TOPOLOGIES ON FAMILIES OF CLOSED SUBSETS

by

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ABSTRACT

TOPOLOGIES ON FAMILIES OF CLOSED SUBSETS

In 1914, *Felix Hausdorff* introduced a metric, called the *Hausdorff metric* on the set of closed subspaces, *Hausdorff space*, of a metric space. This metric and the corresponding topology are the main objects of the study. In algebraic geometry, there is an analogous object called the *Hilbert scheme*, whose points correspond to closed subschemes of a projective variety X. We give a modular interpretation of the Hausdorff space analogous to the one for the Hilbert scheme. The Hilbert scheme is used in various structures in algebraic geometry; using the Hausdorff space we can mimic these constructions in topology. For example, when an algebraic group acts on a projective variety, one can form a quotient $X \mathbin{/\!/}_{Hilb} G$ called the Hilbert quotient. We consider the analogous Hausdorff quotient $X \mathbin{/\!/}_{Haus} G$ associated to a topological group acting on a metric space. We note that this quotient has some desirable properties: When X is compact $X \mathbin{/\!/}_{Haus} G$ is compact and when G is also compact $X \mathbin{/\!/}_{Haus} G$ is the usual quotient X/G. In addition to the Hausdorff topology, one can also topologize the set of closed subspaces of X by considering the compact open topology on the set of continuous maps from X to the Sierpinski space. Although the resulting *Hilbert space* is less well-behaved than the Hausdorff space, it admits a nice modular interpretation when X is locally compact.

ÖZET

KAPALI ALTKÜMELERDEKİ TOPOLOJİLER

1914'te *Felix Hausdorff* bir metrik uzayının kapalı altuzayları, *Hausdorff uzayı*, kümesinde bir metrik tanımladı. Bu metrik ve ona mutabık topoloji, bu çalışmanın esas gayesi olacaktır. Cebirsel geometride, buna analog bir nesne, *Hilbert şeması*, vardır ki bunun noktaları izdüşümsel bir çeşitleme olan X'in kapalı altşemalarına denk gelir. Biz Hausdorff uzayının, Hilbert şemasınınki ile analog bir modüler yorumlamasını vereceğiz. Hilbert şeması cebirsel geometride çeşitli yapılarda kullanılır; Hausdorff uzayını kullanarak biz bu yapıları topolojide taklit edebiliriz. Örneğin, bir cebirsel grup bir izdüşümsel çeşitlemeye etkidiğinde, Hilbert bölümü adı verilen bir bölüm $X /\!\!/_{Hilb} G$ oluşturulabilir. Biz, bir topolojik grubun bir metrik uzayına etkidiği Hausdorff bölümünün $X /\!\!/_{Haus} G$ üzerinde düşüneceğiz. Dikkate değer ki, bu bölüm bazı cazip özelliklere sahiptir: X kompakt olduğunda $X /\!\!/_{Haus} G$ bölümü de kompakttır ve G de kompakt olduğunda $X /\!\!/_{Haus} G$ bölümü alışılagelen X/G bölümüyle aynıdır. Hausdorff topolojisine ilave olarak, X'in kapalı altuzayları üzerine, X'ten Sierpinski uzayına giden sürekli fonksiyonlar kümesindeki kompakt-açık topoloji kullanılarak da topoloji kurulabilir. Her ne kadar ortaya çıkan *Hilbert uzayı*, Hausdorff uzayından daha kötü tabiatlı olsa da, X yerel olarak kompakt olduğunda bu güzel bir modüler yorumlama meydana çıkarıyor.

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LIST OF SYMBOLS

$\overline{A}$	Closure of a topological space A
$\mathring{B}$	Interior of a topological space B
$\mathbb{C}$	The set of complex numbers
f^{-1}	The inverse of a function f
$f\big\|_A$	Restriction of a function f by a set A
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_{\geqslant 0}$	The set of nonnegative real numbers
$\mathbb{Z}$	The set of integers

1. INTRODUCTION

In introductory metric topology, we usually try to find the distance between two points. How about the distance between two *sets*? Is it possible to define a reasonable "distance" between two subsets of, for example, $\mathbb{R}$? What is the distance between the intervals $(0,1)$ and $[0,1]$? It seems to be easier to put a metric between the *closed* subspaces of a metric space. Such a metric -the Hausdorff metric- was defined by *Felix Hausdorff* in 1914 in his book Grundzüge Der Mengenlehre [1]. The space of closed subspaces of a metric space, with the topology determined by the Hausdorff metric is called the *Hausdorff space*. It has many interesting properties such as: If a metric space is compact, then so is the induced Hausdorff space (Corollary 4.21). Furthermore, for a compact metric space X, if two metrics induce the same topology on X, then these two metric determine the same topology on the Hausdorff space of X, denoted by $\mathscr{H}_X$ (Corollary 4.9).

We can describe an ε-neighbourhood in the Hausdorff space of X in terms of the ε-neighbourhoods in X with the equality below:

$$d_{\mathscr{H}}(Z,Z') = \inf\left\{\varepsilon > 0 : Z' \subseteq \bigcup_{z\in Z} B(z,\varepsilon) \text{ and } Z \subseteq \bigcup_{z'\in Z'} B(z',\varepsilon)\right\}.$$

There is another topology on the set of closed subsets of a topological space X given by the basis

$$U(K) = \{Z \subseteq_{\text{closed}} X : Z \cap K = \varnothing\}$$

where K is a compact subspace of X. This topology is called the *Hilbert topology* is the same as the space of continuous functions from X to the Sierpinski space with the compact-open topology. When X is a locally compact space, the Hilbert topology gives a nice modular interpretation of the closed subspaces of X. Since the definitions of local compactness vary book by book, we gave a nice collection of them and described their differences in the second chapter.

In algebraic geometry, the *Hilbert scheme* is a moduli space of closed subschemes of a projective variety. We show in Theorem 4.6 that the Hausdorff space, in fact, has an analogous modular interpretation. In this study, we tried to apply some constructions involving the Hilbert scheme to Hausdorff space. For example in Chapter 6, we present an analog of the *Hilbert Quotient*, called the *Hausdorff Quotient*.

We give a nice definition for the Hausdorff Quotient of a metric space X by a topological group G. We prove in Prop 6.14 that if both X and G are compact, the Hausdorff Quotient is the same as the usual quotient. Furthermore, if X or G is not compact, then the Hausdorff Quotient becomes a more nicely-behaved quotient since it has properties like Hausdorffness that the usual quotient lacks.

Of course, there are some subtleties in the construction of Hausdorff quotient because we don't have "generic flatness" at our disposal as we do in algebraic geometry. We define notions of stability and semi-stability for certain subsets of X as a replacement for generic flatness.

Conventions

Throughout this study, $U \subseteq_\beta X$ will be used for "U is a subset of X with property β." For example $Z \subseteq_{closed} X$ means that "Z is a closed subset of X."

For a metric space (X, d) and a point $x \in X$, we define the set $\{y \in X : d(x, y) < \varepsilon\}$ as the *ε-neighbourhood* of a point x, and it will be denoted by $B_d(x, \varepsilon)$. Similarly, for $Z \in \mathscr{H}_X$ we denote the *ε-neighbourhood* of Z with respect to the Hausdorff metric by

$B_{d_{\mathscr{H}}}(Z,\varepsilon)$. Also, we define

$$B_d(\varepsilon, Z) := \{x \in X : d(x,Z) < \varepsilon\} = \bigcup_{z\in Z} B_d(z,\varepsilon) \quad \text{for } Z \subseteq X.$$

Observe that $B_d(\varepsilon, Z)$ is open for any $Z \subseteq X$ since it is a union of open balls. If the metric is clear from the context, we simply write $B(x,\varepsilon)$, $B(Z,\varepsilon)$ and $B(\varepsilon, Z)$ instead of $B_d(x,\varepsilon)$, $B_d(Z,\varepsilon)$ and $B_d(\varepsilon, Z)$.

By this definition, we can see that $B(\varepsilon, Z)$ is the union of neighbourhoods of elements of Z whereas $B(Z,\varepsilon)$ is the ε-neighbourhood of Z with respect to the Hausdorff metric.

We will write τ_d for a topology generated by a metric d, when the space is clear from the content.

2. LOCAL COMPACTNESS

There are various definitions of a locally compact space. To make a distinction between them, we make some definitions.

Definition 2.1. *A topological space X is called* LC1 *if every point of X has a neighbourhood with a compact closure.*

Definition 2.2. *A topological space X is called* LC2 *if every point of X lies in the interior of a compact subspace K of X.*

Definition 2.3. *A topological space X is called* LC1' *(*LC2' *resp.) if "compact" is replaced by "compact Hausdorff" in the definitions above.*

Unless explicitly mentioned otherwise, "locally compact" means LC2' in this text.

Proposition 2.4. *Let X be a topological space. Then,*

(i) X is LC1' (resp. LC2') $\Longrightarrow$ X is LC1 (resp. LC2).
(ii) X is LC1 $\Longrightarrow$ X is LC2.

Proof. (i) This is an immediate consequence of the fact that a compact Hausdorff subspace of X is compact.

(ii) Assume that X is an LC1 topological space. Pick $x \in X$ arbitrary. By definition of an LC1 space, there is an open U such that $x \in U$ and $\overline{U} = K$ is compact. Now since $U \subseteq_{open} \overline{U} = K$ and the interior $\mathring{K}$ of K is defined as the union of all opens in K, we get $x \in U \subseteq \mathring{K} \subseteq K$. Hence x lies in the interior of a compact subspace of X. Therefore X is LC2. □

Remark 2.5. An LC2' space need not to be LC1'.

Example 2.6. Let $X = \{0,1\} \times \mathbb{R}$ and let $(i_0, x_0) \sim (i_1, x_1)$ whenever $i_0 = i_1$ and $x_0 = x_1 \neq 0$. The quotient space $X/\sim$ is the real line with double origin, say 0 and 0^*. Then $X/\sim$ is LC2'.

To see this fact, pick any $x \in X/\sim$. If $x \neq 0$ or $x \neq 0^*$ then clearly x lies in the interior of the compact Hausdorff space $\overline{B(x, \frac{|x|}{2})}$. Otherwise, the set $K := [-1,0) \cup (0,1] \cup \{x\}$ is compact and Hausdorff since any two points $y_1, y_2 \in K$ can be seperated by epsilon balls around y_1 and y_2 with radius $|y_1 - y_2|$.

However, the space $X/\sim$ is not LC1'. Assume for a contradiction that it is LC1'. Then $x = 0$ has a neighbourhood with a compact Hausdorff closure. The closure of any neighbourhood of x must also contain 0^*. Then x and 0^* should have disjoint neighbourhoods $U := B(x, \varepsilon_x)$ and $V := B(0^*, \varepsilon_{0^*})$. The point $\frac{1}{2}\min\{\varepsilon_x, \varepsilon_{0^*}\}$ lies in both U and V. Contradiction. Hence, $X/\sim$ is not LC1'.

Proposition 2.7. *All definitions of a locally compact space given in* Definition 2.1, Definition 2.2 *and* Definition 2.3 *coincide when* X *is Hausdorff.*

Proof. Every subspace of a Hausdorff space is Hausdorff. So, LC1 $\iff$ LC1' and LC2 $\iff$ LC2' both holds. It suffices to show LC1 $\iff$ LC2. By Proposition 2.4, LC1 $\implies$ LC2. Assume that X is Hausdorff and LC2. Let $x \in X$. Then x lies in the interior of a compact subspace K of X. Since X is Hausdorff, K is closed. $\mathring{K}$ is a neighbourhood of x whose closure is $\overline{\mathring{K}} = K$ since K is closed. Thus X is LC1. The result follows. $\square$

Proposition 2.8. *All definitions of local compactness are inherited by closed subspaces.*

Proof.
- Let X be an LC1 space and Z be a closed subspace of X. Let $x \in Z$. We want to show that x has a neighbourhood with a compact closure. There is a neighbourhood U of x in X so that $\overline{U}$ is compact in X. $U \cap Z$ is a neighbourhood of x in Z with closure $\overline{U} \cap Z$. Also, $\overline{U} \cap Z$ is a closed subspace of $\overline{U}$, so it is compact. Hence x has a neighbourhood $U \cap Z$ with a compact closure $\overline{U} \cap Z$.

- Let X be an LC2 space and Z be a closed subspace of X. Every $x \in Z$ lies in the interior of a compact subspace K of X. Then there is an open U such that $x \in U \subseteq K$. Let $V = Z \cap U$ and $K' = Z \cap K$. Then $V \subseteq_{open} Z$ and $K' \subseteq_{compact} Z$ both holds. So, $x \in V \subseteq K'$ and thus Z is LC2.
- Similar to the LC1 part.
- Similar to the LC2 part. □

Proposition 2.9. *Let $\{U_i : i \in I\}$ be an open cover of a topological space X. If U_i is locally compact* (LC2') *for each i, then X is locally compact.*

Proof. Choose $x \in X$. Since $\{U_i\}$ covers X, there is an i_0 with $x \in U_{i_0}$. U_{i_0} is locally compact, so there is a compact Hausdorff subspace K of U_{i_0} such that $x \in \mathring{K}$. Since K is Hausdorff, there is an open subset V of U_{i_0} such that $x \in V \subseteq \overline{V} \subseteq \mathring{K}$. Observe that $\overline{V}$ is compact in X and thus X is locally compact. □

Remark 2.10. Example 2.6 shows that Proposition 2.9 is not true for an LC1 space.

3. THE HILBERT TOPOLOGY

Definition 3.1. *Let X and Y be two topological spaces. Let $C(X,Y)$ be the set of continuous maps $f : X \to Y$. For $K \subseteq X$, $U \subseteq Y$ define*

$$\mathscr{U}(K,U) := \{f \in C(X,Y) : f(K) \subseteq U\}.$$

The compact-open topology *[2] on $C(X,Y) = \{f : X \to Y\}$ is obtained from the* subbasis

$$\{\mathscr{U}(K,U) : K \subseteq_{compact} X, U \subseteq_{open} Y\}.$$

Definition 3.2. The Sierpinski Space S *is the unique topological space with two elements and three opens. i.e. $S = \{0,1\}$ and $\tau_S = \{\varnothing, \{0\}, \{0,1\}\}$*

Consider the space $C(X,S)$ of continuous functions from a topological space X to the Sierpinski space. The map $f \mapsto f^{-1}(1)$ yields a bijection from $C(X,S)$ to the set $\mathscr{H}_X$ of closed subsets of X.

Definition 3.3. The Hilbert Topology *on $\mathscr{H}_X$ is defined as the topology induced by the compact-open topology on $C(X,S)$ via the bijection above. Explicitly, the sets*

$$U(K) := \{Z \subseteq X : Z \text{ is closed in } X \text{ and } Z \cap K = \varnothing\}$$

for $K \subseteq_{compact} X$ form a subbasis for the Hilbert topology on $\mathscr{H}_X$.

Remark 3.4. The obvious formula

$$U(K_1) \cap U(K_2) = U(K_1 \cap K_2)$$

shows that the subbasis defined above is actually a *basis.*

Proposition 3.5. *Let X be a locally compact* (LC2') *space. Let $\mathscr{H}$ have the Hilbert topology. Then the subset $\mathcal{Z} = \{(Z, x) : x \in Z\}$ is closed in $\mathscr{H} \times X$.*

Proof. We want to show that $(\mathscr{H} \times X) - \mathcal{Z}$ is open. Let $(Z, x) \notin \mathcal{Z}$. Then $x \notin Z$. Since X is LC2' the point x lies in the interior of a compact Hausdorff subspace K of X. Since K is Hausdorff, there are opens U and V such that $x \in U \subseteq \overline{U} \subseteq V \subseteq K$ such that $V \cap Z = \varnothing$. $\overline{U}$ is compact. Now our claim is to show that $(Z, x) \in U(\overline{U}) \times U$ and $(U(\overline{U}) \times U) \cap \mathcal{Z} = \varnothing$. It is clear that $(Z, x) \in U(\overline{U}) \times U$ since $x \in U$ and $\overline{U} \cap Z = \varnothing$. Now we want to show that $x' \notin Z'$ for any $(Z', x') \in U(\overline{U}) \times U$. If $Z' \in U(\overline{U})$ then $Z' \cap \overline{U} = \varnothing$, so, for every $x' \in U \subseteq \overline{U}$ we have $x' \notin Z'$ as desired. $\square$

Theorem 3.6. *Let X be a locally compact space and Z be an arbitrary topological space. Then there is a bijection between the closed subsets of $Z \times X$ and the continuous maps $Z \to \mathscr{H}_X$ where $\mathscr{H}_X$ has the Hilbert topology.*

Proof. Let $C(X, Y)$ be the space of continuous functions $X \to Y$ with the compact-open topology and $\mathscr{H}_X$ have the Hilbert topology. By Theorem 46.11 in Mukres' Topology [3], we know that if X is locally compact, there is a bijection between continuous functions $f : X \times Z \to Y$ and continuous functions $F : Z \to C(X, Y)$ for any topological space Y. Now let $S = Y$ be the Sierpinski space. Clearly, there are bijections

$$\{f : X \times Z \to S : f \text{ is continuous }\} \longleftrightarrow \{\text{ closed subspaces of } X \times Z\}$$
$$\{F : Z \to C(X, S) : F \text{ is continuous }\} \longleftrightarrow \{g : Z \to \mathscr{H}_X : g \text{ is continuous }\}.$$

The result immediately follows. $\square$

Remark 3.7. Consider the functor

$$F : Top^{op} \to Sets$$
$$T \mapsto \{Z \subseteq_{closed} T \times X\}.$$

If X is locally compact, this functor is representable by the Hilbert Topology given in Definition 3.3, that is $Hom(X, S)$ with the compact-open topology where S is the Sierpinski space.

4. THE HAUSDORFF METRIC

Can we define a reasonable metric between *subsets* (instead of points) of a metric space? What if we take $X = \mathbb{R}$? Then what would be the distance between $(0,1)$ and $[0,1]$? In 1914, Felix Hausdorff introduced a metric, called the *Hausdorff metric*, on the set $\mathscr{H}_X$ of *closed* subspaces of a metric space. The topology $\mathscr{H}_X$ determined by the Hausdorff metric is called the Hausdorff topology and it has many exciting features. For instance, if a metric is finer than another metric on compact X, then the induced Hausdorff metric of the former is finer than the latter. Furthermore, if X is compact, so is the Hausdorff topology on $\mathscr{H}_X$.

Observe that if (X,d) is a metric space, and we set $d'(x,y) := \max\{d(x,y),1\}$, then (X,d') is also a metric space satisfying $d'(x,y) \leqslant 1$ for all $x,y \in X$. Furthermore, the topology on X determined by d is the same as the topology on X determined by d'. Thus, there is no major loss of generality if we just assume throughout this work that every metric d satisfies $d(x,y) \leqslant 1$ for all $x,y \in X$.

Definition 4.1. *Let (X,d) be a metric space. A metric $d_{\mathscr{H}}$ on the set $\mathscr{H}$ of closed subsets of X, called* the Hausdorff metric*, will be given by*

$$d_{\mathscr{H}}(A,B) = \frac{1}{2}\left(\sup_{b\in B} d(A,b) + \sup_{a\in A} d(a,B)\right).$$

The topology on $\mathscr{H}$ defined by this metric will be called the Hausdorff topology*.*

Remark 4.2. Alternatively,

$$d_{\mathscr{H}}(A,B) = \max\left\{\sup_{b\in B} d(A,b), \sup_{a\in A} d(a,B)\right\}.$$

These two metrics determine the same topology. We will prefer to use the second definition of the Hausdorff metric. The set $\mathscr{H}$ with the topology determined by $d_{\mathscr{H}}$ will be called the *Hausdorff space* of (X, d).

Proposition 4.3. *[1] The function $d_{\mathscr{H}} : \mathscr{H}_X \times \mathscr{H}_X \to \mathbb{R}_{\geqslant 0}$ defined above is a metric on $\mathscr{H}_X$, called the* Hausdorff metric.

Proof.

- Clearly $d_{\mathscr{H}}(A, B)$ is symmetric in A, B.
- Let us assume that $d_{\mathscr{H}}(A, B) = 0$ for some $A, B \in \mathscr{H}$. Then by definition, we have

$$d(a, B) = 0 \quad \forall a \in A$$
$$d(A, b) = 0 \quad \forall b \in B.$$

  These imply, respectively, that $A \subseteq \bar{B}$ and $B \subseteq \bar{A}$. Since both A and B are closed subsets of X, we get $A \subseteq B$ and $B \subseteq A$. And therefore $A = B$.
- Now assume that $A = B$. Clearly, $d(a, B) = 0$ for every $a \in A$ and $d(A, b) = 0$ for every $b \in B$. So,

$$\max \left\{ \sup_{b \in B} d(A, b), \sup_{a \in A} d(a, B) \right\} = d_{\mathscr{H}}(A, B) = 0.$$

- For the triangle inequality, we want to show that

$$d_{\mathscr{H}}(A, C) \leqslant d_{\mathscr{H}}(A, B) + d_{\mathscr{H}}(B, C).$$

  By the triangle inequality for d, we have

$$d(a, c) \leqslant d(a, b) + d(a, c) \quad \forall a, b, c.$$

So,

$$\begin{aligned} d(a,C) = \inf_{c\in C} d(a,c) &\leqslant \inf_{c\in C}[d(a,b) + d(b,c)] \quad \forall a,b \\ d(a,C) &\leqslant d(a,b) + \inf_{c\in C} d(b,c) \quad \forall a,b \\ d(a,C) &\leqslant d(a,b) + d(b,C) \quad \forall a,b \end{aligned}$$

Now using the fact that $d(b,c) \leqslant \sup_{b\in B} d(b,C)$ for any b, we get

$$d(a,C) \leqslant d(a,b) + \sup_{b\in B} d(b,C) \quad \forall a,b$$

Since the left-hand-side is independent of b,

$$d(a,C) \leqslant d(a,B) + \sup_{b\in B} d(b,C) \quad \forall a$$

Taking the supremum over a we have

$$\sup_{a\in A} d(a,C) \leqslant \sup_{a\in A} d(a,B) + \sup_{b\in B} d(b,C) \tag{4.1}$$

Symmetry gives us that,

$$\sup_{c\in C} d(c,A) \leqslant \sup_{b\in B} d(b,A) + \sup_{c\in C} d(c,B) \tag{4.2}$$

By adding the inequalities (4.1) and (4.2) and dividing by two, we get the desired result:

$$d_{\mathscr{H}}(A,C) \leqslant d_{\mathscr{H}}(A,B) + d_{\mathscr{H}}(B,C)$$

$\square$

Definition 4.4. *Let X be a topological space.*

- *A subset $Z \subseteq X$ is called* sequentially closed *if for every sequence $\{z_n\}$ in Z with a limit point $z \in X$ we have $z \in Z$.*
- *A subset $U \subseteq X$ is called* sequentially open *if every sequence $\{z_n\}$ converging to a point $z \in U$, $\{z_n\}$ is eventually in U.*

Definition 4.5. *A topological space X is called* sequential *if every sequentially closed subset of X is closed. Or equivalently, if every sequentially open subset of X is open.*

Let $SeqTop$ be the category of sequential spaces. Consider the functor

$$\begin{aligned} F : SeqTop^{op} &\to Sets \\ T &\mapsto \{Z \subseteq_{closed} T \times X \text{ and } \pi_1 : Z \to T \text{ is open } \} \end{aligned}$$

Theorem 4.6. *If X is a compact metrizable space, this functor is representable by the metrizable space, $\mathscr{H}_X$, with the Hausdorff topology (Definition 4.1).*

It is natural to ask about the relation between $d_{\mathscr{H}}$ and $d'_{\mathscr{H}}$, if d induces a finer topology than d' on X.

Lemma 4.7. *If $\tau_{d'} \subseteq \tau_d$ and (X, τ_d) is compact (hence $(X, \tau_{d'})$ is also compact), then for* *every* $\varepsilon' > 0$ *there exists* $\varepsilon > 0$ *such that*

$$B_d(x, \varepsilon) \subseteq B_{d'}(x, \varepsilon') \qquad \forall x \in X$$

Proof. d is a finer metric than d' on X. So, for every $x \in X$ and for every $\varepsilon' > 0$ there exists ε_x such that

$$B_d(x, 2\varepsilon_x) \subseteq B_{d'}\left(x, \frac{\varepsilon'}{2}\right) \tag{4.3}$$

Also,

$$\left\{B_d\left(x, \frac{\varepsilon}{2}\right) : x \in X\right\}$$

is an open cover for X. Since (X, τ_d) is compact and

$$B_d\left(x, \frac{\varepsilon_x}{2}\right) \cap B_{d'}\left(x, \frac{\varepsilon'}{2}\right) \in \tau_d \quad \forall x \in X$$

There is a finite set $\{x_1, \ldots, x_n\}$ such that

$$\left\{B_d\left(x_i, \frac{\varepsilon_{x_i}}{2}\right) \cap B_{d'}\left(x_i, \frac{\varepsilon'}{2}\right)\right\}$$

covers X. Now let $\varepsilon := \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_n}\}$

Claim: $B_d(x, \varepsilon) \subseteq B_{d'}(x, \varepsilon') \quad \forall x \in X$.

There exists x_i so that

$$x \in \left\{ B_d\left(x_i, \frac{\varepsilon_{x_i}}{2}\right) \cap B_{d'}\left(x_i, \frac{\varepsilon'}{2}\right) \right\} \tag{4.4}$$

Now pick $y \in B_d(x, \varepsilon)$ arbitrarily.

$$d(x,y) < \varepsilon \quad \& \quad d(x,x_i) < \varepsilon_{x_i} \implies d(x_i,y) < \varepsilon + \varepsilon_{x_i} < 2\varepsilon_{x_i}$$

By (4.3) we have $d'(x_i, y) < \frac{\varepsilon'}{2}$ and by (4.4) we have $d'(x, x_i) < \frac{\varepsilon'}{2}$. Thus,

$$d'(x,y) \leqslant d(x,x') + d(x',y) < \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'$$

The result follows. □

Theorem 4.8. *If $\tau_{d'} \subseteq \tau_d$ and (X, τ_d) is compact (hence $(X, \tau_{d'})$ is also compact), then $\tau_{d'_{\mathscr{H}}} \subseteq \tau_{d_{\mathscr{H}}}$.*

Proof. Let $A \in \mathscr{H}$ and $B_{d'_{\mathscr{H}}}(A, \varepsilon')$ be a neighbourhood of A. We want to find an $\varepsilon > 0$ such that

$$B_{d_{\mathscr{H}}}(A, \varepsilon) \subseteq B_{d'_{\mathscr{H}}}(A, \varepsilon')$$

Choose ε so that (we can choose such ε by Lemma 4.7)

$$d(x, \varepsilon) \subseteq d'\left(x, \frac{\varepsilon'}{2}\right) \quad \forall x \in X$$

holds. Pick $B \in B_{d_{\mathscr{H}}}(A, \varepsilon)$. We have $d_{\mathscr{H}}(A, B) < \varepsilon$. By definition,

$$\max\left\{\sup_{b\in B} d(A,b), \sup_{a\in A} d(a,B)\right\} < \varepsilon$$

and so $d(a, B) < \varepsilon$ $\forall a \in A$ and $d(A, b) < \varepsilon$ $\forall b \in B$. By the choice of ε, we get

$$\begin{aligned} d(a,B) &< \frac{\varepsilon'}{2} \quad \forall a \in A \\ d(A,b) &< \frac{\varepsilon'}{2} \quad \forall b \in B \end{aligned}$$

both holds and thus

$$\max\left\{\sup_{b\in B} d'(A,b), \sup_{a\in A} d'(a,B)\right\} \leqslant \frac{\varepsilon'}{2} < \varepsilon'$$

Therefore

$$B_{d_{\mathscr{H}}}(A, \varepsilon) \subseteq B_{d'_{\mathscr{H}}}(A, \varepsilon')$$

is satisfied. The result follows. $\square$

Corollary 4.9. *Let X be a metric space and let d and d' be two metrics determining the same topology on X. Let X be compact with respect to both metrics. Then, the metrics $d_{\mathscr{H}}$ and $d'_{\mathscr{H}}$ induces the same topology on $\mathscr{H}$.*

Remark 4.10. Observe that Theorem 4.8 and Corollary 4.9 follows from Theorem 4.6. We gave an alternative and direct proof for them above.

Lemma 4.11. *Let Z and Z' be two closed subsets of a metric space X. Then*

$$d_{\mathscr{H}}(Z, Z') < \varepsilon \implies Z' \subseteq B(\varepsilon, Z) \text{ and } Z \subseteq B(\varepsilon, Z')$$

Proof. Assume that $d_{\mathscr{H}}(Z, Z') < \varepsilon$ for $Z, Z' \in \mathscr{H}$. Then,

$$\max\left\{\sup_{z' \in Z'} d(Z, z'), \sup_{z \in Z} d(z, Z')\right\} < \varepsilon$$

So,

$$\sup_{z' \in Z'} d(Z, z') < \varepsilon \quad \text{and} \quad \sup_{z \in Z} d(z, Z') < \varepsilon$$

This yields that $d(z', Z) < \varepsilon$ and $d(z, Z') < \varepsilon$ for any $z' \in Z'$ and $z \in Z$. So,

$$z' \in B_d(Z, \varepsilon) \quad \forall z' \in Z' \quad \text{and} \quad z \in B_d(Z', \varepsilon) \quad \forall z \in Z$$

Then we have

$$Z' \subseteq B(Z, \varepsilon) \text{ and } Z \subseteq B(Z', \varepsilon)$$

as desired. □

Lemma 4.12. *Let Z and Z' be two closed subsets of a metric space X. Then*

$$Z' \subseteq B(\varepsilon, Z) \text{ and } Z \subseteq B(\varepsilon, Z') \implies d_{\mathscr{H}}(Z, Z') \leqslant \varepsilon$$

Proof. Assume that $Z' \subseteq B(Z,\varepsilon)$ and $Z \subseteq B(Z',\varepsilon)$. Then $z' \in B_d(Z,\varepsilon)$ for every $z' \in Z'$ and $z \in B_d(Z',\varepsilon)$ for every $z \in Z$. Since both Z and Z' are closed in X we have $\sup_{z'\in Z'} d(Z,z') \leqslant \varepsilon$ and $\sup_{z\in Z} d(z,Z') \leqslant \varepsilon$. Thus,

$$\max\left\{\sup_{z'\in Z'} d(Z,z'), \sup_{z\in Z} d(z,Z')\right\} \leqslant \varepsilon$$

Therefore $d_{\mathscr{H}}(Z,Z') \leqslant \varepsilon$ for $Z, Z' \in \mathscr{H}$. □

Corollary 4.13. *If Z and Z' are two closed subsets of a metric space, then*

$$d_{\mathscr{H}}(Z,Z') = \inf\{\varepsilon > 0 : Z' \subseteq B(\varepsilon, Z) \textit{ and } Z \subseteq B(\varepsilon, Z')\}$$

Remark 4.14. The Hausdorff topology on the set of closed subspaces of a metrizable space X may depend on the choice of the metric inducing the topology on X even if these metrics are equivalent.

Example 4.15. Consider two metrics determining the same topology on $\mathbb{R}$: one is the usual (Euclidean) metric and the other one is given by $d(x,y) = |\arctan x - \arctan y|$. Define $Z_n := \{x_k^n : k \in \mathbb{Z}_{>0}\}$ where $x_k^n = k + \frac{1}{n}$ when $n \neq k$ and $x_k^n = n + \frac{1}{2}$ otherwise. Let $Z := \mathbb{Z}_{>0}$. Then the Hausdorff distance with respect to the usual metric between the sets Z and Z_n is $\frac{1}{2}$, whereas the Hausdorff distance with respect to d between Z and Z_n tends to 0 as $n \to \infty$.

Theorem 4.16. *The Hausdorff topology on the set of* compact *subspaces of a metrizable space X does not depend on the choice of the metric inducing the topology on X.*

Proof. Let d and d' be two metrics determining the same topology on X. Let K be a compact subspace of X and let $\varepsilon > 0$. We want to show that there exists $\varepsilon' > 0$ satisfying

$$B_{d'_{\mathscr{H}}}(K,\varepsilon') \subseteq B_{d_{\mathscr{H}}}(K,\varepsilon) \tag{4.5}$$

For each $k \in K$ there exists ε_k such that

$$B_{d'}(k, 2\varepsilon_k) \subseteq B_d\left(k, \frac{\varepsilon}{3}\right) \tag{4.6}$$

since d and d' induce the same topology on X. So,

$$\bigcup_{k \in K} B_{d'}(k, \varepsilon_k)$$

is an open cover of K. There is a finite set $\{k_1, k_2, \ldots, k_N\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=1}^{N} B_{d'}(k_i, \varepsilon_{k_i}) \quad \text{and} \quad K \subseteq \bigcup_{i=1}^{N} B_d\left(k_i, \frac{\varepsilon}{3}\right) \tag{4.7}$$

Now let $\varepsilon' := \min\{\varepsilon_{k_i} : 1 \leqslant i \leqslant N\}$. To prove the relation (4.5) above, let $L \in B_{d'_{\mathscr{H}}}(K, \varepsilon')$ and let L be compact. Then $d'_{\mathscr{H}}(L, K) < \varepsilon'$. We have

$$\max\left\{\sup_{k \in K} d'(k, L), \sup_{l \in L} d'(l, K)\right\} < \varepsilon'$$

And thus

$$d'(k, L) < \varepsilon' \quad \text{and} \quad d'(l, K) < \varepsilon' \quad \forall k \in K \quad \forall l \in L$$

Then, $\forall k \in K$ we have $d'(k, L) < \varepsilon'$. So, there exists $l \in L$ such that $d'(k, l) < \varepsilon'$. That is, $l \in B_{d'}(k, \varepsilon')$. So, $l \in B_{d'}(k_i, \varepsilon_{k_i} + \varepsilon')$ for some $l \in L$ and thus $l \in B_{d'}(k_i, 2\varepsilon_{k_i})$.

Using the relation (4.6) we get that $l \in B_d\left(k_i, \frac{\varepsilon}{3}\right)$ for some $l \in L$. Also, by the relation (4.7) we have, $\forall k \in K$ $\exists l \in L$ such that $d(k,l) < \frac{2\varepsilon}{3}$. Hence,

$$\sup_{k \in K} d(k, L) \leqslant \frac{2\varepsilon}{3} \tag{4.8}$$

Similarly, since $d'(l, K) < \varepsilon' \quad \forall l \in L$, we have

$$\begin{aligned}
d'(l,K) < \varepsilon' &\implies \forall l \in L \quad \exists k \in K \text{ such that } d'(l,k) < \varepsilon' \\
&\implies \forall l \in L \quad \exists k \in K \text{ such that } k \in B_{d'}(l, \varepsilon') \\
&\implies \forall l \in L \quad \exists i \in \{1, \ldots, N\} \text{ such that } k_i \in B_{d'}(l, \varepsilon' + \varepsilon_{k_i}) \\
&\implies \forall l \in L \quad \exists i \in \{1, \ldots, N\} \text{ such that } k_i \in B_{d'}(l, 2\varepsilon_{k_i}) \\
&\implies \forall l \in L \quad \exists i \in \{1, \ldots, N\} \text{ such that } l \in B_d\left(k_i, \frac{\varepsilon}{3}\right) \\
&\implies \forall l \in L \quad \exists i \in \{1, \ldots, N\} \text{ such that } d(k_i, l) < \frac{\varepsilon}{3} \\
&\implies \forall l \in L \quad \exists k \in K \text{ such that } d(k,l) < \frac{2\varepsilon}{3} \\
&\implies \forall l \in L \quad d(l,K) < \frac{2\varepsilon}{3}
\end{aligned}$$

Therefore we get

$$\sup_{l \in L} d(l, K) \leqslant \frac{2\varepsilon}{3} \tag{4.9}$$

The inequalities (4.8) and (4.9) give us that

$$d(K,L) = \max\left\{\sup_{k \in K} d(k,L), \sup_{l \in L} d(l,K)\right\} \leqslant \frac{2\varepsilon}{3} < \varepsilon$$

So, the relation (4.5) is proved. Similarly, we can prove that for every $\varepsilon' > 0$ and for every compact $K \subseteq X$ there exists ε such that

$$B_{d_{\mathscr{H}}}(K_\varepsilon) \subseteq B_{d'_{\mathscr{H}}}(K, \varepsilon') \tag{4.10}$$

Combining the relations (4.5) and (4.10) together, the result follows. $\square$

Proposition 4.17. *Let K be a compact and Z be a closed subspace of a metric space X such that $Z \cap K = \varnothing$. Then the minimal distance between Z and K is positive.*

Proof. Assume for a contradiction that $d(Z, K) := \inf\{d(z, k) : z \in Z, k \in K\} = 0$. Then there are sequences $\{z_n\} \in Z$ and $\{k_n\} \in K$ such that $\lim_{n\to\infty} d(z_n, k_n) = 0$. Every sequence in a compact metric space has a convergent subsequence. So, there is a subsequence k_{n_m} such that $k_{n_m} \to k$ for some $k \in K$. Our claim is to show that $\lim_{n\to\infty} z_n = k$.

For every $\varepsilon > 0$ there exists N such that for every $n > N$ we have $d(k_n, z_n) < \frac{\varepsilon}{2}$. Then there exists $M_1 > N$ such that for every $m, n > M_1$ we have $d(k_{n_m}, z_n) < \frac{\varepsilon}{2}$. Also there is a number M_2 such that for every $m > M_2$ we get $d(k_{n_m}, k) < \frac{\varepsilon}{2}$. Thus, for every $n > \max\{M_1, M_2\}$ we have

$$d(z_n, k) < d(k_{n_m}, z_n) + d(k_{n_m}, k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So, z_n converges to k. But since Z is closed, k must lie in K. Contradiction. $d(Z, K) > 0$. $\square$

Lemma 4.18. *Let X be a metric space and K be a compact subspace of X. Then if U is a open subset of X containing K, there exists $\varepsilon > 0$ such that $B(\varepsilon, K) \subseteq U$.*

Proof. Let $Z := X - U$. Then Z is a closed subset of X. By Proposition 4.17, we must have $d(Z, K) := \inf\{d(z, k) : z \in Z, k \in K\} = \varepsilon > 0$.

Now we claim that $B(\varepsilon, K) \subseteq U$. To see this fact, pick $x \in B(\varepsilon, K)$. Then $x \notin Z$ by the choice of ε and therefore $x \in U$. The result follows. □

Proposition 4.19. *[4] If X is a totally bounded metric space, so is $\mathscr{H}$.*

Proof. Let $\varepsilon > 0$. Since X totally bounded, there is a finite subset $K := \{x_1, \ldots, x_n\}$ of X with

$$X = \bigcup_{i=1}^{n} B\left(x_i, \frac{\varepsilon}{2}\right)$$

Now consider the set $\mathcal{P}(K) - \{\varnothing\} =: \{Z_1, \ldots, Z_{2^n-1}\} \subseteq \mathscr{H}$. We want to show that

$$\bigcup_{j=1}^{2^n-1} B(Z_j, \varepsilon) = \mathscr{H}$$

It suffices to show that for every $Z \in \mathscr{H}$ there exists $j_0 \in \{1, 2, \ldots, 2^n - 1\}$ such that $Z \subseteq B\left(\frac{\varepsilon}{2}, Z_{j_0}\right)$ and $Z_{j_0} \subseteq B\left(\frac{\varepsilon}{2}, Z\right)$.

To prove the claim above, let

$$Z_{j_0} := \{x_i : B(x_i, \frac{\varepsilon}{2}) \cap Z \neq \varnothing,\ i = 1, \ldots, n\}$$

For every $z \in Z$ there exists x_{i_0} such that $z \in B(x_{i_0}, \frac{\varepsilon}{2})$ and hence $x_{i_0} \in Z_{j_0}$. Thus, $z \in B(x_{i_0}, \frac{\varepsilon}{2}) \subseteq B(\frac{\varepsilon}{2}, Z_{j_0})$ for every $z \in Z$. Therefore, $Z \subseteq B(\frac{\varepsilon}{2}, Z_{j_0})$.

Now let $x_i \in Z_{j_0}$. Then, $B(x_i, \frac{\varepsilon}{2}) \cap Z \neq \varnothing$. So, there exists $z \in Z$ such that $d(x_i, z) < \frac{\varepsilon}{2}$. Thus, $x_i \in B(z, \frac{\varepsilon}{2}) \subseteq B(\frac{\varepsilon}{2}, Z)$. So, $Z_{j_0} \subseteq B(\frac{\varepsilon}{2}, Z)$.

So, by Lemma 4.12 we have

$$d_{\mathscr{H}}(Z, Z_{j_0}) \leqslant \frac{\varepsilon}{2} < \varepsilon$$

Therefore, $\mathscr{H}$ is totally bounded. $\square$

Proposition 4.20. *[5] If X is a complete metric space, so is $\mathscr{H}$.*

Proof. Let X be complete and $\{Z_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\mathscr{H}, d_{\mathscr{H}})$. Let

$$Z := \{x \in X : \exists \{z_n\}_{n \in \mathbb{N}} \text{ such that } z_n \in Z_n \text{ and } x \text{ is a limit point of } z_n\}$$

We claim that Z is the limit of Z_n with respect to the Hausdorff metric.

We start by showing that Z is closed in X. Let $\{x_n\}$ be a convergent sequence in Z with a limit $x \in X$. We want to show that $x \in Z$. It suffices to show that there exists a sequence $\{z_n : z_n \in Z_n\}$ converging to x. Let $\varepsilon > 0$.

There exists $x_{n_1} \in B\left(x, \frac{\varepsilon}{2}\right)$. Also, since $x_{n_1} \in Z$, there exists $z_{m_1} \in Z_{m_1}$ such that $z_{m_1} \in B\left(x_{n_1}, \frac{\varepsilon}{2}\right)$. So, $d(x, z_{m_1}) < \varepsilon$.

There exists $x_{n_2} \in B\left(x, \frac{\varepsilon}{4}\right)$. Also, since $x_{n_2} \in Z$, there exists $m_2 > m_1$ and $z_{m_2} \in Z_{m_2}$ such that $z_{m_2} \in B\left(x_{n_2}, \frac{\varepsilon}{4}\right)$. So, $d(x, z_{m_2}) < \frac{\varepsilon}{2}$.

Similarly, there exists $x_{n_3} \in B\left(x, \frac{\varepsilon}{8}\right)$. Also, since $x_{n_3} \in Z$, there exists $m_3 > m_2$ and $z_{m_3} \in Z_{m_3}$ such that $z_{m_3} \in B\left(x_{n_3}, \frac{\varepsilon}{8}\right)$. So, $d(x, z_{m_3}) < \frac{\varepsilon}{4}$.

With this progress we can construct a convergent subsequence $\{z_{m_k}\}$ of a sequence

$\{z_m\}$ such that $z_{m_k} \to x$. Thus, $x \in Z$ and hence Z is a closed subspace of X.

Now it remains show that Z is the limit of the sequence $\{Z_n\}$. We want to show that for every $\varepsilon > 0$ there exists N such that $\forall n \geqslant N$ we have $d_{\mathscr{H}}(Z_n, Z) < 3\varepsilon$.

Let $\varepsilon > 0$. There exists N_1 such that $d_{\mathscr{H}}(Z_n, Z_m) < \varepsilon$ for all $m, n > N_1$. So, $Z_n \subseteq B(\varepsilon, Z_m)$ and $Z_m \subseteq B(\varepsilon, Z_n)$. Let $z \in Z$. Since $z \in Z$, there exists a sequence $\{z_n\}$ with $z_n \in Z_n$ and $z_n \to z$.

Consider the sequence $x_{k+1} = z_{n+k}$. We have

$$\begin{aligned}
x_1 &= z_n \in Z_n \subseteq B(\varepsilon, Z_n) \\
x_2 &= z_n + 1 \in Z_{n+1} \subseteq B(\varepsilon, Z_n) \\
x_3 &= z_n + 2 \in Z_{n+2} \subseteq B(\varepsilon, Z_n) \\
&\vdots
\end{aligned}$$

By construction, since $z_n \to z$ we must also have $x_k \to z$. So, z is a limit point of $B(\varepsilon, Z_n)$ and hence

$$Z \subseteq \overline{B(\varepsilon, Z_n)} \subseteq B(2\varepsilon, Z_n) \tag{4.11}$$

It remains to show that $Z_n \subseteq B(2\varepsilon, Z)$. Let $\varepsilon > 0$. There exists $N \geqslant N_1$ such that $\forall m, n > N$ we have $d_{\mathscr{H}}(Z_n, Z_m) < \frac{\varepsilon}{2}$. Also there exists numbers $n_1 < n_2 < n_3 < \ldots$ all bigger than N such that

$$\begin{aligned}
d_{\mathscr{H}}(Z_m, Z_n) &< \frac{\varepsilon}{4} \quad \forall m, n > n_1 \\
d_{\mathscr{H}}(Z_m, Z_n) &< \frac{\varepsilon}{8} \quad \forall m, n > n_2 \\
d_{\mathscr{H}}(Z_m, Z_n) &< \frac{\varepsilon}{16} \quad \forall m, n > n_3 \\
&\vdots
\end{aligned}$$

And hence

$$\begin{aligned}
Z_n &\subseteq B\left(\frac{\varepsilon}{2}, Z_{n_1}\right) \\
Z_{n_1} &\subseteq B\left(\frac{\varepsilon}{4}, Z_{n_2}\right) \\
Z_{n_2} &\subseteq B\left(\frac{\varepsilon}{8}, Z_{n_3}\right) \\
&\vdots
\end{aligned}$$

Let $z_n \in Z_n$. By the subset relationships above we see that

$$\begin{aligned}
&\exists z_{n_1} \in Z_{n_1} \quad \text{such that} \quad d(z_n, z_{n_1}) < \frac{\varepsilon}{2} \\
&\exists z_{n_2} \in Z_{n_2} \quad \text{such that} \quad d(z_{n_1}, z_{n_2}) < \frac{\varepsilon}{4} \\
&\exists z_{n_3} \in Z_{n_3} \quad \text{such that} \quad d(z_{n_2}, z_{n_3}) < \frac{\varepsilon}{8} \\
&\qquad \vdots
\end{aligned}$$

It follows that the sequence $\{z_{n_k}\}$ is Cauchy, so, since X is complete, and $Z \subseteq_{\text{closed}} X$,

$z_{n_k} \to z$ for some $z \in Z$. So,

$$\begin{aligned} d(z_n, z_{n_k}) &< d(z_n, z_{n_1}) + d(z_{n_1}, z_{n_2}) + \cdots + d(z_{n_{k_1}}, z_{n_k}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots + \frac{\varepsilon}{2^k} < \varepsilon \end{aligned}$$

So, $d(z_n, z) \leqslant \varepsilon$ and thus $z_n \in \overline{B(z, \varepsilon)} \subseteq \overline{B(\varepsilon, Z)}$. So, we get

$$Z_n \subseteq \overline{B(\varepsilon, Z)} \subseteq B(2\varepsilon, Z) \tag{4.12}$$

By equations (4.11) and (4.12) and using Lemma 4.12 we have $d(Z_n, Z) \leqslant 2\varepsilon < 3\varepsilon$ when $n \geqslant N$ as desired. □

Corollary 4.21. *If X is a compact metric space, so is $\mathscr{H}$.*

Proof. By Theorem 45.1 in Munkres [3], a metric space is compact if and only if it is totally bounded and complete. So, using Proposition 4.19 and Proposition 4.20 give us the desired result. □

Definition 4.22. *Let $\{Z_t\}_{t \in \mathbb{C}} \subseteq \mathscr{H}$. Then, $Z_{t_0} \in \mathscr{H}$ is the* Hausdorff limit *of Z_t as $t \to t_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|t - t_0| < \delta \implies d_{\mathscr{H}}(Z_t, Z_{t_0}) < \varepsilon$$

Example 4.23. Consider the set $Z_t := \{(x, y) \in \mathbb{C}^2 : xy = t\}$. We will show that $Z_0 = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$ is the Hausdorff limit of Z_t as $t \to 0$.

Let $\varepsilon > 0$ and let $\delta = \min\{\frac{\varepsilon}{2}, \varepsilon^2\}/2$

Assume that $|t| < \delta$. Let $(x_t, y_t) \in Z_t$. Then $d((x_t, y_t), Z_0) = \min\{|x_t|, |y_t|\}$. Observe that for any $(x_t, y_t) \in Z_t$, we have

$$\min\{|x_t|, |y_t|\} \leqslant \sqrt{|t|} < \sqrt{\delta} < \varepsilon$$

So,

$$\sup_{(x_t,y_t)\in Z_t} d((x_t, y_t), Z_0) < \varepsilon \tag{4.13}$$

Now let $(x_0, y_0) \in Z_0$. Either x_0 or y_0 equals to zero. Without loss of generality, assume $y_0 = 0$. Let $x_0 = r_0 e^{i\theta_0}$. Z_t consists of the elements of the form $\left(re^{i\theta_1}, \frac{t}{r}e^{i\theta_2}\right)$. Then,

$$d((x_0, 0), Z_t) = \inf_{r\in\mathbb{R}^+} \left\{ \sqrt{(r_0 - r)^2 + \left(\frac{t}{r}\right)^2} \right\} \leqslant |2t| < 2\delta < \varepsilon$$

So,

$$\sup_{(x_0,y_0)\in Z_0} d((x_0, y_0), Z_t) < \varepsilon \tag{4.14}$$

Using the inequalities (4.13) and (4.14) we get that

$$\max\left\{ \sup_{(x_t,y_t)\in Z_t} d((x_t, y_t), Z_0), \sup_{(x_0,y_0)\in Z_0} d((x_0, y_0), Z_t) \right\} < \varepsilon$$

Proposition 4.24. *Let X be a metric space and $\mathscr{H}$ be the set of closed subsets of X with the Hausdorff topology. Then the subset $\mathcal{Z} = \{(Z, x) : x \in Z\}$ is closed in $\mathscr{H} \times X$ and the projection map $\pi_1^{\mathcal{Z}} : \mathcal{Z} \to \mathscr{H}$ is open.*

Proof. We want to show that $(\mathscr{H} \times X) - \mathcal{Z}$ is open. Let $(Z, x) \notin \mathcal{Z}$. Then $x \notin Z$. We have $B_d(x, \varepsilon) \cap Z = \varnothing$ for some $\varepsilon > 0$. Now we claim that

$$\left\{B_{d_{\mathscr{H}}}\left(Z, \frac{\varepsilon}{2}\right) \times B_d\left(x, \frac{\varepsilon}{2}\right)\right\} \cap \mathcal{Z} = \varnothing$$

To prove the claim, suppose $Z' \in B_{d_{\mathscr{H}}}(Z, \frac{\varepsilon}{2})$ and $x' \in Z'$. Then $d_{\mathscr{H}}(Z', Z) < \frac{\varepsilon}{2}$. There must be some $z \in Z$ so that $d(x', z) < \frac{\varepsilon}{2}$. Then $x' \in B_d(x, \frac{\varepsilon}{2})$ and so $d(x, x') < \frac{\varepsilon}{2}$, so we reach the contradiction

$$d(x, z) < d(x, x') + d(x', z) < \varepsilon.$$

Hence $\mathcal{Z}$ is closed.

It suffices to show that each $\pi_1^{\mathcal{Z}}(U)$ is open for every U in some basis $\mathcal{U}$ for $\mathscr{H}$. Now we want to show that the projection map $\pi_1^{\mathcal{Z}} : \mathcal{Z} \to \mathscr{H}$ is open. So, it is enough to show that

$$\pi_1^{\mathcal{Z}}\left((B_{\mathscr{H}}(Z, \varepsilon) \times B_X(x, \varepsilon)) \cap \mathcal{Z}\right) = B_{\mathscr{H}}(Z, \varepsilon)$$

for any $Z \in \mathscr{H}$, $x \in Z$, $\varepsilon > 0$. Clearly we have

$$B_{\mathscr{H}}(Z, \varepsilon) \supseteq \pi_1\left((B_{\mathscr{H}}(Z, \varepsilon) \times B_X(x, \varepsilon)) \cap \mathcal{Z}\right)$$

Now for the other direction let $Z' \in B_{\mathscr{H}}(Z, \varepsilon)$. Then

$$Z \subseteq B(\varepsilon, Z') = \bigcup_{z' \in Z'} B_X(z', \varepsilon).$$

So, there exists $x' \in Z'$ such that $x \in B_X(x', \varepsilon)$. Hence $x' \in B_X(x, \varepsilon)$ and therefore

$$Z' \in \pi_1 \left((B_{\mathscr{H}}(Z, \varepsilon) \times B_X(x, \varepsilon)) \cap \mathcal{Z} \right).$$

The result follows. $\square$

Remark 4.25. Let X be a topological space. Let $\mathcal{K}$ be the set of compact and metrizable subspaces of X. Then the map

$$\mathscr{H}_X \hookrightarrow \prod_{K \in \mathcal{K}} \mathscr{H}_K$$

is injective.

Since singletons are always compact and metrizable, the image of a closed set under the map above should have its singletons as coordinates. So, if two closed sets are not equal, than they differ in at least one point. Then, their images will differ in at least one coordinate which leads to the injectivity of the map above.

Lemma 4.26. *Let T be a sequential space and Y be an arbitrary topological space. Let $\overline{\mathbb{N}}$ be the one-point-compactification of a countably infinite discrete space. Then, a function $f : T \to Y$ is continuous if and only if $f \circ g$ is continuous for every continuous map $g : \overline{\mathbb{N}} \to T$.*

Proof. ($\Longrightarrow$) Let $f : T \to Y$ and $g : \overline{\mathbb{N}} \to T$ be two continuous maps. We want to show that $f \circ g : \overline{\mathbb{N}} \to Y$ is continuous. Let U be an open subset of Y. $f^{-1}(U)$ is open in T by the continuity of f. Also, $g^{-1}(f^{-1}(U))$ is open in $\overline{\mathbb{N}}$ by the continuity of g. Thus, $(f \circ g)^{-1}(U)$ is open in $\overline{\mathbb{N}}$. Hence, $f \circ g$ is continuous.

($\Longleftarrow$) Assume that $f : T \to Y$ is not continuous. Then since T is a sequential space, there exists an open subset $U \subseteq Y$ such that $A := f^{-1}(U)$ is not open in T. There exists a continuous map $g : \overline{\mathbb{N}} \to T$ such that $g^{-1}(A)$ is not open in $\overline{\mathbb{N}}$. Thus, $(f \circ g)^{-1}(U)$ is not open in $\overline{\mathbb{N}}$. Therefore, $f \circ g$ is not continuous. $\square$

Corollary 4.27. *Let T be a sequential space and X be an arbitrary topological space. A map $f : T \to X$ is continuous at a point $t_0 \in T$ if and only if for every sequence $\{t_n\} \subseteq T$ converging to t_0, the map $f\big|_{\{t_n : n \in \mathbb{N}\} \cup \{t_0\}}$ is continuous.*

Theorem 4.28. *Let T be a topological space and X be a compact metrizable space. Let W be a closed subset of $T \times X$. For $t \in T$, define $W_t := \{x \in X : (t, x) \in W\}$. If the map $F_W : T \to \mathscr{H}_X$ defined by $F_W(t) := W_t$ is continuous, then the projection map $\pi_1^W : W \to T$ is open. The converse holds if T is sequential (Definition 4.5).*

Proof. ($\implies$) Suppose F_W is continuous. From the definition of F_W we have

$$W = \{(t, P) \in T \times \mathscr{Z} : F_W(t) = \pi_1^{\mathscr{Z}}(P)\}$$

It follows that every square in the diagram

$$\begin{tikzcd}
W \arrow[r, "(F_W \times Id_X)\big|_W"'] \arrow[d] \arrow[dd, bend right, "\pi_1^W"'] & \mathscr{Z} \arrow[d, hook] \arrow[dd, bend left, "\pi_1^{\mathscr{Z}}"] \\
T \times X \arrow[r, "F_W \times Id_X"] \arrow[d] & \mathscr{H}_X \times X \arrow[d, "\pi_1"] \\
T \arrow[r, "F_W"] & \mathscr{H}_X
\end{tikzcd}$$

is cartesian. In particular,

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\pi_1^W"'] & \mathcal{Z} \arrow[d, "\pi_1^{\mathcal{Z}}"] \\
T \arrow[r] & \mathscr{H}_X
\end{tikzcd}$$

is cartesian.

We saw in Proposition 4.24 that $\pi_1^{\mathcal{Z}}$ is open and open maps are stable under base change. Therefore, π_1^W is open.

($\Longleftarrow$) Assume that $\pi_1^W : W \to T$ is an open map and T is sequential. Let $t_0 \in T$. We want to show that f_W is continuous at t_0. Let $\{t_n\}$ be a sequence converging to t_0. By the corollary of Lemma 4.26 it suffices to show that the map $f : T' \to X$ is continuous where $T' := \{t_0\} \cup \{t_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$. We want to show that there is a neighborhood U of t_0 in T' satisfying

(i) $\forall t \in U \quad W_t \subseteq B(\varepsilon, W_{t_0})$ and

(ii) $\forall t \in U \quad W_{t_0} \subseteq B(\varepsilon, W_t)$.

It is enough to find U_1 satisfying (i), and U_2 satisfying (ii), for then we can take $U := U_1 \cap U_2$.

First, we want to show that such U_1 exists. Assume for a contradiction that such U_1 does not exist. Then we could find a subsequence $\{m_n\}$ of $\{t_n\}$ and points $w_n \in W_{m_n} \subseteq X$ such that

$$d(W_{t_0}, w_n) \geqslant \varepsilon \quad \forall n \in \mathbb{N} \tag{4.15}$$

By compactness of W, we can assume, after passing to a subsequence, that the points $(m_n, w_n) \in W$ converge to a point $(0, w_0) \in W$. This implies that the w_n converge to $w_0 \in W_0$ in X which contradicts with the inequality (4.15).

Now we want to prove that such U_2 exists. Since T is a sequential space, a singleton is closed. Therefore, W_{t_0} is a closed subset of X, hence it is compact. So, there are points $w_1, \ldots, w_n \in W_{t_0} \subseteq X$ such that

$$W_{t_0} \subseteq \bigcup_{i=1}^{n} B\left(w_i, \frac{\varepsilon}{2}\right) \tag{4.16}$$

Now take

$$U_2 := \bigcap_{i=1}^{n} \pi_1\left(\left(T \times B\left(w_i, \frac{\varepsilon}{2}\right)\right) \cap W\right)$$

To prove that U_2 satisfies (ii), let $w_0 \in W_{t_0}$ and $t \in U_2$. By the relation (4.16) $\exists i \in \{1, \ldots, n\}$ such that $d(w_0, w_i) < \frac{\varepsilon}{2}$. Since $U_2 \subseteq \pi_1((T \times B(w_i, \frac{\varepsilon}{2})) \cap W)$, there exists $w \in W_t$ such that $d(w_i, w) < \frac{\varepsilon}{2}$. Then $d(w_0, w) < \varepsilon$. Since w_0 was arbitrary, we have $W_{t_0} \subseteq B(\varepsilon, W_t)$ as desired.

Now let $U := U_1 \cap U_2$. For every $t \in U$, by Lemma 4.12 we get that $d_{\mathscr{H}}(Z_0, Z_t) \leqslant \varepsilon$. Therefore, F_W is continuous. $\square$

5. THE HILBERT SCHEME AND THE HILBERT QUOTIENT

In this chapter, we begin by recalling some basic properties of the *Hilbert scheme* of a projective variety X (over $\mathbb{C}$). We explain how the Hilbert scheme can be used to define the *Hilbert quotient* $X /\!\!/_{Hilb} G$ of an algebraic group G acting on X. The analogous construction of the "Hausdorff quotient" will be the subject of the next chapter.

Let X be a complex variety. Let Sch and $Sets$ be the category of schemes over $\mathbb{C}$ and the category of sets, respectively. Consider the functor

$$\begin{aligned} F : Sch^{op} &\to Sets \\ T &\mapsto \{Z \subseteq_{closed} T \times X : \pi_1 : Z \to T \text{ is flat}\} \end{aligned}$$

Theorem 5.1 (Grothendieck)**.** *If X is projective, the functor F is representable by a disjoint union of projective schemes, $HilbX$, called the* Hilbert scheme *of X.*

Remark 5.2. [6] One has

$$HilbX = \coprod_p Hilb_p X$$

where $p \in \mathbb{Q}[x]$ and $Hilb_p X$ is the moduli space of closed subschemes of X with "Hilbert polynomial" p (with respect to some ample line bundle $\mathcal{O}_X(1)$ on X). Each $Hilb_p X$ is projective.

Definition 5.3. *Let G be an algebraic group and X be a projective variety. Consider the* *function*

$$f : X \to HilbX$$
$$x \mapsto \overline{Gx}$$

By generic flatness results from algebraic geometry, there is a Zariski open and dense subvariety U of X such that the set

$$\{\overline{Gx} : x \in U\}$$

forms a flat family over U/G and

$$f : U/G \to HilbX$$
$$x \mapsto \overline{Gx}$$

is an embedding. Then we define the Hilbert Quotient *[7] of X by G as*

$$X /\!\!/_{Hilb} G := \overline{f(U/G)}^{HilbX}$$

Remark 5.4. The Hilbert Quotient is independent of the choice of U as above. The set $f(U/G)$ is contained in $Hilb_p X$ for some polynomial p, so $X /\!\!/_{Hilb} G$ is projective.

Finally, we want to construct the *Hilbert-Hausdorff morphism.* For a finite-type $\mathbb{C}$-scheme X, we write X^{an} for the set $X(\mathbb{C})$ of $\mathbb{C}$ points of X, with the analytic topology [8]. The space X^{an} is compact (resp. Hausdorff) iff X is proper (resp. separated) [8]. The map $X \mapsto X^{an}$ is functorial in X, preserves (fibered) products and takes closed embeddings (of schemes) to closed embeddings (of topological spaces). Now let X be a projective variety over $\mathbb{C}$. Let $Z \subseteq (HilbX) \times X$ be the universal closed subscheme. We know that the map

$$\pi_1 : Z \to HilbX \tag{5.1}$$

is flat. A well-known variation of Serre's GAGA results [8] says that flat map of finite-type $\mathbb{C}$-schemes is open in the analytic topology. Such a map is also open in the Zariski topology. So,

$$\pi_1^{an} : Z^{an} \to (HilbX)^{an} \tag{5.2}$$

is open.

By Theorem 4.6 and (5.2) we get the Hilbert-Hausdorff morphism:

$$f_{Z^{an}} : (HilbX)^{an} \to \mathscr{H}_{X^{an}} \tag{5.3}$$

Define $(HilbX)_{red} := \{Z \in HilbX : Z \text{ is reduced}\}$.

Definition 5.5. *Let $(HilbX)^{an}_{red}$ (resp. $(Hilb_pX)^{an}_{red}$) be the subspace of $(HilbX)^{an}$ (resp. $(Hilb_pX)^{an}$) whose points correspond to reduced closed subschemes of X (resp. with Hilbert polynomial p).*

In fact we suspect an even closer relationship when we restrict to *reduced* subschemes. By restriction, the Hilbert-Hausdorff morphism yields a continuous map:

$$f : (Hilb_pX)^{an}_{red} \longrightarrow \mathscr{H}_{X^{an}} \tag{5.4}$$

Conjecture. *The map given in (5.4) is an embedding.*

Remark 5.6. The map given in (5.4) may not be an embedding if it's not restricted to a polynomial p.

6. THE HAUSDORFF QUOTIENT

Definition 6.1. *Let G be a topological group acting continuously on a metric space X. Let $\mathcal{U}$ be the partially ordered family of open, dense, G-invariant subsets of X. For $U \in \mathcal{U}$, consider the map* of sets

$$\begin{aligned} e : U &\to \mathscr{H}_X \\ e(x) &:= \overline{Gx}^X. \end{aligned}$$

The Hausdorff quotient of X by G, *denoted by* $X /\!\!/_{Haus} G$ *is*

$$X /\!\!/_{Haus} G := \bigcap_{U \in \mathcal{U}} \overline{e(U)}^{\mathscr{H}_X}$$

Remark 6.2. The map $e : U \to \mathscr{H}$ is not necessarily continuous.

Example 6.3. An example for which the map e is not continuous can be obtained by taking $X = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and $G = \mathbb{R}^+$. Let the action $G \circlearrowright X$ be defined as $g \cdot (x, y) := (gx, gy)$. Then, $e((x, y))$ is a ray from origin passing through the point (x, y). For any two points $(x_1, y_1), (x_2, y_2) \in X$, if $\frac{x_1}{x_2} \neq \frac{y_1}{y_2}$ then the (Hausdorff) distance between the images of these two points is infinite. So, the map e is nowhere continuous on X.

Remark 6.4. The map e is clearly constant on $G-$orbits, so it yields a map of sets $e : U/G \to \mathscr{H}_X$.

Definition 6.5. *A set $U \in \mathcal{U}$ is called* semi-stable *if*

$$\overline{e(U')}^{\mathscr{H}_X} = \overline{e(U)}^{\mathscr{H}_X}$$

for all $U' \subseteq U$ with $U' \in \mathcal{U}$.

Remark 6.6. For reasonable $G \circlearrowright X$ (such as a Lie group acting on a compact manifold) there is a *semi-stable* $U \in \mathcal{U}$.

Definition 6.7. *We call $U \in \mathcal{U}$* stable *if and only if U is semi-stable, $e : U \to \mathscr{H}$ is continuous, and the induced map $\overline{e} : U/G \to \mathscr{H}$ is an embedding when U/G is given the quotient topology.*

Theorem 6.8. *Let U be an open, dense, G-invariant subset of a metric space X. If U is stable, then U is semi-stable.*

Proof. Assume that the map

$$\begin{aligned} e : U &\to \mathscr{H}_X \\ x &\mapsto \overline{Gx}^X \end{aligned}$$

yields an embedding

$$\overline{e} : U/G \hookrightarrow \mathscr{H}_X$$

so that U is stable. If V is an open, dense, G-invariant subset of X with $V \subseteq U$, then V is open and dense in U. It follows that V/G is open and dense in U/G. Since $\bar{e} : U/G \hookrightarrow \mathscr{H}_X$ is an embedding, we have that $\bar{e}(V/G)$ is an open and dense subset of $\bar{e}(U/G)$. So, $\bar{e}(V/G)$ and $\bar{e}(U/G)$ must have the same closure in $\mathscr{H}_X$. Therefore, U is semi-stable. □

Example 6.9. Consider the 2-simplex

$$\Delta_2 := \frac{\mathbb{R}^3_{\geqslant 0} - \bar{0}}{\mathbb{R}_{>0}}$$

Let $G = \mathbb{R}_{>0}$ under multiplication. G acts on $X := \Delta_2$ by

$$G \times X \to X$$
$$g \cdot [x, y, z] := [gx, g^{-1}y, z]$$

In particular, $\{\{(x, y, z) : z = 0\} \cap \Delta_2\}$ and $\{\{(x, y, z) : x.y = 0\} \cap \Delta_2\}$ are two orbits of this action. So, the orbits are bijective with the closed interval $[0, 1]$. Any $U \in \mathcal{U}$ satisfying $U \subseteq \mathring{\Delta}_2$ gives

$$\overline{e(U)}^{\mathscr{H}_X} = \overline{e(\mathring{\Delta}_2)}^{\mathscr{H}_X}$$

Hence, the semi-stable subset of Δ_2 is its interior $\mathring{\Delta}_2$ for this action.

Remark 6.10. The *Hausdorff quotient* $X /\!\!/_{Haus} G$ depends only on the *topology* of X if X is compact and metrizable. However, if X is not compact, $X /\!\!/_{Haus} G$ may depend on the metric.

Lemma 6.11. *Let G be a compact topological group acting continuously on a metric space X. Then for every $x \in X$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$G \cdot B(x, \delta) \subseteq B(\varepsilon, Gx)$$

Proof. For each $g \in G$, the map $X \to X$ defined by $x \mapsto gx$ is continuous. So, there exists $\delta = \delta_y > 0$ such that

$$g \cdot B(x, \delta) \subseteq B(gx, \varepsilon)$$

Claim: δ is independent of y when G is compact.

Since the action $a : G \times X \to X$ is continuous, for all $g \in G$, there exists $\delta_g > 0$ and a neighbourhood U_g of g in G such that for all $h \in U_g$ we have

$$h \cdot B(x, \delta) \subseteq B(hx, \varepsilon)$$

In other words,

$$U_g \times B(x, \delta) \subseteq a^{-1}(B(gx, \varepsilon)) = a^{-1}(B(a(g, x), \varepsilon))$$

Since G is compact, the cover $\bigcup_{g \in G} U_g$ has a finite subcover $U_{g_1}, \ldots, U_{g_n}$. Now take $\delta := \min\{\delta_{g_i} : 1 \leqslant i \leqslant n\}$. Then for all $h \in G$, we have $h \cdot B(x, \delta) \subseteq B(hx, \varepsilon)$. That is

$$G \cdot B(x, \delta) \subseteq B(Gx, \varepsilon)$$

The result follows. $\square$

Proposition 6.12. *If G is compact, the map*

$$\begin{aligned} X/G &\to \mathscr{H}_X \\ x &\mapsto Gx \end{aligned}$$

is continuous.

Proof. We start by showing that the map

$$X \to \mathscr{H}_X$$
$$x \mapsto Gx$$

is continuous. Fix $x \in X$ and let $\varepsilon > 0$. We need to find $\delta > 0$ such that for every $y \in B_X(x, \delta)$ we have $Gy \in B_{\mathscr{H}_X}(Gx, \varepsilon)$. By lemma 6.11, we can find $\delta_1 > 0$ such that $d(x, y) < \delta_1$ implies

$$Gy \subseteq B\left(\frac{\varepsilon}{2}, Gx\right) \tag{6.1}$$

Now we want to find $\delta_2 > 0$ such that $d(x, y) < \delta_2$ implies $G \cdot x \subseteq B(\frac{\varepsilon}{2}, G \cdot y)$. If there were no such $\delta_2 > 0$, we could find points $y_1, y_2, \ldots$ in X converging to x such that

$$Gx \nsubseteq B\left(\frac{\varepsilon}{2}, Gy_n\right) \quad \forall n$$

For each n, pick $g_n \in G$ such that

$$g_n x \notin B\left(\frac{\varepsilon}{2}, Gy_n\right)$$

So, in particular we have

$$d(g_n x, g_n y_n) > \frac{\varepsilon}{4}$$

Since G is compact, the sequence g_n has a convergent subsequence $g_{n_k} \to g$ for some $g \in G$. Then we must have

$$d(gx, gy_n) > \frac{\varepsilon}{8} \tag{6.2}$$

However, G continuously acts on X and $y_n \to x$. So, the inequality (6.2) cannot hold. Contradiction. Hence,

$$Gx \subseteq B\left(\frac{\varepsilon}{2}, Gy\right) \tag{6.3}$$

Finally, let $\delta = \min\{\delta_1, \delta_2\}$. If $d(x, y) < \delta$, then the relations (6.1) and (6.3) both holds and this imply that

$$d_{\mathscr{H}}(Gx, Gy) \leqslant \frac{\varepsilon}{2} < \varepsilon$$

Therefore, the map

$$\begin{aligned} X &\to \mathscr{H}_X \\ x &\mapsto Gx \end{aligned}$$

is continuous. Observe that this map is constant on G-orbits in X. So, it descends to a map

$$\begin{aligned} X/G &\to \mathscr{H}_X \\ [x] &\mapsto Gx \end{aligned}$$

by the universal property of the quotient topology. $\square$

Proposition 6.13. *The map given in the Proposition 6.12 is an embedding if both G and X are compact.*

Proof. X and G being compact implies that X/G and $\mathscr{H}_X$ are both compact Hausdorff. Furthermore, the map

$$X/G \to \mathscr{H}_X$$

is one-to-one and continuous, and therefore it is an embedding. $\square$

Proposition 6.14. *If X and G are both compact, then the Hausdorff Quotient $X /\!\!/_{Haus} G$ is the same as the usual quotient X/G.*

Proof. The map $X/G \hookrightarrow \mathscr{H}_X$ is an embedding, hence X is stable. $\square$

7. CONCLUSION

In the local compactness chapter, we gave four different definitions of local compactness and showed that local compactness is inherited by closed subspaces.

Then, we gave the definition of the compact-open topology on the space of continuous functions between two topological spaces. Since there is a bijection between the closed subspaces of a topological space X and the space of continuous functions from X to the Sierpinski space, we can put a topology on the set $\mathscr{H}_X$ of closed subspaces of X by using the compact-open topology on this space of functions. In Theorem 3.6 we showed that when X is locally compact this topology on $\mathscr{H}_X$ –called the Hilbert topology– represents the most naive "families of closed subspaces" functor sending a topological space T to the set of closed subspaces of $T \times X$.

Another topology that can be put on $\mathscr{H}_X$ is the Hausdorff topology given by *Hausdorff metric*. If a metric d induces a finer topology than d' on compact X, we showed in Theorem 4.8 that $d_{\mathscr{H}_X}$ induces a finer topology than $d'_{\mathscr{H}_X}$ on $\mathscr{H}_X$. Furthermore, we showed in Corollary 4.13 that there is a more elegant way to express the Hausdorff distance

$$d_{\mathscr{H}}(Z, Z') = \inf\{\varepsilon > 0 : Z' \subseteq B(\varepsilon, Z) \text{ and } Z \subseteq B(\varepsilon, Z')\}$$

A question to ask here was: "Can we express the Hausdorff topology on $\mathscr{H}_X$ in terms of the topology on X?" The answer is yes, if X is compact. Furthermore, if X is compact, then $\mathscr{H}_X$ is also compact (Corollary 4.21). Another interesting theorem is about the necessary and sufficient conditions for a function $T \to \mathscr{H}_X$ to be continuous where T is any sequential space (Theorem 4.28). This leads to the "modular interpretation" of the Hausdorff topology established in Theorem 4.6.

In Chapter 5, we introduced the Hilbert functor, the Hilbert scheme and the Hilbert

Quotient. Modifying the Hilbert functor a little gave us two analogs of the Hilbert scheme on the closed subspaces of a topological space equipped with the Hilbert topology and the Hausdorff topology. The analog of the Hilbert Quotient is the Hausdorff Quotient and we established some nice properties of the Hausdorff Quotient in Proposition 6.14 and Remark 6.10.

The Hausdorff Quotient chapter gives a nice definition of the Hausdorff Quotient of a metric space X by a topological group G as an *arbitrary* intersection. We proved that the usual quotient is the same as the Hausdorff Quotient when both X and G are compact. An open question is that: “When is the Hausdorff Quotient a *finite* intersection? *i.e.* When does X have a semi-stable subset?” Furthermore, it would be interesting to investigate circumstances under which “stable” and “semi-stable” are equivalent yet we know of no example of a semi-stable set which is not stable.

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