AN OPTIMAL CHANGE OF VARIABLES SCHEME FOR SINGLE SCATTERING PROBLEMS

by

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ABSTRACT

AN OPTIMAL CHANGE OF VARIABLES SCHEME FOR SINGLE SCATTERING PROBLEMS

In this work we are concentrated on the direct obstacle scattering problem for convex bodies in two dimensions. In order to calculate the scattered field, we first need to compute the normal derivative of the total field on the object's surface. This quantity is the unique solution of a combined field integral equation which we solve using Galerkin method wherein the approximation spaces depend on the wave number and the geometry of the scatterer. We are particularly focused on the large wave numbers in which the solution has highly oscillating behavior. In order to analyze this solution accurately, we separate the highly oscillating part of it and then study the derivatives of the acquired function. This derivative study gives us the information about the smoothness of the solution and an idea about how to approximate it. As for the geometry of the scatterer, we divide the boundary of the object into subregions regarding where we expect high oscillations. In each region, in order to achieve improved approximations, we choose different polynomial bases. In various scenarios, we examine the polynomial bases such as monomial, Lagrange, and Chebyshev. As the wave number increases, in order to obtain better results one needs to formulate these approximation spaces with higher polynomial degrees. However, it includes enormous computational cost and the condition numbers of Galerkin matrices elevate dramatically. The goal of this research is to optimize the choice of approximation spaces so as to improve accuracy of numerical solutions while keeping the number of degrees of freedom independent of frequency, and reduce the condition numbers of the related Galerkin matrices.

ÖZET

TEKİL SAÇILMA PROBLEMLERİ İÇİN OPTİMAL DEĞİŞKEN DEĞİŞTİRME ŞEMASI

Bu çalışmada dışbükey nesneler üzerindeki doğrudan saçılma problemi üzerinde yoğunlaştık. Saçılan dalgayı çözebilmek için, öncelikle nesnenin üzerinde oluşan toplam dalganın normal türevini hesaplamamız gerekliydi. Bu değer ise Galerkin yaklaştırım uzayları ile çözmeye çalıştığımız kombine alan integral denklemin tek çözümüdür. Burada kurmaya çalıştığımız yaklaştırım uzaylarının, gelen dalga boyuna ve nesnenin geometrisine bağlı olduğunu biliyoruz. Bizim asıl ilgilendiğimiz kısım ise yüksek dalga boyuna sahip dalgaların oluşturduğu çok salınımlı çözüme sahip olan denklemlerdir. İşte bu çözümü daha iyi çalışabilmek için, onun yüksek salınımlı kısmını ayırıp kalan kısmının türevlerini analiz ediyoruz. Bu türev analizi bize çözüm fonksiyonunun asimptotları hakkında bilgi veriyor ve ona nasıl yaklaşacağımızı anlamamızı sağlıyor. Nesnenin geometrisine baktığımız zaman ise, çözüm fonksiyonundan yüksek salınım beklediğimiz yerleri işaretleyip nesnenin yüzeyini küçük kısımlara ayırıyoruz. Daha sonra bu kısımlarda çalışan farklı polinom bazları seçerek daha iyi bir yaklaştırım uzayı bina etmeye çalışıyoruz. Bize nümerik olarak en iyi yakınsaklığı verecek polinom bazlarını bulmak maksadı ile tek terimli polinom bazı, Lagrange, Chebyshev bazları ve trigonometrik polinom bazlarını tek tek inceliyoruz. Fakat dalga boyu arttıkça, elde ettiğimiz sonucu korumak için daha yüksek dereceli polinom uzayları seçmemizin gerekliliği çok masraflı hesaplamaları karşımıza çıkıyor. Ayrıca bu büyük dereceli uzaylarda Galerkin matrisi hesaplamak bize çok büyük kondisyon sayılarına mal oluyor. İşte bu araştırmadaki hedefimiz, bir yandan yaklaştırma uzaylarının inşasını optimize edecek bir algoritma oluşturmak, diğer yandan ise çözümün bağımsızlık derecesini frekanstan serbest yapmak ve Galerkin matrisinden doğabilecek kondisyon sayılarını mümkün olduğu kadar küçük tutabilmek.

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$\|\|.\|\|_2$	L^2 norm
$\langle ., . \rangle$	Sesquilinear form
C^n	n times continuously differentiable functions
D_s^n	n^{th} serivative operator with respect to s
$\mathcal{G}$	Galerkin space
$H_n^{1,2}$	Hankel function of the first and second kind of order n
i	Imaginary unit
K	Obstacle
k	wave number
$\mathcal{K}'$	Normal derivative of single layer operator
L^2	Square integralle functions
$\mathbb{N}$	Natural numbers
$\mathcal{O}$	Big-O notation
$\mathbb{P}_d$	Polinomials of degree not more than n
$\mathbb{R}$	Real numbers
$\mathcal{R}$	Combined field integral operator
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathfrak{Re}$	Real part of a number
$\mathcal{S}$	Single layer operator
$S^{\mu}_{\varrho,\delta}$	Symbol class of Höremainder of order μ and type ϱ, δ
$\mathbb{T}$	Trigonometric polynomials of degree not more than n
u	Scattered field
u^{inc}	incident wave
α	Direction of incident wave
γ	Parametrization of ∂K
Δ	The laplacian
η	Normal derivative of the total field

η^{slow}	Slowly oscillating part of η
ν	Unit normal vector
$\boldsymbol{\xi}$	an eight-tuple of parameters
Φ	Fundamental solution of the Helmholz equation
ϕ	Change of variables functions
$\chi_{(a,a',b',b)}$	Bump function defined based on (a, a', b', b)
$\chi_{\sigma(I_j)}$	Bump function defined on I_j
∂K	Boundary of K
∇	The gradient
$\bigoplus$	Direct sum

LIST OF ACRONYMS/ABBREVIATIONS

DS	Deep shadow
IL	Illimunated region
IT	Illuminated transition
SB	Shadow boundary
ST	Shadow transition

1. INTRODUCTION

In this thesis, our main problem is determination of a scattered field u for a given incident field $e^{ik\,\alpha\cdot x}$ and a smooth compact obstacle K in two dimensions. This problem can be represented by the well known Helmholtz equation

$$(\Delta + k^2)u = 0$$

in exterior domain of K. In order to find the solution u, using the Green's identities as well as single and double layer potentials we aim to solve an equivalent linear integral equation

$$\mathcal{R}\eta = f$$

on the boundary of K, where $\mathcal{R}$ is a combined field integral operator and f is a function related to the incident field, definitions of which will be discussed in the next chapter. Our main tool for solving this linear integral equation is Galerkin method.

Moreover, independent of the adopted numerical method, because of the asymptotical behaviour of the solution, when the wave number k of the incident field $e^{ik\,\alpha\cdot x}$ increases, in order to attain the same error, one has to expand the size of the discretization space by $\mathcal{O}(k)$. Hence the computational time for solving the matrix which arise from the discretization grows by $\mathcal{O}(k^3)$. It is obvious that when it comes to the high frequency problems, the required computational complexity becomes dramatically enormous, and theoretical findings appears to be impractical.

It is important to point the fact that Galerkin method highly depends on the clever choice of the finite dimensional subspace of $L^2(\partial K)$ otherwise known as Galerkin approximation space. In this thesis, we aim to find an optimal scheme to construct this Galerkin approximation space, in order to achieve better numerical solutions. Here we take into consideration several objectives when we describe a numerical solution as a better one. *(1) Small relative error*: We examine a theoretically described scheme's numerical results and observe the number of digits of accuracy when computing the relative L^2 error of the function η. *(2) Convergence for a fixed frequency*, *(3) Reduced dependence on the wave number k*: In the previous works low frequency scattering problem has been discussed thoroughly. However their schemes are not suitable for high frequency incident fields. In order to describe a stable numerical algorithm for the latter scattering problem, one has to build up a scheme which requires less degrees of of freedom while the wave number k elevates. *(4) Better condition numbers*: As well as interpreting a convergent numerical solution, we intend our scheme to be well-conditioned and possess maximum accuracy.

Bearing these objectives in our mind, we rewrite our solution as

$$\eta = \eta^{slow} e^{ik\,\alpha\cdot x} \tag{1.1}$$

for x belongs to ∂K. Separating the highly oscillating part $e^{ik\,\alpha\cdot x}$ from the function η is more convenient when studying the high frequency problems. As a result of this reformulation, approximation spaces require less than $\mathcal{O}(k)$ degrees of freedom while wave number increases. Thanks to the rigorous asymptotic behavior of the normal derivative of the total field, which corresponds to our solution η, studied in Melrose and Taylor's appreciated work [1], the asymptotic expansion of η^{slow} can be also derived.

Observing this expansion, we notice that η^{slow} behaves differently in different regions of ∂K. Although in the most of the previous works, these regions are described to be the illuminated region, shadow region and shadow boundaries, in order to construct the optimal basis we divide ∂K into five different types of subregions (see Figure 1.1).

The determination of the length of these intervals depends on the choice of the utilized basis functions related to the Galerkin approximation space. However, in order to achieve the finest error analysis, we compute the optimized eight-tuple $\boldsymbol{\xi}$ which will be defined throughout the thesis and is responsible of the shape of the subregions. After subtly designing each subregion we define a Galerkin approximation space that consists

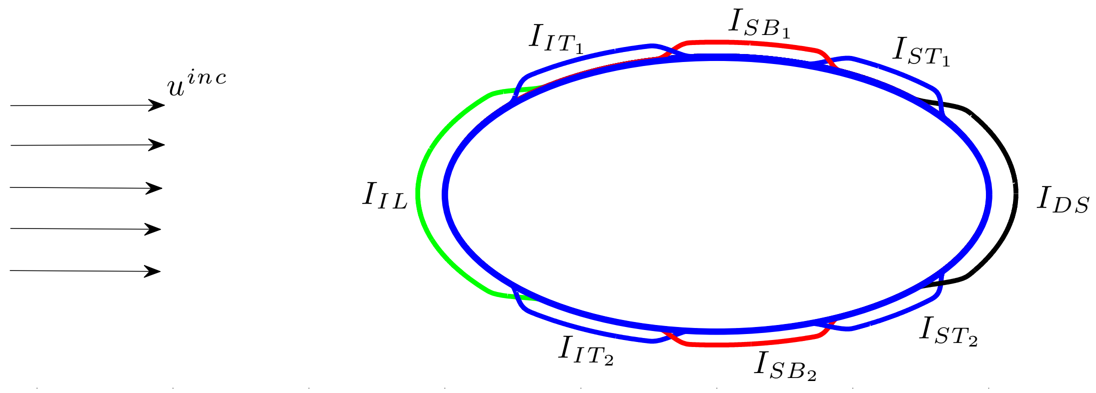

Figure 1.1. Interval scheme: Illuminated region (I_{IL}), shadow boundaries (I_{SB_l}), deep shadow region (I_{DS}), illuminated transitions (I_{IT_l}), shadow transitions (I_{ST_l}).

of algebraic or trigonometric polynomials, then by taking the direct sum of these spaces we define the global Galerkin approximation space. In order to mimic the nature of the solution η of the integral equation, we elegantly construct the approximation spaces for each subregion. More importantly with a cleverly invented change of variables scheme we manage to attain a requirement of $\mathcal{O}(\log k)$ increase in the number of degrees of freedom, which is a highly notable result considering the early works in this area.

When we investigate the relevant literature among those concentrated on the problem of the computational inefficiency in the high frequency cases and regarded the separation (1.1), we see many scholars proposed serious solutions. In this area, perhaps the work of the Abboud *et al.* [2], [3] (1994,1995) can be considered as a pioneer. In their work, using the method of stationary phases, they constructed Galerkin approximation spaces, dimension of which requires an $\mathcal{O}(k^{1/3})$ increase (instead of $\mathcal{O}(k)$) for larger wave number k, in order to achieve the same precision.

Later, between 2004-2007 several academicians presented some attempts with a few similarities. For example Bruno *et al.* designed an approach [4] by extending the ideas of Abboud *et al.* In this case they used the Nyström method and introduced a change of variables scheme where the solution of the integral equation has faster oscillations like shadow boundaries. They further broadened their analysis in [5], [6] and [7]. Following similar footsteps on one hand, we see Giladi *et al.* [8] offered a numerical method with the usage of boundary element collocation and a set of basis functions which mimic the asymptotical behavior of the solution (also see [9]). On the other hand we also examine the work of Huybrechs *et al.* [10] in which they used very effective quadrature rules in order to discretize the integrals. These three researches raised the question of an error analysis independent of the wave number k. Although, from some point of view their approach requires an $\mathcal{O}(1)$ increase in the number of degrees of freedom, their analysis are not rigorous and whether or not the application of this approach is successful in the high frequency numerical experiments is unclear.

Although these approaches are introduced for single scattering problems, they have been extended to multiple scattering scenarios in [5]. See Ecevit *et al.* [11], [12] for an analysis of these approaches.

Afterwards, Graham *et al.* introduced a more rigorous technique [13] in 2007. With the implications of Galerkin method and regarding the behavior of the solution in shadow boundaries they deliberately formed their approximation spaces. Hence they managed to attain an $\mathcal{O}(k^{1/9})$ requirement in the degrees of freedom. On the contrary to [4], [8] and [10] Graham *et al.*'s method is discussed rigorously in their article. However since they did not use an approximation space for the deep shadow region and assumed the numerical solution to be zero in there, they could not manage to obtain a converging scheme.

Recently, Ecevit and Ozen [14] manifested a more compact and carefully analyzed regime. Comparing to [13], they improved the requirement $\mathcal{O}(k^{1/9})$ to $\mathcal{O}(k^{\epsilon})$ where, via escalating the variable m (or the number of the intervals), ϵ can be chosen arbitrarily small. Moreover, in their work (in contrary to [13]) since they adopted an approximation space in the deep shadow region, the matrix related to the Galerkin approximation space appeared to be more stable which yielded improved condition numbers. Also we have to remark that, the numerical solution of η occurred to be convergent thanks to the approximation in the deep shadow region and their brilliant choice of subregions as well as the idea of transition regions facilitated the speed of this convergence. For recent improvements on the related scattering problems we refer to [15] and [16].

This thesis is an improvement of [14], in which they prescribed $4(m-1)$ transition regions and then raised the number m when wave number k increased, in order to achieve the same correctness. However in this thesis, while keeping the role of the transition region idea, we adopt a change of variables function that enables us 4 united transition regions instead of $4(m-1)$ ones. Especially the estimation in the deep shadow region is significantly enhanced. Comparing to other attempts it can be seen that our scheme mimic the behavior of the solution better. For example in the recent work of Huybrechs *et al.* [17] examining figures 7, 10, 11, 12 we can see this enhancement. Thanks to our new scheme, we have less number of subregions to handle, and a better error analysis. Moreover, one of the most important feature of this change of variables idea is that other than proportional to an exponential of k, the number of the degree's of freedom should be increased by $\mathcal{O}(\log k)$ as k increases for a desired numerical precision.

The thesis organized as follows, we begin in Chapter 2 by introducing the combined field integral equation equivalent to our scattering problem, and defining the Galerkin method as well as approximation space relations.

In Chapter 3, we give the details of the construction of the Galerkin approximation spaces and how the division of the subregions of ∂K decided. In this chapter we also state our main theorems about the best approximations of the η^{slow} related to the approximation spaces consisting of both algebraic and trigonometric polynomials. Chapter 4 however, in order to present a favorable error analysis, contains the study of the asymptotic and derivatives of η^{slow}.

In Chapter 5 we prove the main theorems stated in Chapter 3, by using both the asymptotical behavior and the estimates of the derivatives of the η^{slow}. In this chapter we also give the details of the change of variables procedure and describe how it is adopted to unite the $m-1$ transition regions as one.

Finally, Chapter 6 is devoted to the numerical experiments which depicts the improved results of our optimal scheme of change of variables under the choice of different approximation spaces and comparing to them those achieved in the previous work of [14].

2. SCATTERING PROBLEM, INTEGRAL EQUATIONS, AND GALERKIN METHOD

We consider the problem of scattering by a smooth compact obstacle K of a time harmonic incident plane wave of unit amplitude, $u^{\text{inc}} = e^{ik\,\alpha\cdot x}$, in $\mathbb{R}^2$. Here α is a two dimensional vector of magnitude 1 which represents the direction of the incident wave, and as mentioned earlier k is the wave number. In this thesis we look for a scattered field u which is a solution to

$$(\Delta + k^2)u = 0 \quad \text{in } \mathbb{R}^2 \setminus K \tag{2.1}$$

otherwise known as the Helmholtz equation. Then we seek our scattered field to satisfy the Dirichlet condition on the boundary and Summerfeld radiation condition uniformly for all directions, at infinity which are implemented as follows

$$u = -u^{inc} \quad \text{on } \partial K \tag{2.2}$$

$$\lim_{r\to\infty} r^{1/2}\left[\frac{\partial u}{\partial r} - iku\right] = 0, \quad r = |x|. \tag{2.3}$$

If we represent the scattered field u as a single layer potential then we have

$$u(x) = \int_{\partial K} \Phi(x,y)\eta(y)ds(y), \quad x \in \mathbb{R}^2 \setminus K,$$

where

$$\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|)$$

is the fundamental solution of the Helmholtz equation (2.1) and $H_0^{(1)}$ is the Hankel function of the first kind and order zero. Furthermore the density function η satisfies the uniquely solvable integral equation

$$\mathcal{R}\eta = f \quad \text{on } \partial K \tag{2.4}$$

Here the integral operator and the right hand side f are given as

$$\mathcal{R} = I + \mathcal{K}' - ik\mathcal{S} \quad \text{and} \quad f = 2\left\{\frac{\partial u^{\text{inc}}}{\partial \nu} - ik\ u^{\text{inc}}\right\}$$

where

$$(\mathcal{S}\eta)(x) = 2\int_{\partial K} \Phi(x,y)\eta(y)ds(y), \quad x \in \partial K,$$
$$(\mathcal{K}'\eta)(x) = 2\int_{\partial K} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\eta(y)ds(y), \quad x \in \partial K,$$

are the acoustic single-layer operator and its normal derivative, $\nu(x)$ is the unit normal vector to ∂K directed into the exterior of K. Here the normal derivative on the boundary is given as

$$\frac{\partial u}{\partial \nu} = \lim_{h \to 0^+} \nu(x) \cdot \nabla u(x - h\nu(x)), \quad x \in \partial D$$

so our right hand side function in (2.4) becomes

$$f = ik\left\{\alpha \cdot \nu(x) - 1\right\} e^{ik\,\alpha \cdot x}$$

Before giving the Galerkin formulation let us define the sesquilinear form and the bounded linear functional

$$\langle \mathcal{R}\cdot, \cdot \rangle_{L^2(\partial K)} : L^2(\partial K) \times L^2(\partial K) \to \mathbb{C}$$
$$\langle f, \cdot \rangle_{L^2(\partial K)} : L^2(\partial K) \to \mathbb{C}$$

where $\langle g, h\rangle_{L^2(\partial K)} := \int_{\partial K} \overline{g(s)} h(s) ds$. Then according to [18], the unique solution of the combined field (2.4) coincides with the solution of its weak formulation where we are looking for an $\eta \in L^2(\partial K)$ that satisfies

$$\langle \mathcal{R}\eta, \mu\rangle_{L^2(\partial K)} = \langle f, \mu\rangle_{L^2(\partial K)}, \quad \forall \mu \in L^2(\partial K). \tag{2.5}$$

We should note that the representation of the boundary of K appeared in these formulations will be clarified in the next chapter.

Now in order to solve (2.4) numerically, applying the Galerkin method to its weak formulation (2.5), where given a finite dimensional subspace $\mathcal{G}$ of $L^2(\partial K)$, one finds an approximate solution $\hat{\eta} \in \mathcal{G}$ to equation (2.5) requiring that

$$\langle \mathcal{R}\hat{\eta}, \hat{\mu}\rangle_{L^2(\partial K)} = \langle f, \hat{\mu}\rangle_{L^2(\partial K)}, \quad \forall \hat{\mu} \in \mathcal{G}. \tag{2.6}$$

From [19] and [20] we can see the unique solvability of (2.6) and approximation properties of its solution, which are given in the following famous lemma.

Lemma 2.1. *(Céa's lemma) If $\langle \mathcal{R}\cdot, \cdot\rangle_{L^2(\partial K)} : X \times X \to \mathbb{C}$ is a bounded sesquilinear form on a Hilbert space X such that*

$$\begin{aligned} |\langle \mathcal{R}\eta, \mu\rangle_{L^2(\partial K)}| &\le C||\eta||||\mu|| && ,\forall \eta, \mu \in X \\ \mathfrak{Re}\ \langle \mathcal{R}\eta, \mu\rangle_{L^2(\partial K)} &\ge c||\eta||^2 && ,\forall \mu \in X, \end{aligned}$$

holds for some positive constants C, c and $\langle f, \cdot\rangle_{L^2(\partial K)} : X \to \mathbb{C}$ is a bounded linear functional, then given a finite dimensional subspace $\mathcal{G}$ of X, there exists a unique $\hat{\eta} \in \mathcal{G}$ such that

$$\langle \mathcal{R}\hat{\eta}, \hat{\mu}\rangle_{L^2(\partial K)} = \langle f, \hat{\mu}\rangle_{L^2(\partial K)}, \quad \forall \hat{\mu} \in \mathcal{G}$$

and we have the error estimate

$$||\eta - \hat{\eta}|| \leq \frac{C}{c} \inf_{\hat{\mu} \in \mathcal{G}} ||\eta - \hat{\mu}|| \tag{2.7}$$

where η is the solution of

$$\langle \mathcal{R}\eta, \mu \rangle_{L^2(\partial K)} = \langle f, \mu \rangle_{L^2(\partial K)}, \quad \forall \mu \in X$$

Here the space $\mathcal{G}$, and the unique solution $\hat{\eta}$ are called Galerkin approximation space and Galerkin solution, respectively. Furthermore the constants C and c are called *continuity* and *coercivity* constants and there are many articles about their properties and estimations. For our scattering problem, the sesquilinear form given in (2.6) which is related to the combined field integral operator $\mathcal{R}$ appearing in (2.4) satisfies the conditions of the Céa's Lemma. As it is shown in [13] for circular obstacles the ratio $C/c = \mathcal{O}(k^{1/3})$ as $k \to \infty$.

With the light of Céa's lemma, if we choose a basis for our Galerkin approximation space, we can design a linear system the solution of which gives us the numerical solution $\hat{\eta}$ of (2.6).

Remark 2.2. *Suppose $\{\hat{\mu}_1, \cdots, \hat{\mu}_{\dim(\mathcal{G})}\}$ is a basis of the Galerkin approximation space $\mathcal{G}$ with dimension $\dim(\mathcal{G})$, and $\hat{\eta}$ satisfies the condition (2.6). If we write $\hat{\eta} = \sum_{i=1}^{\dim(\mathcal{G})} \lambda_i \, \hat{\mu}_i$ for some scalers λ_i. Then for every $1 \leq i \leq \dim(\mathcal{G})$ we have*

$$\sum_{j=1}^{\dim(\mathcal{G})} \overline{\lambda}_j \langle \mathcal{R}\hat{\mu}_j, \hat{\mu}_i \rangle_{L^2(\partial K)} = \langle f, \hat{\mu}_i \rangle_{L^2(\partial K)}$$

Clearly this set of equations represents the following system

$$\begin{bmatrix} \langle \mathcal{R}\hat{\mu}_1, \hat{\mu}_1 \rangle & \cdots & \langle \mathcal{R}\hat{\mu}_{\dim(\mathcal{G})}, \hat{\mu}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathcal{R}\hat{\mu}_1, \hat{\mu}_{\dim(\mathcal{G})} \rangle & \cdots & \langle \mathcal{R}\hat{\mu}_{\dim(\mathcal{G})}, \hat{\mu}_{\dim(\mathcal{G})} \rangle \end{bmatrix} \begin{bmatrix} \overline{\lambda}_1 \\ \vdots \\ \overline{\lambda}_{\dim(\mathcal{G})} \end{bmatrix} = \begin{bmatrix} \langle f, \hat{\mu}_1 \rangle \\ \vdots \\ \langle f, \hat{\mu}_{\dim(\mathcal{G})} \rangle \end{bmatrix} \tag{2.8}$$

Considering this remark, our Galerkin method aims to find the unknown vector $\boldsymbol{\lambda}^{\mathcal{G}} = (\overline{\lambda}_1, \cdots \overline{\lambda}_{\dim(\mathcal{G})})$ for a given basis of the Galerkin approximation space $\mathcal{G}$. The construction and the numerical solution of the matrix appears in (2.8) will be discussed in Chapter 6.

In the next chapter, using the results of Céa's Lemma, we will be constructing our Galerkin approximation spaces. Our aim will be obtaining an error estimate occurring in inequality (2.7) which should be less dependent on the wave number k as much as possible.

3. GALERKIN APPROXIMATION SPACES

In the process of constructing the Galerkin approximation spaces, our aim is locally mimicking the behavior of the normal derivative of the total field, i.e. η. In order to achieve this goal, we divide the surface of the scatterer into subregions where the behavior of the asymptotics of η changes. Considering the work of [13], we will take approximation spaces in the $\mathcal{O}(k^{1/3})$ neighborhood of shadow boundaries. Then following the foot steps of the previous work of [14], we will also construct an approximation space in deep shadow region. Moreover we will use their idea of transition regions, in which the behavior of η changes considerably. Especially the illuminated transition regions, which is defined in [14] by dividing the region between illuminated and shadow boundaries to $m-1$ subregions, provided improved error estimate. Although their work gives us a stable error analysis, defining a number of, $4m$ in total, different subregions makes it harder to construct the approximation spaces. Instead of that we will further define single transition regions each of which depicts the properties of $m-1$ old ones. The details of this analysis will be discussed in Section 5.2.

In this chapter and so on, we will use L to denote the perimeter of the scatterer. In Chapter 4 we have introduced our results when $s \in [0, 2\pi]$, however it is clear that those findings remain also valid when $s \in [0, L]$. Moreover we define the smooth natural parametrization $\gamma : [0, L] \to \partial K$ in the counterclockwise orientation such that

- $\gamma(s + L) = \gamma(s)$ satisfies for all $s \in [0, L]$.
- $\gamma(0)$ belongs to deep shadow region, i.e. I_{DS}.
- $\gamma(t_1), \gamma(t_2)$ represent the two shadow boundary points as described in Chapter 1.
- (t_1, t_2) corresponds to the illuminated region, i.e. I_{IL}.
- $(t_2, t_1 + L)$ corresponds to the deep shadow region, I_{DS}.

Now, we will describe the construction of Galerkin approximation spaces both for the algebraic and trigonometric polynomials.

3.1. Weighted Algebraic Polynomials

Given $m \in \mathbb{N}$, $0 \le \epsilon_m < \epsilon_{m-1} < \cdots < \epsilon_1 \le 1/3$, and constants ξ_1, ξ_2 we divide the interval $[0, L]$ into $4m$ subregions as follows:

(i) Illuminated region: $I_{IL} = [t_1 + \xi_1 k^{-1/3+\epsilon_1}, t_2 - \xi_2 k^{-1/3+\epsilon_1}]$,

(ii) Deep shadow region: $I_{DS} = [0, t_1 - \xi_1 k^{-1/3+\epsilon_1}] \cup [t_2 + \xi_2 k^{-1/3+\epsilon_1}, L]$,

(iii) Shadow boundaries ($l = 1, 2$): $I_{SB_l} = [t_l - \xi_l k^{-1/3+\epsilon_m}, t_l + \xi_l k^{-1/3+\epsilon_m}]$,

(iv) Illuminated transitions: For $j = 1, 2, \cdots, m-1$,

$$I_{IT_1}^j = [t_1 + \xi_1 k^{-1/3+\epsilon_{j+1}}, t_1 + \xi_l k^{-1/3+\epsilon_j}],$$
$$I_{IT_2}^j = [t_2 - \xi_2 k^{-1/3+\epsilon_j}, t_2 - \xi_2 k^{-1/3+\epsilon_{j+1}}],$$

(v) Shadow transitions: For $j = 1, 2, \cdots, m-1$,

$$I_{ST_1}^j = [t_1 - \xi_1 k^{-1/3+\epsilon_j}, t_1 - \xi_1 k^{-1/3+\epsilon_{j+1}}],$$
$$I_{ST_2}^j = [t_2 + \xi_2 k^{-1/3+\epsilon_{j+1}}, t_2 + \xi_2 k^{-1/3+\epsilon_j}].$$

Noting that $\mathbb{P}_d = \text{span}\{x^r : r = 0, \cdots, d\}$ and above definitions of intervals, for $j = 1, \cdots, 4m$ in I_j (j^{th} interval), we choose the approximation space to be $\mathbb{1}_{I_j}\, e^{ik\,\alpha\cdot\gamma}\, \mathbb{P}_{d_j}$ which is of dimension $d_j + 1$. We will denote this $4m$ tuple of integers with $\mathbf{d} := (d_1, \cdots, d_{4m})$. Now for a $4m$ tuple $\mathbf{d}$ we define our global approximation space as

$$\mathscr{P}_{\mathbf{d}} = \bigoplus_{j=1}^{4m} \mathbb{1}_{I_j}\; e^{ik\,\alpha\cdot\gamma}\; \mathbb{P}_{d_j}$$
$$\dim(\mathscr{P}_{\mathbf{d}}) = 4m + \sum_{j=1}^{4m} d_j$$

so that in each region we can mimic the behavior of η with the complex exponential function. Then our Galerkin formulation (2.6) is equivalent finding the unique $\hat{\eta} \in \mathscr{P}_{\mathbf{d}}$

such that

$$B(\hat{\eta}, \hat{\mu}) = F(\hat{\mu}), \quad \forall \hat{\mu} \in \mathscr{P}_{\mathbf{d}} \tag{3.1}$$

Theorem 3.1. *For all $n_j \in 0, 1, \cdots, d_j + 1$ $(j = 1, \cdots, 4m)$ and all sufficiently large $k \geq 1$, we have*

$$||\eta - \hat{\eta}||_{L^2(\partial K)} \lesssim_{n_1, \cdots, n_{4m}} \frac{C}{c} k \sum_{j=1}^{4m} M_j^{\mathbb{P}}(k)$$

for the Galerkin solution $\hat{\eta}$ to equation (3.1) where

$$M_j^{\mathbb{P}}(k) = \begin{cases} \dfrac{1 + k^{-(1+3\epsilon_1)/2}\left(k^{(1/3-\epsilon_1)/2}\right)^{n_j}}{(d_j)^{n_j}}, & \text{for } I_{IL} \text{ and } I_{DS}, \\[2ex] \dfrac{1 + k^{-1/2}\left(k^{\epsilon_m}\right)^{n_j}}{(d_j)^{n_j}}, & \text{for } I_{SB_1} \text{ and } I_{SB_2}, \\[2ex] \dfrac{1 + k^{-(1+3\epsilon_{r+1})/2}\left(k^{(\epsilon_r-\epsilon_{r+1})/2}\right)^{n_j}}{(d_j)^{n_j}}, & \text{for } I_{IT_l}^r \text{ and } I_{ST_l}^r. \end{cases} \tag{3.2}$$

Although this theorem with some suitable choice of ϵ_j and ξ_l's as manifested in [14] gives us stable error analysis, we will give an alternative version of this theorem. With this purpose in mind, we set $m = 2$. Now for the illuminated region, deep shadow region and shadow boundaries we choose the same polynomial spaces as we do above, but for the transition regions we choose polynomials composed with a change of variables function ϕ^{-1} the definition of which will be given in Section 5.2. The important feature of this function ϕ^{-1} is that it maps each transition region to itself. Then we define $\mathbb{P}_d \circ \phi^{-1} = \text{span}\{(\phi^{-1})^r : r = 0, \cdots, d\}$. Now for the transition regions instead of the previous one we choose the approximation space as $\mathbb{1}_{I_j}\ e^{ik\,\alpha\cdot\gamma}\ \mathbb{P}_{d_j} \circ \phi^{-1}$

$$\mathscr{P}_{d_j} = \begin{cases} \mathbb{1}_{I_j}\ e^{ik\,\alpha\cdot\gamma}\ \mathbb{P}_{d_j}, & \text{if it is not a transition region} \\ \mathbb{1}_{I_j}\ e^{ik\,\alpha\cdot\gamma}\ \mathbb{P}_{d_j} \circ \phi^{-1}, & \text{if it is a transition region} \end{cases}$$

Similarly, we define our ($8+\sum_{j=1}^{8} d_j$ dimensional) global Galerkin approximation space as

$$\tilde{\mathscr{P}}_{\mathbf{d}} = \bigoplus_{j=1}^{4m} \mathscr{P}_{d_j}.$$

So, our Galerkin formulation is replaced by

$$\langle \mathcal{R}\hat{\eta}, \hat{\mu} \rangle_{L^2(\partial K)} = \langle f, \hat{\mu} \rangle_{L^2(\partial K)}, \quad \forall \hat{\mu} \in \tilde{\mathscr{P}}_{\mathbf{d}} \tag{3.3}$$

Theorem 3.2. *If we set $\epsilon_1 = 1/3$ and $\epsilon_2 = 0$, then for all $n_j \in 0, 1, \cdots, d_j + 1$ ($j = 1, \cdots, 8$) and all sufficiently large $k \geq 1$, we have*

$$||\eta - \hat{\eta}||_{L^2(\partial K)} \lesssim_{n_1, \cdots, n_8} \frac{C}{c} k \sum_{j=1}^{8} \tilde{M}_j^{\mathbb{P}}(k)$$

for the Galerkin solution $\hat{\eta}$ to our new formulation (3.3) where

$$\tilde{M}_j^{\mathbb{P}}(k) = \begin{cases} \dfrac{1+k^{-1}}{(d_j)^{n_j}}, & \text{for } I_{IL} \text{ and } I_{DS}, \\[2ex] \dfrac{1+k^{-1/2}}{(d_j)^{n_j}}, & \text{for } I_{SB_1} \text{ and } I_{SB_2}, \\[2ex] \dfrac{(\log k)^{1/2}(\log k)^{n_j}}{(d_j)^{n_j}}, & \text{for } I_{IT_l}^r \text{ and } I_{ST_l}^r. \end{cases} \tag{3.4}$$

Here it is clear that the upper bound function $\tilde{M}_j^{\mathbb{P}}(k)$ becomes $\mathcal{O}(1)$ in each region except the transition ones, in other words the error becomes independent of the variable k. Also in the transition regions, choosing the number of degrees of freedom d proportional to $\log k$ gives us highly stable error analysis while the wave number k increases. We will discuss the proof of Theorem 3.1 and 3.2 in Section 5.1. However, we have a more simplified result.

Using the fact that $\eta = \mathcal{O}(k)$ (see [1]), if we assign the same polynomial degree for each region's Galerkin approximation spaces, then we have the following relative estimates:

Corollary 3.3. *For all $n \in \{0, \cdots, d+1\}$ and all sufficiently large $k \geq 1$, we have*

$$\frac{||\eta - \hat{\eta}||_{L^2(\partial K)}}{||\eta||_{L^2(\partial K)}} \lesssim_n \frac{C}{c} \frac{(\log k)^{n+1/2}}{d^n}$$

3.2. Weighted Trigonometric Polynomials

For given $m \in \mathbb{N} \cup \{0\}$, $0 \leq \epsilon_{2m} < \epsilon_{2m-1} \leq \epsilon_{2m-2} < \cdots < \epsilon_5 \leq \epsilon_4 < \epsilon_3 \leq \epsilon_2 < \epsilon_1 \leq 1/3$, and constants $\xi_1, \xi_2, \xi_3, \xi_4$, we divide the interval $[0, L]$ into $4m$ subregions as follows:

(i) Illuminated region: $I_{IL} = [t_1 + \xi_1 k^{-1/3+\epsilon_2}, t_2 - \xi_2 k^{-1/3+\epsilon_2}]$,

(ii) Deep shadow region: $I_{DS} = [0, t_1 - \xi_4 k^{-1/3+\epsilon_2}] \cup [t_2 + \xi_3 k^{-1/3+\epsilon_2}, L]$,

(iii) Shadow boundaries

$$I_{SB_1} = [t_1 - \xi_4 k^{-1/3+\epsilon_{2m-1}}, t_1 + \xi_1 k^{-1/3+\epsilon_{2m-1}}]$$
$$I_{SB_2} = [t_2 - \xi_2 k^{-1/3+\epsilon_{2m-1}}, t_2 + \xi_3 k^{-1/3+\epsilon_{2m-1}}]$$

(iv) Illuminated transitions: For $j = 1, 2, \cdots, m-1$,

$$I^j_{IT_1} = [t_1 + \xi_1 k^{-1/3+\epsilon_{2j+2}}, t_1 + \xi_1 k^{-1/3+\epsilon_{2j-1}}]$$
$$I^j_{IT_2} = [t_2 - \xi_2 k^{-1/3+\epsilon_{2j-1}}, t_2 - \xi_2 k^{-1/3+\epsilon_{2j+2}}]$$

(v) Shadow transitions: For $j = 1, 2, \cdots, m-1$,

$$I^j_{ST_1} = [t_1 - \xi_4 k^{-1/3+\epsilon_{2j-1}}, t_1 - \xi_4 k^{-1/3+\epsilon_{2j+2}}]$$
$$I^j_{ST_2} = [t_2 + \xi_3 k^{-1/3+\epsilon_{2j+2}}, t_2 + \xi_3 k^{-1/3+\epsilon_{2j-1}}]$$

Next, for $j = 1, \cdots, 4m$ recalling I_j and d_j from the previous section let us define the space of trigonometric polynomials of degree at most d on the interval $I = [a, b]$ as follows

$$\mathbb{T}_d(I) = \mathbb{T}_d[a,b] = \operatorname{span}\left\{\exp\left(2\pi\ i\ r\ \frac{s-a}{b-a}\right) : r = -\frac{d}{2}, \cdots, \frac{d}{2}\right\}.$$

As we did in the algebraic case, we denote $\mathbf{d} = (d_1, \cdots, d_{4m})$ as this $4m$ tuple of integers and choose our global approximation space in the following way.

$$\mathscr{T}_{\mathbf{d}} = \bigoplus_{j=1}^{4m} \mathbb{1}_{I_j}\ e^{ik\,\alpha\cdot\gamma}\ \mathbb{T}_{d_j}(I_j)$$
$$\dim(\mathscr{T}_{\mathbf{d}}) = 4m + \sum_{j=1}^{4m} d_j$$

Then Galerkin formulation becomes finding the function $\hat{\eta} \in \mathscr{T}_{\mathbf{d}}$ such that

$$\langle \mathcal{R}\hat{\eta}, \hat{\mu}\rangle_{L^2(\partial K)} = \langle f, \hat{\mu}\rangle_{L^2(\partial K)}, \quad \hat{\mu} \in \mathscr{T}_{\mathbf{d}}. \tag{3.5}$$

The fundamental result of this section is given in the next theorem.

Theorem 3.4. *For all $n_j \in \mathbb{N}$ $(j = 1, \cdots, 4m)$ and all sufficiently large $k \geq 1$, we have*

$$||\eta - \hat{\eta}||_{L^2(\partial K)} \lesssim_{n_1,\cdots,n_{4m}} \frac{C}{c} k \sum_{j=1}^{4m} M_j^{\mathbb{T}}(k)$$

for the Galerkin solution $\hat{\eta}$ to (3.5) where

$$M_j^{\mathbb{T}}(k) = \begin{cases} \left(\dfrac{k^{(1/3-\epsilon_1)/2}}{d_j}\right)^{n_j}, & \textit{for } I_{IL} \textit{ and } I_{DS}, \\[2ex] \left(\dfrac{k^{\epsilon_{2m-1}}}{d_j}\right)^{n_j}, & \textit{for } I_{SB_1} \textit{ and } I_{SB_2}, \\[2ex] \left(\dfrac{k^{\epsilon_{2j-1}-\epsilon_{2j+2}}}{d_j}\right)^{n_j}, & \textit{for } I^r_{IT_l} \textit{ and } I^r_{ST_l}. \end{cases} \tag{3.6}$$

We will discuss the proof of Theorem 3.4 in Section 5.3. However, we have a more simplified result.

In order to obtain an optimal error, its more suitable to choose ϵ_j such that

$$\frac{1}{3} - \epsilon_2 = \epsilon_{2m-1} = \epsilon_{2j-1} - \epsilon_{2j+2} \tag{3.7}$$

for $j = 1, \cdots, m-1$. However further requiring that

$$\epsilon_{2j} - \epsilon_{2j-1} = \epsilon_{2j+2} - \epsilon_{2j+1} = \frac{1}{\kappa}(\epsilon_{2j+1} - \epsilon_{2j})$$

for $j = 1, \cdots, m-1$ and for some $\kappa > 0$ makes the construction of the spaces easier. With this purpose in our mind, for each $j = 1, \cdots, 4m$ we choose

$$\epsilon_j = \frac{1}{3} - \frac{\lfloor \frac{j+1}{2} \rfloor (\kappa+1) + \frac{(-1)^{j+1}-1}{2}}{3[(\kappa+1)m + (\kappa+2)]}$$

Then the values on (3.7) becomes equal to $\frac{1}{3m+3\frac{\kappa+1}{\kappa+2}}$, let us call that ε. In this case setting the same degree of freedom for each region's Galerkin approximation spaces, we will have the following relative error estimate

Corollary 3.5. *If ϵ_j's are chosen as above, then for all $n \in \mathbb{N}$ and all sufficiently large $k \geq 1$, we have*

$$\frac{||\eta - \hat{\eta}||_{L^2(\partial K)}}{||\eta||_{L^2(\partial K)}} \lesssim_n \frac{C}{c} m \left(\frac{k^\varepsilon}{d+1} \right)^n$$

4. ASYMPTOTIC AND DERIVATIVE ESTIMATES OF η^{slow}

In this chapter, in order to present a converging numerical solution, we aim to study the asymptotic behavior of η^{slow}. We also have to examine the nature of its derivatives in different subregions of the boundary of K, which will be useful in following chapters. Throughout the chapter and only in this chapter when we talk about I_{IL} we mean I_{IL} together with I_{IT_1} and I_{IT_2}. Similarly we consider I_{DS} with I_{ST_1} and I_{ST_2}. Also only in this chapter variable L will be different from the one defined in Chapter 3. Now we give some relavant definitions first.

Definition 4.1. *(Symbol classes of Hörmander [12, Definition 2.1]) Let $\mathcal{M}$ be an open subset of $\mathbb{R}^p$, and let Γ be an open conic subset of $\mathcal{M} \times \mathbb{R}^q$ (i.e. $(x, \xi) \in \Gamma$ implies $(x, t\xi) \in \Gamma$ when $t > 0$). The symbol class of order $\mu \in \mathbb{R}$ and type $\varrho, \delta \in [0, 1]$ of Hörmander, denoted as $S^{\mu}_{\varrho,\delta}(\Gamma)$, is defined to be the collection of all complex-valued functions $a \in C^{\infty}(\Gamma)$ such that, for any compact set $W \subset \Gamma$ and all multi-indices β, γ, the estimate*

$$|D_x^{\beta} D_{\xi}^{\gamma} a(x, \xi)| \lesssim_{\beta,\gamma,W} (1 + |\xi|)^{\mu - \varrho|\gamma| + \delta|\beta|}, \quad (x, \xi) \in W^c \tag{4.1}$$

holds, where $W^c = \{(x, t\xi) : (x, \xi) \in W, t \geq 1\}$.

We should note that this definition is quite general and throughout this thesis we only consider the case when $p = q = 1$ therefore $\Gamma \subset \mathbb{R}^2$, and the multi-indices β, γ becomes ordinary indices.

Definition 4.2. *(Asymptotic expansion in the sense of Hörmander [12, Definition 2.2]) Let $a_j \in S^{\nu_j}_{\varrho,\delta}(\Gamma)$ where $\nu_j \to -\infty$ as $j \to \infty$. We say that $a \in S^{\mu}_{\varrho,\delta}(\Gamma)$ admits the asymptotic expansion*

$$a \sim \sum_{j=0}^{\infty} a_j$$

if

$$a - \sum_{i<j} a_i \in S^{\mu_j}_{\varrho,\delta}(\Gamma)$$

for every $j = 0, 1, 2, \cdots$ *where* $\mu_j = \max_{i \geq j} \nu_j$ *and* $\mu = \mu_0$.

With the light of these definitions, in order to prove Theorem 3.1, 3.2 and 3.4 we need to estimate the derivates of η^{slow}. For this reason we will have a look at the asymptotic expansion of η^{slow} which is carefully studied in [1] and [13]. From there, we know that the function η^{slow} has the following expansion:

Theorem 4.1. *For a compact smooth object K which is parametrized as in the beginning of Chapter 3 we have the following asymptotical expansions for* η^{slow}

- *In a small subset of* $I_\Delta \subset (I_{SB_1} \cup I_{SB_2})$ *(see [13, Theorem 5.1])*

$$\eta^{slow} \sim \sum_{l,m \geq 0} k^{2/3 - 2l/3 - m} b_{l,m}(s) \Psi^{(l)}(k^{1/3} Z(s)). \tag{4.2}$$

  valid for $s \in I_\Delta$. *Here* $b_{l,m}$ *and* Ψ *are a complex valued* C^∞ *functions defined on* I_Δ. *Moreover* $Z(s) = \omega(s) h(s)$, *where* $h(s)$ *is a smooth positive function and* $\omega(s) = (s - t_1)(t_2 - s)$ *on* $[0, 2\pi]$.
- *In the illuminate region* I_{IL}*(see [1, equation 1.15])*

$$\eta^{slow}(s,k) \sim \sum_{j \geq 0} k^{1-j} d_j(s)$$

  for complex valued C^∞ *functions* $d_j(s)$.
- *Furthermore, the asymptotical behavior of* Ψ *is given as [1, Lemma 9.9]:*

$$\Psi(\tau) \;=\; a_0 \tau + a_1 \tau^{-2} + a_2 \tau^{-5} + \cdots + a_N \tau^{1-3N} \;+\; \mathcal{O}(\tau^{1-3(N+1)}) \tag{4.3}$$

  as $\tau \to \infty$ *and this expansion remains valid for all derivatives of* Ψ *by formally* *differentiating each term on the right hand side, including the error term.*

Although this theorem is useful from many aspects, the following one aims to compactify the first two items of it.

Theorem 4.2. *[13] The function $b_{l,m}$ can be extended to 2π-periodic C^∞ functions such that for all $L, M \in \mathbb{N} \cup \{0\}$, the decomposition*

$$\eta^{slow}(s,k) = \left[\sum_{l,m=0}^{L,M} k^{2/3-2l/3-m} b_{l,m}(s) \Psi^{(l)}(k^{1/3} Z(s)) \right] + R_{L,M}(s,k) \tag{4.4}$$

holds for all $s \in [0, 2\pi]$, with remainder term satisfying, for all $n \in \mathbb{N} \cup \{0\}$

$$|D_s^n R_{L,M}| \lesssim_{L,M,n} (1+k)^{\mu+n/3},$$

where

$$\mu := -\min\{\frac{2}{3}(L+1), (M+1)\}. \tag{4.5}$$

For simplicity lets define, $a_{l,m}(s,k) := k^{2/3-2l/3-m} b_{l,m}(s) \Psi^{(l)}(k^{1/3} Z(s))$. Before giving the proof of Theorem 4.2 we will prove some useful propositions.

Proposition 4.3. *For some sufficiently large $\tau \in \mathbb{R}$ and Ψ as given in Theorem 4.2 we have:*

(i) $|\Psi(\tau)| \leq C_0(1 + |\tau|)$

(ii) $|\Psi'(\tau)| \leq C_1$

(iii) $|\Psi^{(l)}(\tau)| \leq C_l(1 + |\tau|)^{-2-l}$, *for* $l \geq 2$

Proof. Assume that $|\tau| \geq |\tau_0|$ for some $|\tau_0| > 1$. Then, according to the asymptotics of function $\Psi(\tau)$ given in Theorem 4.1 we have

$$\Psi(\tau) \;\lesssim\; |a_0||\tau| + \sum_{n=1}^{N} |a_n||\tau|^{1-3n} \;\lesssim\; |\tau| + \sum_{n=1}^{\infty} |\tau_0|^{1-3n} \;\leq\; C_0(|\tau|+1)$$

Again, according to Theorem 4.1 after differentiating the right hand side of the expansion (4.3) we have

$$|\Psi'(\tau)| \;\lesssim\; |a_0| + \sum_{n=1}^{N} |a_n||1-3n||\tau|^{-3n} \;\lesssim\; 1 + \sum_{n=1}^{N} |1-3n||\tau_0|^{-3n} \;\leq\; C_1$$

Moreover again by (4.3):

$$\begin{aligned}
|\Psi^{(l)}| \;&\lesssim\; \sum_{n=1}^{N} (1-3n)(-3n)(-1-3n)\cdots((-l+2-3n))|a_n||\tau|^{1-3n-l} \\
&\lesssim\; |\tau|^{-2-l} \sum_{n=0}^{\infty} (1-3n)(-3n)(-1-3n)\cdots((-l+2-3n))|\tau_0|^{-3n} \\
&\lesssim\; |\tau|^{-2-l}\, D^l_{|\tau_0|} \sum_{n=0}^{\infty} |\tau_0|^{-3n} \\
&\leq\; C_l(1+|\tau|^{-2-l})
\end{aligned}$$

This finishes the proof. $\square$

Proposition 4.4. *For some arbitrary $\mu_1, \mu_2 \in \mathbb{R}$ and $m, l \in \mathbb{N} \cup \{0\}$ with $l \geq 1$, we have:*

(i) $a \in S^{\mu_1}_{2/3,1/3}$ *and* $\mu_1 < \mu_2 \Rightarrow a \in S^{\mu_2}_{2/3,1/3}$

(ii) $a_{0,m} \in S^{1-m}_{2/3,1/3}$

(iii) $a_{l,m} \in S^{2/3-2l/3-m}_{2/3,1/3}$.

where $S^{\mu}_{\varrho,\delta}$ denotes the Hörmander Class as we defined in Definition 4.1

Proof.

(i) This follows from the definition of Hörmander Classes.

(ii) All we need to show is $|D_k^\alpha D_s^n a_{0,m}| \lesssim_{\alpha,n,m} (1+k)^{1-m+n/3-2\alpha/3}$, for all multi-indices α, n. Now using the Leibnitz rule and the boundedness of function $b_{0,m}$, we will take derivatives D_s and D_k in that order.

$$\begin{aligned}
|D_k^\alpha D_s^n a_{0,m}| &= |D_k^\alpha k^{2/3-m} D_s^n b_{0,m}(s)\Psi(k^{1/3}Z(s))| \\
&\lesssim_{\alpha,n,m} \sum_{i=0}^{n} |D_k^\alpha k^{2/3-m} k^{i/3}\Psi^{(i)}(k^{1/3}Z(s))| \\
&\lesssim_{\alpha,n,m} k^{2/3-m} \sum_{i=0}^{n}\sum_{j=0}^{\alpha} k^{-2j/3+i/3}|\Psi^{(i+j)}(k^{1/3}Z(s))| \\
&= k^{2/3-m}\Big[|\Psi(k^{1/3}Z(s))| + k^{1/3}|\Psi'(k^{1/3}Z(s))| + k^{-2/3}|\Psi'(k^{1/3}Z(s))|\Big] \\
&\qquad + k^{2/3-m} \sum_{i=1}^{n}\sum_{j=1}^{\alpha} k^{-2j/3+i/3}|\Psi^{(i+j)}(k^{1/3}Z(s))| :
\end{aligned}$$

In the last identity we separated the $(i=0, j=0), (i=1, j=0)$ and $(i=0, j=1)$ cases from the sum. After applying Proposition 4.3 to those sums, we will simplify them using the boundedness of $|Z(s)|$

$$\begin{aligned}
|D_k^\alpha D_s^n a_{0,m}| \;\lesssim_{\alpha,n,m}\; & k^{2/3-m}\Big[1 + k^{1/3}|Z(s)| + k^{1/3} + k^{-2/3}\Big] \\
& + \; k^{2/3-m} \sum_{i=1}^{n}\sum_{j=1}^{\alpha} k^{-2j/3+i/3}(1+k^{1/3}|Z(s)|)^{-2-i-j} \\
\lesssim_{\alpha,n,m}\; & k^{1-m} + k^{2/3-m} \sum_{i=1}^{n}\sum_{j=1}^{\alpha} k^{i/3}k^{-2j/3} \\
= \; & k^{1-m} + k^{2/3-m} \sum_{i=1}^{n} k^{i/3}k^{-2/3}\frac{1-(k^{-2/3})^\alpha}{1-(k^{-2/3})} \\
\lesssim_{\alpha,n,m}\; & k^{1-m} + k^{2/3-m}\left(nk^{n/3}\right)k^{-2/3}\frac{(1+k)^{-2\alpha/3}}{1-(k_0^{-2/3})} \\
\lesssim_{\alpha,n,m}\; & (1+k)^{1-m+n/3-2\alpha/3}
\end{aligned}$$

(iii) Same as the previous case we need to show

$$|D_k^\alpha D_s^n a_{l,m}| \lesssim_{\alpha,n,m,l} (1+k)^{-m+2/3-2l/3+n/3-2\alpha/3}$$

for all multi-indices α, n. By the same arguments and examining the cases $(i=1, j=0), (i=0, j=1)$ separated from the sum gives us:

$$\begin{aligned}|D_k^\alpha D_s^n a_{l,m}| &\lesssim k^{2/3-2l/3-m} \sum_{i=0}^{n} \sum_{j=0}^{\alpha} k^{-2j/3+i/3} |\Psi^{(l+i+j)}(k^{1/3}Z(s))| \\ &\lesssim k^{2/3-2l/3-m}(1+k)^{n/3-2\alpha/3} \\ &\lesssim_{\alpha,n,m,l} (1+k)^{-m+2/3-2l/3+n/3-2\alpha/3}\end{aligned}$$

This prints the desired result. □

Proposition 4.5. *If we define $r_N(s,k) := \eta^{slow}(s,k) - \sum_{j=0}^{N} k^{1-j} d_j(s)$, then we have:*

$$\begin{aligned} r_N(s,k) &\in S_{1,0}^{-N}, \\ \eta^{slow}(s,k) &\in S_{1,0}^{1} \quad \textit{on } I_{IL}. \end{aligned}$$

Proof. Since $d_j(s)$ are smooth functions their derivates are bounded in $[0, 2\pi]$. Thus $k^{1-j}d_j(s) \in S_{1,0}^{1-j}$, for all non-negative j. By Proposition 4.4 we have $k^{1-j}d_j(s) \in S_{1,0}^{-N}$, for all $j \geq N+1$ and $k^{1-j}d_j(s) \in S_{1,0}^1$, for all $j \geq 0$. This implies $r_N \in S_{1,0}^{-N}$ and $\eta^{slow}(s,k) \in S_{1,0}^1$. □

Now using these propositions we are ready to give the proof of Theorem 4.2.

Proof. (of Theorem 4.2) First of all, we will show that $R_{L,M}(s,k) \in S^{\mu}_{2/3,1/3}$. Then using Theorem 4.1 we will extend the smooth functions $b_{l,m}$'s, which are defined only on I_{SB}, to the regions I_{IL} and I_{DS}, and we will complete the extension of $b_{l,m}$'s to the whole interval $[0, 2\pi]$ as in the way described in the theorem.

By [13, Remark 5.2] we can choose $L' > L$,and $M' > M$ such that $R_{L',M'}(s,k) \in S^{\mu}_{2/3,1/3}$ and:

$$R_{L,M} = \left[\sum_{l=L+1}^{L'} \sum_{m=0}^{M'} a_{l,m}(s,k) + \sum_{l=0}^{L'} \sum_{m=M+1}^{M'} a_{l,m}(s,k) \right] + R_{L',M'}(s,k) \tag{4.6}$$

In order to prove that $R_{L,M}(s,k) \in S^{\mu}_{2/3,1/3}$, using Proposition 4.4 we will show that the summands of the double sums in above equality belongs to $S^{\mu}_{2/3,1/3}$.

In the first sum, since $-l \le -(L+1)$, $-m \le 0$, then by its definition $\mu \ge -\frac{2}{3}(L+1) - \frac{1}{3}$. So $a_{l,m} \in S^{-1/3-2(L+1)/3}_{2/3,1/3} \subset S^{\mu}_{2/3,1/3}$. For the second sum, when $l = 0$, since $-m \le -(M+1) \le \mu$, we know that $a_{0,m} \in S^{\mu}_{2/3,1/3}$. For $l \ge 1$ since $1/3 - 2l/3 - m \le \mu$, we have also $a_{l,m} \in S^{\mu}_{2/3,1/3}$.

Thus, since Hörmander Class is closed under summation and remainder is in $S^{\mu}_{2/3,1/3}$ we have $R_{L,M}(s,k) \in S^{\mu}_{2/3,1/3}$, for all $s \in I_{IL}$.

Now, since $d_j(s)$ and $Z(s) \in C^{\infty}$, for $j \in \mathbb{N} \cup \{0\}$ and $s \in [0, 2\pi]$, we can find $\tilde{b}_{l,m} \in C^{\infty}$ such that for every $s \in I_{IL}$:

$$\begin{aligned} d_0(s) &= a_0(s)\tilde{b}_{0,0}(s)Z(s) \\ d_1(s) &= a_0(s)(\tilde{b}_{0,1}(s)Z(s) + \tilde{b}_{1,0}(s)) + a_1(s)\tilde{b}_{0,0}(s)Z^{-2}(s) \\ d_j(s) &= a_0(s)(\tilde{b}_{0,j}(s)Z(s) + \tilde{b}_{1,j-1}(s)) \\ &\quad + \sum_{\substack{l+m+n=j \\ n \ge 1}} \left[\prod_{p=0}^{l-1} (1 - 3n - p) \right] a_n(s)\tilde{b}_{l,m}(s)Z^{1-3n-l}(s), \quad j \ge 2, \end{aligned} \tag{4.7}$$

where a_i's are from equation (4.3).

If we set $\tilde{b}_{l,m}$ s as in equation (4.7), then on I_{IL} we will clearly have

$$\eta^{slow}(s,k) \sim \sum_{l,m \ge 0} \tilde{b}_{l,m}(s)\Psi^{(l)}(k^{1/3}Z(s))$$

Now, let's define the following:

$$b_{l,m}^{\text{new}} := \mathbb{1}_{I_{IL}} \tilde{b}_{l,m} + \mathbb{1}_{I_{SB}} b_{l,m}$$

By the same arguments we have at the beginning of the proof, it can be seen that $R_{L,M}(s,k) \in S^{\mu}_{2/3,1/3}$ for $s \in I_{SB} \cup I_{IL}$. So we have defined C^∞ functions $b_{l,m}^{\text{new}}$ on $[0,2\pi] \setminus I_{DS}$ such that the conditions of the theorem hold for $b_{l,m}^{\text{new}}$ and the remainder functions $R_{L,M}$.

Finally with the similar work we can complete the extension of $b_{l,m}$'s on I_{DS} and therefore on the whole interval $[0,2\pi]$. □

Theorem 4.6. *[13] We have*

$$|D_s^n \eta^{slow}| \lesssim_n k + \sum_{m=4}^{n+2} (k^{-1/3} + |\omega(s)|)^{-m}$$

for all $n \in \mathbb{N} \cup \{0\}$, $s \in [0,2\pi]$ *and all sufficiently large* k.

Proof. For a given $n \in \mathbb{N}$, we can choose $L, M \in \mathbb{N}$ such that

$$\min\{2/3(L+1), (M+1)\} \geq n/3 \tag{4.8}$$

Now using the asymptotics of η^{slow} given in (4.2) and the results of Theorem 4.2, we define:

$$B_{l,M}(s,k) := \sum_{m=0}^{M} k^{-m} b_{l,m}(s), \quad A_{L,M}(s,k) := k^{2/3} \sum_{l=0}^{L} k^{-2l/3} B_{l,M}(s) \Psi^{(l)}(k^{1/3} Z(s))$$

so that we can write:

$$\eta^{slow}(s,k) = A_{L,M}(s,k) + R_{L,M}(s,k)$$

By Theorem 4.2 and inequality (4.8) we know that there exist a constant C_n such that $|D_s^n R_{L,M}(s,k)| \leq C_n$ holds for all k, s. In order to complete the proof, we need to bound $|D_s^n A_{L,M}(s,k)|$.

By the linearity of the operator D_s^n and the triangle inequality:

$$\begin{aligned}
|D_s^n A_{L,M}(s,k)| &= \left| k^{2/3} D_s^n \sum_{l=0}^{L} k^{-2l/3} B_{l,M}(s) \Psi^{(l)}(k^{1/3} Z(s)) \right| \\
&\leq k^{2/3} \sum_{l=0}^{L} k^{-2l/3} \left| D_s^n B_{l,M}(s) \Psi^{(l)}(k^{1/3} Z(s)) \right| \qquad (4.9)
\end{aligned}$$

By Leibniz integral rule and boundedness of both function $Z(s)$ and its derivatives, we have:

$$|D_s^n B_{l,m}(s) \Psi^{(l)}(k^{1/3} Z(s))| \lesssim_n \sum_{i=0}^{n} |D_s^{n-i}(B_{l,m}(s))| |k^{i/3} \Psi^{(l+i)}(k^{1/3} Z(s))|$$

Knowing that derivatives of $B_{l,M}(s)$ are bounded above, we can bound the right hand side of inequality (4.9) by the following way:

$$\begin{aligned}
|D_s^n A_{L,M}(s,k)| &\lesssim_n k^{2/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{i=0}^{n} \left| k^{i/3} \Psi^{(l+i)}(k^{1/3} Z(s)) \right| \\
&= k^{2/3} \sum_{i=0}^{n} \sum_{l=0}^{L} \left| k^{(i-2l)/3} \Psi^{(l+i)}(k^{1/3} Z(s)) \right| \\
&= k^{2/3} \sum_{l=0}^{L} \left| k^{-2l/3} \Psi^{(l)}(k^{1/3} Z(s)) \right| \\
&\quad + k^{2/3} \sum_{l=0}^{L} \left| k^{(1-2l)/3} \Psi^{(l+1)}(k^{1/3} Z(s)) \right| \\
&\quad + k^{2/3} \sum_{i=2}^{n} \sum_{l=0}^{L} \left| k^{(i-2l)/3} \Psi^{(l+i)}(k^{1/3} Z(s)) \right| \qquad (4.10)
\end{aligned}$$

Now our aim is estimating the three sums in (4.10) separately.

(i) From Theorem 4.1 we know we can find constants A, B independent of s such that $A \le h(s) \le B$, for all $s \in [0, 2\pi]$. Together with the fact that $k \ge k_0$ for some fixed $k_0 > 1$ gives us:

$$(1 + k^{1/3}|Z(s)|) = (1 + k^{1/3}|h(s)||\omega(s)|) \gtrsim (1 + k^{1/3}|\omega(s)|)$$

By Proposition 4.3 for $l \ge 2$ we have: $|\Psi^{(l)}(\tau)| \le C_{l+2}(1 + |\tau|)^{-2-l}$. Combining these two estimates we can write:

$$\begin{aligned}
k^{2/3} \sum_{i=2}^{n} \sum_{l=0}^{L} \left| k^{(i-2l)/3} \Psi^{(l+i)}(k^{1/3} Z(s)) \right| &\lesssim_n k^{2/3} \sum_{i=2}^{n} \sum_{l=0}^{L} k^{(i-2l)/3} C_l (1 + k^{1/3}|Z(s)|)^{-2-l-i} \\
&\lesssim_n k^{2/3} \sum_{i=2}^{n} \sum_{l=0}^{L} k^{(i-2l)/3} (1 + k^{1/3}|\omega(s)|)^{-2-l-i} \\
&\lesssim_n k^{2/3} \sum_{i=2}^{n} k^{i/3} (1 + k^{1/3}|\omega(s)|)^{-2-i} \sum_{l=0}^{L} k^{-2l/3} \\
&\lesssim_n k^{2/3} \sum_{i=2}^{n} k^{i/3} (1 + k^{1/3}|\omega(s)|)^{-2-i} \sum_{l=0}^{\infty} k_0^{-2l/3} \\
&\lesssim_n k^{2/3} \sum_{i=2}^{n} k^{i/3} (1 + k^{1/3}|\omega(s)|)^{-2-i} \qquad (4.11)
\end{aligned}$$

(ii) By (5.11) and (5.12) we know $|\Psi'(\tau)| \le C_1$, $|\Psi^{(l)}(\tau)| \le C_l(1+|\tau|)^{-2-l}$, for $l \ge 2$. So we can write:

$$\begin{aligned}
&k^{2/3} \sum_{l=0}^{L} \left| k^{(1-2l)/3} \Psi^{(l+1)}(k^{1/3} Z(s)) \right| \\
&\quad = k^{2/3} k^{1/3} \left| \Psi^{(1)}(k^{1/3} Z(s)) \right| + k^{2/3} \sum_{l=1}^{L} k^{(1-2l)/3} \left| \Psi^{(l+1)}(k^{1/3} Z(s)) \right| \\
&\quad \lesssim_n k + k^{2/3} \sum_{l=1}^{L} k^{(1-2l)/3} C_{l+1} (1 + k^{1/3}|\omega(s)|)^{-3-l} \\
&\quad \lesssim_n k + k^{2/3} \sum_{l=1}^{L} k^{(1-2l)/3} \ \lesssim_n \ k + k^{2/3} \sum_{l=1}^{\infty} k_0^{(1-2l)/3} \\
&\quad \lesssim_n k + k^{2/3} \ \lesssim_n k \qquad (4.12)
\end{aligned}$$

(iii) Again by (5.11) and (5.12) we know $|\Psi(\tau)| \le C_0(1+|\tau|)$, $|\Psi'(\tau)| \le C_1$, $|\Psi^{(l)}(\tau)| \le$

$C_l(1+|\tau|)^{-2-l}$, for $l \geq 2$. So we can write:

$$\begin{aligned}
& k^{2/3}\sum_{l=0}^{L}\left|k^{-2l/3}\Psi^{(l)}(k^{1/3}Z(s))\right| \\
& = k^{2/3}\left|\Psi(k^{1/3}Z(s))\right| + k^{2/3}k^{-2/3}\left|\Psi'(k^{1/3}Z(s))\right| + k^{2/3}\sum_{l=2}^{L}k^{-2l/3}\left|\Psi^{(l)}(k^{1/3}Z(s))\right| \\
& \lesssim_n k^{2/3}(1+k^{1/3}|\omega(s)|) + k^{2/3}k^{-2/3} + k^{2/3}\sum_{l=2}^{L}k^{-2l/3}C_l(1+k^{1/3}|\omega(s)|)^{-2-l} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } \omega \text{ is bounded in } [0,2\pi]) \\
& \lesssim_n k^{2/3}(1+k^{1/3}) + 1 + k^{2/3}\sum_{l=2}^{\infty}k_0^{-2l/3} \\
& \lesssim_n k
\end{aligned} \tag{4.13}$$

Now combining inequalities (4.10) - (4.13) we can write:

$$|D_s^n A_{L,M}(s,k)| \lesssim_n \begin{cases} k, & n=0 \\ k, & n=1 \\ k + \sum_{j=2}^{n} k^{j+2/3}(1+k^{1/3}|\omega|)^{-j-2}, & n \geq 2 \end{cases}$$

We have already stated that $|D_s^n R_{L,M}(s,k)| \leq C_n$. Using triangle inequality and the fact that $\eta^{slow}(s,k) = A_{L,M}(s,k) + R_{L,M}(s,k)$ we complete the proof. $\square$

5. CONVERGENCE ANALYSIS

In this chapter, we will provide the proof of Theorem 3.1. First we will discuss the related best approximation theorem in [14, Theorem 3]. Then we will give the parts where we pose the improved contributions.

5.1. Approximation by weighted algebraic polynomials

Theorem 5.1 (Best approximation of η^{slow} by algebraic polynomials [14]). *Given $d \in \mathbb{N}$, for all $n \in \{0, \cdots, d+1\}$, all sufficiently large $k > k_0 > 1$, and $0 \leq \epsilon, \delta \leq 1/3$ we have:*

(i) *[Illuminated region] If* $I_{IL} = [t_1 + \xi_1 k^{-1/3+\epsilon},\, t_2 - \xi_2 k^{-1/3+\epsilon}]$, *then*

$$\inf_{p \in \mathbb{P}_d} ||\eta^{slow} - p||_{L^2(I_{IL})} \lesssim_n k \frac{1 + k^{-(1+3\epsilon)/2} k^{(1/3-\epsilon)n/2}}{d^n}$$

(ii) *[Deep shadow region] If* $I_{DS} = [t_2 + \xi_1 k^{-1/3+\epsilon},\, L + t_1 - \xi_2 k^{-1/3+\epsilon}]$, *then*

$$\inf_{p \in \mathbb{P}_d} ||\eta^{slow} - p||_{L^2(I_{DS})} \lesssim_n k \frac{1 + k^{-(1+3\epsilon)/2} k^{(1/3-\epsilon)n/2}}{d^n}$$

(iii) *[Shadow boundaries] If* $I_{SB_l} = [t_1 - \xi k^{-1/3+\delta},\, t_1 + \xi k^{-1/3+\epsilon}]$, *or if* $I_{SB_l} = [t_2 - \xi k^{-1/3+\epsilon},\, t_2 + \xi k^{-1/3+\delta}]$, *then*

$$\inf_{p \in \mathbb{P}_d} ||\eta^{slow} - p||_{L^2(I_{SB_l})} \lesssim_n k \frac{1 + k^{-1/2} k^{(\epsilon+\delta)n/2}}{d^n}$$

(iv) *[Illuminated transitions] If* $I_{IT_l} = [t_1 + \xi k^{-1/3+\delta},\, t_1 + \xi k^{-1/3+\epsilon}]$, *or if* $I_{IT_l} = [t_2 - \xi k^{-1/3+\epsilon},\, t_2 - \xi k^{-1/3+\delta}]$, *then*

$$\inf_{p \in \mathbb{P}_d} ||\eta^{slow} - p||_{L^2(I_{IT_l})} \lesssim_n k \frac{1 + k^{-(1+3\delta)/2} k^{(\epsilon-\delta)n/2}}{d^n}$$

(v) [Shadow transitions] If $I_{ST_l} = [t_1 - \xi k^{-1/3+\epsilon},\, t_1 - \xi k^{-1/3+\delta}]$*, or if* $I_{ST_l} = [t_2 + \xi k^{-1/3+\delta},\, t_2 + \xi k^{-1/3+\epsilon}]$*, then*

$$\inf_{p \in \mathbb{P}_d} ||\eta^{slow} - p||_{L^2(I_{ST_l})} \lesssim_n k \frac{1 + k^{-(1+3\delta)/2} k^{(\epsilon-\delta)n/2}}{d^n}$$

Proof. If we recall the well known semi-norm for a given interval (a, b)

$$|f|_{n,(a,b)} := \Big[\int_a^b |D^n f(s)|^2 (s-a)^n (b-s)^n ds \Big]^{1/2}. \tag{5.1}$$

Then we have [21]:

$$\inf_{p \in \mathbb{P}_d} ||f - p|| \lesssim_n |f|_{n,(a,b)} d^{-n} \tag{5.2}$$

for all $n \in \{0, 1, \cdots, d+1\}$ where $\mathbb{P}_d$ is the space of univariate polynomials of degree at most d.

Using this inequality and Theorem 4.6 we have:

$$\begin{aligned} |\eta^{slow}(s,k)|^2_{n,(a,b)} &\lesssim_n \int_a^b |D_s^n \eta^{slow}(s,k)|^2 (s-a)^n (b-s)^n ds \\ &\lesssim_n \int_a^b \Big[k^2 + \sum_{m=8}^{2n+4} \big[k^{-1/3} + |\omega(s)| \big]^{-m} \Big] (s-a)^n (b-s)^n ds \\ &\lesssim_n k^2 + \sum_{m=8}^{2n+4} \int_a^b \frac{(s-a)^n (b-s)^n}{[k^{-1/3} + |\omega(s)|]^m} ds \end{aligned} \tag{5.3}$$

Now we will give the estimates of the right hand side of (5.3) and together with (5.2) we will complete the proof.

First, without loss of generality, assume that $(t_2 - t_1)/2 > 1$, $k > k_0 \geq 1$ and

$0 < \xi, \xi_1, \xi_2 < \min\{t_1, \frac{t_2-t_1}{2}\}$, and define:

$$
\begin{aligned}
c_1 &:= \frac{t_2+t_1}{2} - \sqrt{\left(\frac{t_2-t_1}{2}\right)^2 + k^{-1/3}} \\
d_1 &:= \frac{t_2+t_1}{2} + \sqrt{\left(\frac{t_2-t_1}{2}\right)^2 + k^{-1/3}} \\
c_2 &:= \frac{t_2+t_1}{2} - \sqrt{\left(\frac{t_2-t_1}{2}\right)^2 - k^{-1/3}} \\
d_2 &:= \frac{t_2+t_1}{2} + \sqrt{\left(\frac{t_2-t_1}{2}\right)^2 - k^{-1/3}}
\end{aligned}
$$

Also note that the L-periodic function $k^{-1/3} + |\omega(s)|$ factors as:

$$
k^{-1/3} + |\omega(s)| = \begin{cases} (s-c_1)(d_1-s), & s \in [t_1, t_2], \\ -(s-c_2)(d_2-s), & s \in [0, L] \backslash [t_1, t_2], \end{cases} \tag{5.4}
$$

In order to estimate the integrals on the right hand side of inequality (5.3) we will use the above separation trick, then using the exact calculation given in Lemma A.3 we will present an estimate on those integrals, therefore an estimate to the semi-norm $|\eta^{slow}|_{n,(a,b)}$. Since the steps of the calculations for the cases (i) and (iv) are symmetric to (ii) and (v) respectively, we will only give the proof for the cases (i), (iii), and (iv). In all cases we will take into consideration that,

$$
\begin{aligned}
t_2 - t_1 \ < \ d_1 - c_1 \ \leq 2\sqrt{\left(\frac{t_2-t_1}{2}\right)^2 + 1} \\
0 < 2\sqrt{\left(\frac{t_2-t_1}{2}\right)^2 - 1} \ \leq \ d_2 - c_2 \ < t_2 - t_1
\end{aligned}
$$

Keeping those in mind, for the case (i) illuminated region, using the exact calculation of the integral given in Lemma A.3 we have the following estimate

$$\int_a^b \frac{(s-a)^n(b-s)^n}{(k^{-1/3}+|\omega(s)|)^m}ds = \int_a^b \frac{(s-a)^n(b-s)^n}{(s-c_1)^m(d_1-s)^m}ds$$
$$\lesssim_n \sum_{r=1}^{m}\Bigg\{ \sum_{\substack{0\le p,q\le n\\ p+q\neq 2n+1-r}} \Big|(c_1-a)^p(c_1-b)^q[(b-c_1)^{2n-(p+q+r)+1}-(a-c_1)^{2n-(p+q+r)+1}]$$
$$+(a-d_1)^p(b-d_1)^q[(d_1-a)^{2n-(p+q+r)+1}-(d_1-b)^{2n-(p+q+r)+1}]\Big|\Bigg\}$$

Considering the inequalities,

$$\xi_1 k^{-1/3}k^{\epsilon} < a-c_1 < (1+\xi_1)k^{-1/3}k^{\epsilon}$$
$$\frac{t_2-t_1}{2} < b-c_1 < \frac{t_2-t_1}{2} + \sqrt{\left(\frac{t_2-t_1}{2}\right)^2+1}$$
$$\frac{t_2-t_1}{2} < d_1-a < \frac{t_2-t_1}{2} + \sqrt{\left(\frac{t_2-t_1}{2}\right)^2+1}$$
$$\xi_2 k^{-1/3}k^{\epsilon} < d_1-b < (1+\xi_2)k^{-1/3}k^{\epsilon}$$

our integral has the following estimate

$$\int_a^b \frac{(s-a)^n(b-s)^n}{(k^{-1/3}+|\omega(s)|)^m}ds$$
$$\lesssim_n \sum_{r=1}^{m}\sum_{\substack{0\le p,q\le n\\ p+q\neq 2n+1-r}} \Big[(k^{-1/3+\epsilon})^p|1-(k^{-1/3+\epsilon})^{2n-(p+q+r)+1}|+(k^{-1/3+\epsilon})^q|1-(k^{-1/3+\epsilon})^{2n-(p+q+r)+1}|\Big]$$
$$\lesssim_n \sum_{r=1}^{m}\sum_{\substack{0\le p,q\le n\\ p+q\neq 2n+1-r}} \Big[(k^{-1/3+\epsilon})^p+(k^{-1/3+\epsilon})^{2n-(q+r)+1}+(k^{-1/3+\epsilon})^q+(k^{-1/3+\epsilon})^{2n-(p+r)+1}\Big]$$
$$\lesssim_n \sum_{r=1}^{m}\sum_{\substack{0\le p,q\le n\\ p+q\neq 2n+1-r}} \Big[(k^{-1/3+\epsilon})^p+(k^{-1/3+\epsilon})^{2n-(q+r)+1}+(k^{-1/3+\epsilon})^q+(k^{-1/3+\epsilon})^{2n-(p+r)+1}\Big]$$
$$\lesssim_n 1+(k^{-1/3+\epsilon})^{2n-(n+m)+1}+1+(k^{-1/3+\epsilon})^{2n-(n+m)+1} \lesssim_n (k^{-1/3+\epsilon})^{n-m+1}$$

Inserting this upperbound in inequality (5.3) we have

$$|\eta^{slow}(s,k)|^2_{n,(a,b)} \lesssim_n k^2 + \max_{1\le m\le 2n+4}(k^{-1/3+\epsilon})^{n-m+1}$$
$$\lesssim_n k^2+(k^{1/3}k^{-\epsilon})^{n+3} \quad \lesssim_n \quad \Big\{k\Big(1+k^{-(1+3\epsilon)/2}k^{(1/3-\epsilon)n/2}\Big)\Big\}^2$$

On the other hand, for the case (iii) of shadow boundaries with the same tricks we have

$$\begin{aligned}
\int_a^b \frac{(s-a)^n(b-s)^n}{\left[k^{-1/3}+|\omega(s)|\right]^m}ds &= (-1)^m\int_a^{t_1}\frac{(s-a)^n(b-s)^n}{(s-c_2)^m(d_2-s)^m}ds+\int_{t_1}^b\frac{(s-a)^n(b-s)^n}{(s-c_1)^m(d_1-s)^m}ds\\
\lesssim_n \sum_{r=1}^m\Bigg\{ &\sum_{\substack{0\le p,q\le n\\ p+q\ne 2n+1-r}}\Big|(c_2-a)^p(c_2-b)^q[(b-c_2)^{2n-(p+q+r)+1}-(a-c_2)^{2n-(p+q+r)+1}]\\
&\qquad +(a-d_2)^p(b-d_2)^q[(d_2-a)^{2n-(p+q+r)+1}-(d_2-b)^{2n-(p+q+r)+1}]\Big|\\
&+\sum_{\substack{0\le p,q\le n\\ p+q\ne 2n+1-r}}\Big|(c_1-a)^p(c_1-b)^q[(b-c_1)^{2n-(p+q+r)+1}-(a-c_1)^{2n-(p+q+r)+1}]\\
&\qquad +(a-d_1)^p(b-d_1)^q[(d_1-a)^{2n-(p+q+r)+1}-(d_1-b)^{2n-(p+q+r)+1}]\Big|\Bigg\}
\end{aligned}$$

Using the definitions of d_1, d_2 and the definition of end points of the interval I_{SB_l} we have

$$\frac{t_1+t_2}{2} < d_2, d_1 < 2t_2$$
$$t_1-\xi < t_1, a, b < t_1+\xi$$

Hence for all $x \in \{(d_2-t_1),(d_2-b),(d_2-a),(d_1-t_1),(d_1-b),(d_1-a)\}$ we have

$$0 < \frac{t_2+t_1}{2}-(t_1+\xi) \ < \ x \ < \ 2t_2-t_1+\xi$$

Using this idea we can bound our integral as follows

$$\begin{aligned}
\int_a^b \frac{(s-a)^n(b-s)^n}{\left[k^{-1/3}+|\omega(s)|\right]^m}ds &= (-1)^m\int_a^{t_1}\frac{(s-a)^n(b-s)^n}{(s-c_2)^m(d_2-s)^m}ds+\int_{t_1}^b\frac{(s-a)^n(b-s)^n}{(s-c_1)^m(d_1-s)^m}ds\\
\lesssim_n \sum_{r=1}^m\Bigg\{ &\sum_{\substack{0\le p,q\le n\\ p+q\ne 2n+1-r}}\Big|(c_2-a)^p(c_2-b)^q[(b-c_2)^{2n-(p+q+r)+1}-(a-c_2)^{2n-(p+q+r)+1}]\Big|\\
&+\Big|(c_1-a)^p(c_1-b)^q[(b-c_1)^{2n-(p+q+r)+1}-(a-c_1)^{2n-(p+q+r)+1}]\Big|\Bigg\}
\end{aligned}$$

Now considering the inequalities

$$
\begin{aligned}
\xi_1 k^{-1/3}k^{\delta} - 2\frac{k^{-1/3}}{t_2-t_1} &< (c_2-a),(c_1-a) < (1+\xi_1)k^{-1/3}k^{\delta},\\
\frac{k^{-1/3}}{\frac{t_2-t_1}{2}(1+\sqrt{2})} &< (t_1-c_1),(c_2-t_1) < 2\frac{k^{-1/3}}{t_2-t_1},\\
\xi_2 k^{-1/3}k^{\epsilon} - 2\frac{k^{-1/3}}{t_2-t_1} &< (b-c_1),(b-c_2) < (1+\xi_2)k^{-1/3}k^{\epsilon},
\end{aligned}
$$

we can bound our integral in following way

$$
\begin{aligned}
&\int_a^b \frac{(s-a)^n(b-s)^n}{[k^{-1/3}+|\omega(s)|]^m}ds\\
&\quad\lesssim_n \sum_{r=1}^{m}\sum_{\substack{0\le p,q\le n\\ p+q\neq 2n+1-r}} \left\{(k^{-1/3+\delta})^p(k^{-1/3+\epsilon})^q[(k^{-1/3+\epsilon})^{2n-(p+q+r)+1}+(k^{-1/3+\delta})^{2n-(p+q+r)+1}]\right\}\\
&\quad\lesssim_n (k^{-1/3+\delta})^n(k^{-1/3+\epsilon})^n[(k^{-1/3+\epsilon})^{2n-(n+n+m)+1}+(k^{-1/3+\delta})^{2n-(n+n+m)+1}]
\end{aligned}
$$

Considering again inequality (5.3) we have

$$
\begin{aligned}
&|\eta^{slow}(s,k)|^2_{n,(a,b)}\\
&\qquad\lesssim_n \max_{1\le m\le 2n+4}(k^{-1/3+\delta})^n(k^{-1/3+\epsilon})^n[(k^{-1/3+\epsilon})^{2n-(n+n+m)+1}+(k^{-1/3+\delta})^{2n-(n+n+m)+1}\\
&\qquad\lesssim_n k^2+kk^{(\epsilon+\delta)n} \lesssim_n \left\{k\left(1+k^{-1/2}k^{(\epsilon+\delta)n/2}\right)\right\}^2
\end{aligned}
$$

When it comes to the (iv) illuminated transitions, we have

$$
\begin{gathered}
\xi k^{-1/3}k^{\delta} < a-c_1 < (1+\xi)k^{-1/3}k^{\delta}\\
\xi k^{-1/3}k^{\epsilon} < b-c_1 < (1+\xi)k^{-1/3}k^{\epsilon}\\
\frac{t_2-t_1}{2} < d_1-a < \frac{3(t_2-t_1)}{2}\\
\frac{t_2-t_1}{2} < d_1-b < \frac{3(t_2-t_1)}{2}
\end{gathered}
$$

with the similar manipulations done in (i) we conclude

$$|\eta^{slow}(s,k)|^2_{n,(a,b)} \lesssim_n k\left(1 + k^{-(1+3\delta)/2} k^{(\epsilon-\delta)n/2}\right)$$

Finally, applying our semi-norm estimate (5.2) to these inequalities, we complete the proof. □

5.2. Estimate on the Transition Regions and the Change of Variables

In this section we aim to improve the estimates on the transition regions given in Theorem 5.1. Upon completion of this section the proof of Theorem 3.1 will also be obtained.

In the previous work [14], in order to mimic the behavior of η^{slow}, the four main regions that separates illuminated and deep shadow regions from the shadow boundaries are treated carefully. They divided each of those four main regions into $m-1$ small regions, the transition regions, and defined different sets of basis functions in each of them. So, together with the $4m-4$ transition regions they had $4m$ regions in total. This approach gave them an enhanced approximation. However it is possible to unite the each set of $m-1$ little transition regions and obtain four main regions instead of $4(m-1)$ ones. Yet, in order to unite them we need to present a clever change of variables in each set of transition regions. Now let us denote the four main transition regions in the following way:

$$\begin{aligned} I_{IT_1} &= [t_1 + \xi k^{-1/3}, t_1 + \xi] = [a_1, b_1] \\ I_{IT_2} &= [t_2 - \xi, t_2 - \xi k^{-1/3}] = [a_2, b_2] \\ I_{ST_1} &= [t_1 - \xi, t_1 - \xi k^{-1/3}] = [a_3, b_3] \\ I_{ST_2} &= [t_2 + \xi k^{-1/3}, t_2 + \xi] = [a_4, b_4] \end{aligned}$$

Now in each interval we inroduce the following change of variables:

$$\phi(s) = \begin{cases} t_1 + \xi k^{\psi(s)}, & s \in I_{IT_1} \\ t_2 - \xi k^{\psi(s)}, & s \in I_{IT_2} \\ t_1 - \xi k^{\psi(s)}, & s \in I_{ST_1} \\ t_2 + \xi k^{\psi(s)}, & s \in I_{ST_2}, \end{cases}$$

where the linear function ψ is defined such that

$$\psi(s) = \begin{cases} \frac{1}{3(b_1-a_1)}(s-a_1) - \frac{1}{3}, & s \in I_{IT_1} \\ \frac{1}{3(a_2-b_2)}(s-b_2) - \frac{1}{3}, & s \in I_{IT_2} \\ \frac{1}{3(a_3-b_3)}(s-b_3) - \frac{1}{3}, & s \in I_{ST_1} \\ \frac{1}{3(b_4-a_4)}(s-a_4) - \frac{1}{3}, & s \in I_{ST_2}. \end{cases}$$

With this clever choice of ψ, our function ϕ maps each of the transition regions to itself.

Now, we intent to give the detailed analysis of this change of variables function ϕ, but without loss of generality we will only discuss the part of it restricted to the interval I_{IT_1}, the Illuminated Region-1. Throughout this section we will assume that the functions $\phi : [a, b] \to [a, b]$ and $\psi : [a, b] \to [-\frac{1}{3}, 0]$ are defined as

$$\begin{aligned} \phi(s) &= t_1 + \xi k^{\psi(s)} \\ \psi(s) &= \frac{1}{3(b-a)}(s-a) - \frac{1}{3} \end{aligned}$$

where $a := t_1 + \xi k^{-1/3}, b := t_1 + \xi$, and $I_{IT_1} = [a, b]$. We also have, $k > k_0$ for some fixed $k_0 > 1$ and $|\xi| < (t_2 - t_1)/2$. Since, $\psi(a) = -1/3, \psi(b) = 0, \phi(a) = a,$ and $\phi(b) = b$, ϕ preserves the interval $I_{IT_1} = [a, b]$, as we mentioned earlier.

With the aid of this change of variables we will prove the main result of this section:

Theorem 5.2. *For the interval $I_{IT_1} = [a, b]$ and the change of variable function ϕ*

defined as above, we have

$$\inf_{p\in\mathbb{P}^d} ||\eta^{slow} - p(\phi^{-1})||_{L^2(I_{IT_1})} \lesssim_n k(\log k)^{1/2}\left(\frac{\log k}{d}\right)^n$$

where $\mathbb{P}^d$ represents the univariate polynomials of degree at most than $d \in \mathbb{N}$.

Before giving the proof of this theorem, we should verify some propositions.

Proposition 5.3. *For ϕ, ψ defined as above we have*

$$|\psi'(s)| \lesssim 1, \quad \forall s \in [0, 2\pi]$$
$$|\phi^{(i)}(s)| \lesssim (\log k)^i, \quad \forall s \in [0, 2\pi], \forall i \in \mathbb{N}.$$

Proof. Since $k > k_0$ implies $1 - k^{-1/3} > 1 - k_0^{-1/3}$, we have,

$$|\psi'(s)| = \left|\frac{1}{3(b-a)}\right| = \left|\frac{1}{3\xi(1-k^{-1/3})}\right| < \frac{1}{3|\xi|(1-k_0^{-1/3})} \lesssim 1$$

On the other hand,

$$s \in [a,b] \quad \Rightarrow \quad 0 \le s-a \le b-a \quad \Rightarrow \quad \frac{s-a}{b-a} \le 1 \quad \Rightarrow \quad \frac{s-a}{3(b-a)} - \frac{1}{3} \le 0$$

Obviously as s increases $\frac{s-a}{3(b-a)} - \frac{1}{3}$ also increases. Thus we have,

$$-1/3 \le s \le 0 \quad ,\forall s \in [a,b] \quad \Rightarrow \quad k^{-1/3} \le k^{\psi(s)} \le 1 \quad ,\forall s \in [a,b]$$

Now, calculating the derivatives of ϕ gives us,

$$\phi'(s) = \xi\psi'(s)(\log k)k^{\psi(s)}$$
$$\phi^{(i)}(s) = (\xi\psi'(s)(\log k))^i k^{\psi(s)}$$
$$|\phi^{(i)}(s)| \lesssim ((\log k))^i k^{\psi(s)} \lesssim ((\log k))^i$$

printing the desired result. $\square$

The bounds calculated above for the derivatives of ψ and ϕ are important for the derivative analysis of η^{slow}, which is given in the next proposition.

Proposition 5.4. *For ϕ defined as above and for all $n \in \mathbb{N}$ we have*

$$|D_s^n(\eta^{slow} \circ \phi)(s)| \lesssim_n k(\log k)^n$$

Proof. By Faà Di Bruno's formula [22],

$$\begin{aligned}
|D_s^n(\eta^{slow} \circ \phi(s))| &= \left| \sum \left\{ i! D^i \eta^{slow}(\phi(s)) \prod_{j=1}^{n} \frac{(D^j \phi(s))^{i_j}}{j! i_j!} : i = \sum_{j=1}^{n} i_j, \quad n = \sum_{j=1}^{n} j i_j, \quad i_j \geq 0 \right\} \right| \\
&\lesssim_n \sum \left\{ |D^i \eta^{slow}(\phi(s))| \prod_{j=1}^{n} |(D^j \phi(s))^{i_j}| : i = \sum_{j=1}^{n} i_j, \quad n = \sum_{j=1}^{n} j i_j, \quad i_j \geq 0 \right\} \\
&\lesssim_n \sum \left\{ |(\eta^{slow})^{(i)}(\phi(s))| \prod_{j=1}^{n} |((\log k))^{j i_j} k^{j\psi(s)}| : i = \sum_{j=1}^{n} i_j, n = \sum_{j=1}^{n} j i_j, i_j \geq 0 \right\} \\
&\lesssim_n \sum \left\{ |(\eta^{slow})^{(i)}(\phi(s))| |((\log k))^{\sum j i_j} k^{\sum j \psi(s)}| : i = \sum_{j=1}^{n} i_j, n = \sum_{j=1}^{n} j i_j, i_j \geq 0 \right\}
\end{aligned}$$

Hence we have

$$|D_s^n(\eta^{slow} \circ \phi(s))| \lesssim_n \sum_{i=0}^{n} |(\eta^{slow})^{(i)}(\phi(s))| k^{i\psi(s)} (\log k)^n \tag{5.5}$$

Next, we will give the upper bound for the summand. For this purpose let us recall the definiton of the function $\omega(s) = (s - t_1)(t_2 - s)$ together with Theorem 4.6 gives us:

$$k^{i\psi(s)} |\eta^{(i)}(\phi(s))| \lesssim k^{i\psi(s)} \left[k + \sum_{j=4}^{i+2} (k^{-1/3} + |\omega(\phi(s))|)^{-j} \right].$$

Now, instead of the summation in the right hand side, we will insert the following estimate, for all $j \leq i + 2$ and $s \in [a, b]$

$$(k^{-1/3} + |\omega(\phi)|)^{-j} \lesssim_n (k^{-1/3} + |\omega(\phi)|)^{-(i+2)}.$$

This estimate can be seen from,

$$
\begin{aligned}
(k^{-1/3}+|\omega(\phi(s))|)^{-j} &= (k^{-1/3}+|\omega(\phi(s))|)^{-(i+2)}(k^{-1/3}+|\omega(\phi(s))|)^{(i+2)-j} \\
&\leq (k^{-1/3}+|\omega(\phi(s))|)^{-(i+2)}(k_0^{-1/3}+(2\pi)^2)^{(i+2)-j} \\
&\lesssim_n (k^{-1/3}+|\omega(\phi(s))|)^{-(i+2)}
\end{aligned}
$$

Therefore we have:

$$
\begin{aligned}
k^{i\psi(s)}|\eta^{(i)}(\phi(s))| &\lesssim k^{i\psi(s)}\left[k+\sum_{j=4}^{i+2}(k^{-1/3}+|\omega(\phi(s))|)^{-j}\right] \\
&\lesssim_n k^{i\psi(s)}k+k^{i\psi(s)}(k^{-1/3}+|\omega(\phi(s))|)^{-(i+2)} \\
&\lesssim_n k+\Big(\frac{k^{\psi(s)}}{k^{-1/3}+|\omega(\phi(s))|}\Big)^{i}(k^{-1/3}+|\omega(\phi(s))|)^{-2}
\end{aligned}
$$

The estimate on $(k^{-1/3}+|\omega(\phi(s))|)$ can be established as the following way:

$$
\begin{aligned}
(k^{-1/3}+|\omega(\phi(s))|)^{-1} &= (k^{-1/3}+|(t_2-t_1)\xi k^{\psi(s)}-\xi^2k^{2\psi(s)}|)^{-1} \\
&= k^{-\psi(s)}(k^{-1/3-\psi(s)}+|(t_2-t_1)\xi-\xi^2k^{\psi(s)}|)^{-1} \\
&\leq k^{-\psi(s)}\left(|(t_2-t_1)\xi-\xi^2|\right)^{-1} \\
&\lesssim k^{-\psi(s)}
\end{aligned}
$$

So we can conclude,

$$
\begin{aligned}
k^{i\psi(s)}|\eta^{(i)}(\phi(s))| &\lesssim_n k+\Big(\frac{k^{\psi(s)}}{k^{-1/3}+|\omega(\phi(s))|}\Big)^{i}(k^{-1/3}+|\omega(\phi(s))|)^{-2} \\
&\lesssim k+\,(1)^i\,k^{-2\psi(s)}\;\leq\;k+k^{2/3}\;\lesssim_n\;k
\end{aligned}
$$

Hence inserting this result in the estimate (5.5) we complete the proof. □

After constructing the necessary propositions now we are ready to prove this section's main theorem.

Proof. (of Theorem 5.2) Using the estimate on the semi-norm that is given in (5.2), and the boundedness of ϕ' on $I_{IT_1} = [a, b]$,

$$\begin{aligned}
\inf_{p\in\mathbb{P}^d} ||\eta^{slow} - p(\phi^{-1})||_{L^2[a,b]} &= \inf_{p\in\mathbb{P}^d} \Big\{ \int_a^b |\eta^{slow}(s) - p(\phi^{-1}(s))|^2 ds \Big\}^{1/2} \\
&\le \inf_{p\in\mathbb{P}^d} \Big\{ \int_a^b |\eta^{slow}(\phi(s)) - p(s)|^2 |\phi'(s)| ds \Big\}^{1/2} \\
&\le \inf_{p\in\mathbb{P}^d} \big\{ ||(\eta^{slow} \circ \phi) - p||_{L^2} (\max_{s\in[a,b]} |\phi'(s)|)^{1/2} \big\} \\
&\lesssim_n |\eta^{slow} \circ \phi|_{n,[a,b]} \; d^{-n} (\max_{s\in[a,b]} |\phi'(s)|)^{1/2},
\end{aligned}$$

where in the last line estimate holds for all $n \in \mathbb{N}$ such that $n \le d + 1$. Moreover, in the last line we will impose the definition of the semi-norm in Definition 5.1, and the upper bound for ϕ', then with the aid of Proposition 5.4:

$$\begin{aligned}
\inf_{p\in\mathbb{P}^d} ||\eta^{slow} - p(\phi^{-1})||_{L^2[a,b]} &\lesssim_n \Big\{ \int_a^b |D^n(\eta \circ \phi)(s)|^2 (s-a)^n (b-s)^n ds \Big\}^{1/2} d^{-n} (\log k)^{1/2} \\
&\lesssim_n (b-a)^{\frac{2n+1}{2}} \; k \; (\log k)^n d^{-n} (\log k)^{1/2} \\
&\lesssim_n k (\log k)^{1/2} \left(\frac{\log k}{d} \right)^n
\end{aligned}$$

This completes the argument. $\square$

Hence, with the help of Theorem 5.2 we have established an improved upper bound for the error in the transition regions. Together with Theorem 5.1 we have completed the proof of Theorem 3.1.

5.3. Approximation by weighted trigonometric polynomials

In this section we will use approximation spaces constructed by trigonometric polynomials. However, our convergence analysis depends on the periodicity of the functions in those spaces. In order to attain the periodicity we will introduce the smooth bump functions defined on the $4m$ subregions we described on Section 5.1. The crucial point here is, the set of these $4m$ bump functions will be also representing a smooth partition of the interval $[0, L]$. So multiplying each of the bump functions with η^{slow} gives us periodic smooth functions and summation of all gives us η^{slow} itself.

To begin with, let us define the smooth functions

$$\lambda(x) = \begin{cases} 1, & x \leq 0, \\ \exp\left(\frac{2\exp(-\frac{1}{x})}{x-1}\right), & 0 < x < 1, \\ 0, & 1 \leq x. \end{cases}$$
$$\mu(x) = \frac{1}{2}[\lambda(x) + 1 - \lambda(1-x)]$$

Here note that $\mu(x)$ is a positive non-increasing smooth function (see Figure 5.1).

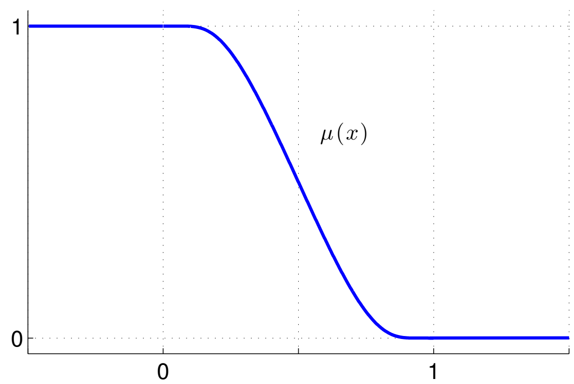

Figure 5.1. $\mu(x)$ on $[0, 1]$.

Now for given real numbers $a < a' \leq b' < b$, we set up the bump function

$$\chi_{(a,a',b',b)}(x) := \mu\left(\frac{x-a}{a'-a}\right)\mu\left(\frac{x-b}{b'-b}\right)$$

On $[a, a']$ the first multiplicand increases from 0 to 1, and on $[b', b]$ the second one decreases from 1 to 0. Hence the smooth function $\chi_{(a,a',b',b)}$ is equal to 1 on $[a', b']$ and vanishes outside $[a, b]$(See Figure 5.2). On the other hand, since

$$\mu(x) + \mu(1-x) = 1, \quad x \in \mathbb{R}$$
$$\left(\frac{x-a}{b-a}\right) + \left(\frac{x-b}{a-b}\right) = 1, \quad x \in \mathbb{R}$$

then for given real numbers $a_1 < a_2 \leq a_3 < a_4 \leq a_5 < a_6$ we have

$$\chi_{(a_1,a_2,a_3,a_4)}(x) + \chi_{(a_3,a_4,a_5,a_6)}(x) = 1, \quad x \in \mathbb{R} \tag{5.6}$$

Figure 5.2. Plot of $\chi_{(a,a',b',b)}$.

In order to build up the bump functions $\chi_{(a,a',b',b)}$, we need to identify the reals $a < a' \leq b' < b$ for each region. Recalling the definitions of the intervals given in Section 3.2 we set the variable a' of an interval to be equal to the end point of the previous interval, in the counter clockwise orientation. Then we set the variable b' of an interval to be equal to the starting point of the next interval, in the same orientation. From now on we will symbolize the quadruple (a, a', b', b) of an interval I with $\sigma(I)$.

Regarding these identifications, we denote the jth interval as I_j, its bump function as $\chi_{\sigma(I_j)}$ and its characteristic function as $\mathbb{1}_j$. With the light of the identity (5.6), for $x \in [0, L]$ we have $\sum_{j=1}^{4m} \chi_{\sigma(I_j)}(x) = 1$ Then, from Definition 1.1 we have

$$\eta = \sum_{j=1}^{4m} e^{ik\,\alpha\cdot\gamma}\eta^{slow}\chi_{\sigma(I_j)} \tag{5.7}$$

and each $\hat{\mu} \in \dim(\mathscr{T}_{\mathbf{d}})$ can be written in the following form

$$\hat{\mu} = \sum_{j=1}^{4m} e^{ik\,\alpha\cdot\gamma} \mathbb{1}_j p_j,$$

where $p_j \in \mathbb{T}_{d_j}$. Using these facts, Céa's lemma gives us

$$||\eta - \hat{\eta}||_{L^2(\partial K)} \leq \frac{C}{c} k \sum_{j=1}^{4m} \inf_{p \in \mathbb{T}_{d_j}} ||\eta^{slow} \chi_{\sigma(I_j)} - p||_{L^2(I_j)} \tag{5.8}$$

This inequality together with Theorem 5.5 complete the proof of Theorems 3.4

Theorem 5.5 (Best approximation by trigonometric polynomials [14])**.** *For all $n \in \mathbb{N} \cup \{0\}$ and all sufficiently large $k \geq 1$, there holds the following estimates:*

(i) [Illumunated region] If $0 \leq \epsilon_1 < \epsilon_2 \leq 1/3$ and $0 \leq \epsilon_4 < \epsilon_3 \leq 1/3$, and

$$\sigma(I_{IL}) = (t_1 + \xi_1 k^{-1/3} k^{\epsilon_1},\, t_1 + \xi_1 k^{-1/3} k^{\epsilon_2},\, t_2 - \xi_2 k^{-1/3} k^{\epsilon_3},\, t_2 - \xi_2 k^{-1/3} k^{\epsilon_4})$$

then we have

$$\inf_{p \in \mathbb{T}_d(I_{IL})} ||\eta^{slow} \chi_{\sigma(I_{IL})} - p||_{L^2(I_{IL})} \lesssim_n k \left(\frac{k^{1/3 - \min\{\epsilon_1, \epsilon_4\}}}{d+1} \right)^n.$$

(ii) [Deep shadow region]If $0 \leq \epsilon_1 < \epsilon_2 \leq 1/3$ and $0 \leq \epsilon_4 < \epsilon_3 \leq 1/3$, and

$$\sigma(I_{DS}) = (t_2 + \xi_1 k^{-1/3} k^{\epsilon_1}, t_2 + \xi_1 k^{-1/3} k^{\epsilon_2},\, L + t_1 - \xi_2 k^{-1/3} k^{\epsilon_3},\, L + t_1 - \xi_2 k^{-1/3} k^{\epsilon_4})$$

then we have

$$\inf_{p \in \mathbb{T}_d(I_{DS})} ||\eta^{slow} \chi_{\sigma(I_{DS})} - p||_{L^2(I_{DS})} \lesssim_n k \left(\frac{k^{1/3 - \min\{\epsilon_1, \epsilon_4\}}}{d+1} \right)^n.$$

(iii) [Shadow boundaries] If $0 \leq \epsilon_2 < \epsilon_1 \leq 1/3$ and $0 \leq \epsilon_3 < \epsilon_4 \leq 1/3$, and

$$\sigma(I_{SB_l}) = (t_1 - \xi_1 k^{-1/3}k^{\epsilon_1},\, t_1 - \xi_1 k^{-1/3}k^{\epsilon_2},\, t_1 + \xi_2 k^{-1/3}k^{\epsilon_3},\, t_1 + \xi_2 k^{-1/3}k^{\epsilon_4})$$

or

$$\sigma(I_{SB_l}) = (t_2 - \xi_1 k^{-1/3}k^{\epsilon_1},\, t_2 - \xi_1 k^{-1/3}k^{\epsilon_2},\, t_2 + \xi_2 k^{-1/3}k^{\epsilon_3},\, t_2 + \xi_2 k^{-1/3}k^{\epsilon_4})$$

then we have

$$\inf_{p\in\mathbb{T}_d(I_{SB_l})} ||\eta^{slow}\chi_{\sigma(I_{SB_l})} - p||_{L^2(I_{SB_l})} \lesssim_n k\left(\frac{k^{\max\{\epsilon_1,\epsilon_4\}}}{d+1}\right)^n.$$

(iv) [*Illuminated transitions:*] *If* $0 \le \epsilon_1 < \epsilon_2 \le \epsilon_3 < \epsilon_4 \le 1/3$*, and*

$$\sigma(I_{IT_l}) = (t_1 + \xi k^{-1/3}k^{\epsilon_1},\, t_1 + \xi k^{-1/3}k^{\epsilon_2},\, t_1 + \xi k^{-1/3}k^{\epsilon_3},\, t_1 + \xi k^{-1/3}k^{\epsilon_4})$$

or

$$\sigma(I_{IT_l}) = (t_2 - \xi k^{-1/3}k^{\epsilon_4},\, t_2 - \xi k^{-1/3}k^{\epsilon_3},\, t_2 - \xi k^{-1/3}k^{\epsilon_2},\, t_2 - \xi k^{-1/3}k^{\epsilon_1})$$

then

$$\inf_{p\in\mathbb{T}_d(I_{IT_l})} ||\eta^{slow}\chi_{\sigma(I_{IT_l})} - p||_{L^2(I_{IT_l})} \lesssim_n k\left(\frac{k^{\epsilon_4-\epsilon_1}}{d+1}\right)^n.$$

(v) [*Shadow transitions:*] *If* $0 \le \epsilon_1 < \epsilon_2 \le \epsilon_3 < \epsilon_4 \le 1/3$*, and*

$$\sigma(I_{ST_l}) = (t_1 - \xi k^{-1/3}k^{\epsilon_4},\, t_1 - \xi k^{-1/3}k^{\epsilon_3},\, t_1 - \xi k^{-1/3}k^{\epsilon_2},\, t_1 - \xi k^{-1/3}k^{\epsilon_1})$$

or

$$\sigma(I_{ST_l}) = (t_2 + \xi k^{-1/3}k^{\epsilon_1},\, t_2 + \xi k^{-1/3}k^{\epsilon_2},\, t_2 + \xi k^{-1/3}k^{\epsilon_3},\, t_2 + \xi k^{-1/3}k^{\epsilon_4})$$

then

$$\inf_{p\in\mathbb{T}_d(I_{ST_l})} ||\eta^{slow}\chi_{\sigma(I_{ST_l})} - p||_{L^2(I_{ST_l})} \lesssim_n k\left(\frac{k^{\epsilon_4-\epsilon_1}}{d+1}\right)^n.$$

In order to prove this theorem we need to establish some estimates about periodic trigonometric polynomials as in the next Corollary.

Corollary 5.6. *If the periodic function f of period $b-a$ possesses an $(n-1)^{th}$ absolutely continuous derivative and $f^{(n)} \in L^2[a,b]$, then for any $d \in \mathbb{N} \cup \{0\}$ we have*

$$\inf_{p \in \mathbb{T}_d[a,b]} ||f - p||_{L^2[a,b]} \lesssim_n \left(\frac{b-a}{d+1}\right)^n \sup_{|h| \leq \frac{b-a}{2\pi(d+1)}} \left\{ \int_a^b |f^{(n)}(x+h) - f^{(n)}(x)|^2 dx \right\}^{1/2} \tag{5.9}$$

Proof. This corollary simply is a more general version of the the best approximation theorem by trigonometric polynomials in [23, 5.1.32], in which the estimate holds for $a = 0, b = 2\pi$. In the proof we will only apply couple of maps of change of variables to make sure that theorem still holds in the interval $[a,b]$. For this purpose let us define the affine map $[0, 2\pi] \to [a,b]$

$$\rho(x) := \frac{b-a}{2\pi} x + a \tag{5.10}$$

Then applying the change of variables $y = \rho(x)$ we have

$$\begin{aligned}
\inf_{p_d \in \mathbb{T}^d[a,b]} ||f - p_d||_{L^2[a,b]} &= \inf_{p_d \in \mathbb{T}^d[a,b]} \left\{ \int_a^b |f(y) - p_d(y)|^2 dy \right\}^{1/2} \\
&= \inf_{p_d \in \mathbb{T}^d[a,b]} \left(\frac{b-a}{2\pi}\right)^{1/2} \left\{ \int_0^{2\pi} |f(\rho(x)) - p_d(\rho(x))|^2 dx \right\}^{1/2} \\
&= \inf_{\tilde{p}_d \in \mathbb{T}^d[0,2\pi]} \left(\frac{b-a}{2\pi}\right)^{1/2} ||f \circ \rho - \tilde{p}_d||_{L^2[0,2\pi]}
\end{aligned}$$

Now using [23, 5.1.32] we can write

$$\begin{aligned}
\inf_{p_d \in \mathbb{T}^d[a,b]} ||f - p_d||_{L^2[a,b]} &= \inf_{\tilde{p}_d \in \mathbb{T}^d[0,2\pi]} \left(\frac{b-a}{2\pi}\right)^{1/2} ||f \circ \rho - \tilde{p}_d||_{L^2[0,2\pi]} \\
&\lesssim_n (d+1)^{-n} \sup_{|h| \leq 1/(d+1)} \left\{ \frac{b-a}{2\pi} \int_0^{2\pi} |f^{(n)}(\rho(x+h))(\rho'(x+h))^n - f^{(n)}(\rho(x))(\rho'(x))^n|^2 dx \right\}^{1/2}
\end{aligned}$$

Considering $\rho'(y) = \frac{b-a}{d+1}$ and applying back the change of variables $\rho(x) = y$ give us

$$\begin{aligned}
\inf_{p_d \in \mathbb{T}^d[a,b]} ||f - p_d||_{L^2[a,b]} &\lesssim_n \left(\frac{b-a}{d+1}\right)^n \sup_{|h| \leq 1/(d+1)} \left\{ \int_a^b |f^{(n)}(y + \frac{b-a}{2\pi} h) - f^{(n)}(y)|^2 dy \right\}^{1/2} \\
&= \left(\frac{b-a}{d+1}\right)^n \sup_{|h'| \leq \frac{b-a}{2\pi(d+1)}} \left\{ \int_a^b |f^{(n)}(y + h') - f^{(n)}(y)|^2 dy \right\}^{1/2}
\end{aligned}$$

Thus the result. $\square$

We want to note that, although the periodic extension of the function η^{slow} is not smooth, by the careful construction of the bump functions, the product $\eta^{slow}\chi_{(a,a',b',b)}$ is a smooth $(b-a)$-periodic function. Thus we can apply Corollary 5.6 to them. With these ideas in our mind, we are ready to present the proof of Theorem 5.5

Proof. (of Theorem 5.5) Since the cases (ii) and (v) are symmetric to (i) and (iv), respectively, we will only prove the cases (i), (iii), and (iv). Now for $\sigma(I) = (a, a', b', b)$ we have

$$\begin{aligned} D_s^n \chi_{\sigma(I)} &= \sum_{j=0}^{n} \binom{n}{j} D_s^j \mu \left(\frac{s-a}{a'-a} \right) D_s^{n-j} \mu \left(\frac{s-b}{b'-b} \right) \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{a'-a} \right)^j \left(\frac{1}{b-b'} \right)^{n-j} \mu^j \left(\frac{s-a}{a'-a} \right) \mu^{(n-j)} \left(\frac{s-b}{b'-b} \right) \end{aligned}$$

Since $\mu(x)$ is a smooth function the derivatives appearing in the last line are bounded with some constants depending only on n. Hence

$$||D_s^n \chi_{\sigma(I)}||_{L^\infty[a,b]} \lesssim_n \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{a'-a} \right)^j \left(\frac{1}{b-b'} \right)^{n-j}.$$

In the case (i) of illuminated region, without loss of generality we may assume that $\epsilon_1 \leq \epsilon_4$. Then

$$\begin{aligned} ||D_s^n \chi_{\sigma(I)}||_{L^\infty[a,b]} &\lesssim_n \sum_{j=0}^{n} \left(\frac{1}{k^{-1/3}(k^{\epsilon_2} - k^{\epsilon_1})} \right)^j \left(\frac{1}{k^{-1/3}(k^{\epsilon_3} - k^{\epsilon_4})} \right)^{n-j} \\ &\lesssim_n \left(\frac{k^{1/3}}{k^{\epsilon_4}(k^{\epsilon_3-\epsilon_4} - 1)} \right)^n \sum_{j=0}^{n} \left(\frac{k^{\epsilon_4}(k^{\epsilon_3-\epsilon_4} - 1)}{k^{\epsilon_1}(k^{\epsilon_2-\epsilon_1} - 1)} \right)^j \\ &\lesssim_n \left(\frac{k^{1/3}}{k^{\epsilon_4}} \right)^n \sum_{j=0}^{n} \left(\frac{k^{\epsilon_4}}{k^{\epsilon_1}} \right)^j \lesssim_n \left(k^{1/3-\epsilon_1} \right)^n \end{aligned}$$

where in the last row we have used the fact that $\left(\frac{k^{\epsilon_4}}{k^{\epsilon_1}}\right)^j \leq \left(\frac{k^{\epsilon_4}}{k^{\epsilon_1}}\right)^n$.

On the other hand, Theorem 4.6 gives us,

$$
\begin{aligned}
|D_s^n \eta^{slow}(s,k)| &\lesssim_n k + \sum_{m=4}^{n+2} (k^{-1/3} + |\omega(s)|)^{-m} \lesssim_n k + \sum_{m=4}^{n+2} (k^{\epsilon_1 - 1/3})^{-m} \\
&\lesssim_n k + k^{(n+2)(1/3-\epsilon_1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||D_s^n \eta^{slow} \chi_{\sigma(I)}||_{L^\infty[a,b]} &= || \sum_{i=0}^{n} \binom{n}{i} D_s^i \eta^{slow} D_s^{n-i} \chi_{[a,a'],[b,b']} ||_{L^\infty[a,b]} \\
&\lesssim_n k \sum_{i=0}^{n} \left(1 + k^{-1} k^{(i+2)(1/3-\epsilon_1)}\right) \left(k^{1/3-\epsilon_1}\right)^{n-i} \;\; \lesssim_n \;\; k \left(k^{1/3-\epsilon_1}\right)^n
\end{aligned}
$$

Recalling Corollary 5.6 with $\xi = \min\{\xi_1, \xi_2\}$, and noting that $||.||_{L^2[a,b]} \leq (b-a)^{1/2} ||.||_{L^\infty[a,b]}$, we have the following:

$$
\begin{aligned}
&\inf_{p \in \mathbb{T}_d[a,b]} ||\eta^{slow} \chi_{\sigma(I)} - p||_{L^2[a,b]} \\
&\quad \lesssim_n \left(\frac{b-a}{d+1}\right)^n \sup_{|h| \leq \frac{b-a}{2\pi(d+1)}} \left\{ \int_a^b |D_s^n \eta^{slow} \chi_{\sigma(I)}(x+h) - D_s^n \eta^{slow} \chi_{\sigma(I)}(x)|^2 dx \right\}^{1/2} \\
&\quad \lesssim_n \left(\frac{b-a}{d+1}\right)^n ||D_s^n \eta^{slow} \chi_{\sigma(I)}||_{L^2[a,b]} \\
&\quad \lesssim_n \frac{(b-a)^{n+1/2}}{(d+1)^n} k \left(k^{1/3-\epsilon_1}\right)^n \;\; \lesssim_n \;\; k \left(\frac{k^{1/3-\min\{\epsilon_1,\epsilon_4\}}}{d+1}\right)^n
\end{aligned}
$$

Secondly, in the case (iii) of shadow boundaries, without loss of generality we may assume that $\epsilon_2 \leq \epsilon_3$. Then with the same manipulations as we did in the previous case

$$
\begin{aligned}
||D_s^n \chi_{\sigma(I)}||_{L^\infty[a,b]} &\lesssim_n \sum_{j=0}^{n} \left(\frac{1}{k^{-1/3}(k^{\epsilon_1} - k^{\epsilon_2})}\right)^j \left(\frac{1}{k^{-1/3}(k^{\epsilon_4} - k^{\epsilon_3})}\right)^{n-j} \\
&\lesssim_n \left(\frac{k^{1/3}}{k^{\epsilon_3}(k^{\epsilon_4-\epsilon_3} - 1)}\right)^n \sum_{j=0}^{n} \left(\frac{k^{\epsilon_3}(k^{\epsilon_4-\epsilon_3} - 1)}{k^{\epsilon_2}(k^{\epsilon_1-\epsilon_2} - 1)}\right)^j \\
&\lesssim_n \left(\frac{k^{1/3}}{k^{\epsilon_3}}\right)^n \sum_{j=0}^{n} \left(\frac{k^{\epsilon_3}}{k^{\epsilon_2}}\right)^j \;\; \lesssim_n \;\; \left(k^{1/3-\epsilon_2}\right)^n.
\end{aligned}
$$

Again, as an immediate consequence of Theorem 4.6,

$$
\begin{aligned}
|D_s^n \eta^{slow}(s,k)| &\lesssim_n k + \sum_{m=4}^{n+2} (k^{-1/3} + |\omega(s)|)^{-m} \lesssim_n k + \sum_{m=4}^{n+2} (k^{-1/3})^{-m} \\
&\lesssim_n k + k^{(n+2)(1/3)},
\end{aligned}
$$

and combination of these estimates gives us:

$$
\begin{aligned}
||D_s^n \eta^{slow} \chi_{\sigma(I)}||_{L^\infty[a,b]} &= || \sum_{i=0}^{n} \binom{n}{i} D_s^i \eta^{slow} D_s^{n-i} \chi_{[a,a'],[b,b']} ||_{L^\infty[a,b]} \\
&\lesssim_n k \sum_{i=0}^{n} \left(1 + k^{-1/3} k^{i/3}\right) k^{(n-i)(1/3-\epsilon_2)} \ \lesssim_n \ k k^{n/3}.
\end{aligned}
$$

Implementing the results of Corollary 5.6, we get

$$
\begin{aligned}
&\inf_{p \in \mathbb{T}_d[a,b]} ||\eta^{slow} \chi_{\sigma(I)} - p||_{L^2[a,b]} \\
&\quad \lesssim_n \left(\frac{b-a}{d+1}\right)^n \sup_{|h| \le \frac{b-a}{2\pi(d+1)}} \left\{ \int_a^b |D_s^n \eta^{slow} \chi_{\sigma(I)}(x+h) - D_s^n \eta^{slow} \chi_{\sigma(I)}(x)|^2 dx \right\}^{1/2} \\
&\quad \lesssim_n \left(\frac{b-a}{d+1}\right)^n ||D_s^n \eta^{slow} \chi_{\sigma(I)}||_{L^2[a,b]} \\
&\quad \lesssim_n \frac{(b-a)^{n+1/2}}{(d+1)^n} k k^{n/3} \ \lesssim_n \ k \left(\frac{k^{\max\{\epsilon_1,\epsilon_4\}}}{d+1}\right)^n
\end{aligned}
$$

Lastly, in the case (iv) of illuminated transitions, analogous to the previous ones we have the following estimates:

$$
\begin{aligned}
||D_s^n \chi_{\sigma(I)}||_{L^\infty[a,b]} &\lesssim_n \sum_{j=0}^{n} \left(\frac{1}{k^{-1/3}(k^{\epsilon_2} - k^{\epsilon_1})}\right)^j \left(\frac{1}{k^{-1/3}(k^{\epsilon_3} - k^{\epsilon_4})}\right)^{n-j} \\
&\lesssim_n \left(\frac{k^{1/3}}{k^{\epsilon_3}(k^{\epsilon_4-\epsilon_3} - 1)}\right)^n \sum_{j=0}^{n} \left(\frac{k^{\epsilon_3}(k^{\epsilon_4-\epsilon_3} - 1)}{k^{\epsilon_1}(k^{\epsilon_2-\epsilon_1} - 1)}\right)^j \\
&\lesssim_n \left(\frac{k^{1/3}}{k^{\epsilon_3}}\right)^n \sum_{j=0}^{n} \left(\frac{k^{\epsilon_3}}{k^{\epsilon_1}}\right)^j \ \lesssim_n \ \left(k^{1/3-\epsilon_1}\right)^n
\end{aligned}
$$

The estimate from Theorem 4.6 is

$$\begin{aligned}|D_s^n \eta^{slow}| &\lesssim_n k + \sum_{m=4}^{n+2} (k^{-1/3} + |\omega(s)|)^{-m} \lesssim_n k + \sum_{m=4}^{2n+4} (k^{\epsilon_1 - 1/3})^{-m} \\ &\lesssim_n k + k^{(n+2)(1/3-\epsilon_1)},\end{aligned}$$

and as a consequence,

$$\begin{aligned}||D_s^n \eta^{slow} \chi_{\sigma(I)}||_{L^\infty[a,b]} &= || \sum_{i=0}^{n} \binom{n}{i} D_s^i \eta^{slow} D_s^{n-i} \chi[a,a'],[b,b']||_{L^\infty[a,b]} \\ &\lesssim_n k \sum_{i=0}^{n} \left(1 + k^{-1} k^{(i+2)(1/3-\epsilon_1)}\right) \left(k^{1/3-\epsilon_1}\right)^{n-i} \ \lesssim_n \ k \left(k^{1/3-\epsilon_1}\right)^n\end{aligned}$$

Finally, utilizing similar arguments,

$$\begin{aligned}&\inf_{p \in \mathbb{T}_d[a,b]} ||\eta^{slow} \chi_{\sigma(I)} - p||_{L^2[a,b]} \\ &\quad \lesssim_n \left(\frac{b-a}{d+1}\right)^n \sup_{|h| \le \frac{b-a}{2\pi(d+1)}} \left\{ \int_a^b |D_s^n \eta^{slow} \chi_{\sigma(I)}(x+h) - D_s^n \eta^{slow} \chi_{\sigma(I)}(x)|^2 dx \right\}^{1/2} \\ &\quad \lesssim_n \left(\frac{k^{-1/3}(k^{\epsilon_4 - \epsilon_1})}{d+1}\right)^n ||D_s^n \eta^{slow} \chi_{\sigma(I)}||_{L^2[a,b]} \\ &\quad \lesssim_n \frac{(k^{-1/3}(k^{\epsilon_4-\epsilon_1}))^{n+1/2}}{(d+1)^n} k \left(k^{1/3-\epsilon_1}\right)^n \ \lesssim_n \ k \left(\frac{k^{\epsilon_4-\epsilon_1}}{d+1}\right)^n,\end{aligned}$$

we complete the proof. □

6. NUMERICAL EXPERIMENTS

In this chapter we test our Galerkin method in several settings. In the cases of unit circle we use the smooth parametrization $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, L]$.

Throughout the chapter for given eight-tuples $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8)$, $\boldsymbol{\xi}' = (\xi_1', \xi_2', \xi_3', \xi_4', \xi_5', \xi_6', \xi_7', \xi_8')$ and a triple $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$ we have constructed our intervals in the following way

$$
\begin{aligned}
I_{IL} &= [t_1 + \xi_1 k^{-1/3+\epsilon_1}, t_2 - \xi_1' k^{-1/3+\epsilon_1}],\\
I_{DS} &= [t_2 + \xi_2 k^{-1/3+\epsilon_3}, L + t_1 - \xi_2' k^{-1/3+\epsilon_3}],\\
I_{SB_1} &= [t_1 - \xi_3 k^{-1/3+\epsilon_2}, t_1 + \xi_4 k^{-1/3+\epsilon_2}],\\
I_{SB_2} &= [t_2 - \xi_4' k^{-1/3+\epsilon_2}, t_2 + \xi_3' k^{-1/3+\epsilon_2}],\\
I_{IT_1} &= [t_1 + \xi_5 k^{-1/3+\epsilon_2}, t_1 + \xi_6 k^{-1/3+\epsilon_1}],\\
I_{IT_2} &= [t_2 - \xi_6' k^{-1/3+\epsilon_1}, t_2 - \xi_5' k^{-1/3+\epsilon_2}],\\
I_{ST_1} &= [t_1 - \xi_7 k^{-1/3+\epsilon_1}, t_1 - \xi_8 k^{-1/3+\epsilon_2}],\\
I_{ST_2} &= [t_2 + \xi_8' k^{-1/3+\epsilon_2}, t_2 + \xi_7' k^{-1/3+\epsilon_1}],
\end{aligned}
$$

and sometimes we divide the deep shadow region into two subregions

$$
\begin{aligned}
I_{DS_1} &= [L - L/32, L + t_1 - \xi_2 k^{-1/3+\epsilon_2}],\\
I_{DS_2} &= [t_2 + \xi_2' k^{-1/3+\epsilon_2}, L + L/32].
\end{aligned}
$$

In the examples if we do not mention $\boldsymbol{\xi}'$, we consider it is equal to $\boldsymbol{\xi}$. However in some elliptical settings where we lost the symmetry we take $\boldsymbol{\xi}'$ slightly different from $\boldsymbol{\xi}$, for better results.

Since in the case of change of variables scheme (See Section 5.2) we set $\boldsymbol{\epsilon} = (1/3, 0, 1/3)$, it can be easily seen that for a given wave number k, the only unknown parameter in above set of definitions of the intervals is the parameters represented by $\boldsymbol{\xi}$ (in this analysis we think of $\boldsymbol{\xi}$ together with $\boldsymbol{\xi}'$). As we mentioned earlier because of the choice of the approximation spaces highly depends on the construction of the intervals, we need to choose our $\boldsymbol{\xi}$ wisely in order to obtain the best accuracy. For this purpose we devise an algorithm, in which we start from a sensible $\boldsymbol{\xi}$ and set two variables w(width) and Δ(increment) to a desired precision. After that as j ranges over the indices of $\boldsymbol{\xi}$ at the jth step we pick the element ξ_j and instead of ξ_j we try the set of variables $\{\xi_j - w + r\Delta \ : \ r = 0, \cdots, 2w/\Delta\}$ one by one. Then we examine the error related to each of these variables. Then we set ξ_j to the variable from that set which is responsible from the minimal error. Cycling couple of times over the elements of $\boldsymbol{\xi}$, gives us the optimal one so that increasing or decreasing any element of it by a multiple of Δ would give us a larger error. We also need to point that the development of the intervals is done in a way so that the adjoint intervals do intersect. This intersection is particularly important for the construction of the bump function related to each interval.

In the above formulations we only define the variables a, b of an interval $I = [a, b]$. In order to construct the bump function $\chi_{\sigma(I)}$ described in Section 3.2 we also need to clarify the variables a', b' of the interval I. The definitions of those variables are attained by the same method in Section 3.2 so that the summation of all the $\chi_{\sigma(I)}$ functions of all intervals add up to 1.

Using the ideas discussed in Keller's Geometrical Theory of Diffraction [24], in some cases separating the deep shadow region into two subregions offers us better approximation in the deep shadow region (See the rightmost plots in Figures 6.1 and 6.2). Furthermore, assigning the polynomial degree of approximation space on single deep shadow region to $2d + 1$ enhances these results. In double deep shadow cases, we assign d for both of the subregions.

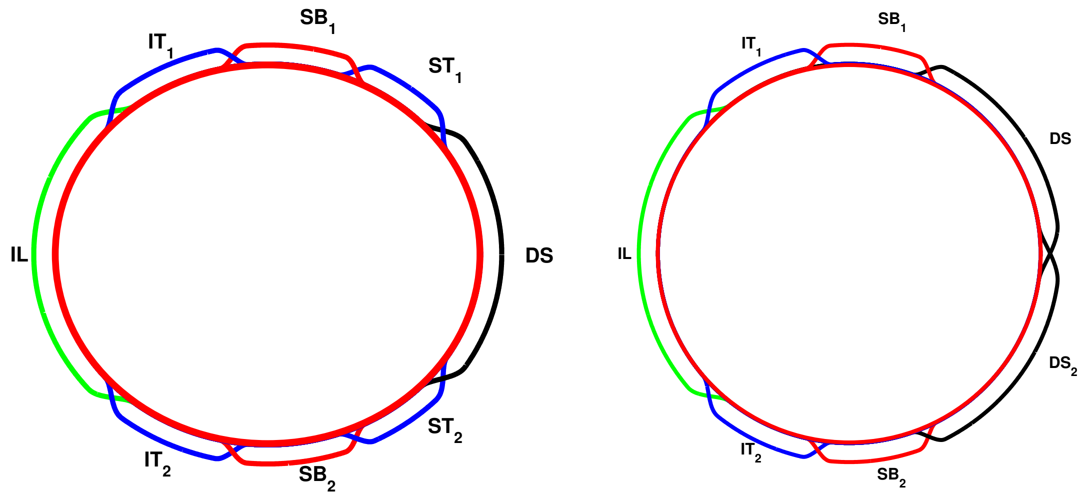

Figure 6.1. Interval schemes: 8 and 7 subregions.

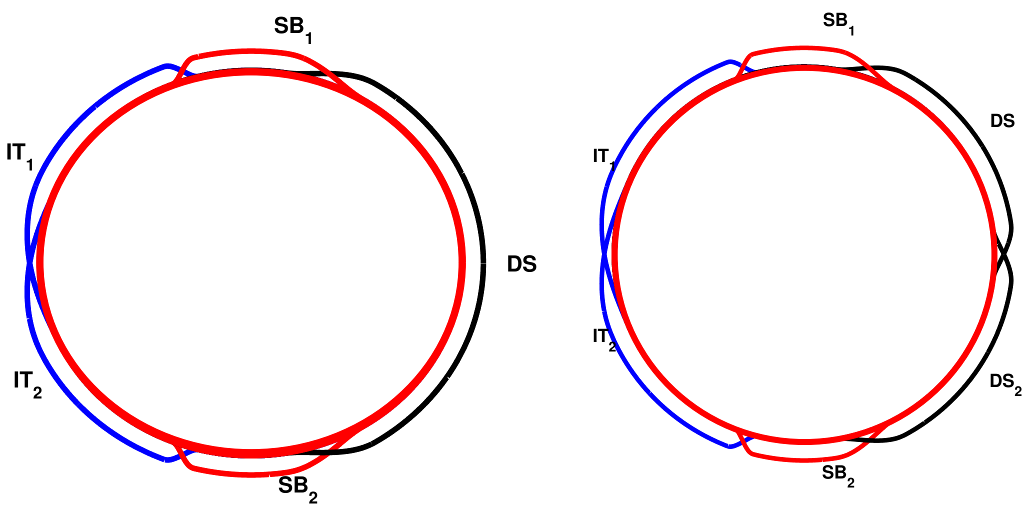

Figure 6.2. Interval schemes: 5 and 6 subregions.

Since our new approach mimic the behavior of the function η on both illuminated region and illuminated transitions, in the cases based on our change of variables model, we prefer not to define an illuminated region at all (See Figure 6.2). Moreover observing the related work of [25] and [26] we did not resort to the idea of shadow transition in the numerical experiments.

During our experiments, we used univariate polynomial, trigonometric and cosines bases. We also experimented with Lagrange and Chebyshev bases but they produced larger condition numbers and slightly worse error behavior. Hence in the examples we will not discuss the later ones.

In order to produce the numerical solution of Equation (2.4), we solved the linear system (2.8). We used Nyström method to numerically calculate the functions $\mathcal{R}\hat{\mu}_j$ and quadrature rule to evaluate the elements of the matrices appearing on both sides. Since the exact solution is known for the circular obstacle case, we used that to test the accuracy of our numerical solution.

In each example we have the following format: First presenting the numerical experiment using the techniques given in [14], then demonstrating the ones based on our new scheme described in Section 5.2. The latter set consists of single and double deep shadow regions. The figures showing the error analysis includes three parts, in each part for a given degrees of freedom d, the leftmost part shows the relative $\log_{10}$ error of $||\eta - \hat{\eta}||/||\eta||_{L^2}$, the middle one is the relative $\log_{10}$ error in the deep shadow region, and the rightmost part presents the $\log_{10}$ of the condition numbers of the Galerkin matrices arisen from the related Galerkin approximation spaces. So clearly in these three graphs the x-axis depicts the degrees of freedom for each approximation.

In the examples with circular object we have the parametrization $\{(\cos(t), \sin(t)) : t \in [0, 2\pi]\}$ and incident field direction $\alpha = (0, 1)$. On the other hand the ones with elliptical object we have the parametrization $\{(2\cos(t), \sin(t)) : t \in [0, 2\pi]\}$ and incident field direction $\alpha = \frac{1}{\sqrt{10}}(3, 1)$. In both settings we denote L by the circumference of the object.

6.1. Weighted Algebraic Polynomials and Change of Variables

While constructing the Galerkin matrix related to the Galerkin formulation described in (2.6) one has to be careful in order to decrease the computational error. If the matrix contains unstable range of elements, then the solution based on that matrix will be more likely to be inaccurate, which is also known as ill-conditioning problem. To solve this problem we try to make the basis elements of the global Galerkin approximation space belong to a more narrow interval. With this purpose in our mind we define the change of variable $\rho(s) = \frac{2s-a-b}{b-a}$ which maps the interval $[a, b]$ onto $[-1, 1]$. Then instead of $\mathbb{P}_d$ and $\mathbb{P}_d \circ \phi^{-1}$ in Section 5.1 we use

$$\mathbb{P}_d \circ \rho = \text{span}\{(\rho(s))^r : r = 0, \cdots, d\}$$
$$\mathbb{P}_d \circ \phi^{-1} \circ \rho = \text{span}\{(\phi^{-1}(\rho(s)))^r : r = 0, \cdots, d\}$$

respectively.

Example 6.1. : *Circular Object*

(i) Using the classical basis of monomial algebraic polynomials from [14]

- *For 8 subregions (See the leftmost plot in Figure 6.1)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (L/4, 3L/10, 7L/15, 7L/30, L/15, 7L/20, 2L/5, 3L/10)$, $\quad \boldsymbol{\epsilon} = (1/5, 1/15, 1/5)$

the error behavior is given in Figure 6.3.

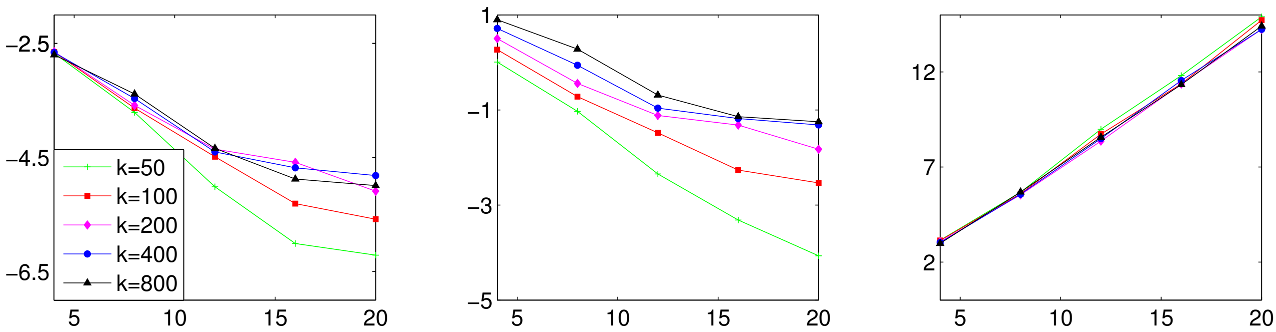

Figure 6.3. Algebraic polynomials for 8 subregions.

(ii) Using our new basis of monomial algebraic polynomials composed with change of variables,

- *Demonstrating for first* 6 *then* 5 *subregions (See the plots in Figure 6.2)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d)$ *then* $\mathbf{d} = (2d + 1, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (0, 1.0, 5.15, 3.65, 2.10, 2.0, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/3, 0, 1/3)$ *then*
  $\boldsymbol{\xi} = (0, 0.3, 5.25, 3.70, 1.70, 2.0, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/3, 0, 1/3)$*, respectively.*

The error behavior is given in Figure 6.4

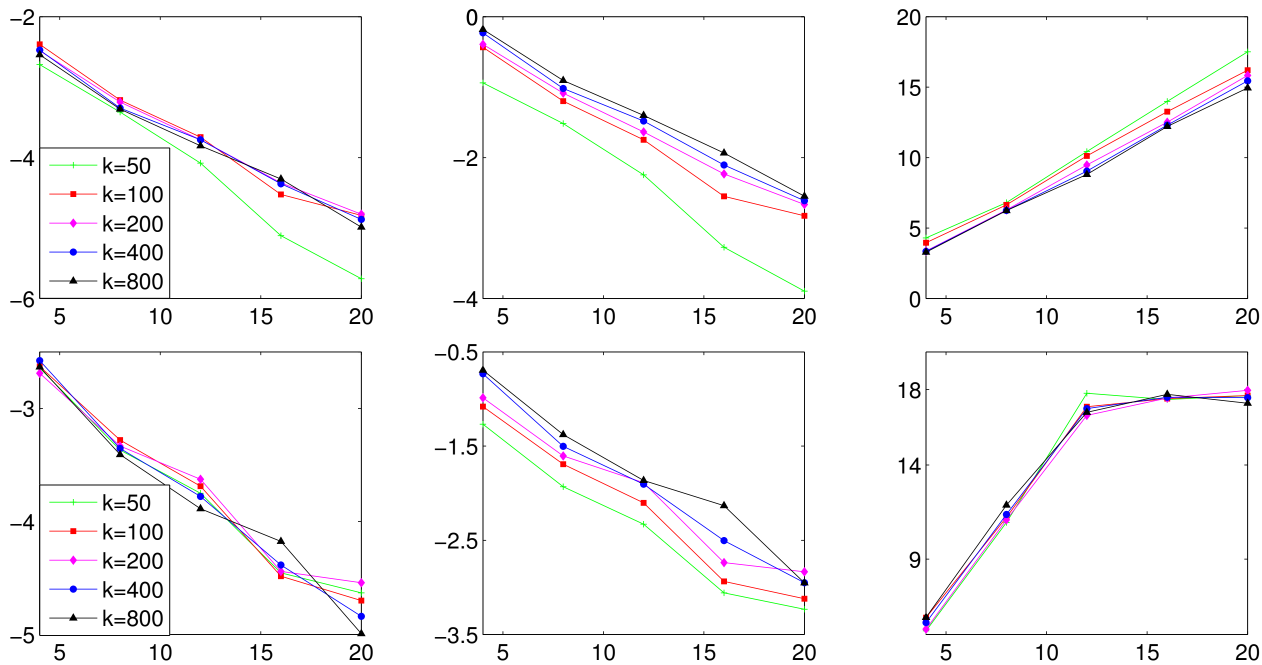

Figure 6.4. Our new basis with algebraic polynomials for 6 and 5 subregions, upper and lower row respectively.

Comparing Figures 6.3 and 6.4 we see a fast convergence in the relative L^2 norm difference. However more importantly, we observe a remarkable convergence in the deep shadow region. On the other hand our new approximation regime slightly disturb the conditioning of the Galerkin matrix, but the drawback is not significant. Let us examine the similar improvement in the elliptical case where we lost the symmetries.

Example 6.2. *: Elliptical Object*

(i) Using the classical basis of monomial algebraic polynomials from [14]

- *For 8 subregions (See the leftmost plot in Figure 6.1)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (3L/20, L/5, L/4, 3L/20, L/20, L/5, 7L/30, L/5)$, $\quad \boldsymbol{\epsilon} = (2/9, 1/9, 2/9)$

the error behavior is given in Figure 6.5.

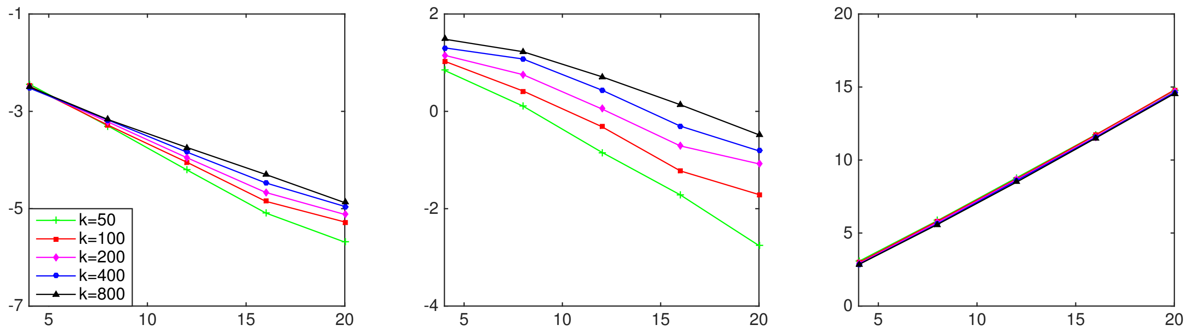

Figure 6.5. Algebraic polynomials for 8 subregions.

(ii) Using our new basis of monomial algebraic polynomials composed with change of variables,

- *For 6 subregions (See the rightmost plot in Figure 6.2)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (0, 1.05, 5.75, 3.45, 2.30, 0, 0, 0)$, $\quad \boldsymbol{\xi}' = (0, 0.8, 5.25, 3.35, 2.30, 0, 0, 0)$, $\boldsymbol{\epsilon} = (1/3, 0, 1/3)$

The error behavior is given in Figure 6.6

6.2. Weighted Trigonometric Polynomials and Change of Variables

As it is discussed in the beginning of the previous section, we also need to stabilize the Galerkin matrix when it is constructed depending on trigonometric polynomials. With the same idea, we note that the linear function $\tilde{\rho}(s) = \frac{2\pi(s-a)}{b-a}$ maps the interval

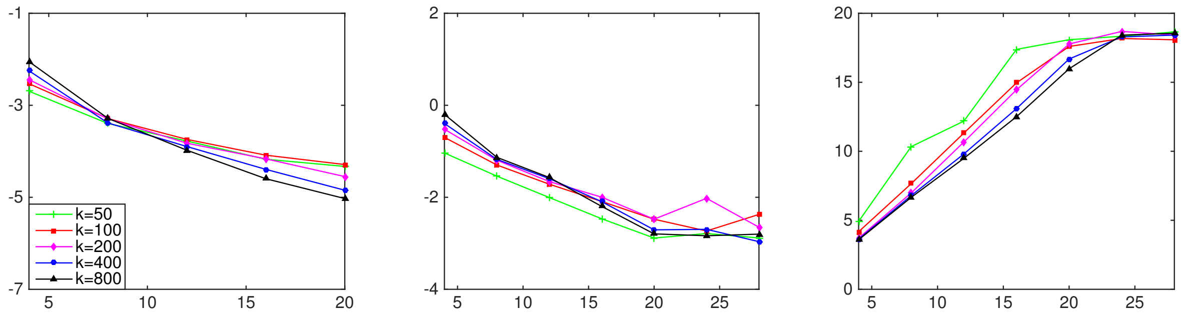

Figure 6.6. Our new basis with algebraic polynomials for 6 subregions.

$[a,b]$ onto $[0,2\pi]$. Then instead of $\mathbb{T}_d$ in the Section 5.3 we use

$$\mathbb{T}_d \circ \tilde{\rho} = \text{span}\{\exp(ir\ \tilde{\rho}(s)) : r = -\frac{-d}{2}, \cdots, \frac{d}{2}\}$$

Example 6.3. *: Circular Object*

(i) Using the basis of trigonometric polynomials from [14]

- *For 7 subregions(See the rightmost plot in Figure 6.1)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d,d,d,d,d,d,d)$ *and*
- $\boldsymbol{\xi} = (L/4, 3L/10, 7L/15, 7L/30, L/15, 7L/20, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/5, 1/15, 1/15)$

The errors are given in Figure 6.7

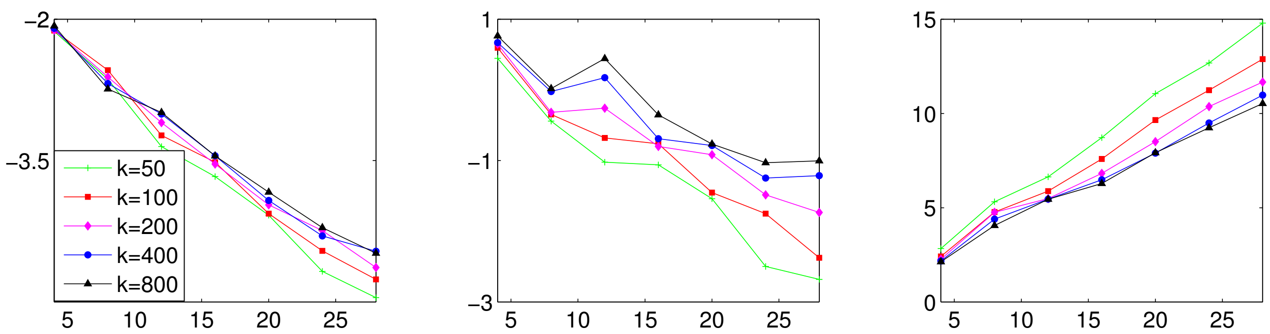

Figure 6.7. Trigonometric polynomials on 7 subregions.

(ii) Using our new basis for trigonometric polynomials composed with change of variables

- *Demonstrating for first* 6 *then* 5 *subregions (See the plots in Figure 6.2)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d)$ *then* $\mathbf{d} = (2d + 1, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (0, 1.55, 5.80, 3.55, 2.15, 2.0, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/3, 0, 1/3)$ *then* $\boldsymbol{\xi} = (0, 0.90, 5.75, 3.75, 2.15, 2.0, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/3, 0, 1/3)$, *respectively.*

the error behavior is given in Figure 6.8

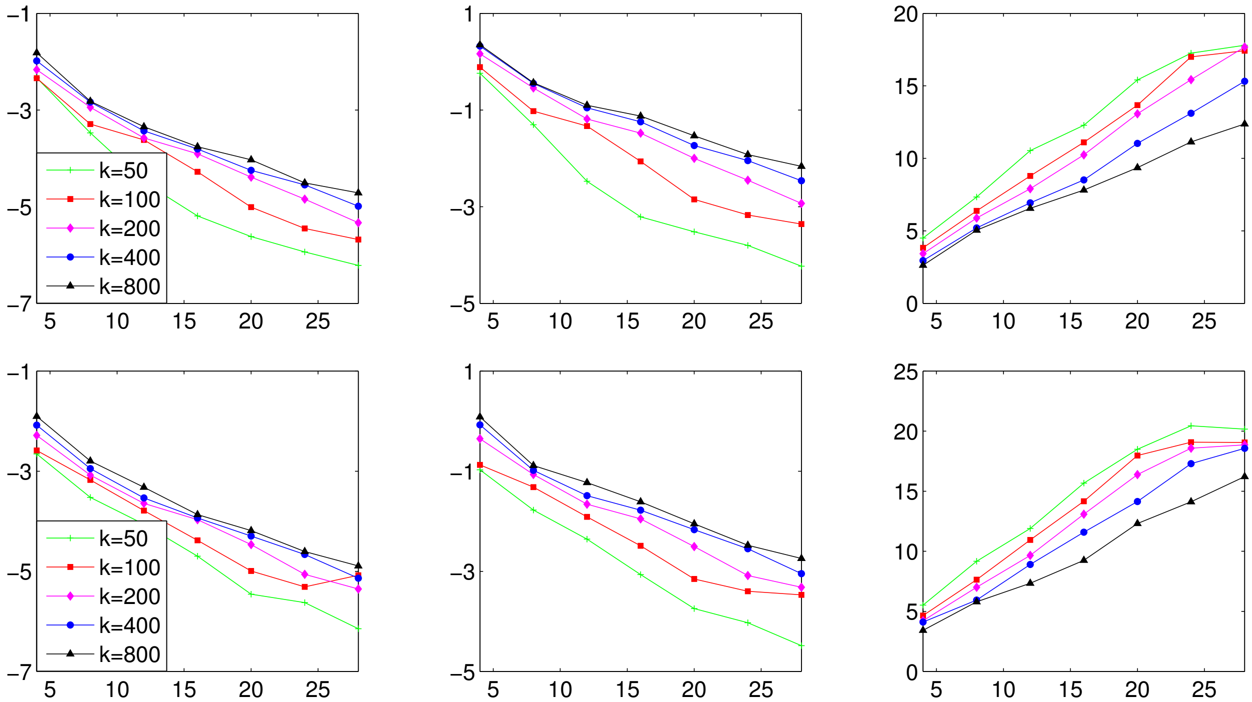

Figure 6.8. Our new basis with trigonometric polynomials for 6 and 5 subregions, upper and lower row respectively.

Example 6.4. *: Elliptical Object*

(i) Using the basis of trigonometric polynomials from [14]

- *For* 7 *subregions (See the rightmost plot in Figure 6.1)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (3L/20, L/5, L/4, 3L/20, L/20, L/5, 7L/30, L/5)$, $\quad \boldsymbol{\epsilon} = (2/9, 1/9, 1/9)$

the error behavior is given in Figure 6.9.

(ii) Using our new basis for trigonometric polynomials composed with change of vari-

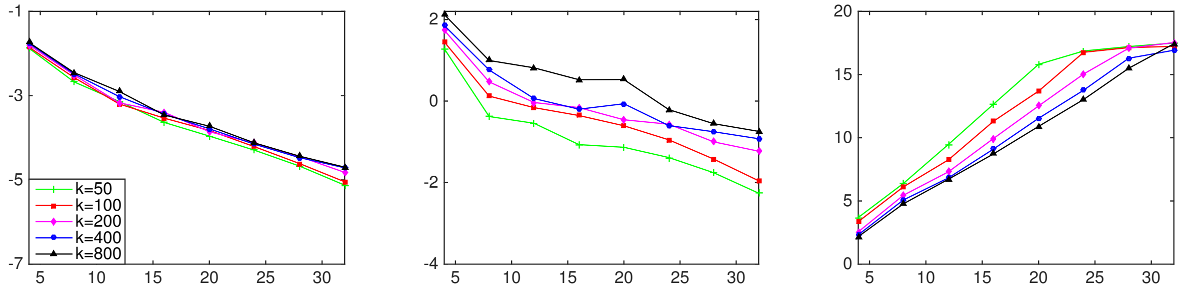

Figure 6.9. Algebraic polynomials for 8 subregions.

bles

- *For 6 subregions (See the rightmost plot in Figure 6.2)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (0, 1.25, 5.75, 3.45, 2.20, 0, 0, 0)$, $\quad \boldsymbol{\xi}' = (0, 0.6, 5.25, 3.35, 2.30, 0, 0, 0)$, $\boldsymbol{\epsilon} = (1/3, 0, 1/3)$

The error behavior is given in Figure 6.10

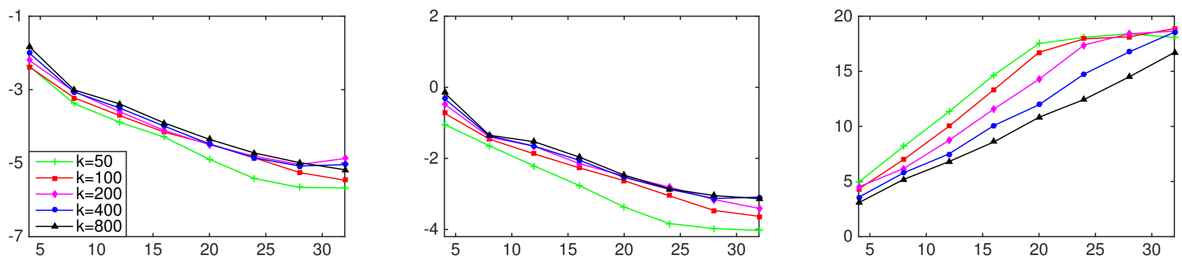

Figure 6.10. Our new basis with algebraic polynomials for 6 and 5 subregions, upper and lower row respectively.

Investigating these results, we see the condition numbers related to our new method exceeds the machine precision, however our scheme enables us a fast converging algorithm. Furthermore it gives us a numerical solution mimicking η^{slow} in the deep shadow region a lot more accurately which is always harder target to attain.

Although we did not give the theoretical developments of the change of variables scheme related to the trigonometric polynomials, our numerical experiments also indicate this composition's improved results.

Finally, the approximation space taken into consideration in [14] and consisting of cosines bases will be examined. It is similar to the above trigonometric polynomials other than having its half periodicity. The normalized approximation space is given as

$$\mathrm{span}\{\cos(r\frac{1}{2}\tilde{\rho}(s)) : r = 0, \cdots, d\}$$

Hence we have the following results.

Example 6.5. *: Circular Object*

(i) Using the basis of cosines from [14]

- *For 7 subregions (See the rightmost plot in Figure 6.1)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (L/4, 3L/10, 7L/15, 7L/30, L/15, 7L/20, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/5, 1/15, 1/15)$

the errors are given in Figure 6.11.

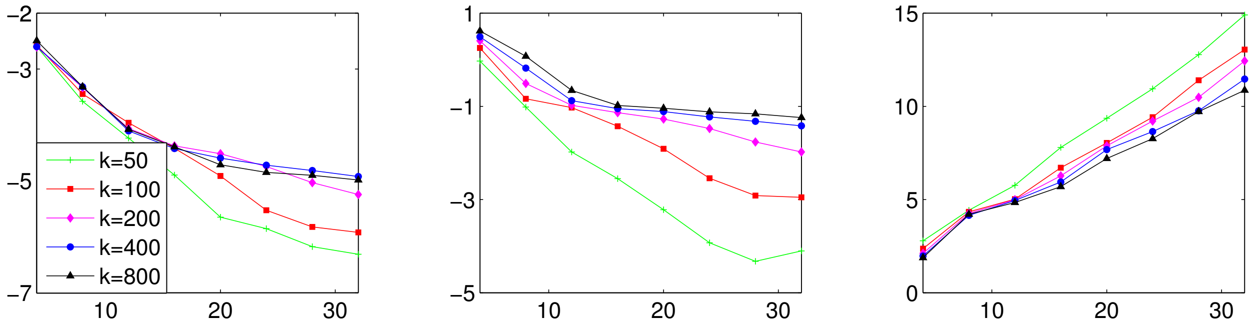

Figure 6.11. Cosines on 7 subregions.

(ii) Using our new basis for cosines composed with change of variables,

- *Demonstrating for first* 6 *then* 5 *subregions (See the plots in Figure 6.2)*
- *Assigning the degrees of freedom to* $\mathbf{d} = (d, d, d, d, d, d)$ *then* $\mathbf{d} = (2d + 1, d, d, d, d)$ *and*
- $\boldsymbol{\xi} = (0, 1.75, 5.85, 3.35, 1.95, 2.0, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/3, 0, 1/3)$ *then* $\boldsymbol{\xi} = (0, 0.55, 5.45, 3.60, 2.00, 2.0, 0, 0)$, $\quad \boldsymbol{\epsilon} = (1/3, 0, 1/3)$

the error behavior is given in Figure 6.12

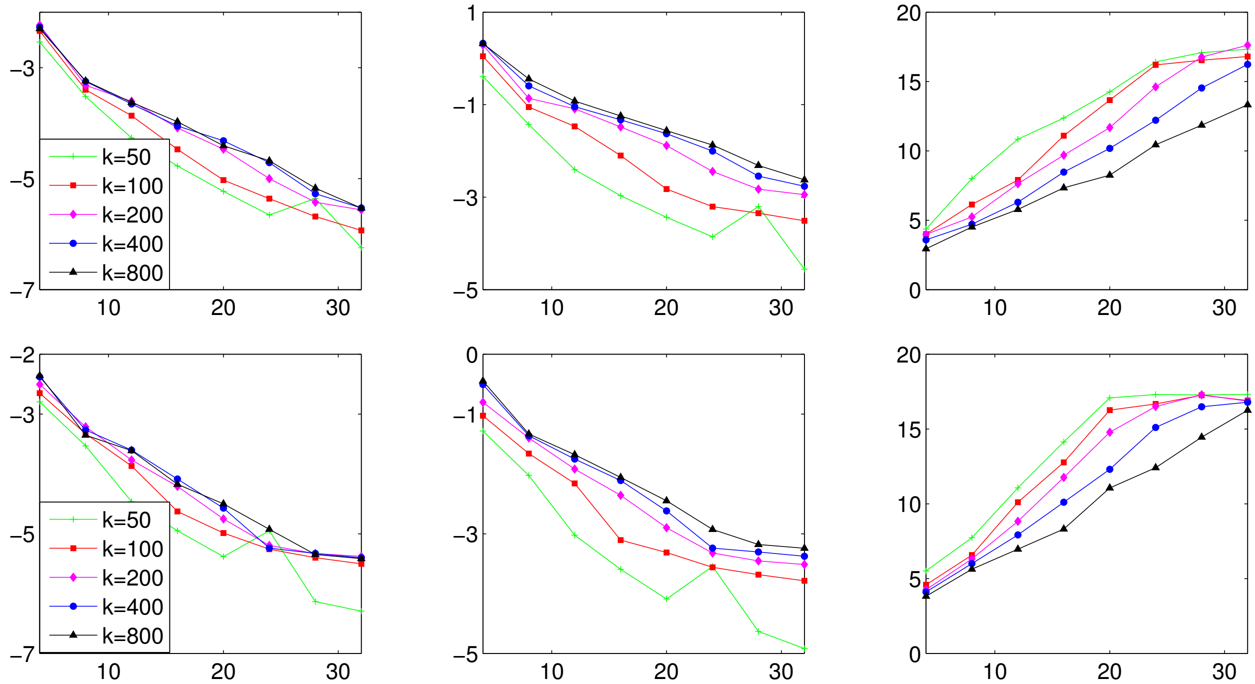

Figure 6.12. Our new basis with cosines for 6 and 5 subregions, upper and lower row respectively.

Comparing to the trigonometric polynomials this basis seems to provide better accuracy and condition numbers.

7. CONCLUSION

In this thesis, we aimed to solve the Helmholtz equation (2.1) for a given incident $e^{ik\,\alpha\cdot x}$ and a compact object K. We studied the equivalent integral equation version of this problem, i.e. the combined field integral equation.

In order to make our numerical algorithm stable as the wave number increases, instead of the solution η of the integral equation we tried to numerically solve the unknown function η^{slow} via the Galerkin method. For the purpose of mimicking η^{slow} we divided the surface of the obstacle into several subregions and constructed our Galerkin approximation spaces in each region.

In this subregion division process, we extended the idea of transition regions and using a clever change of variables scheme we obtained a better approximation in whole domain as well as in the deep shadow region.

Although we only applied this change of variables idea in the case of algebraic polynomials, the numerical results corresponding to the case of trigonometric polynomials also demonstrate the feasibility of change of variables for improved accuracy. The theoretical analysis of that part is currently an ongoing work.

We also want to point that, these improved results may also be applied to the cases of multiple scattering scenarios with some suitable iteration techniques. The possible numerical results would have been improved comparing to the previous methods.

All numerical experiments are conducted on a computer with 2.4 GHz Intel Core I5 processor using the MATLAB software. Thanks to the ideas in [26], all the algorithms are implemented in a vectorized fashion in order to obtain the fastest results.

APPENDIX A: AUXILIARY RESULTS

Lemma A.1 (A binomial identity)**.** *For all $t \in \mathbb{R}$ and $m \in \mathbb{N}$, we have*

$$(t+1)^{2m} = \sum_{r=1}^{m} \binom{2m-r-1}{m-r} (1+t)^r (1+t^r) t^{(m-r)}$$

Proof. We will prove this identity by checking the coefficient of t^k for $k = 0, \cdots, 2m$. Considering the right hand side when $k \leq m$, the contribution comes from $(1+t^r)t^{(m-r)}$, so it is $\binom{2m-r-1}{m-r}\binom{r}{k-(m-r)}$. However when $k > m$, in this case contribution comes from $(1+t^r)t^m$, therefore $\binom{2m-r-1}{m-r}\binom{r}{k-m}$. Combining these cases, our lemma becomes equivalent to

$$\binom{2m}{k} = \sum_{r=|m-k|}^{m} \binom{2m-r-1}{m-r} \binom{r}{|m-k|}, \quad k = 0, \cdots, 2m$$

Since the cases are symmetric, we will only prove the part when $m - k > 0$. If we change the index of the summation with $j = m - r$, then we have

$$\binom{2m}{k} = \sum_{j=0}^{k} \binom{m+j-1}{m-1} \binom{m-j}{m-k}. \tag{A.1}$$

Equation (A.1) is a well known binomial identity which can be found in [27, Example 2.6.2] as in the following form

$$\binom{p+q+r+1}{q+r+1} = \sum_{j=0}^{p} \binom{p+q-j}{q} \binom{r+j}{r} \tag{A.2}$$

In this identity inserting $p = k, q = m - k, r = m - 1$ delivers our desired result. $\square$

Corollary A.2. *For real numbers $c \neq d$ and $s \in \mathbb{R} \setminus \{c, d\}$ and natural number $m \geq 1$*

we have the following decomposition

$$\frac{1}{(s-c)^m(d-s)^m}=\sum_{r=1}^{m}\binom{2m-r-1}{m-r}\frac{1}{(d-c)^{2m-r}}\left(\frac{1}{(s-c)^r}+\frac{1}{(d-s)^r}\right)$$

Proof. In Lemma A.1 after a few manipulations

$$\begin{aligned}(t+1)^{2m}&=\sum_{r=1}^{m}\binom{2m-r-1}{m-r}(1+t)^r(1+t^r)t^{(m-r)}\\ t^m(1+\frac{1}{t})^{2m}&=\sum_{r=1}^{m}\binom{2m-r-1}{m-r}(1+\frac{1}{t})^r(1+t^r)\\ t^m&=\sum_{r=1}^{m}\binom{2m-r-1}{m-r}\frac{1}{(1+\frac{1}{t})^{2m-r}}(1+t^r),\end{aligned}$$

if we insert $t=x/y$, then we have

$$(x/y)^m=\sum_{r=1}^{m}\binom{2m-r-1}{m-r}\frac{x^{2m-r}}{(x+y)^{2m-r}}(1+\frac{x^r}{y^r}),$$

After dividing both sides with x^{2m}, and plugging $x=s-c,\ y=d-s$ we have our desired result. $\square$

Lemma A.3. *For any $a,b\in\mathbb{R}$, $n\in\mathbb{N}\cup\{0\},m\in\mathbb{N}$, if either,*

- $c\le t_1\le\alpha<\beta\le t_2\le d$, *or*
- $t_1<c<d<t_2$ *and* $[\alpha,\beta]\cap(t_1,t_2)=\emptyset$,

then we can compute the following integral.

$$\int_\alpha^\beta\frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m}ds=\sum_{r=1}^{m}\binom{2m-r-1}{m-r}\frac{(-1)^n}{(d-c)^{2m-r}}\Big\{A+B+C\Big\}$$

where A,B and C are as follows

$$A := \sum_{\substack{0\le p,q\le n\\ p+q=2n+1-r}} \binom{n}{p}\binom{n}{q}\left[(c-a)^p(c-b)^q \log\left(\frac{\beta-c}{\alpha-c}\right) + (a-d)^p(b-d)^q \log\left(\frac{d-\alpha}{d-\beta}\right)\right]$$

$$B := \sum_{\substack{0\le p,q\le n\\ p+q\neq 2n+1-r}} \binom{n}{p}\binom{n}{q}(c-a)^p(c-b)^q \frac{(\beta-c)^{2n-(p+q+r)+1}-(\alpha-c)^{2n-(p+q+r)+1}}{2n-(p+q+r)+1}$$

$$C := \sum_{\substack{0\le p,q\le n\\ p+q\neq 2n+1-r}} \binom{n}{p}\binom{n}{q}(a-d)^p(b-d)^q \frac{(d-\alpha)^{2n-(p+q+r)+1}-(d-\beta)^{2n-(p+q+r)+1}}{2n-(p+q+r)+1}$$

Proof. The decomposition in Corollary A.2 gives us

$$\begin{aligned}&\int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m}ds\\ &\quad= \sum_{r=1}^m \binom{2m-r-1}{m-r}\frac{1}{(d-c)^{2m-r}}\int_\alpha^\beta \left(\frac{(s-a)^n(b-s)^n}{(s-c)^r}+\frac{(s-a)^n(b-s)^n}{(d-s)^r}\right)ds\end{aligned}$$

Now, we apply the change of variables inside the first and second integrals, by setting $s-c=t$ and $d-s=t$ respectively.

$$\begin{aligned}\int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m}ds &= \sum_{r=1}^m \binom{2m-r-1}{m-r}\frac{(-1)^n}{(d-c)^{2m-r}}\\ &\times\left\{\int_{\alpha-c}^{\beta-c} t^{-r}(t+c-a)^n(t+c-b)^n dt + \int_{d-\alpha}^{d-\beta} t^{-r}(t+a-d)^n(t+b-d)^n dt\right\}\end{aligned}$$

Then we use the binomial identity to have a computable integral.

$$\begin{aligned}\int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m}ds &= \sum_{r=1}^m \binom{2m-r-1}{m-r}\frac{(-1)^n}{(d-c)^{2m-r}}\sum_{0\le p,q\le n}\binom{n}{p}\binom{n}{q}\\ &\times\left\{(c-a)^p(c-b)^q\int_{\alpha-c}^{\beta-c} t^{2n-(p+q+r)}dt + (a-p)^p(b-d)^q\int_{d-\alpha}^{d-\beta} t^{2n-(p+q+r)}dt\right\}\end{aligned}$$

Calculating the integrals for $2n-(p+q+r)=-1$ and $2n-(p+q+r)\neq -1$ separately, we complete the proof. $\square$

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