \in G$. We can calculate the corresponding *Noether charge* by taking its integral over a *Cauchy surface* Σ, a codimension 1 surface in $\mathcal{M}$ which is intersected by every maximal causal curve exactly once [21].

There is a special type of global symmetries, called the *gauge symmetry*, which is a global symmetry of L whose generator arbitrarily depends on the coordinates.

Theorem 4.2 (Noether's first theorem)**.** *If a physical theory described by a Lagrangian L admits global symmetries, there exists a bijection between equivalence classes of continuous global symmetries of L and the equivalence classes of conserved currents J^μ. J^μ are called the* Noether currents*. [21]*

Here, two global symmetries are said to be equivalent if and only if their difference is given by a gauge transformation and some symmetry whose generator vanishes *on shell*, i.e. when equations of motion are satisfied. Similarly, two currents J_1^μ, J_2^μ are said to be equivalent if and only if they obey

$$J_2^\mu = J_1^\mu + \partial_\nu k^{\mu\nu} + t^\mu, \tag{4.14}$$

where $k^{\mu\nu}$ is a skew symmetric tensor and t^μ vanishes on shell, hence the conservation of one current implies the conservation of the other if they are equivalent. We would like to emphasize that the equivalent currents do not necessarily give the same charge, and that will be used later to define lower degree conserved quantities.

The Noether currents of gauge transformations have exact Hodge duals, i.e. $\mathbf{J}[a] = d\mathbf{k}_a$ for some $(n-2,0)$-form $\mathbf{k}_a = k_a^{[\mu\nu]}(d^{n-2}x)_{\mu\nu}$. Then $d\mathbf{J}[a] = 0$ even when the equations of motions are not satisfied. Such quantities are said to be *trivially conserved.* Then its integral, the associated Noether charge, reduces to the flux of $\mathbf{k}$ through the boundary $\partial\Sigma$

$$\int_\Sigma \mathbf{J}[a] = \int_{\partial\Sigma} \mathbf{k}_a. \tag{4.15}$$

This shows why one can calculate the energy as an integral at the spatial infinity in general relativity, for which we must discard two coordinates, time and a radial spacelike coordinate and integrate over a codimension 2 surface, such as $\partial\Sigma$ in (4.15).

Now consider a closed $(n-2,0)$-form $\mathbf{k}$. Then the surface charge $\int_S \mathbf{k}$ is independent of S, since we can write $\int_{S_1} \mathbf{k} - \int_{S_2} \mathbf{k} = \int_V d\mathbf{k} = 0$ where V is the volume between S_1 and S_2. In this case, S can be deformed continuously, assuming we do not cross over some singularity of $\mathbf{k}$. We would expect this charge to be associated to some symmetry of the theory. This correspondence was stated in the following theorem by Barnich, Brandt and Henneaux [35].

Theorem 4.3 (Generalized Noether theorem)**.** *For any physical theory described by a Lagrangian density L defined on a spacetime manifold $(\mathcal{M}, g)$ which admits global symmetries, some of which might be gauge transformations, there exists a bijection between:*

- *The equivalence class of gauge parameters $\lambda(x^\mu)$ that are field symmetries, i.e. the variations of all fields Φ^i under the transformation generated by λ vanish on shell, and*
- *The equivalence class of $(n-2,0)$-forms $\mathbf{k}$ that are closed on shell. [21]*

Here, two gauge parameters are considered to be equivalent if they are equal on shell and two $(n-2,0)$-forms are equivalent if they differ on shell by an exact $(n-2)$-form. In contrast to the Noether's first theorem, Theorem 4.2, an equivalence class of $(n-2,0)$-forms corresponds to a single charge now, because the integral of the exact $(n-2,0)$-form would become in an integral over $\partial(\partial\Sigma)$ via Stokes' Theorem and thus vanish. The $(n-2,0)$-forms mentioned in the theorem are called *surface charge densities* and the charges associated with them are *surface charges.*

In general relativity, gauge transformations are diffeomorphisms due to the general covariance. A *field symmetry* is a transformation under which the variations of fields vanish on shell. A diffeomorphism which is also a field symmetry of the metric must be an isometry of the spacetime manifold. These diffeomorphisms are generated by the Killing vectors, but in general, a metric does not have to admit any isometries. Therefore it is not possible to use this theorem directly, instead we are going to look at the symmetries of a set of metrics that share a common feature, and that feature will be being asymptotically AdS. We will elaborate on that in Section 4.4.1.

To explain what we aim in this chapter, let us introduce the variation of the presymplectic potential mentioned in Section 4.2. The presymplectic potential $\mathbf{\Theta}[\delta\Phi;\Phi]$ is an $(n-1,1)$-form, therefore its variation $\delta\mathbf{\Theta}$ is an $(n-1,2)$-form

$$\omega[\delta\Phi,\delta\Phi;\Phi]=\delta\mathbf{\Theta}[\delta\Phi;\Phi], \tag{4.16}$$

which is called the *presymplectic form*. We are going to link this form with the surface charge densities, but we need Noether's second theorem to do that.

Theorem 4.4 (Noether's second theorem)**.** *Let* $\mathbf{L}=Ld^nx$ *be a generally covariant Lagrangian form and* ξ^μ *an arbitrary diffeomorphism. Then*

$$\frac{\delta\mathbf{L}}{\delta\Phi^i}\delta_\xi\Phi^i=d\mathbf{S}_\xi\left[\frac{\delta\mathbf{L}}{\delta\Phi^i};\Phi^i\right], \tag{4.17}$$

where $\mathbf{S}_\xi$ *is an* $(n-1,0)$*-form proportional to the equations of motion and its derivatives. The equality also holds for other types of gauge transformations where* ξ^μ *is then replaced by an arbitrary gauge parameter of the other type. [21]*

We show this theorem holds for the Einstein-Hilbert Lagrangian density $L=\frac{1}{16\pi G}R\sqrt{-g}$ in (4.31).

4.3. Noether-Wald Surface Charge

Now we are ready to define Noether-Wald surface charges. Take the variation of $\mathbf{L}$ along any infinitesimal diffeomorphism ξ^μ

$$\begin{aligned}\delta_\xi\mathbf{L}&=\mathcal{L}_\xi\mathbf{L}\\&=d(i_\xi\mathbf{L})+i_\xi d\mathbf{L}\\&=d(\xi^\mu L(d^{n-1}x)_\mu)\\&=\partial_\mu(\xi^\mu L)d^nx.\end{aligned} \tag{4.18}$$

Here, we used Cartan's magic formula $\mathcal{L}_\xi \omega = d(i_\xi \omega) + i_\xi(d\omega)$, where d and i_ξ stand for the exterior derivative and the interior product, respectively. The second term in the second line vanishes as Lagrangian $\mathbf{L}$ is a top form and therefore $d\mathbf{L} = 0$.

On the other hand, the same quantity can be written using the Second Noether Theorem. For some $(n-1,0)$-form $\mathbf{S}_\xi$ proportional to the equations of motion and its derivatives one obtains

$$\begin{aligned}
\delta_\xi \mathbf{L} &= \frac{\delta \mathbf{L}}{\delta \Phi^i}(\Phi^i)\delta_\xi \phi^i + d\mathbf{\Theta}[\mathcal{L}_\xi \Phi; \Phi] \\
&= d\mathbf{S}_\xi + d\mathbf{\Theta}[\mathcal{L}_\xi \Phi; \Phi] \\
&= \partial_\mu S^\mu_\xi d^n x + \partial_\mu \Theta^\mu [\mathcal{L}_\xi \Phi; \Phi] d^n x. \qquad (4.19)
\end{aligned}$$

Combining (4.18) and (4.19) we get

$$\begin{aligned}
d(i_\xi \mathbf{L}) &= d\mathbf{S}_\xi + d\mathbf{\Theta}[\mathcal{L}_\xi \Phi; \Phi] \\
\partial_\mu(\xi^\mu L) &= \partial_\mu S^\mu_\xi + \partial_\mu \Theta^\mu[\mathcal{L}_\xi \Phi; \Phi] \\
0 &= \partial_\mu(\xi^\mu L - S^\mu_\xi - \Theta^\mu[\mathcal{L}_\xi \Phi; \Phi]) \qquad (4.20)
\end{aligned}$$

The Noether current is defined as J^μ, the Hodge dual of the conserved $(n-1,0)$-form [36]

$$\mathbf{J}_\xi \coloneqq i_\xi \mathbf{L} - \mathbf{\Theta}[\mathcal{L}_\xi \Phi; \Phi]. \qquad (4.21)$$

It is seen from (4.20) that $d\mathbf{J}_\xi = d\mathbf{S}_\xi$ and it vanishes since $\mathbf{S}$ vanishes when equations of motion hold.

Hence, $d(\mathbf{J}_\xi - \mathbf{S}_\xi) = 0$. Then the vector $J^\mu_\xi - S^\mu_\xi$ is trivially conserved, i.e. conserved even without imposing the equations of motion. When equations of motion are imposed $J^\mu_\xi - S^\mu_\xi \approx J^\mu_\xi$.

Since $J^\mu_\xi - S^\mu_\xi$ is trivially conserved, it can be written using an exact form, assuming we can apply the Poincaré Lemma. Then $\mathbf{J}_\xi = \mathbf{S}_\xi + d\mathbf{Q}_\xi$ for some $(n-2,0)$-form $\mathbf{Q}_\xi$. A short proof which shows that the Poincaré Lemma can be used is given in [21]. $\mathbf{Q}_\xi = Q^{\mu\nu}_\xi (d^{(n-2)}x)_{\mu\nu}$ is called *Noether potential* or *Noether-Wald surface charge*, and it satisfies that $J^\mu_\xi - S^\mu_\xi = \partial_\nu Q^{\mu\nu}_\xi$.

We define the operator I_ξ such that $\mathbf{Q}_\xi = I_\xi(\mathbf{J}_\xi - \mathbf{S}_\xi)$ as follows. For any $(k,0)$-form ω,

$$I_\xi \omega_\xi = \frac{1}{n-k} \xi^\alpha \frac{\partial}{\partial \partial_\mu \xi^\alpha} \frac{\partial}{\partial dx^\mu} \omega_\xi + \text{(higher derivative terms)} \tag{4.22}$$

Then the Noether-Wald charge can be written as

$$\begin{aligned} \mathbf{Q}_\xi &= I_\xi(\mathbf{J}_\xi - \mathbf{S}_\xi) \\ &= I_\xi i_\xi \mathbf{L} - I_\xi \boldsymbol{\Theta}[\mathcal{L}_\xi \Phi; \Phi] - I_\xi \mathbf{S}_\xi \\ &= -I_\xi \boldsymbol{\Theta}[\mathcal{L}_\xi \Phi; \Phi] \end{aligned} \tag{4.23}$$

since $\mathbf{S}_\xi$ and $i_\xi \mathbf{L}$ do not contain any derivatives of ξ.

Let us give the final result of this section as stated in [21].

Theorem 4.5 (Fundamental theorem of the covariant phase formalism)**.** *In the Grassmann odd convention for δ, contracting the presymplectic form with a gauge transformation $\delta_\xi \Phi^i$, it exists an $(n-2,1)$-form $\mathbf{k}_\xi[\delta\Phi;\Phi]$ that satisfies the identity*

$$\boldsymbol{\omega}[\delta_\xi \Phi, \delta\Phi; \Phi] \approx d\mathbf{k}_\xi[\delta\Phi;\Phi], \tag{4.24}$$

where Φ^i solves the equations of motion, and $\delta\Phi^i$ solves the linearized equations of motion around the solution Φ^i. The infinitesimal surface charge $\mathbf{k}_\xi[\delta\Phi;\Phi]$ is unique, up to a total derivative that does not affect the equality above, and it is given in terms *of the Noether-Wald surface charge and the presymplectic potential by the following relation*

$$\mathbf{k}_\xi = -\delta \mathbf{Q}_\xi[\delta\Phi;\Phi] + i_\xi \mathbf{\Theta}[\delta\Phi;\Phi] + \text{ \textit{total derivative.}} \tag{4.25}$$

We would like to emphasize that the above mentioned $\mathbf{k}_\xi$ is not equal to the surface charge density $\mathbf{k}$ in (4.15), which is an $(n-2,0)$-form and does not contain $\delta\Phi$ terms.

4.3.1. Noether-Wald Surface Charge of the Einstein Gravity

Now, let's consider the Einstein-Hilbert Lagrangian in n dimensions

$$L[g_{\mu\nu}] = \frac{1}{16\pi G}\sqrt{-g}R. \tag{4.26}$$

To calculate an arbitrary variation of the Lagrangian density one needs the following. Denoting $h_{\mu\nu} := \delta g_{\mu\nu}$,

$$\begin{aligned}
\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}, \\
\delta g^{\alpha\beta} &= -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}, \\
\delta R_{\alpha\beta} &= \nabla_\lambda(\delta\Gamma^\lambda_{\alpha\beta}) - \nabla_\beta(\delta\Gamma^\lambda_{\lambda\alpha}), \\
\delta\Gamma^\lambda_{\alpha\beta} &= \frac{1}{2}g^{\lambda\rho}(\nabla_\beta h_{\rho\alpha} + \nabla_\alpha h_{\rho\beta} - \nabla_\rho h_{\alpha\beta}).
\end{aligned} \tag{4.27}$$

$$
\begin{aligned}
\delta L &= \frac{1}{16\pi G}(\delta\sqrt{-g}R + \sqrt{-g}\delta(R_{\alpha\beta}g^{\alpha\beta})) \\
&= \frac{\sqrt{-g}}{16\pi G}\left[\frac{1}{2}g^{\mu\nu}R\delta g_{\mu\nu} + [\nabla_\lambda(\delta\Gamma^\lambda_{\beta\alpha}) - \nabla_\beta(\delta\Gamma^\lambda_{\lambda\alpha})]g^{\alpha\beta} + R_{\alpha\beta}(-g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu})\right] \\
&= \frac{\sqrt{-g}}{16\pi G}\left[\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu} + \frac{1}{2}\left(\nabla_\lambda\nabla_\beta h^{\lambda\beta} + \nabla_\lambda\nabla_\alpha h^{\lambda\alpha} - \nabla_\lambda\nabla^\lambda h^\alpha_\alpha\right.\right. \\
&\qquad\qquad \left.\left. -\nabla_\beta\nabla_\lambda h^{\lambda\beta} - \nabla_\beta\nabla^\beta h^\lambda_\lambda + \nabla_\beta\nabla_\rho h^{\beta\rho}\right)\right] \\
&= \frac{\sqrt{-g}}{16\pi G}\left[-G^{\mu\nu}\delta g_{\mu\nu} + \nabla_\lambda(\nabla_\beta h^{\lambda\beta} - \nabla^\lambda h^\beta_\beta)\right] \\
&= \frac{\delta L}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\sqrt{-g}}{16\pi G}\nabla_\lambda(\nabla_\beta h^{\lambda\beta} - \nabla^\lambda h). \qquad (4.28)
\end{aligned}
$$

Now the variation is written in the form of (4.19). Under an infinitesimal diffeomorphism ξ^μ the equation (4.28) becomes

$$
\delta_\xi L = \frac{\delta L}{\delta g_{\mu\nu}}\delta_\xi g_{\mu\nu} + \frac{\sqrt{-g}}{16\pi G}\nabla_\lambda(\nabla_\beta h^{\lambda\beta} - \nabla^\lambda h), \qquad (4.29)
$$

and in this case, $h = \delta_\xi g$ is given by

$$
\begin{aligned}
h_{\mu\nu} &= \delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}, \\
h^{\mu\nu} &= 2\nabla^{(\mu}\xi^{\nu)}, \\
h &= h^\mu_\mu = 2\nabla_\mu\xi^\mu. \qquad (4.30)
\end{aligned}
$$

Since ξ is a diffeomorphism we can apply the Noether's second theorem, Theorem (4.4), and write the first part of the equation (4.29) as a total derivative term. Its derivation is given below, where we use the symmetry of the Einstein tensor $G^{\mu\nu}$ and the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$.

$$\begin{aligned} \frac{\delta L}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} &= -\frac{\sqrt{-g}}{16\pi G} G^{\mu\nu} h_{\mu\nu} \\ &= -\frac{\sqrt{-g}}{8\pi G} \nabla_\mu (G^{\mu\nu} \xi_\nu) \\ &= \partial_\mu (-\frac{\sqrt{-g}}{8\pi G} G^{\mu\nu} \xi_\nu). \end{aligned} \tag{4.31}$$

Thus, in this case $\mathbf{S}_\xi$ in Noether's second theorem is given by $\mathbf{S}_\xi = -\frac{\sqrt{-g}}{8\pi G} G^{\mu\nu} \xi_\nu (d^{n-1})_\mu$.

The second term in the equation (4.29) is already in the form $\partial_\mu \Theta^\mu$, yet we can make some modifications and introduce the Riemann tensor using $(\nabla_c \nabla_b - \nabla_b \nabla_c)\xi_a = R^d_{abc} \xi_d$ in.

$$\begin{aligned} \Theta^\mu_\xi &= \frac{\sqrt{-g}}{16\pi G} (\nabla_\nu h^{\mu\nu} - \nabla^\mu h^\nu_\nu) \\ &= \frac{\sqrt{-g}}{8\pi G} (\nabla_\nu (\nabla^\mu \xi^\nu - \nabla^{[\mu} \xi^{\nu]}) - \nabla^\mu (\nabla_\nu \xi^\nu)) \\ &= \frac{\sqrt{-g}}{8\pi G} (R^\nu{}_{\alpha\nu}{}^\mu \xi^\alpha + \nabla_\nu \nabla^{[\nu} \xi^{\mu]}) \\ &= \frac{\sqrt{-g}}{8\pi G} (R^{\mu\nu} \xi_\nu + \nabla_\nu \nabla^{[\nu} \xi^{\mu]}). \end{aligned} \tag{4.32}$$

Hence

$$\begin{aligned} \delta_\xi L &= \partial_\mu (S^\mu_\xi + \Theta^\mu_\xi) \\ &= \partial_\mu \left(-\frac{\sqrt{-g}}{8\pi G} G^{\mu\nu} \xi_\nu + \frac{\sqrt{-g}}{8\pi G} (R^{\mu\nu} \xi_\nu + \nabla_\nu \nabla^{[\nu} \xi^{\mu]}) \right) \\ &= \partial_\mu \left\{ \frac{\sqrt{-g}}{8\pi G} \left(\frac{1}{2} R \xi^\mu + \nabla_\nu \nabla^{[\nu} \xi^{\mu]} \right) \right\}, \end{aligned} \tag{4.33}$$

or equivalently, as seen in (4.19)

$$\delta_\xi \mathbf{L} = \partial_\mu \left\{ \frac{\sqrt{-g}}{8\pi G} \left(\frac{1}{2} R \xi^\mu + \nabla_\nu \nabla^{[\nu} \xi^{\mu]} \right) \right\} d^n x. \tag{4.34}$$

One can also compute the same quantity as in (4.18) to get

$$\delta_\xi \mathbf{L} = d(i_\xi \mathbf{L}) = \partial_\mu(\xi^\mu L) d^n x = \partial_\mu \left(\frac{\sqrt{-g}}{16\pi G} R \xi^\mu \right) d^n x. \tag{4.35}$$

Then the standard Noether current described in (4.21) is

$$\begin{aligned}
\mathbf{J}_\xi &= i_\xi \mathbf{L} - \mathbf{\Theta}_\xi \\
&= \frac{\sqrt{-g}}{16\pi G} R \xi^\mu (d^{n-1}x)_\mu - \frac{\sqrt{-g}}{8\pi G}(R^{\mu\nu}\xi_\nu + \nabla_\nu \nabla^{[\nu}\xi^{\mu]})(d^{n-1}x)_\mu \\
&= \frac{\sqrt{-g}}{8\pi G}(-G^{\mu\nu}\xi_\nu + \nabla_\nu \nabla^{[\nu}\xi^{\mu]})(d^{n-1}x)_\mu.
\end{aligned} \tag{4.36}$$

Either from (4.34) and (4.35) or from the result $d(\mathbf{J} - \mathbf{S}) = 0$ in the previous section it is seen

$$\partial_\nu \partial_\mu \left(\frac{\sqrt{-g}}{8\pi G} \nabla^{[\nu}\xi^{\mu]} \right) d^n x = 0. \tag{4.37}$$

Applying the Poincaré Lemma, the Noether-Wald surface charge with $(J-S)^\mu = \partial_\nu Q^{\mu\nu}$ is calculated as

$$\mathbf{Q}_\xi = \frac{\sqrt{-g}}{8\pi G} \nabla^{[\mu}\xi^{\nu]} (d^{(n-2)}x)_{\mu\nu}. \tag{4.38}$$

To compute the Noether-Wald surface charge density $\mathbf{k}_\xi = -\delta \mathbf{Q}_\xi + i_\xi \mathbf{\Theta}$ two more derivations are needed:

$$\begin{aligned}
i_\xi \mathbf{\Theta} &= (\xi^\nu \Theta^\mu - \xi^\mu \Theta^\nu)(d^{n-2}x)_{\mu\nu} \\
&= \frac{\sqrt{-g}}{16\pi G}[\xi^\nu(\nabla_\alpha h^{\mu\alpha} - \nabla^\mu h) - \xi^\mu(\nabla_\alpha h^{\nu\alpha} - \nabla^\nu h)](d^{n-2}x)_{\mu\nu}
\end{aligned} \tag{4.39}$$

$$
\begin{aligned}
-\delta\mathbf{Q}_\xi &= \delta\left(\frac{\sqrt{-g}}{8\pi G}g^{\alpha\nu}\nabla_\alpha\xi^\mu\right)(d^{n-2}x)_{\mu\nu} \\
&= \frac{1}{8\pi G}[\delta\sqrt{-g}\nabla^\nu\xi^\mu + \sqrt{-g}\delta g^{\alpha\nu}\nabla_\alpha\xi^\mu + \sqrt{-g}g^{\alpha\nu}\delta(\nabla_\alpha\xi^\mu)](d^{n-2}x)_{\mu\nu} \\
&= \frac{1}{8\pi G}\left[\frac{1}{2}\sqrt{-g}g^{\sigma\rho}\delta g_{\sigma\rho}\nabla^\nu\xi^\mu + \sqrt{-g}(-g^{\alpha\sigma}g^{\nu\rho}\delta g_{\sigma\rho})\nabla_\alpha\xi^\mu\right. \\
&\qquad\qquad\qquad\qquad\qquad\left. + \sqrt{-g}g^{\alpha\nu}\delta(\partial_\alpha\xi^\mu + \Gamma^\mu_{\alpha\lambda}\xi^\lambda)\right](d^{n-2}x)_{\mu\nu} \\
&= \frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}h^\rho_\rho\nabla^\nu\xi^\mu - h^{\alpha\nu}\nabla_\alpha\xi^\mu + g^{\alpha\nu}\delta(\Gamma^\mu_{\alpha\lambda})\xi^\lambda\right](d^{n-2}x)_{\mu\nu} \\
&= \frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}h\nabla^\nu\xi^\mu - h^{\alpha\nu}\nabla_\alpha\xi^\mu + \frac{1}{2}(\nabla^\nu h^{\mu\lambda} + \nabla^\lambda h^{\mu\nu} - \nabla^\mu h^{\nu\lambda})\xi_\lambda\right](d^{n-2}x)_{\mu\nu} \\
&= \frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}h\nabla^\nu\xi^\mu - h^{\alpha\nu}\nabla_\alpha\xi^\mu + \frac{1}{2}(\nabla^\nu h^{\mu\lambda} - \nabla^\mu h^{\nu\lambda})\xi_\lambda\right](d^{n-2}x)_{\mu\nu} \qquad (4.40)
\end{aligned}
$$

We use the antisymmetry of $(d^{n-2}x)_{\mu\nu}$ at the beginning and the symmetry of $h^{\mu\nu}$ in the last step. Again by using the antisymmetry of $(d^{n-2}x)_{\mu\nu}$, the surface charge density can be written as

$$
\begin{aligned}
\mathbf{k}_\xi &= -\delta\mathbf{Q}_\xi + i_\xi\mathbf{\Theta} \\
&= \frac{\sqrt{-g}}{16\pi G}[h\nabla^\nu\xi^\mu - 2h^{\alpha\nu}\nabla_\alpha\xi^\mu + (\nabla^\nu h^{\mu\lambda} - \nabla^\mu h^{\nu\lambda})\xi_\lambda \\
&\qquad\qquad + \xi^\nu(\nabla_\alpha h^{\mu\alpha} - \nabla^\mu h) - \xi^\mu(\nabla_\alpha h^{\nu\alpha} - \nabla^\nu h)](d^{n-2}x)_{\mu\nu} \\
&= \frac{\sqrt{-g}}{16\pi G}(h\nabla^\nu\xi^\mu - 2h^{\alpha\nu}\nabla_\alpha\xi^\mu + 2\nabla^{[\nu}h^{\mu]\lambda}\xi_\lambda - 2\xi^{[\mu}\nabla_\alpha h^{\nu]\alpha} + 2\xi^{[\mu}\nabla^{\nu]}h)(d^{n-2}x)_{\mu\nu} \\
&= \frac{\sqrt{-g}}{8\pi G}\left(\frac{1}{2}h\nabla^\nu\xi^\mu - h^{\alpha\nu}\nabla_\alpha\xi^\mu + \nabla^\nu h^{\mu\lambda}\xi_\lambda - \xi^\mu\nabla_\alpha h^{\nu\alpha} + \xi^\mu\nabla^\nu h\right)(d^{n-2}x)_{\mu\nu}. \qquad (4.41)
\end{aligned}
$$

4.4. Conserved charges

By integrating the $(n-2,1)$-form $\mathbf{k}_\xi[\delta\Phi;\Phi]$ over a closed surface of codimension 2, for example on an $(n-2)$-sphere where the coordinates t and r are fixed, we obtain the *surface charge*

$$
\not{\delta} H_\xi[\delta\Phi;\Phi] = \oint_S \mathbf{k}_\xi[\delta\Phi;\Phi]. \qquad (4.42)
$$

$\not\delta H_\xi$ is a $(0,1)$-form, which corresponds to the local variation of some charge between the two solutions Φ^i and $\Phi^i + \delta\Phi^i$. This means that are now on shell, i.e. Φ^i solves the equations of motion and $\delta\Phi^i$ solves the linearized equations of motion around Φ^i.

By using $\not\delta$ we underline that we do not know if this expression is a variation of some charge. If $\not\delta H_\xi$ is an exact 1-form in the space of fields then a surface charge H_ξ can be defined. A necessary condition is

$$\delta\not\delta H = \delta\left(\oint_S \mathbf{k}_\xi[\delta\Phi;\Phi]\right) = \delta_1 \oint_S \mathbf{k}_\xi[\delta_2\Phi;\Phi] - \delta_2 \oint_S \mathbf{k}_\xi[\delta_1\Phi;\Phi] = 0, \tag{4.43}$$

for all $\delta_1\Phi, \delta_2\Phi \in T^*_\Phi F,$ for all $\Phi \in F$, and it is called the *integrability condition.* If this holds and the use the Poincaré Lemma is allowed in the fields space, which we will assume from now on, we say the charge is *integrable*, i.e. H_ξ exists such that $\delta H_\xi = \not\delta H_\xi[\delta\Phi;\Phi]$.

To define H_ξ explicitly let $\bar{\Phi}^i$ be a reference field configuration, for example a background metric $\bar{g}_{\mu\nu}$ if we consider gravity. Define H_ξ at the point Φ^i as

$$H_\xi[\Phi;\bar{\Phi}] = \int_\gamma \oint_S \mathbf{k}_\xi[\delta\Phi;\Phi] + N_\xi[\bar{\Phi}], \tag{4.44}$$

where γ is some path between $\bar{\Phi}^i$ and Φ^i, and $N_\xi[\bar{\Phi}]$ depends only on the reference field configuration. The integrability condition guarantees that the integral is independent of the path γ.

If such an H_ξ exists and remains invariant under any continuous deformation of S, it is said to be *conserved.* H_ξ is conserved if and only if $\boldsymbol{\omega}[\delta_\xi\Phi, \delta\Phi;\Phi] \approx d\mathbf{k}_\xi[\delta\Phi;\Phi] \approx 0$ so that

$$H_\xi|_{S_1} - H_\xi|_{S_2} = \int_\gamma \oint_{S_1} \mathbf{k}_\xi - \int_\gamma \oint_{S_2} \mathbf{k}_\xi = \int_\gamma \int_C d\mathbf{k}_\xi \approx \int_\gamma \int_C \boldsymbol{\omega}. \tag{4.45}$$

4.4.1. Constructing Charges in Gravity and Asymptotic Symmetry Group

In this section we aim to construct charges for gravitational theories and follow closely the lecture notes [21]. Since general relativity is non-linear and hard to handle, we will prefer to use an infinitesimal linearization about a known solution, a *background metric*, which preferably admits some number of symmetries. Then, thanks to the variational bicomplex structure, we take an integral of this infinitesimal charge in the field space as given in (4.44) to calculate charges for another metric. A more detailed explanation can be found in [21]. There are three types of symmetries that are used to define conserved charges of the gravity theory:

(i) Exact symmetries, symmetries generated by Killing vectors, which imply directly $\boldsymbol{\omega}[\delta_\xi \Phi, \delta\Phi; \Phi] = 0$ and give conserved surface charges in the bulk of the spacetime.

(ii) Asymptotic symmetries generated by asymptotically Killing vectors, which imply $\boldsymbol{\omega}[\delta_\xi \Phi, \delta\Phi; \Phi] \to 0$ as $r \to \infty$ and give surface charges conserved at the spatial infinity $r \to \infty$.

(iii) Symplectic symmetries, transformations generated by vector fields under which the presymplectic form $\boldsymbol{\omega}$ vanishes but are not isometries or asymptotic isometries of the metric. They give surface charges conserved in the bulk.

We want to make use of the generalized Noether theorem, Theorem (4.3), to define these surface charges. Start with a Lagrangian density L defined on $(\mathcal{M}, g_{\mu\nu})$ and consider the gauge transformations. Gauge transformations are given by diffeomorphisms, since the theory is assumed to be generally covariant. To apply the generalized Noether theorem we need to select the field symmetries. A field symmetry, defined in Section 4.2, of $g_{\mu\nu}$ is a transformation generated by a vector field ξ^μ such that $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \approx 0$, by definition. This is satisfied by the Killing vectors of $g_{\mu\nu}$, however, $g_{\mu\nu}$ is just a generic metric, hence we can not specify any set of Killing vectors in general and construct charges.

Instead considering a single metric $g_{\mu\nu}$, we may restrict ourselves to a set of field configurations, i.e. a family of metrics, that have some common property. Consider a metric $g_{\mu\nu}$ close enough to a background metric $\bar{g}_{\mu\nu}$ such that we can linearize the theory. Assume $\bar{g}_{\mu\nu}$ solves the equations of motion and $h_{\mu\nu}$ solves the linearized equations of motion. The Lagrangian density of the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ is gauge invariant under any diffeomorphism ξ^μ that satisfies $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$ [37]. If ξ is also a Killing vector of $\bar{g}_{\mu\nu}$ then it is a global symmetry of the linearized theory for $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where $\delta_\xi h_{\mu\nu} = 0$. Such metrics $g_{\mu\nu}$ give a family of metrics that share an exact symmetry ξ, and the corresponding $(n-2)$ surface charge given by the generalized Noether theorem, Theorem 4.3.

What we are interested in is using the asymptotic symmetries and the generalized Noether theorem. Similar to the case above, we start by defining a set G of field configurations, i.e. metrics, by restricting them to a specific set of conditions, the so-called boundary conditions. We will explain how these boundary conditions are chosen and which metrics to be included in G in the Section 5.2. The *allowed diffeomorphisms* in G are the vector fields ξ^μ that sends a metric in G to some metric in G, or equivalently its action is tangential to G. This way it is guaranteed that when acting on the metrics in G with ξ we preserve the chosen boundary conditions, hence they preserve G. The allowed diffeomorphisms will form a Lie algebra, as we will see in Sections 5.2 and 5.3.

If we assume that we define G such that its allowed diffeomorphisms are asymptotic Killing vectors, defined in (5.44), a conserved charge H_ξ can be defined. If H_ξ integrable and gives a finite charge, either it is zero for all $g_{\mu\nu} \in G$, and in this case the corresponding diffeomorphism ξ is called a gauge transformation and it does not change the physical state but merely defines a coordinate transformation, or it is nonzero for some $g_{\mu\nu} \in G$ and therefore it corresponds to a change in the physical state. The *asymptotic symmetry group* is then defined as the following quotient group that gives the "state-changing transformations".

$$\text{Asymptotic symmetry group} = \frac{\text{Allowed diffeomorphisms}}{\text{Gauge transformations}} \tag{4.46}$$

Until now, we developed a systematic approach to define conserved charges, but let us mention some ambiguities our definitions suffer from. Our definition of charges depend on the vanishing of the presymplectic form $\boldsymbol{\omega}[\delta\Phi, \delta\Phi; \Phi]$. One is allowed to add an exact n-form $\mathbf{K}$ to the Lagrangian n-form $\mathbf{L}[\Phi]$, but this would correspond to total variation to the presymlectic potential $\Theta \to \Theta + \delta\mathbf{K}$, so it does not change the presymplectic form $\boldsymbol{\omega} = \delta\Theta$. Similarly, an exact $n-1$-form $\mathbf{B}$ can be added to the presymplectic potential form $\boldsymbol{\Theta}$, but the corresponding term $\delta d\mathbf{B}$ added to the presymplectic form $\boldsymbol{\omega}$ vanishes when $\mathbf{k}_\xi$ is calculated for an exact symmetry ξ. Notice that the Noether-Wald charge $\mathbf{Q}$ is also ambiguous, one can add a closed $n-2$-form so the surface charge densities $\mathbf{k}_\xi$ are defined uniquely up to a total derivative, though this does not affect the charges H_ξ. They change if we change the representative field $\bar{\Phi}$ and hence the part $N_\xi[\bar{\Phi}]$ in the (4.44). For a more detailed discussion the reader may check the works [21, 36].

4.4.2. Charge Algebra and the Representation Theorem

There is a similar concept, an algebra that contains the corresponding charges. The set of charges form an algebra under the Poisson bracket defined by

$$\{H_\chi, H_\xi\} \coloneqq \delta_\xi H_\chi = i_{\delta_\xi}\delta H_\chi = i_{\delta_\xi}\oint_S \mathbf{k}_\chi[\delta\Phi; \Phi] = \oint_S \mathbf{k}_\chi[\delta_\xi\Phi; \Phi], \tag{4.47}$$

where χ and ξ are arbitrary infinitesimal diffeomorphisms. Note the use of the operator i_{δ_a} given in (4.2). Yet, one needs to check if this gives a conserved charge and show that the algebra is closed under this Poisson bracket. We refer the reader for details to [21] and state the following theorem without proving.

Theorem 4.6 (Charge representation theorem)**.** *Assuming integrability* (4.43)*, the conserved charges associated to a Lie algebra of diffeomorphisms also form an algebra under the Poisson bracket* $\{H_\chi, H_\xi\} \coloneqq \delta_\xi H_\chi$*, which is isomorphic to the Lie algebra of diffeomorphisms up to a central extension, a term commutes with all the elements in the algebra. It gives a Lie algebra only when this term is zero.*

4.4.3. A Conserved Charge for BTZ Black Hole

We will give an examples now to demonstrate the calculation of the conserved charges. Let us show that the conserved charge corresponding to the Killing vector ∂_t is M for BTZ black hole, which is interpreted as its mass. Here J is taken to be zero to simplify the calculations.

If $J = 0$ and $8G = 1$, the BTZ black hole in (3.37) is given by the metric

$$ds^2 = \left(M - \frac{r^2}{l^2}\right) dt^2 + \frac{1}{-M + \frac{r^2}{l^2}} dr^2 + r^2 d\phi^2. \tag{4.48}$$

Some necessary quantities are the following

$$g^{tt} = \frac{1}{M - \frac{r^2}{l^2}}, \qquad g^{rr} = -M + \frac{r^2}{l^2}, \qquad g^{\phi\phi} = \frac{1}{r^2}, \qquad \sqrt{-g} = r, \tag{4.49}$$

$$\begin{aligned} &\Gamma^r_{tt} = \frac{r}{l^2}\left(-M + \frac{r^2}{l^2}\right), \qquad &&\Gamma^r_{\phi\phi} = \left(M - \frac{r^2}{l^2}\right) r, \qquad &&\Gamma^\phi_{r\phi} = \frac{1}{r}, \\ &\Gamma^r_{rr} = \frac{\frac{r}{l^2}}{M - \frac{r^2}{l^2}}, &&\Gamma^t_{rt} = \frac{-\frac{r}{l^2}}{M - \frac{r^2}{l^2}}. && \end{aligned} \tag{4.50}$$

Now, let $h_{\mu\nu} = \delta g_{\mu\nu}$ be an arbitrary variation of the metric. Here, we can only vary M, then using $g^{\mu\nu}$ to raise indices we write

$$\begin{aligned} &h_{tt} = \delta M, \qquad &&h_{rr} = \frac{-\delta M}{\left(-M + \frac{r^2}{l^2}\right)^2}, \qquad &&h_{\phi\phi} = 0, \\ &h^{rr} = -\delta M, &&h^{tt} = \frac{\delta M}{\left(M - \frac{r^2}{l^2}\right)^2}, &&h^{\phi\phi} = 0, \end{aligned} \tag{4.51}$$

$$h = h^t_t + h^r_r + h^\phi_\phi = \frac{\delta M}{M - \frac{r^2}{l^2}} - \frac{\delta M}{-M + \frac{r^2}{l^2}} = \frac{2\delta M}{M - \frac{r^2}{l^2}}. \tag{4.52}$$

With h as given above and $\xi = \partial_t$, we calculate the Noether-Wald surface charge density (4.41), taking $8G = 1$ and including only the nonvanishing terms, as follows

$$
\begin{aligned}
\mathbf{k}_\xi =& \frac{\sqrt{-g}}{\pi} \left(\frac{1}{2} h \nabla^\nu \xi^\mu - h^{\alpha\nu} \nabla_\alpha \xi^\mu + \nabla^\nu h^{\mu\lambda} \xi_\lambda - \xi^\mu \nabla_\alpha h^{\nu\alpha} + \xi^\mu \nabla^\nu h \right) (d^{n-2}x)_{\mu\nu} \\
=& \frac{\sqrt{-g}}{\pi)} \left[\left(\frac{1}{2} h \nabla^r \xi^t - h^{rr} \nabla_r \xi^t + \nabla^r h^{tt} \xi_t - \xi^t (\nabla_r h^{rr} + \nabla_t h^{rt} + \nabla_\phi h^{r\phi}) + \xi^t \nabla^r h \right) (dx)_{tr} \right. \\
& \left. \qquad + \left(\frac{1}{2} h \nabla^t \xi^t - h^{tt} \nabla_t \xi^r + \nabla^t h^{rt} \xi_t \right) (dx)_{rt} \right] \\
=& \frac{\sqrt{-g}}{\pi)} \left[\frac{1}{2} h g^{rr} \Gamma^t_{rt} \xi^t - h^{rr} \Gamma^t_{rt} \xi^t + g^{rr} (\partial_r h^{tt} + \Gamma^t_{rt} h^{tt} + \Gamma^t_{rt} h^{tt}) \xi^t g_{tt} \right. \\
& \qquad - \xi^t (\partial_r h^{rr} + \Gamma^r_{rr} h^{rr} + \Gamma^r_{tt} h^{tt} + \Gamma^r_{\phi\phi} h^{\phi\phi} + \Gamma^r_{rr} h^{rr} + \Gamma^t_{tr} h^{rr} + \Gamma^\phi_{\phi r} h^{rr}) \\
& \qquad \left. + \xi^t g^{rr} \partial_r h - \frac{1}{2} h g^{tt} \Gamma^r_{tt} \xi^t + h^{tt} \Gamma^r_{tt} \xi^t - g^{tt} (\Gamma^r_{tt} h^{tt} + \Gamma^t_{tr} h^{rr} \xi^t g_{tt}) \right] (dx)_{tr} \\
=& \frac{\sqrt{-g}}{\pi} \left(\frac{1}{r} \delta M \right) (dx)_{tr} \\
=& \frac{\delta M}{2\pi} d\phi.
\end{aligned}
\tag{4.53}
$$

where we directly calculated using the previous results (4.49), (4.50), (4.51), (4.52) and used our convention for $(dx)_{tr} = \frac{1}{2!1!} d\phi$ taking the orientation as $\epsilon_{tr\phi} = 1$ in the last step. Then the corresponding Noether charge

$$
\delta\!\!\!/ H = \int_0^{2\pi} \frac{\delta M}{2\pi} d\phi = \delta M \tag{4.54}
$$

is integrable and $H = M$.

5. ASYMPTOTICALLY AdS_3 SPACETIMES

In this chapter we introduce the notion of asymptotically anti-de Sitter spacetimes for which there are two basic approaches in literature. We will briefly review some historical works and end the chapter with a glimpse at the recent work of Compere, Song and Strominger [7].

Asymptotically locally AdS spacetimes (*AlAdS*) are solutions of the vacuum Einstein equation (2.2) with $\Lambda < 0$ that can be conformally compactified as explained in Section 3.4. This approach is used in [13, 14, 16]. Here, the conformal structure is not restricted, hence the boundary metric and the boundary topology are free. It gives the most general definition of such spacetimes.

The second approach defines *asymptotically AdS spacetimes* by setting boundary conditions for the metrics. These conditions should satisfy a list of properties we will see in Section 5.2. They may give specific restrictions on the topology of the boundary as in [38, 39] or choose specific boundary coordinates and boundary metric [1, 2, 7]. Also by giving a set of allowed variations they determine the behavior of the metric components at infinity, and hence they are called the *boundary conditions.* Note that such boundary conditions fix the boundary topology as a result. As seen in Section 3.4 the conformal boundary of AdS_{d+1} has the topology $\mathbb{R} \times S^{d-1}$, and the definitions we consider in this thesis include this as a boundary condition. The metrics then approach to this topology. The difference between these two approaches is emphasized by adding "locally" in the first definition, since it does not restrict the topology of the boundary. Note that *AAdS* spacetimes are *AlAdS*, but the converse is not always true.

We will start with *AlAdS* spacetimes and present a result proven in eighties by Fefferman and Graham [3], which is a very useful tool in studying *AlAdS* and *AAdS* spacetimes. We will give a simplified version of the proof that Fefferman-Graham expansion terminates in three dimensions. In the next sections, we will review the properties of the $AAdS$ metrics that obey Brown-Henneaux [1] and Compere-Song-Strominger boundary conditions [7], find the most general metric that obeys these conditions, their asymptotic symmetry groups and corresponding conserved charge algebras.

5.1. Fefferman-Graham Expansion

In 1985 [3], Fefferman and Graham asked if one can find a Poincaré metric on a manifold, given a conformal structure. Their set up is as follows.

Let N be an d-dimensional manifold with conformal structure $[g]$ of arbitrary signature. Create a new manifold $M = N \times [0,1]$ by adding a new coordinate r such that $r = 0$ gives $N = \partial M$. The problem is to find a Poincaré metric $\tilde{g}$ on M such that

1) $[g]$ is the conformal structure on the boundary for the metric $\tilde{g}$,

2) $\tilde{R}_{\mu\nu} = -\lambda \tilde{g}_{\mu\nu}$.

They concluded such a metric can be always written in a suitable coordinate system $(x^1, \ldots, x^d, r)$ as

$$\tilde{g} = r^{-2}\left[dr^2 + g_{ij}(x,r)dx^i dx^j\right], \tag{5.1}$$

and wanted to find the explicit form of $g_{ij}(x,r)$ as an expansion in r.

Note that if any diffeomorphism f of M that fixes N and $\tilde{g}$ is a solution then so is $f^*\tilde{g}$, the pushforward of $\tilde{g}$ under f. They added a new condition to narrow the scope of their search:

3) For all $1 < i,j < d$, $g_{ij}(x,r)$ is an even function of r, when written in the form (5.1).

We refer the reader to Theorem 2.3 in [3] and write it following [14, 21]:

Theorem 5.1. *Any asymptotically locally* AdS_{d+1} *metric can be brought into the following form near the asymptotic boundary*

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{l^2}{r^2}g_{(0)ij}dx^i dx^j + \mathcal{O}(r), \tag{5.2}$$

where r *is a spacelike coordinate and the asymptotic boundary of this spacetime is at* $r = 0$. *The metric* $g_{(0)ab}$ *is a representative of the chosen conformal structure and determines the behavior of the metric at the boundary.*

The expansion of $g_{ij}(x, r)$ in (5.1) is given explicitly in [40] for an arbitrary d. Letting $\rho = r^2$, the metric (5.1) becomes

$$ds^2 = l^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j \right). \tag{5.3}$$

They showed by imposing the Einstein field equations to (5.3) one gets

$$g(x, \rho) = g_{(0)} + \rho g_{(2)} + \cdots + \rho^{d/2} g_{(d)} + h_{(d)} \rho^{d/2} \log \rho + \ldots, \tag{5.4}$$

where $g_{(0)}, \ldots, g_{(d)}, h_{(d)}$ depend only on x^i and the logarithmic term appears when d is even. Given a metric $g_{(0)}$ one can determine $g_{(2)}, \ldots, g_{(d-2)}$ and $h_{(d)}$ in terms of $g_{(0)}$ by solving Einstein field equations order by order in ρ [40]. Here the boundary metric $g_{(0)}$ is free and when the boundary metric and the set of boundary coordinates are chosen they determine the topology of the boundary.

The Fefferman-Graham metric (5.3) takes the form

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2} \left(g_{(0)ij} + \frac{l^2}{r^2} g_{(2)ij} + \ldots \right) dx^i dx^j, \tag{5.5}$$

if we use $\rho = \dfrac{l^4}{r^2}$. The spacelike boundary of this metric is now at $r \to \infty$.

In three dimensions the Weyl tensor vanishes, as we showed in Section 2.2. Skenderis and Solodukhin showed in [41] the following theorem, which we will use for $AlAdS_3$ case.

Theorem 5.2. *If the Weyl tensor is identically zero, the Fefferman-Graham expansion* (5.3) *of an* $AlAdS_{d+1}$ *metric terminates at the second order*

$$g(x,p) = g_{(0)} + \rho g_{(2)} + \rho^2 g_{(4)} \tag{5.6}$$

or

$$g = \left(1 + \frac{\rho}{2} g_{(2)} g_{(0)}^{-1}\right) g_{(0)} \left(1 + \frac{\rho}{2} g_{(0)}^{-1} g_{(2)}\right) \tag{5.7}$$

with

$$g_{(2)ij} = \frac{l^2}{d-2}(R_{(0)ij} - \frac{1}{2(d-1)} R_{(0)} g_{(0)ij}) \tag{5.8}$$

if $d \neq 2$ *and*

$$\begin{aligned} Tr(g_{(0)}^{-1} g_{(2)}) &= \frac{l^2}{2} R_{(0)}, \\ g_{(2)ij} &= \frac{l^2}{2}(\frac{1}{2} R_{(0)} g_{(0)ij} - R_{(0)ij}), \end{aligned} \tag{5.9}$$

when $d = 2$. *Here,* $R_{(0)}$ *denotes the curvature of the metric* $g_{(0)}$ *and* $Tr(M)$ *denotes the trace of the matrix* M.

Proof. This result is given in [41] with only few computational details. We demonstrate this result in detail using their method for $d = 2$ and a boundary metric $g_{(0)}$ which is Ricci-flat, i.e. $R_{(0)} = 0$.

Start by picking a specific representative $g_{(0)}$ of the boundary conformal structure. Since the examples we will consider in this chapter are Ricci-flat, that will be enough for our purposes. We choose to work with $r = \sqrt{\rho}$ coordinate in (5.2). Let $l = 1$ for simplicity, which corresponds to taking $\Lambda = -1$ in (2.2) and it can be brought back through dimensional analysis. Using (5.4) we start with the metric below.

$$ds^2 = \frac{l^2}{r^2}(dr^2 + g_{ij}(x,r)dx^i dx^j), \tag{5.10}$$

$$g = g_{(0)} + g_{(1)}r + g_{(2)}r^2 + hr^2 \log r + g_{(3)}r^3 + g_{(4)}r^4 + \dots \tag{5.11}$$

Christoffel symbols, Riemann curvature tensor and Ricci tensor of metric (5.10) can be calculated in terms of the g_{ij} and r. In the following, prime denotes $\partial/\partial r$, ∇^g is the covariant derivative with respect to g for r fixed.

$$\begin{aligned}
&\Gamma^r_{rr} = -\frac{1}{r} && \Gamma^r_{ri} = 0 = \Gamma^i_{rr} \\
&\Gamma^r_{ij} = \frac{1}{r}g_{ij} - \frac{1}{2}g'_{ij} && \Gamma^i_{rj} = -\frac{1}{r}\delta^i_j + \frac{1}{2}g^{ik}g'_{kj} \\
&\Gamma^i_{jk} = \Gamma^i_{jk}(g)
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
R^r_{irj} &= -\frac{1}{r^2}g_{ij} + \frac{1}{2r}g'_{ij} - \frac{1}{2}g''_{ij} + \frac{1}{4}g'_{jk}g^{kl}g'_{li} \\
R^r_{ijk} &= -\frac{1}{2}(\nabla^g_j g'_{ik} - \nabla^g_k g'_{ij}) \\
R^i_{jkl} &= R^i_{jkl}(g) - \frac{1}{r^2}(\delta^i_k g_{jl} - \delta^i_l g_{jk}) + \frac{1}{2r}(\delta^i_k g'_{jl} - \delta^i_l g'_{jk} + g^{im}g'_{mk}g_{jl} - g^{im}g'_{ml}g_{jk}) \\
&\quad - \frac{1}{4}(g^{im}g'_{mk}g'_{ij} - g^{im}g'_{ml}g'_{jk}) \\
R_{ij} &= -\frac{n}{r^2}g_{ij} + \frac{1}{2r}((n-1)g'_{ij} + g^{kl}g'_{lk}g_{ij}) + R_{ij}(g) - \frac{1}{2}g''_{ij} + \frac{1}{2}g'_{ik}g^{kl}g'_{lj} - \frac{1}{4}g^{kl}g'_{kl}g'_{ij} \\
R_{rr} &= -\frac{n}{r^2} + \frac{1}{2r}g^{ij}g'_{ij} - \frac{1}{2}g^{ij}g''_{ij} + \frac{1}{4}g^{ij}g'_{jk}g^{kl}g'_{li} \\
R_{rk} &= \frac{1}{2}g^{ij}(\nabla^g_j g'_{ik} - \nabla^g_k g'_{ij})
\end{aligned} \tag{5.13}$$

The Einstein field equations (2.2) then becomes

$$\begin{aligned}\frac{1}{2r}g^{ij}g'_{ij} - \frac{1}{2}g^{ij}g''_{ij} + \frac{1}{4}g^{ij}g'_{jk}g^{kl}g'_{li} =& 0,\\ \frac{1}{2}g^{ij}(\nabla^g_j g'_{ik} - \nabla^g_k g'_{ij}) =& 0,\\ \frac{1}{2r}(g'_{ij} + g^{kl}g'_{lk}g_{ij}) + R_{ij}(g) - \frac{1}{2}g''_{ij} + \frac{1}{2}g'_{ik}g^{kl}g'_{lj} - \frac{1}{4}g^{kl}g'_{kl}g'_{ij} =& 0.\end{aligned} \tag{5.14}$$

We will solve these equations order by order in r. This is done by differentiating the equations with respect to r and then setting $r = 0$. To do this (5.14) can be written in the following form for convenience

$$g^{ij}g'_{ij} + r(-g^{ij}g''_{ij} + \frac{1}{2}g^{ij}g'_{jk}g^{kl}g'_{li}) = 0, \tag{5.15a}$$

$$g^{ij}(\nabla^g_j g'_{ik} - \nabla^g_k g'_{ij}) = 0, \tag{5.15b}$$

$$g'_{ij} + g^{kl}g'_{lk}g_{ij} + r(2R_{ij}(g) - g''_{ij} + g'_{ik}g^{kl}g'_{lj} - \frac{1}{2}g^{kl}g'_{lk}g'_{ij}) = 0. \tag{5.15c}$$

We can replace ∇^g by $\nabla^{(0)}$, the covariant derivative of $g_{(0)}$, since $\nabla^g - \nabla^{(0)}$ vanish when $r = 0$ and does not affect our calculations. The inverse metric can also be taken to be equal to the inverse of the boundary metric, $g^{kl} = g^{kl}_{(0)}$, for the same reason.

We will eliminate a term in the expansion and move on with the new form of the metric, then eliminate another term and repeat this process until reaching (5.6). It is easy to see that $g_{(1)} = 0$. With $r = 0$ the above equations give

$$\begin{aligned}g^{ij}g_{(1)ij} =& 0,\\ g^{ij}(\nabla^{(0)}_j g_{(1)ik} - \nabla^{(0)}_k g_{(1)ij}) =& 0,\\ g_{(1)ij} + g^{kl}g_{(1)lk}g_{(0)ij} =& 0.\end{aligned} \tag{5.16}$$

The first and the last line lines together show $g_{(1)ij} = 0$, so g is in form

$$g = g_{(0)} + g_{(2)}r^2 + hr^2\log r + g_{(3)}r^3 + g_{(4)}r^4 + \dots, \tag{5.17}$$

$$
\begin{aligned}
g' &= 2r(g_{(2)} + h) + 2hr\log r + 3r^2 g_{(3)} + 4r^3 g_{(4)} + \dots, \\
g'' &= 2(g_{(2)} + 2h) + 2h\log r + 6r g_{(3)} + 12r^2 g_{(4)} + \dots, \\
g''' &= \frac{2}{r}h + 6g_{(3)} + 24r g_{(4)} + \dots
\end{aligned}
$$

Differentiating (5.15a) we get

$$
-\frac{1}{2}Tr(g^{-1}g'g^{-1}g') + r[2Tr(g^{-1}g'g^{-1}g'') - Tr(g^{-1}g''') - Tr(g^{-1}g'g^{-1}g'g^{-1}g')] = 0, \quad (5.18)
$$

which gives for $r = 0$

$$
g^{ij}h_{ij} = 0. \quad (5.19)
$$

Thus, h is traceless, which will be useful in the following calculation. We differentiate (5.15c)

$$
\begin{aligned}
g''_{ij} - Tr(g^{-1}g'g^{-1}g')g_{ij} + Tr(g^{-1}g'')g_{ij} + Tr(g^{-1}g')g'_{ij} \\
+2R_{ij} - g''_{ij} + g'_{ik}g^{kl}g'_{lj} - \frac{1}{2}Tr(g^{-1}g')g'_{ij} \\
+r[2R'_{ij} - g'''_{ij} + g''_{ik}g^{kl}g'_{lj} + g'_{ik}g^{kl}g'_{lm}g^{mn}g'_{nj} \\
-\frac{1}{2}Tr(g^{-1}g'g^{-1}g')g'_{ij} - \frac{1}{2}Tr(g^{-1}g'')g'_{ij} - \frac{1}{2}Tr(g^{-1}g')g'_{ij}] = 0.
\end{aligned} \quad (5.20)
$$

Then for $r = 0$

$$
g^{kl}g_{(2)kl}g_{(0)ij} + R_{(0)ij} - h_{ij} = 0. \quad (5.21)
$$

The last term vanishes when contracted with the inverse metric, hence we get

$$
g^{kl}g_{(2)kl} = -\frac{1}{2}R_{(0)}, \quad (5.22)
$$

Now we write (5.22) back in (5.21) to get

$$R_{(0)ij} - \frac{1}{2} R_{(0)} g_{(0)ij} = h_{ij}. \tag{5.23}$$

This shows h_{ij} is the traceless Ricci tensor in two dimensions, but that is identically zero. Note that we did not use our condition $R_{(0)} = 0$ yet.

Until now, $g_{(1)}$ and the logarithmic part h are eliminated from the metric (5.10) for any boundary metric $g_{(0)}$, then g can be written as

$$g = g_{(0)} + g_{(2)} r^2 + g_{(3)} r^3 + g_{(4)} r^4 + \ldots. \tag{5.24}$$

Taking the derivative of (5.15b) at $r = 0$ we obtain

$$g^{ij} (\nabla_j^{(0)} 2 g_{(2)ik} - \nabla_k^{(0)} 2 g_{(2)ij}) = 0. \tag{5.25}$$

By using (5.22) and $R_{(0)} = 0$ condition, the second term vanishes and we have

$$g^{ij} \nabla_j^{(0)} (2 g_{(2)})_{ik} = 0. \tag{5.26}$$

Hence, the covariant divergence of $g_{(2)}$ with respect to $g_{(0)}$ is zero. Note that for $R_{(0)} \neq 0$ there would be a nonzero term containing the covariant derivative of $R_{(0)}$ on the right hand side.

Differentiating (5.15a) twice we get

$$Tr(g^{-1} g' g^{-1} g'') - Tr(g^{-1} g''') - Tr(g^{-1} g' g^{-1} g' g^{-1} g') + r[\ldots] = 0, \tag{5.27}$$

and by evaluating it at $r = 0$ we see

$$g^{ij} g_{(3)ij} = 0, \tag{5.28}$$

thus $g_{(3)}$ is traceless. We take the second derivative of (5.15c)

$$
\begin{aligned}
&[2Tr(g^{-1}g'g^{-1}g'g^{-1}g') - 3Tr(g^{-1}g'g^{-1}g'') + Tr(g^{-1}g''')]g_{ij} \\
&+\frac{3}{2}[Tr(g^{-1}g'') - Tr(g^{-1}g'g^{-1}g')]g'_{ij} + \frac{1}{2}Tr(g^{-1}g')g''_{ij} \\
&+2R'_{ij} - g'''_{ij} + g''_{ik}g^{kl}g'_{lj} + g'_{ik}g^{kl}g'_{lm}g^{mn}g'_{nj} + g'_{ik}g^{kl}g''_{lj} + r[\dots] = 0,
\end{aligned}
$$

and plug in $r = 0$ to have

$$g^{kl}g_{(3)kl}g_{ij} - g_{(3)ij} = 0. \tag{5.29}$$

But the first part is zero by (5.28), therefore $g_{(3)} = 0$.

So far, we eliminated $g_{(1)}$, h and $g_{(3)}$. Now g is in the form

$$g = g_{(0)} + g_{(2)}r^2 + g_{(4)}r^4 + \dots \tag{5.30}$$

Third derivative of (5.15a) evaluated at $r = 0$ gives

$$g_{(2)jk}g^{kl}g_{(2)li} - 4g_{(4)ji} = 0, \tag{5.31}$$

which implies $g_{(4)ij} = \frac{1}{4}g_{(2)ik}g^{kl}g_{(2)lj}$.

To see that the terms after $g_{(4)}$ are zero, we will switch to another coordinate system. But we could have keep on calculating using r, and we would get $g_{(5)} = 0$ from the fourth derivatives of (5.15a) and (5.15c), and the rest would also vanish after continuing this process. However, it is easier to see that this expansion stops at $g_{(4)}$ if the radial coordinate $\rho = r^2$ is used. From now on, prime will denote derivative with respect to ρ. In the analogue of the equations (5.15) for ρ, (5.15a) becomes [41]

$$g'' - \frac{1}{2}g'g^{-1}g' = 0. \tag{5.32}$$

By differentiating it and using $(g^{-1})' = -g^{-1}g'g^{-1}$ it is obtained

$$g''' = 0. \tag{5.33}$$

This shows the last term in the expansion is ρ^2, which is the main result shown by Skenderis and Solodukhin in [41] for an arbitrary boundary metric $g_{(0)}$. □

Summing it up, for a boundary metric $g_{(0)}$ with $R_{(0)} = 0$, the expansion (5.10) stops at the second order and

$$g_{(4)ij} = \frac{1}{4} g_{(2)ik} g^{kl} g_{(2)lj}, \tag{5.34a}$$

$$g^{kl} g_{(2)kl} = 0, \tag{5.34b}$$

$$g^{ij} \nabla_j^{(0)} g_{(2)ik} = 0, \tag{5.34c}$$

must hold for any choice of boundary coordinates x^i. We will use this result in the following chapters.

5.2. Brown-Henneaux Boundary Conditions

In this section we want to present how Brown and Henneaux [1] defined asymptotically AdS_3 spacetimes, similar to the four dimensional case studied in Henneaux and Teitelboim's paper in 1985 [2]. In this approach $AAdS_3$ geometry is defined by setting a set of boundary conditions that satisfy the following:

(i) The boundary conditions must be invariant under the symmetry group of AdS, i.e. AdS Killing vectors should send an *allowed metric*, a metric that obeys the boundary conditions, to another allowed metric.

(ii) The symmetries of this set of metrics (the asymptotic symmetries) must give well defined conserved charges, and their generators should obey the symmetry algebra of AdS.

(iii) Interesting solutions must be included in the set of metrics, such as BTZ black hole (3.37) in three dimensional case.

By setting boundary conditions we define a set G of allowed metrics. In the set G, Brown and Henneaux wanted to include the metrics of the form

$$ds^2 = -\left(\alpha^2 + \frac{r^2}{l^2}\right) dt^2 + 2A\alpha dt d\phi + \left(\alpha^2 + \frac{r^2 - A^2}{l^2}\right)^{-1} dr^2 + (r^2 - A^2)d\phi^2. \quad (5.35)$$

Here, A and α are arbitrary contants parametrizing this family of metrics. Note that these metrics behave like the static AdS_3 metric (3.7) as $r \to \infty$.

Starting with (5.35) in G, we want to find a set of boundary conditions. For them to satisfy condition (i) we should get a metric in G when we act on the metrics (5.35) with symmetry transformations of AdS_3, which are generated by $\mathfrak{so}(2,2)$ elements in (3.21). In static coordinates (3.6) these are

$$\begin{aligned}
J_{uv} &= -l\partial_t, \\
J_{ux} &= \left(1 + \frac{r^2}{l^2}\right)^{1/2} \left[l \sin\frac{t}{l} \cos\phi \partial_r + r\left(1 + \frac{r^2}{l^2}\right)^{-1} \cos\frac{t}{l}\cos\phi\partial_t - \frac{l}{r}\sin\frac{t}{l}\sin\phi\partial_\phi \right], \\
J_{uy} &= \left(1 + \frac{r^2}{l^2}\right)^{1/2} \left[l \sin\frac{t}{l} \sin\phi \partial_r + r\left(1 + \frac{r^2}{l^2}\right)^{-1} \cos\frac{t}{l}\sin\phi\partial_t + \frac{l}{r}\sin\frac{t}{l}\cos\phi\partial_\phi \right], \\
J_{vx} &= \left(1 + \frac{r^2}{l^2}\right)^{1/2} \left[l \cos\frac{t}{l} \cos\phi \partial_r - r\left(1 + \frac{r^2}{l^2}\right)^{-1} \sin\frac{t}{l}\cos\phi\partial_t - \frac{l}{r}\cos\frac{t}{l}\sin\phi\partial_\phi \right], \\
J_{vy} &= \left(1 + \frac{r^2}{l^2}\right)^{1/2} \left[l \cos\frac{t}{l} \sin\phi \partial_r - r\left(1 + \frac{r^2}{l^2}\right)^{-1} \sin\frac{t}{l}\sin\phi\partial_t + \frac{l}{r}\cos\frac{t}{l}\cos\phi\partial_\phi \right], \\
J_{xy} &= \partial_\phi. \qquad (5.36)
\end{aligned}$$

J_{uv} and J_{xy} are the Killing vectors of any metric in this family, so the metrics (5.35) remain invariant. When we calculate the Lie derivatives of the metric components of (5.35) under $\xi = J_{ux}, J_{uy}, J_{vx}, J_{vy}$ we see they change as follows

$$\begin{aligned}
\mathcal{L}_\xi g_{tt} &= \mathcal{O}(1),\\
\mathcal{L}_\xi g_{tr} &= \mathcal{O}(r^{-3}),\\
\mathcal{L}_\xi g_{t\phi} &= \mathcal{O}(1),\\
\mathcal{L}_\xi g_{rr} &= \mathcal{O}(r^{-4}),\\
\mathcal{L}_\xi g_{r\phi} &= \mathcal{O}(r^{-3}),\\
\mathcal{L}_\xi g_{\phi\phi} &= \mathcal{O}(1).
\end{aligned} \tag{5.37}$$

where the big-oh notation $\mathcal{O}(r^k)$ represents terms up to order r^k.

Then any metric in the form (5.35) is sent to a metric with components

$$\begin{aligned}
g_{tt} &= -\frac{r^2}{l^2} + \mathcal{O}(1),\\
g_{tr} &= \mathcal{O}(r^{-3}),\\
g_{t\phi} &= \mathcal{O}(1),\\
g_{rr} &= \frac{l^2}{r^2} + \mathcal{O}(r^{-4}),\\
g_{r\phi} &= \mathcal{O}(r^{-3}),\\
g_{\phi\phi} &= r^2 + \mathcal{O}(1),
\end{aligned} \tag{5.38}$$

and note that the metrics (5.35) already obey these conditions. These are called the *Brown-Henneaux boundary conditions*, and the set G of allowed metrics contains all metrics that satisfy 5.38 which actually can be explicitly parametrized as we do in the next section.

5.2.1. Bañados Metric

Here, we will derive the most general metric that obeys the Brown-Henneaux boundary conditions (5.38). Let $g_{\mu\nu}$ be such a metric and write it using Fefferman-

Graham expansion (5.5) with boundary coordinates t and ϕ we used above. The Brown-Henneaux boundary conditions then imply

$$g_{(0)} = \text{diag}(-1, l^2), \qquad \delta g_{(0)\mu\nu} = 0, \qquad \delta g_{(2)\mu\nu} = \text{arbitrary}. \tag{5.39}$$

The boundary coordinates are $t \in \mathbb{R}$ and $0 \leq \phi < 2\pi$, so these conditions also restrict the boundary in the shape of a cylinder.

Let us introduce the light cone coordinates on the boundary $x^{\pm} = \dfrac{t}{l} \pm \phi$ such that $g_{(0)ij}dx^i dx^j = -dt^2 + l^2 d\phi = -l^2 dx^+ dx^-$. Using Theorem 5.2 the metric (5.5) is written in the form

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}\left(g_{(0)ij} + \frac{l^2}{r^2}g_{(2)ij} + \frac{l^4}{r^4}g_{(4)ij}\right)dx^i dx^j, \tag{5.40}$$

where $g_{(2)ij}$ and $g_{(4)ij}$ satisfy (5.34).

We already know $g_{(0)}$ and we only need to solve for $g_{(2)}$ since $g_{(4)}$ will then be given by (5.34a). The equation (5.34b) gives $0 = 2g^{+-}g_{(2)+-}$ and implies $g_{(2)+-} = 0$. The Christoffel symbols of $g_{(0)}$ are all zero, so by (5.34c) we write

$$\begin{aligned} g^{+-}\nabla^{(0)}_{-}g_{(2)++} &= 0, & g^{-+}\nabla^{(0)}_{+}g_{(2)--} &= 0, \\ \partial_{-}g_{(2)++} &= 0, & \partial_{+}g_{(2)--} &= 0, \end{aligned} \tag{5.41}$$

and see that $g_{(2)\pm\pm}$ must be a function of $x^{\pm}$. We can then define $g_{(2)\pm\pm} \coloneqq l^2 L_{\pm}(x^{\pm})$ and after calculating $g_{(4)}$ in (5.34a) we prove the following theorem.

Theorem 5.3. *A three dimensional asymptotically AdS metric defined by Brown-Henneaux boundary conditions* (5.38) *can be written as*

$$ds^2 = \frac{l^2}{r^2}dr^2 + l^2 L_{-}(dx^-)^2 + l^2 L_{+}(dx^+)^2 + (-r^2 - \frac{l^4}{r^2}L_{-}L_{+})dx^- dx^+, \tag{5.42}$$

using Fefferman-Graham coordinates (5.5) *with* $x^{\pm} = \frac{t}{l} \pm \phi$*. In a more compact form* *we can write*

$$ds^2 = \frac{l^2}{r^2}dr^2 - \left(rdx^+ - l^2\frac{L_-}{r}dx^-\right)\left(rdx^- - l^2\frac{L_+}{r}dx^+\right), \tag{5.43}$$

where r is the radial coordinate, $0 \leq \phi < 2\pi$, $t \in \mathbb{R}$ and L_+ and L_- are functions of x^+ and x^-, respectively. This metric is called the Bañados metric.

It is worth mentioning that the BTZ black hole (3.37) is represented here when $L_\pm$ are taken to be constants. So the Brown-Henneaux boundary conditions (5.38) satisfy condition (iii). It is again seen here that each metric in this form has a cylindrical boundary at infinity, therefore they approach globally to the AdS_3 spacetime.

5.2.2. Asymptotic Symmetry Algebra

Recall that a Killing vector ξ of a spacetime is defined by $\mathcal{L}_\xi g_{\mu\nu} = 0$. They generate the continuous symmetries of the spacetime. Similarly we can define vector fields which does not preserve a specific metric as a Killing vector does, but preserves our set of metrics G, or equivalently the boundary conditions. A vector field ξ is said to be an *asymptotic Killing vector* if it satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{O}(\delta g_{\mu\nu}), \tag{5.44}$$

Note that this is what was defined as an "allowed diffeomorphism" in Section 4.4.1. Under the flow created by an asymptotic vector field the metric does not remain the same as in standard Killing vectors but it may move to another metric within the set G. Hence the number of asymptotic Killing vectors is much more as compared to the Killing vectors whose maximum number are limited by $\frac{1}{2}n(n+1)$. Our aim here is to find the set of asymptotic Killing vectors of the set G, the set of metrics obeying the Brown-Henneaux boundary conditions (5.38).

Using Gaussian normal coordinates at the boundary we can write the Fefferman-

Graham expansion (5.5) using $\rho = l \log \frac{r}{l}$ as

$$ds^2 = d\rho^2 + \left(e^{2\rho/l} g_{(0)ij} + g_{(2)ij} + e^{-2\rho/l} g_{(4)ij}\right) dx^i dx^j. \tag{5.45}$$

The boundary is now reached when $\rho \to \infty$.

In these coordinates Brown-Henneaux boundary conditions (5.38) are given by

$$g_{(0)++} = g_{(0)--} = 0, \qquad g_{(0)+-} = g_{(0)-+} = -1/2, \qquad \delta g_{(0)\mu\nu} = 0, \qquad \delta g_{(2)\mu\nu} = \text{arbitrary}, \tag{5.46}$$

where the light cone coordinates $x^\pm$ in (5.43) are used. An asymptotic Killing vector ξ should satisfy

$$\mathcal{L}_\xi g_{\rho\rho} = 0, \tag{5.47a}$$

$$\mathcal{L}_\xi g_{\rho\pm} = 0, \tag{5.47b}$$

$$\mathcal{L}_\xi g_{\pm\pm} = \mathcal{O}(1), \tag{5.47c}$$

$$\mathcal{L}_\xi g_{+-} = \mathcal{O}(1). \tag{5.47d}$$

Note that the equations (5.47a) and (5.47b) preserve the Fefferman-Graham form of the metric, the components $g_{\rho\rho}$ and $g_{\rho\pm}$ are kept unchanged. By (5.47c) and (5.47d), the other Brown-Henneaux boundary conditions in (5.46) remain invariant.

Now we solve for ξ. From (5.47a) it is seen ξ^ρ does not depend on ρ coordinate, hence it can be written as a function of $x^\pm$, then $\xi^\rho = f(x^+, x^-)$ for some function $f(x^+, x^-)$. Write (5.47b) by denoting the boundary coordinates x^+ and x^- with $i, j, k \ldots$

$$0 = \partial_i \xi^\rho + g_{ij} \partial_\rho \xi^j = g^{ik} \partial_i \xi^\rho + \partial_\rho \xi^k. \tag{5.48}$$

Then we can write $\xi^k = -\int g^{ik} \partial_i \xi^\rho \, d\rho + \epsilon^k(x^+, x^-)$ for some functions $\epsilon^\pm(x^+, x^-)$. The

$\partial_i \xi^\rho$ term in the integral is of order ρ^0, and the inverse metric is in the form

$$g^{++} = \mathcal{O}(e^{-4\rho/l}) = g^{--}, \quad \text{and} \quad g^{+-} = \mathcal{O}(e^{-2\rho/l}), \tag{5.49}$$

therefore $\xi^\pm = \epsilon^\pm(x^+, x^-) + \mathcal{O}(e^{-2\rho/l})$.

Until now we found the most general diffeomorphism that preserves the Fefferman-Graham coordinates and that satisfies

$$\begin{aligned} \xi^\rho &= \xi^\rho(x^+, x^-), \\ \xi^\pm &= \epsilon^\pm(x^+, x^-) + \mathcal{O}(e^{-2\rho/l}). \end{aligned} \tag{5.50}$$

Writing (5.47c) explicitly

$$\xi^\rho \partial_\rho g_{\pm\pm} + \xi^+ \partial_+ g_{\pm\pm} + \xi^- \partial_- g_{\pm\pm} + 2 g_{\pm\pm} \partial_\pm \xi^\pm + 2 g_{+-} \partial_\pm \xi^\mp = \mathcal{O}(1), \tag{5.51}$$

we see the middle terms on the left hand side are of order ρ^0. We have $g_{ij} = \mathcal{O}(e^{2\rho/l})$ easily seen from (5.45), where i, j denote boundary coordinates. Hence we write

$$\partial_\pm \xi^\mp = \mathcal{O}(e^{-2\rho/l}), \tag{5.52}$$

Combined with (5.50) this means $\partial_\mp \epsilon^\pm(x^+, x^-) = 0$ and therefore

$$\xi^\pm = \epsilon^\pm(x^\pm)\partial_\pm + \mathcal{O}(e^{-2\rho/l}). \tag{5.53}$$

The equation (5.47d) gives

$$\xi^\mu \partial_\mu g_{+-} + g_{++} \partial_- \xi^+ + g_{--} \partial_+ \xi^- + g_{+-}(\partial_+ \xi^+ + \partial_- \xi^-) = \mathcal{O}(1), \tag{5.54}$$

Here, we see the second and third terms does not contribute by (5.52). Recalling $\xi^\rho = \mathcal{O}(1) = \xi^\pm$ and $g_{+-} = e^{2\rho/l} + g_{+-}^{(2)} + \mathcal{O}(e^{-2\rho/l})$, the contributing terms in (5.54) are

$$\begin{aligned} &\xi^\rho \frac{2}{l} e^{2\rho/l} + e^{2\rho/l}(\partial_+\xi^+ + \partial_-\xi^-) = \mathcal{O}(1) \\ &\Rightarrow \xi^\rho = -\frac{l}{2}(\partial_+\epsilon^+ + \partial_-\epsilon^-) + \mathcal{O}(e^{-2\rho/l}). \end{aligned} \tag{5.55}$$

From (5.53) and (5.55) we derive that the asymptotic Killing vectors of asymptotically AdS spacetimes that obey Brown-Henneaux boundary conditions are given in Gaussian normal coordinates (5.45) as

$$\xi(\epsilon^+, \epsilon^-) = \epsilon^+(x^+)\partial_+ + \epsilon^-(x^-)\partial_- - \frac{l}{2}(\partial_+\epsilon^+ + \partial_-\epsilon^-)\partial_\rho + \mathcal{O}(e^{-2\rho/l}). \tag{5.56}$$

We can calculate the Lie brackets of these asymptotic Killing vectors ignoring the $\mathcal{O}(e^{-2\rho/l})$ terms

$$\begin{aligned} [\xi(\epsilon_1^+, \epsilon_1^-), \xi(\epsilon_2^+, \epsilon_2^-)] &= [\epsilon_1^+\partial_+ + \epsilon_1^-\partial_- - \frac{l}{2}(\partial_+\epsilon_1^+ + \partial_-\epsilon_1^-)\partial_\rho,\ (1 \to 2)] \\ &= [\epsilon_1^+\partial_+,\ \epsilon_2^+\partial_+] - \frac{l}{2}[\epsilon_1^+\partial_+,\ \partial_+\epsilon_2^+\partial_\rho] - \frac{l}{2}[\epsilon_1^+\partial_+,\ \partial_-\epsilon_2^-\partial_\rho] \\ &\quad + [\epsilon_1^-\partial_-,\ \epsilon_2^-\partial_-] - \frac{l}{2}[\epsilon_1^-\partial_-,\ \partial_+\epsilon_2^+\partial_\rho] - \frac{l}{2}[\epsilon_1^-\partial_-,\ \partial_-\epsilon_2^-\partial_\rho] \\ &\quad - \frac{l}{2}\left\{[\partial_+\epsilon_1^+\partial_\rho,\ \epsilon_2^+\partial_+] + [\partial_-\epsilon_1^-\partial_\rho,\ \epsilon_2^+\partial_+] + [\partial_+\epsilon_1^+\partial_\rho,\ \epsilon_2^-\partial_-]\right. \\ &\quad \left. + [\partial_-\epsilon_1^-\partial_\rho,\ \epsilon_2^-\partial_-]\right\} \\ &= (\epsilon_1^+\partial_+\epsilon_2^+ - \epsilon_2^+\partial_+\epsilon_1^+)\partial_+ + (\epsilon_1^-\partial_-\epsilon_2^- - \epsilon_2^-\partial_-\epsilon_1^-)\partial_- \\ &\quad - \frac{l}{2}(\epsilon_1^+\partial_+^2\epsilon_2^+ - \epsilon_2^+\partial_+^2\epsilon_1^+ + \epsilon_1^-\partial_-^2\epsilon_2^- - \epsilon_2^-\partial_-^2\epsilon_1^-)\partial_\rho \\ &= \xi((\epsilon_1^+\partial_+\epsilon_2^+ - \epsilon_2^+\partial_+\epsilon_1^+),\ (\epsilon_1^-\partial_-\epsilon_2^- - \epsilon_2^-\partial_-\epsilon_1^-)). \end{aligned} \tag{5.57}$$

This shows that these generators form an algebra as $\rho \to \infty$ and it is called the *asymptotic symmetry algebra*.

Notice that $\xi^+ = \xi(\epsilon^+, 0)$ and $\xi^- = \xi(0, \epsilon^-)$ commute with each other. This means they give two infinite families of independent generators. As the functions ϵ^+ and ϵ^- live on the cylindrical boundary of AdS_3, if $x^\pm \sim x^\pm + 2\pi$, we may introduce their Fourier modes

$$\begin{aligned} \epsilon^+(x^+) &\rightsquigarrow \epsilon_n^+ = ie^{inx^+}\partial_+, \\ \epsilon^-(x^-) &\rightsquigarrow \epsilon_n^- = ie^{inx^-}\partial_-, \end{aligned} \tag{5.58}$$

and use them to write two families of vector fields

$$\begin{aligned} \xi_n^+ &= \xi(\epsilon_n^+ = ie^{inx^+}) = ie^{inx^+}\partial_+ + \frac{l}{2}ne^{inx^+}\partial_\rho + \mathcal{O}(e^{-2\rho/l}), \\ \xi_n^- &= \xi(\epsilon_n^- = ie^{inx^-}) = ie^{inx^-}\partial_- + \frac{l}{2}ne^{inx^-}\partial_\rho + \mathcal{O}(e^{-2\rho/l}). \end{aligned} \tag{5.59}$$

Since ξ^+ and ξ^- commute for any ϵ^+ and ϵ^- their Fourier modes will commute, too,

$$[\xi_n^+, \xi_m^-] = 0. \tag{5.60}$$

Thus with $\mathcal{O}(e^{-2\rho/l})$ terms omitted, there are two algebras each satisfying

$$\begin{aligned} [\xi_n^\pm, \xi_m^\pm] &= [ie^{inx^\pm}\partial_\pm,\, ie^{imx^\pm}\partial_\pm] + \frac{l}{2}[ie^{inx^\pm}\partial_\pm,\, me^{imx^\pm}\partial_\rho] \\ &\quad + \frac{l}{2}[ne^{inx^\pm}\partial_\rho,\, ie^{imx^\pm}\partial_\pm] + \frac{l^2}{4}[ne^{inx^\pm}\partial_\rho,\, me^{imx^\pm}\partial_\rho] \\ &= (n-m)ie^{i(n+m)x^\pm}\partial_\pm. \end{aligned} \tag{5.61}$$

This is called the *Witt algebra*. Thus the families of vector fields in (5.59) form a Lie algebra which is a direct sum of two subalgebras isomorphic to the Witt algebra. By investigating the subsets $\{\xi^{\pm}_{-1}, \xi^{\pm}_{0}, \xi^{\pm}_{1}\}$ it is seen each subalgebra contains the $\mathfrak{sl}(2,\mathbb{R})$

$$[\xi^{\pm}_{-1}, \xi^{\pm}_{0}] = -\xi^{\pm}_{-1}, \qquad [\xi^{\pm}_{0}, \xi^{\pm}_{1}] = -\xi^{\pm}_{1}, \qquad [\xi^{\pm}_{1}, \xi^{\pm}_{-1}] = 2\xi^{\pm}_{0}, \tag{5.62}$$

thus their union forms an algebra isomorphic to

$$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1) \simeq \mathfrak{so}(2,2), \tag{5.63}$$

and it contains the exact symmetry group of AdS_3, so it satisfies condition (ii) partially. The conserved charges should also be calculated to show their algebra satisfies condition (ii). We will use (4.42). For that, the Noether-Wald charge density (4.41) is needed. An arbitrary variation of the Bañados metric (5.43) is given by

$$h_{\mu\nu} = \delta g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial L_+}\delta L_+ + \frac{\partial g_{\mu\nu}}{\partial L_-}\delta L_-. \tag{5.64}$$

Using coordinates in metric (5.43), the asymptotic Killing vectors ξ^+ and ξ^- are given by

$$\xi^+ = \epsilon^+(x^+)\partial_+ - \frac{r}{2}\partial_+\epsilon^+\partial_r + +\frac{1}{2r^2}\partial_+^2\epsilon^+\partial_- + \mathcal{O}(r^{-4}), \tag{5.65}$$
$$\xi^- = \epsilon^-(x^-)\partial_- - \frac{r}{2}\partial_-\epsilon^-\partial_r + +\frac{1}{2r^2}\partial_-^2\epsilon^-\partial_+ + \mathcal{O}(r^{-4}), \tag{5.66}$$

and we get

$$\delta_{\xi^\pm} g_{rr} = 0, \tag{5.67}$$
$$\delta_{\xi^\pm} g_{r\pm} = 0, \tag{5.68}$$
$$\delta_{\xi^+} g_{++} = \mathcal{L}_{\xi^+} g_{++} = l^2\delta_{\xi^+}L_+ + \text{(subleading)}, \tag{5.69}$$
$$\delta_{\xi^-} g_{--} = \mathcal{L}_{\xi^-} g_{--} = l^2\delta_{\xi^-}L_- + \text{(subleading)}, \tag{5.70}$$
$$\delta_{\xi^\pm} g_{+-} = \text{(subleading)}. \tag{5.71}$$

When we calculate charges densities that correspond to these vectors and integrate them over a codimension 2 surface with r and t coordinates fixed, i.e. on a circle S^1 at fixed t and r, we get the following infinitesimal charges (4.42)

$$\delta H^{\pm} = \oint_{S^1} \mathbf{k}_{\xi^{\pm}}[h; g] = \frac{l}{8\pi G} \int_0^{2\pi} \delta L_{\pm} \epsilon^{\pm} d\phi, \tag{5.72}$$

which are clearly integrable. We can introduce their Fourier modes as we did for the asymptotic vectors. But here we need to make a decision. We need to choose a reference metric and assign some charges to it. The Fourier modes of the charges are given by

$$H_m^{\pm} = \frac{l}{8\pi G} \int_0^{2\pi} L_{\pm} e^{imx^{\pm}} d\phi, \tag{5.73}$$

when the charges of the Bañados metric with $L_+ = L_- = 0$ are taken to be to zero. Their Poisson bracket can be calculated using (4.47) as

$$\{H_m^{\pm}, H_n^{\pm}\} = \delta_{\xi_n^{\pm}} H_m^{\pm} = \oint_{S^1} \mathbf{k}_{\xi_m^{\pm}}[\delta_{xi_n^{\pm}}; g] = \frac{l}{8\pi G} \int_0^{2\pi} \delta_{\xi_n^{\pm}} L_{\pm} e^{imx^{\pm}} d\phi, \tag{5.74}$$

but we need $\delta_{\xi_n^{\pm}} L_{\pm}$ to calculate them. We can check how the metric components $g_{\mu\nu}$ vary under $\xi^{\pm}$ in (5.65) and deduce $\delta_{xi^{\pm}} L_{\pm}$ from there. This way we find out that the variations are

$$\delta_{\xi^{\pm}} L_{\pm} = \epsilon^{\pm} \partial_{\pm} L_{\pm} + 2 L_{\pm} \partial_{\pm} \epsilon_{\pm} - \frac{1}{2} \partial_{\pm}^3 \epsilon_{\pm}, \tag{5.75}$$

$$\delta_{\xi^{\pm}} L_{\mp} = 0, \tag{5.76}$$

hence we have, as explicitly calculated in [21],

$$\{H_m^+, H_n^-\} = 0, \tag{5.77}$$

$$i\{H_m^{\pm}, H_n^{\pm}\} = (m-n) H_{m+n}^{\pm} + \frac{l}{8G} m^3 \delta_{m+n,0}, \tag{5.78}$$

for all $n, m \in \mathbb{Z}$. This is a direct sum of two copies of *Virasoro algebras*. Note that as mentioned in the charge representation theorem, Theorem 4.6, this is a central extension of the Witt algebra(5.61) with the additional term $\frac{l}{8G}m^3\delta_{m+n,0}$. Conventionally the coefficient is taken as $\frac{c}{12}m^3\delta_{m+n,0}$ where c is called the *Brown-Henneaux central charge*

$$c = \frac{3l}{2G}. \tag{5.79}$$

So we see our boundary conditions give nontrivial integrable finite charges, so they satisfy condition (ii), too.

This result is a precursor of the AdS/CFT duality [4] since the algebra of the Fourier modes of the conserved currents on the 2-dimensional CFT defined on the worldsheet of strings is also given by the Virasoro algebra (5.77).

5.3. Compère-Song-Strominger Boundary Conditions

Now we will study the boundary conditions for AdS_3 that was considered by Compère, Song and Strominger [7] in 2013, using their notations.

Let $\mathcal{M}$ be a three dimensional manifold with coordinates (r, t, ϕ) with $\phi \sim \phi + 2\pi$, and switch to the light cone coordinates $t^{\pm} = t \pm \phi$. The *Compère-Song-Strominger (CSS) boundary conditions* are given by

$$g_{rr} = \frac{l^2}{r^2} + \mathcal{O}(r^{-4}), \tag{5.80a}$$

$$g_{r\pm} = \mathcal{O}(r^{-3}), \tag{5.80b}$$

$$g_{+-} = -\frac{l^2 r^2}{2} + \mathcal{O}(r^0), \tag{5.80c}$$

$$g_{++} = \partial_+ \bar{P}(t^+) l^2 r^2 + \mathcal{O}(r^0), \tag{5.80d}$$

$$g_{--} = 4Gl\Delta + \mathcal{O}(r^{-1}), \tag{5.80e}$$

where Δ denotes an arbitrary constant and the function $\partial_+\bar{P}(t^+)$ is periodic. These conditions can be relaxed by allowing Δ to vary, and this case is explained in Appendix B of [7]. It is easily noted how CSS boundary conditions (5.80) differ from the Brown-Henneaux boundary conditions (5.38), considering (5.80d) and (5.80e). We also notice that they both have a flat boundary metric $g_{(0)}$, the Christoffel symbols for $g_{(0)}$ are all zero in Brown-Henneaux case, whereas $\Gamma^{(0)-}_{++} = -\partial_+^2 \bar{P}$ in CSS.

5.3.1. The General Solution

For pure gravity in three dimensions, one can choose the following Fefferman-Graham coordinates to work with

$$ds^2 = \frac{l^2}{r^2} dr^2 + l^2 r^2 \left(g_{(0)ab} + \frac{1}{r^2} g_{(2)ab} + \frac{1}{r^4} g_{(4)ab} \right), \tag{5.81}$$

then the boundary conditions (5.80) fix

$$g_{(0)--} = 0, \qquad g_{(2)--} = \frac{4G}{l}\Delta, \qquad g_{(0)++} = \partial_+ \bar{P}, \qquad g_{(0)+-} = -\frac{1}{2}. \tag{5.82}$$

Recalling the relations given in (5.34), we can calculate each term in the metric (5.81). Using (5.34c) and (5.80e) it is seen

$$\partial_- g_{(2)+-} = 0, \tag{5.83a}$$

$$\partial_- g_{(2)++} + \partial_+ g_{(2)+-} + \partial_+^2 \bar{P} \frac{4G}{l} \Delta = 0. \tag{5.83b}$$

We conclude that $g_{(2)+-}$ must be a function of t^+ by (5.83a). Since $g_{(2)}$ is traceless by (5.34b), $g_{(2)+-} = -\frac{4G}{l}\Delta \partial_+ \bar{P}$. Plugging this result in (5.83b) we get $\partial_- g_{(2)++} = 0$, hence it is also a function depending only on t^+. This function is conventionally given as $g_{(2)++} = \frac{4G}{l}(\bar{L}(t^+) + \Delta(\partial_+ \bar{P})^2)$ in [7]. The remaining terms are computed from (5.34a) as

$$g_{(4)++} = \frac{16G^2}{l^2} \Delta \bar{L} \partial_+ \bar{P}, \qquad g_{(4)+-} = -\frac{1}{2}\Delta \bar{L}, \tag{5.84}$$

and the metric (5.81) becomes

$$ds^2 = \frac{l^2}{r^2}dr^2 - l^2r^2dt^+(dt^- - \partial_+\bar{P}dt^+) + 4Gl[\bar{L}(dt^+)^2 + \Delta(dt^- - \partial_+\bar{P}dt^+)^2]$$
$$- \frac{16g^2}{r^2}\Delta\bar{L}dt^+(dt^- - \partial_+\bar{P}dt^+). \tag{5.85}$$

This family of metrics contain the BTZ black hole with $M = \frac{\Delta+\bar{L}}{l}$ and $J = \Delta - \bar{L}$, when $\partial_+\bar{P}$ vanishes and $\bar{L}$ is constant. The interpretation of this metric as a BTZ black hole and the physical meaning of $\bar{P}$ and $\bar{L}$ is explained in [7] for the interested reader.

5.3.2. Asymptotic Symmetry Algebra

We calculate the asymptotic Killing vectors using (5.44) as follows:

$$\mathcal{L}_\xi g_{rr} = 2\frac{l^2}{r^2}[-\frac{1}{r}\xi^r + \partial_r\xi^r] = \mathcal{O}(r^{-4})$$
$$\Rightarrow \quad -\frac{1}{r}\xi^r + \partial_r\xi^r = \mathcal{O}(r^{-2}). \tag{5.86}$$

Then $\xi^r = rf(t^\pm) + \mathcal{O}(r^{-1})$, for some function f of $t^\pm$. Let $a, b, c \in \{+, -\}$.

$$\mathcal{L}_\xi g_{ra} = g_{ab}\partial_r\xi^b + \frac{l^2}{r^2}\partial_a(rf(t^\pm)) = \mathcal{O}(r^{-3})$$
$$\Rightarrow \quad \delta^c_b\partial_r\xi^b + \frac{l^2}{r}g^{ac}\partial_a f = \mathcal{O}(r^{-3}). \tag{5.87}$$

where the first line is contracted with g^{ac}. Then $\xi^c = h^c(t^\pm) - \int \frac{l^2}{r}g^{ac}\partial_a f dr + \mathcal{O}(r^{-2})$ for some 2-vector h^c whose components depend only on $t^\pm$. We will investigate the following equation for each order with respect to r

$$\mathcal{L}_\xi g_{ab} = \xi^r\partial_r g_{ab} + \xi^c\partial_c g_{ab} + g_{ac}\partial_b\xi^c + g_{bc}\partial_a\xi^c = \mathcal{O}(r^0). \tag{5.88}$$

Then the r^2 terms must cancel each other, hence

$$\begin{aligned} 0 &= rf\partial_r(r^2 g_{(0)ab}) + h^c\partial_c(r^2 g_{(0)ab}) + r^2 g_{(0)ac}\partial_b h^c + r^2 g_{(0)bc}\partial_a h^c \\ &= 2f g_{(0)ab} + h^c\partial_c g_{(0)ab} + g_{(0)ac}\partial_b h^c + g_{(0)bc}\partial_a h^c \\ &= 2f g_{(0)ab} + \mathcal{L}_h g_{(0)ab} \\ &= 2f g_{(0)ab} + (\nabla^{(0)}_a h_b + \nabla^{(0)}_b h_a). \end{aligned} \tag{5.89}$$

After contracting the last line with $g^{ab}_{(0)}$ we get $f = -\frac{1}{2}\nabla^{(0)}_a h_a$. It can be plugged it in (5.89) again to see

$$\nabla^{(0)}_a h_b + \nabla^{(0)}_b h_a = \nabla^{(0)}_c h^c g_{(0)ab}. \tag{5.90}$$

Hence for $a, b = -$, recalling $\Gamma^{(0)\pm}_{--} = 0$ we have

$$\begin{aligned} 0 &= \partial_- h_- - \Gamma^{(0)c}_{--} h_c \\ &= \partial_-(g_{(0)-+} h^+), \end{aligned} \tag{5.91}$$

so that $h^+ = h^+(t^+)$.

Let us look at the r^0 terms in (5.88) for $a, b = -$. Only the last two terms in (5.88) contribute:

$$\begin{aligned} 0 &= 2g_{-c}\partial_-\xi^c \\ &= g_{(2)--}\partial_- h^-, \end{aligned} \tag{5.92}$$

hence $h^- = h^-(t^+)$. We can now calculate f in terms of $h^\pm$ as

$$f = -\frac{1}{2}\nabla^{(0)}_a h_a = -\frac{1}{2}\partial_+ h^+. \tag{5.93}$$

Then the components of an asymptotic Killing vector are

$$\xi^r = -\frac{r}{2}\partial_+ h^+ + \mathcal{O}(r^-1), \tag{5.94a}$$

$$\xi^+ = h^+(t^+) - \int \frac{l^2}{r} g^{a+}\partial_a f dr, \tag{5.94b}$$

$$\xi^- = h^-(t^+) - \int \frac{l^2}{r} g^{a-}\partial_a f dr. \tag{5.94c}$$

We can calculate the subleading terms of ξ^+ and ξ^- using the inverse metric g^{ab}

$$g^{ab} = \frac{1}{l^2 r^2}(g^{ab}_{(0)} - \frac{1}{r^2} g^{ac}_{(0)} g_{(2)cd} g^{db}_{(0)} + \dots). \tag{5.95}$$

Then we see

$$\begin{aligned} \xi^+ &= h^+ - \int \frac{l^2}{r}\left[\frac{1}{l^2 r^2} g^{a+}_{(0)} + \mathcal{O}(r^{-4})\right]\partial_a\left(-\frac{1}{2}\partial_+ h^+\right) dr \\ &= h^+ + \mathcal{O}(r^{-4}), \end{aligned} \tag{5.96}$$

as $g^{++}_{(0)}$ and $\partial_-\partial_+ h^+$ vanish.

$$\begin{aligned} \xi^- &= h^- - \int \frac{l^2}{r}\left[\frac{1}{l^2 r^2} g^{a-}_{(0)} + \mathcal{O}(r^{-4})\right]\partial_a\left(-\frac{1}{2}\partial_+ h^+\right) dr \\ &= h^- + \int \frac{1}{2r^3}[g^{-+}_{(0)}\partial_+^2 h^+ + g^{--}_{(0)}\partial_-\partial_+ h^+]dr + \mathcal{O}(r^{-4}) \\ &= h^- + \frac{1}{2r^2}\partial_+^2 h^+ + \mathcal{O}(r^{-4}), \end{aligned} \tag{5.97}$$

since $g^{-+}_{(0)} = -2$ and $\partial_-\partial_+ h^+ = 0$.

In [7], the functions h^+ and h^- are denoted by ϵ and σ, respectively. The asymptotic symmetries are then generated by

$$\xi(\epsilon) = \epsilon(t^+)\partial_+ - \frac{r}{2}\epsilon'(t^+)\partial_r + \frac{1}{2r^2}\epsilon''(t^+)\partial_- + \mathcal{O}(r^{-4}), \tag{5.98}$$

$$\eta(\sigma) = \sigma(t^+)\partial_- + \mathcal{O}(r^{-4}). \tag{5.99}$$

These vector fields give a Virasoro algebra and a $U(1)$ current algebra [7].

Using (4.41) and (4.44) we calculate the corresponding charges given in [7]

$$Q_\xi = \frac{1}{2\pi}\int_0^{2\pi} \epsilon(t^+)(\bar{L}(t^+) - \Delta(\partial_+\bar{P}(t^+))^2)d\phi,$$
$$Q_\eta = \frac{1}{2\pi}\int_0^{2\pi} \sigma(t^+)(\Delta + 2\Delta\partial_+\bar{P}(t^+))d\phi. \tag{5.100}$$

We can use the Fourier modes for the functions ϵ and σ to write two families of vector fields and charges

$$\xi_n = e^{int^+}\partial_+ - \frac{r}{2}ie^{int^+}\partial_r - \frac{1}{2r^2}e^{int^+}\partial_- + \mathcal{O}(r^{-4}) \tag{5.101}$$
$$\eta_n = e^{int^+}\partial_- + \mathcal{O}(r^{-4}) \tag{5.102}$$
$$\bar{\mathcal{L}}_n = \frac{1}{2\pi}\int_0^{2\pi} e^{int^+}(\bar{L}(t^+) - \Delta(\partial_+\bar{P}(t^+))^2)d\phi, \tag{5.103}$$
$$\bar{\mathcal{P}}_n = \frac{1}{2\pi}\int_0^{2\pi} e^{int^+}(\Delta + 2\Delta\partial_+\bar{P}(t^+))d\phi \tag{5.104}$$

and calculate the charge algebra using the variations

$$\delta_{\xi_n}\bar{L} = \partial_+\bar{L} + 2in\bar{L} + \frac{in^3 l}{8G} \tag{5.105}$$
$$\delta_{\xi_n}(\partial_+\bar{P}) = e^{int^+}[\partial_+^2\bar{P} + in\partial_+\bar{P}], \tag{5.106}$$

that we derived from the variation of metric components as we did for Brown-Henneaux case. We see the charges obey

$$i\{\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\} = (m-n)\bar{\mathcal{L}}_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}, \tag{5.107}$$
$$i\{\bar{\mathcal{L}}_m, \bar{\mathcal{P}}_n\} = -n\bar{\mathcal{P}}_{m+n}, \tag{5.108}$$
$$i\{\bar{\mathcal{P}}_m, \bar{\mathcal{P}}_n\} = \frac{k_{KM}}{2}m\delta_{m+n,0}, \tag{5.109}$$

where c is the Brown-Henneaux central charge (5.79) and k_{KM} is the Kac-Moody level

$k_{KM} = -4\Delta$. The relation (5.107) shows that $\bar{\mathcal{L}}_m$ form a Virasoro algebra as in (5.77) and (5.109) gives a *Kac-Moody algebra*. For a review see [42]. Hence we get a semidirect sum of these algebras as the charge algebra for CSS boundary conditions, and showed that CSS boundary conditions also satisfy conditions (i), (ii) and (iii).

6. CONCLUSION

Two fundamental ways are used in literature to define asymptotically *AdS* spacetimes. It is common to use the Fefferman-Graham expansion to derive the form of the metric of the spacetime near the boundary. In this method, one starts at the boundary and extends it smoothly into the interior. However, there is no guarantee that it closes to a smooth interior, in general. It might also expand to another boundary at the infinity. In three dimensions, it is much easier to find the explicit form of the metrics since the expansion stops at the second order. We have shown the most general metrics for Brown-Henneaux and Compère-Song-Strominger boundary conditions. On the other hand, this question remains to be investigated for other sets of boundary conditions and dimensions, as a recent example see [43].

By setting the boundary conditions one puts restrictions on the metrics, hence it creates an obstacle against the search for the most general metric for *AAdS* spacetimes. Some works aim to relax the conditions on the metrics [44,45] or investigate the relations between the possible coordinate systems and how they affect the results [46].

Instead of the pure gravity case, one can investigate the asymptotic properties of spacetimes in other theories like Topological Massive Gravity (TMG) to find the most general metric family. These theories differ from pure gravity in the asymptotic region. For example, in TMG one demands the spacetimes to get close to a warped *AdS* spacetime [47–49]. Another direction of research could be handling supersymmetric theories of gravity.

The study of *AAdS* spacetimes has a side benefit regarding their role in the investigation of the asymptotically flat spacetimes since they approach asymptotically flat spacetimes as $\Lambda \to 0$. There is ongoing research [50] on this subject that we know of. This approach has the advantage of offering a new point of view to flat spacetimes and it can reveal some properties that are missed in the usual approach.

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APPENDIX A: Conventions

In this thesis, all manifolds, fields and functions are assumed to be smooth unless mentioned otherwise. We take $c = 1$. There are some proofs where we take $8G = 1$ or $l = 1$, and these are noted at the beginning of that part. We use Einstein summation convention throughout the thesis, which means whenever the same index appears as an upper and lower index, the summation is understood as

$$A^i B_i = \sum_i A^i B_i. \tag{A.1}$$

The dimensions are denoted by n or d. The Greek indices denote the indices for a generic spacetime, whereas the Latin indices are used for flat spacetime or denote the coordinates of some subspace, such as spacelike coordinates or coordinates on the boundary. We us mostly plus convention, that is the metric signature is $(- + \ldots +)$. The Christoffel symbols and Riemann curvature tensors are calculated using

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}), \tag{A.2}$$

$$R^{\kappa}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\kappa}_{\lambda\nu} + \Gamma^{\xi}_{\lambda\nu}\Gamma^{\kappa}_{\xi\mu} - \ (\mu \leftrightarrow \nu). \tag{A.3}$$

We use the normalized symmetrization and antisymmetrization of tensors as in

$$T^{(\mu\nu)} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) \qquad T^{[\mu\nu]} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) \tag{A.4}$$

When we use differential forms we write them in bold letters. We use the following to denote the differential forms in short

$$(\boldsymbol{d}^{n-k})_{\mu_1 \ldots \mu_k} = \frac{1}{k!(n-k)!}\epsilon_{\mu_1 \ldots \mu_k \mu_{k+1} \ldots \mu_n} dx^{\mu_{k+1}} \wedge \cdots \wedge dx^{\mu_n}. \tag{A.5}$$