CENTRAL-FIRING OF TYPE A_{2n} WITH INITIAL WEIGHT 0

by

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ABSTRACT

CENTRAL-FIRING OF TYPE A_{2n} WITH INITIAL WEIGHT 0

In this thesis a variant of the chip-firing game introduced by Hopkins, McConville and Propp in [1], called the labeled chip-firing on $\mathbb{Z}$, is studied. We will first explore the basic properties and examples of this game. We will then show, how one can see this game as a binary relation on the weight lattice of Type A root system. It is then a natural step to generalize it to other root systems, which was done by Galashin, Hopkins, McConville and Postnikov in [2] and [3]. After reviewing the basics of central-firing introduced in these papers, we examine the central-firing of type A_{2n} with initial weight 0 in Chapter 5. Finally, we study the restrictions in Lemma 12 of [1] in more detail, and conjecture that the number of permutations with maximum number of inversions allowed by this lemma is given by the Catalan numbers.

ÖZET

0'DAN BAŞLAYAN A_{2n} TİPLİ MERKEZİ ATEŞLEME

Bu tezde, çip ateşleme oyununun bir versiyonu olan, ve ilk olarak Hopkins, McConville and Propp [1] tarafından incelenen $\mathbb{Z}$ üzerinde etiketli çip ateşleme oyunu incelenmiştir. İlk olarak bu oyunun temel özellikleri ve örnekleri incelenmiştir. Daha sonra bu oyunu A tipi kök sistemlerinin ağırlık kafesinde bir ikili bağıntı olarak nasıl görülebileceği gösterilmiştir. Oyunun diğer kök sistemlerine genelleştirilmesi Galashin, Hopkins, McConville ve Postnikov tarafından [2] ve [3]'te yapılmıştır. Bu makalelerde tanımlanan merkez ateşleme'nin temel özelliklerini hatırlandıktan sonra, başlangıç ağırlığı 0 olan A_{2n} tipli merkez ateşleme incelenmiştir. Son olarak, [1]'de verilen Lemma 12'nin kısıtlamaları anlamaya çalışılmış olup, bu kısıtlamalara göre maksimum evirtime sahip olan permütasyonların sayısının Katalan sayılarıyla verildiği sanı olarak verilmiştir.

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LIST OF SYMBOLS

$O(n)$	Group of orthogonal transformations of $\mathbb{R}^n$
$[n]$	Finite set $\{1, 2, \ldots, n-1, n\}$
$\mathrm{Span}_{\mathbb{Z}}(A)$	Set of all integer linear combinations of elements in A
δ_i	Unlabeled chip configuration on $\mathbb{Z}$ with a single chip at the vertex i, and no other chips
$\delta_{[i,j]}$	Unlabeled chip configuration on $\mathbb{Z}$ with a single chip at each vertex k for $i \leqslant k \leqslant j$, and no other chips
c°	Stabilization of c
$\xrightarrow[\Phi^+]{}$	Central-firing
Δ^n	Labeled chip configuration on $\mathbb{Z}$ with n chips at the origin and no other chips
$\widetilde{\Delta^{2n+1}}$	Set of permutations corresponding to possible stabilizations of Δ^{2n+1}
$\mathbb{P}(A)$	Probability of an event A
C_n	nth Catalan number

1. INTRODUCTION

Chip-firing process can be thought of as a discrete diffusion process on a graph. It was independently introduced in the late 80's in mathematics and physics. Mathematicians first analyzed the concept on $\mathbb{Z}$ [4, 5], while physicists analyzed it on $\mathbb{Z}^2$ [6]. Chip-firing has connections to diverse range of fields including tropical geometry, combinatorics, probability theory, commutative algebra, representation theory, algebraic geometry, number theory and graph theory. Luckily, there are two recent advanced undergraduate and graduate books on chip-firing that overview these developments [7, 8].

To understand the idea of chip-firing, we start with a finite, connected and undirected graph. A *chip configuration* is an assignment of nonnegative integers to the vertices of the graph. We think of these integers as counting the number of chips on the corresponding vertex. To define the process, we require that the difference between two consecutive chip configurations in the process is given by the multiplication of the discrete Laplacian by a vector that has only one 1 and zeros elsewhere. This dictates that whenever a vertex has as many chips as the number of its neighbors (such a vertex is said to be *unstable*) it gives (*fires*) one chip to each of its neighbors. The whole process is defined by starting with an initial chip configuration, finding an unstable vertex and firing it (it does not matter which one), and repeating this process until there is no unstable vertex.

In this thesis, we will study chip-firing process on $\mathbb{Z}$, but with a slight modification: the chips will be distinguishable. This modification causes the process to become less intuitive physically, but the mathematics involved is interesting enough for exploration.

The thesis is outlined as follows. In Chapter 2, we will give the necessary definitions and theorems for the rest of the paper. In Chapter 3, we will define both the classical chip-firing on $\mathbb{Z}$, which is called the unlabeled chip-firing on $\mathbb{Z}$, and its labeled analogue. In Chapter 4, we will define the same process, but by using root-theoretic terminology. Almost all of Chapter 3 and 4 can be found in [1–3].

In Chapter 5, we will try to understand two related problems. Both of these problems is about understanding some set of permutations. The first of these sets gives rise to a probability distribution on the symmetric group. We will call this distribution the central distribution. This name requires almost no explanation: It is a distribution arising from what we call central-firing, and is central in the sense that the identity element is the most likely outcome. The second set consists of permutations where each letter is allowed to be placed only at certain locations. Even though this set comes from chip-firing, we are interested in it with independent interest.

2. PRELIMINARIES

In this chapter, we give the necessary definitions and results that are needed for the rest of the thesis. More information on root systems can be found in [9].

2.1. Binary Relations

Let S be any set. A *binary relation* on S, is any subset R of $S \times S$. We will write $a \to b$ for $(a, b) \in R$. A relation $\to$ is said to be *reflexive* if $a \to a$ for all $a \in S$, and transitive if $a \to b$ and $b \to c$ implies $a \to c$, for any $a, b, c \in S$. If $\to$ is any relation on S, we will write $\xrightarrow{*}$ to denote the reflexive, transitive closure of $\to$, i.e, we write $a \xrightarrow{*} b$ to mean that there exists a sequence $a = a_0 \to a_1 \to a_2 \to \ldots \to a_{n-1} \to a_n = b$, for some $n \in \mathbb{N}$. If $a \xrightarrow{*} b$, we say that b is *reachable* from a.

We say that $\to$ is *terminating* if there are no infinite sequence of relations: $a_0 \to a_1 \to a_2 \to \ldots$. For any $a \in S$, we say that $\to$ is *confluent from* a, if whenever $a \xrightarrow{*} b$ and $a \xrightarrow{*} c$, there exists $d \in S$ such that $b \xrightarrow{*} d$ and $c \xrightarrow{*} d$. We say that a is *stable* if there does not exist any $b \neq a$ with $a \to b$.

Proposition 2.1. *Let S be a set, and let $\to$ be a binary relation on S. If $\to$ is terminating and is confluent from $a \in S$, then there is a unique stable $b \in S$ such that $a \xrightarrow{*} b$.*

Proof. Since $\to$ is terminating, there exists a stable element b such that $a \xrightarrow{*} b$. Now assume $c \in S$ is stable and $a \xrightarrow{*} c$. Since $\to$ is confluent from a, there must exists some $d \in S$ such that $b \xrightarrow{*} d$ and $c \xrightarrow{*} d$. This can only happen if $b = d = c$, since b and c are stable. $\square$

A relation $\to$ is said to be *confluent*, if it is confluent from all $a \in S$. If the relation $\to$ is known to be terminating, then there is a weaker notion of confluence called local confluence that guarantees unique stabilization. A relation $\to$ is *locally* *confluent from* a, if whenever $a \to b$ and $a \to c$, there exists $d \in S$ such that $b \xrightarrow{*} d$ and $c \xrightarrow{*} d$. A relation $\to$ is *locally confluent* if it is locally confluent from all $a \in S$.

Lemma 2.2. *(Diamond lemma, (Theorem 3, [10]). Let S be any set and $\to$ a binary relation on S. If $\to$ is terminating, then $\to$ is confluent if and only if $\to$ is locally confluent.*

Corollary 2.3. *Let S be any set and $\to$ a binary relation on S. If $\to$ is terminating and is locally confluent, then there exists a unique stable b such that $a \xrightarrow{*} b$, for any $a \in S$.*

Proof. Follows directly from Proposition 2.1 and Lemma 2.2. □

If b is the unique stable element of S reachable from a, we say b is the *stabilization* of a, and we denote it by $a^\circ = b$.

Let S be a set, $\to$ a binary relation on S, and G a group acting on S. For any $a \in S$, we write $G \cdot a$ to denote the orbit of a under the action of G. Then $\to$ reduces to a binary relation on S/G, the set of orbits in S under the action of G, as follows: $G \cdot a \to G \cdot b$ if and only if there exists an element $a' \in G \cdot a$, and an element $b' \in G \cdot b$ such that $a' \to b'$.

2.2. Root Systems

Let α be any vector in $\mathbb{R}^n$. Then the map $s_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$s_\alpha(\beta) = \beta - 2\tfrac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

for all $\beta \in \mathbb{R}^n$, where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^n$, is the *reflection with respect to* α. Note that s_α maps α to $-\alpha$ and the hyperplane orthogonal to α to itself.

For any $\alpha \in \mathbb{R}^n$, write

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}$$

Definition. A finite set of nonzero vectors Φ in an Euclidean vector space $(E, (\cdot, \cdot))$ is called a *root system*, if it satisfies the following properties:
1. Φ spans E.
2. If $\alpha \in \Phi$ and $c \in \mathbb{R}$, then $c\alpha \in \Phi$ if and only if $c \in \{-1, +1\}$.
3. If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) \in \Phi$.
4. If $\alpha, \beta \in \Phi$, then $(\alpha, \beta^\vee) \in \mathbb{Z}$.

The elements α and $\alpha^\vee$ for $\alpha \in \Phi$ are called *roots* and *coroots*, respectively. The dimension of E is called the *rank* of Φ. If Φ is a root system of rank n, the *Weyl group* W of Φ is defined to be the subgroup of the orthogonal group $O(n)$ generated by the reflections s_α, $\alpha \in \Phi$.

Suppose Φ_1 and Φ_2 are two root systems living in E_1 and E_2, respectively. For each $v \in \Phi_1$, let $\tilde{v} \in E_1 \oplus E_2$ be the image of v under the natural injection from E_1 to $E_1 \oplus E_2$ and let $\widetilde{\Phi_1} = \{\tilde{v} : v \in \Phi_1\}$. Likewise, let $\widetilde{\Phi_2}$ be the set of vectors obtained from Φ_2. Then, it is an exercise to show that $\Phi = \widetilde{\Phi_1} \cup \widetilde{\Phi_2}$ forms a root system that lives in $E_1 \oplus E_2$. Φ is called the *direct sum* of Φ_1 and Φ_2.

If a root system Φ can be written as a direct sum of two non-trivial root systems, then it is said to be *reducible*. Otherwise, it is called *irreducible*.

It is a fundamental fact in the theory of root systems that for any root system Φ, we can choose a set of *simple roots* $\Delta \subseteq \Phi$ such that Δ is a basis for E, and every $v \in \Phi$ can be written as either all non-negative or all non-positive combinations of simple roots. Hence, depending on this choice of simple roots, we can uniquely write $\Phi = \Phi^+ \cup \Phi^-$. We write $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, where n is the dimension of the Euclidean space E.

If Φ is any root system, then the set of integer linear combinations of all roots in Φ, $Q := \mathrm{Span}_{\mathbb{Z}}(\Phi)$, is called the *root lattice* of Φ. The root lattice lives inside of a usually denser lattice called the *weight lattice*, defined as $P := \{v \in E : (v, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. For each $i \in [n] = \{1, 2, \ldots, n-1, n\}$, we define the *fundamental weights* by $(\omega_i, \alpha_j^\vee) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. The sum of all weights $\rho = \sum_{i=1}^n \omega_i$ is given a special name, the *Weyl vector*.

The following theorem gives the classification of irreducible root systems, and consequently gives the classification of complex simple Lie algebras.

Theorem 2.4. *(Theorem 8.49, [11]) Every irreducible root system is isomorphic to one of the following:*

- A_n, $n \geq 1$
- B_n, $n \geq 2$
- C_n, $n \geq 3$
- D_n, $n \geq 4$
- *One of the exceptional root systems* G_2, F_4, E_6, E_7, *and* E_8.

Let us now define the classical root systems A_n, B_n, C_n, and D_n. For exceptional types, see [12].

2.2.1. The A_n root system

Let E be the n-dimensional subspace of $\mathbb{R}^{n+1}$ consisting of all vectors whose entries sum to 0. Then A_n is the root system consisting of vectors of the form

$$\Phi = \{e_i - e_j, \ i \neq j\}$$

where $\{e_i\}$ is the standard basis for $\mathbb{R}^{n+1}$. The vectors

$$e_i - e_{i+1}, \ i \in [n]$$

can be taken as simple roots Δ for A_n. Note that for $j < k$, we can write

$$e_j - e_k = (e_j - e_{j+1}) + (e_{j+1} - e_{j+2}) + ... + (e_{k-1} - e_k),$$

so that every root is either all non-negative combination of base vectors, or all non-positive. The fundamental weights of A_n are given by

$$\omega_i = e_1 + e_2 + ... + e_i - ih,$$

for all $i \in [n]$, where

$$h = \frac{e_1 + e_2 + ... + e_n}{n}.$$

2.2.2. The B_n root system

Let $E = \mathbb{R}^n$. Then B_n is the root system consisting of vectors of the form

$$\Phi = \{\pm e_i \pm e_j,\ i < j\} \cup \{\pm e_i,\ i \in [n]\}$$

where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. The vectors $e_i - e_{i+1},\ i \in [n-1]$, together with e_n can be taken as simple roots Δ for B_n. The fundamental weights of B_n are given by

$$\omega_i = e_1 + e_2 + ... + e_i,$$

for all $i \in [n-1]$, and

$$\omega_n = \frac{e_1 + e_2 + ... + e_n}{2}.$$

2.2.3. The C_n root system

Let $E = \mathbb{R}^n$. Then C_n is the root system consisting of vectors of the form

$$\Phi = \{\pm e_i \pm e_j,\ i < j\} \cup \{\pm 2e_i,\ i \in [n]\}$$

where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. The vectors $e_i - e_{i+1},\ i \in [n-1]$ together with $2e_n$ can be taken as simple roots Δ for C_n. The fundamental weights of C_n are given by

$$\omega_i = e_1 + e_2 + ... + e_i,$$

for all $i \in [n]$.

2.2.4. The D_n root system

Let $E = \mathbb{R}^n$. Then D_n is the root system consisting of vectors of the form

$$\Phi = \{\pm e_i \pm e_j,\ i < j\}$$

where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. The vectors $e_i - e_{i+1},\ i \in [n-1]$ together with $e_{n-1} + e_n$ can be taken as simple roots Δ for D_n. The fundamental weights of D_n are given by

$$\omega_i = e_1 + e_2 + ... + e_i,$$

for all $i \in [n-2]$, together with

$$\omega_{n-1} = \frac{e_1 + e_2 + ... + e_{n-1} - e_n}{2}$$

and

$$\omega_n = \frac{e_1 + e_2 + ... + e_{n-1} + e_n}{2}.$$

3. CHIP-FIRING ON $\mathbb{Z}$

In this chapter, we'll define unlabeled and labeled chip-firing on $\mathbb{Z}$. Let $G = (\mathbb{Z}, E)$ be the infinite undirected graph with the vertex set $\mathbb{Z}$ and the edge set

$$E = \{\{i, j\} : |i - j| = 1 \text{ and } i, j \in \mathbb{Z}\}$$

3.1. Unlabeled Chip-Firing on $\mathbb{Z}$

In this section, we define the unlabeled chip-firing process on $\mathbb{Z}$ and study its properties.

Definition. *A chip configuration on* $\mathbb{Z}$ is a function $c : \mathbb{Z} \to \mathbb{N}$ with $\sum_{i\in\mathbb{Z}} c(i) < \infty$. We think of $c(i)$ as recording the number of chips at the vertex i.

For two chip configurations c and d on $\mathbb{Z}$, we can define the *sum* of c and d by $(c + d)(i) = c(i) + d(i)$. One can also define

$$(c^n)(i) = \underbrace{c(i) + c(i) \cdots + c(i)}_{n \text{ times}}$$

for any $n \in \mathbb{N}$. Let δ_i denote the configuration that has a single chip at i, and no other chips. So, δ_0^n is the configuration with n chips at the origin. Let $\delta_{[i,j]}$ denote the configuration that has a single chip at each vertex k for $i \leqslant k \leqslant j$, and no other chips.

Definition. *Chip-firing process on* $\mathbb{Z}$ is a binary relation $\to$ on the set of all chip configurations on $\mathbb{Z}$: We write $c \to d$, if for some $i \in \mathbb{Z}$,

$$\begin{cases} d(k) = c(k) - 2 & k = i \\ d(k) = c(k) + 1 & k = i + 1 \\ d(k) = c(k) + 1 & k = i - 1 \\ d(k) = c(k) & \text{otherwise.} \end{cases}$$

If the above conditions hold, we say d can be obtained from c by a *firing move at* i. If d is obtained from c by a single firing move at i, we will write $c \underset{i}{\to} d$.

Example 1. Let $c = \delta_0^2 + \delta_1$. Then the chip-firing process with initial configuration c is visualized in Figure 2.1. The first step is a firing move at 0, and the second step is a firing move at 1.

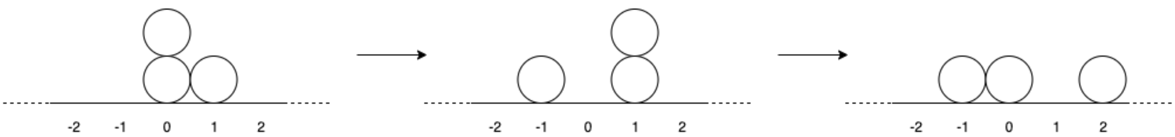


Figure 3.1. Chip-firing process with initial configuration $\delta_0^2 + \delta_1$.

When there are no firing moves left, the resulting configuration is said to be *stable*. It is clear that d is stable if and only if $d(i) \leqslant 1$ for all $i \in \mathbb{Z}$. If a stable configuration d is a result of the chip-firing process starting from c, we say that d is a *stabilization* of c and we denote it by $d = c^\circ$. For instance, we have

$$(\delta_0^2 + \delta_1)^\circ = (\delta_{-1} + \delta_1^2)^\circ = \delta_{[-1,0]} + \delta_2$$

and we write

$$\delta_0^2 + \delta_1 \underset{0}{\to} \delta_{-1} + \delta_1^2 \underset{1}{\to} \delta_{[-1,0]} + \delta_2.$$

Proposition 3.1. *For any chip configuration c on $\mathbb{Z}$, there is a unique stable configuration d such that $c^\circ = d$.*

Proof. Let the chip-firing process start at c. We assign each edge e the first chip that crosses e. In all subsequent firings, we either move the assigned chip across e, or do not move the chip. Since the number of chips was finite, only finitely many edges will have an assigned chip. Hence, there are infinitely many edges with no assigned chips, hence an infinite number of vertices that can never fire. By Corollary 2.3.3 of [7], this shows that the chip-firing process on $\mathbb{Z}$ is terminating. To show that it is also locally confluent, let c, d_1, and d_2 be three configurations such that $c \underset{i_1}{\longrightarrow} d_1$ and $c \underset{i_2}{\longrightarrow} d_2$. This means, c is fireable from i_1 and i_2. Since a firing move at i_1 cannot decrease the number of chips at i_2, we conclude that d_1 is fireable from from i_2, and likewise, d_2 is fireable from i_1. This implies that there exists a configuration b such that $d_1 \underset{i_2}{\longrightarrow} b$ and $d_2 \underset{i_1}{\longrightarrow} b$. By Corollary 2.3, the result follows. $\square$

3.2. Labeled Chip-Firing on $\mathbb{Z}$

We have defined the chip-firing process where the chips were indistinguishable. What happens when we label the chips, and modify the process accordingly? It turns out that the confluence property does not continue to hold for all initial configurations. In Chapters 4 and 5, we will try to understand this labeled chip-firing process in a more detailed way, but for now let us give the definition of labeled chip-firing process on $\mathbb{Z}$. Remember that $[n] = \{1, 2, \ldots, n-1, n\}$.

Definition. A *labeled chip configuration* on $\mathbb{Z}$ is a function $C : [n] \to \mathbb{Z}$. We think of $C(i)$ as the position of the ith chip.

Example 2. Let $C(1) = 1$, $C(2) = 0$, and $C(3) = -1$. Then the labeled chip configuration C is visualised in Figure 3.2.

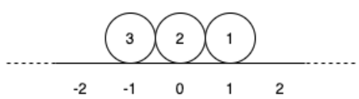


Figure 3.2. An example of a labeled configuration.

We will denote the unlabeled chip configuration corresponding to C by $[C]$. So, $[C](i) = \#\{j \in [n] : C^{-1}(i) = j\}$, for $i \in \mathbb{Z}$.

Definition. *Labeled chip-firing process on* $\mathbb{Z}$ is a binary relation on the set of all labeled configurations on $\mathbb{Z}$: We write $C \to D$, if for some $i, j \in [n]$ with $i < j$ and $C(i) = C(j)$, we have

$$\begin{cases} D(k) = C(k) - 1 & k = j \\ D(k) = C(k) + 1 & k = i \\ D(k) = C(k) & \text{otherwise.} \end{cases}$$

In other words, whenever a firing move is made, the chip with a bigger label moves to the left, and the chip with the smaller label moves to the right. It turns out that not every initial configuration stabilizes to a unique labeled configuration with this definition. For example, if C is the configuration with 3 chips at the origin and no other chips, then all three stable configuration in Figure 3.3 can be the result of the chip-firing process starting from C.

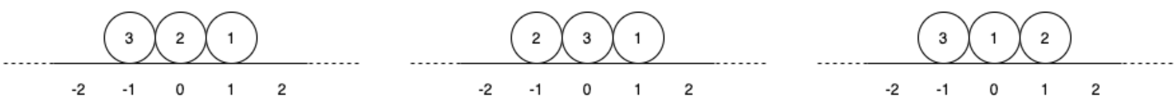


Figure 3.3. Three possible stabilization of the chip-firing process for the same initial configuration.

On the other hand, there are many examples of configurations that stabilizes to a unique configuration. For example, if C is the configuration with 4 chips at the origin, and no other chips, then the final configuration will necessarily be sorted (Figure 3.4).

Note that any labeled chip configuration $C : [n] \to \mathbb{Z}$ can be seen as a vector $\overline{C} \in \mathbb{Z}^n$ by $\overline{C} := (C(1), C(2), ..., C(n-1), C(n))$. Hence, the following is one possible sequence of stabilization in vector form for the labeled chip configuration with 4 chips at the origin: $(0,0,0,0) \to (1,-1,0,0) \to (1,-1,1,-1) \to (2,-1,0,-1) \to (2,0,0,-2) \to (2,1,-1,-2)$.

In terms of the standard basis for $\mathbb{R}^4$, the above sequence can be written as:

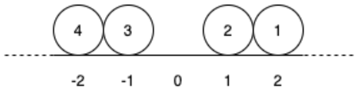


Figure 3.4. An example of a unique stabilization.

$$0 \to e_1 - e_2 \to e_1 - e_2 + e_3 - e_4 \to 2e_1 - e_2 - e_4 \to 2e_1 - 2e_4 \to 2e_1 + e_2 - e_3 - 2e_4.$$

Observe that choosing 2 chips that occupy the same vertex i in C corresponds to finding a vector of the form $e_i - e_j$ with $i < j$ such that

$$\overline{C} \perp e_i - e_j$$

And sending these chips to neighboring vertices corresponds to the move

$$\overline{C} \to \overline{C} + e_i - e_j.$$

We saw in Chapter 2 that the set of vectors of the form $e_i - e_j$ with $i < j$ is the positive roots of the A_n root system.

4. ROOT SYSTEM CHIP-FIRING: CENTRAL-FIRING

Motivated by the labeled chip-firing on $\mathbb{Z}$, Galashin, McConville, Hopkins and Postnikov introduced and developed *root system chip-firing* in [2, 3]. This chapter is a short review of [2].

4.1. Definition and Examples of Central-Firing

Definition. Let Φ be any irreducible root system. *Central-firing* is a binary relation $\xrightarrow[\Phi^+]{}$ defined on the weight lattice P of Φ by the following relation:

$$v \xrightarrow[\Phi^+]{} v + \alpha, \text{ if } v \perp \alpha \text{ for any } v \in P \text{ and } \alpha \in \Phi^+.$$

Note that after the firing move $v \to v + \alpha$, v is no longer orthogonal to α, but it might become orthogonal after some additional moves.

We will now try to understand what the central-firing means for classical types in terms of labeled chip-firing on $\mathbb{Z}$.

Definition. Given a labeled chip configuration C, we define the following 4 types of moves:

(a) for $i < j$ if chips i and j are in the same position (i.e, if $C(i) = C(j)$), move chip i one step to the right, and move chip j one step to the left.

(b) for $i \in [n]$, if chip i is at the origin (i.e, if $C(i) = 0$), move it one step to the right.

(c) for $i \in [n]$, if chip i is at the origin (i.e, if $C(i) = 0$), move it two steps to the right.

(d) for $i < j$, if chips (i) and (j) are in the opposite positions (i.e, if $C(i) = -C(j)$), then move both chips to one step right.

While the first move corresponds to translation by the vectors of the form $\{e_i - e_j : i < j\}$, the other moves corresponds respectively to translation by the vectors of the form $\{e_i\}$, $\{2e_i\}$, and $\{e_i + e_j : i < j\}$, where $i, j \in [n]$.

Proposition 4.1. *Two labeled chip configurations C and D satisfy $C \to D$ if and only if D can be obtained from C by a sequence of firing moves*

(1) of the form (a), if $\Phi = A_{n-1}$,

(2) of the form (a), (b), or (d), if $\Phi = B_n$,

(3) of the form (a), (c), or (d), if $\Phi = C_n$,

(4) of the form (a) or (d), if $\Phi = D_n$.

Proof. There is a one-to-one correspondence between the set of moves and the set of positive roots for each type. □

Let us illustrate these definitions with examples.

Example 3. Let $\Phi = A_3$ and let $\alpha_i = e_i - e_{i+1}$ for $i \in [3]$. Let $v = 0$ be the initial point. We saw in Chapter 3 that $(0,0,0,0) \to (1,-1,0,0) \to (1,-1,1,-1) \to (2,-1,0,-1) \to (2,0,0,-2) \to (2,1,-1,-2)$ was one possible sequence of stabilization. We can also write this sequence in terms of simple roots:

$$0 \xrightarrow[A_3^+]{} \alpha_1 \xrightarrow[A_3^+]{} \alpha_1 + \alpha_3 \xrightarrow[A_3^+]{} 2\alpha_1 + \alpha_2 + \alpha_3 \xrightarrow[A_3^+]{} 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \xrightarrow[A_3^+]{} 2\alpha_1 + 3\alpha_2 + 2\alpha_3$$

Since this process started with an element of the root lattice, we were able to write all terms in the sequence as a combination of simple roots (by definition). However, given any initial point in the weight lattice, one can write each point in the sequence in terms of fundamental weights. Note that

$$\alpha_1 = 2\omega_1 - \omega_2, \alpha_2 = -\omega_1 + 2\omega_2 - \omega_3, \text{ and } \alpha_3 = -\omega_2 + \omega_3.$$

So, the above sequence can also be written as:

$$0 \underset{A_3^+}{\longrightarrow} 2\omega_1 - \omega_2 \underset{A_3^+}{\longrightarrow} 2\omega_1 - 2\omega_2 + 2\omega_3 \underset{A_3^+}{\longrightarrow} 3\omega_1 - \omega_2 + \omega_3 \underset{A_3^+}{\longrightarrow} 2\omega_1 + 2\omega_3 \underset{A_3^+}{\longrightarrow} \omega_1 + 2\omega_2 + \omega_3.$$

The following proposition generalizes this example.

Proposition 4.2. *Let $\Phi = A_{2n-1}$. Then the central-firing starting from 0 stabilizes to $\omega_n + \rho$ where ρ is the Weyl vector.*

Proof. The main result of [1] says that the central-firing starting from 0 of A_{2n-1}, stabilizes to a unique configuration given by

$$(n, n-1, \ldots, 1, -1, \ldots, -(n-1), -n).$$

It can be computed that $\omega_n + \rho$ is also given by the same vector (see 2.2). $\square$

Example 4. Let $\Phi = B_2$. Let $v = 0$ be the initial point. Then the following is one possible sequence of stabilization

$$(0,0) \to (1,-1) \to (2,0) \to (2,1)$$

In terms of the fundamental weights of B_2, the above sequence corresponds to

$$0 \underset{B_2^+}{\longrightarrow} 2\omega_1 - 2\omega_2 \underset{B_2^+}{\longrightarrow} 2\omega_1 \underset{B_2^+}{\longrightarrow} \omega_1 + 2\omega_2.$$

Example 5. Let $\Phi = C_3$. Let $v = 0$ be the initial point. Then the following is one possible sequence of stabilization

$$(0,0,0) \to (1,-1,0) \to (2,0,0) \to (2,0,2) \to (3,0,1) \to (3,2,1).$$

In terms of the fundamental weights of C_3, the above sequence corresponds to

$$0 \xrightarrow[C_3^+]{} 2\omega_1 - \omega_2 \xrightarrow[C_3^+]{} 2\omega_1 \xrightarrow[C_3^+]{} 2\omega_1 - 2\omega_2 + 2\omega_3 \xrightarrow[C_3^+]{} 3\omega_1 - \omega_2 + \omega_3 \xrightarrow[C_3^+]{} \omega_1 + \omega_2 + \omega_3.$$

Example 6. Let $\Phi = D_3$. Let $v = 0$ be the initial point. Then the following is one possible sequence of stabilization

$$(0,0,0) \to (1,-1,0) \to (2,0,0) \to (2,1,1) \to (2,2,0) \to (3,1,0).$$

In terms of the fundamental weights of D_3, the above sequence corresponds to

$$0 \xrightarrow[D_3^+]{} 2\omega_1 - \omega_2 - \omega_3 \xrightarrow[D_3^+]{} 2\omega_1 \xrightarrow[D_3^+]{} \omega_1 + 2\omega_3 \xrightarrow[D_3^+]{} 2\omega_2 + 2\omega_3 \xrightarrow[D_3^+]{} 2\omega_1 + \omega_2 + \omega_3.$$

All of these examples and Proposition 4.2 can be seen as a special case of a more general result, which we state here for completeness. Basically, this proposition tells us that if we start with an initial weight inside the convex hull of the Weyl orbit of $\rho + \omega$, where ω is some minuscule weight or the zero vector, then the process ends at a vector on the Weyl orbit of $\rho + \omega$.

Proposition 4.3. *(Proposition 4.10, [2]) Suppose that $\lambda \in \Pi^Q(\rho + \omega)$, for some $\omega \in \Omega_m^0$. Then $W.(\rho + \omega)$ is the $\xrightarrow[\Phi^+]{}$ stabilization of $W.\lambda$.*

4.2. The Confluence Conjecture

Understanding which initial weights give rise to confluent process is intricate. We give the conjectural classification given in [2] below.

The confluence conjecture. (Conjecture 7.1, [2]) Let Φ be a classical root system, and let ω be a fundamental weight or the zero vector. Then $\xrightarrow[\Phi^+]{}$ is confluent starting from ω if and only if $\omega \notin Q+\rho$, unless one of the four exceptional cases happens:
(1) $\Phi = A_n$ in which case $\xrightarrow[\Phi^+]{}$ is confluent from ω if and only if

$$\begin{cases} \omega = 0, \omega_1, \omega_n, & \text{if } n \text{ is odd;} \\ \omega = \omega_{n/2}, \omega_{n/2+1}, & \text{if } n \text{ is even.} \end{cases}$$

(2) $\Phi = B_n$ in which case $\omega = \omega_n$ is confluent despite the fact that $\omega_n \in Q+\rho$;
(3) $\Phi = D_{4n+2}$ for $n \geqslant 1$ in which case $\omega = 0$ is not confluent even though $0 \notin Q+\rho$.

Partial results to this conjecture and to other conjectures given in [1] are obtained in [13].

4.3. Central-Firing Modulo Weyl Group

Recall that for any root system Φ, there is an associated finite group called the Weyl group W, generated by the reflections with respect to roots, i.e., $W = \langle s_\alpha \rangle$, $\alpha \in \Phi$. One can show that the action of W on P is well-defined. Hence, we can talk about the binary relation induced from P to P/W (see section 2.1) as follows: we write $W.v_1 \xrightarrow[\Phi^+]{} W.v_2$ if and only if there exist $v_1' \in W.v_1$ and $v_2' \in W.v_2$ such that $v_1' \xrightarrow[\Phi^+]{} v_2'$. To understand what this means in terms of chip-firing on $\mathbb{Z}$, we list the Weyl groups of classical types.

Theorem 4.4. *1.* $W(A_n) \simeq S_{n+1}$
2. $W(B_n) \simeq W(C_n) \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$
3. $W(D_n) \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$.

Proof. See the first section in [14]. □

S_{n+1} acts on vectors in A_n by simply permuting the entries of $n+1$ tuples. Consequently, when we mod out by the action of this group, we exactly get the unlabeled chip-firing on $\mathbb{Z}$, since permuting the entries corresponds to permuting the labels on chips.

$(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ acts on vectors in B_n and C_n by both permuting and changing signs of the entries of n tuples. In terms of chip-firing on $\mathbb{Z}$, moding out the action of Weyl group corresponds to not only deleting the labels, but also moving any chip from its position to its negative.

$(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ acts on vectors in D_n by both permuting and changing even number of signs of the entries of n tuples. In terms of chip-firing on $\mathbb{Z}$, moding out the action of Weyl group corresponds to not only deleting the labels, but also moving any even number of chips from its position to its negative.

It turns out that the central-firing modulo Weyl group is confluent for all root systems.

Theorem 4.5. *(Corollary 4.8, [2]) The relation $\xrightarrow[\Phi^+]{}$ on P/W is confluent and terminating for any root system Φ.*

5. CENTRAL-FIRING OF TYPE A_n

In this chapter, we'll first compute the stabilizations of all unlabeled configurations corresponding to fundamental weights of A_n and the the total number of firings for each of these cases. We'll then look at the case of odd number of chips at the origin more closely, and prove that 'dual' permutations are equally likely to happen. Finally, we'll study the permutations restricted by Lemma 12 of [1] and give two conjectures related to that Lemma.

5.1. Computations on Central-Firing of Type A_n Modulo Weyl Group

As we saw in the previous chapter, central-firing of type A modulo Weyl group is the same process as unlabeled chip firing on $\mathbb{Z}$. In order to find the stabilizations of unlabeled configurations corresponding to fundamental weights the following lemma is useful.

Lemma 5.1. *(Proposition 3, [1])* $(\delta_{[a,b]} + \delta_i)^\circ = \delta_{[a-1,a+b-i-1]} + \delta_{[a+b-i+1,b+1]}$*, for all* $a, b, i \in \mathbb{Z}$ *such that* $a \leqslant i \leqslant b$*.*

Proof. Proof is by induction on $b - a$. If $b - a = 0$, the statement becomes $(\delta_a + \delta_a)^\circ = \delta_{a-1} + \delta_{a+1}$, which is clearly true. Now assume that the statement holds for all values of $b, a \in \mathbb{Z}$ with $b - a < n$, and let $b - a = n$, for some positive integer n. Without loss of generality, assume $i \neq b$. Then,

$$\begin{aligned}
(\delta_{[a,b]} + \delta_i)^\circ &= ((\delta_{[a,b-1]} + \delta_i)^\circ + \delta_b)^\circ && (5.1)\\
&= (\delta_{[a-1,a+b-i-2]} + \delta_{[a+b-i,b]} + \delta_b)^\circ && (5.2)\\
&= (\delta_{[a-1,a+b-i-2]} + (\delta_{[a+b-i,b]} + \delta_b)^\circ)^\circ && (5.3)\\
&= (\delta_{[a-1,a+b-i-2]} + \delta_{[a+b-i-1,a+b-i-1]} + \delta_{[a+b-i+1,b+1]})^\circ && (5.4)\\
&= \delta_{[a-1,a+b-i-1]} + \delta_{[a+b-i+1,b+1]} && (5.5)
\end{aligned}$$

where we have used the confluence property in (5.1) and (5.3) and the induction hypothesis in (5.2) and (5.4). □

We'll now list stabilizations of all unlabeled configurations corresponding to fundamental weights of A_n. Correspondence between weights of A_n and the unlabeled configurations is given by finding the labeled chip configuration corresponding to the given weight and deleting the labels.

Proposition 5.2. *1.* $(\delta_0^{2n})^\circ = \delta_{[-n,-1]} + \delta_{[1,n]}$

2. $(\delta_0^{2n+1})^\circ = \delta_{[-n,n]}$

3. $(\delta_0^{2n-i} + \delta_1^{i+1})^\circ = \delta_{[-n,n-1-i]} + \delta_{[n+1-i,n+1]}$ *for all* $0 \leqslant i \leqslant 2n-1$.

4. $(\delta_0^{2n-i} + \delta_1^{i})^\circ = \delta_{[-n,-i-1]} + \delta_{[-i+1,n]}$ *for all* $1 \leqslant i \leqslant n-1$

5. $(\delta_0^{2n-i} + \delta_1^{i})^\circ = \delta_{[-n+1,n]}$ *for* $i = n$

6. $(\delta_0^{2n-i} + \delta_1^{i})^\circ = \delta_{[-n+1,2n-i]} + \delta_{[2n+2-i,n+1]}$ *for all* $n+1 \leqslant i \leqslant 2n-1$.

Proof. The first two is proved in [1]. The rest can be proven using Lemma 5.1 and mathematical induction. □

One of the most interesting property of the classical chip-firing game is the fact that the number of firings each vertex made and the number of total firings is independent from the stabilization sequence [7]. Thus, we can compute the total number of firings for each case in Proposition 5.2. First, we define the following statistic introduced in [1].

Proposition 5.3. *Let* $\varphi_\infty^2(c) = \sum_{i\in\mathbb{Z}} i^2 \cdot c(i)$ *and suppose* $c \underset{i}{\rightarrow} d$. *Then* $\varphi_\infty^2(d) = \varphi_\infty^2(c) + 2$.

Proof. c and d differ only at $i-1$, i, $i+1$, and we have $(i-1)^2 + (i+1)^2 - 2(i)^2 = 2$. □

So, every firing move increases the quantity φ_∞^2 by two. We can use this fact to compute the following.

Proposition 5.4. *The following numbers are the total number of firings for each of the cases in Proposition 5.2, with the same order.*

1. $\frac{n(n+1)(2n+1)}{6}$
2. $\frac{n(n+1)(2n+1)}{6}$
3. $\frac{n(n+1)(2n+1)}{6} + \frac{2n+2in-i^2-i}{2}$
4. $\frac{n(n+1)(2n+1)}{6} - \frac{i^2+i}{2}$
5. $\frac{n(n+1)(2n+1)}{6} - \frac{n^2+n}{2}$
6. $\frac{n(n+1)(2n+1)}{6} - \frac{4n^2+2n-4in-i+i^2}{2}$

Proof. Total number of firings in the sequence $c \to d$ is given by $\frac{1}{2}(\varphi^2_\infty(d) - \varphi^2_\infty(c))$. $\square$

5.2. Understanding the set $\widetilde{\Delta^{2n+1}}$

In [1] (and recently in [13]), it was proved that when we start with an even number of labeled chips at the origin, we always end up at the same configuration. This was the content of Proposition 4.2. In other words, if we let Δ^n to denote the labeled analogue of δ^n_0, then

$$(n, n-1, \ldots, 1, -1, \ldots, -n+1, -n).$$

is the only possible final configuration for the labeled chip-firing process that starts at Δ^{2n}.

What if we start with an odd number of labeled chips at the origin? We already saw that when $n = 3$, there were 3 possible final configuration (Figure 3.3). It is easy to see that, in the odd case we don't have the confluence property, since the final move necessarily occurs at a vertex with 3 chips. We'll now try to understand this case. Let $\widetilde{\Delta^{2n+1}}$ be the set of all possible final configurations of Δ^{2n+1}. We'll think of elements of $\widetilde{\Delta^{2n+1}}$ as reverse permutations of $2n+1$ letters. So, $\widetilde{\Delta^3} = \{321, 312, 231\}$. Let S_n be the symmetric group on n letters, and let $s_i = (i, i+1)$ be the adjacent transpositions of S_n for $i \in [n-1]$. The fact $S_n = \langle s_i \rangle$ is well-known. In the following, we'll sometimes use the notation S_n for the set of n-permutations, even if we don't use the underlying group structure. So, $\widetilde{\Delta^{2n+1}} \subset S_{2n+1}$. It is also worth noting that we use 'reverse' permutations, for example 54321 or $(5,4,3,2,1)$ denote the identity element of S_5, so that it aligns with the labeled chip-firing game. This was more convenient for us since we wrote our programs using reverse permutations. Also, we'll try to write the permutations in terms of s_i's so that it becomes easier to see patterns. Thus, $\widetilde{\Delta^3} = \{e, s_1, s_2\}$. The following lemma is useful for studying $\widetilde{\Delta^{2n+1}}$.

Lemma 5.5. *(Lemma 12, [1]) Suppose $\Delta^n \to C$. Then $-\left\lfloor \frac{k}{2} \right\rfloor \leqslant C(k) \leqslant \left\lfloor \frac{n+1-k}{2} \right\rfloor$ for all $1 \leqslant k \leqslant n$.*

We can easily understand this lemma with a matrix. For instance, when $n = 9$, the following matrix shows the restrictions in Lemma 5.5.

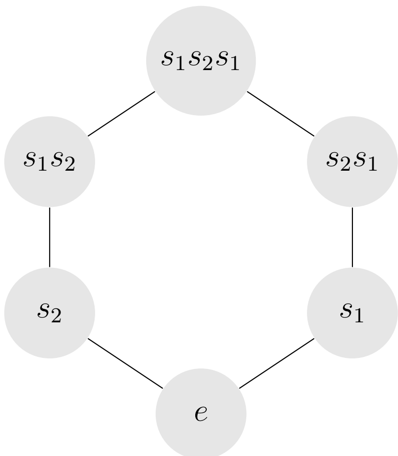
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We enumerate the rows and columns by $i \in \{9,8,7,6,5,4,3,2,1\}$ and $j \in \{-4,-3,-2,-1,0,1,2,3,4\}$. Thus $A_{ij} = 1$ if and only if ith chip can end up at the jth coordinate after some number of moves.

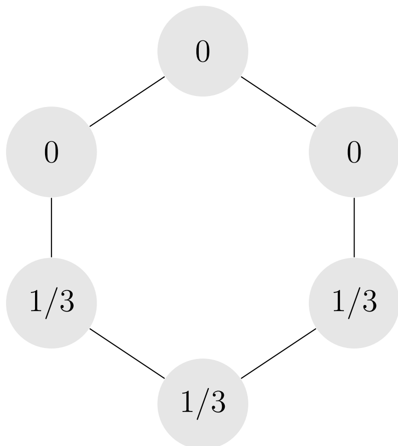
We can also try to understand $\widetilde{\Delta^{2n+1}}$ probabilistically. To do that, we need to define the transition probabilities. There are several ways of doing this. What we will do is to assume that at any step, the firing move is chosen among all possible moves uniformly. This will give us a probability distribution on $\widetilde{\Delta^{2n+1}}$. We can extend this distribution to all of S_{2n+1} by setting the probabilities on $S_{2n+1} \setminus \widetilde{\Delta^{2n+1}}$ equal to 0. We call this distribution *the central distribution* on S_{2n+1}.

5.2.1. $\widetilde{\Delta^3}$

This case is trivial, but let us draw the poset of weak Bruhat order of S_3 anyway. The second figure shows the central distribution on S_3. We like to think the central distribution as a heat diffusing into the Bruhat poset from the bottom.



(a) Weak Bruhat order.



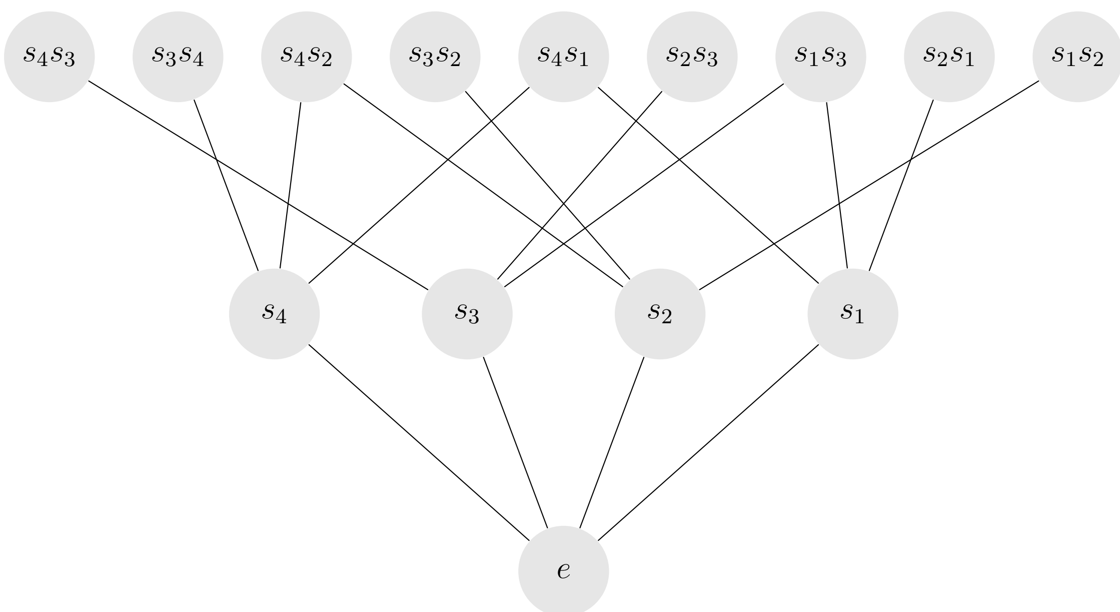
(b) The central distribution on S_3.

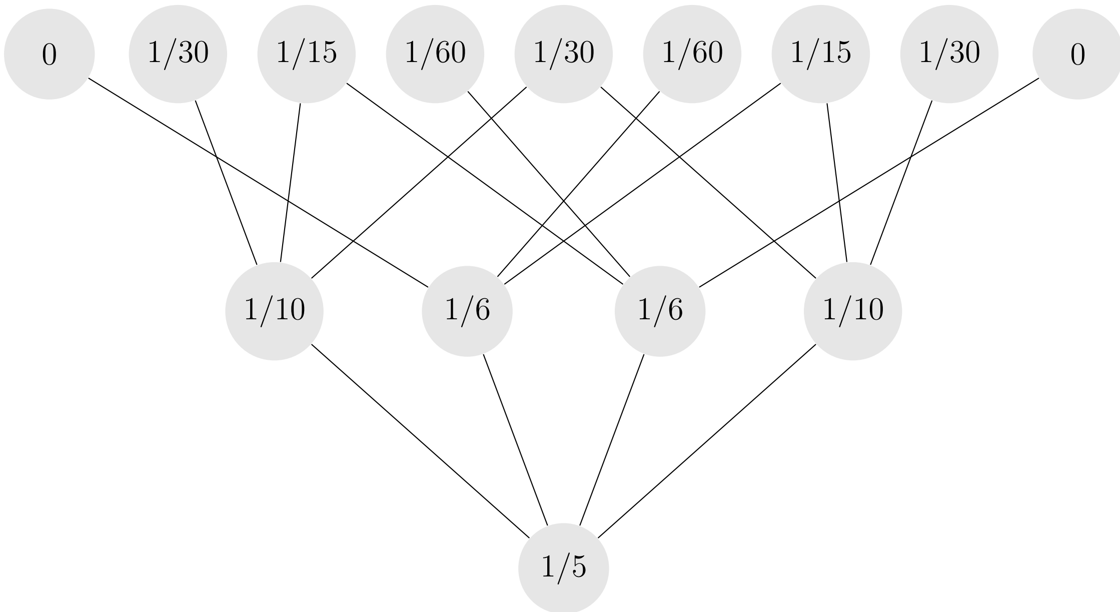
5.2.2. $\widetilde{\Delta^5}$

The following table shows the results of a simulation we run 25 million times on a computer for 5 chips at the origin. This table includes all permutations with positive probability.

Computations on S_5			
Element	Permutation	Count	$\approx$ Probability
e	(5,4,3,2,1)	4998504	$\frac{1}{5}$
s_2	(5,4,2,3,1)	4170084	$\frac{1}{6}$
s_3	(5,3,4,2,1)	4165612	$\frac{1}{6}$
s_1	(5,4,3,1,2)	2502621	$\frac{1}{10}$
s_4	(4,5,3,2,1)	2500657	$\frac{1}{10}$
$s_1 s_3$	(5,3,4,1,2)	1665167	$\frac{1}{15}$
$s_2 s_4$	(4,5,2,3,1)	1664361	$\frac{1}{15}$
$s_4 s_1$	(4,5,3,1,2)	833939	$\frac{1}{30}$
$s_2 s_1$	(5,4,1,3,2)	833249	$\frac{1}{30}$
$s_3 s_4$	(4,3,5,2,1)	832128	$\frac{1}{30}$
$s_2 s_3$	(5,3,2,4,1)	417972	$\frac{1}{60}$
$s_3 s_2$	(5,2,4,3,1)	415706	$\frac{1}{60}$

The missing elements with inversion 2 are $s_1 s_2$ and $s_4 s_3$. Observe that these elements corresponds to (5,4,2,1,3) and (3,5,4,2,1), respectively. But we already knew these permutations were not allowed by Lemma 5.5. The weak Bruhat order and the central distribution on S_5 are shown below.





5.2.3. $\widetilde{\Delta^7}$

There are 54 elements in $\widetilde{\Delta^7}$. So, we have only listed the most likely 11 permutations below. The number of simulations again 25 million.

Computations on S_7			
Element	Permutation	Count	$\approx$ Probability
e	(7, 6, 5, 4, 3, 2, 1)	6279082	0.25
s_4	(7, 6, 4, 5, 3, 2, 1)	3975396	0.159
s_3	(7, 6, 5, 3, 4, 2, 1)	3955145	0.159
s_5	(7, 5, 6, 4, 3, 2, 1)	2384262	0.095
s_2	(7, 6, 5, 4, 2, 3, 1)	2375100	0.095
s_1	(7, 6, 5, 4, 3, 1, 2)	881613	0.035
s_6	(6, 7, 5, 4, 3, 2, 1)	869907	0.035
s_3s_5	(7, 5, 6, 3, 4, 2, 1)	433880	0.017
s_2s_4	(7, 6, 4, 5, 2, 3, 1)	423681	0.017
s_1s_3	(7, 6, 5, 3, 4, 1, 2)	362105	0.014
s_6s_4	(6, 7, 4, 5, 3, 2, 1)	353289	0.014

A word of caution here: These tables are not graded by the number of inversions, for example the element $s_4s_2s_1$ seems to be more likely than s_2s_3 when we run the simulations. And we have certainly observed more convincing examples when we have increased the number of chips.

5.2.4. $\widetilde{\Delta^{2n+1}}$

For the general case, it can be proven that the dual permutations are equally likely to happen, which we'll define now.

Definition. Let $\pi = a_1a_2 \dots a_{n-1}a_n$ be a permutation on $[n]$. Let $a_i^* = n+1-a_{n+1-i}$ for all $i \in [n]$. Then *the dual permutation* is defined as $\pi^* := a_1^*a_2^* \dots a_{n-1}^*a_n^*$.

Example 7. If $\pi = 7632154$, then $\pi^* = 4376521$.

Dual of a permutation π can also be defined as $\pi^* = \omega\pi\omega$ where $\omega = 12...(n-1)n$ is the unique permutation with maximum number of inversions (remember we write permutations from the right). Before proving that the dual permutations are equally likely events, we need the following definition from [1]. If C is a chip configuration with n chips, then the *dual* C^* is defined as follows: first we reflect the configuration with respect to origin, then we replace each label i by $n+1-i$.

Proposition 5.6. *Let $\mathbb{P}$ be the central distribution on S_{2n+1}. Then,*

$$\mathbb{P}(\{\pi\}) = \mathbb{P}(\{\pi^*\})$$

for any $\pi \in S_{2n+1}$.

Proof. Let C_π be the chip configuration corresponding to the permutation π, and let

$$\Delta^{2n+1} \to C_1 \to \cdots \to C_\pi$$

be any sequence of firing moves that ends up at C_π. If we take the dual of each entry in the above sequence, we get

$$(\Delta^{2n+1})^* \to C_1^* \to \cdots \to C_\pi^*.$$

All the moves are well-defined by symmetry. Thus, we have an equally probable sequence that starts at $(\Delta^{2n+1})^* = \Delta^{2n+1}$ and ends up at C_π^*. But the permutation corresponding to C_π^* is π^*. Hence,

$$\mathbb{P}(\{\pi\}) \leqslant \mathbb{P}(\{\pi^*\}).$$

Since $(C^*)^* = C$ for any chip configuration, the reverse inequality also holds and the result follows. □

Corollary 5.7. *Let $\mathbb{P}$ be the central distribution on S_{2n+1}. Then*

$$\mathbb{P}(\{s_{i_1}s_{i_2}\ldots s_{i_{k-1}}s_{i_k}\}) = \mathbb{P}(\{s_{2n+1-i_1}s_{2n+1-i_2}\ldots s_{2n+1-i_{k-1}}s_{2n+1-i_k}\}).$$

5.3. Permutations with Restricted Positions

In this section, we'll try to understand the set of permutations allowed by Lemma 5.5. The idea of restricting the positions of letters in a permutation is of course not new. For example, derangements are sets of permutations where the letters are not allowed to be in their original position.

More rigorously, let A be an $n \times n$ matrix consisting of zeros and ones, which is called a *board* [15]. We say P *is restricted by* A, if P is a permutation matrix satisfying $(1 - a_{ij})(p_{ij}) = 0$ for all i and j; that is, zero entries of A should be a subset of the zero entries of P. Hence, if A is the matrix with zeros on the diagonal, and ones everywhere else, then the permutations restricted by A is exactly the derangements.

Now, let us analyze the permutations restricted by the matrix of Lemma 5.5 for small n.

5.3.1. S_3

For $n = 3$, we have the following board.

$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The set of permutations restricted by A_3 is $\{321, 312, 231\}$. Remember that we write permutations in reverse order. We are interested in two problems:
(1) What is the maximum number of inversion for any of these permutations?
(2) How many permutations are in this set with maximum inversion?

Let max and count be the functions that answers these questions. So, $\max(A_3) = 1$ and $\text{count}(A_3) = 2$.

5.3.2. S_5

For $n = 5$, we have the following board.

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

It can be computed that following permutations have the maximum number of inversions among permutations restricted by A_5: $\{53142, 52341, 45132, 43512, 42531\}$.

Hence, $\max(A_5) = 3$ and $\mathrm{count}(A_5) = 5$.

5.3.3. S_7

For $n = 7$, we have the following board.

$$A_7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The following permutations have the maximum number of inversions among permutations restricted by A_7: $\{7531642, 7523641, 7451632, 7435612, 7425631, 6731542, 6723541, 6571342, 6471532, 6537142, 6527341, 6457132, 6437512, 6427531\}$. Hence, $\max(A_7) = 6$ and $\mathrm{count}(A_7) = 14$.

5.3.4. S_{2n+1}

For the general case, we have the following table verified by a computer.

Restricted Permutations		
Group	Max	Count
S_3	1	2
S_5	3	5
S_7	6	14
S_9	10	42
S_{11}	15	132
S_{13}	21	429
S_{13}	28	1430

Conjecture 1. $\max(A_{2n+1}) = \binom{n+1}{2}$.

Conjecture 2. $\text{count}(A_{2n+1}) = C_{n+1}$, where C_n is the nth Catalan number.

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