SUPERSYMMETRIC NON-LINEAR SIGMA MODELS IN D=2+1 DIMENSIONAL SPACETIME

by

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B.S., Mathematics, Boğaziçi University, 2019

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2021

ACKNOWLEDGEMENTS

First of all I would like to thank G-d above for making all of this possible for me; it has truly been *"living the dream"* for me throughout my studies. For this I am eternally grateful. This was my dream.

I would like to thank my friend *Tolga Domurcukgül* from physics department. It has been an ongoing joke among us that he has become my *"unofficial advisor"* since I started taking up learning some mathematical physics. If not for his help, I would have given up trying to carry the burden of mathematics and physics altogether. I am very happy for that classical mechanics midterm night when I happened to meet him.

I am blessed with a beautiful family; my dad *Yasef*, my mom *Sibel*, and my dear sister *Tina*. I can never thank them enough for their warmth. I recall a conversation I had with my father when academically things were going tough for me: he told me exactly this, "You do not have to be the best mathematician in the world; **be a mathematician**." Since that day I live by those words. As the saying goes; *"Family is not an important thing. It's everything."*

I would like to express my highest gratitude to my dear advisor professor *Nihat Sadık Değer*. Throughout this project for roughly a year, he has been giving me the ultimate inspiration to continue day after day. I am flattered that his efforts have allowed me to grow outside of the math that I have known into the beautiful world of mathematical physics. He showed me that everyone could understand, enjoy, and add something[1] to theoretical physics if they had the will to do it, even me. I thank him from the depth of my heart for always believing in me.

I gratefully acknowledge Scientific and Technological Research Council of Turkey (**TÜBİTAK**) for their support via the project *"Üç Boyutlu Süperkütleçekim Kuramlarının Süpersimetrik Çözümleri"* under the project number **116F137**.

[1]No matter how small.

I learned a great deal from the scientists who worked on these fields for their lifetime. For this I say that *Newton* was right, we are really standing on the shoulders of giants.

And lastly I am very grateful if my insights into the theory of differential geometry and Non-linear $\sigma-$models have given you any pleasure. For this I am thankful to you.

Abstract

SUPERSYMMETRIC NON-LINEAR SIGMA MODELS IN D=2+1 DIMENSIONAL SPACETIME

In this thesis we study the theory of Supersymmetric Non-Linear $\sigma-$Models. We first review the free scalar field theory which can be considered as a Linear $\sigma-$model. We then discuss the Non-Linear $\sigma-$model as well as the symmetries of the model with an emphasis on the role of Killing vector fields. After that we turn our attention to study the possible target spaces of these models; these being Kähler and homogeneous manifolds. Finally we introduce the $3-$dimensional Non-Linear $\sigma-$Model with $\mathcal{N} = 1, 2$ and 4 supersymmetries. We show that the $D = 3$ and $\mathcal{N} = 2$ NLσM has to admit a Kähler target manifold while $D = 3$ and $\mathcal{N} = 4$ model has to admit a HyperKähler target manifold.

ÖZET

D=2+1 BOYUTLU UZAY-ZAMANDA SÜPERSİMETRİK NON-LİNEER SİGMA MODEL

Bu tezde Süpersimetrik Non-Lineer $\sigma-$Modellerin teorisini konu alacağız. İlk olarak $\sigma-$modellerin temeli olacak Lineer bir model olan bağımsız skaler alan teorisini çalışacağız. Daha sonrasında Non-Lineer $\sigma-$modelleri, modelin simetrilerini, ve bunların Killing vektör alanları ile olan ilişkilerini inceleyeceğiz. Ardından dikkatimizi bu modellerin olası hedef uzaylarına çevireceğiz; bunlar Kähler ve homojen manifoldlar olacak. Son kısımda 3$-$boyuttaki Non-Lineer $\sigma-$modeli $\mathcal{N} = 1, 2$, ve 4 süpersimetri altında tanıtacağız. $D = 3$ ve $\mathcal{N} = 2$ NLσM için görüntü uzayının Kähler olması gerektiğini, ve $D = 3$ ve $\mathcal{N} = 4$ modelde ise görüntü uzayının HyperKähler olması gerektiğini ispatlayacağız.

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LIST OF SYMBOLS

NLσM	Non-Linear Sigma Model
GLσM	Gauged Non-Linear Sigma Model
D	Dimension of the Minkowski spacetime
$\mathcal{N}$	Number of supersymmetries
$\eta_{\mu\nu}$	Mostly plus signature Minkowski metric
$\mathbb{A}^T$	Transpose of a matrix $\mathbb{A}$
$O(n)$	Group of orthogonal transformations of $\mathbb{R}^n$
$SO(n)$	Special Orthogonal group
$O(p,q;\mathbb{F})$	Indefinite Orthogonal group with entries from $\mathbb{F}$
$U(n)$	Unitary matrices of size $n \times n$
$SU(n)$	Special Unitary group
$GL_n(\mathbb{F})$	General Linear transformations with elements from a field $\mathbb{F}$
$SL_n(\mathbb{F})$	Special Linear transformations with elements from a field $\mathbb{F}$
$PSL_n(\mathbb{F})$	Projective Special Linear group of $n \times n$
$\mathbb{H}$	Ring of Quaternions
$T_p\mathcal{M}$	The tangent space of a manifold $\mathcal{M}$ at the point p
$\mathbb{R}P^n$	Real Projective Space
$\mathbb{C}P^n$	Complex Projective Space
S^n	$n-$dimensional sphere
V^*	Dual space to a vector space V
$\mathfrak{g}$	Lie Algebra corresponding to the Lie Group G
$\wedge$	Wedge product
$\otimes$	Tensor product
$\oplus$	Direct sum
$\times$	Direct product of groups
$\mathbb{I}_m$, or $\mathbb{1}_m$	The identity matrix of $m \times m$
$d\Omega$	The exterior derivative of an arbitrary differential form Ω
∂M	The boundary of a set M

$\mathfrak{X}(\mathcal{M})$	Set of smooth vector fields on a manifold $\mathcal{M}$
$f^*\Omega$	The pull-back of a form Ω by a map f
$f_*\vec{V}$	The push-forward of a vector $\vec{V}$ by a map f
$\mathcal{L}_X(T)$	Lie derivative of T with respect to X
∇	Connection on a manifold
∇_μ	Covariant derivative
D_μ	Lorentz or Gauge covariant derivative
$\mathbb{D}_\mu$	Kähler covariant derivative
$\Gamma^\rho_{\mu\nu}$	Connection coefficients
$\Box$	d'Alembertian operator
$\mathrm{Ker}(\Psi)$	Kernel of a map Ψ
$[.,.]$	The Lie bracket
$\langle .,. \rangle$	An inner product defined on a vector space
δ^ν_μ	Kronecker Delta
$\|\vec{V}\|$	Norm of a vector $\vec{V}$

1. INTRODUCTION

The Non-Linear $\sigma-$Models have been remarkable mathematical models for use in physics in the past half century. These models provide a theory of bosonic scalar fields under a quantum field theory setting; this allows us to introduce interacting field theories. In NLσM's, these scalar fields are maps from some base manifold to a number field, and hence they can be viewed as coordinates of some abstract manifold. Interestingly there are many possibilities for this abstract target manifold in a NLσM. The target manifold could be compact, non-compact, Riemannian, or Kähler. The target manifold could represent the world in string theory or in the case of minimal surface problems; or could be an abstract construction. We can classify the applications of NLσM's by the dimension of the base manifold,

- $1-$Dimensional base manifold:
  - (i) Action on a charged particle,
  - (ii) Curved motion/ path of a particle (possibly relativistic)
  - (iii) Quantum mechanics of a wave function.
- $2-$Dimensional base manifold:
  - (i) Soap bubbles, i.e., minimal surface/ area problems,
  - (ii) Surfaces traced out by relativistic strings in String Theory.

For the list of applications, see [1, 2].

In supersymmetric $\sigma-$models, a target manifold is introduced by the coupling of a metric tensor in the Lagrangian; the metric is then restricted by the conditions proposed by the SuperPoincaré invariant theory. This allows us to understand the underlying mathematics hidden within the theory - which is dependent on the space-time dimension, and the number of supersymmetries. The geometric consequences of introducing supersymmetry in $\sigma-$models were first realized by the papers of [3], and described thoroughly in [4, 5]; this study later on grew massively as a field of research. Referencing [6], we can give the following table for the target manifolds of $\sigma-$models with arbitrary spacetime dimensions (denoted by D) and rigid supersymmetries (denoted by $\mathcal{N}$),

D	4	3 & 2	**Geometry**
$\mathcal{N}$	2	4	HyperKähler
$\mathcal{N}$	1	2	Kähler
$\mathcal{N}$		1	Riemannian

Table 1.1: Target spaces of supersymmetric NLσM's with $D \leq 4$.

In addition to complex properties of the target space, these target manifolds can also carry the structure of a quotient manifold, or specifically a symmetric space. We will first introduce the free scalar field theory, and then introduce the possible target spaces. We then follow the $D = 3$ and $\mathcal{N} = 1, 2$ and 4 supersymmetric $\sigma-$model of [5] with an emphasis on the geometric picture. We will show the results of the table 1.1. Some introductory information on complex manifolds and supersymmetry is given in the appendices.

We will be using Einstein Summation Convention for the repeated subscript and superscript indices throughout the thesis.

2. FREE SCALAR FIELD THEORY: A LINEAR $\sigma-$MODEL

"Newton was right, we are really standing on the shoulders of giants."

The $\sigma-$model is an interacting field theory of scalar fields. But before talking about the complexities brought by such a model, we start by considering a simplified version; this is by eliminating the interactions. This system is the free scalar field theory of Minkowski spacetime. After understanding this system, we will move on to the generalization of it to Non-Linear $\sigma-$Models.

Definition 2.1. *A **scalar field** is a real or complex valued function with domain from the Minkowski spacetime.*

We start by considering fields $\phi^i(x)$, $1 \leq i \leq n$, on a flat $D-$dimensional Minkowski spacetime. Flat Minkowski spacetime means that the metric is $\eta_{\mu\nu} := (-1, \underbrace{+1, .., +1}_{D-1 \text{ many}})$ on each point of the manifold. We assume that our fields satisfy the ***massless Klein-Gordon equation*** which is a relativistic wave equation:

$$\Box\phi^i(x) = 0 \tag{2.0.1}$$

where the d'Alembertian is reduced to $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ since the metric is flat Minkowski. We will always consider field configurations that vanish at large spacetime distances, that is, as $x \to \infty$, $\phi^i(x) \to 0$ which will be used in partial integrations.

Proposition 2.2. *The Klein-Gordon equation can be derived by examining the variational derivative of the following action with respect to* $\phi^i(x)$*,*

$$S = \int d^D x\mathcal{L}(x) = -\frac{1}{2}\int d^D x\, \{\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\}\,\delta_{ij}. \tag{2.0.2}$$

where $\mathcal{L}$ is the Lagrangian density.

Proof. We distribute the Kronecker delta inside the integrand, so we contract the Latin indices. With that, taking the variation gives,

$$\begin{aligned}
\delta S &= -\frac{1}{2}\int d^D x \left\{2\eta^{\mu\nu}\partial_\mu(\delta\phi^i)\partial_\nu\phi_i\right\} \\
&= -\frac{1}{2}\int d^D x \left\{-\delta\phi_i 2\eta^{\mu\nu}\partial_\mu\partial_\nu(\phi^i)\right\} + 2\eta^{\mu\nu}\partial_\nu(\phi^i)\cancel{\delta\phi_i|_{\text{limits}}} \\
&= -\int d^D x \left\{-\Box\phi^i(x)\right\}\delta\phi_i
\end{aligned}$$

here we used integration by parts in the second step. Thus the variation vanishes $\Leftrightarrow$ the Klein-Gordon equation is satisfied. This proof is also equivalent to the following statement. $\square$

Proposition 2.3. *The action $S = \int \mathcal{L}(\phi^i, \partial_\mu\phi)d^D x$ is extremal when the Klein-Gordon equation is satisfied. This is a direct result of Euler-Lagrange equations;*

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\delta\partial_\mu\phi^i(x)} - \frac{\delta\mathcal{L}}{\delta\phi^i(x)} = 0. \tag{2.0.3}$$

2.1. Symmetries

Definition 2.4. *A transformation of the scalar field $\phi^i(x) \mapsto \phi'^i(x)$ that leaves the action* (2.0.2) *invariant is called a* ***symmetry*** *of the system. A symmetry transformation must satisfy $S[\phi^i] = S[\phi'^i]$ so that $\delta S = 0$.*

If ϕ^i is a solution to the Klein-Gordon equation, a symmetry is a mapping $\phi \mapsto \phi'^i$ where ϕ'^i is also a solution to the equation of motion.

There are two basic types of symmetries; an **internal symmetry** is a transformation acting only on the fields themselves, and not transforming the spacetime variables. On the other hand **external symmetries** are global spacetime symmetries.

2.1.1. Internal Symmetries

We now would like to show that the internal symmetry of the action (2.0.2) is given in terms of an $O(n)$ transformation and constant translations of ϕ^i's.
Let the matrix $\mathbb{A} \in O(n)$ with $\mathbb{A}^T\mathbb{A} = \mathbb{A}\mathbb{A}^T = \mathbb{1}_{n\times n}$. Now consider $n-$many fields $\{\phi^i(x)\}$ in vector form;

$$\Phi(x) = \begin{bmatrix} \phi^1(x) \\ \phi^2(x) \\ \vdots \\ \phi^n(x) \end{bmatrix} \tag{2.1.1}$$

such a transformation acts on Φ by, $\mathbb{A}\Phi(x) = \Phi'(x)$.

Proposition 2.5. *Any transformation of the form $\Phi \mapsto \mathbb{A}\Phi + \mathbb{B}$ where $\mathbb{A} \in O(n)$ and $\mathbb{B}$ is an arbitrary $(n \times 1)-$constant matrix leaves the action (2.0.2) invariant.*

Proof. The action (2.0.2) can be rewritten using (2.1.1) as $\mathcal{L} = -\frac{1}{2}\partial_\mu\Phi^T\partial^\mu\Phi$. It is now mapped to,

$$\begin{aligned}
\mathcal{L}_{new}(\Phi, \partial\Phi) &= -\frac{1}{2}\partial_\mu(\Phi^T\mathbb{A}^T + \mathbb{B}^T)\partial^\mu(\mathbb{A}\Phi + \mathbb{B}) \\
&= -\frac{1}{2}(\partial_\mu\Phi^T\mathbb{A}^T + \cancel{\partial_\mu\mathbb{B}^T})(\mathbb{A}\partial^\mu\Phi + \cancel{\partial^\mu\mathbb{B}}) \\
&= -\frac{1}{2}\partial_\mu\Phi^T \underbrace{\mathbb{A}^T\mathbb{A}}_{\mathbb{1}} \partial^\mu\Phi = \mathcal{L}_{old}.
\end{aligned}$$

□

Remark. *Note that the internal symmetry transformation $\Phi \mapsto \mathbb{A}\Phi + \mathbb{B}$ is actually the most general isometry transformation on $\mathbb{R}^n$ if we think of ϕ^i to be coordinates of $\mathbb{R}^n$ with metric δ_{ij}. Since this symmetry transformation is linear in fields, the model defined by the action (2.0.2) is an example of a* ***Linear $\sigma-$Model.***

2.1.2. External Symmetries

The external (spacetime) symmetries of the $\sigma-$model defined by the action (2.0.2) correspond to isometries of the Minkowski spacetime, namely Lorentz transformations and translations.

Definition 2.6. ***Spacetime translations*** *are transformations of the form* $\phi^i(x) \mapsto \phi'^i(x) := \phi^i(x+a)$ *where* $a = \{a^\mu\}$ *are constants.*

Definition 2.7. *We define the* ***Lorentz transformations*** *in* $D-$*dimensions to be the set of matrices such that* $\{\Lambda \in GL_D(\mathbb{R}) | \, \Lambda^T \eta \Lambda = \eta\}$, *in index notation that is* $\Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\sigma = \eta_{\rho\sigma}$. *The metric is the Minkowski metric.*

Remark. *Notice that changing* $\eta_{\mu\nu} \mapsto \delta_{\mu\nu}$ *gives us the orthogonal group* $O(D)$. *We call the Minkowski* η *case as a group called the* ***pseudo-orthogonal group***, *denoted by* $O(D-1,1)$.

Theorem 2.8. *The action in* (2.0.2) *is left invariant under Lorentz transformations and translations.*

Proof. To show the Lorentz invariance, we start by noting that $\phi^i(x)$ are scalars and hence do not themselves transform under Lorentz transformations. Note that the transformations we need are as follows;

$$\text{Coordinates: } x^\mu \mapsto x^{\mu'} = \Lambda^{\mu'}_\mu x^\mu \tag{2.1.2}$$

$$\text{Partial derivatives: } \partial_\nu \mapsto \partial_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\nu} = \Lambda^\nu_{\mu'} \frac{\partial}{\partial x^\nu} \tag{2.1.3}$$

with this setup the new Lagrangian turns out;

$$\begin{aligned} \mathcal{L} = \eta^{\mu'\nu'} \partial_{\mu'} \phi'^i(x') \partial_{\nu'} \phi'^j(x') \delta_{ij} &= \eta^{\mu'\nu'} \Lambda^\kappa_{\mu'} \partial_\kappa \phi^i(x') \Lambda^\lambda_{\nu'} \partial_\lambda \phi^j(x') \delta_{ij} \\ &= \eta^{\kappa\lambda} \partial_\kappa \phi^i(x) \partial_\lambda \phi^j(x) \delta_{ij} = \mathcal{L}_{\text{old}}. \end{aligned}$$

For the proof of the invariance of (2.0.2) under translations, we refer to [7]. ☐

3. NON-LINEAR $\sigma-$MODEL

The main reference for this chapter is [7], a more rudimentary introduction to $\sigma-$models and the derivation of harmonic maps can be found in [1]. The relation of the Lagrangian and actions of $\sigma-$models and String theory are briefly covered in [8], and the types of $\sigma-$models in physical theories are classified in [2].

The ***Non-Linear Sigma model*** (shortened to NLσM) is a physical theory of maps that work on dynamics of scalar fields in flat spacetime. It constructs a theory of interacting fields by bringing in a geometric approach to the study of field theory in a fundamental manner. This way it generalizes the free field theory described in the previous chapter. The rigor of NLσM's lies under its mathematical framework for providing numerous applications of field theories. Mathematically NLσM's are important for the study of Harmonic maps, as those are the maps that ***extremize*** the kinetic action of scalars.

The Non-Linear $\sigma-$Model considers the scalar fields as coordinates on a Riemannian manifold $\mathcal{M}$ with a Levi-Civita connection and Euclidean signature. The $\sigma-$model is specified by scalar fields $\{\phi^i(x)\}$ where $1 \leq i \leq n$; if we think of scalar fields as maps from some base space to reals, we can think of them as defining coordinates. This way we consider an abstract target manifold formed by the scalar fields themselves, this allows us to study a target space geometry from a field theory setting.

$$\phi^i : \quad \text{Minkowski Spacetime}_{\eta_{\mu\nu}} \longrightarrow \text{Target manifold} \;\; \mathcal{M}_{g_{ij}}. \tag{3.0.1}$$

The target manifold is denoted by $(\mathcal{M}, g)$ throughout this thesis. We postulate that the dynamics of these maps is governed by the (kinetic) action,

$$S[\phi] := -\frac{1}{2}\int d^D x g_{ij}(\phi)\eta^{\mu\nu}\partial_\mu\phi^i(x)\partial_\nu\phi^j(x) \tag{3.0.2}$$

the spacetime indices are raised and lowered via the flat Minkowski metric $\eta_{\mu\nu}$.

Remark. *The above is the standard definition of Non-Linear $\sigma-$Models when gravitation is neglected; however when gravity is present instead of the Minkowski metric one needs to consider a curved spacetime metric. Also note that the signature of the target manifold can be Lorentzian, as in the case of String Theory.*

Remark. *In the action of* (2.0.2), *we are summing over the indices $i, j \in \{1, 2, .., n\}$; that is by through multiplying the whole expression by Kronecker delta δ_{ij}. In fact by doing so we are introducing a ghost target manifold with Kronecker metric, and hence a baby $\sigma-$model. This is why scalar field theories are simplistic $\sigma-$models. The complexity and the beauty of $\sigma-$models lies in the fact that the Kronecker delta could be an arbitrary metric and the scalar fields could interact with each other.*

Theorem 3.1. *The equation of motion corresponding to the action* (3.0.2) *is given by*

$$\Box\phi^i + \Gamma^i_{jk}\partial^\mu\phi^j\partial_\mu\phi^k = 0. \tag{3.0.3}$$

Proof. To obtain the field equations that follow from (3.0.2), we extremize the action integral in (3.0.2). Note that the metric $g_{ij}(\phi)$ depends on the target space coordinates (i.e., the scalar fields), hence one should also consider its variations. The variation of the action is,

$$\begin{aligned}
\delta S[\phi] &= -\frac{1}{2}\int d^Dx \left\{2g_{ij}(\phi)\delta(\partial_\mu\phi^i)\partial^\mu\phi^j + \partial_k g_{ij}\delta\phi^k\partial_\mu\phi^i\partial^\mu\phi^j\right\} \\
&= -\frac{1}{2}\int d^Dx \left\{-2g_{ij}(\phi)\delta\phi^i\partial_\mu\partial^\mu\phi^j - 2\partial_k g_{ij}\delta\phi^i\partial_\mu\phi^k\partial^\mu\phi^j + \partial_i g_{kj}\partial_\mu\phi^k\partial^\mu\phi^j\delta\phi^i\right\} \\
&= -\frac{1}{2}\int d^Dx \left\{-2\partial_\mu\partial^\mu\phi^j g_{ij}(\phi) - 2\partial_k g_{ij}\partial_\mu\phi^k\partial^\mu\phi^j + \partial_i g_{kj}\partial_\mu\phi^k\partial^\mu\phi^j\right\}\delta\phi^i
\end{aligned}$$

where we used integration by parts and renamed the dummy indices. The extra terms proportional to scalar fields are thrown away since they vanish as fields approach to infinity. To evaluate $\partial_k g_{ij}$ that appear, we use the metricity condition, i.e., $\nabla_k g_{ij} = 0$.

$$\nabla_k g_{ij} = 0 \;\Leftrightarrow\; \partial_k g_{ij} = \Gamma^m_{ki}g_{jm} + \Gamma^m_{kj}g_{im}. \tag{3.0.4}$$

Therefore the integrand vanishes if and only if we satisfy the condition that,

$$-2\partial_\mu\partial^\mu\phi^j g_{ij} + \left(-2\Gamma^m_{ki}g_{jm} + \Gamma^m_{ik}g_{mj} + \Gamma^m_{ij}g_{km} - 2\Gamma^m_{kj}g_{im}\right)\partial_\mu\phi^k\partial^\mu\phi^j = 0. \tag{3.0.5}$$

Note that since we are summing k and j, the first three Christoffel coefficients add up to zero. Renaming the dummy indices in the remaining terms, we get:

$$g_{ij}\left(\partial_\mu\partial^\mu\phi^j + \Gamma^j_{\alpha\beta}\partial_\mu\phi^\alpha\partial^\mu\phi^\beta\right) = 0. \tag{3.0.6}$$

Since the metric is invertible, we arrive at the field equation (3.0.3). ☐

Remark. *Note that $\Box = \eta_{\mu\nu}\partial^\mu\partial^\nu$ is the analogue of the Laplacian in Euclidean signature. Its solutions are generalized Harmonic maps (see [1]). Another important observation about* (3.0.3) *is that now the scalar fields are interacting unlike the Linear $\sigma-$Model* (2.0.1).

Remark. *When the spacetime (that is the base manifold) is* 1*-dimensional (i.e., $D = 1$) with a single time coordinate "t", the equation of motion* (3.0.3) *reduces to the Geodesic equation*

$$\frac{d^2\phi^i}{dt^2} + \Gamma^i_{jk}\frac{d\phi^j}{dt}\frac{d\phi^k}{dt} = 0.$$

So this $\sigma-$model is describing a particle moving on a geodesic line, which shows that $\sigma-$model can be a very useful tool in studying dynamics of particles and extended objects (branes) such as strings (see [8]). For applications of $\sigma-$models, see [2].

3.1. Symmetries of Non-Linear σ-Models

As in the Linear $\sigma-$Model, the symmetries of the Non-Linear $\sigma-$Model can be divided as internal and external. The only difference between the two models, defined by the actions (2.0.2) and (3.0.2), is the appearance of a general metric in the latter. Hence external (spacetime) symmetries of both models are the same, namely Lorentz transformations and spatial translations. Therefore we will focus on the internal symmetries of the Non-Linear $\sigma-$Model which as we will see will be non-linear in fields in general.

Now let us consider a continuous coordinate transformation in the form; $\phi \mapsto \phi'$, this changes the metric by,

$$g_{ij} \mapsto g'_{ij}. \tag{3.1.1}$$

Because of the continuity requirement, we must have the coordinates $\phi'^i(\phi)$ depend continuously on constant parameters θ^A with $0 < \theta^A \ll 1$. Because θ^A is very small, in computations we will neglect terms of order higher than 1 in θ^A. Here the index A is the Lie algebra index that corresponds to the vector fields $\{k_A\}$ which are the generators of the coordinate transformation. We can Taylor expand ϕ' up to first order in θ^A as,

$$\phi'^i = \phi^i + \theta^A k^i_A, \tag{3.1.2}$$

which is in general a non-linear transformation.

Proposition 3.2. *Under the infinitesimal coordinate transformation* (3.1.2)*, the metric changes as,*

$$\Delta g := g'_{ij}(\phi) - g_{ij}(\phi) = -\theta^A \mathcal{L}_{k_A} g_{ij}. \tag{3.1.3}$$

where $\mathcal{L}_{k_A}$ is the Lie derivative corresponding to the vector field k_A.

Proof. Under the transformation (3.1.2), to be able to find the change in the metric tensor, we have to expand the metric and use the Lorentz transformation as follows,

$$\left.\begin{array}{ll} g'_{ij}(\phi) & = \frac{\partial\phi^i}{\partial\phi^{i'}} g_{ij}(\phi') \frac{\partial\phi^j}{\partial\phi^{j'}} \\ g_{ij}(\phi') & = g_{ij}(\phi) - \theta^A k^\lambda_A \frac{\partial g_{ij}}{\partial\phi^\lambda} + \cancel{(\theta^A)^2}(..) + .. \end{array}\right\}$$

therefore we examine the total difference in the metric,

$$
\begin{aligned}
\Delta g &:= g'_{mn}(\phi) - g_{mn}(\phi) \\
&= \left(\delta^i_m - \theta^A\partial_m(k^i_A)\right)\left(g_{ij}(\phi) - \theta^A k^\lambda_A \partial_\lambda g_{ij}\right)\left(\delta^j_n - \theta^A\partial_n(k^j_A)\right) - g_{mn}(\phi) \\
&= \left(\delta^i_m\delta^j_n - \theta^A\partial_m(k^i_A)\delta^j_n - \theta^A\partial_n(k^j_A)\delta^i_m + \cancel{(\theta^A)^2(..)}\right)\left(g_{ij}(\phi) - \theta^A k^\lambda_A\partial_\lambda g_{ij}\right) - g_{mn}(\phi) \\
&= \cancel{g_{mn}}(\phi) - \theta^A\partial_m(k^i_A)g_{ij}(\phi)\delta^j_n - \theta^A\partial_n(k^j_A)g_{ij}(\phi)\delta^i_m - \theta^A k^\lambda_A\partial_\lambda g_{mn}(\phi) - \cancel{g_{mn}}(\phi) \\
&= -\theta^A\underbrace{\left(\partial_m(k^i_A)g_{in}(\phi) + \partial_n(k^j_A)g_{mj}(\phi) + k^\lambda_A\partial_\lambda g_{mn}(\phi)\right)}_{\mathcal{L}_{k_A} g_{mn}(\phi)} \\
&= -\theta^A\mathcal{L}_{k_A} g_{mn}(\phi).
\end{aligned}
$$

□

Note. *By making use of* (3.0.4), *the Lie derivative of the metric tensor can also be given as,*

$$\mathcal{L}_{k_A} g_{ij} = \nabla_i k_{jA} + \nabla_j k_{iA}, \tag{3.1.4}$$

where $\nabla_i k_{jA} = \partial_i k_{jA} - \Gamma^m_{ij}(g)k_{mA}$.

Definition 3.3. *When* $\mathcal{L}_{k_A} g_{ij} = \nabla_i k_{jA} + \nabla_j k_{iA} = 0$, *such* k_A *is called a* ***Killing vector field****. An* ***isometry map*** *is a diffeomorphism on a manifold* $\mathcal{M} \to \mathcal{M}$ *which preserves the metric on all points on the manifold. Killing vector fields are the generators of the isometry transformations.*

Remark. *The isometries on a manifold form a group under composition which is called the* ***isometry group*** *of a manifold* $(\mathcal{M}, g)$. *This group can be identified by studying the Lie algebra of the linearly independent Killing vector fields:*

$$[k_A, k_B] = f^C_{AB} k_C = f^C_{AB} k^i_C \frac{\partial}{\partial \phi^i}$$

where f^C_{AB} *are the structure coefficients. We note that the Lie bracket of Killing vectors is also a Killing vector field itself; this can be shown as we are given that* $\mathcal{L}_{k_A} g_{ij} =$

$0, \; and \;\; \mathcal{L}_{k_B} g_{ij} = 0 \quad \Rightarrow$

$$\begin{aligned} \mathcal{L}_{[k_A, k_B]} g_{ij} &= \mathcal{L}_{k_A} \mathcal{L}_{k_B} g_{ij} - \mathcal{L}_{k_B} \mathcal{L}_{k_A} g_{ij} \\ &= \mathcal{L}_{k_A} (\cancel{\mathcal{L}_{k_B} g_{ij}}) - \mathcal{L}_{k_B} (\cancel{\mathcal{L}_{k_A} g_{ij}}) = 0. \end{aligned} \tag{3.1.5}$$

hence Killing vectors form a proper Lie algebra, which is oftentimes non-abelian.

Proposition 3.4. *The transformation* (3.1.2) *leave the $\sigma-$model action* (3.0.2) *invariant when k_A is a Killing vector field.*

Proof. We want to show that $\Delta S = S[\phi'] - S[\phi] = 0$. The original action of the $\sigma-$model is given by $S[\phi] = -\frac{1}{2} \int d^D x \eta^{\mu\nu} g_{ij} \partial_\mu \phi^i \partial_\nu \phi^j$, we are sending $\phi \mapsto \phi'$ and consequently $g_{ij}(\phi) \mapsto g'_{ij}(\phi)$. We will apply Taylor expansion again,

$$\begin{aligned} &\phi'^i = \phi^i + \theta^A k^i_A, \;\; \Rightarrow \;\; \frac{\partial \phi' i}{\partial x^\mu} = \frac{\partial \phi^i}{\partial x^\mu} + \theta^A \frac{\partial k^i_A}{\partial x^\mu} = \frac{\partial \phi^i}{\partial x^\mu} + \theta^A \frac{\partial k^i_A}{\partial \phi^m} \frac{\partial \phi^m}{\partial x^\mu} \\ &\text{therefore, } \mathcal{L}(\phi') = \underbrace{(g_{ij} + \theta^A \frac{\partial g_{ij}}{\partial \phi^m} k^m_A)}_{\mathbf{g}} \, \partial_\mu \phi'^i \partial_\nu \phi'^j \eta^{\mu\nu} \\ &\qquad = \mathbf{g} \left(\partial_\mu \phi^i + \theta^A \frac{\partial k^i_A}{\partial \phi^m} \frac{\partial \phi^m}{\partial x^\mu} \right) \left(\partial_\nu \phi^j + \theta^A \frac{\partial k^j_A}{\partial \phi^m} \frac{\partial \phi^m}{\partial x^\nu} \right) \eta^{\mu\nu} \\ &\qquad = \mathbf{g} \left(\partial_\mu \phi^i \partial_\nu \phi^j + \theta^A \partial_\mu \phi^i \frac{\partial k^j_A}{\partial \phi^m} \frac{\partial \phi^m}{\partial x^\nu} + \theta^A \partial_\nu \phi^j \frac{\partial k^i_A}{\partial \phi^m} \frac{\partial \phi^m}{\partial x^\mu} + \cancel{(\theta^A)^2 (\dots)} \right) \eta^{\mu\nu}. \end{aligned}$$

Note that derivatives with respect to Greek indices $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ are with respect to spacetime coordinates, while we left the target space derivatives of coordinates ϕ^m as is. From now on derivatives with respect to Latin indices will represent $\partial_m \equiv \frac{\partial}{\partial \phi^m}$ derivatives with respect to the target space. We will also make use of the symmetry of the metrics $g_{ij} = g_{ji}$, & $\eta^{\mu\nu} = \eta^{\nu\mu}$.

$$\begin{aligned}
\mathcal{L}(\phi') &= (g_{ij} + \theta^A \partial_m g_{ij} k^m_A)\left(\partial_\mu\phi^i\partial_\nu\phi^j + \theta^A\partial_\mu\phi^i\partial_m k^j_A\partial_\nu\phi^m + \theta^A\partial_\nu\phi^j\partial_m k^i_A\partial_\mu\phi^m\right)\eta^{\mu\nu} \\
&= \Big\{ g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + \theta^A g_{ij}\partial_\mu\phi^i\partial_m k^j_A\partial_\nu\phi^m + \theta^A g_{ij}\partial_\nu\phi^j\partial_m k^i_A\partial_\mu\phi^m \\
&\qquad\qquad + \theta^A\partial_m g_{ij} k^m_A\partial_\mu\phi^i\partial_\nu\phi^j + \cancel{(\theta^A)^2\ldots}\Big\}\eta^{\mu\nu} \\
&= g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + \partial_m g_{ij} k^m_A\partial_\mu\phi^i\partial^\mu\phi^j + g_{ij}\partial_m k^j_A\partial^\mu\phi^m\partial_\mu\phi^i + g_{ij}\partial_m k^i_A\partial_\mu\phi^m\partial^\mu\phi^j
\end{aligned}$$

we can rename the dummy indices in their respective contractions, namely that is to replace $m \leftrightarrow j$ and $m \leftrightarrow i$ in 3^{rd} and 4^{th} terms. Also symmetry of $\eta^{\mu\nu}$ implies that the $\partial_\mu \leftrightarrow \partial^\mu$ interchange is free. Hence,

$$\begin{aligned}
\mathcal{L}(\phi') &= g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + k^m_A\partial_m g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \partial_i k^m_A g_{mj}(\partial_\mu\phi^i\partial^\mu\phi^j) + \partial_j k^m_A g_{im}\partial^\mu\phi^i\partial_\mu\phi^j \\
&= g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + \partial_\mu\phi^i\partial^\mu\phi^j\ \underbrace{(k^m_A\partial_m g_{ij} + \partial_i k_{jA} + \partial_j k_{iA})}_{\mathcal{L}_{k_A} g_{ij}}
\end{aligned}$$

the right hand term on the final result is simply the Lie derivative of the metric in the direction of the vector field k_A, we assume that this vanishes as k_A is chosen to be a Killing vector field. In this case $\mathcal{L}(\phi') = g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j = \mathcal{L}(\phi)$, and the action integral is $S[\phi'] = -\frac{1}{2}\int d^D x\mathcal{L}(\phi) = S[\phi]$, this proves that $\Delta S = 0$. $\square$

Definition 3.5. *To every differentiable continuous symmetry generated by local actions, there corresponds a* ***conserved current****. For an action integral S, if the variation δS vanishes without imposing any conditions, S has a symmetry.*

Theorem 3.6. *The current corresponding to an infinitesimal isometry transformation,*

$$J^\mu_A := (\partial^\mu\phi^i)k_{iA}(\phi) = g_{ij}(\partial^\mu\phi^i)k^j_A(\phi), \tag{3.1.6}$$

is conserved, i.e., $\partial_\mu J^\mu_A = 0$.

Proof. We will make use of the equation of motion (3.0.3) of the Non-Linear $\sigma-$Model action where $k_i A$ is a Killing vector field. We have,

$$
\begin{aligned}
\partial_\mu J^\mu_A = \partial_\mu \cdot [\partial^\mu \phi^i k_{iA}] &= \partial_\mu \partial^\mu \phi^i k_{iA} + \partial^\mu \phi^i \partial_\mu k_{iA} \\
&= \Box \phi^i k_{iA} + \partial^\mu \phi^i \partial_k k_{iA} \partial_\mu \phi^k \\
&= \partial_\mu \phi^k \Big(- \Gamma^i_{jk} \partial^\mu \phi^j k_{iA} + \partial^\mu \phi^i \partial_k k_{iA} \Big) \\
&= \partial_\mu \phi^k \partial^\mu \phi^j \big(\underbrace{-\Gamma^i_{jk} k_{iA} + \partial_k k_{jA}} \big) \\
&= \partial_\mu \phi^k \partial^\mu \phi^j (\nabla_k k_{jA}) \\
&= \partial_\mu \phi^k \partial^\mu \phi^j \Big(\frac{1}{2} (\nabla_k k_{jA} + \nabla_j k_{kA}) + \frac{1}{2} (\nabla_k k_{jA} - \nabla_j k_{kA}) \Big)
\end{aligned}
$$

The second portion vanishes as the coefficient of $\partial_\mu \phi^k \partial^\mu \phi^j$ is symmetric under $k \leftrightarrow j$. Hence this is conserved if and only if k_{jA} satisfies (3.1.4), which is an implicit assumption of the theorem. It is stated in [7] that this is the Noether current corresponding to the Non-Linear $\sigma-$Model action under an isometry transformation. $\Box$

The target space geometry is arbitrary, but it becomes restricted when we assume supergravity theories. The field equations of the theory (coming from conformal invariance at quantum level) constrain the target space. Imposing integrability also restricts the target space geometry. As of rigid **supersymmetry**, it turns out that most of the time the target spaces have to be Kähler manifolds or homogeneous spaces; that is why in the upcoming chapters we will be focusing on these cases.

4. KÄHLER MANIFOLDS AS TARGET SPACES

We will first review some basic properties of Kähler manifolds, then we consider $\mathbb{C}P^n$ as an example of a Kähler manifold and build a $\sigma-$model. For this section we will rely on the introduction of the complex manifolds which are presented in A.1. For some examples we refer to [7]. We then discuss the Kähler potential which is fairly important for the essence of many papers like [5, 9–11], and [12] .

As we are studying in a Kähler manifold, which will be defined as a complex manifold in definition 4.2, the target manifold of such a theory should be even dimensional. For this, we consider $2n-$many scalar fields. We begin with the real coordinate patch provided by these fields; and switch to complex coordinates[2] by defining,

$$z^\alpha := \phi^\alpha + i\phi^{\alpha+n} \;\Rightarrow\; \bar{z}^{\bar{\alpha}} = \phi^\alpha - i\phi^{\alpha+n} = \overline{z^\alpha}$$

where $1 \leq \alpha \leq n$. We take the target space $\mathcal{M}$ to have Levi-Civita connection as before, and assume that its metric has Euclidean signature $(+, +, .., +)$.

The Riemannian positive-definite metric is $ds^2 = g_{ij} d\phi^i d\phi^j$ and can be rewritten in the complex basis by change of transform,

$$\begin{aligned} ds^2 &= g_{ij} \frac{\partial \phi^i}{\partial z^a} \frac{\partial \phi^j}{\partial z^b} dz^a dz^b \\ &= g_{\alpha\beta} dz^\alpha dz^\beta + 2 g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}} + g_{\bar{\alpha}\bar{\beta}} d\bar{z}^{\bar{\alpha}} d\bar{z}^{\bar{\beta}} \end{aligned} \tag{4.0.1}$$

where we used the symmetry of the metric. The connection coefficients transform $\{\Gamma^k_{ij}\} \mapsto \{\Gamma^\delta_{\alpha\beta}\}$ analogously, which means $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho}(\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\beta\rho} - \partial_\rho g_{\beta\gamma})$.

[2] In this chapter the complex coordinates carry Greek $\{\alpha, \beta, \gamma, ..\}$ indices, while real basis coordinates carry Latin $\{i, j, k, ..\}$.

4.1. Kähler Manifolds

Definition 4.1. *For a Hermitian manifold* $(\mathcal{M}, g)$*, define a* 2*-form* Ω *called the* ***Kähler form****,*

$$\Omega_p(\vec{X}, \vec{Y}) := g_p(\mathcal{J}_p\vec{X}, \vec{Y}) \quad \vec{X}, \vec{Y} \in T_p\mathcal{M}. \tag{4.1.1}$$

The Kähler form is sometimes called the fundamental form. Notice that,

$$\Omega(\vec{X}, \vec{Y}) = g(\mathcal{J}\vec{X}, \vec{Y}) = g(\mathcal{J}^2\vec{X}, \mathcal{J}\vec{Y}) = -g(\mathcal{J}\vec{Y}, \vec{X}) = -\Omega(\vec{Y}, \vec{X}), \tag{4.1.2}$$

so anti-symmetric. Also Ω is invariant under the action by $\mathcal{J}$

$$\Omega(\mathcal{J}\vec{X}, \mathcal{J}\vec{Y}) = g(\mathcal{J}^2\vec{X}, \mathcal{J}\vec{Y}) = g(\mathcal{J}^3\vec{X}, \mathcal{J}^2\vec{Y}) = \Omega(\vec{X}, \vec{Y}). \tag{4.1.3}$$

By its definition, in local complex coordinates the Kähler form in tensorial notation is given by $\Omega := -2ig_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^{\bar{\nu}}$. We will consider exterior derivative action on the Kähler $2-$form Ω to define Kähler manifolds.

Definition 4.2. ***Kähler manifolds*** *are manifolds with the property that* $d\Omega = 0$ *for the symplectic* $2-$*form defined on them. We call the metric tensor making up the* 2*-form as the* ***Kähler metric****.*

Theorem 4.3. *Kähler manifold axiom, i.e.,* $d\Omega = 0$*, is equivalent to the the metric relations*

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial z^\alpha} \quad \& \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{z}^\gamma} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{z}^\beta} \tag{4.1.4}$$

Proof. Let g be a Kähler metric. Then $d\Omega = 0$ implies $(\partial + \bar{\partial}) \cdot (ig_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^{\bar{\beta}}) = 0$.

i.e.,

$$i\partial_\gamma g_{\alpha\bar{\beta}} dz^\gamma \wedge dz^\alpha \wedge d\bar{z}^\beta + i\partial_{\bar{\gamma}} g_{\alpha\bar{\beta}} d\bar{z}^\gamma \wedge dz^\alpha \wedge d\bar{z}^\beta = 0. \tag{4.1.5}$$

Since the left part is a $(2,1)$-form and right $(1,2)$-form, they must vanish separately. This translates to,

$$\frac{1}{2} i \left(\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}}\right) dz^\gamma \wedge dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + \frac{1}{2} i \left(\partial_{\bar{\gamma}} g_{\alpha\bar{\beta}} - \partial_{\bar{\beta}} g_{\alpha\bar{\gamma}}\right) d\bar{z}^{\bar{\gamma}} \wedge dz^\alpha \wedge d\bar{z}^{\bar{\beta}} = 0$$

We require the coefficients to vanish and thus obtain the equations; $\partial_\gamma g_{\alpha\bar{\beta}} = \partial_\alpha g_{\gamma\bar{\beta}}$ and also $\partial_{\bar{\gamma}} g_{\alpha\bar{\beta}} = \partial_{\bar{\beta}} g_{\alpha\bar{\gamma}}$. □

In the real basis $\{\partial/\partial\phi^i\}$ and the corresponding complex basis, the Kähler form can be written as in the following,

$$\Omega = -2i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}} \text{ (in complex basis)} \tag{4.1.6}$$

$$= -\mathcal{J}_i^k g_{kj} d\phi^i \wedge d\phi^j \text{ (in real basis).} \tag{4.1.7}$$

Kähler manifolds can also be defined as manifolds with covariantly constant complex structure defined on them, i.e., $\nabla_m \mathcal{J}_j^i = 0$. This is shown in the theorem below.

Theorem 4.4. *([13] Theorem 8.5) A Hermitian manifold $(\mathcal{M}, g)$ is a Kähler manifold if and only if the almost complex structure $\mathcal{J}$ is covariantly constant.*

Proof. We start with a Hermitian manifold, we have an almost complex $\mathcal{J}$ and a Kähler form $\Omega := -\mathcal{J}_{ij} d\phi^i \wedge d\phi^j$. We want to show that $\nabla_m \mathcal{J}_j^i = 0$ if and only if Ω is closed, i.e., $d\Omega = 0$.

Fact. *If w is any given $r-$form, then*

$$dw = \nabla w \equiv \frac{1}{r!} \nabla_\mu w_{\nu_1 \ldots \nu_r} dx^\mu \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_r}$$

Proof. We will show the fact for $r = 2$,

$$
\begin{aligned}
\nabla\Omega &= \frac{1}{2}\nabla_\gamma \Omega_{\mu\nu} dx^\gamma \wedge dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\gamma \Omega_{\mu\nu} - \Gamma^\kappa_{\gamma\mu}\Omega_{\kappa\nu} - \Omega^\kappa_{\gamma\nu}\Omega_{\mu\kappa}) dx^\gamma \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}\partial_\gamma \Omega_{\mu\nu} dx^\gamma \wedge dx^\mu \wedge dx^\nu - \Gamma^\kappa_{\gamma\mu}\Omega_{\kappa\nu} dx^\gamma \wedge dx^\mu \wedge dx^\nu - \Gamma^\kappa_{\gamma\nu}\Omega_{\mu\kappa} dx^\gamma \wedge dx^\mu \wedge dx^\nu \\
&= d\Omega - X_{\gamma\mu\nu} dx^\gamma \wedge dx^\mu \wedge dx^\nu - X_{\gamma\nu\mu} dx^\gamma \wedge dx^\mu \wedge dx^\nu \\
&= d\Omega - \cancel{X_{\gamma\mu\nu} dx^\gamma \wedge dx^\mu \wedge dx^\nu} + \cancel{X_{\gamma\mu\nu} dx^\gamma \wedge dx^\mu \wedge dx^\nu} = d\Omega
\end{aligned}
$$

where we renamed the dummy indices. □

($\Rightarrow$) Assume that the Hermitian manifold is a Kähler manifold. Then $d\Omega = d(\mathcal{J}_i^k g_{kj}) = -\partial_\mu \mathcal{J}_{ij} d\phi^\mu \wedge d\phi^i \wedge d\phi^j = 0$. By the fact, this sum becomes zero only when $\nabla_m \mathcal{J} = 0$ is satisfied.
($\Leftarrow$) If $\nabla_m \mathcal{J} = 0$, it is automatic that the Kähler form $\Omega = -\mathcal{J}_{ij} d\phi^i \wedge d\phi^j$ vanishes under the action of the exterior derivative. Defining such a Kähler form promotes $\mathcal{M}$ to a Kähler manifold. This finishes the proof. □

Together with theorem A.16, this theorem implies the following:

Theorem 4.5. *Any complex Riemannian manifold with a covariantly constant complex structure is Kähler.*

Definition 4.6. *The* ***Kähler potential*** *is defined to be function $\mathcal{K}$ such that,*

$$
\partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z, \bar{z}) := g_{\alpha\bar{\beta}}. \tag{4.1.8}
$$

These potentials are not unique, they admit integration constants of holomorphic and antiholomorphic functions. So a Kähler potential lies in an equivalence class of potentials such that,

$$
\mathcal{K}(z, \bar{z}) \sim \mathcal{K}(z, \bar{z}) + f(z) + \bar{f}(\bar{z}). \tag{4.1.9}
$$

for some holomorphic function $f = f(z)$.

Remark. *For a Kähler manifold, due to the Hermicitiy condition (i.e.,* $g_{\alpha\beta} = 0$ *and* $g_{\bar{\alpha}\bar{\beta}} = 0$*), the connection coefficients* $\Gamma^{\alpha}_{\bar{\beta}\bar{\gamma}}$ *and the conjugates* $\Gamma^{\bar{\alpha}}_{\beta\gamma}$ *all vanish. Moreover the Kähler condition (that is* $2-$*form* Ω *being closed) sets Christoffel symbols* $\Gamma^{\alpha}_{\beta\bar{\gamma}} = \Gamma^{\bar{\alpha}}_{\bar{\beta}\gamma} = 0$*. So the only non-vanishing Christoffel coefficients turn out to be,*

$$\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\bar{\rho}}\partial_{\beta}g_{\gamma\bar{\rho}}, \qquad \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = g^{\rho\bar{\alpha}}\partial_{\bar{\beta}}g_{\rho\bar{\gamma}}. \tag{4.1.10}$$

Now we would like to look at some examples of Kähler manifolds.

Example 4.7. *All complex manifolds* $\mathcal{M}$ *with* $dim_{\mathbb{R}}\mathcal{M} = 2$ *are Kähler manifolds. This is due to the fact that complex manifolds are Hermitian manifolds and any* $2-$*form* Ω *on a* $2-$*manifold has a vanishing exterior derivative. Another very important example is the complex projective space which we work out and study next.*

Definition 4.8. *For any* $\lambda \in \mathbb{C} - \{0\}$*, the* ***complex projective space****, denoted by* $\mathbb{C}P^n$*, is the set defined by:*
$\mathbb{C}P^n := \{$*points in* $\mathbb{C}^{n+1}$ *with equivalence relation* $(z^1, ..., z^{n+1}) \sim (\lambda z^1, ..., \lambda z^{n+1})\}$.

Theorem 4.9. *Complex projective space is a complex manifold, moreover it is Kähler.*

Proof. On $\mathbb{C}P^n$ we can define the following atlas that is given by the neighborhoods $U_\kappa := \{z^\mu \,:\, \mu = 1, 2, .., n+1 |\, z^\kappa \neq 0\}$. Now on these sets of U_κ's we define the ***non-homogeneous coordinates***

$$\zeta^{\mu}_{\kappa} := \frac{z^\mu}{z^\kappa} \qquad \forall \mu \in \{1, 2, ..., n\} \tag{4.1.11}$$

these ζ^{μ}_{κ} are well-defined since $z^\kappa \neq 0$. Moreover these coordinates respect the equivalence relation; if we divide each coordinate by a fixed non-zero coordinate z^κ, then

$$\begin{aligned}\left(\frac{z^\mu}{z^\kappa}\right) \Rightarrow \left[\frac{z^1}{z^\kappa} : \frac{z^2}{z^\kappa} : .. : \frac{z^\kappa}{z^\kappa} = 1 : .. \frac{z^{n+1}}{z^\kappa}\right] &\sim \left[\frac{\lambda z^1}{\lambda z^\kappa} : \frac{\lambda z^2}{\lambda z^\kappa} : .. : \frac{\lambda z^\kappa}{\lambda z^\kappa} = 1 : .. \frac{\lambda z^{n+1}}{\lambda z^\kappa}\right] \\ &= \frac{\cancel{\lambda}}{\cancel{\lambda}}\left[\frac{z^1}{z^\kappa} : \frac{z^2}{z^\kappa} : .. : \frac{z^\kappa}{z^\kappa} = 1 : .. : \frac{z^{n+1}}{z^\kappa}\right]\end{aligned}$$

meaning that the ratios are still the same. For another coordinate chart we can define new coordinates in a neighborhood again with a non-vanishing γ^{th} coordinate by,

$$U_\gamma := \{z^\mu; \mu = 1, 2, ..., n+1 |\ z^\gamma \neq 0\}, \qquad \zeta^\mu_\gamma := \frac{z^\mu}{z^\gamma}. \tag{4.1.12}$$

For two patches $(U_\kappa, \zeta^\mu_\kappa)$ and $(U_\gamma, \zeta^\mu_\gamma)$ existence of intersection implies both $z^\kappa \neq 0$ and $z^\gamma \neq 0$. Also the transition function is $\zeta^\mu_\kappa \equiv \frac{z^\mu}{z^\gamma}\frac{z^\gamma}{z^\kappa} = \frac{\zeta^\mu_\gamma}{\zeta^\kappa_\gamma}$ which is holomorphic. This proves that $\mathbb{C}P^n$ is a **complex manifold**. Therefore defining a potential of the following form is reasonable, we cite [14] for this particular choice of the potential,

$$exp\{\mathcal{K}_\kappa\} := \sum_{\mu=1}^{n+1} |\zeta^\mu_\kappa|^2 \qquad \Rightarrow \mathcal{K}_\kappa := \log\left(\sum_{\mu=1}^{n+1} |\zeta^\mu_\kappa|^2\right) \tag{4.1.13}$$

$$\text{i.e.,} \quad \left(\frac{z_\kappa}{z_\gamma}\right)^2 \sum_{\mu=1}^{n+1} |\zeta^\mu_\kappa|^2 = \sum_{\mu=1}^{n+1} |\zeta^\mu_\gamma|^2 = exp\{\mathcal{K}_\gamma\}. \tag{4.1.14}$$

Therefore taking the logarithm of this expression gives,

$$\log(\zeta^\kappa_\gamma) + \log(\bar{\zeta}^\kappa_\gamma) + \mathcal{K}_\kappa = \mathcal{K}_\gamma, \tag{4.1.15}$$

which meets the Kähler potential axiom (4.1.9). Kähler potential for the neighborhood U_κ is given by;

$$exp\{\mathcal{K}_\kappa\} := \sum_{\mu=1}^{n+1} |\zeta^\mu_\kappa|^2 \ \Rightarrow \quad \mathcal{K} = ln(1 + |z|^2) = ln(1 + \delta_{\mu\bar{\nu}} z^\mu \bar{z}^{\bar{\nu}}). \tag{4.1.16}$$

Inside the logarithm we have $+1$ appearing since when $\mu = \nu$ we get $z^\mu / z^\mu = 1$. We obtain the metric tensor by taking the holomorphic and anti-holomorphic partial derivatives of this Kähler potential.

$$ds^2 = g_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}} = \frac{1}{1 + \bar{z}z}\left(\delta_{\mu\bar{\nu}} - \frac{\bar{z}_\mu z_{\bar{\nu}}}{1 + \bar{z}z}\right) dz^\mu d\bar{z}^{\bar{\nu}} \tag{4.1.17}$$

Hence we obtain a Hermitian metric as desired. It is also possible to construct the Kähler 2−form corresponding to this particular coordinate chart. This metric is the

Fubini-Study metric. □

The standard $\sigma-$model on $\mathbb{C}P^n$ is completely determined by the metric tensor of (4.1.17) according to the action (3.0.2). The Non-Linear $\sigma-$Model Lagrangian of $\mathbb{C}P^n$ is then given by,

$$\mathcal{L} = \kappa \frac{1}{1+\bar{z}z}\left(\delta_{\mu\bar{\nu}} - \frac{\bar{z}_\mu z_{\bar{\nu}}}{1+\bar{z}z}\right)\partial_\rho z^\mu \partial^\rho \bar{z}^{\bar{\nu}}. \tag{4.1.18}$$

for some constant $\kappa \in \mathbb{R}$.

4.2. Symmetries of Kähler Manifolds

In section 3.1, we found that for a Non-Linear $\sigma-$Model on a general Riemannian manifold, the symmetries that leave the NLσM action invariant are given by the Killing vector fields of the target space metric. Now we would like to analogously study the Killing symmetries of Kähler manifolds. Let the change in complex coordinates be as follows,

$$z^\alpha \mapsto z^\alpha + \delta z^\alpha = z^\alpha + \theta^A k_A^\alpha(z,\bar{z}). \tag{4.2.1}$$

Notice that this is an isometry transformation if the Killing equation, (3.1.4) $\mathcal{L}_{k_A} g_{ij} = \nabla_i k_{jA} + \nabla_j k_{iA}$, is satisfied. On a complex manifold (and hence on any Kähler manifold), Killing equation splits into two distinct conditions in terms of the complex basis coordinates,

$$\bullet\ \nabla_\mu k_{\nu A} + \nabla_\nu k_{\mu A} = 0 \qquad \bullet\ \nabla_\mu k_{\bar{\nu} A} + \bar{\nabla}_{\bar{\nu}} k_{\mu A} = 0. \tag{4.2.2}$$

The Kähler metric is invariant under isometry transformations, yet this might not be the case for the Kähler potential $\mathcal{K}$. It satisfies a weaker condition. Say we act on local coordinates by an isometry transformation as in (4.2.1), this leads to the following change in the Kähler potential,

$$g_{\alpha\bar{\beta}} =: \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z, \bar{z}) \tag{4.2.3}$$

$$\mathcal{K}(z, \bar{z}) \mapsto \mathcal{K}(z + \delta z, \bar{z} + \delta \bar{z}) = \mathcal{K}(z^\alpha + \theta^A k_A^\alpha(z)\,,\, \bar{z}^{\bar{\alpha}} + \theta^A k_A^{\bar{\alpha}}(\bar{z})) \tag{4.2.4}$$

$$\mathcal{K}(z'^\alpha, \bar{z}'^{\bar{\alpha}}) \mapsto \mathcal{K}(z^\alpha, \bar{z}^{\bar{\alpha}}) + \partial_\alpha \mathcal{K}(z^\alpha, \bar{z}^{\bar{\alpha}}) \theta^A k_A^\alpha + \partial_{\bar{\alpha}} \mathcal{K}(z^\alpha, \bar{z}^{\bar{\alpha}}) \theta^A k_A^{\bar{\alpha}} + \ \dots \tag{4.2.5}$$

we applied first order expansion to a function of two variables in the last step, while assuming locality with $0 < \theta \ll 1$.

We know that, $\mathcal{K}(z, \bar{z})$ lies in an equivalence class of functions such that $\mathcal{K} \sim \mathcal{K} + f(z) + \bar{f}(\bar{z})$, but now we see that $f(z)$ is no longer arbitrary:

$$\delta \mathcal{K} = \Delta \mathcal{K} = \theta^A (k_A^\alpha \partial_\alpha \mathcal{K} + k_A^{\bar{\alpha}} \partial_{\bar{\alpha}} \mathcal{K}) =: F(z) + \bar{F}(\bar{z}). \tag{4.2.6}$$

Actually we still have an extra freedom, namely we can shift F with a constant as follows; $F(z) \mapsto F(z) + i\xi$, and hence $\bar{F}(\bar{z}) \mapsto \bar{F}(\bar{z}) - i\xi$. So that $F(z) + \bar{F}(\bar{z})$ does not change. This gives a symmetry of the Kähler potential.

Definition 4.10. *A Killing vector satisfying,*

$$\mathcal{L}_k \mathcal{J}_j^i = 0, \tag{4.2.7}$$

where $\mathcal{J}$ is the complex structure on a complex manifold, is called a ***real-holomorphic Killing vector*** *.*

Proposition 4.11. *(see [15] proposition 9.5) For compact Kähler manifolds, all vector fields (in particular Killing vector fields) are real-holomorphic.*

Proof. Now the Lie derivative of the complex structure with respect to a vector field k_A is,

$$\mathcal{L}_{k_A} \mathcal{J}_j^i \quad = {k_A}^m (\partial_m \mathcal{J}_j^i) + (\partial_j {k_A}^m) \mathcal{J}_m^i - (\partial_m {k_A}^i) \mathcal{J}_j^m. \tag{4.2.8}$$

In the complex basis, we have (A.0.5) $\mathcal{J}_\alpha^\beta = i\delta_\alpha^\beta$, using this in the above we immediately get that $\mathcal{L}_{k_A}\mathcal{J}_j^i = 0$. ☐

4.2.1. Non-Linear σ−Model on $\mathbb{C}P^1$

Theorem 4.12. *The smooth manifolds $\mathbb{C}P^1$ and S^2 are diffeomorphic, i.e., $\mathbb{C}P^1 \cong S^2$.*

Proof. We take the non-homogeneous coordinates on $\mathbb{C}P^1$ on the neighborhoods U_1 and U_2. Notice that the following are differentiable invertible maps between $\mathbb{C}P^1$ and $\mathbb{C}$,

$$\Psi_1 : U_1 \to \mathbb{C} : \ [z_1 : z_2] \mapsto z_1/z_2 = \zeta_2^1 \tag{4.2.9}$$

$$\Psi_2 : U_2 \to \mathbb{C} : \ [z_1 : z_2] \mapsto z_2/z_1 = \zeta_1^2 \tag{4.2.10}$$

the charts U_1 and U_2 cover the whole input of $\mathbb{C}P^1$ as the origin $\vec{0}$ is not included in the domain. The inverses of the maps Ψ_i are between $\Psi_1^{-1} : \mathbb{C} \to U_1$ given by $z \mapsto [1 : z]$ and $\Psi_2^{-1} : \mathbb{C} \to U_2$ with $z \mapsto [z : 1]$. As also discussed above, the transition between coordinate charts for the neighborhoods of $U_1 \cap U_2$ (this is when none of $z_1 \neq 0$ and $z_2 \neq 0$) are given by the ratios of ζ's which is a complex holomorphic function.
We recall the stereographic projection of S^2; call $\vec{n} := (0,0,1)$ the north pole and $\vec{s} := (0,0,-1)$ the south pole. We have the following two charts to parametrize the sphere,

$$\pi_n : S^2 - \{\vec{n}\} \to \mathbb{C} : (x_1, x_2, x_3) \mapsto \Big(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\Big) \tag{4.2.11}$$

$$\pi_s : S^2 - \{\vec{s}\} \to \mathbb{C} : (x_1, x_2, x_3) \mapsto \Big(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\Big). \tag{4.2.12}$$

With that, we claim the following map is a diffeomorphism,

$$\Phi : \mathbb{C}P^1 \to S^2, \quad \Phi([z^1 : z^2]) := \begin{cases} \pi_n(z^1/z^2) & \text{if } z^2 \neq 0, \\ \vec{n} & \text{if } z^2 = 0 \end{cases} \tag{4.2.13}$$

This map is onto as the points on the 2−sphere can be covered by the domain. If we can show that the charts $\pi_n \circ \Phi \circ \Psi_1^{-1}$ and $\pi_s \circ \Phi \circ \Psi_2^{-1}$ are differentiable maps with a differentiable inverse, then we are done. It is easy to see that the first chart is nothing but the identity mapping of $\mathbb{C} \to \mathbb{R}^2$, hence holds true. The second map for $z \neq 0$ is the conjugation map of $\mathbb{C} \to \mathbb{C}$ by $z \mapsto \bar{z}$ which also fits our requirements. Notice that $z = 0$ implies that $\pi_s \circ \Phi \circ \Psi_2(0) = 0$. Thus we conclude that $\mathbb{C}P^1$ and S^2 are diffeomorphic. □

Now since $\mathbb{C}P^1$ and S^2 are diffeomorphic, we would like to show that the Fubini-Study metric is equivalent to the standard metric on S^2. We use the stereographic projection to identify the points on the sphere by complex coordinates $Z = X + iY$, those lie on the complex plane embedded in the ambient manifold $\mathbb{R}^3$. The coordinates are $X := \cos\varphi \tan\frac{\theta}{2}$, and $Y := \sin\varphi \tan\frac{\theta}{2}$. The standard metric on S^2 parametrized by spherical coordinates (θ, φ) is

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{4.2.14}$$

From these relations, we obtain differential 1−forms dX and dY, and express the metric under this transformation by,

$$ds^2 = \frac{4(dX^2 + dY^2)}{(1 + X^2 + Y^2)^2}. \tag{4.2.15}$$

Now we wish to express the metric in the complex basis $\{Z, \bar{Z}\}$. This can be achieved by switching from the real basis to the complex via the relations $X = \frac{Z+\bar{Z}}{2}, \quad Y = \frac{Z-\bar{Z}}{2i}$. We apply these transformations to the metric;

$$ds^2 = \frac{4\left(\left(\frac{dZ+d\bar{Z}}{2}\right)^2 + \left(\frac{dZ-d\bar{Z}}{2i}\right)^2\right)}{\left(1 + \left(\frac{Z+\bar{Z}}{2}\right)^2 + \left(\frac{Z-\bar{Z}}{2i}\right)^2\right)^2} = \frac{4dZd\bar{Z}}{\left(\frac{Z\bar{Z}}{4} + 1\right)^2} \to \frac{16\, dZd\bar{Z}}{(1 + |Z|^2)^2}. \tag{4.2.16}$$

where the last arrow denotes equality when we send $Z \mapsto 2Z$ and $\bar{Z} \mapsto 2\bar{Z}$. The final result is exactly the Fubini-Study metric in 2−dimensions.

In the light of previous results, we will search for Killing vectors of the Kähler manifold $\mathbb{C}P^1$, they will prove to span the Lie algebra of the Lie group $SU(2)$.[3] Elementary $SU(2)$ transformations of coordinates (ϕ^1, ϕ^2) give a holomorphic global symmetry of the $\mathbb{C}P^1$ metric tensor. The Kähler potential is as (4.1.16); we copy the metric in (4.1.17) for $n = 1$,

$$g_{z\bar{z}} = \frac{1}{(1+z\bar{z})^2}, \tag{4.2.17}$$

for this metric the connection coefficients are evaluated by (4.1.10) which gives the only remaining coefficients to be $\Gamma^z_{zz} = g^{z\bar{z}}\partial_z g_{z\bar{z}} = (1+z\bar{z})^2(-2)\bar{z}(1+z\bar{z})^{-3} = -2\bar{z}(1+z\bar{z})^{-1}$ and similarly $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = -2z(1+z\bar{z})^{-1}$. The isometries of $\mathbb{C}P^1$ with the Fubini-Study metric (4.1.17) are generated by the vector fields, we cite [7] here for the derivation of the Killing vector fields,

$$\begin{aligned} k_1 &= \frac{1}{2}i\left(\phi^1\frac{\partial z}{\partial\phi^2} + \phi^2\frac{\partial z}{\partial\phi^1}\right)\frac{\partial}{\partial z} = -\frac{i}{2}(1-z^2)\frac{\partial}{\partial z}, \\ k_2 &= -\frac{1}{2}\left(\phi^1\frac{\partial z}{\partial\phi^2} - \phi^2\frac{\partial z}{\partial\phi^1}\right)\frac{\partial}{\partial z} = \frac{1}{2}(1+z^2)\frac{\partial}{\partial z}, \\ k_3 &= -\frac{1}{2}i\left(\phi^1\frac{\partial z}{\partial\phi^1} - \phi^2\frac{\partial z}{\partial\phi^2}\right)\frac{\partial}{\partial z} = -iz\frac{\partial}{\partial z}. \end{aligned} \tag{4.2.18}$$

It is easy to very that these vector fields solve the Killing equation (4.2.2). Notice that each of the vector fields in (4.2.18) are holomorphic as expected from theorem 4.11. Observe that the vector field k_3 corresponds to the generator for the Lie group $U(1)$ on the $\mathbb{C}P^1$ target space. Let us show that it satisfies the Killing conditions of (4.2.2),

$$k_3^z\partial_z = -iz\partial_z \quad \Rightarrow \quad k_{\bar{z}3} = g_{z\bar{z}}k_3^z = \frac{1}{(1+z\bar{z})^2}(-iz),$$

and we have that $k_{z3} = 0$. As we have only the anti-holomorphic component of the vector field, we only have to satisfy $\bar{\nabla}_{\bar{z}}k_{\bar{z}3} = 0$. This is equivalent to $\partial_{\bar{z}}(k_{\bar{z}3}) - \Gamma^{\bar{z}}_{\bar{z}\bar{z}}k_{\bar{z}3} = 0$. Plugging in we have $\partial_{\bar{z}}(k_{\bar{z}3}) = \frac{2izz}{(1+z\bar{z})^3} = \frac{(-iz)}{(1+z\bar{z})^2}\frac{-2z}{1+z\bar{z}} = \Gamma^{\bar{z}}_{\bar{z}\bar{z}}k_{\bar{z}3}$. Hence k_3 is a Killing vector.

[3] $SU(2)$ indeed gives an isometry of the 2−sphere, and hence of $\mathbb{C}P^1$.

Example 4.13. *Let us examine the Lie brackets of these vector fields; this will help understand the Lie algebra generated by* $\{k_1, k_2, k_3\}$.

$$\begin{aligned}
[k_1, k_3] &= k_1 \cdot k_2 - k_2 \cdot k_1 = \left[-i\Big(-\frac{i}{2} + i\frac{z^2}{2}\Big) - (-iz)(zi)\right]\partial_z = \Big(\frac{-1-z^2}{2}\Big)\partial_z = -k_2,\\
[k_1, k_2] &= \left[-\frac{i}{2}\Big(1-z^2\Big)z - \Big(\frac{1+z^2}{2}\Big)iz\right]\partial_z = -iz\partial_z = k_3,\\
[k_2, k_3] &= \left[-i\Big(\frac{1+z^2}{2}\Big) - (-iz)z\right]\partial_z = -i\Big(\frac{1-z^2}{2}\Big)\partial_z = k_1,
\end{aligned}$$

It can be seen that the Lie brackets obey the structure equation $[k_A, k_B] = \epsilon_{ABC} k_C$, *which is exactly the structure of* $SU(2)$.

Remark. *The NLσM on* $\mathbb{C}P^1$ *is given by the metric* (4.1.17) *which gives out the Lagrangian,*

$$\mathcal{L} = \kappa \frac{1}{(1+z\bar{z})^2}\partial_\rho z \partial^\rho \bar{z}. \tag{4.2.19}$$

5. HOMOGENEOUS MANIFOLDS AS TARGET SPACES

In the previous chapters, we discussed the symmetries of NLσM's, those are the isometries of the Target manifold. For homogeneous spaces, the manifold itself is characterized by isometries, and hence they are appropriate target spaces for Non-Linear $\sigma-$Models, and actually they appear frequently in supergravity theories. We will start by giving basic properties and constructions of homogeneous spaces, for more details see [16]. A very important example of homogeneous spaces is the $n-$dimensional sphere [17] which will also be discussed. Symmetric spaces are analyzed in rigor in the books [18, 19] while for the derivation of the homogeneous space connection we follow [20].

Through this section G will be a Lie group and $\mathcal{M}$ will denote a manifold endowed with a left $G-$action. In this case we put the following axioms; given e (identity element), $g_1, g_2 \in G$, and $p, q \in \mathcal{M}$,

$$g_1 \cdot (g_2 \cdot p) = (g_1 \cdot g_2) \cdot p, \; \textit{and} \; e \cdot p = p.$$

We have such properties of group actions;

- An action is ***continuous*** if the defining map $\theta^g : \mathcal{M} \to g \cdot \mathcal{M}$ is continuous.
- For any element g and corresponding map θ^g on $\mathcal{M}$, we know the existence of $g^{-1} \in G$ and the map $\theta^{g^{-1}}$, therefore group actions are ***invertible*** and smooth group actions give *automorphisms* of $\mathcal{M}$.
- The ***isotropy group*** of an element $p \in \mathcal{M}$ is defined by $G_p := \{g \in G |\; g \cdot p = p\}$. It is not hard to show that G_p forms a subgroup of G.
- The action is called ***transitive*** if for any two points p, q in $\mathcal{M}$, there exists $g \in G$ that connects them, i.e., $g \cdot p = q$.

Definition 5.1. *A smooth manifold endowed with a transitive smooth action by a Lie group G is called a* ***homogeneous*** *$G-$**space** (or a homogeneous manifold if to specify* *the group is not of importance).*

Note. *Basically a homogeneous space is a manifold in which any point can be reached from any other point on the manifold by a symmetry transformation. These operations are provided by Lie groups. For the case of target manifolds of NLσM's, we showed that Killing vectors are the generators of the isometry transformations. They form a Lie algebra. For example for a manifold of dimension n, we would need $n-$many linearly independent Killing vectors for such symmetries.*

Definition 5.2. *For a Lie group G and $H < G$ a Lie subgroup, define a subset of G of the form; $gH := \{gh|\, h \in H\}$ (left cosets of H). This forms a partition of the group G; these left cosets with the quotient topology is called the left coset space of G modulo H.*

In the case that g_1H and g_2H give the same partition, it must be that $g_1 \sim g_2$ lying in the same class. Equivalently this can mean $g_1 \equiv g_2 \;(\mathrm{mod} H)$. This is the case when $g_1H = g_2H$, and that is if and only if $g_2^{-1}g_1H = H$, true only when $g_2^{-1}g_1 \in H$.

Theorem 5.3. ***(Homogeneous Space Construction Theorem)*** *Say G is a Lie group, H a closed subgroup of G. The left coset space G/H is a topological manifold of dimension $dimG - dimH$ and has a unique smooth structure s.t. $\pi : G \to G/H$ is a smooth submersion (i.e., onto). The action of G on G/H is given by $g_1 \cdot (g_2 \cdot H) = (g_1g_2) \cdot H$, turning G/H into a homogeneous $G-$space.*

Theorem 5.4. *Every homogeneous space is of the type described in the Homogeneous Space Construction Theorem 5.3.*

If we have a manifold $\mathcal{M}$ which is homogeneous $G-$space, then we have a strong theorem to identify $\mathcal{M}$ equivalently with a coset space of the form G/H.

Corollary. *A homogeneous space $\mathcal{M}$ with a Lie group G and an isotropy subgroup H is diffeomorphic to the quotient manifold G/H. Moreover G/H admits a unique smooth structure provided by the smooth map $G \times G/H \to G/H$ with $(g, g'H) \mapsto g \cdot g'H$.*

5.1. Sphere as a Coset Space

Our desire in this section is to derive the identification,

$$S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1). \tag{5.1.1}$$

This is done by the following method; when constructing a homogeneous manifold as a coset space of groups (the quotient itself may not be a group) we consider the isometry group of the manifold $\mathcal{M}$, call it G. Isometry group consists of automorphisms that preserve the metric; a subgroup of isometries is the isotropy group of a point, call it H. The manifold $\mathcal{M}$ is then identifiable by the coset set G/H. As an example, we will apply the procedure on a good candidate; the *sphere.*

Theorem 5.5. *The Lie group $O(n)$ acts transitively on S^{n-1} for $n \geq 2$.*

Proof. We want to work out the action of the orthogonal group $O(n)$ on S^{n-1}. The choice of S^{n-1} is because we consider the sphere lying in the ambient manifold $\mathbb{R}^n$, and the coordinates are $n-$tuples,

$$S^{n-1} = \{(x^1, x^2, .., x^n) \in \mathbb{R}^n \ s.t. \ \|x\|^2 = \sum_{i=1}^{n} (x^i)^2 = 1\}$$

Consider the simplest situation, that we want to map the north pole $\vec{n} = (1, 0, 0.., 0)$ to any other point on the sphere. Notice that under action of $\mathbb{A} \in O(n)$, the image $\mathbb{A}\vec{n}$ is the first column of the matrix $\mathbb{A}$,

$$\vec{n} \mapsto \mathbb{A}\vec{n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \tag{5.1.2}$$

For transitivity, take a destination $x' \in S^{n-1}$, therefore the first column $(a_{11}, a_{21}, .., a_{n1})^T$ is fixed. The column $\mathbb{A}_{i1}$ has norm 1, and so can be a component of an orthonormal basis. Find the rest of this basis by Gram-Schmidt orthogonalization process; and we obtain $\mathbb{A}_{i1}, v_2, v_3, .., v_n$ $n-$many vectors corresponding to a basis for $\mathbb{R}^n$. A matrix that places this basis in columns give an orthogonal matrix

$$\begin{pmatrix} | & | & & | \\ \mathbb{A}_{i1} & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}_{n\times n}$$

and it takes the north pole to the desired point. It is possible to generalize this to any $x \mapsto x'$ rather than the north pole, as assured to us by the spherical symmetry of a sphere. Hence the action is transitive. □

Theorem 5.6. *The isotropy group of the north pole is the subgroup $O(n-1)$ of $O(n)$.*

Proof. The restriction now is $\mathbb{A} \cdot (1, 0, 0, .., 0)^T = (1, 0, 0, .., 0)^T$, from this we have $a_{11} = 0$, and $a_{21} = a_{31} = .. = a_{n1} = 0$. We next apply orthogonality condition to this matrix, namely we must obtain,

$$\mathbb{A} = \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & * & \dots & * \\ \vdots & \vdots & \mathbb{B} & \vdots \\ 0 & * & \dots & * \end{pmatrix}, \text{ while also } \mathbb{A}^T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ a_{12} & * & \dots & * \\ \vdots & \vdots & \mathbb{B}^T & \vdots \\ a_{1n} & * & \dots & * \end{pmatrix},$$

$$\mathbb{A}\mathbb{A}^T = \mathbb{1} \Rightarrow \mathbb{1}_{11} = 1 = 1 + (a_{12})^2 + (a_{13})^2 + ... + (a_{1n})^2.$$

The only solution is when the first row of $\mathbb{A}$ consists of all zero entries. Hence the isotropy group is simply the matrices of the form,

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \mathbb{B} & \vdots \\ 0 & * & \dots & * \end{pmatrix}_{n\times n} \qquad \text{where } \mathbb{B} \in O(n-1)$$

As $\mathbb{B}$ is the unique matrix in this case, the isotropy group of $\vec{n}$ is $O(n-1)$. We can embed the subgroup $O(n-1)$ inside $O(n)$ by the above prescription. And due to spherical symmetry, $\vec{n}$ could be taken as well as any other point; leading to the same conclusions. $\square$

Remark. *We could have shown the transitive isometry group to be $SO(n)$. We would require a positive determinant as an additional property for the column matrix $(\mathbb{A}_{i1}, v_2, .., v_n)$. Say the determinant is -1, then send any $v_j \mapsto -v_j$ which brings a coefficient of (-1) for the determinant. And orthonormality is preserved. And the isometry group turns out to be $SO(n)$.*
Moreover while working on $\det = +1$ matrices, the isotropy group also can be shown to be $SO(n-1)$. Hence (5.1.1) is a valid identification of spheres.

Note. *The spheres which are also Lie groups are S^1, and S^3. For these manifolds, the group $SO(n-1)$ is normal in $SO(n)$ (or similarly $O(n-1)$ in $O(n)$), and consequently the coset space forms a well-defined group under coset multiplication (see [16]).*

5.2. Symmetric Spaces

Say we have a connected Lie group G with corresponding Lie algebra $\mathfrak{g}$ and a Lie subgroup H with Lie algebra $\mathfrak{h}$. Define a complementary space to $\mathfrak{h}$ by $\mathfrak{m}$, then with this setup any $g \in \mathfrak{g}$ can be written in the form,

$$g = h + m, \quad h \in \mathfrak{h}, \;\; m \in \mathfrak{m}. \tag{5.2.1}$$

This gives a direct sum decomposition on $\mathfrak{g}$ as we can construct projection maps from $\mathfrak{g}$ onto $\mathfrak{h}$ and $\mathfrak{m}$. Therefore $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In certain cases it happens that the commutations of $\mathfrak{h}$ and $\mathfrak{m}$ satisfy the following relations,

$$\bullet\ [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad \bullet\, [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad \bullet\, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \tag{5.2.2}$$

$$\{\text{Generators of } \mathfrak{g}\} = \{\text{Generators of } \mathfrak{h}\} \oplus \{\text{Generators of } \mathfrak{g} - \mathfrak{h}\} \tag{5.2.3}$$

this looks very much like the structure of a graded Lie algebra.[4] The spaces that satisfy such a particular structure are of importance in the study of homogeneous spaces.

Remark. *The first condition is a rudimentary criteria of* $\mathfrak{h} \subseteq \mathfrak{g}$. *The second condition implies that* $\mathfrak{m}$ *is an* $\mathfrak{h}-$*invariant complement to* $\mathfrak{h} \subseteq \mathfrak{g}$, *however the third property is of interest as it is the defining property for the algebras which correspond to the symmetric space.*

Definition 5.7. *Any simply connected homogeneous space* $\mathcal{M} = G/H$ *for which the isomorphism and isotropy algebra admit the structure in* (5.2.2) *is called a* ***symmetric space****. One of the most important feature of these manifolds is that they inherit a covariantly constant curvature tensor.*

5.3. Connection on a Homogeneous Space

We want to explore the geometrical quantities which we can make use of in coupling spinors in a NLσM; especially when the target is taken to be a homogeneous space, it can be useful to examine the invariance of quantities under global $G-$transformations. For this, we again consider a homogeneous space G/H and tensor fields we can define on this quotient. It will be important to analyze the Lie algebra valued differential forms.

Definition 5.8. *The* ***left invariant*** $1-$***forms*** *are differential forms on a manifold such that* $(\mathcal{L}_a)^* w = w$ *for all* $a \in G$. *On a homogeneous space* G/H, *the left invariant* $1-$*forms under the actions of* G *are elements of the form*

$$w = g^{-1} dg$$

where $g \in G$ *corresponds to a coset in* G/H.

The left invariant one-forms are differential forms which are left untouched by global constant transformations coming from G. Hence the defined object $g^{-1}dg$ inherits a ***rigid*** $G-$***action invariance***. However in this definition there is an implicit

[4]As in the case of examples towards the SuperPoincaré algebra.

statement implied; that such 1−forms are in fact invariant under left actions by G. Let us show how this unfolds:

Proof. The 1−forms $g^{-1}dg$ take their value in Lie algebra of the Lie group G. Say we act by $h \in G$ by sending $g \mapsto h \cdot g$, then

$$(h \cdot g)^{-1}d(h \cdot g) = g^{-1}\underbrace{h^{-1}h}\,dg = g^{-1}dg$$

This proves the statement. Moreover, we can see that such a form satisfies the "torsion" Maurer-Cartan equation as well,

$$d(g^{-1}dg) = dg^{-1} \wedge dg \overset{*}{=} -g^{-1}dg\,g^{-1} \wedge dg = -g^{-1}dg \wedge g^{-1}dg = -\frac{1}{2}[g^{-1}dg, g^{-1}dg]$$

The starred equality is derived by the following trick,

$$gg^{-1} = \mathbb{1} \;\; \Rightarrow \;\; d(gg^{-1}) = d(\mathbb{1}) = 0 \tag{5.3.1}$$

$$\Rightarrow dg \wedge g^{-1} + g \wedge dg^{-1} = 0 \;\; \Rightarrow \;\; g \wedge dg^{-1} = -dg \wedge g^{-1}. \tag{5.3.2}$$

$$\Rightarrow dg^{-1} = -g^{-1} \wedge dg \wedge g^{-1} \tag{5.3.3}$$

These observations are mainly covered in [20] and [18]. □

The differential form $g^{-1}dg$ lies inside the vector space $\mathfrak{g}^*$; that is the dual space of the Lie algebra $\mathfrak{g}$ of G. That is because by definition a Lie algebra consists of left invariant vector fields on the manifold, which is a subset of $\mathfrak{X}(G/H)$
We can always decompose such an element to elements from the Lie algebra of H and the Lie algebra of the remaining $G - H$. This means that indeed we can find w, and e such that,

$$g^{-1}dg = w + e \qquad w \in \mathfrak{h}, \quad e \in \mathfrak{m} \tag{5.3.4}$$

We have the desired algebraic properties when G/H satisfies the conditions for a *sym-*

metric space. In this case we make use of the conditions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as in (5.2.2),

$$\bullet\, [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad \bullet\, [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad \bullet\, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h},$$

hence in a symmetric space we have,

$$[w_\mu, w_\nu] \subset \mathfrak{h}, \qquad [e_\mu, e_\nu] \subset \mathfrak{h}.$$

Rewriting the torsion equation using this decomposition gives,

$$d(w + e) = -(w + e) \wedge (w + e). \tag{5.3.5}$$

Assume that we consider a right action on the left invariant forms on G/H. That is sending $g \mapsto gh$, which gives a different parametrization of the coset space. This time we assume not a global constant action by members of G, yet possibly coordinate dependent (and therefore local) transformations from the subgroup H. How are the components $w + e$ are affected by this transformation (note that now $\mathrm{d}h \neq 0$)?

$$\begin{aligned} g^{-1}dg \mapsto (gh)^{-1}d(gh) &= h^{-1}g^{-1}\left(dg \cdot h + g \cdot dh\right) = h^{-1}g^{-1}dgh + h^{-1}\mathbb{1}dh \\ &= h^{-1}(g^{-1}dg)h + h^{-1}dh \end{aligned} \tag{5.3.6}$$

meaning that,

$$(w + e) \mapsto h^{-1}(w + e)h + h^{-1}dh \tag{5.3.7}$$

as the elements e, and w belong in non-intersecting sets, we can further decompose these relations into

$$w_\mu \mapsto h^{-1}w_\mu h + h^{-1}\partial_\mu h, \text{ and } \quad e_\mu \mapsto h^{-1}e_\mu h \tag{5.3.8}$$

The objects w_μ and e_μ transform as covariant vectors under transformations, however they are not proper tensors. We will uncover how they play a role in the quotient G/H, first we state that we can define even Lie algebraic notions of curvatures. Put,

$$R_{\mu\nu}(H) := \partial_\mu w_\nu - \partial_\nu w_\mu + [w_\mu, w_\nu] \tag{5.3.9}$$

$$R_{\mu\nu}(G/H) := \partial_\mu e_\nu - \partial_\nu e_\mu + [w_\mu, e_\nu] - [w_\nu, e_\mu] \tag{5.3.10}$$

furthermore, the ***H-covariant derivatives*** can be defined to satisfy,

$$[D_\mu, D_\nu] = -R_{\mu\nu}(H), \text{ and } \quad R_{\mu\nu}(G/H) := D_\mu e_\nu - D_\nu e_\mu$$

meaning that we indeed put $D_\mu e_\nu := \partial_\mu e_\nu + [w_\mu, e_\nu]$ on G/H. We can express this equations by decomposing w and e into their respective Lie algebra generators. This will be used in proving the next proposition.

Proposition 5.9. *For $w \in \mathfrak{h}$ and $e \in \mathfrak{m}$ in a symmetric space, the curvatures reduce down to $R_{\mu\nu}(H) = -[e_\mu, e_\nu]$, and $R_{\mu\nu}(G/H) = 0$.*

Proof. Let us elaborate on what happens to the Lie algebraic curvatures when G/H is chosen to be a *symmetric space.* In equation (5.3.5) we can expand the terms and separate the components into the sets they belong,

$$\begin{aligned} d(w+e) &= -(w+e)\wedge(w+e) \\ \underbrace{dw}_{\in\mathfrak{h}} + \underbrace{de}_{\in\mathfrak{m}} &= -\Big(\underbrace{w\wedge w}_{\in\mathfrak{h}} + \underbrace{w\wedge e}_{\in\mathfrak{m}} + \underbrace{e\wedge w}_{\in\mathfrak{m}} + \underbrace{e\wedge e}_{\in\mathfrak{h}}\Big) \Rightarrow \end{aligned} \tag{5.3.11}$$

the equations of $\mathfrak{h}$ and $\mathfrak{m}$ are split since these are completely disjoint sets. This way we end up with two equations,

$$\mathrm{d}w = -w\wedge w - e\wedge e \tag{5.3.12}$$

$$\mathrm{d}e = -w\wedge e - e\wedge w \tag{5.3.13}$$

• The first equation describes the following,

$$\begin{aligned}
&dw_\mu = -w_\nu \wedge w_\mu - e_\nu \wedge e_\mu, \quad \text{and,} \quad dw_\nu = -w_\mu \wedge w_\nu - e_\mu \wedge e_\nu \\
&dw_\nu - dw_\mu = [w_\nu, w_\mu] + [e_\nu, e_\mu] \\
\Rightarrow\ &\partial_\mu w_\nu - \partial_\nu w_\mu + [w_\mu, w_\nu] = R_{\mu\nu}(H) = -[e_\mu, e_\nu].
\end{aligned}$$

• The second equation tells us that the H-covariant derivative of the vielbein is symmetric in two indices.

$$\begin{aligned}
dw_\mu + de_\mu &= -(w_\nu \wedge w_\mu + w_\nu \wedge e_\mu + e_\nu \wedge w_\mu + e_\nu \wedge e_\mu) \\
\Rightarrow\ de_\mu &= -w_\nu \wedge e_\mu - e_\nu \wedge w_\mu, \qquad de_\nu = -w_\mu \wedge e_\nu - e_\mu \wedge w_\nu
\end{aligned}$$

and so that we have,

$$\begin{aligned}
&\Rightarrow de_\mu - de_\nu = -w_\nu \wedge e_\mu + e_\mu \wedge w_\nu - e_\nu \wedge w_\mu + w_\mu \wedge e_\nu = -[w_\nu, e_\mu] + [w_\mu, e_\nu] \\
&\Rightarrow \partial_\nu e_\mu - \partial_\mu e_\nu - [w_\mu, e_\mu] + [w_\nu, e_\mu] = 0 \\
&\Rightarrow D_\nu e_\mu - D_\mu e_\nu = 0.
\end{aligned}$$

this completes the proof. □

The decomposition of the fields e_μ in $\mathfrak{m}$ give us a set of ***vielbein***, it reduces the curved spacetime metric into flat Lorentzian metric at each point on the tangent space of the manifold. It helps diagonalize the metric and in a sense it is the "square root" of the metric.

The third condition of a symmetric space, namely $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ tells us that the generators of $\mathfrak{m}$ of $G - H$ give a representation of the group H. These elements rotate the tangent frames in G/H while giving an isometry of an induced metric on the target space. The members $\{w_\mu\}$ appearing here are called ***spin connection***.

5.4. NLσM's with Homogeneous Target Space

In this chapter we focus on the $\sigma-$model survey of [20]. We again consider a field theory where the set $\{\phi^i(x)\}$ are mappings from the spacetime "*the world*" to a target space; yet now the target manifold is a homogeneous space G/H. The points on the target space are the fields $\phi^i(x)$. In the quotient space, the points can be represented by elements $g(\phi^i(x))$ as they correspond to coset representatives. That is we send

$$g(\phi) \in G \to g(\phi) \cdot H \text{ lying inside } G/H.$$

Throughout this section, for the $\sigma-$models; adopting the notation of [20], we will call $\mathcal{V}(\phi) := g(\phi)$ as the target space coordinates.
Just like we decomposed the left invariant forms by the Lie algebra $\mathfrak{g}$ of G and $\mathfrak{h}$ of H, in the same manner we can further define the Lie algebraic quantities:

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = \mathcal{Q}_\mu + \mathcal{P}_\mu, \quad \mathcal{Q}_\mu \in \mathfrak{h}, \text{ and } \mathcal{P}_\mu \in \mathfrak{m} \tag{5.4.1}$$

this equation is a direct analogue of $g^{-1}dg = w + e$ for the left invariant forms.

Definition 5.10. *The quantities $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ satisfying* (5.4.1) *are the* ***pull-backs*** *of the target space connection coefficients $\{w_\mu\}$ and vielbein $\{e_\mu\}$ discussed in* (5.3.4). *Explicitly they are defined to be,*

$$\mathcal{Q}_\mu(\phi) := w_i(\phi)\partial_\mu \phi^i \tag{5.4.2}$$

$$\mathcal{P}_\mu(\phi) := e_i(\phi)\partial_\mu \phi^i \tag{5.4.3}$$

Suppose that we consider a right action as in (5.3.8), and we would like to find out the transformations of $\mathcal{Q}$ and $\mathcal{P}$ under such an action, in fact that is equivalent to considering local $H-$transformations. Say that we consider $\mathcal{V}(x) \mapsto \mathcal{V}(x)h(x)$ then,

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} \mapsto h^{-1}\mathcal{V}^{-1}\left(\partial_\mu \mathcal{V} \cdot h + \mathcal{V} \cdot \partial_\mu h\right) = h^{-1}\mathcal{V}^{-1}\partial_\mu \mathcal{V} h + h^{-1}\mathbb{1}\partial_\mu h \quad \Rightarrow \tag{5.4.4}$$

$$\mathcal{Q}_\mu \mapsto h^{-1}(x)\mathcal{Q}_\mu(x)h(x) + h^{-1}(x)\partial_\mu h(x) \tag{5.4.5}$$

$$\mathcal{P}_\mu \mapsto h^{-1}(x)\mathcal{P}_\mu(x)h(x) \tag{5.4.6}$$

these equations are the analogues of (5.3.8). It can be observed that the components $\mathcal{Q}_\mu$ transform as the gauge field associated with the local H transformation, whoever is under consideration. This will lead to a new covariant derivative for the terms fundamentally. This is because we are looking for a gauge invariant (gauge field here being $\mathcal{Q}_\mu$) Lagrangian which consists of the important kinetic term, and hence we need a way of taking a derivative as the classical $\sigma-$model Lagrangians include contractions of the derivative: $\partial_\mu \phi^i \partial^\mu \phi^j g_{ij}$. This way, the Lagrangian will be invariant under actions of $G_{rigid} \times H_{local}$.

Definition 5.11. *The* ***H-covariant derivative*** *D is a linear derivation acting on G/H by,*

$$D_\mu \mathcal{V} := \partial_\mu \mathcal{V} - \mathcal{V}\mathcal{Q}_\mu. \tag{5.4.7}$$

The H-covariant derivative is defined as such, the equation (5.4.1) now becomes,

$$\mathcal{V}^{-1}D_\mu \mathcal{V} = \mathcal{P}_\mu. \tag{5.4.8}$$

This is a quite important relation, if the gauge field in the theory changes, then the covariant differentiation changes in respect, however we still have the validity of (5.4.8). There is also the analogue of the structure equations under the context of spin connection. The Cartan structure equations are analogues of the same equations in the Lie algebra theory, known as Maurer-Cartan equations. In a non-field theory context we gave them in (5.3.9), here we adjust them accordingly, put

$$\begin{aligned} &\bullet F_{\mu\nu}(\mathcal{Q}) := D_{[\mu},\mathcal{Q}_{\nu]} = \partial_\mu \mathcal{Q}_\nu - \partial_\nu \mathcal{Q}_\mu + [\mathcal{Q}_\mu, \mathcal{Q}_\nu] = -[\mathcal{P}_\mu, \mathcal{P}_\nu], \\ &\bullet D_{[\mu}, \mathcal{P}_{\nu]} = D_\mu \mathcal{P}_\nu - D_\nu \mathcal{P}_\mu = \partial_\mu \mathcal{P}_\nu - \partial_\nu \mathcal{P}_\mu - P_\nu \mathcal{Q}_\mu + \mathcal{P}_\mu \mathcal{Q}_\nu = 0. \end{aligned} \tag{5.4.9}$$

The first one is the analogue of the 1^{st} ***Cartan structure*** equation, while the second is the one regarding the curvature 2^{nd} ***Cartan structure*** equation.

Definition 5.12. *We define the* ***class of Lagrangians*** *for NLσM's with homogeneous target space. The Lagrangian must be left fixed under the global actions of G and local H transformations, such a class is given by,*

$$\mathcal{L} := \frac{1}{2} tr \left[D_\mu \mathcal{V}^{-1} D^\mu \mathcal{V} \right],$$

where $tr[.]$ stands for the trace over the Lie algebra.

Remark. *Actually this reduces to the Non-Linear $\sigma-$Model action given in* (3.0.2) *when the target space is a Lie group. To see this, let $H = \{e\}$, and hence $G/H \cong G$ be a Lie group. The corresponding Lagrangian will be invariant under linear transformations of G_{rigid}. The gauge fields $\mathcal{Q}_\mu \sim w_\mu$ will fall under the Lie algebra $\mathfrak{h} \cong \{0\}$ (the trivial Lie algebra) and therefore it will vanish, and the $H-$covariant derivative reduces to $D_\mu \mathcal{V} \equiv \partial_\mu \mathcal{V}$,*

$$D_\mu \mathcal{V} = \partial_\mu \mathcal{V} - \mathcal{V} \cancel{\mathcal{Q}_\mu} \;\Rightarrow\; \mathcal{V}^{-1} \partial_\mu \mathcal{V} = \mathcal{V}^{-1} D_\mu \mathcal{V} = \mathcal{P}_\mu = e_i \partial_\mu \phi^i.$$

Then we get,

$$\frac{1}{2} tr \left[\partial_\mu \mathcal{V}^{-1} \partial^\mu \mathcal{V} \right] \equiv -\frac{1}{2} tr \left[\partial_\mu \phi^i \partial^\mu \phi^j e_i e_j \right] = -\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^j \, g_{ij}.$$

Proposition 5.13. *The Lagrangian given in definition 5.12 can also be written as,*

$$\mathcal{L} = -\frac{1}{2} tr \left[\mathcal{P}_\mu \mathcal{P}^\mu \right] \tag{5.4.10}$$

Proof. We need an alternative way of writing $D_\mu\mathcal{V}^{-1}D^\mu\mathcal{V}$, we will make use of equation (5.4.8). Also notice that $D_\mu(\mathcal{V}\mathcal{V}^{-1}) = D_\mu(\mathbb{1}) = 0$, expanding this in accordance with Leibniz's theorem,

$$\Leftrightarrow D_\mu\mathcal{V}^{-1}\mathcal{V} + \mathcal{V}^{-1}D_\mu\mathcal{V} = 0 \tag{5.4.11}$$

$$\Leftrightarrow D_\mu\mathcal{V}^{-1} = -\mathcal{V}^{-1}(D_\mu\mathcal{V})\mathcal{V}^{-1} \tag{5.4.12}$$

therefore the new Lagrangian turns out to be,

$$\mathcal{L} = \frac{1}{2}tr\left[D_\mu\mathcal{V}^{-1}D^\mu\mathcal{V}\right] = \frac{1}{2}tr\left[-\mathcal{V}^{-1}(D_\mu\mathcal{V})\mathcal{V}^{-1}D^\mu\mathcal{V}\right] \tag{5.4.13}$$

$$= -\frac{1}{2}tr[\mathcal{V}^{-1}\underbrace{D_\mu\mathcal{V}}_{\mathcal{V}\mathcal{P}_\mu}\mathcal{V}^{-1}\underbrace{D^\mu\mathcal{V}}_{\mathcal{V}\mathcal{P}^\mu}] = -\frac{1}{2}tr\left[\mathcal{P}_\mu\mathcal{P}^\mu\right] \tag{5.4.14}$$

this finishes the proof. □

Proposition 5.14. *The Lagrangian in definition 5.12 is invariant under rigid G and local coordinate dependent $H-$transformations. Hence the equations of motion preserve $G_{rigid} \times H_{local}$ symmetry.*

Proof. Say that we act on the Lagrangian by a constant $\mathcal{U} \in G$, this is a transformation sending $\mathcal{V} \mapsto \mathcal{U}\mathcal{V}$, then the new Lagrangian turns out,

$$\mathcal{L}_{new} = \frac{1}{2}\left[D_\mu(\mathcal{V}^{-1}\mathcal{U}^{-1})D^\mu(\mathcal{U}\mathcal{V})\right] = \frac{1}{2}\left[D_\mu\mathcal{V}^{-1}\cdot\mathcal{U}^{-1}\mathcal{U}\cdot D^\mu\mathcal{V}\right] \tag{5.4.15}$$

$$= \frac{1}{2}\left[D_\mu\mathcal{V}^{-1}D^\mu\mathcal{V}\right] = \mathcal{L}_{old} \tag{5.4.16}$$

this shows the invariance of Lagrangian under G transformations. Moreover if we were to map $\mathcal{V}(x) \mapsto \mathcal{V}(x)h(x)$, i.e., consider a local $H-$transformation, using the transformation (5.4.6) the new Lagrangian turns out to be,

$$\mathcal{L}_{old} = -\frac{1}{2}tr\left[\mathcal{P}_\mu\mathcal{P}^\mu\right] \mapsto \mathcal{L}_{new} = -\frac{1}{2}tr\left[h^{-1}(x)\mathcal{P}_\mu(x)h(x)\cdot h^{-1}(x)\mathcal{P}_\mu(x)h(x)\right]$$

$$= -\frac{1}{2}tr\left[h^{-1}(x)\mathcal{P}_\mu(x)\mathcal{P}^\mu(x)h(x)\right]$$

recall that the trace operator conserves cyclic permutations of the arguments. That is, $tr(\mathbb{ABCD}) = tr(\mathbb{BCDA}) = ... = tr(\mathbb{DABC})$, therefore we obtain,

$$\mathcal{L}_{new} = -\frac{1}{2} tr\left[\mathcal{P}_\mu \mathcal{P}^\mu \cdot h(x)h^{-1}(x)\right] = \mathcal{L}_{old}. \tag{5.4.17}$$

When we do not impose any gauge to restrict the $\mathcal{V}$ to a coset representative, the Lagrangian (and hence the theory) is invariant under linear transformations of $G_{rigid} \times H_{local}$. $\square$

5.5. The $\sigma-$Model on $2-$Sphere

We will consider the parametrization for the ungauged $\sigma-$model given in ref. [21, 22]; the sphere is modeled as a quotient manifold of the form $S^2 \cong SU(2)/U(1)$.[5] The target manifold is given by the parametrization,

$$\mathcal{V} = \frac{1}{\sqrt{1 + \|\phi\|^2}} \begin{pmatrix} 1 & \phi \\ -\bar{\phi} & 1 \end{pmatrix}, \tag{5.5.1}$$

the target manifold is a $2-$manifold, therefore a single complex valued scalar field ϕ is sufficient. Yet keep in mind that it is not always possible to cover a whole manifold, example here being the S^2, with a single parametrization. We compute the left hand side of (5.4.1) with the parametrization of (5.5.1),

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = \frac{1}{(1 + \|\phi\|^2)} \begin{pmatrix} 1/2(\phi\partial_\mu\bar{\phi} - \bar{\phi}\partial_\mu\phi) & \partial_\mu\phi \\ -\partial_\mu\bar{\phi} & 1/2(\bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi}) \end{pmatrix}. \tag{5.5.2}$$

[5]The $SU(2)$ gives isometries of $S^2 \cong \mathbb{C}P^1$ (see section 4.12), and the isotropy group corresponding is $U(1)$. Hence likewise the example of (5.1.1), we can also identify the $2-$sphere as the quotient manifold $SU(2)/U(1)$.

Referring to [21], to compute the right hand side of (5.4.1) we need to use the $SU(2)$ generators as follows: we compute the equation (3.7) of [21],

$$\mathcal{V}^{-1}(\partial_\mu \mathcal{V}) = \frac{1}{2}\mathcal{P}_\mu(T_1 - iT_2) + \frac{1}{2}\bar{\mathcal{P}}_\mu(T_1 + iT_2) + \mathcal{Q}_\mu T_3 \tag{5.5.3}$$

where the $(T_i)_{2\times 2}-$matrices are represented through the Pauli $\sigma-$matrices by,

$$T_1 := \frac{-i}{2}\sigma_1 = \begin{pmatrix} 0 & \frac{-i}{2} \\ \frac{-i}{2} & 0 \end{pmatrix}, \quad T_2 := \frac{i}{2}\sigma_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{-1}{2} & 0 \end{pmatrix}, \quad T_3 := \frac{-i}{2}\sigma_3 = \begin{pmatrix} \frac{-i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}.$$

which gives out,

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = \begin{pmatrix} 0 & \frac{\mathcal{P}_\mu}{2} \\ -\frac{\bar{\mathcal{P}}_\mu}{2} & 0 \end{pmatrix} + \mathcal{Q}_\mu \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}. \tag{5.5.4}$$

And so we obtain that,

$$\mathcal{P}_\mu = \frac{(2\partial_\mu \phi)}{1 + \|\phi\|^2}, \qquad \mathcal{Q}_\mu = \frac{i}{1 + \|\phi\|^2}(\phi \partial_\mu \bar{\phi} - \bar{\phi}\partial_\mu \phi). \tag{5.5.5}$$

The Non-Linear $\sigma-$Model Lagrangian is then given by plugging P_μ in (5.4.10),

$$\mathcal{L} = 2\frac{\partial_\mu \phi}{1 + \|\phi\|^2}\frac{2\partial^\mu \bar{\phi}}{1 + \|\phi\|^2} = 4\frac{\partial_\mu \phi \partial^\mu \bar{\phi}}{(1 + \|\phi\|^2)^2}, \tag{5.5.6}$$

which is exactly the result of (4.2.19) with the proper $\kappa \in \mathbb{R}$.

Remark. *The $\mathcal{Q}_\mu$ is used in supersymmetric models to define a covariant derivative for the fermionic fields so that they remain invariant under local-H transformations when the target space is a homogeneous space G/H. Under H_{local}, a spinor field is mapped to $\psi(x) \mapsto h^{-1}(x)\psi(x)$. The corresponding $H-$covariant derivative for a spinor field is then given by, $D_\mu \psi := (\partial_\mu + \mathcal{Q}_\mu)\psi$. This allows us to incorporate local $H-$invariant spinorial terms in the Lagrangian.*

6. SUPERSYMMETRIC NON-LINEAR $\sigma-$MODELS

Supersymmetric $\sigma-$models have been the keystone in the research for supersymmetric field and supergravity theories for the past 50 years. The first geometric interpretations of the $D = 2$, and 3 model was given in [12]. Later on, the same model was studied in various dimensions in more detail in [3–5]. The more recent papers of [10, 11, 21–23] have all studied supergravity theories with this model to determine the target manifolds. The classification of target manifolds in various models are provided in [24]. As of this chapter we will stick to the component approach of the Lagrangian formalism while following [5] and [9].

The NLσM's we analyzed until the Supersymmetric $\sigma-$Model chapter were of the bosonic type of $\sigma-$models; this is because we had not yet included any fermionic fields. Bosonic is a term used to refer to the forces governing the nature, and the fermions are the constituents of matter in the nature. We give a short review of supersymmetry in the appendix B.1. Since we now wish to consider supersymmetric NLσM's, we need to add fermionic partners to scalar fields; in this case the multiplet we need will be called the chiral multiplet. As of NLσM's, we interested in the general structure of the target manifold when supersymmetry and fermion fields are existent in the theory. For this part we refer to [7, 25]. For a more comprehensive outlook on supersymmetry see [26–28].

6.1. The Chiral Model

Supersymmetric multiplets contain bosons and fermions with equal number of degrees of freedom. Since we are interested in supersymmetric $\sigma-$models, we need a supermultiplet that contains a scalar field. This is achieved by adding a spin$-1/2$ spinor field.

Definition 6.1. *The multiplet which consists of a spinor field $\psi(x)$ and a scalar field $\phi(x)$ is called the* ***chiral multiplet****. In $D = 3$, the scalar field is complex for $\mathcal{N} > 1$*

and the fermionic field is a Majorana spinor.

The *on-shell supersymmetric* theory of a chiral multiplet in $D = 4$ was constructed by Wess and Zumino in 1973 [29]. Adapting this to $D = 3$ with $n-$chiral multiplets (ϕ^i, ψ^i, $1 \leq i \leq n$), we obtain the action,

$$S := -\frac{1}{2}\int d^3x\{\partial_\mu\phi^i\partial^\mu\bar{\phi}^j + \bar{\psi}^i\partial\!\!\!/\psi^j\}\delta_{ij}. \tag{6.1.1}$$

It is straightforward to show that this action remains invariant under the following supersymmetry transformations,

$$\begin{aligned}\delta\phi^i &= \bar{\epsilon}\psi^i \\ \delta\psi^i &= \partial\!\!\!/\phi^i\epsilon,\end{aligned} \tag{6.1.2}$$

and moreover $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi^i = -2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\phi^i$ and $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi^i = -2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\psi^i$ where field equations are used. For details, see [27].

6.2. $D = 3$, and $\mathcal{N} = 1$ Supersymmetric $\sigma-$Model

There is even a greater class brought in by the inclusion of a non-generic metric, and hence the inclusion of **target geometry** in the picture. The Non-Linear $\sigma-$Model brings this significant upgrade; we couple the Lagrangian with a metric tensor $g_{ij} = g_{ij}(\phi)$, and we modify the Wess-Zumino action (6.1.1) accordingly. By doing so, we introduce the notion of an abstract target manifold formed by the scalar fields $\{\phi^i\}$. This target manifold has the dimension of the supersymmetric partners in the theory. This will remark the true beauty of the $\sigma-$Model, as now different fields are in interaction non-trivially.

In this section we sketch the derivation of [5] for the construction of a supersymmetric Lagrangian, and supersymmetry variations. The case of $D = 3$ Non-Linear $\sigma-$Models is paralleling the $D = 2$ models; everything we prove for $D = 3$ in these sections will also valid for $D = 2$ Non-Linear $\sigma-$Models as the properties of spinor in these two dimensions are similar. Note that in $D = 2$ scalar fields need to be real-valued.

We take $n-$copies of the chiral multiplet, namely $\{\phi^i, \psi^i\}$ where $1 \leq i \leq n$ and consider generalization of the Wess-Zumino model (6.1.1) as follows,

$$S := -\frac{1}{2}\int d^3x\{\partial_\mu\phi^i\partial^\mu\bar{\phi}^j + \bar{\psi}^i\partial\!\!\!/\psi^j + ...\}g_{ij}, \tag{6.2.1}$$

we have to uncover the necessary extra terms of this action. To find out necessary modifications, we apply the **Noether procedure**; this requires minimal adjustments so that supersymmetry is retained up to all orders of fermionic fields. The details of the procedure depend on the spacetime dimension.

Notice that the metric is a function of the the scalar fields, i.e., $g_{ij} = g_{ij}(\phi)$. And hence under supersymmetry it changes as,

$$\delta_\epsilon \cdot g_{ij} = \partial_k g_{ij}\delta_\epsilon\phi^k = \big(\Gamma^m_{ki}g_{jm} + \Gamma^m_{kj}g_{im}\big)\delta_\epsilon\phi^k. \tag{6.2.2}$$

Moreover when we do partial integrations, the derivatives will also act on the metric. Hence the action is not going to remain invariant under (6.1.2). To get rid of the extra Christoffel symbols that appear from (6.2.2), we replace the ordinary derivative on the Majorana spinor ψ^i in the action with the following covariant derivative,

$$D_\mu\psi^i := \partial_\mu\psi^i + \Gamma^i_{kl}\partial_\mu\phi^k\psi^l. \tag{6.2.3}$$

It is remarkable that such a connection term is needed and works as a part of the Noether procedure. This is actually not a surprise as the fermions carry the index of the target space, and it is very natural that their derivative should be covariantized with respect to the target space connection.

By introducing such a covariant derivative in the action, although this helps us get rid of the extra terms that come from the action of scalar fields, we end up with the spinorial terms of higher orders in the supersymmetry variation. Notice that under

(6.1.2), the terms $\delta_\epsilon[\bar{\psi}^i \not{D}\psi^j]$ contain connection coefficients coming from the covariant derivatives (6.2.3) according to $\delta_\epsilon \Gamma^i_{kl} = \partial_m \Gamma^i_{kl} \delta_\epsilon \phi^m$. We recognize $\partial_m \Gamma^i_{kl}$ as part of the Riemann tensor since,

$$R^k{}_{lij} = \partial_i(\Gamma^k_{jl}) - \partial_j(\Gamma^k_{il}) + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}. \tag{6.2.4}$$

This results in adding a quadratic 4−spinor term contracted with the Riemann curvature tensor $R_{ijkl} = R_{ijkl}(\phi)$ to the action. In summary we have the following additive terms in the action,

$$g_{ij} \xrightarrow{\delta_\epsilon} \Gamma^k_{ij} \xrightarrow{\delta_\epsilon} R_{ijkl}. \tag{6.2.5}$$

Moreover also the supersymmetry transformation of the spinor needs to be adjusted as,

$$\begin{aligned} \delta\phi^i &= \bar{\epsilon}\psi^i, \\ \delta\psi^i &= \not{\partial}\phi^i\epsilon - \Gamma^i_{kl}\delta\phi^k\psi^l. \end{aligned} \tag{6.2.6}$$

It turns out that no further modifications are needed in the action. The final form of the action is given in [5],

$$S := -\frac{1}{2}\int d^3x \Big\{ g_{ij}\partial_\mu\phi^i\partial^\mu\bar{\phi}^j + g_{ij}\bar{\psi}^i \not{D}\psi^j - \frac{1}{6}R_{ijkl}(\bar{\psi}^i\psi^k)(\bar{\psi}^j\psi^l) \Big\}, \tag{6.2.7}$$

where $\not{D} := \gamma^\kappa D_\kappa$. It is remarkable that the Noether procedure finally terminates with the action (6.2.7), where the covariant derivatives are as in (6.2.3). This action is invariant under the supersymmetry transformations (6.2.6).

Lastly for the $\mathcal{N} = 1$ case, let us also verify that the supersymmetry algebra closes on the scalar fields of the NLσM.

Proposition 6.2. *The scalar fields $\{\phi^i\}$ close the on-shell supersymmetry algebra.*

Proof. For the scalar fields ϕ^i,

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \cdot \phi^i &= \delta_{\epsilon_1}[\delta_{\epsilon_2} \cdot \phi^i] - \delta_{\epsilon_2}[\delta_{\epsilon_1} \cdot \phi^i] \\
&= \delta_{\epsilon_1}[\delta^i{}_j \epsilon_2 \bar{\psi}^j] - \delta_{\epsilon_2}[\delta^i{}_j \epsilon_1 \bar{\psi}^j] \\
&= \delta^i{}_j \epsilon_2 \delta_{\epsilon_1}(\bar{\psi}^j) - \delta^i{}_j \epsilon_1 \delta_{\epsilon_2}(\bar{\psi}^j) \\
&= \delta^i{}_j \epsilon_2 \Big((\delta^{-1})^j{}_m \partial\!\!\!/ \phi^m \bar{\epsilon}_1 - \Gamma^j_{kl}(\bar{\epsilon}_1 \psi^k)\bar{\psi}^l \Big) - \delta^i{}_j \epsilon_1 \Big((\delta^{-1})^j{}_m \partial\!\!\!/ \phi^m \bar{\epsilon}_2 + \Gamma^j_{kl}(\bar{\epsilon}_2 \psi^k)\bar{\psi}^l \Big) \\
&= \epsilon_2 \partial\!\!\!/ \phi^m \bar{\epsilon}_1 \delta^i_m - \epsilon_1 \partial\!\!\!/ \phi^m \bar{\epsilon}_2 \delta^i_m + \delta^i{}_j \Gamma^j_{kl} \Big(- \epsilon_2 (\bar{\epsilon}_1 \psi^k)\bar{\psi}^l + \epsilon_1 (\bar{\epsilon}_2 \psi^k)\bar{\psi}^l \Big) \\
&= \epsilon_2 \partial\!\!\!/ \phi^i \bar{\epsilon}_1 - \epsilon_1 \partial\!\!\!/ \phi^i \bar{\epsilon}_2 + \delta^i{}_j \Gamma^j_{kl} \underbrace{(-\bar{\epsilon}_2 \epsilon_1 \psi^k \bar{\psi}^l + \epsilon_1 \bar{\epsilon}_2 \psi^k \bar{\psi}^l)}_{vanishes} \\
&= -2\bar{\epsilon}_2 \partial\!\!\!/ \phi^i \epsilon_1 = -2\big(\bar{\epsilon}_2 \gamma^\mu \epsilon_1\big)\partial_\mu \phi^i.
\end{aligned}
$$

so the algebra also closes on the field ϕ^i. This proves that the proposed variations in (6.2.6) do provide valid supersymmetry transformations on ϕ^i. □

As highlighted in ref. [12], this Lagrangian (6.2.7) has a purely geometric meaning to it, which proves the motivation to study the Non-Linear $\sigma-$Models from a geometric point of view. Notice the target manifold of the NLσM with $\mathcal{N} = 1$ supersymmetry is simply a Riemannian manifold with Levi-Civita connection. However $\mathcal{N} = 2$ case brings stronger restrictions on the target manifold. This is what we unravel in the next chapter.

6.3. $D = 3$, and $\mathcal{N} = 2$ Supersymmetric $\sigma-$Model

We wish to add more supersymmetry to our NLσM and to the transformations of (6.2.6). The fundamental ideas must again be implemented. Having extended supersymmetry means that we have more than a single spinor parameter; to achieve this note that supersymmetry transformations (6.2.6) can be generalized as follows [5]:

$$
\begin{aligned}
\delta\phi^i &= \mathcal{J}^i_j\, \bar{\epsilon}\psi^j, \\
\delta\psi^i &= (\mathcal{J}^{-1})^i{}_j \partial\!\!\!/ \phi^j \epsilon - \Gamma^i_{kl}\delta\phi^k \psi^l.
\end{aligned}
\tag{6.3.1}
$$

Here $\mathcal{J}$ is a $(1,1)-$tensor field; when $\mathcal{J}^i_j = \delta^i_j$, we get back to (6.2.6) transformations. The square matrix $[\mathcal{J}\bar{\epsilon}]^i_j$ acts as our new supersymmetry parameter on the target manifold. Tensor $\mathcal{J}$ is globally defined as we assume the existence of rigid, i.e., global supersymmetry. The necessary and sufficient conditions for the supersymmetry invariance of the action (6.2.7) under the new supersymmetry variations are (as outlined in [5]),

$$\mathcal{J}^k_i g_{kl} \mathcal{J}^l_j = g_{ij} \quad \& \quad \nabla_k \mathcal{J}^i_j = 0. \tag{6.3.2}$$

6.3.1. Extra Conditions raised by Extended Supersymmetry

In (6.3.1) we have generalized our supersymmetry variations in accordance with the existence of exactly two supercharges. We now wish to further expand this discussion without specifying $\mathcal{N} > 1$. The extended SuperPoincaré relations are mentioned in (B.2.1), this too must be realized in the system. For this, we will generalize (6.3.1) to carry the supersymmetry index $\{A, B, ..\}$ to denote the number of supersymmetries $\mathcal{N}$; so that $\mathcal{J} \mapsto \mathcal{J}^{(A)}$ and $\epsilon \mapsto \epsilon^A$.

In extended supersymmetry, the supersymmetry algebra becomes,

$$\{Q^{(A)}, \bar{Q}^{(B)}\} = 2\gamma^\mu P_\mu \delta^{AB}, \qquad A, B = 1, 2, .., \mathcal{N} \tag{6.3.3}$$

hence each supersymmetry brings its own tensor ${\mathcal{J}^{(A)}}^i{}_j$, and these are covariantly constant due to (6.3.2).

Proposition 6.3. *The supercharge algebra implies that the following relation holds true,*

$$\mathcal{J}^{(A)} \mathcal{J}^{(B)\,-1} + \mathcal{J}^{(B)} \mathcal{J}^{(A)\,-1} = 2\delta^{AB}. \tag{6.3.4}$$

Proof. To show the (6.3.4) relation, we again check the closure of the algebra, yet this time keep the supersymmetry indices A and B. So we evaluate,

$$
\begin{aligned}
&[\delta^{(A)}_{\epsilon_1}, \delta^{(B)}_{\epsilon_2}] \cdot \phi^i \\
&= \delta^{(A)}_{\epsilon_1}[\delta^{(B)}_{\epsilon_2}\phi^i] - \delta^{(B)}_{\epsilon_2}[\delta^{(A)}_{\epsilon_1}\phi^i] \\
&= \delta^{(A)}_{\epsilon_1}[(\mathcal{J}^{(B)})^i{}_j\bar{\epsilon}_2\psi^j] - \ \big(1 \leftrightarrow 2 \text{ and } A \leftrightarrow B\big) \\
&= (\mathcal{J}^{(B)})^i{}_j\epsilon_2\delta^{(A)}_{\epsilon_1}[\bar{\psi}^j] - \ \big(1 \leftrightarrow 2 \text{ and } A \leftrightarrow B\big) \\
&= (\mathcal{J}^{(B)})^i{}_j\epsilon_2\big\{(\mathcal{J}^{(A)\ -1})^j{}_m\partial\!\!\!/\phi^m\bar{\epsilon}_1 - \Gamma^j_{kl}(\bar{\epsilon}_1\psi^k)\bar{\psi}^l\big\} - \ \big(1 \leftrightarrow 2 \text{ and } A \leftrightarrow B\big) \\
&= (\mathcal{J}^{(B)})^i{}_j(\mathcal{J}^{(A)\ -1})^j{}_m\epsilon_2\partial\!\!\!/\phi^m\bar{\epsilon}_1 - (\mathcal{J}^{(B)})^i{}_j\Gamma^j_{kl}(\bar{\epsilon}_1\psi^k)\bar{\psi}^l - \ \big(1 \leftrightarrow 2 \text{ and } A \leftrightarrow B\big) \\
&= (\mathcal{J}^{(B)})^i{}_j(\mathcal{J}^{(A)\ -1})^j{}_m\epsilon_2\partial\!\!\!/\phi^m\bar{\epsilon}_1 - (\mathcal{J}^{(A)})^i{}_j(\mathcal{J}^{(B)\ -1})^j{}_m\epsilon_1\partial\!\!\!/\phi^m\bar{\epsilon}_2 + \underbrace{(\mathcal{J})\Gamma(\bar{\epsilon}\psi)\psi + ..}_{\text{terms with } \psi}
\end{aligned}
$$

referencing [5] we will just be focusing on the terms without ψ of this calculation. The terms with ψ should vanish. The left hand side of this calculation is postulated by the SuperPoincaré relation of (6.3.3). For the infinitesimal transformations, (6.3.3) implies that we have $[\delta^{(A)}_{\epsilon_1}, \delta^{(B)}_{\epsilon_2}] \cdot \phi^i = 2\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\phi^i\delta^{AB}$ which has no spinor terms. The non-zero remainders on the right hand side gives,

$$
[\delta^{(A)}_{\epsilon_1}, \delta^{(B)}_{\epsilon_2}] \cdot \phi = \Big(\mathcal{J}^{(B)}\mathcal{J}^{(A)\ -1} + \mathcal{J}^{(A)}\mathcal{J}^{(B)\ -1}\Big)\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\phi \tag{6.3.5}
$$

which implies that the tensors proposed in the infinitesimal supersymmetry variations of (6.3.1) obey $\mathcal{J}^{(A)}\mathcal{J}^{(B)\ -1} + \mathcal{J}^{(B)}\mathcal{J}^{(A)\ -1} = 2\delta^{AB}$. This finishes the proof. □

Now it is important to observe geometric meanings of these equations. Equations (6.3.2) and (6.3.4) imply that the tensor $\mathcal{J}^i_j$ would be like an almost complex tensor if its square was $-\mathbb{1}$. As there are $\mathcal{N}-$many supersymmetries, there must exist $(\mathcal{N}-1)$ many such $\mathcal{J}$'s; that is one for each additional supersymmetry. In the next section we will analyze the $\mathcal{N}=2$ case.

6.3.2. Restrictions on the Target Manifold

By solving the variation of the Lagrangian under supersymmetry transforms, we found the relations of (6.3.2) regarding $\mathcal{J}'$'s. These will prove to provide a **Kähler manifold structure** on the target space.

For supersymmetry to be manifest, the supercharge commutation relations must be satisfied by the tensor fields $\mathcal{J}^{(A)}$, that is the highlight of equation (6.3.4). If there is a single supersymmetry, i.e. $A = B = 1$, then (6.3.4) tells nothing original. However say that we consider $\mathcal{N} = 2$ supersymmetries, then we have two nonidentical $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ which provide (6.3.1). The first supersymmetry is when ${\mathcal{J}^{(1)}}^a{}_b = \delta^a_b$ the Kronecker delta; yet the second tensor $\mathcal{J}^{(2)}$ is more interesting. We solve (6.3.4) for $\mathcal{J}^{(2)}$,

$$\begin{aligned} &\mathcal{J}^{(1)}\mathcal{J}^{(2)\,-1} + \mathcal{J}^{(2)}\mathcal{J}^{(1)\,-1} = 0 \;\Rightarrow \\ &\mathbb{1}\mathcal{J}^{(2)\,-1} + \mathcal{J}^{(2)}\mathbb{1} = 0. \end{aligned} \tag{6.3.6}$$

Proposition 6.4. *In a $D = 3$, $\mathcal{N} = 2$ supersymmetric Non-linear $\sigma-$Model, the target manifold admits exactly one complex structure.*

Proof. The relation (6.3.6) implies that $\mathcal{J}^{(2)} = -\mathcal{J}^{(2)\,-1}$, i.e., $\left(\mathcal{J}^{(2)}\right)^2 = -\mathbb{1}$. Such $\mathcal{J}$ is defined throughout the target space as we assume rigid supersymmetry. Hence it is a complex structure. $\square$

Proposition 6.5. *In a $D = 3$, $\mathcal{N} = 2$ supersymmetric Non-linear $\sigma-$Model, the target manifold is Kähler.*

Proof. We know the existence of a globally defined complex structure, and it is covariantly constant (6.3.2), and we have a complex Riemannian manifold in $(\mathcal{M}, g, \mathcal{J})$. By theorem 4.5, the target manifold of a $D = 3$, $\mathcal{N} = 2$ supersymmetric $\sigma-$model is a Kähler manifold. $\square$

6.4. $D = 3$, and $\mathcal{N} = 4$ Supersymmetric $\sigma-$Model

Now we would like to discuss the theory with more supersymmetries. The number of supersymmetries we can include in the theory is not arbitrary, this is the highlight of our next proposition.

Proposition 6.6. *In a $D = 3$ supersymmetric Non-linear $\sigma-$Model, $\mathcal{N} = 3$ supersymmetry implies the existence of a 4^{th} supersymmetry.*

Proof. We assume that we have two complex structures $\{\mathcal{J}^{(1)} = \mathbb{1}, \mathcal{J}^{(A)}, \mathcal{J}^{(B)}\}$ corresponding to $\mathcal{N} = 3$ supersymmetries. We write (6.3.4),

$$\begin{aligned}
&\mathcal{J}^{(A)}\mathcal{J}^{(B)\,-1} + \mathcal{J}^{(B)}\mathcal{J}^{(A)\,-1} = 0 \\
\Leftrightarrow\ &\mathcal{J}^{(A)}\mathcal{J}^{(B)\,-1} = -\mathcal{J}^{(B)}\mathcal{J}^{(A)\,-1} \\
\Leftrightarrow\ &\mathcal{J}^{(B)\,-1}\mathcal{J}^{(A)}\mathcal{J}^{(B)\,-1}\mathcal{J}^{(A)} = -\mathbb{1} \\
\Leftrightarrow\ &\mathring{\mathcal{J}}\mathring{\mathcal{J}} = -\mathbb{1},
\end{aligned}$$

where $\mathring{\mathcal{J}} := \mathcal{J}^{(B)\,-1}\mathcal{J}^{(A)}$. Let us also show that such $\mathring{\mathcal{J}}$ satisfies the supercharge relation in (6.3.4),

$$\begin{aligned}
\text{for } (\mathring{\mathcal{J}}, \mathcal{J}^{(A)}) \Rightarrow\ &\mathring{\mathcal{J}}\mathcal{J}^{(A)\,-1} + \mathcal{J}^{(A)}\mathring{\mathcal{J}}^{-1} \\
&= \mathcal{J}^{(B)\,-1}\mathcal{J}^{(A)}\mathcal{J}^{(A)\,-1} + \mathcal{J}^{(A)}\mathcal{J}^{(A)\,-1}\mathcal{J}^{(B)} = 0.\ \checkmark \\
\text{for } (\mathring{\mathcal{J}}, \mathcal{J}^{(B)}) \Rightarrow\ &\mathring{\mathcal{J}}\mathcal{J}^{(B)\,-1} + \mathcal{J}^{(B)}\mathring{\mathcal{J}}^{-1} \\
&= \mathcal{J}^{(B)\,-1}\mathcal{J}^{(A)}\mathcal{J}^{(B)\,-1} + \mathcal{J}^{(B)}\mathcal{J}^{(A)\,-1}\mathcal{J}^{(B)} \\
&= \mathcal{J}^{(B)}\mathcal{J}^{(A)}\mathcal{J}^{(B)} - \mathcal{J}^{(B)}\mathcal{J}^{(A)}\mathcal{J}^{(B)} = 0.\ \checkmark
\end{aligned}$$

Therefore supposing the existence of unique (and non-trivial) complex structures, we can construct a fourth complex tensor. Physically this implies that a fourth supersymmetry transformation also exists; hence $\mathcal{N} = 3$ implies that $\mathcal{N} = 4$. □

Proposition 6.7. *In a $D = 3$, and $\mathcal{N} = 4$ NLσM, the three complex structures satisfy the quaternion algebra.*

Proof. In a $D = 3$, $\mathcal{N} = 4$ supersymmetric Non-linear $\sigma-$Model, we have exactly three complex structures $\{\mathcal{J}_{(i)}\}_{i=1}^{3} := \{\mathcal{J}^{(A)}, \mathcal{J}^{(B)}, \mathring{\mathcal{J}}\}$. They are covariantly constant. Recall that for any almost complex tensor $\mathcal{J}^{-1} = -\mathcal{J}$; hence $\mathring{\mathcal{J}} = \mathcal{J}^{(A)\,-1}\mathcal{J}^{(B)} = -\mathcal{J}^{(A)}\mathcal{J}^{(B)}$, which means that $\mathring{\mathcal{J}}$ is simply the multiplication of two previous complex structures. In quaternion language this corresponds to the relation $\mathbf{i}\,\mathbf{j} = \mathbf{k}$. Hence the hypercomplex algebra consisting of elements $\{\pm\mathbb{1}, \pm\mathcal{J}^{(A)}, \pm\mathcal{J}^{(B)}, \pm\mathring{\mathcal{J}}\}$ is now closed. □

With the above proposition we now have three complex structures defined on $\mathcal{M}$ that satisfy the quaternionic algebra which are moreover covariantly constant. This means that the target manifold $\mathcal{M}$ is a **Hyperkähler manifold** (see [30]). Hyperkähler manifolds are a generalization of Kähler manifolds. All Hyperkähler manifolds are Kähler manifolds themselves and are necessarily of real dimension $4m$ for $m \in \mathbb{N}$ (for more information on HyperKähler manifolds, see [31]).

Remark. *In the light of Supersymmetric $\sigma-$Model analysis in $D = 3$, we can infer the results of 1.1. The way supersymmetry works in $D = 3$ and $D = 4$ dimensions are related; namely we can show by dimensional reduction that $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry in $D = 4$ is equivalent to $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry in $D = 3$ respectively. Hence $\mathcal{N} = 1$ and $\mathcal{N} = 2$ NLσM's in $D = 4$ require the target spaces to be Kähler and HyperKähler manifolds respectively (see [7]).*

Remark. *In supergravity theories, the supersymmetry is local, hence the complex structures are defined locally. When supergravity is coupled, the target manifold is no longer HyperKähler; the target spaces fall in a larger class of manifolds called **quaternionic manifolds**. For quaternionic manifolds the complex structures still provide a quaternion algebra, however this is only locally defined. HyperKähler and quaternionic Kähler manifolds are special classes of Einstein manifolds, whose Ricci curvature is proportional to the metric tensor. More information can be found at [32].*

In the case of $D = 3$ Non-Linear $\sigma-$Model, we showed the existence of at least $(\mathcal{N}-1)$ complex structures living on the target manifold. These follow from global/rigid supersymmetry transformation rules. Proper supersymmetries also satisfy the supercharge algebra of (6.3.4) which gives out a Clifford algebra. Through propositions made in this section we investigated the algebraic structure of these tensor fields.

The recent research including Supersymmetric $\sigma-$Models include the study of target spaces with $U(1)$ isometries such as $\mathbb{C}P^1$ and the Poincaré plane. Moreover these $\sigma-$models are coupled to gravity to form supergravity theories. Examples of these can be found at the papers of [10, 11, 21–23].

7. FUTURE RESEARCH AND REFLECTIONS

In this thesis we studied various properties of NLσM's and looked at supersymmetric versions in 3$-$dimensions. The models we explored were global supersymmetric models, however we could have studied local supersymmetry and taken a sigma model coupled to gravitation. In fact, the final Lagrangian (6.2.7) is the base for global and local supersymmetric models in various spacetime dimensions; hence it has also been studied for supergravity theories. Such examples can be found in the literature [10, 11, 21, 23]. The general classes of target manifolds in such models are classified, yet not specifically all models corresponding to each of these target manifolds have been written. The explicit models in 3-dimensions are still ongoing problems under current research.

There is also a possibility to describe $\sigma-$models under generalized geometry formulation. Generalized geometry captures tangent and cotangent spaces under the same construction, the $\sigma-$models in this methodology could or could not inherit supersymmetry. For more resources in this topic, we refer to [6].

Lastly we could have dropped the assumptions on the target manifold; in Non-Linear $\sigma-$Models we assume that the target space admits Levi-Civita connection and a Euclidean signature metric. However for example this is not the case in string theories. Therefore we could have started with a target space that does not admit Levi-Civita connection and has non-vanishing torsion. Such generalizations might also be considered as an improvement in the theory.

Bibliography

1. Fecko M., *Differential Geometry and Lie Groups for Physicists*, Cambridge University Press, 2006.

2. Lindström U., *"Uses of Sigma Models"*, Corfu Summer Institute 2017 "School and Workshops on Elementary Particle Physics and Gravity", arXiv:1803.08873v1, 2017.

3. Alvarez-Gaume L., Freedman D. Z., *"Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model"*, Communications in Mathematical Physics, Vol.80, 1981.

4. Townsend P. K., *"Finite Field Theory"*, Lectures given at the 18^{th} Winter School of Theoretical Physics, Poland, 1981.

5. Bagger J., *"Supersymmetric Sigma Models"*, Lectures given at the Bonn-NATO Advanced Study Institute on Supersymmetry, 1984.

6. Lindström U., *"Supersymmetric Nonlinear Sigma Model Geometry"*, 2012, Retrieved from arXiv:1207.1241v1, Accessed on 15.09.2021.

7. Freedman D. Z. and Proeyen A. V., *Supergravity*, Cambridge University Press, ISBN 978-0-521-19401-3, 2012.

8. Zwiebach B., *A First Course in String Theory, Cambridge University Press*, ISBN-13 978-0-521-88032-9, 2009.

9. Hitchin N. J., Karlhede A., Lindström U., and Roček M., *"Hyperkähler Metrics and Supersymmetry"*, Communications in Mathematical Physics, Vol. 108, pp. 535–589, 1987.

10. Abou-Zeid M. and Samtleben H., *"Chern-Simons Vortices in Supergravity"*, Physical Review D., Vol. 65, 085016, 2002.

11. Izquierdo J. M., Townsend P.K., *"Supersymmetric spacetimes in 2+1 adS-supergravity models"* , Classical and Quantum Gravity, Vol. 12, pp. 895-924, 1995.

12. Zumino B., *"Supersymmetry and Kähler Manifolds"*, Physics Letters B, Vol. 87, 1979.

13. Nakahara M., *Geometry & Topology and Physics*, Institute of Physics Publishing, Bristol and Philadelphia, ISBN 0 7503 0606.

14. Vandoren S., *"Lectures on Riemannian Geometry Part II: Complex Manifolds"*, Utrecht, 2008.

15. Moroianu A., *"Lectures on Kähler Geometry"*, Cambridge University Press, January 2010.

16. Lee J. M., *Introduction to Smooth Manifolds*, 2^{nd} edition, Springer, ISBN 978-1-4419-9981-8, 2013.

17. Garrett P., *"Classical homogeneous spaces"*, 2010, Retrieved from `http://www-users.math.umn.edu/~garrett/m/mfms/notes/08_homogeneous.pdf`, Accessed on 15.09.2021.

18. Helgason S., *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, 1978.

19. Kobayashi S. and Nomizu K., *Foundations of Differential Geometry*, Volumes I & II, Interscience Publishers, 1969.

20. de Wit B., *"Supergravity"*, 2001 Les Houches Summer school *"Unity from Duality: Gravity, Gauge Theory and Strings"*, arXiv:hep-th/0212245v1, 2002.

21. Deger N. S., Kaya A., Sezgin E., Sundell P., *"Matter Coupled AdS_3 Supergravities and Their Black Strings"*, Nuclear Physics B., Vol. 573, pp. 275-290, arXiv:hep-th/9908089v2, 2000.

22. Deger N. S., Kaya A., Sezgin E., Sundell P., Tanii Y., *"(2,0) Chern-Simons Supergravity Plus Matter Near the Boundary of AdS_3"*, Nuclear Physics B., Vol. 604, pp. 343-366, arXiv:hep-th/0012139v2, 2001.

23. de Wit B., Herger I., Samtleben H., *"Gauged locally supersymmetric D=3 nonlinear sigma models"*, Nuclear Physics B., Vol. 671, pp. 175-216, 2003.

24. de Wit B., Tollsten A. K., Nicolai H., *"Locally supersymmetric D=3 non-linear sigma models"*, arXiv:hep-th/9208074v1, Nuclear Physics B., Vol. 392, pp. 3-38, 1993.

25. Gates Jr S. J., Grisaru M.T., Roček M., Siegel W., *"Superspace, or One thousand and one lessons in supersymmetry"*, Frontiers in Physics, Vol. 58, pp. 1-548, 1983.

26. Müller-Kirsten H. J. W. & Wiedemann A., *Introduction to Supersymmetry* (Second Edition), World Scientific Lecture Notes in Physics, Vol.80, 2010.

27. Nastase H., *"Introduction to supergravity"*, Brazil, December 2011, Retrieved from arXiv:1112.3502v3, Accessed on 15.09.2021.

28. Wess J. and Bagger J., *Supersymmetry and Supergravity*, Princeton University Press, Second edition, 1992.

29. Wess J., Zumino B., *"Supergauge transformations in four dimensions"*, Nuclear Physics B., Vol. 70, pp. 39-50, 1974.

30. Hitchin N., *"Hyperkähler manifolds"*, Séminaire Bourbaki, Astérisque tome 206, pp. 137-166 1992.

31. Galicki K., *"Quaternionic Kähler and HyperKähler Manifolds"*, Retrieved

from `http://www.galicki.com/math/courses/pdf/Notes.pdf`, Accessed on 15.09.2021.

32. Swann A. F., 1990, *"HyperKähler and Quaternionic Kähler Geometry"*, Degree of D. Phil., Oriel College, Oxford.

33. Candelas P., *"Lectures on Complex Manifolds"*, Department of Physics at the University of Texas, 1988, Retrieved from `http://www.math.toronto.edu/mgualt/courses/MAT477-2017/docs/Candelas-delaOssa.pdf`, Accessed on 15.09.2021.

34. van Holten J.W., *"Kähler manifolds and supersymmetry"*, arXiv:hep-th/0309094v1, Acta Physica Polonica B., Vol. 34, pp. 5983-6004, 2003.

35. Tanii Y., *Introduction to Supergravity*, SpringerBriefs in Mathematical Physics, Vol. 1, ISBN 978-4-431-54827-0, 2014.

Appendices

A. MANIFOLD GROUNDWORK

We cover the necessary background for the smooth and complex manifolds that arise as target manifolds in supersymmetric Non-Linear $\sigma-$models. The references are [13, 14, 16, 33].

Definition A.1. *A **real Smooth/Differentiable Manifold** is a space $\mathcal{M}$ that satisfies the following axioms,*

(i) *$\mathcal{M}$ is a topological space.*
(ii) *$\mathcal{M}$ is given with an atlas $\{(U_i, \varphi_i)\}$.*
(iii) *For$\{U_i\}$ is a family of open sets that provide a cover for the space $\mathcal{M}$, φ_i are homeomorphisms; they are one-to-one and continuous maps with a continuous inverse (if both differentiable called **diffeomorphisms**) and provide a chart of $U_i \to U \subset \mathbb{R}^m$.*
(iv) *The differentiable structure: The transition map between U_i and U_j where $U_i \cap U_j \neq \emptyset$ is naturally given by $f := \varphi_j(\varphi_i^{-1})$. Such a function f must be differentiable.*

Definition A.2. *A **Riemannian Manifold** is a real smooth manifold (it admits a differentiable structure for intersections of open domains) with a positive definite metric tensor $g_{\mu\nu}$ defined on $T_p\mathcal{M}$. Riemannian Manifolds are denoted by the pair $(\mathcal{M}, g)$.*

Definition A.3. *An **Almost complex structure** on a real manifold is a globally defined $(1,1)-$tensor field $\mathcal{J} : T_p\mathcal{M} \to T_p\mathcal{M}$ s.t. $\mathcal{J}^2 = -\mathbb{1}$. In local real coordinates $\{\frac{\partial}{\partial x^\mu}, dx^\mu\}$ we can write $\mathcal{J}_p = \mathcal{J}_\mu^\nu(p)\frac{\partial}{\partial x^\nu} \otimes dx^\mu$. For a fixed point p on $\mathcal{M}$, this translates in local real coordinates to*

$$\mathcal{J}_\mu^\rho(p)\mathcal{J}_\rho^\nu(p) = -\delta_\mu^\nu.$$

Definition A.4. *An **almost complex manifold** is a real manifold with an almost complex tensor field defined on it. An almost complex manifold is given by the pair $(\mathcal{M}, \mathcal{J})$, existence of a metric tensor g is implied.*

Corollary. *Almost complex manifolds have even dimension.*

Proof. Take $\mathcal{M}$ as the base manifold with dimension m; we have a globally defined $\mathcal{J}$ such that $\mathcal{J}^2 = -\mathbb{1}_{m\times m}$. Taking the determinant with multiplicative property gives $(det\mathcal{J})^2 = (-1)^m$. An almost complex manifold is a real manifold, thus we can pick a real basis to have real entries in $\mathcal{J}_\mu^\nu$. In this basis $(det\mathcal{J})^2$ must be positive, thus $(-1)^m$ must be $+1$ rather than -1. Concluding that $m = 2n$ an even integer. □

Definition A.5. *The almost complex structure acting on the space $T_p\mathcal{M}$ can be defined by transformations of basis vectors in the following way,*

$$\mathcal{J}\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial}{\partial y^\mu}, \quad \mathcal{J}\left(\frac{\partial}{\partial y^\mu}\right) := -\frac{\partial}{\partial x^\mu}$$

In a given basis, $\mathcal{J}_p^2 = -\mathbb{1}$ is satisfied due to action on basis vectors. Indeed $\mathcal{J}_p$ applied to $\partial/\partial z^\mu$ and $\partial/\partial \bar{z}^\mu$ is just multiplication by complex i and we obtain a representation of the almost complex operator in real basis;

$$\mathcal{J}_p = \begin{bmatrix} 0 & -\mathbb{I}_m \\ \mathbb{I}_m & 0 \end{bmatrix}_{2m\times 2m} \Rightarrow \mathcal{J}_p^2 = \begin{bmatrix} -\mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_m \end{bmatrix} = -\mathbb{1}_{2m\times 2m} \tag{A.0.1}$$

Thus $\mathcal{J}^4 \equiv id|_{T_p\mathcal{M}}$ acting on real basis $\{\partial/\partial x^\mu; \partial/\partial y^\nu\}$. This means when complex dimension is $\dim_{\mathbb{C}}\mathcal{M} = m$, the real dimension is $\dim_{\mathbb{R}}\mathcal{M} = 2m$. The natural principle with dimension is when working on the tangent plane of the manifold, one needs dimension many basis vectors - for a generating set. A complex manifold is a good example for having concrete basis vectors; for locally it looks like $\mathbb{R}^{2m}$. This means at any point $p \in \mathcal{M}$, tangent plane $T_p\mathcal{M}$ is spanned by $2m$-many vectors;

$$\left\{\frac{\partial}{\partial x^{\mu_1}}, \ldots, \frac{\partial}{\partial x^{\mu_m}}, \frac{\partial}{\partial y^{\mu_1}}, \ldots, \frac{\partial}{\partial y^{\mu_m}}\right\}$$

While the above is for the vectors on $T_p\mathcal{M}$; for covectors living in T_p^*M we consider the basis

$$\{dx^{\mu_1}, ..., dx^{\mu_m}, dy^{\mu_1}, ..., dy^{\mu_m}\}$$

Definition A.6. *The following sets give a basis for the vector space $T_p\mathcal{M}^{\mathbb{C}}$ where $1 \leq \mu \leq m$*

$$\frac{\partial}{\partial z^\mu} := \frac{1}{2}\left\{\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right\} \quad \& \quad \frac{\partial}{\partial \bar{z}^{\bar{\mu}}} := \frac{1}{2}\left\{\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\right\}$$

We call the set $\{\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\mu}\}$ as the ***local complex coordinate basis*** *for $\mathcal{M}$. Similarly for one-forms we can define a basis of $T_p^*\mathcal{M}^{\mathbb{C}}$;*

$$dz^\mu := dx^\mu + i\, dy^\mu \quad \& \quad d\bar{z}^{\bar{\mu}} := dx^\mu - i\, dy^\mu$$

Proposition A.7. *The contractions of the above defined basis vectors are summed up by the two equations:*

$$\langle dz^\mu, \partial/\partial \bar{z}^{\bar{\nu}}\rangle = \langle d\bar{z}^{\bar{\mu}}, \partial/\partial z^\nu\rangle = 0 \tag{A.0.2}$$

$$\langle dz^\mu, \partial/\partial z^\nu\rangle = \langle d\bar{z}^{\bar{\mu}}, \partial/\partial \bar{z}^{\bar{\nu}}\rangle = \delta^\mu_\nu \tag{A.0.3}$$

Proof. Let us show the first one,

$$\begin{aligned}\langle dz^\mu, \partial/\partial \bar{z}^\mu\rangle = \langle dx^\mu + idy^\mu; \frac{1}{2}(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu})\rangle \quad \Rightarrow \\ = \frac{1}{2}\langle dx^\mu, \partial/\partial x^\mu\rangle + \frac{i}{2}\langle dx^\mu, \partial/\partial y^\mu\rangle + \frac{i}{2}\langle dy^\mu, \partial/\partial x^\mu\rangle - \frac{i^2}{2}/2\langle dy^\mu, \partial/\partial y^\mu\rangle\end{aligned}$$

where we used bilinearity, and this vanishes. The second equality is similar. □

Remark. *The almost complex operator satisfying the above properties can be written* *in the **complex basis** as a* $(1,1)-$*tensor of the form*

$$\mathcal{J} = \begin{bmatrix} i\mathbb{I}_m & 0 \\ 0 & -i\mathbb{I}_m \end{bmatrix} \quad i.e., \qquad \mathcal{J}_p = i\frac{\partial}{\partial z^a} \otimes dz^a - i\frac{\partial}{\partial \bar{z}^{\bar{a}}} \otimes d\bar{z}^{\bar{a}} \tag{A.0.4}$$

Remark. *To distinguish tensors written in real basis and complex basis coordinates, we will use Latin indices* $\{a, b, c, ..\}$ *for the complex basis, and Greek indices* $\{\mu, \nu, \rho, ..\}$ *for the real basis. So in complex basis from* (A.0.4)*,*

$$\mathcal{J}_b^a = i\delta_b^a. \tag{A.0.5}$$

A.1. Complex Manifolds

Definition A.8. *A **holomorphic function** is a complex function that satisfies the Cauchy-Riemann equations and therefore is differentiable. This allows us to work with analytic functions on the complex plane, that is, for* $f(z^\mu) = u(x^\mu, y^\mu) + iv(x^\mu, y^\mu)$

$$\text{Holomorphic: } \frac{\partial u}{\partial x^\mu} = \frac{\partial v}{\partial y^\mu}, \quad \& \quad \frac{\partial u}{\partial y^\mu} = -\frac{\partial v}{\partial x^\mu}.$$

Definition A.9. *A **complex manifold** is a real smooth manifold satisfying the axioms in definition A.1 with an additional stronger version of condition (4);*
$(4)_2$*: The transition map for non-vanishing intersections of neighborhoods,* $f := \varphi_j(\varphi_i^{-1})$ *is a holomorphic function.*

For two different atlases constructed on the same manifold, the Cauchy-Riemann requirement for composition of different atlases is not broken; thus the holomorphic property is independent of the choice of chart on the manifold. This means our definition is well-defined.

Definition A.10. *We define the Nijenhuis tensor field to be the mapping* $N : \mathfrak{X}(\mathcal{M}) \times$

$\mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$

$$N(\vec{U},\vec{V}) := [\vec{U},\vec{V}] + \mathcal{J}[\mathcal{J}\vec{U},\vec{V}] + \mathcal{J}[\vec{U},\mathcal{J}\vec{V}] - [\mathcal{J}\vec{U},\mathcal{J}\vec{V}].$$

Theorem A.11. *(see [13]) Nijenhuis tensor gives a very precise condition for testing integrability on a complex manifold. It can be shown that the followings are equivalent:*

(i) Integrable almost complex structure.

(ii) Vanishing Nijenhuis tensor field.

(iii) $(\mathcal{M},\mathcal{J})$ *is a Complex manifold.*

Nijenhuis tensor is an important tool for understanding when an almost complex manifold can be made a complex manifold.

Theorem A.12. *Complex manifolds are almost complex.*

Proof. Say we have a complex manifold $\mathcal{M}$; it has a holomorphic atlas and holomorphic transition functions. Define the almost complex structure on $\mathcal{M}$ via the coordinate patch (U,z) by $\mathcal{J} = i\partial/\partial z^\mu \otimes dz^\mu - i\partial/\partial\bar{z}^{\bar{\mu}} \otimes d\bar{z}^{\bar{\mu}}$. We need to analyze $\mathcal{J}$ on the overlap of two patches (U,z) and (V,w). We know that the following transition is analytic

$$\frac{\partial}{\partial w^\mu} = \frac{\partial z^\nu}{\partial w^\mu}\frac{\partial}{\partial z^\nu}, \qquad \mathcal{J}_{(V,w)} = i\frac{\partial}{\partial w^\mu} \otimes dw^\mu - i\frac{\partial}{\partial \bar{w}^{\bar{\mu}}} \otimes d\bar{w}^{\bar{\mu}} \tag{A.1.1}$$

$$\text{Therefore} \quad \frac{\partial}{\partial z^\mu} \otimes dz^\mu = \frac{\partial z^\kappa}{\partial w^\mu}\frac{\partial w^\nu}{\partial z^\kappa}\frac{\partial}{\partial w^\nu} \otimes dw^\mu = \frac{\partial}{\partial w^\mu} \otimes dw^\mu \tag{A.1.2}$$

Thus switching between coordinate patches would give $\mathcal{J}_{(U,z)} \mapsto \mathcal{J}_{(V,w)}$. We obtain a global well-defined almost complex structure, so $\mathcal{M}$ is almost complex. □

Theorem A.13. *Any orientable two-dimensional Riemannian manifold is a complex manifold.*

Proof. Riemannian manifold $(\mathcal{M}, g)$ comes with a positive-definite metric $g_{\mu\nu}$. Around the neighborhood of a point, we can pick coordinates x, y so that the metric tensor takes on the form $ds^2 = \lambda^2(x,y)(dx^2+dy^2)$. Using the complex basis, where $dz = dx+idy$, we get $dzd\bar{z} = dx^2-i^2dy^2 = dx^2+dy^2$. Therefore metric can be written $ds^2 = \lambda^2(z,\bar{z})dzd\bar{z}$. Now take another coordinate pair u, v and define the complex coordinates (similarly) $w = u+iv$. The metric tensor becomes $ds^2 = \mu^2(w,\bar{w})dwd\bar{w}$. Because the manifold is orientable, we need to have the Jacobian $\frac{\partial(u,v)}{\partial(x,y)} > 0$. Moreover, the metric must be the same on the points living in the overlap; thus $\lambda^2 dzd\bar{z} = \mu^2 dwd\bar{w}$.
We want to show that the transition function on the overlapping neighborhood is holomorphic. The change of coordinates is given by

$$dw = \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \bar{z}}d\bar{z} \tag{A.1.3}$$

Plugging dw into metric equality gives

$$\frac{\partial w}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial \bar{z}} = 0 \tag{A.1.4}$$

This means either $w = w(z)$ or $w = w(\bar{z})$ (w is holomorphic or anti-holomorphic). Supposing the latter, it must satisfy the anti-Cauchy Riemann equations, resulting in

$$det\mathbb{J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} < 0 \qquad \text{i.e.,} \quad \frac{\partial(u,v)}{\partial(x,y)} = -\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 < 0 \tag{A.1.5}$$

This contradicts the orientability assumption of Jacobian property. Thus $w = w(z)$, and the transition functions are holomorphic. □

Definition A.14. *A non-degenerate closed two-form is called a* ***symplectic form****; its an antisymmetric tensor by construction. A manifold that admits a symplectic 2-form is called a* ***symplectic manifold****. In local covector basis* $\{dx^\mu\}$ *as* $1 \le \mu \le dim\mathcal{M}$ *it is defined as* $\Omega := \Omega_{\mu\nu}dx^\mu \wedge dx^\nu$ *with* $d\Omega = 0$.

A bilinear form is non-degenerate if for some $\vec{X} \in T_pM$, $\Omega(\vec{X}, \vec{Y}) = 0$ for all $\vec{Y}$

implies that $\vec{X} = 0$. This means that the form is 'honest' in the sense that it is not identically vanishing. Moreover non-degenerate forms are invertible with the use of metric; and taking the wedge product $m-$many times, we obtain a $2m-$form

$$\underbrace{\Omega \wedge \Omega \wedge \ldots \wedge \Omega}_{m}$$

that is nowhere vanishing. Thus a good candidate for a volume form.

A.1.1. Hermitian Manifolds

Definition A.15. *Given a complex manifold $\mathcal{M}$ with a Riemannian metric g; if*

$$g_p(\mathcal{J}_p\vec{X}, \mathcal{J}_p\vec{Y}) = g_p(\vec{X}, \vec{Y}) \quad \forall p \in M$$

for any $X, Y \in T_p\mathcal{M}$, then g is called a ***Hermitian metric****, and $(\mathcal{M}, g)$ is a Hermitian manifold. In the complex basis, there is a choice of coordinate domain in which $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$ and the metric obtains the form $ds^2 = 2g_{\mu\bar{\nu}}dz^{\mu}d\bar{z}^{\bar{\nu}}$.*

Hermicity axiom is a condition for the metric defined on the manifold; it is not a condition on the manifold itself. And not every metric can be made Hermitian. In a local coordinate patch, a Hermitian metric must satisfy,

$$g_{\mu\nu} = \mathcal{J}_{\mu}{}^{\rho} g_{\rho\sigma} \mathcal{J}_{\nu}{}^{\sigma}, \text{i.e.,} \quad \mathcal{J}\mathbf{g}\mathcal{J}^{T} = \mathbf{g} \ \ \text{(in Matrix Notation).} \tag{A.1.6}$$

Theorem A.16. *A complex Riemannian manifold always admits a Hermitian metric.*

Proof. Take the Riemannian metric g already on the manifold; and define

$$\tilde{g}(\vec{X}, \vec{Y}) := \frac{1}{2}[g(\vec{X}, \vec{Y}) + g(\mathcal{J}_p\vec{X}, \mathcal{J}_p\vec{Y})] \tag{A.1.7}$$

$\tilde{g}$ is positive definite since g is, and $\tilde{g}(\mathcal{J}\vec{X}, \mathcal{J}\vec{Y}) = \tilde{g}(\vec{X}, \vec{Y})$ meeting the Hermitian axiom as,

$$\tilde{g}(\mathcal{J}\vec{X},\mathcal{J}\vec{Y}) = \frac{1}{2}\Big(g(\mathcal{J}\vec{X},\mathcal{J}\vec{Y}) + g(\underbrace{\mathcal{J}^2}_{-\mathbb{1}}\vec{X},\underbrace{\mathcal{J}^2}_{-\mathbb{1}}\vec{Y})\Big) = \frac{1}{2}\Big(g(\mathcal{J}\vec{X},\mathcal{J}\vec{Y}) + g(\vec{X},\vec{Y})\Big)$$

$= \tilde{g}(\vec{X},\vec{Y})$. This completes the proof. □

B. A REVIEW OF SUPERSYMMETRY

B.1. SuperPoincaré Algebra

Supersymmetry is one of the important symmetries that was discovered in recent years in physics which relates fermions and bosons. It is very much like the previous symmetries that we have mentioned, yet it is very unique in the sense that it is a combination of external/spacetime symmetries and internal symmetries; it has been a fundamental tool in many research in theoretical physics, and it plays a very important role in string theory. We will give a brief review of supersymmetry in this section. The references for this section are [7, 26–28].

Supersymmetry is constructed by forming a **Superalgebra**. This is done by relaxing one condition of Lie algebras; particularly the "defining relations" of the algebra includes anti-commutators (those are $\{,\}$) in addition to commutator brackets ([,]). The Poincaré algebra is defined by the following equations,

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}),\\
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu),\\
[P_\mu, P_\nu] &= 0.
\end{aligned} \tag{B.1.1}$$

where the $\{M_{\mu\nu}\}$ are Lorentz group generators and $\{P_\mu\}$ are the spatial translation operators.

Supersymmetry generalizes the Poincaré algebra to a superalgebra by adding Majorana spinor charges, denoted $\{Q_\alpha\}$, with spinor indices are running from $\alpha = 1, 2, .., x$. These charges satisfy the following axioms:

$$\begin{aligned}
&\{Q, \bar{Q}\} \sim P \\
&[Q, P] = 0 \\
&[Q, M] \sim Q \\
&\{Q, Q\} \sim P
\end{aligned} \tag{B.1.2}$$

The Lorentz group generators $M^{\mu\nu}$ and momentum operators P_μ are already contained in the Poincaré algebra; SuperPoincaré algebra extends this with the (anti)commutation postulates of (B.1.2). The new operators are the spinor supercharges $\{Q_\alpha\}$, and their action on the fields provide an interchange map between bosonic and spinor sectors. Supersymmetry postulates that,

$$Q_\alpha \cdot \phi = \psi_\alpha \qquad Q^\alpha \cdot \psi_\beta = \partial\!\!\!/ \phi = (\gamma^\mu)^\alpha{}_\beta \partial_\mu \phi. \tag{B.1.3}$$

Definition B.1. *The action of the supercharges (an overall look of* (B.1.3)*) gives us the* ***infinitesimal supersymmetry variations****. As in the case of Lie groups, we call* $\delta_\epsilon := \epsilon^\alpha Q_\alpha$ *for a spinor parameter* ϵ*, the action of* $\epsilon^\alpha Q_\alpha \cdot \varphi = \delta_\epsilon[\varphi]$ *is the supersymmetry variation of an arbitrary field* φ*.*

The fundamental principles of a supersymmetric theory are the following two very important premises;

(i) The theory must be expressed by a supersymmetry invariant action. For this, $\delta_\epsilon \cdot \mathcal{L}$ need not be zero, yet $\delta_\epsilon \cdot S = \int d^D x \delta_\epsilon[\mathcal{L}]$ has to vanish. This condition is satisfied also when the integrand is a total derivative.

(ii) The supersymmetry algebra has to be realized for all field configurations in the system, i.e., the commutator brackets $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ must be as postulated by the algebra.

Definition B.2. *The bosonic and fermionic fields that define a supersymmetric model are called a* ***supermultiplet****. This means that the supersymmetry algebra* ***closes*** *on* *these fields. If this closure requires the use of field equations, then supersymmetry is said to be* ***on-shell****. Otherwise it is called* ***off-shell****.*

B.2. Extended Supersymmetry

In basic $\mathcal{N} = 1$ supersymmetry, we proposed the inclusion of supersymmetry generators $\{Q_\alpha\}$ in the superalgebra; this structure is extended when we assume that $\mathcal{N} > 1$, i.e., there is more than a single supersymmetry. This brings in another index for the supercharges, this is, they carry the capital Latin index $\{A, B, ..\}$ running from $\{1, 2, ..., \mathcal{N}\}$. This is to denote the number of supersymmetries in the system. Hence the new supersymmetry generators are $\{Q^A_\alpha\}$. They satisfy the following anticommutation relation,

$$\{Q^A_\alpha, \bar{Q}^B_\beta\} = 2(\gamma^\mu)_{\alpha\beta} P_\mu \delta^{AB}. \tag{B.2.1}$$

B.3. Spinors in $D = 2, 3, 4$ Spacetime

This section serves to mention some of the technical aspects of spinor variables and related objects that pop out in calculations of supersymmetric field theories.
We are working in a field theory setting; this is to axiomatize the massless Klein-Gordon equation $\Box\phi = 0$ and the massless Dirac equation $\partial\!\!\!/\psi = 0$ where $\partial\!\!\!/ := \gamma^\mu \partial_\mu$. The square $\gamma-$matrices that appear in this equation satisfy what is called a **Clifford algebra**,

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbb{1} \qquad 0 \leq \mu, \nu \leq D - 1. \tag{B.3.1}$$

This leads to the following relationship between the d'Alembertian operator $\Box$ and $\partial\!\!\!/$,

$$\Box = \partial^\mu \partial_\mu = \eta_{\mu\nu}\partial^\mu\partial^\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)\partial^\mu\partial^\nu \tag{B.3.2}$$

$$= \frac{1}{2}(\partial\!\!\!/\partial\!\!\!/ + \partial\!\!\!/\partial\!\!\!/) = \partial\!\!\!/^2. \tag{B.3.3}$$

this is the reason why Dirac equation is in a sense the square root of the Klein-Gordon equation. The $\gamma-$matrices act on spinor variables ψ_α, hence they must admit an applicable size. A Dirac spinor in $D = 2m$ or $D = 2m + 1$ spacetime dimensions are $2^m \times 1$ *column matrices.* From this we refer that Dirac $\gamma-$matrices are of $2^m \times 2^m$.

Definition B.3. *We define the* ***higher rank*** $\gamma-$***matrices*** *by complete antisymmetrization as in the following,*

$$\gamma^{\mu_1 \dots \mu_r} := \gamma^{[\mu_1} \dots \gamma^{\mu_r]} \tag{B.3.4}$$

the antisymmetrization brackets contain a normalization factor, hence a $1/r!$ *for a* $\gamma-$*matrix of rank* r.

Definition B.4. *There exists a unitary matrix* C*, also called the* ***Charge conjugation matrix*** *which allows us to label higher rank* $\gamma-$*matrices as symmetric or antisymmetric when multiplied by* C*. That is to say,*

$$(C\gamma^{\mu_1 \dots \mu_r})^T = -t_r C\gamma^{\mu_1 \dots \mu_r}, \qquad t_r = \pm 1. \tag{B.3.5}$$

Such C *exists in all spacetime dimensions. Moreover it is not hard to show that the coefficients* t_r *are equal modulo* 4*. i.e.,* $t_r = t_{r+4}$.

The values of the constants $\{t_i\}_{i=0}^{3}$ in supersymmetry for spacetimes in $D = 2, 3$ and 4 are given in B.1.

Dimension	t_0	t_1	t_2	t_3
$D = 2$	+1	-1	-1	+1
$D = 3$	+1	-1	-1	+1
$D = 4$	+1	-1	-1	+1

Table B.1: Constants for $D \leq 4$ spacetime dimensions.

For the details of the table, see [7]. The most important relation that follows from this is the contraction of two spinors (which gives out a spinor bilinear) via an $r-$rank

γ-matrix. In supersymmetric calculations for two spinors ψ and χ, we will make use of,

$$\bar{\psi}\gamma_{\mu_1\ldots\mu_r}\chi = t_r\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi \tag{B.3.6}$$

these are called **Majorana flip relations**. In dimensions $D = 2$, and 3, for two spinors ϵ_1 and ϵ_2, Majorana flips will allow us to derive relations such as

- $\bar{\epsilon}_1\epsilon_2 = t_0\bar{\epsilon}_2\epsilon_1 = \bar{\epsilon}_2\epsilon_1$
- $\bar{\epsilon}_1(\gamma^\mu)\epsilon_2 = t_1\bar{\epsilon}_2(\gamma^\mu)\epsilon_1 = -\bar{\epsilon}_2(\gamma^\mu)\epsilon_1$
- $\bar{\epsilon}_1(\gamma^{\mu\nu})\epsilon_2 = t_2\bar{\epsilon}_2(\gamma^{\mu\nu})\epsilon_1 = -\bar{\epsilon}_2(\gamma^{\mu\nu})\epsilon_1$

Definition B.5. *We define a unitary matrix B via the Charge conjugation matrix and 0^{th} Clifford algebra element $B := it_0C\gamma^0$. Such B satisfies $BB^* = -t_1\mathbb{1}$.*

Definition B.6. *We define the **Charge conjugate** of a spinor by ψ^C by $\psi^C := B^{-1}\psi^*$.*

The **Majorana spinors** are defined to be Dirac spinors that satisfy reality condition. They are spinors such that,

$$\psi = \psi^C = B^{-1}\psi^* \tag{B.3.7}$$

relation is satisfied. Taking the charge conjugate once again to get back to ψ, we find the condition that $BB^* = \mathbb{1}$, which is only possible in spacetime dimensions with $t_1 = -1$. That is, the reality condition gives consistent result only when the spacetime admits $t_0 = 1$ and $t_1 = -1$; from this we conclude that $D = 2, 3, 4 \pmod 8$ are the only spacetime dimensions where Majorana spinors can exist.

This extra condition divides the degrees of freedom of a Majorana spinor by half, so in total a Majorana spinor has 2^{m-1} components as opposed to 2^m of a regular Dirac spinor.