MASLOV INDICES FOR LAGRANGIAN TRIPLETS

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ABSTRACT

MASLOV INDICES FOR LAGRANGIAN TRIPLETS

In this thesis, we study the properties and an application of Maslov ternary index. By finding some special elements which generate symplectic group, we show that Maslov ternary index becomes a very important tool in the process of calculating the signature of a Lefschetz fibration. Besides, we will provide easier proofs of some known theorems by using those special elements.

ÖZET

LAGRANGE ÜÇLÜLERİ İÇİN MASLOV İNDİSLERİ

Bu tezde, Maslov üçlü indeksinin özellikleri ve bir uygulaması incelenmiştir. Simplektik grubu oluşturan bazı özel elemanları keşfederek Maslov üçlü indeksinin Lefschetz liflenmesinin işaretinin bulma sürecinde önemli bir araç olduğu gösterilmiştir. Ayrıca, bu özel elemanlarla bazı bilinen teoremlerin daha basit ispatları verilmiştir.

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LIST OF SYMBOLS

$\mathbb{C}$	The set of all complex numbers
$\mathbb{D}^2$	2-disc
D_γ	Dehn twist about the simple closed curve γ
$H_1(\Sigma, \mathbb{Z})$	First homology group with integers
$H_1(\Sigma, \mathbb{R})$	First homology group with real coefficients
$\hat{i}(.,.)$	Algebraic intersection form
$\mathcal{L}(V)$	The set of all Lagrangian subspaces of V
P_{ij}	Projection map on a Lagrangian subspace perpendicular to another Lagrangian subspace
$\mathbb{R}$	The set of all real numbers
$\mathcal{T}$	The set of all transvections on a symplectic vector space
$T_{a,u}$	Transvection with coefficient a, and vector u
$\Gamma(\Psi)$	Graph of a map Ψ
$\widetilde{\Gamma(\Psi)}$	Reversed graph of a map Ψ
δ_{ij}	Kronecker delta which is 1 if $i = j$ and 0 otherwise
$\partial\Sigma_g^b$	Boundary of Σ_g^b
$\mu(L_1, L_2, L_3)$	Maslov ternary index of L_1, L_2, L_3
$\mu_{Wall}(L_1, L_2, L_3)$	Wall's index of L_1, L_2, L_3
$\sigma(X)$	Signature of a 4 dimensional manifold X
Σ	A compact connected oriented surface
Σ_g	A closed compact connected oriented surface with genus g
Σ_g^b	A compact connected oriented surface with genus g and b boundary parts
$\omega(.,.)$	Symplectic form
W^ω	Symplectic complement of W

LIST OF ACRONYMS/ABBREVIATIONS

det	Determinant of a matrix
id	Identity map
ker	Kernel of a map
Mod	Mapping class group of a surface
n-manifold	n dimensional manifold
Q	Quadratic form of a Maslov ternary index
rank	Rank of a matrix
sign	Signature of a matrix
Sp	Symplectic group of a vector space

1. INTRODUCTION

Topologists have been interested in topological 4 dimensional manifolds, especially in the last 40 years. For dimensions up to 3, each topological manifold has unique smooth structure, and for dimensions 5 or higher, it is known that topological manifolds have finitely many smooth structures. For dimension 4, there is no information about the finiteness of smooth structures.

Let us consider $\mathbb{R}^n$, it has unique smooth structure up to diffeomorphism for $n < 4$, see [1] and [2] for the case $n = 2$ and 3 respectively. This statement is valid also for $n > 4$, see [3, Theorem 5.1]. However, for $n = 4$ there are many smooth structures. Indeed, Taubes showed that $\mathbb{R}^4$ admits uncountably many smooth structures, see [4, Theorem 1.1].

Different smooth structures have given rise to different categories of manifolds. From mathematical physics, a new category emerged in 80s. This is symplectic manifolds and they are smooth manifolds admitting a closed nondegenerate 2-form. The properties of these manifolds have attracted the attentions of many topologists to study in this area. There was a breaktrough which led us to examine the topological properties of these manifolds. Donaldson and Gompf showed that any 4-manifold admits a symplectic structure if and only if it admits a Lefschetz fibration. This implies that the topology of a symplectic 4-manifold is determined by a monodromy factorization which can be understood by 2-dimensional methods.

Topologists are mainly interested in the invariants of topological manifolds. For symplectic 4-manifolds, to compute the invariants one can look at the monodromy factorization of the corresponding Lefschetz fibration. Signature is the known simplest non-trivial invariant. The signature $\sigma(X)$ of a compact oriented 4-manifold X is the signature of the intersection form on the second homology group $\mathrm{H}_2(X;\mathbb{Z})$.

Firstly, Moishezon [5] defined the smooth Lesfchetz fibrations and studied their monodromies in the case of fiber genus 1. Matsumoto benefited from Moishezon's study and classified Lefschetz fibrations up to isomorphism [6]. Then he studied the case of fiber genus 2, and gave a signature formula (see [7]) by using Meyer's signature formula ([8]). Endo [9] extended local signature formula for genus 2 fibrations due to Matsumoto to that of hyperelliptic Lefschetz fibrations of arbitrary genus g and calculated its values.

Moreover, Endo and Nagami [10] introduced the signature for relations in mapping class groups and gave a signature formula for Lefschetz fibrations over 2-sphere by using that notion. They found that the signature is the sum of the signatures for basic relations appearing in its monodromy. Then Endo, Hasegawa, Kamada and Tanaka [11] generalized the signature formula of Endo and Nagami for Lefschetz fibrations over the 2–sphere to that for Lefschetz fibrations over a closed oriented surface of arbitrary genus.

Furthermore, Ozbagci [12] developed an algorithm to compute the signature for Lefschetz fibrations over 2-disc or 2-sphere with closed fibers by using handlebody decomposition by Kas [13] and Wall's non-additivity formula [14]. By Kas's handlebody description, the topology of symplectic 4-manifold is determined by its vanishing cycles. Ozbagci showed that the signature of a symplectic 4-manifold depends only on the algebraic properties of the vanishing cycles. This means that although vanishing cycles are defined up to isotopy, their homology classes are the only elements which determine the signature.

Currently, Çengel and Karakurt [15] developed a new algorithm for computing the signature of Lefschetz fibrations over 2-disc of any genus g with fibers which can be closed or not. They introduced the notion of *partial fiber sum decomposition* and by using this notion, they remodified Wall's non-additivity formula. In this thesis, we rewrote this formula by putting Maslov ternary index instead of Wall's index.

Cappell, Lee and Miller [16] showed that Maslov ternary index is the minus one times Wall's index. We will show that using Maslov ternary index leads to an easier calculation of the signature.

By the way, in all calculations we assume that Lefschetz fibrations have closed fibers. If we encounter a bordered Lefschetz fibration, we can make some manipulations to make fibers closed without any change in signature thanks to Çengel and Karakurt's study, see [15, Theorem 3.1].

Maslov index is an invariant for Lagrangian subspaces of symplectic vector spaces, it maps some Lagrangians to an integer. There are different definitions of Maslov index in literature. Some are defined for pairs and one is defined for triplets. Cappell, Lee and Miller [16] showed that all these definitions are equivalent. The reason why we chose ternary (triple) index is that it is directly related to the defect of the Wall's non-additivity formula, which is called Wall's index.

Kashiwara defined Maslov ternary index in a very simple way. By using this definition to the Çengel and Karakurt's formula, we will see that the signature of Lefschetz fibrations will be calculated in an easier way with the use of some simple maps, which are *positive transvections*. In this thesis, we benefited from positive transvections to prove some theorems about the signature of Lefschetz fibrations. Additionally, we prove some known theorems by using positive transvections. The first theorem is that determinant of any symplectomorphism is 1. The second one is the surjectivity of symplectic representation of mapping class groups.

In chapters 2 and 3, we will give necessary preliminaries.

In chapter 4, we define transvections and show that they generate symplectic group. We classify transvections into two, which are positive and negative transvections. We provide an easier proof of the fact that determinant of symplectomorphisms is 1 in that chapter.

In chapter 5, we define Maslov ternary index and give its properties. We give the reasons why we are interested in that index.

In chapter 6, we give information about mapping class group and its generating elements, which are Dehn twists. We show that Dehn twists are transvections on homology. We provide an easier proof of surjectivity of symplectic representation.

In chapter 7, we give information about Lefschetz fibration.

In chapter 8, we rewrite Çengel and Karakurt's formula for the signature of Lefschetz fibration by putting Maslov ternary index instead. Then, it will turn out that the positive transvections play the most important role in the process of computing the signature.

2. SYMPLECTIC VECTOR SPACES

In this chapter, we will give background to this study. For more information about this chapter, we refer the reader to Chapter 2 in [17].

Definition 2.1. *Let V be a vector space and ω be a nondegenerate skew symmetric bilinear form on V. Then the vector space V is symplectic vector space, and denoted by (V, ω).*

Definition 2.2. *The symplectic complement of a subspace W of V is defined as*

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \quad \forall w \in W\}$$

Definition 2.3. *Let u and v be elements of V. Then u and v are symplectically orthogonal if $\omega(u, v) = 0$. For any two subspaces U and W, we say U and W are symplectically orthogonal if $\omega(u, v) = 0$ for all $u \in U$ and $v \in W$.*

Theorem 2.4. *Symplectic vector spaces are even dimensional.*

Proof. Suppose that V is an m dimensional symplectic vector space. For the case $m = 1$, all vectors have to be a multiple of any nonzero vector, i.e. for any two nonzero vectors v and w, there exists a scalar number k such that $v = kw$. This implies that $\omega(v, w) = k\omega(w, w) = 0$, nondegeneracy of the symplectic form is violated. For the case $m = 2$ is trivial. Assume that $m > 2$, then there must be a symplectic vector space with dimension $m - 2$ by the following way. Nondegeneracy of the symplectic form shows that there exist u and v satisfying $\omega(u, v)$ is nonzero. Let W be the space spanned by u and v. It is easy to see that W^ω is $m - 2$ dimensional symplectic vector space. After applying this method many times, we will get a 2 dimensional symplectic vector space if m is even, and a one dimensional symplectic vector space if m is odd, which is impossible. $\square$

Theorem 2.5. [17, THEOREM 2.1.3] *Let (V, ω) be a symplectic vector space with dimension $2n$. Then there exists a basis $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ such that $\omega(u_i, u_j) = 0$, $\omega(v_i, v_j) = 0$ and $\omega(u_i, v_j) = \delta_{ij}$. This basis is called a symplectic basis for V.*

Proof. Assume V is a nonzero vector space. So there exist u_1 and $\widetilde{v_1}$ such that $\omega(u_1, \widetilde{v_1}) \neq 0$. Let it call k and if we take $v_1 = \widetilde{v_1}/k$, then $\omega(u_1, v_1) = 1$. Let W be span of u_1 and v_1. Then W^ω is $2n-2$ dimensional symplectic vector space and we can choose u_2 and v_2 by the same way. After repeating, we will find the basis which satisfy the conditions in the theorem. $\square$

Definition 2.6. $\mathbb{R}^{2n}$ *has a symplectic structure and* $(\mathbb{R}^{2n}, \omega_0)$ *denotes the symplectic vector space with the symplectic basis* $\{e_1, ..., e_n, f_1, ..., f_n\}$ *such that* $\omega_0\{e_i, f_j\} = \delta_{ij}$ *and* $\omega_0\{e_i, e_j\} = \omega_0\{f_i, f_j\} = 0$ *for all* $i, j \in \{1, ..., n\}$.

By Theorem 2.5, we can say that all symplectic vector spaces with same dimensions are isomorphic. This lead us to consider $(\mathbb{R}^{2n}, \omega_0)$ as a representation of all symplectic vector spaces with dimension $2n$.

Let W be a subspace of the symplectic vector space V. By its structure form, W^ω can be in different shapes. It can be included in W. On the contrary, it can cover W or can be equal to W.

Definition 2.7. *Let W be a subspace of symplectic vector space V, then W is*

- *Symplectic if* $W \cap W^\omega = \{0\}$
- *Isotropic if* $W \subset W^\omega$
- *Coisotropic if* $W \supset W^\omega$
- *Lagrangian if* $W = W^\omega$.

By the way, the next theorem is also very important to consider for understanding whether the subspaces are Lagrangian or not.

Theorem 2.8. [17, Lemma 2.1.1] *dim W + dim W^ω = dim V for all subspaces W of symplectic vector space V.*

Proof. Let f be the map from V to dual space V^* by defining

$$f(v) = \omega(v, .)$$

By nondegeneracy of ω, ker$f = 0$. This means f is isomorphism and dim $f(W) =$ dim W. In the dual vector space, $f(W)$ is a subspace and $f(W)^\perp$ is annihilator of W. This implies dim $f(W) +$ dim$f(W)^\perp =$ dim V^* by rank nullity. Let $\widetilde{W}$ be the subspace of V whose dual is $f(W)^\perp$. So, the elements of $\widetilde{W}$ are kernel of $f(W)$, this means $W^\omega = \widetilde{W}$ and isomorphic to $f(W)^\perp$, so dimensions are equal. Therefore,

$$dimW + dimW^\omega = dimf(W) + dimf(W)^\perp = dimV^* = dimV$$

□

An important result emerges after this theorem: Lagrangian subspaces of $2n$-dimensional symplectic vector spaces are always n-dimensional. In other words, let W be a subspace of symplectic vector space V with dim $V = 2n$, if for all v and w in W, $\omega(v, w) = 0$, and dim $W = n$, then W is a Lagrangian subspace of V.

Example 2.9. *Let (V, ω) be $2n$-dimensional symplectic vector space and has the symplectic basis $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$. Then*

(i) span$\{u_1, v_1\}$ and span$\{u_3, u_n, v_3, v_n\}$ are symplectic subspaces,

(ii) span$\{u_1\}$ and span$\{v_2, v_3\}$ are isotropic subspaces,

(iii) span$\{v_1, v_2, ..., v_n\}$ is a Lagrangian subspace,

(iv) The symplectic complements of isotropic subspaces are coisotropic subspaces.

3. SYMPLECTOMORPHISMS

In this chapter, we will give information about symplectomorphisms, which are one of the core elements of this study. For more detailed information, we refer the reader to Chapter 2 in [17].

Let (V, ω) be a symplectic vector space and $\Psi : V \to V$ be linear isomorphism. Then Ψ be a symplectomorpism if it preserves the form structure, $\Psi^{*}\omega = \omega$, in other words

$$\omega(\Psi v, \Psi w) = \omega(v, w) \quad \text{for all } \ v, w \in V$$

Theorem 3.1. *Let (V,ω) be a symplectic vector space. The set of all symplectomorphisms of (V,ω) is a group under composition of maps, called symplectic group and denoted by Sp(V).*

Proof. Since identity map preserves the form, it is included in that group. So $\mathrm{Sp}(V)$ is not empty set. Let Ψ and Φ be in $\mathrm{Sp}(V)$, then

$$\omega(\Psi\Phi v, \Psi\Phi w) = \omega(\Phi v, \Phi w) = \omega(v, w)$$

This implies $\Psi\Phi \in \mathrm{Sp}(V)$. Associativity and identity properties are satisfied obviously. All elements in $\mathrm{Sp}(V)$ are linear isomorphisms, so their inverses are also linear isomorphisms and it can be easily shown that they preserve the form structure. So, inverse property is also satisfied. □

Corollary 3.2. *Because any $2n$ dimensional symplectic vector space is isomorphic to $(\mathbb{R}^n, \omega_0)$, Sp(V) is isomorphic to the symplectic group of $\mathbb{R}^{2n}$, and denoted by Sp(2n).*

Theorem 3.3. *Sp(2n)* $= \{\Psi \in GL(2n,\mathbb{R}) \mid \Psi^T J \Psi = J\}$ *where J is* $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

Proof. Let $\Psi \in \mathrm{Sp}(2n)$ be chosen arbitrarily. For any $\alpha, \beta \in \mathbb{R}^{2n}$, we know that $\omega_0(\alpha, \beta) = \omega_0(\Psi\alpha, \Psi\beta)$. Due to $\omega_0(\alpha, \beta) = \alpha^T J \beta$, $\Psi \in \mathrm{Sp}(2n)$ is equivalent to $\Psi^T J \Psi = J$. □

3.1. The structure of Sp$(2n)$

Let Ψ be in $\mathrm{Sp}(2n)$, and a block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C and D are $n \times n$ matrices. $\Psi^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$ and $J\Psi = \begin{bmatrix} C & D \\ -A & -B \end{bmatrix}$. Then,

$$\Psi^T J \Psi = \begin{bmatrix} A^T C - C^T A & A^T D - C^T B \\ B^T C - D^T A & B^T D - D^T B \end{bmatrix} = J$$

Thus,

$$A^T C = C^T A,\ B^T D = D^T B,\ A^T D - C^T B = I. \tag{3.1}$$

Ψ is also invertible matrix but there is no need for modifying A, B, C and D. The next theorem shows determinant is nonzero, so Ψ is invertible.

Theorem 3.4. *det* $\Psi = 1$

There are many complicated proofs of Theorem 3.4 in the literature. We will see a very simple proof of this theorem later. But now, it must be seen that determinant of any symplectomorphism must be 1 or -1 by the Theorem 3.3 and preserving the structure implies 1.

Definition 3.5. *O(2n) is the group of orthogonal* $2n \times 2n$ *matrices. If* $A \in O(2n)$ *then* $A^T A = I$.

Definition 3.6. *$GL(n,\mathbb{C})$ is the group of all matrices* $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$ *in $GL(2n,\mathbb{R})$.*

Definition 3.7. *$U(n)$ is a subgroup of $GL(n,\mathbb{C})$ with the condition $X^TX + Y^TY = I$ and $X^TY = Y^TX$.*

As you noticed that we wrote a bit different definitions of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(n)$. These are isomorphisms of the actual groups. Let h be the map from $\mathrm{GL}(n,\mathbb{C})$ to $\mathrm{GL}(2n,\mathbb{R})$ as defined

$$h([X+iY]) = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$$

Obviously, h is injective and all matrices in the image of h under $\mathrm{GL}(n,\mathbb{C})$ are invertible. The decomposition below show invertibility by looking the determinants.

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}\begin{bmatrix} X+iY & 0 \\ 0 & X-iY \end{bmatrix}\begin{bmatrix} I & -iI \\ I & iI \end{bmatrix}$$

For $\mathrm{U}(n)$, if $X+iY \in U(n)$ then, $(X+iY)^H(X+iY) = I$, this means $(X^T - iY^T)(X + iY) = I$. So $X^TX + Y^TY = I$ and $X^TY = Y^TX$. Now, we are ready for the next lemma.

Lemma 3.8. [17, Lemma 2.2.1] *$Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n,\mathbb{C}) = O(2n) \cap GL(n,\mathbb{C}) = U(n)$*

Proof. Let $\Psi \in \mathrm{GL}(2n,\mathbb{R})$, then

(i) $\Psi \in \mathrm{GL}(n,\mathbb{C}) \iff \Psi J = J\Psi$ notice that J is image of iI in $\mathrm{GL}(n,\mathbb{C})$.

(ii) $\Psi \in \mathrm{Sp}(2n) \iff \Psi^T J \Psi = J$, and

(iii) $\Psi \in O(2n) \iff \Psi^T\Psi = I$.

Recognize that any two of them implies the third one. So,

(i) $\mathrm{GL}(n,\mathbb{C}) \cap \mathrm{Sp}(n) \subset O(2n)$

(ii) $\mathrm{GL}(n,\mathbb{C}) \cap O(2n) \subset \mathrm{Sp}(2n)$

(iii) $\mathrm{Sp}(2n) \cap O(2n) \subset \mathrm{GL}(n,\mathbb{C})$

One can easily show that $\mathrm{GL}(n,\mathbb{C}) \cap \mathrm{Sp}(n)$, $\mathrm{GL}(n,\mathbb{C}) \cap O(2n)$ and $\mathrm{Sp}(2n) \cap O(2n)$ are same. $\Psi \in \mathrm{GL}(n,\mathbb{C})$, so it is in the form of $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$, and also in $\mathrm{Sp}(2n)$. By putting the conditions at 3.1, we will get that any matrix in $U(n)$ satisfy all conditions and all such matrices are in $U(n)$. □

3.2. Lagrangian Subspaces

The set of all Lagrangian subspaces of (V,ω) is denoted by $\mathcal{L}(V)$. If the vector space is $(\mathbb{R}^{2n},\omega_0)$, $\mathcal{L}(V)$ is denoted by $\mathcal{L}(n)$.

Theorem 3.9. [17, Lemma 2.3.1] *Let X and Y be $n \times n$ matrices and $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$. Let Λ be the column space of Z. Then*

$$\Lambda \in \mathcal{L}(n) \iff \mathit{rankZ}{=}n \;\; \text{and} \;\; X^TY = Y^TX$$

Proof. Suppose that Λ is a Lagrangian subspace, so dimension of Λ is n. This is equivalent to $rankZ = n$ when Λ is the column space of $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$. Let us take z and z' in Λ arbitrarily. So there is some u and u' such that $z = \begin{bmatrix} X \\ Y \end{bmatrix} u$ and $z' = \begin{bmatrix} X \\ Y \end{bmatrix} u'$. Then.

$$\omega_0(z,z') = z^TJz' = u^T \begin{bmatrix} X^T & Y^T \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} u'$$

$$\omega_0(z, z') = u^T(X^T Y - Y^T X)u' = 0$$

So, $\omega_0(z, z') = 0$ if and only if $X^T Y - Y^T X = 0$, which means $X^T Y = Y^T X$. □

Definition 3.10. *The Z matrices ($2n \times n$) satisfying the conditions above are called Lagrangian frame. If the columns of Z constitute an orthonormal basis, then they are called unitary Lagrangian frame.*

Notice that

$$z^T z = \begin{bmatrix} X^T & Y^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = X^T X + Y^T Y = I \iff U = X + iY \text{ is unitary matrix.}$$

There are some properties of $\mathcal{L}(n)$.

Lemma 3.11. *If $\Lambda \in \mathcal{L}(n)$ and $\Psi \in Sp(2n)$, then $\Psi\Lambda \in \mathcal{L}(n)$.*

Proof. Since Ψ is invertible, $\Psi\Lambda$ has rank n. Let v_1 and v_2 be arbitrary elements in Λ. So, $\omega_0(v_1, v_2) = 0$ and this implies $\omega_0(\Psi v_1, \Psi v_2) = 0$. Thus $\Psi\Lambda \in \mathcal{L}(n)$. □

This lemma says that image of any Lagrangian space under any symplectomorphism is a Lagrangian space.

Lemma 3.12. [17, Lemma 2.3.2] *Assume Λ and $\Lambda' \in \mathcal{L}(n)$, then there exists a symplectic matrix $\Psi \in U(n)$ such that $\Lambda' = \Psi\Lambda$.*

Proof. Assume that Λ and Λ' are given and their unitary Lagrangian frames are Z and Z' respectively. Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$, so $X^T Y = Y^T X$ and $X^T X + Y^T Y = I$. This means if $\Psi = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$, then $\Psi \in Sp(2n) \cap O(2n)$.

For Z', similarly, if we put K and T instead of X and Y, and Θ instead of Ψ, then $\Theta = \begin{bmatrix} K & T \\ -T & K \end{bmatrix} \in Sp(2n) \cap O(2n)$. So $\Theta\Psi^{-1}(\Lambda) = \Lambda'$. $\square$

This lemma says that for any two Lagrangian spaces, there exists a symplectomorphism mapping from the one to the other one.

Some Lagrangian subspaces are created by using symplectomorphisms. We will use such spaces later in this study.

Definition 3.13. *Let (V, ω) be a symplectic vector space and $\Psi \in Sp(V)$, then $\Gamma(\Psi)$ and $\widetilde{\Gamma(\Psi)}$ denotes the graph and reversed graph of Ψ respectively. In other words,*

$$\Gamma(\Psi) = \{(u, \Psi u) | u \in V\} \quad \textit{and} \quad \widetilde{\Gamma(\Psi)} = \{(\Psi u, u) | u \in V\}$$

Theorem 3.14. *Assume (V, ω) is symplectic vector space and $\Psi \in Sp(V)$. Then $\Gamma(\Psi)$ and $\widetilde{\Gamma(\Psi)}$ are Lagrangian spaces of $V \times V$ with the symplectic form $\omega \oplus (-\omega)$.*

Proof. $V \times V$ is $4n$ dimensional symplectic vector space and dimension of the graphs is $2n$. Thus it is rest to show that any two elements are symplectically orthogonal. Let u and v be chosen arbitrarily. Then

$$(\omega \oplus (-\omega))((u, \Psi u), (v, \Psi v)) = \omega(u, v) - \omega(\Psi u, \Psi v) = 0$$

Similarly,

$$(\omega \oplus (-\omega))((\Psi u, u), (\Psi v, v)) = \omega(\Psi u, \Psi v) - \omega(u, v) = 0$$

$\square$

4. TRANSVECTIONS

In this chapter, we give information about transvections, which are the other core elements of this study. The definition and properties of transvections can be found in Chapter 3 of the book [18].

Definition 4.1. *Let $T : V \to V$ be a linear map. T is transvection with fixed hyperplane W if $T|_W = id_W$ and $T(v) - v \in W$ for all $v \in V$.*

Theorem 4.2. *Let W be a hyperplane, then there exists a nonzero vector u in V such that $W = u^{\omega}$.*

Proof. By Theorem 2.8, $\dim(W^{\omega}) = 1$. This means $W^{\omega} = \operatorname{span}\{u\}$ for some nonzero $u \in V$. Thus $u^{\omega} = (W^{\omega})^{\omega} = W$. □

Theorem 4.3. [18, P. 22,23] *Any transvection in a symplectic group can be written in the following way*

$$T_{a,u} := v \longrightarrow v + a\omega(v, u)u.$$

Here $W = u^{\omega}$ and $T(v) - v = a\omega(v, u)u \in u^{\omega}$. Conversely, for any nonzero a, and any nonzero vector u, $T_{a,u}$ is a transvection in symplectic group.

Throughout this study, we use the term *transvection* as a transvection in symplectic group, so $T_{a,u}$.

Definition 4.4. *For any transvection $T_{a,u}$, a is called the coefficient of $T_{a,u}$ and u is called the vector of $T_{a,u}$.*

Theorem 4.5. [18, THEOREM 3.4] *Sp(V) is generated by transvections.*

Let $\mathcal{T}$ be the group generated by all transvections. It must be shown that $\mathcal{T} = Sp(V)$. Firstly, we want to show that $\mathcal{T} \subset Sp(V)$.

Let $T(v) = v + a\omega(v,u)u$ be given and v_1 and v_2 be arbitrarily chosen elements of V. Then

$$\begin{aligned}
\omega(T(v_1), T(v_2)) &= \omega(v_1 + a\omega(v_1,u)u, v_2 + a\omega(v_2,u)u) \\
&= \omega(v_1, v_2 + a\omega(v_2,u)u) + \omega(a\omega(v_1,u)u, v_2 + a\omega(v_2,u)u) \\
&= \omega(v_1,v_2) + a\omega(v_2,u)\omega(v_1,u) + a\omega(v_1,u)\omega(u,v_2) \\
&= \omega(v_1,v_2) + a\omega(v_2,u)\omega(v_1,u) - a\omega(v_1,u)\omega(v_2,u) \\
&= \omega(v_1,v_2)
\end{aligned}$$

So, any transvection is actually a symplectomorphism. By induction, one can show that composition of transvections is also a symplectomorphism.

Now, we want to show $Sp(V) \subset \mathcal{T}$. Let $v \neq w \in V \backslash \{0\}$ be given. It will be shown that there is a composition of transvections which map v to w. Actually with at most 2 transvections, one can do this.

Case 1: $\omega(v,w) \neq 0$

Let a be $\frac{1}{\omega(v,w)}$ and u be $v - w$. Then

$$\begin{aligned}
T_{a,u}(v) &= v + a\omega(v,u)u \\
&= v + \frac{1}{\omega(v,w)}\omega(v, v-w)(v-w) \\
&= v + \frac{1}{\omega(v,w)}\omega(v,-w)(v-w) \\
&= v + \frac{\omega(v,-w)}{\omega(v,w)}(v-w) \\
&= v - (v-w) \\
&= w.
\end{aligned}$$

So, in this case, we have a transvection mapping v to w.

Case 2: $\omega(v, w) = 0$

In this case, we want to select $z \in V$ such that $\omega(v, z) \neq 0$ and $\omega(w, z) \neq 0$. By nondegeneracy of ω, there exist z_1 and z_2 in V such that $\omega(v, z_1) \neq 0$ and $\omega(w, z_2) \neq 0$. If $\omega(v, z_2) \neq 0$, then z can be z_2. Assume not, and $\omega(w, z_1) \neq 0$, then we can choose $z = z_1$. There is one case left, which is $\omega(v, z_2) = 0 = \omega(w, z_1)$. Then we can choose $z = z_1 + z_2$. In all cases, such z can be found. By Case 1, there exist T_1 and T_2 such that $T_1(v) = z$ and $T_2(z) = w$. $T_2T_1(v) = w$. Thus, we proved the next lemma.

Lemma 4.6. [18, Proposition 3.2] *For all v and w in V, there exists an element in $\mathcal{T}$ mapping v to w.*

Lemma 4.7. [18, Proposition 3.3] *$\mathcal{T}$ is transitive on hyperbolic pairs.*

Proof. Let (α_1, β_1) and (α_2, β_2) be hyperbolic pairs in V ($\omega(\alpha_1, \beta_1) = \omega(\alpha_2, \beta_2) = 1$). We want to find a composition of transvections such that α_1 will be mapped to α_2 and β_1 will be mapped to β_2. By Lemma 4.4, there is $T_1 \in \mathcal{T}$ such that $T_1(\alpha_1) = \alpha_2$. Now our aim is to find $T_2 \in \mathcal{T}$ such that

$$T_2(\alpha_2) = \alpha_2 \quad \text{and} \quad T_2(T_1(\beta_1)) = \beta_2$$

Case 1: $\omega(T_1(\beta_1), \beta_2) \neq 0$

Let $a = \frac{1}{\omega(T_1(\beta_1),\beta_2)}$ and $u = T_1(\beta_1) - \beta_2$, then

$$
\begin{aligned}
T_{u,a}(\alpha_2) &= \alpha_2 + a\omega(\alpha_2, u)u \\
&= \alpha_2 + \frac{\omega(\alpha_2, u)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(\alpha_2, T_1(\beta_1) - \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(\alpha_2, T_1(\beta_1)) - \omega(\alpha_2, \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(T_1(\alpha_1), T_1(\beta_1)) - \omega(\alpha_2, \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(\alpha_1, \beta_1) - \omega(\alpha_2, \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{0}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2
\end{aligned}
$$

and

$$
\begin{aligned}
T_{u,a}(T_1(\beta_1)) &= T_1(\beta_1) + a\omega(T_1(\beta_1), u)u \\
&= T_1(\beta_1) + \frac{\omega(T_1(\beta_1), u)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= T_1(\beta_1) + \frac{\omega(T_1(\beta_1), T_1(\beta_1) - \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= T_1(\beta_1) + \frac{\omega(T_1(\beta_1), -\beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= T_1(\beta_1) - u \\
&= T_1(\beta_1) - (T_1(\beta_1) - \beta_2) \\
&= \beta_2.
\end{aligned}
$$

Let this transvection be denoted by T_2. In this case, $T_2T_1 \in \mathcal{T}$ and $T_2T_1(\alpha_1) = \alpha_2$, $T_2T_1(\beta_1) = \beta_2$.

Case 2: $\omega(T_1(\beta_1), \beta_2) = 0$

By the similar way we have made before, we want to find a z such that $\omega(T_1(\beta_1), z) \neq 0$ and $\omega(\beta_2, z) \neq 0$. For this $\alpha_2 + T_1(\beta_1)$ is very good choice since

$$\omega(T_1(\beta_1), \alpha_2 + T_1(\beta_1)) = \omega(T_1(\beta_1), T_1(\alpha_1)) = -1 \quad \text{and}$$
$$\omega(\beta_2, \alpha_2 + T_1(\beta_1)) = \omega(\beta_2, \alpha_2) = -1.$$

We want to find the transvections T_3 and T_4 such that

$$\begin{array}{ll} T_3(\alpha_2) = \alpha_2 & \qquad T_3(T_1(\beta_1)) = T_1(\beta_1) + \alpha_2 \\ T_4(\alpha_2) = \alpha_2 & \qquad T_4(T_1(\beta_1) + \alpha_2) = \beta_2 \end{array}$$

Finally, we will see that $T_4T_3T_1 \in \mathcal{T}$ and $T_4T_3T_1(\alpha_1) = \alpha_2$, $T_4T_3T_1(\beta_1) = \beta_2$.

For T_3, let a be -1 and u be $-\alpha_2$, then

$$\begin{aligned} T_3(\alpha_2) &= \alpha_2 + (-1).\omega(\alpha_2, -\alpha_2)(-\alpha_2) \\ &= \alpha_2 \qquad \text{and} \end{aligned}$$

$$\begin{aligned} T_3(T_1(\beta_1)) &= T_1(\beta_1) + (-1).\omega(T_1(\beta_1), -\alpha_2)(-\alpha_2) \\ &= T_1(\beta_1) + (-1).\omega(T_1(\beta_1), -T_1(\alpha_1))(-\alpha_2) \\ &= T_1(\beta_1) + (-1).(-1).(-1)(-\alpha_2) \\ &= T_1(\beta_1) + \alpha_2. \end{aligned}$$

For T_4, let a be 1 and u be $\alpha_2 + T_1(\beta_1) - \beta_2$, then

$$\begin{aligned}
T_4(\alpha_2) &= \alpha_2 + \omega(\alpha_2, \alpha_2 + T_1(\beta_1) - \beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + \omega(\alpha_2, T_1(\beta_1) - \beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + [\omega(T_1(\alpha_1), T_1(\beta_1)) - \omega(\alpha_2, \beta_2)](\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + 0.(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
T_4(\alpha_2 + T_1(\beta_1)) &= \alpha_2 + T_1(\beta_1) + \omega(\alpha_2 + T_1(\beta_1), \alpha_2 + T_1(\beta_1) - \beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + \omega(\alpha_2 + T_1(\beta_1), -\beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + [\omega(\alpha_2, -\beta_2) + \omega(T_1(\beta_1), -\beta_2)](\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + \omega(\alpha_2, -\beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + (-1)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \beta_2.
\end{aligned}$$

□

Remark 4.8. *Let T_1 be a transvection, then there is T_2 such that $T_2T_1 = id$. If $T_1 = T_{a,u}$, then T_2 must be $T_{-a,u}$. Assume θ_1 is an element of $\mathcal{T}$ and θ_2 is inverse of θ_1. So if $\theta_1 = T_1T_2...T_n$, then θ_2 must be $T_n^{-1}T_{n-1}^{-1}...T_1^{-1}$. Thus, every element in $\mathcal{T}$ has an inverse in $\mathcal{T}$.*

Proof of Theorem 4.5. If $\Psi \in Sp(V)$ and $\{u_1, v_1, u_2, v_2, ..., u_n, v_n\}$ is a symplectic basis of V, then there is an element in $\mathcal{T}$, let it be denoted by θ_1, such that $\theta_1(\Psi u_1) = u_1$ and $\theta_1(\Psi v_1) = v_1$. Let $W_1 := span\{u_1, v_1\}$. $\theta_1\Psi$ is identity on W_1. W_1^{ω} is symplectic vector space with basis $\{u_2, v_2, ..., u_n, v_n\}$. So, $\theta_1\Psi \in Sp(W_1^{\omega})$.

Assume that there are similar k elements such that $\theta_k...\theta_1\Psi$ is identity on $W_k := span\{u_1, ..., u_k, v_1, ..., v_k\}$. So, $\theta_k...\theta_1\Psi \in \mathrm{Sp}(W_k^\omega)$. There is an element in $\mathcal{T}$, $\theta_{k+1} \in \mathrm{Sp}(W_k^\omega)$ mapping Ψu_{k+1} to u_{k+1}, and Ψv_{k+1} to v_{k+1}. Notice that any vector in W_k^ω is symplectically orthogonal to W_k, this means hyperplanes of the transvections in W_k^ω cover W_k. This implies that these transvections are identity on W_k. So $\theta_{k+1}\theta_k...\theta_1\Psi$ is identity on $W_{k+1} := \{u_1, ..., u_{k+1}, v_1, ..., v_{k+1}\}$. By induction, we showed $\theta_n\theta_{n-1}...\theta_1\Psi$ is identity. By Remark 4.6, $\Psi \in \mathcal{T}$. □

Theorem 4.9. *Let $T_{a,u}$ be a transvection. Then the only eigenvalue of $T_{a,u}$ is* 1*.*

Proof. Let x be a nonzero vector such that $T_{a,u}(x) = \lambda x$. After applying the definition of transvections, we get that

$$T_{a,u}(x) = \lambda x \rightarrow x + a\omega(x,u)u = \lambda x \rightarrow a\omega(x,u)u = (\lambda - 1)x \tag{4.1}$$

If u and x are linearly independent, then λ must be 1. If they are linearly dependent, then $\omega(x,u) = 0$, so λ is again 1. In all cases, eigenvalue is 1. □

Corollary 4.10. *All transvections in Sp(2n) have determinant 1.*

Proof. By Theorem 4.9, product of all eigenvalues is 1, so determinant is 1. □

Remark 4.11. *Now, we are ready to give a very simple proof of Theorem 3.4. By Theorem 4.5, determinant of a symplectomorphism is equal to determinant of some transvections, which is always 1.*

4.1. Some Properties of Transvections

Theorem 4.12 (Conjugacy)**.** *Assume (V, ω) is symplectic vector space and $\Psi \in Sp(V)$, then for any transvection $T_{a,u}$*

$$T_{a,\Psi u} = \Psi T_{a,u} \Psi^{-1} \tag{4.2}$$

Proof. $T_{a,\Psi u}(v) = v + a\omega(v, \Psi u)\Psi u = \Psi\Psi^{-1}(v + a\omega(v, \Psi u)\Psi u) = \Psi(\Psi^{-1}v + a\omega(v, \Psi u)u)$ $\omega(v, \Psi u) = \omega(\Psi^{-1}v, u)$ implies that $v + a\omega(v, \Psi u)\Psi u = \Psi(T_{a,u}\Psi^{-1}u)$. Because v is chosen arbitrarily in V, the equation 4.2 holds. □

Theorem 4.13 (Commutativity)**.** *Assume that (V, ω) is symplectic vector space, T_{a_1,u_1} and T_{a_2,u_2} are transvections. If $\omega(u_1, u_2) = 0$, then*

$$T_{a_1,u_1}T_{a_2,u_2} = T_{a_2,u_2}T_{a_1,u_1}$$

Proof. This theorem can be proven easily by using the definition of transvections. □

Corollary 4.14. *By Theorem 4.13, k transvections commute if their vectors are symplectically orthogonal.*

If any two transvections are the same maps, then, their vectors must be linearly dependent, i.e. one of them is a multiple of the other one. However, for the coefficients of transvections, we can do any change but the sign of the numbers cannot be changed, since any multiple of the vector contributes to the coefficient by square of that number. The next theorem clarifies this situation. Yet first, we should define the types of the transvections.

Definition 4.15. *The transvections T_{-1,u_1} and T_{1,u_2} are called positive and negative transvections respectively.*

Although the signs of coefficients and the names of this transvections are opposite, there is a fair reason for it. We will see this reason later.

Theorem 4.16. *Any transvection on the symplectic vector space is a positive transvection or a negative transvection.*

Proof. If $a = 0$, $T_{a,u}$ is identity, so not transvection. Assume that a is nonzero.

Let $a > 0$ and $u \in V$, then

$$T_{a,u}(v) = v + a\omega(v,u)u = v + \omega(v, \sqrt{a}u)\sqrt{a}u = T_{1,\sqrt{a}u}(v)$$

Similarly,

$$T_{-a,u}(v) = v - a\omega(v,u)u = v - \omega(v, \sqrt{-a}u)\sqrt{-a}u = T_{-1,\sqrt{-a}u}(v)$$

Thus, the proof is completed. $\square$

Theorem 4.17. *The symplectic vector space V is generated by positive transvections.*

Proof. Theorem 4.5 and 4.16 implies that $\mathrm{Sp}(V)$ is generated by positive and negative transvections. Remark 4.8 says that negative transvections are inverse of positive transvections. $\square$

5. MASLOV INDEX

In the literature, there are different definitions of Maslov index which is an invariant for Lagrangian spaces. Some of them are defined on pairs and one of them is defined on triplets, which we will focus on completely. This is called *Maslov ternary index* or *Maslov triple index*. In their comprehensive study [16], Cappell, Lee and Miller showed that these different definitions satisfy the same system of axioms and hence is equivalent to each other. Kashiwara defined the Maslov ternary index and his definition looks very simple. For the details of Maslov ternary index, we refer the reader to Chapter 1.5 of [19].

Definition 5.1. [19, Definition 1.5.1 (Kashiwara)] *Assume (V, ω) is symplectic vector space and L_1, L_2 and L_3 are Lagrangian subspaces of V. The Maslov index $\mu(L_1, L_2, L_3)$ is an integer and defined as the signature of the quadratic form*

$$Q : L_1 \oplus L_2 \oplus L_3 \to \mathbb{R}$$
$$Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$$

Theorem 5.2. $\mu(L_1, L_2, L_3) = \epsilon(\sigma)\mu(L_{\sigma(1)}, L_{\sigma(2)}, L_{\sigma(3)})$ *where σ is any permutation on $\{1, 2, 3\}$, and ϵ is $+1$ if number of elementary permutations is even and -1 otherwise.*

Proof. Without loss of generality, assume we change places of L_1 and L_2, then

$$Q(x_2, x_1, x_3) = \omega(x_2, x_1) + \omega(x_1, x_3) + \omega(x_3, x_2) = -Q(x_1, x_2, x_3)$$

This is equaivalent to say that

$$Q(x_1, x_1, x_3) = \epsilon(\sigma)Q(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$$

which proves theorem. ∎

Remark 5.3. *There are two equivalent definitions of the signature of quadratic forms. From the given variables of quadratic forms, we can create new variables* $A_1, ..., A_k$ *such that the quadratic form is equal to* $\sum_{i=1}^{k} a_i A_i^2$ *where* $a_i = \pm 1$*. First definition is that the number of* $a_i = +1$*'s minus number of* $a_i = -1$*'s is signature. This definition help us to understand Theorem 5.2. The second definition is that the signature of the correspondent symmetric matrix of the quadratic form is signature. The latter will be more helpful to understand the next theorems, indeed the all theorems.*

Another proof of Theorem 5.2. Assume A is the correspondent symmetric matrix of Q, then if we permute arbitrarily two elements, then $-A$ become the correspondent matrix of the new quadratic form. If Λ is the diagonal matrix of A by applying simultaneous row and column operations, then

$$A = P^T \Lambda P \;\Rightarrow\; -A = P^T(-\Lambda)P \;\Rightarrow\; \text{all signs of diagonal elements change.}$$

This proves the theorem. □

Theorem 5.4. *Assume that* (V, ω_1) *and* (W, ω_2) *are symplectic vector spaces and* $L_i \subset V$*,* $L_i^\# \subset W$ *are Lagrangian subspaces for* $i = 1, 2, 3$*. Then*

$$\mu_{V\oplus W}(L_1 \oplus L_1^\#, L_2 \oplus L_2^\#, L_3 \oplus L_3^\#) = \mu_V(L_1, L_2, L_3) + \mu_W(L_1^\#, L_2^\#, L_3^\#) \tag{5.1}$$

Proof. If $\{u_1, ..., u_n, v_1, ..., v_n\}$ is a symplectic basis of V and $\{u'_1, ..., u'_m, v'_1, ..., v'_m\}$ is a symplectic basis of W. Then $V \oplus W$ is symplectic vector space with the form

$$\omega(\alpha_1 + \beta_1, \alpha_2 + \beta_2) = \omega_1(\alpha_1, \alpha_2) + \omega_2(\beta_1, \beta_2)$$

where $\alpha_1, \alpha_2 \in V$ and $\beta_1, \beta_2 \in W$. The basis of $V \oplus W$ become $\{u_i + u'_j, v_i + v'_j \mid i = 1, 2, ..., n,\;\; j = 1, 2, ..., m\}$.

Let Q_1 and Q_2 be the quadratic forms on V and W respectively, then the quadratic form on $V \oplus W$ become

$$Q(x_1 + x_1', x_2 + x_2', x_3 + x_3') = Q_1(x_1, x_2, x_3) + Q_2(x_1', x_2', x_3')$$

If A is symmetric matrix of Q_1 and B is symmetric matrix of Q_2, then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is symmetric matrix of Q. So after diagonalization we will get $\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$. This implies 5.1. $\square$

Theorem 5.5. *For all* $\Psi \in Sp(V)$, *then*

$$\mu(L_1, L_2, L_3) = \mu(\Psi L_1, \Psi L_2, \Psi L_3) \tag{5.2}$$

Proof. Under any symplectomorphism, quadratic form cannot change, so the signature too. $\square$

Theorem 5.6. *Assume* $(\mathbb{C}, \omega)$ *is symplectic vector space such that* $\omega(1, i) = 1$, *then*

$$\mu_{\mathbb{C}}(\mathbb{R}, \mathbb{R}(1+i), \mathbb{R}(i)) = 1 \tag{5.3}$$

where $\mathbb{R}(z) := \{az \mid a \in \mathbb{R}\}$.

Proof. Assume $a \in \mathbb{R}$, $b + bi \in \mathbb{R}(1+i)$ and $ci \in \mathbb{R}(i)$ where a, b and $c \in \mathbb{R}$. Then the quadratic formula of the three Lagrangian subspaces $\mathbb{R}, \mathbb{R}(1+i)$ and $\mathbb{R}(i)$ is,

$$\begin{aligned} Q(a, b+bi, ci) &= \omega(a, b+bi) + \omega(b+bi, ci) + \omega(ci, a) \\ &= \omega(a, bi) + \omega(b, ci) + \omega(ci, a) \\ &= ab + bc - ac \end{aligned}$$

The matrix representation of this quadratic form is

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{5.4}$$

So the similar matrix of the quadratic form is

$$\begin{bmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

To get diagonal matrix, we should apply all row operations and column operations at the same time,

$$\begin{bmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} \xrightarrow{R_2+R_3=R_3} \begin{bmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \xrightarrow{C_2+C_3=C_3} \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_1+R_3=R_3} \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-C_1+C_3=C_3} \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1+R_2=R_1]{C_1+C_2=C_1}$$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{R_1}{2}-R_2=R_2} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{C_1}{2}-C_2=C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After applying same row and column operations simultaneously, we get our diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The diagonal entries are $1, -1/4$ and 1.

There are 2 positive numbers and 1 negative number. So the signature is $2-1=1$. $\square$

Theorem 5.7. [19, Proposition 1.5.8] *Assume L_1, L_2, L_3 and L_4 are Lagrangian spaces, then*

$$\mu(L_1, L_2, L_3) = \mu(L_1, L_2, L_4) + \mu(L_1, L_4, L_3) + \mu(L_4, L_2, L_3) \tag{5.5}$$

The proof of this theorem is not straightforward. Therefore we will sketch the proof and show some lemmas which directly affect the theorem. Transversality of spaces means that they have only one common element which is zero vector. We will first assume that L_4 is transverse to L_1, L_2 and L_3 and prove the theorem. Then we will find a Lagrangian space which is transverse to L_1, L_2, L_3, L_4 and by using the first case, we will prove this theorem for all Lagrangian spaces.

Assume L_1, L_2, L_3 and L_4 are Lagrangian subspaces of V such that $L_i \cap L_4 = \{0\}$ for $i \in \{1, 2, 3\}$. So $L_i \oplus L_4 = V$.

Suppose $x_1 \in L_1$, $x_2 \in L_2$, $x_3 \in L_3$ and y_1, y_2, y_3 are constructed in the following way

$$\begin{aligned}
y_1 &= \frac{1}{2}(x_1 - P_{14}x_2 + P_{14}x_3) \\
y_2 &= \frac{1}{2}(x_2 - P_{24}x_3 + P_{24}x_1) \\
y_3 &= \frac{1}{2}(x_3 - P_{34}x_1 + P_{34}x_2)
\end{aligned}$$

where P_{i4} is the projection map on L_i perpendicular to L_4 for $i \in \{1, 2, 3\}$. Notice that $y_1 \in L_1, y_2 \in L_2$, and $y_3 \in L_3$.

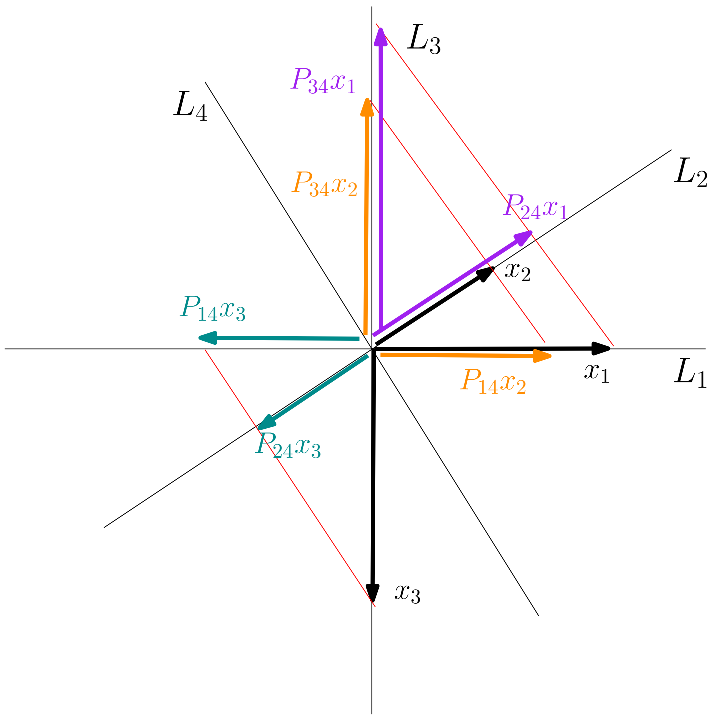

Figure 5.1. The vectors and their projections on L_i's

$$
\begin{aligned}
P_{14}y_2 &= P_{14}(\frac{1}{2}(x_2 - P_{24}x_3 + P_{24}x_1)) \\
&= \frac{1}{2}P_{14}(x_2 - P_{24}x_3 + P_{24}x_1) \\
&= \frac{1}{2}(P_{14}x_2 - P_{14}P_{24}x_3 + P_{14}P_{24}x_1) \\
&= \frac{1}{2}(P_{14}x_2 - P_{14}x_3 + x_1)
\end{aligned}
$$

So, $x_1 = y_1 + P_{14}y_2$.

$$
\begin{aligned}
P_{24}y_3 &= P_{24}(\frac{1}{2}(x_3 - P_{34}x_1 + P_{34}x_2)) \\
&= \frac{1}{2}P_{24}(x_3 - P_{34}x_1 + P_{34}x_2) \\
&= \frac{1}{2}(P_{24}x_3 - P_{24}P_{34}x_1 + P_{24}P_{34}x_2) \\
&= \frac{1}{2}(P_{24}x_3 - P_{24}x_1 + x_2)
\end{aligned}
$$

So, $x_2 = y_2 + P_{24}y_3$.

$$\begin{aligned} P_{34}y_1 &= P_{34}(\frac{1}{2}(x_1 - P_{14}x_2 + P_{14}x_3)) \\ &= \frac{1}{2}P_{34}(x_1 - P_{14}x_2 + P_{14}x_3) \\ &= \frac{1}{2}(P_{34}x_1 - P_{34}P_{14}x_2 + P_{34}P_{14}x_3) \\ &= \frac{1}{2}(P_{34}x_1 - P_{34}x_2 + x_3) \end{aligned}$$

So, $x_3 = y_3 + P_{34}y_1$.

Let Q' be a quadratic form on $L_1 \oplus L_2 \oplus L_3$ constructed in the following way

$$Q'(y_1, y_2, y_3) = \omega(P_{14}y_2, y_2) + \omega(P_{24}y_3, y_3) + \omega(P_{34}y_1, y_1)$$

Lemma 5.8. $\omega(y_1, y_2) + \omega(y_2, P_{34}y_1) + \omega(P_{34}y_1, P_{14}y_2) = 0$

Proof. $y_2 = P_{14}y_2 + P_{41}y_2$ implies that

$$\begin{aligned} &\omega(y_1, y_2) + \omega(y_2, P_{34}y_1) + \omega(P_{34}y_1, P_{14}y_2) \\ &= \omega(y_1, P_{14}y_2 + P_{41}y_2) + \omega(P_{14}y_2 + P_{41}y_2, P_{34}y_1) + \omega(P_{34}y_1, P_{14}y_2) \\ &= \omega(y_1, P_{41}y_2) + \omega(P_{14}y_2, P_{34}y_1) + \omega(P_{41}y_2, P_{34}y_1) + \omega(P_{34}y_1, P_{14}y_2) \\ &= \omega(y_1, P_{41}y_2) + \omega(P_{41}y_2, P_{34}y_1) \\ &= \omega(y_1, P_{41}y_2) + \omega(P_{41}y_2, P_{34}y_1 + P_{43}y_1) \\ &= \omega(y_1, P_{41}y_2) + \omega(P_{41}y_2, y_1) \\ &= 0 \end{aligned}$$

□

By Lemma 5.8, we can write

$$\omega(y_2, y_3) + \omega(y_3, P_{14}y_2) + \omega(P_{14}y_2, P_{24}y_3) = 0$$
$$\omega(y_3, y_1) + \omega(y_1, P_{24}y_3) + \omega(P_{24}y_3, P_{34}y_1) = 0$$

So now, we can state the next proposition

Proposition 5.9. *$Q'(y_1, y_2, y_3)$ is equivalent to $Q(x_1, x_2, x_3)$.*

Proof. $Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$ implies that

$$\begin{aligned}
&Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)\\
&= \omega(y_1 + P_{14}y_2, y_2 + P_{24}y_3) + \omega(y_2 + P_{24}y_3, y_3 + P_{34}y_1) + \omega(y_3 + P_{34}y_1, y_1 + P_{14}y_2)\\
&= \omega(y_1, y_2) + \omega(y_2, P_{34}y_1) + \omega(P_{34}y_1, P_{14}y_2)\\
&+ \omega(y_2, y_3) + \omega(y_3, P_{14}y_2) + \omega(P_{14}y_2, P_{24}y_3)\\
&+ \omega(y_3, y_1) + \omega(y_1, P_{24}y_3) + \omega(P_{24}y_3, P_{34}y_1)\\
&+ \omega(y_1, y_2) + \omega(y_2, y_3) + \omega(y_3, y_1)\\
&= \omega(y_1, y_2) + \omega(y_2, y_3) + \omega(y_3, y_1)\\
&= Q'(y_1, y_2, y_3)
\end{aligned}$$

□

If one of the Lagrangian subspaces is L_4, Q' becomes,

- $Q'(y_1, y_2, y_3) = \omega(P_{14}y_2, y_2) + \omega(P_{34}y_1, y_1)$
- $Q'(y_1, y_4, y_3) = \omega(P_{24}y_3, y_3) + \omega(P_{34}y_1, y_1)$
- $Q'(y_4, y_2, y_3) = \omega(P_{14}y_2, y_2) + \omega(P_{24}y_3, y_3)$

Then, their sum is

$$
\begin{aligned}
&Q'(y_1,y_2,y_3)+Q'(y_1,y_4,y_3)+Q'(y_4,y_2,y_3)\\
&=\omega(P_{14}y_2,y_2)+\omega(P_{34}y_1,y_1)+\omega(P_{24}y_3,y_3)+\omega(P_{34}y_1,y_1)+\omega(P_{14}y_2,y_2)+\omega(P_{24}y_3,y_3)\\
&=2(Q'(y_1,y_2,y_3))
\end{aligned}
$$

By using Proposition 5.9, the above equation implies that

$$\mu(L_1,L_2,L_3)=\mu(L_1,L_2,L_4)+\mu(L_1,L_4,L_3)+\mu(L_4,L_2,L_3)$$

if L_4 is transverse to the other ones.

Now assume that L_4 is not transverse to the other Lagrangian spaces. Then, Theorem 5.11 guarantees the existence of a Lagrangian space which is transverse to L_i for $i \in \{1,2,3,4\}$.

Lemma 5.10. *Let $L_1,...,L_n$ be some Lagrangian spaces. Then there is an element $v \in V$ such that $v \notin L_i$ for all $i \in \{1,...,n\}$.*

Proof. This lemma can be proven by induction, $k=1$ is obvious. Assume that $v \notin L_i$ for all $i \in \{1,...,k\}$. We want to show that if we add L_{k+1}, then we can still find such vector. If $v \notin L_{k+1}$, then we are done. Assume $v \in L_{k+1}$, by nondegeneracy of the form, there exists $u \in V$ such that $\omega(v,u)=1$. This means that $u \notin L_{k+1}$. In fact, $tv+u \notin L_{k+1}$ for all real numbers t. Thus, this vector can be in the form of $tv+u$, but firstly this vector must not be in the previous Lagrangian spaces.

If $v+u$ is in one of the Lagrangian spaces, then we can choose $(v+u)+v$. If this vector is in another Lagrangian space, we can choose $(2v+u)+v$. Since there are finitely many Lagrangian spaces, we can find a vector $tv+u$ which is not in any L_i for all $i \in \{1,...,k\}$.

Thus there exists a vector which is not included in one of the Lagrangian spaces $L_1, ..., L_{k+1}$. $\square$

Theorem 5.11. *For the Lagrangian spaces $L_1, L_2, ..., L_n$, there exists another Lagrangian space L_{n+1} which is transverse to them.*

Proof. By the previous lemma, there exists $v \in V$ such that $v \notin L_i$ for all $i \in \{1, ..., n\}$ and let R denote span$\{v\}$. So R is an isotropic subspace of V which is transversal to $\{L_1, ..., L_n\}$. Let K be the set of all isotropic subspaces including R and transversal to $\{L_1, ..., L_n\}$. Since $R \in K$, then K is not empty set. Let $L' \in K$ such that there is no element in K whose proper subset is L'. This means L' is maximal isotropic subspace including R and transversal to $\{L_1, ..., L_n\}$. The definition of isotropic subspace implies $L' \subset (L')^\omega$. Suppose that $L' \neq (L')^\omega$.

Let us choose $\alpha \in (L')^\omega \backslash L'$ arbitrarily. This implies $\omega(x, \alpha) = 0$ for all $x \in L'$. Let us consider $L' \oplus \{\alpha\}$. This is isotropic subspace containing R and L' is proper subspace of it. Maximality of L' implies that $L' \oplus \{\alpha\}$ is not in K. So this subspace is not transverse to $\{L_1, ..., L_n\}$. There exists at least one $j \in \{1, ..., n\}$ such that

$$(L' \oplus \{\alpha\}) \cap L_j \neq \{0\}$$

This implies that

$$v + \alpha \in L_j \text{ for some } v \in L'$$

So $\omega(v + \alpha, x) = 0$ for all $x \in L_j$. L' is transverse to L_j, so $v \notin L_j$. This means there exist $y \in L_j$ such that $\omega(v, y) = 1$. Then we get that

$$0 = \omega(v + \alpha, y) = \omega(v, y) + \omega(\alpha, y) = 1 + \omega(\alpha, y) \to \omega(\alpha, y) = -1 \neq 0$$

So $\alpha \notin L_j$. Since α can be choosen in $(L')^\omega \backslash L'$ arbitrarily, we can say that

$$((L')^\omega \backslash L') \cap L_j = \{0\}$$

This is equivalent to

$$(L')^\omega \cap L_j = \{0\}$$

But $v + \alpha \in L_j$ and $v + \alpha \in (L')^\omega$ since $v \in L' \subset (L')^\omega$ and $\alpha \in (L')^\omega$. Contradiction! So $L' = (L')^\omega$ this means L' is Lagrangian subspace. We can choose $L_{k+1} = L'$. $\square$

Proof of Theorem 5.7. The case in which L_4 is transversal to L_i for all $i \in \{1, 2, 3\}$ is proven. Assume transversality condition is not hold. Then we can choose L_5 to be transverse to L_1, L_2, L_3 and L_4 by the Theorem 5.11. Then

$$\mu(L_1, L_2, L_4) = \mu(L_1, L_2, L_5) + \mu(L_1, L_5, L_4) + \mu(L_5, L_2, L_4)$$
$$\mu(L_1, L_4, L_3) = \mu(L_1, L_4, L_5) + \mu(L_1, L_5, L_3) + \mu(L_5, L_4, L_3)$$
$$\mu(L_4, L_2, L_3) = \mu(L_4, L_2, L_5) + \mu(L_4, L_5, L_3) + \mu(L_5, L_2, L_3)$$

Their sum is

$$\begin{aligned}
&\mu(L_1, L_2, L_4) + \mu(L_1, L_4, L_3) + \mu(L_4, L_2, L_3) \\
&= \mu(L_1, L_2, L_5) + \mu(L_1, L_5, L_3) + \mu(L_5, L_2, L_3) \\
&= \mu(L_1, L_2, L_3)
\end{aligned}$$

$\square$

By the study of Cappell, Lee and Miller [16, Theorem 8.1], there is a unique system of functions $\mu(L_1, L_2, L_3)$ which satisfies the properties which are stated in Theorem 5.2, 5.4, 5.5 and 5.6. Any such system equals Kashiwara's definition of Maslov ternary index, and so the property which stated in Theorem 5.7 is also satisfied.

Let L_1, L_2 and L_1', L_2' be transverse Lagrangian pairs of same symplectic vector space V. Since these pairs give some symplectic basis for V, we find a symplectomorphism mapping L_i to L_i' for $i \in \{1, 2\}$. This means that $\mathrm{Sp}(V)$ is transitive on transverse Lagrangian pairs. However, $\mathrm{Sp}(V)$ is not transitive on Lagrangian triplets. By Theorem 5.5, there is no symplectomorphism which maps one triplet to the other one if their indices are not same. Maslov ternary index determines the configuration of these Lagrangian subspaces.

6. MAPPING CLASS GROUPS

In this chapter, we will give some information about mapping class group. We will see the definition and some properties of Dehn twists which are another core elements of this study. We will recognize a correspondence between Dehn twists and transvections. For the definitions and theorems, we follow this book [20].

Let Σ be a compact connected orientable surface. This implies that Σ has some holes and some cut parts. These are called *genus* and *boundary parts*, respectively. We denote such a surface by Σ_g^b, g genus b boundary parts. Throughout this study, we use the term *surface* in referring to the compact connected orientable surfaces, and we are mainly concerned about the closed (no boundary parts) surfaces. Therefore we usually use the notation Σ_g instead of Σ_g^0.

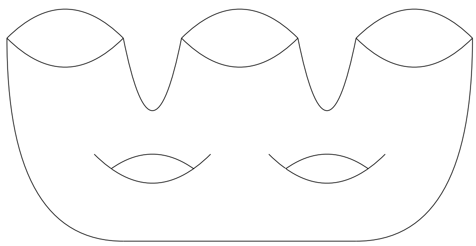

Figure 6.1. Σ_2^3 surface

Definition 6.1. *Let $f : \Sigma_g^b \to \Sigma_g^b$ be an orientation preserving homeomorphism and induced map on the boundary is identity. Such maps create a group called mapping class group up to isotopy. The mapping class group of Σ_g^b is denoted by Mod(Σ_g^b).*

In other words, $\mathrm{Mod}(\Sigma_g^b)$ is the group of all isotopy classes of orientation preserving self diffeomorphisms of Σ_g^b, which is identity on $\partial\Sigma_g^b$. We use the term diffeomorphism instead of homeomorphism due to the fact that any homeomorphism is homotopic to a diffeomorphism in the surface.

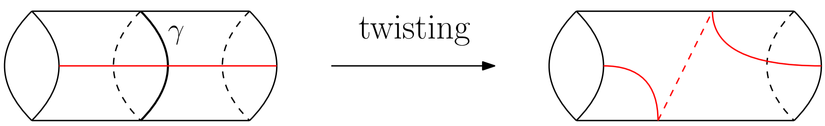

Figure 6.2. Dehn twist D_γ

In this study, we are mainly interested in the generator set of the mapping class groups, which are called *Dehn twists*. Therefore we will talk more about Dehn twists under the next heading.

6.1. Dehn Twists

Definition 6.2. *Consider a simple closed curve γ on a surface. In the closed tubular neighborhood of γ, Dehn twist about γ is a self-homeomorphism on that surface, made by regluing after $360°$ twisting of that closed tubular neigborhood, as in Figure 6.2.*

Remark 6.3. *Twisting right or left gives two different Dehn twists. The former is called right Dehn twist and the latter is called left Dehn twist. Twisting right and left gives identity map in the mapping class group. So we can say that left Dehn twists are inverse of right Dehn twists.*

Definition 6.4. *Let γ be a simple closed curve on a surface. Then γ is called separating if the new surface is disconnected after cutting γ, and nonseparating if the new surface is still connected.*

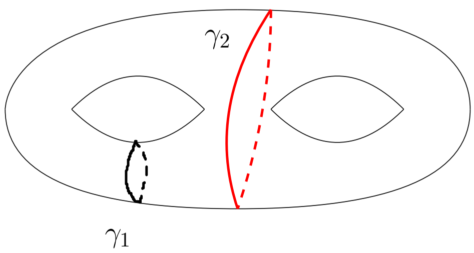

Figure 6.3. γ_1 is nonseparating curve and γ_2 is separating curve

For any simple closed curve γ in a surface, D_γ represents right Dehn twists and D_γ^{-1} represents left Dehn twists. We are mainly interested in right Dehn twists.

By Dehn-Lickorish theorem [20, Theorem 4.1], Dehn twists generate the mapping class group of a surface. For a closed surface, although there are many studies giving a finite number of Dehn twists, Humphries [21] showed that $2g+1$ Dehn twists generate $\mathrm{Mod}(\Sigma_g)$ and this number is the most favorable one in the literature.

There are some properties of Dehn twists which deserve attention. For the details and proofs, see [20]. First one is that for disjoint simple closed curves γ_1 and γ_2, $D_{\gamma_1}D_{\gamma_2} = D_{\gamma_2}D_{\gamma_1}$. By induction, we conclude that Dehn twists about disjoint curves commute. The second property is conjugation. For any $f \in \mathrm{Mod}(\Sigma_g)$, we have $fD_\gamma f^{-1} = D_{f(\gamma)}$.

6.2. Symplectic Representation

Definition 6.5. *For given Σ_g, the algebraic intersection map $\hat{i}(.,.)$ is defined on the first homology group of Σ_g, denoted by $H_1(\Sigma_g, \mathbb{Z})$ and satisfy the following properties*

(i) For all $a, b \in H_1(\Sigma_g, \mathbb{Z})$, $\hat{i}(a,b) = -\hat{i}(b,a)$, $\hat{i}$ is skew symmetric.
(ii) For all $a, b \in H_1(\Sigma_g, \mathbb{Z})$, let α and β represent a and b respectively. Then $\hat{i}(a,b)$ is the sum of signed intersections of α and β curves.

Theorem 6.6. *Let Σ_g be given and γ be separating curve, then the sum of the signed intersection with any curve in the surface is zero.*

For convention, let $[a] \in \mathrm{H}_1(\Sigma_g, \mathbb{Z})$ is the homology class representing the oriented simple closed curve a. For any separating curve γ, $[\gamma] = 0$. By Theorem 6.6, we can get a result that $\hat{i}$ is nondegenerate.

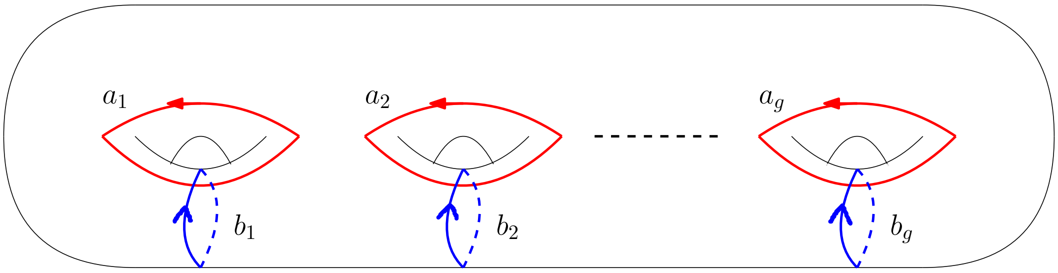

Figure 6.4. Geometric symplectic basis for $\mathrm{H}_1(\Sigma_g, \mathbb{R})$

Let us consider the ordered basis $\{[a_1], [b_1], ..., [a_g], [b_g]\}$ for $\mathrm{H}_1(\Sigma_g, \mathbb{R})$ in Figure 6.4. The algebraic intersection map extends to a nondegenerate skew symmetric bilinear form

$$\hat{i} : \mathrm{H}_1(\Sigma_g, \mathbb{R}) \oplus \mathrm{H}_1(\Sigma_g, \mathbb{R}) \to \mathbb{R} \tag{6.1}$$

With this structure, $(\mathrm{H}_1(\Sigma_g, \mathbb{R}), \hat{i})$ is $2g$ dimensional symplectic vector space and the basis $\{[a_1], [b_1], ..., [a_g], [b_g]\}$ is symplectic basis.

This collection of the curves $a_1, ..., a_g, b_1, ..., b_g$ is called *geometric symplectic basis* for $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$.

The action of $\mathrm{Mod}(\Sigma_g)$ on $\mathrm{H}_1(\Sigma_g, \mathbb{R})$ preserves the structure of $\hat{i}$. This yields a representation

$$Mod(\Sigma_g) \to Sp(2g)$$

and this representation is called the symplectic representation of $\mathrm{Mod}(\Sigma_g)$.

Theorem 6.7. *The representation $Mod(\Sigma_g) \to Sp(2g)$ is surjective.*

Before proving this theorem, we must look at the actions of the generators of $\mathrm{Mod}(\Sigma_g)$ on $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$. By [20, Proposition 6.3],

$$D_\gamma(a) = a - \hat{i}(a, [\gamma]).[\gamma] \text{ where } a \in \mathrm{H}_1(\Sigma_g, \mathbb{Z}) \tag{6.2}$$

This is exactly what transvections do in a symplectic vector space. We get the result that a right Dehn twist about a nonseparating curve is a positive transvection and left Dehn twist about nonseparating curve is a negative transvection on $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$. Besides, Dehn twists and transvections have some common properties, which are conjugacy and commutativity properties.

Remark 6.8. *Dehn twists about separating curves are trivial on $H_1(\Sigma_g, \mathbb{Z})$. By Theorem 6.6 and the equation in 6.2 gives this result.*

Proof of Theorem 6.7. Theorem 4.16 says that the images of all Dehn twists about nonseparating curves generate $\mathrm{Sp}(V)$. $\square$

Although the Dehn twists about separating curves are trivial on homology, they are nontrivial and infinite order in the mapping class group. The kernel of the symplectic representation is nontrivial for $g > 1$ and this kernel is called *Torelli group* of $\mathrm{Mod}(\Sigma_g)$. The Dehn twists about separating curves are not the only elements in Torelli group, Torelli group is not finitely generated like $\mathrm{Mod}(\Sigma_g)$. See Chapter 6 in [20] for details.

7. LEFSCHETZ FIBRATIONS

We now come to the core topic of this study, Lefschetz fibrations. Relevant theorems of Donaldson and Gompf show that any 4-manifold admits a symplectic structure if and only if it admits a Lefschetz fibration. For details the reader can look at Chapter 10 of this book [22]. Studying Lefschetz fibrations is more preferable because of their properties. We will see these properties and more in the light of Fuller's study [23] in this chapter.

Definition 7.1. *Let X be a compact, oriented smooth 4-manifold and A be a compact, oriented smooth 2-manifold. A Lefschetz fibration on X is a smooth surjective map $f : X \to A$ such that:*

(i) $\{x_1, x_2, ..., x_n\}$ *are the critical points of f and $a_i = f(x_i)$ are the distinct critical values of f inside A, and*

(ii) about each x_i and a_i, f has an orientation preserving chart on which $f : \mathbb{C}^2 \to \mathbb{C}$ is given by $f(w, z) = w^2 + z^2$.

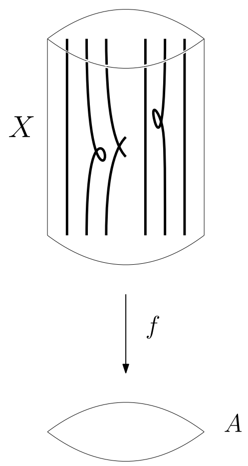

Figure 7.1. Lefschetz Fibration

The points in A which are different from $\{a_1, a_2, ..., a_n\}$ are called *regular values* of f and $f^{-1}(b)$ for any regular value b is called *regular fiber* of f. By Sard's theorem, regular fibers of f are diffeomorphic to Σ_g with fixed genus g. Therefore, we can call f as a *genus g Lefschetz fibration.*

The points $\{a_1, a_2, ..., a_n\}$ are called *singular values* of f and their preimages are called *singular fibers* of f. At Definition 3.1, we assumed that all critical points are in different singular fibers.

7.1. The Topology of Lefschetz Fibrations

It is always best to start from the simplest case, which is looking at the neighborhood of a critical point. Let $f : X \to \mathbb{D}^2$ be a Lefschetz fibration with only one singular fiber. Let the singular fiber be denoted by $F_1 = f^{-1}(a_1)$. From the assumptions of Definition 7.1, we assumed that a_1 is in the interior of $\mathbb{D}^2$. Moreover, let a_0 be a regular value near to a_1, and F_0 be regular fiber of it with genus g surface Σ_g as in Figure 3.2. We can visualize F_1 as a new shape of F_0 by shrinking a simple closed curve γ in F_0 to a point. This curve γ is called the *vanishing cycle* of that fiber. We will see that all vanishing cycles give us information about the topology of Lefschetz fibrations.

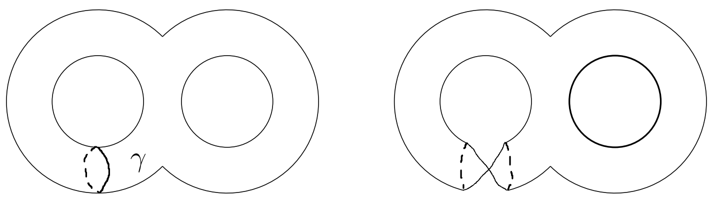

Figure 7.2. F_0 and F_1

What is the shape of X? Intuitively, if there is no any critical point, then X is diffeomorphic to $\Sigma_g \times \mathbb{D}^2$. In the existence of one singular fiber, there must be a change in the structure of $\Sigma_g \times \mathbb{D}^2$ around the vanishing cycle γ. To shrink for γ to a point, 2-handle must be attached around it with some framing conditions which are stated in [13].

When we think Morse theoretically, we can say that the boundary of X is diffeomorphic to the boundary of the neigborhood of the singular fiber. In the absense of the singular fiber, the boundary is automatically diffeomorphic to $\Sigma_g \times S^1$. The existense of the singular fiber enable us to cut S^1 to interpret about the boundary of X. After cutting S^1, we get I and the edges of I must be attached by a map. Figure 7.3 helps us to understand this cutting-gluing process.

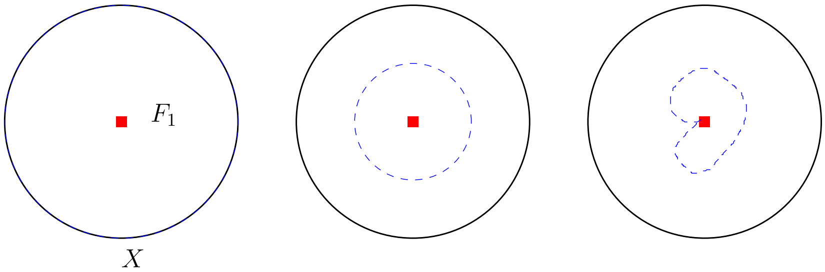

Figure 7.3. Blue dashed lines are diffeomorphic

As a result, we can describe the boundary of X as

$$\partial X = \frac{\Sigma_g \times I}{(\psi(x), 0) \sim (x, 1)}$$

where $\psi : \Sigma_g \to \Sigma_g$ is a homeomorphism and is called the *monodromy* of the singular fiber. This monodromy is right Dehn twist about the vanishing cycle.

Due to the local charts of the fibration around the critical points, we have to preserve the orientation of this charts. This enable us to use only right twists as monodromies of Lefschetz fibrations.

Now we can generalize the simplest case. Assume that $f : X \to \mathbb{D}^2$ is a Lefschetz fibration with singular fibers $F_1 = f^{-1}(a_1), F_2 = f^{-1}(a_2), ..., F_n = f^{-1}(a_n)$ in the interior of X. Let V_i be a small disk centered at a_i and contain only one critical value which is a_i and all V_i's are disjoint for all $i \in \{1, 2, .., n\}$. Then we can say that f is a Lefschetz fibration on $f^{-1}(V_i)$ with one singular fiber and we know its topology from the previous paragraphs. But now the question is how can we relate this topologies determined by different vanishing cycles, say $\{\gamma_1, \gamma_2, ..., \gamma_n\}$.

Let a_0 be a regular value of f and so $F_0 = f^{-1}(a_0)$ be a regular fiber of f, and let s_i be an arc from a_0 to a_i for all $i \in \{1, 2, ..., n\}$. We assume that these arcs are indexed in order to move counterclockwise about a_0 and they never intersect to each other and the other V_i's. Figure 3.5 helps us to illustrate this construction.

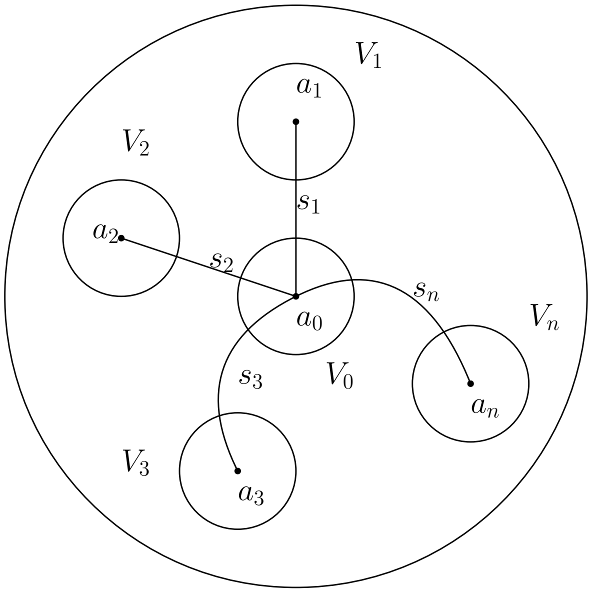

Figure 7.4. Fibration over the disk

We already know that $f^{-1}(V_0) \cong \Sigma_g \times \mathbb{D}^2$. Let $\nu(s_i)$ denote a regular neighborhood of the arc s_i, this is diffeomorphic to $\Sigma_g \times I$, and $f^{-1}(V_0 \cup \nu(s_1) \cup V_1)$ is diffeomorphic to $\Sigma_g \times \mathbb{D}^2$ with a 2-handle H_1 attached along γ_i with the framing conditions, and the boundary is diffeomorphic to $\Sigma_g \times S^1$ with the monodromy D_{γ_1}. After moving counterclockwise about a_0, we collected the other a_i's and we can say that

$$X \cong f^{-1}\left(V_0 \cup \left(\bigcup_{i=1}^{n} \nu(s_i)\right) \cup \left(\bigcup_{i=1}^{n} V_i\right)\right)$$

and this is diffeomorphic to $\Sigma_g \times \mathbb{D}^2$ with 2-handle H_i's are attached along γ_i's with the framing conditions. The boundary of X is diffeomorphic to $\Sigma_g \times S^1$ with the monodromy $D_{\gamma_n} D_{\gamma_{n-1}} ... D_{\gamma_2} D_{\gamma_1}$, which is a composition of Dehn twists. This is called the *global monodromy* of f.

Now, a new question arises. Does any change in choice of paths or regular value affect this global monodromy? A Lefschetz fibration does not completely determine the ordered collection of vanishing cycles. Choosing different regular value gives a conjugation of the Dehn twist in the global monodromy by the same element in the mapping class group.

Changing regular fiber gives another vanishing cycle but this new cycle is diffeomorphic to the first one and an element in the mapping class group maps the first cycles to the new ones. This can be shown that $D_{\gamma_i} \to \psi \circ D_{\gamma_i} \circ \psi^{-1}$, which is $D_{\psi(\gamma_i)}$, a new Dehn twist. Moreover, different choice of arcs will give different monodromies. For a given two choices, it is possible to get one of them from the other one by applying some moves in finite steps. These moves are called *elementary transformations*.

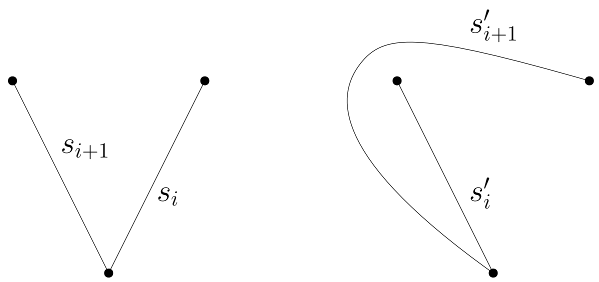

Figure 7.5. Elementary transformation

The Figure 7.6 shows an example of getting different order of vanishing cycle by applying elementary transformations, represented by right rows.

Each elementary transformation change the vanishing cycles $(..., \gamma_i, \gamma_{i+1}, ...)$ to $(..., \gamma_{i+1}, D_{\gamma_{i+1}}(\gamma_i), ...)$. If we look at the new global monodromies, $...D_{\gamma_{i+1}} D_{\gamma_i}...$ become $...(D_{\gamma_{i+1}} D_{\gamma_i} D^{-1}_{\gamma_{i+1}}) D_{\gamma_{i+1}}....$ This implies that the global monodromy is not affected. Thus, any choice of paths never affect the global monodromy.

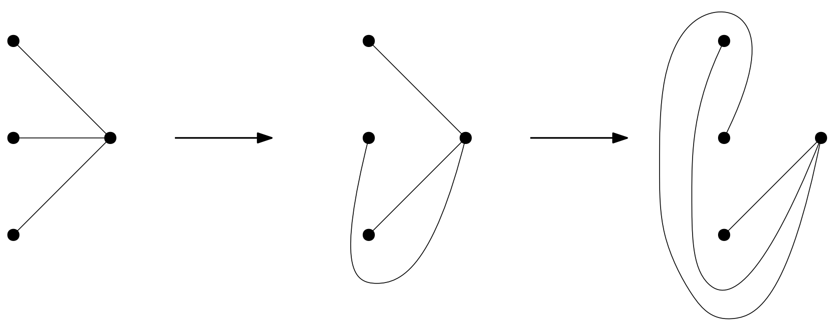

Figure 7.6. A different choice of paths

Now let us look at the new case, $f : X \to S^2$ is Lefschetz fibration over S^2. Since we assumed that there are finitely many critical values in S^2, we can divide S^2 to two hemispheres, $S^2 = \mathbb{D}^2_1 \cup \mathbb{D}^2_2$ in which $\mathbb{D}^2_1$ contains all the critical values. The topology of induced f on $f^{-1}(\mathbb{D}^2_1)$ is described above.

However in this case, the boundary must be the trivial $\Sigma_g \times S^1$ to attach the other hemisphere. So the global monodromy must be isotopic to the identity.

7.2. Bordered Lefschetz Fibrations

In the previous paragraphs, we assumed that all regular fibers are closed, i.e. diffeomorphisc to Σ_g. In some cases, Let $f : X \to A$ be a Lefschetz fibration where X is compact oriented smooth 4-manifold with non-closed regular fiber. Such fibrations are called Bordered Lefschetz fibration.

Assume that Σ_g^b is regular fiber of the Lefschetz fibration f. By adding 2-dimensional 1 handles to the boundary parts, we reduce b to 1 and then by capping off a disk to the left boundary part, we will get Σ_{g+b-1}. The figure below helps us to illustrate this modification.

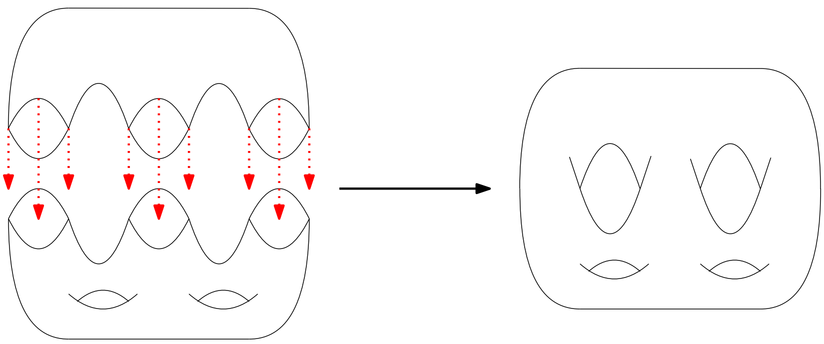

Figure 7.7. Σ_2^3 becomes Σ_4

From this view, we get a fact that $\mathrm{Mod}(\Sigma_g^b)$ is a subgroup of $\mathrm{Mod}(\Sigma_{g+b-1})$. So there is a homeomorphism from $\mathrm{Mod}(\Sigma_g^b)$ to $\mathrm{Sp}(2(g+b-1))$.

Definition 7.2. *Let $f : X \to \mathbb{D}^2$ be a bordered Lefschetz fibration whose regular fiber is Σ_g^b. Then $\overline{f} : \overline{X} \to \mathbb{D}^2$ new Lefschetz fibration whose regular fiber is closed surface Σ_{g+b-1}, constructed in the way explained above. The monodromy of the original fibration extend to the 1-handles in the new closed fibers. This new fibration $\overline{f}$ is called the closure of f.*

The next theorem lead us consider only closed fibers to find signature of total spaces of Lefschetz fibrations.

Theorem 7.3. [15, Theorem 3.1] *The signature of the total space of a bordered Lefschetz fibration is equal to that of its closure.*

8. THE SIGNATURES OF LEFSCHETZ FIBRATIONS

Signature is the simplest invariant of a 4-manifold and denoted by σ. Let X_1 and X_2 be two compact oriented 4-manifolds. If their boundaries are glued via an orientation reversing diffeomorphism and then we get a closed 4-manifold X. Then the signature of X is the sum of the signatures of X_1 and X_2, this is called Novikov additivity, see [24, p. 587-589] for the proof. However, if X_1 and X_2 are glued along their common submanifold X_0, the signature of X is not sum of the other signatures. In his study [14], Wall gave a formula for the signature in all cases. This formula is called *Wall's non-additivity formula*, which is sum of the signatures of X_1 and X_2 minus Wall's index. By using this formula and defining new terms, Çengel and Karakurt gave a new formula for the signature of 4-manifold which admit a Lefschetz fibration in their study [15]. Cappell, Lee and Miller showed that Wall's index coincides with Maslov ternary index, see [16, Proposition 8.2]. In this chapter, we rewrite Çengel and Karakurt's formula by putting Maslov ternary index instead of Wall's index. We will see that the key elements of this rewritten formula are positive transvections, which makes the operations in proofs and examples easier. Then, we will write some theorem and corollaries. At the end of this chapter, we will give an example.

8.1. Wall's Formula and Partial Fiber Sum Decompositions

Suppose we have a Lefschetz fibration $f : X \to \mathbb{D}^2$, bordered or not. If bordered, take $\overline{f} : \overline{X} \to \mathbb{D}^2$, then we will get the same signature by Theorem 7.3. Therefore assume that $f : X \to \mathbb{D}^2$ is not bordered, and let ϕ be its monodromy and Σ_g be its regular fiber. Divide $\mathbb{D}^2$ to two hemispheres $\mathbb{D}^+$ and $\mathbb{D}^-$ in which no singular value on the diameter.

$\mathbb{D}^2 = \mathbb{D}^+ \cup \mathbb{D}^-$ and $X = X^+ \cup X^-$ where $f^\pm$ is the induced map of f on $X^\pm$, and $X^\pm = {f^{\pm}}^{-1}(D^\pm)$. The pair (f^+, f^-) is called a partial fiber sum decomposition of the Lefschetz fibration f.

We denote the monodromies of f^+ and f^- by ϕ^+ and ϕ^- respectively. Let $\phi_*^{\pm}$ be the image of $\phi^{\pm}$ under the symplectic representation. Çengel and Karakurt [15] created the descriptions above and then remodified Wall's non-additivity formula.

Theorem 8.1. [15, Theorem 4.1] *For the partial fiber decomposition $f = (f^+, f^-)$*

$$\sigma(X) = \sigma(X^+) + \sigma(X^-) - \mu_{Wall}(\Gamma(\phi_*^-), \Gamma(id), \widetilde{\Gamma(\phi_*^+)}) \tag{8.1}$$

As we said before, this study [16, Proposition 8.2] showed that

$$\mu_{Wall}(\Gamma(\phi_*^-), \Gamma(id), \widetilde{\Gamma(\phi_*^+)}) = -\mu(\Gamma(\phi_*^-), \Gamma(id), \widetilde{\Gamma(\phi_*^+)})$$

Hence, 8.1 becomes

$$\sigma(X) = \sigma(X^+) + \sigma(X^-) + \mu(\Gamma(\phi_*^-), \Gamma(id), \widetilde{\Gamma(\phi_*^+)}) \tag{8.2}$$

In the rest of this study we focus on the equation 8.2. Assume that $\sigma(X)$ is known, and its monodromy is ϕ. Via that formula, we can interpret about the new signature of X if we add a Dehn twist to ϕ, say δ. Let us construct D^+ and D^- such that D^- contains the new singular value and others are in D^+. Then the formula become

$$\sigma(X) = \sigma(X^+) + \sigma(X^-) + \mu(\Gamma(\delta_*), \Gamma(id), \widetilde{\Gamma(\phi_*)})$$

$\sigma(X^-)$ is the signature of a Lefschetz fibration whose monodromy is a unique Dehn twist. It is well known that if the vanishing cycle is nonseparating, then $\sigma(X^-) = 0$, and $\sigma(X^-) = -1$ if the vanishing cycle is separating, see [15, p. 10-11] for proofs. Thus if we know the Maslov index, then we know the new signature.

Example 8.2. *Let $f : X \to \mathbb{D}^2$ be a Lefschetz fibration and its monodromy is product of same two Dehn twists about nonseparating curves. Then we know that the signature is -1.*

If we apply the partial fiber sum decomposition, $\mathbb{D}^+$ contains one singular value and $\mathbb{D}^-$ contains the other one, then we will get that

$$\sigma(X) = \mu(\Gamma(S), \Gamma(id), \widetilde{\Gamma(S)}) \tag{8.3}$$

where S is a positive transvection, and signatures of preimages of hemispheres vanish. Let $S = T_{-1,u}$, so $\mu(\Gamma(S), \Gamma(id), \widetilde{\Gamma(S)})$ is the signature of the quadratic form

$$\begin{aligned} Q((a, Sa), (b, b), (Sc, c)) = (\omega \oplus (-\omega))((a, Sa), (b, b)) \\ + (\omega \oplus (-\omega))((b, b), (Sc, c)) \\ + (\omega \oplus (-\omega))((Sc, c), (a, Sa)) \end{aligned}$$

This equals to

$$= \omega(a - Sa, b) + \omega(b, Sc - c) + \omega(Sc, a) + \omega(c, Sa) \tag{8.4}$$

In this quadratic form, u is given and a, b and c can be choosen arbitrarily, so we can choose the following variables in the quadratic form

- $X = \omega(a, y)$
- $Y = \omega(b, y)$
- $Z = \omega(c, y)$

Then 8.4 becomes $-XY - YZ + 2XZ$. The symmetric matrix of this quadratic form is

$$\begin{bmatrix} 0 & -1/2 & 1 \\ -1/2 & 0 & -1/2 \\ 1 & -1/2 & 0 \end{bmatrix}$$

After applying simultaneous row and column operations, we will get that

$$\begin{bmatrix} 0 & -1/2 & 1 \\ -1/2 & 0 & -1/2 \\ 1 & -1/2 & 0 \end{bmatrix} \xrightarrow{-R_1+R_3=R_3} \begin{bmatrix} 0 & -1/2 & 1 \\ -1/2 & 0 & -1/2 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{-C_1+C_3=C_3}$$

$$\begin{bmatrix} 0 & -1/2 & 1 \\ -1/2 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow[2R_2+R_3=R_3]{2C_2+C_3=C_3} \begin{bmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[R_2+R_1=R_1]{C_2+C_1=C_1}$$

$$\begin{bmatrix} -1 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[-R_1/2+R_2=R_2]{-C_1/2+C_2=C_2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

So the signature is -1.

Theorem 8.3. *If we add a Dehn twist about separating curve to the monodromy of a Lefschetz fibration whose signature is* h*, then the new signature is* $h-1$.

Proof. If we apply the partial fiber sum decomposition, then we get

$$\sigma(X) = \sigma(X^+) + \sigma(X^-) + \mu(\Gamma(\delta_*), \Gamma(id), \widetilde{\Gamma(\phi_*)})$$

where δ is the added Dehn twist, X^- is the preimage of the hemisphere including only one singular value which is the last one. Since δ is trivial homologically, δ_* is identity implying that the Maslov index vanishes by Theorem 5.2. Hence

$$\sigma(X) = \sigma(X^+) + \sigma(X^-) = h - 1$$

□

Corollary 8.4. *If the monodromy is the product of k Dehn twists and all vanishing cycles are separating curves, then the signature is $-k$.*

Proof. By induction, Theorem 8.3 implies this result. □

Theorem 8.5. *Let $f : X \to \mathbb{D}^2$ be a Lefschetz fibration with monodromy which is trivial homologically. Suppose we applied partial fiber sum decomposition $f = (f_1, f_2)$ with correspondent total space X^+ and X^-, then $\sigma(X) = \sigma(X^-) + \sigma(X^+)$.*

Proof. Let $\phi^{\pm}$ denote the monodromy of $f^{\pm}$, then $\phi^-_* \phi^+_* = id$. Let Ψ denote ϕ^-_* and Ψ^{-1} denote ϕ^+_*, then the signature of X is

$$\sigma(X) = \sigma(X^-) + \sigma(X^+) + \mu(\Gamma(\Psi), \Gamma(id), \widetilde{\Gamma(\Psi^{-1})})$$

Notice that $\Gamma(\Psi)$ and $\widetilde{\Gamma(\Psi^{-1})}$ are same Lagrangian spaces. By theorem 5.2, Maslov index vanishes. □

Theorem 8.6. *Let $f : X \to \mathbb{D}^2$ be a Lefschetz fibration such that all vanishing cycles nonseparating curves and their homology classes are linearly independent. Then the signature of the total space is zero.*

Proof. By partial fiber sum decomposition, it is enough to show that for all k

$$\mu(\Gamma(S_{k+1}), \Gamma(id), \Gamma(\widetilde{S_k ... S_1})) = 0 \tag{8.5}$$

For $k = 1$, let $S_1 = T_{-1,u_1}$ and $S_2 = T_{-1,u_2}$ be given and u_1 and u_2 are linearly independent. Thus, we have 6 variables in the quadratic formula. The variables are the followings

- $X = \omega(a, u_1)$
- $Y = \omega(b, u_1)$
- $Z = \omega(c, u_1)$
- $X' = \omega(a, u_2)$
- $Y' = \omega(b, u_2)$
- $Z' = \omega(c, u_2)$

After applying the definition of the quadratic form, that for $k = 1$ is

$$-X'Y' - YZ + XZ + X'Z' \tag{8.6}$$

The symmetric matrix (with the order X, Y, Z, X', Y', Z') of this quadratic form is

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ where } A = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & -1/2 \\ 1/2 & -1/2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1/2 & 1/2 \\ -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

Since signatures of A and B are zero, the total signature is also zero.

Assume that the next equation is true.

$$\mu(\Gamma(S_k), \Gamma(id), \widetilde{\Gamma(\Psi)}) = 0 \text{ where } \Psi = S_{k-1}...S_1 \tag{8.7}$$

We want to prove the next statement

$$\mu(\Gamma(S_{k+1}), \Gamma(id), \widetilde{\Gamma(S_k\Psi)}) = 0 \tag{8.8}$$

Since all homology classes of vanishing cycles are linearly independent, the vectors of transvections are also linearly independent. Therefore we have the next variables for quadratic forms for all $i \in \{1, 2, ..., k+1\}$

- $X_i = \omega(a, u_i)$
- $Y_i = \omega(b, u_i)$
- $Z_i = \omega(c, u_i)$

The quadratic form of 8.7 is

$$-X_k Y_k + X_k Z_k + \omega(b, \Psi c - c) + \omega(\Psi c - c, a) \tag{8.9}$$

and the quadratic form of 8.8 is

$$-X_{k+1}Y_{k+1} + X_{k+1}Z_{k+1} + X_k\omega(\Psi c, u_k) - Y_k\omega(\Psi c, u_k) + \omega(b, \Psi c - c) + \omega(\Psi c - c, a) \tag{8.10}$$

by the way

$$\omega(\Psi c, u_k) = Z_k + \sum_{i=1}^{k-1} A_i Z_i \text{ where } A_i\text{'s are some coefficients} \tag{8.11}$$

Then,

$$X_k\omega(\Psi c, u_k) = X_k Z_k + \sum_{i=1}^{k-1} A_i X_k Z_i \text{ and } Y_k\omega(\Psi c, u_k) = Y_k Z_k + \sum_{i=1}^{k-1} A_i Y_k Z_i \tag{8.12}$$

Let us write the symmetric matrices (with the order $X_1, Y_1, Z_1, ..., X_{k+1}, Y_{k+1}, Z_{k+1}$) of the quadratic forms 8.9 and 8.10 respectively

$$F_1 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} A & C^T & 0 \\ C & D & 0 \\ 0 & 0 & B \end{bmatrix}$$

where A is $3(k-1)\times 3(k-1)$ matrix, $B = \begin{bmatrix} 0 & -1/2 & 1/2 \\ -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$, C is $3 \times 3(k-1)$ matrix whose first and second rows are same, which is

$$C = \begin{bmatrix} 0 & 0 & A_1 & 0 & 0 & A_2 & ... & A_{k-1} \\ 0 & 0 & A_1 & 0 & 0 & A_2 & ... & A_{k-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix} \tag{8.13}$$

For matrix F_2, applying $-R_{3k} + R_{3k+1} \to R_{3k+1}$ and $-C_{3k} + C_{3k+1} \to C_{3k+1}$, we get

$$\text{sign}\left(\begin{bmatrix} A & C^T & 0 \\ C & D & 0 \\ 0 & 0 & B \end{bmatrix}\right) = \text{sign}\left(\begin{bmatrix} A & C'^T & 0 \\ C' & D & 0 \\ 0 & 0 & B \end{bmatrix}\right) \tag{8.14}$$

where C'^T is same matrix in 8.16 but the second row is zero. If we apply the related simultaneous column and row operations by using 1/2s in matrix D, we get that

$$\text{sign}\left(\begin{bmatrix} A & C'^T & 0 \\ C' & D & 0 \\ 0 & 0 & B \end{bmatrix}\right) = \text{sign}\left(\begin{bmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B \end{bmatrix}\right) \tag{8.15}$$

Thus

$$\mu(\Gamma(S_{k+1}), \Gamma(id), \widetilde{\Gamma(S_k\Psi)}) = \text{sign}A + \text{sign}D + \text{sign}B = \text{sign}A \tag{8.16}$$

By the assumption,

$$\mu(\Gamma(S_k), \Gamma(id), \widetilde{\Gamma(\Psi)}) = \text{sign}A = 0$$

Thus $\mu(\Gamma(S_{k+1}), \Gamma(id), \widetilde{\Gamma(S_k\Psi)}) = 0$. $\square$

8.2. An Example

In this section, we apply our method on an example which has a well known result. Consider the elliptic surface $E(1)$ whose signature is -8. Its monodromy is $\phi^6 = id$, where $\phi = D_\beta D_\alpha$ and α, β are the curves in Figure 8.1. Let X_ϕ denote the total space whose monodromy is ϕ to ease the next calculation. So we want to find $\sigma(X_{\phi^6})$.

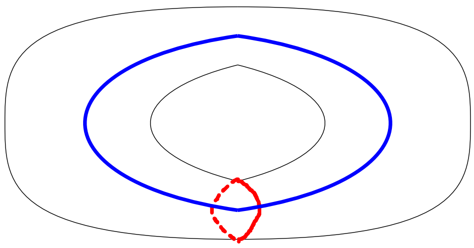

Figure 8.1. The red curve is α and the blue one is β

By Theorem 8.6, $\phi_*^6 = id$ implies that $\sigma(X_{\phi^6}) = 2\sigma(X_{\phi^3})$. Then,

$$\sigma(X_{\phi^3}) = \sigma(X_{\phi^2}) + \sigma(X_\phi) + \mu(\Gamma(\phi_*), \Gamma(id), \widetilde{\Gamma(\phi_*^2)}) \tag{8.17}$$

By Theorem 8.6, $\sigma(X_\phi)$ vanishes. Thus $\sigma(X_{\phi^2}) = \mu(\Gamma(\phi_*), \Gamma(id), \widetilde{\Gamma(\phi_*)})$. If we combine all of these results above, we will get that

$$\sigma(E(1)) = 2\left(\mu(\Gamma(\phi_*), \Gamma(id), \widetilde{\Gamma(\phi_*)}) + \mu(\Gamma(\phi_*), \Gamma(id), \widetilde{\Gamma(\phi_*^2)})\right)$$

Let S_1, S_2 be positive transvections representing $(D_\alpha)_*, (D_\beta)_*$, respectively. Then, $T_{-1,u_i} = S_i$ implies $\omega(u_1, u_2) = 1$.

$$\phi_*(v) = S_2(S_1 v) = S_2(v - \omega(v, u_1)u_1) = v - \omega(v, u_1)u_1 - \omega(v, u_2)u_2 + \omega(v, u_1)u_2$$

and

$$\begin{aligned}\phi_*^2(v) &= \phi_*(\phi_*(v)) = \phi_*(v - \omega(v,u_1)u_1 - \omega(v,u_2)u_2 + \omega(v,u_1)u_2) \\ &= v - \omega(v,u_1)u_1 - \omega(v,u_2)u_1 + 2\omega(v,u_1)u_2 - 3\omega(v,u_2)u_2\end{aligned}$$

Let X, X', Y, Y' be defined as the previous paragraphs. Then, the quadratic form of $\mu(\Gamma(\phi_*), \Gamma(id), \widetilde{\Gamma(\phi_*)})$ is that

$$-XY - X'Y' + XY' - YZ - Y'Z' + Y'Z + 2XZ + 2X'Z' - X'Z - XZ' \tag{8.18}$$

The symmetric matrix (with order X, Y, Z, X', Y', Z') of 8.18 is that

$$\begin{bmatrix} 0 & -1/2 & 1 & 0 & 1/2 & -1/2 \\ -1/2 & 0 & -1/2 & 0 & 0 & 0 \\ 1 & -1/2 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & -1/2 & 0 & -1/2 & 1 \\ 1/2 & 0 & 1/2 & -1/2 & 0 & -1/2 \\ -1/2 & 0 & 0 & 1 & -1/2 & 0 \end{bmatrix} \tag{8.19}$$

and signature of this matrix is -2. The quadratic form of $\mu(\Gamma(\phi_*), \Gamma(id), \widetilde{\Gamma(\phi_*^2)})$ is that

$$-XY - X'Y' + XY' - YZ - YZ' + 2Y'Z - 3Y'Z' + 2XZ - 2X'Z + 4X'Z' \tag{8.20}$$

The symmetric matrix (with order X, Y, Z, X', Y', Z') of 8.20 is that

$$\begin{bmatrix} 0 & -1/2 & 1 & 0 & 1/2 & -1/2 \\ -1/2 & 0 & -1/2 & 0 & 0 & -1/2 \\ 1 & -1/2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1/2 & 2 \\ 1/2 & 0 & 1 & -1/2 & 0 & -3/2 \\ -1/2 & -1/2 & 0 & 2 & -3/2 & 0 \end{bmatrix} \tag{8.21}$$

and the signature of this matrix is also -2. Thus signature of total space is $2(-2-2) = -8$.

9. CONCLUSION

As we saw from the proof techniques, calculating the signature by using Maslov ternary index looks easy. Only elementary linear algebra is enough to calculate the signatures. Indeed, positive transvections are the key elements of the process of calculating the signature of Lefschetz fibrations. We saw that positive transvections enabled us to prove some known theorems more easily.

By this method, we found some important results like Theorem 8.6. This method may open new doors in this area. For a given monodromy, if a new Dehn twist is added, whose homology class is linearly independent from the others, it is highly likely that one can show the total signature stays unchanged by using this method. Furthermore, one may prove that the signature of a Lefschetz fibration, whose monodromy is trivial homologically, is nonpositive. This remains as an open problem.

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