ZEROS OF ORTHOGONAL POLINOMIALS AND UNIVERSALITY LIMITS

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ABSTRACT

ZEROS OF ORTHOGONAL POLINOMIALS AND UNIVERSALITY LIMITS

It has been discovered that "old style" techniques from orthogonal polynomials have been very useful in establishing universality results for quite general measures. The main goal of this master thesis is to present some methods recently introduced by D. S. Lubinsky for establishing universality limits of random matrices, in the unitary case, based on orthogonal polynomials and some Hilbert spaces of entire functions. Let μ be a measure defined on the real line with compact support. Assume that μ is absolutely continuous in a neighbourhood of some point x in the support, and that μ' is positive and continuous in a compact subset of that neighbourhood. Theorem 1.1 shows that universality at x is equivalent to universality "along the diagonal". The same equivalence is obtained when the hypothesis involve a Lebesgue type condition, instead of continuity of μ' on a compact subset. Such universality limits can be also described by the reproducing kernel of a de Branges space of entire functions that equals a Paley-Wiener space (Theorem 1.4). In order to study this assertion, we use the theory of entire functions of exponential type and de Branges spaces as background.

ÖZET

DİK POLİNOMLARIN SIFIRLARI VE EVRENSELLİK LİMİTLERİ

Dik polinomlardaki "eski stil" tekniklerin oldukça genel ölçüler için evrensel neticeleri göstermekte çok işe yaradığı tespit edilmiştir. Bu tezin ana amacı, D. S. Lubinsky tarafından dik polinomlar ve bazı tüm düzlemde analitik Hilbert uzayları baz alınarak, birimsel durumda, rastgele matrislerin evrensellik limitlerini belirlemek amacıyla ortaya konulan yeni metodları sunmaktır. μ, gerçel sayı ekseni üzerine tanımlı tıkız destekli bir ölçü olsun. Destekteki bir x noktasının etrafında μ'nün mutlakça sürekli olduğunu; ve μ''in o komşuluğun tıkız bir alt kümesinde pozitif ve sürekli olduğunu varsayalım. Theorem 1.1, x'teki evrenselliğin köşegende evrenselliğe denk olduğunu kanıtlar. μ''in tıkız bir alt kümedeki sürekliliği yerine Lebesgue tipi koşulunu sağladığını varsayarsak aynı denkliği elde ederiz. Bu tip evrensellik limitleri bir Paley-Wiener uzayına eşit de Branges'ın tüm düzlemde analitik fonksiyonlar uzayının doğuran çekirdeğiyle de tanımlanabilirler (Teorem 1.4). Bu iddiayı çalışmak için, arka plan olarak üssel tipteki tüm düzlemde analitik fonksiyonlar teorisi ve de Branges uzaylarını kullanıyoruz.

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LIST OF SYMBOLS

$\square$	End of the proof
$\|.\|_E$	The norm on de Brange's space $\mathcal{H}(E)$
$\arg z$	The argument of a complex number z
$\overline{A}$	Closure of the set A
∂A	Boundary of the set A
C	Cartwright class
$\mathbb{C}$	The set of complex numbers
$C^2(\Omega)$	The space of twice continuously differentiable functions on the domain Ω
$\mathbb{C}^+$	The set of complex numbers with positive imaginary part
$\bar{f}$	The conjugate of function f
h_f	The indicator function of f
$H^2(\Omega)$	Hardy space of a domain Ω
$\overline{HB}$	Hermite-Biehler class
$\mathcal{H}(E)$	De Branges space of functions associated to the function $E \in \overline{HB}$
$\text{Im } f$	The imaginary part of function f
$L_2(\mathbb{R})$	The space of square integrable functions on the real axis
M_n	An $n \times n$ matrix
$n(f, r)$	The number of zeros of a function f in the ball with center 0 and radius r
$N(\Omega)$	Nevanlinna class of domain Ω, the space of functions of bounded type on Ω
$N^+(\Omega)$	The space of functions of bounded type and non-positive mean type on Ω
$\mathbb{P}$	The set of polynomials
$\mathbb{P}_k$	The set of polynomials of degree less than or equal to k
$\mathbb{P}_k^\circ$	The set of monic polynomials of degree k
PW_σ	The space of entire functions of exponential type $\leq \sigma$ that are square integrable along the real axis

Re f	The real part of function f
sign$f$	Sign function of f
$supp(\mu)$	Support of a measure μ
δ_{kl}	Kronecker delta function
Δf	Laplacian of f
Δ_n	Also written by $\det M_n$, denotes the determinant of M_n
π_k	Orthogonal polynomial of degree k
$\tilde{\pi}_k$	Orthonormal polynomial of degree k
ρ_f	The order of an entire function f
σ_f	The exponential type of an entire function f

1. INTRODUCTION

Discussions on the zeros of orthogonal polynomials on the real line dates back to Gauss' discovery that the best discrete approximations of Riemann integrals involve zeros of Legendre polynomials, and has generated a huge work in many areas including general theoretical and mathematical physics communities who study "eigenvalue statistics." Much of the work on random matrices deals with this subject. Interested readers should refer to Mehta [1]. An approach generating orthogonal polynomials that has turned out to be of great importance was given by Fokas, Its and Kitaev in the early 1990's [2]. This new approach concerns with the matrix valued Riemann-Hilbert problem. For details on this Riemann-Hilbert approach, see Deift [3].

Over the past few years, Barry Simon provided two reviews related to the subject which survey the recent progress and the open questions [4]. The latter one includes some works of Lubinsky and Levin on universality limits. The approach introduced by Lubinsky was followed by many other authors, even though the tactics differ. Levin's rediscovery of universality limits for some measures, which was implicit in Freud's book [5], yields the associated work of Avila, Last, and Simon on clock spacing [6]. Maltsev transferred the theory of universality limits from orthogonal polynomials to half-line Schrödinger operators [7]. In this new framework, Maltsev deduced the "clock behaviour" of eigenvalues and the zeros of the solution of eigenvalue problem for Schrödinger operators subject to Dirichlet or Neumann boundary conditions at 0.

In order to take a closer look to the approach introduced by Lubinsky, we begin by setting notation and terminology. Let μ be a finite positive Borel measure on $\mathbb{R}$ with all moments $\int_{\mathbb{R}} x^k d\mu(x)$, $k \geq 0$, finite; and with infinitely many points in its support. By applying the Gram-Schmidt process to $1, x, x^2, \ldots$, we may define the orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

for $n \in \mathbb{N}$, satisfying the orthonormality conditions

$$\int_{\mathbb{R}} p_n p_m d\mu(x) = \delta_{nm}.$$

Throughout we use $\mu'(x) = \frac{d\mu}{dx}$ to denote the Radon-Nikodym derivative of μ. The nth reproducing kernel for μ is

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y).$$

And the normalized kernel is

$$\tilde{K}_n(x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x, y).$$

K_n satisfies the very useful extremal property

$$K_n(\xi, \xi) = \inf_{\deg(P) \leq n-1} \frac{P^2(\xi)}{\int P^2 d\mu}.$$

The simplest case of the *universality limit* is the limit

$$\lim_{n \to \infty} \frac{\tilde{K}_n(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)})}{\tilde{K}_n(\xi, \xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}, \tag{1.1}$$

with the sinc kernel on the right hand side. It describes the distribution of spacing of eigenvalues of random matrices. Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a technical limit involving orthogonal polynomials. The term universal is quite justified: the limit on the right-hand side of (1.1) is independent of ξ, but more importantly is independent of the underlying measure.

Typically, the limit (1.1) is established uniformly for a, b in compact subsets of the real line, but if we remove the normalization from the outer K_n, we can also establish its validity for complex a, b, that is

$$\lim_{n\to\infty} \frac{K_n(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)})}{K_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}.$$

The most obvious approach is to use Christoffel-Darboux formula,

$$K_n(u,v) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(u)p_{n-1}(v) - p_{n-1}(u)p_n(v)}{u-v}, \quad u \neq v,$$

and

$$K_n(u,u) = \frac{\gamma_{n-1}}{\gamma_n}(p_n'(u)p_{n-1}(u) - p_{n-1}'(u)p_n(u)).$$

This leads to (for $a \neq b$),

$$\begin{aligned} &\frac{K_n(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)})}{K_n(\xi,\xi)} \\ &= w(\xi)\frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)})p_{n-1}(\xi + \frac{b}{\tilde{K}_n(\xi,\xi)}) - p_{n-1}(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)})p_n(\xi + \frac{b}{\tilde{K}_n(\xi,\xi)})}{a-b}. \end{aligned}$$

It has been seen that if we have sufficient knowledge of the asymptotic behaviour of p_n as $n \to \infty$, then we can substitute in these asymptotics, and deduce universality.

In this thesis, we first present a method of Lubinsky, based on the theory of entire functions of exponential type, that works for arbitrary, possibly non-regular, measures with compact support.

Theorem 1.1. *Let μ be a finite positive Borel measure on the real line with compact support. Let $J \subset supp(\mu)$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume that μ' is positive and continuous at each point of J. The following are equivalent:*

(i) Uniformly for $\xi \in J$ and a in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{K_n(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a}{\tilde{K}_n(\xi,\xi)})}{K_n(\xi,\xi)} = 1. \tag{1.2}$$

(ii) Uniformly for $\xi \subset J$ and a,b in compact subsets of the complex plane, we have

$$\lim_{n\to\infty} \frac{K_n(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)})}{K_n(\xi,\xi)} = \frac{sin\pi(a-b)}{\pi(a-b)}. \tag{1.3}$$

Instead of assuming continuity on J, we can assume a Lebesgue point type condition.

Theorem 1.2. *Let μ be a finite positive Borel measure with compact support. Let $J \subset supp(\mu)$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume that w is bounded above and below by positive constants in that open set. Assume, moreover, that uniformly for $\xi \in J$, we have*

$$lim_{s\to 0+} \frac{1}{s} \int_{\xi-s}^{\xi+s} |w(t) - w(\xi)| dt = 0. \tag{1.4}$$

Then the equivalence of $(i), (ii)$ *in Theorem* 1.1 *remains valid.*

Lubinsky obtained these results by exploring the possible limits of subsequences of the sequence $\{f_n\}$, where

$$f_n(a,b) = \frac{K_n(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n,\xi_n)},$$

and $\{\xi_n\}$ is a sequence of real numbers. Since $\{K_n\}$ are reproducing kernels for polynomials, it is quite probable that the limits of subsequences of $\{f_n\}$ produces reproducing kernels for suitable spaces of entire functions. It turns out that such spaces are de Branges spaces.

Definition 1.3. *The de Branges space $\mathcal{H}(E)$ corresponding to the entire function $E \in$*

$\overline{HB}$, *is the set of all entire functions g such that g/E and g*/E belong to* $H^2(\mathbb{C}^+)$, *with*

$$||g||_E = \left(\int_{\mathbb{R}} \left|\frac{g}{E}\right|^2\right)^{1/2} < \infty. \tag{1.5}$$

$\mathcal{H}(E)$ *is a Hilbert space with inner product*

$$(g,h) = \int_{\mathbb{R}} \frac{g\bar{h}}{|E|^2}. \tag{1.6}$$

One may construct a reproducing kernel for $\mathcal{H}(E)$ from E. Indeed, if we let

$$\mathcal{K}(\zeta, z) = \frac{i}{2\pi}\frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{z - \bar{\zeta}} \tag{1.7}$$

then for all ζ, $\mathcal{K}(\zeta, .) \in \mathcal{H}(E)$ and for all complex ζ and all $g \in \mathcal{H}(E)$,

$$g(\zeta) = \int_{\mathbb{R}} \frac{g(t)\overline{\mathcal{K}(\zeta,t)}}{|E(t)|^2} dt. \tag{1.8}$$

The classical de Branges spaces are Paley-Wiener spaces PW_σ, consisting of entire functions of exponential type $\leq \sigma$ that are square integrable along the real axis. There one may take $E(z) = exp(-i\sigma z)$, and the norm is just

$$||g||_{L_2(\mathbb{R})} = \left(\int_{\mathbb{R}} |g|^2\right)^{1/2}.$$

We write

$$\mathcal{H}(E) = PW_\sigma$$

if the two spaces are equal as sets, and have equivalent norms (we do not imply isometric isomorphism). Let us recall that having equivalent norms means that for some $C > 1$

independent of $g \in PW_\sigma$,

$$C^{-1}||g||_{L_2(\mathbb{R})} \le ||g||_E \le C||g||_{L_2(\mathbb{R})} \tag{1.9}$$

In much of this thesis, we will be concerned with the proof of the following result.

Theorem 1.4. *Let μ be a measure with compact support. Let J be a compact set such that μ is absolutely continuous in an open set O containing J, and for some $C > 1$,*

$$C^{-1} \le \mu' \le C \text{ in } O.$$

Choose $\{\xi_n\} \subset J$ and define $\{f_n\}$ by (1.7).

(i) $\{f_n(.,.)\}$ is a normal family in compact subsets of $\mathbb{C}^2$.

(ii) Let $f(.,.)$ be the limit of some subsequence $\{f_n(.,.)\}_{n\in S}$. Then f is an entire function of two variables, that is real valued in $\mathbb{R}^2$ and has $f(0,0) = 1$. Moreover, for some $\sigma > 0$, $f(.,.)$ is entire of exponential type σ in each variable.

(iii) Define

$$L(u,v) = (u - v)f(u,v), \quad u, v \in \mathbb{C}. \tag{1.10}$$

Let $a \in \mathbb{C}$ have $\operatorname{Im} a > 0$ and let

$$E_a(z) = \sqrt{2\pi}\frac{L(\bar{a}, z)}{|L(a,\bar{a})|^{1/2}}. \tag{1.11}$$

Then f is a reproducing kernel for $\mathcal{H}(E_a)$. In particular, for all z, ζ,

$$f(z,\bar{\zeta}) = \frac{i}{2\pi}\frac{E_a(z)\overline{E_a(\zeta)} - E_a^*(z)\overline{E_a^*(\zeta)}}{z - \bar{\zeta}}. \tag{1.12}$$

(iv) Moreover,

$$\mathcal{H}(E_a) = PW_\sigma \tag{1.13}$$

and the norms $||.||_{E_a}$ *of* $\mathcal{H}(E_a)$ *and* $||.||_{L_2(\mathbb{R})}$ *of* PW_σ *are equivalent.*

The fundamental aim of this thesis is to analyze the new approach of Lubinsky [8], [9] in establishing universality limits in the bulk; and present the proofs of above theorems in a self contained manner. *Chapter 2* is devoted to the theory of classical orthogonal polynomials. In *Chapter 3*, our main issue is to define *de Branges* spaces of entire functions, and cite some basic properties of this space. Then, we give some preliminary results from the theory of entire functions of exponential type that will be needed in proving main theorems. Ultimately, we will be concerned with proving those theorems in a great detail in *Chapter 4*.

2. ORTHOGONAL POLYNOMIALS

Let $\mu(t)$ be a non-decreasing function on the real line $\mathbb{R}$ with finite limits as $t \to -\infty$ and $t \to \infty$, and assume that the positive measure $d\mu$ has finite moments of all orders,

$$\mu_r := \int_{\mathbb{R}} t^r d\mu(t), \quad r = 0, 1, 2, \ldots, \quad \text{with } \mu_0 > 0. \tag{2.1}$$

Let $\mathbb{P}$ be the space of real polynomials and $\mathbb{P}_k \subset \mathbb{P}$ the space of polynomials of degree $\leq k$. For any pair u, v in $\mathbb{P}$, an *inner product* is defined as follows:

$$(u, v) = \int_{\mathbb{R}} u(t)v(t)d\mu(t). \tag{2.2}$$

Setting $v = u$, we obtain

$$||u|| = \sqrt{(u, u)} = \left(\int_{\mathbb{R}} u^2(t)d\mu(t) \right)^{\frac{1}{2}}, \tag{2.3}$$

which is called the *norm* of u. One may easily see that $||u|| \geq 0$ for all $u \in \mathbb{P}$.

Definition 2.1. *The inner product* (2.2) *is said to be positive definite on* $\mathbb{P}$ *if* $||u|| > 0$ *for all* $u \in \mathbb{P}$, $u \not\equiv 0$. *Similarly, it is said to be positive definite on* $\mathbb{P}_k$ *if* $||u|| > 0$ *for any* $u \in \mathbb{P}_k$, $u \not\equiv 0$.

To decide whether the inner product (2.2) is positive definite or not, we firstly look at the points of increase of the function $\mu(t)$. A point $t \in \mathbb{R}$ is a *point of increase* of $\mu(t)$, if $\mu(t_1) < \mu(t_2)$ holds for every pair of numbers t_1, t_2 with $t_1 < t < t_2$.

Definition 2.2. *The support of a measure* $d\mu$ *is the set of all points of increase of* $\mu(t)$, *and is denoted by* $supp(\mu)$.

Proposition 2.3. *Let* $\mu(t)$ *be a function as described at the beginning of this chapter whose support* $supp(\mu)$ *contains infinitely many points and let* $p(t)$ *be a non-zero* *polynomial, taking non-negative values for $t \in supp(\mu)$. Then*

$$\int_{\mathbb{R}} p(t)d\mu(t) > 0.$$

Proof. The existence of this integral is satisfied by hypothesis. Since $p(t)$ can have only a finite number of zeros and $supp(\mu)$ is an infinite set, there must exist a point $x_0 \in supp(\mu)$ with $p(x_0) > 0$; then there also exists an interval $[x_1, x_2]$, containing x_0, such that $p(x) \geq \frac{1}{2}p(x_0)$ holds for $x \in [x_1, x_2]$. Then

$$\int_{\mathbb{R}} p(x)d\mu(x) \geq \int_{x_1}^{x_2} p(x)d\mu(x) \geq \frac{1}{2}p(x_0)[\mu(x_2) - \mu(x_1)] > 0,$$

in accordance with our statement. $\square$

The next theorem demonstrates another criterion for positive definiteness using the *Hankel determinants* in the moments μ_r;

$$\Delta_n = det\mathbf{M}_n, \quad \mathbf{M_n} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{pmatrix}, \quad n = 1, 2, 3, \ldots. \tag{2.4}$$

Theorem 2.4. *The inner product* (2.2) *is positive definite on $\mathbb{P}$ if and only if*

$$\Delta_n > 0, \quad n = 1, 2, 3, \ldots. \tag{2.5}$$

It is positive definite on $\mathbb{P}_k$ if and only if $\Delta_n > 0$, for $n = 1, 2, \ldots, k+1$.

Proof. Let us first consider the finite dimensional case. Let $u \in \mathbb{P}_k$, $u = a_0 + a_1 t +$

$\cdots + a_d t^d$ for some $d \leq k$. We have

$$||u||^2 = \int_{\mathbb{R}} \sum_{l,m=0}^{k} a_m a_l t^{m+l} d\mu(t) = \sum_{l,m=0}^{k} \mu_{m+l} a_m a_l. \tag{2.6}$$

Hence, positive definiteness on $\mathbb{P}_k$ is equivalent to the Hankel matrix $\mathbf{M_{k+1}}$ being positive definite. By definition, this is equivalent to $\Delta_n > 0$ for $n = 1, 2, \ldots, k+1$. Positive definiteness on $\mathbb{P}$, in turn, is equivalent to $\Delta_n > 0$ for $n = 1, 2, 3, \ldots$. □

Definition 2.5. *Monic real polynomials $\pi_k(t) = t^k + \cdots$, $k = 0, 1, 2, \ldots$, are called monic orthogonal polynomials with respect to the measure $d\mu$, if they satisfy the following properties:*

(i) $(\pi_k, \pi_l) = 0$ *for* $k \neq l$, $k, l = 0, 1, 2, \ldots,$

(ii) $||\pi_k|| > 0$ *for* $k = 0, 1, 2, \ldots$.

Normalization $\tilde{\pi}_k = \frac{\pi_k}{||\pi_k||}, k = 0, 1, 2, \ldots,$ *yields the orthonormal polynomials, which satisfy*

$$(\tilde{\pi}_k, \tilde{\pi}_l) = \delta_{kl} \doteq \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases} \tag{2.7}$$

Lemma 2.6. *Let π_k, $k = 0, 1, \ldots, n$, be monic orthogonal polynomials. If $p \in \mathbb{P}_n$ satisfies $(p, \pi_k) = 0$ for $k = 0, 1, \ldots, n$, then $p \equiv 0$.*

Proof. If we write $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, then

$$0 = (p, \pi_n) = a_n (t^n, \pi_n) = a_n (\pi_n, \pi_n).$$

As $(\pi_n, \pi_n) > 0$, we have $a_n = 0$. Similarly, it can be shown that $a_{n-1} = a_{n-2} = \cdots = a_0 = 0$. □

Theorem 2.7. *If the inner product* (2.2) *is positive definite on $\mathbb{P}$, there exists a unique* *infinite sequence π_k of monic orthogonal polynomials.*

Proof. In order to generate the polynomials π_k, we apply *Gram-Schmidt process* to the sequence, $e_k(t) = t^k, k = 0, 1, 2, \ldots$. Taking $\pi_0 = 1$, we recursively obtain

$$\pi_k = e_k - \sum_{l=0}^{k-1} c_l k_l, \quad c_l = \frac{(e_k, \pi_l)}{(\pi_l, \pi_l)}. \tag{2.8}$$

By Theorem 2.4, $(\pi_l, \pi_l) > 0$, and therefore the polynomials π_k are uniquely defined and, by construction, is orthogonal to all polynomials π_j, $j < k$. □

Lemma 2.8. *A set $\pi_0, \pi_1, \ldots, \pi_n$ of monic orthogonal polynomials is linearly independent. Moreover; any polynomial $p \in \mathbb{P}_n$ can be uniquely represented in the form*

$$p = \sum_{k=0}^{n} c_k \pi_k \tag{2.9}$$

for some real constants c_k, i.e., $\pi_0, \pi_1, \ldots, \pi_n$ form a basis of $\mathbb{P}_n$.

Proof. Suppose $\sum_{k=0}^{n} \gamma_k \pi_k \equiv 0$. Then, taking the inner product of both sides with π_j, $j = 0, 1, \ldots, n$, yields γ_j =0 by orthogonality. This proves linear independence. If $p = a_n x^n + \cdots$, then the degree of $p - a_n \pi_n$ is less than n. By repeated applications of this fact, p can be represented in the form

$$p = \sum_{k=0}^{n} c_k \pi_k$$

with certain real constants c_k. Taking the inner product of both sides with π_j gives $c_j = (p, \pi_j)/(\pi_j, \pi_j)$, $j = 0, 1, \ldots, n$. Uniqueness of this representation follows from the linear independence of $\pi_0, \pi_1, \ldots, \pi_n$. □

Theorem 2.9. *If the inner product* (2.2) *is positive definite on $\mathbb{P}_k$ but not on $\mathbb{P}_n$ for any $n > k$, there exists only $k + 1$ of orthogonal polynomials $\pi_0, \pi_1, \ldots, \pi_k$.*

Proof. The Gram- Schmidt procedure is applicable as long as the denominators (π_l, π_l) in (2.8) is positive. In this case, it is, for $l \leq k+1$, and the last polynomial π_{k+1} is orthogonal to the all π_j, $j \leq k$. The set $\pi_0, \pi_1, \pi_2, \ldots, \pi_k$ consists of mutually orthogonal polynomials with positive norm. The norm of π_{k+1} on the other hand is zero. Indeed, by hypothesis there exists a monic polynomial $u \in \mathbb{P}_{k+1}$ such that $||u|| = 0$. As $u - \pi_{k+1}$ has degree k, there holds

$$u = \pi_{k+1} + \sum_{j=0}^{k} \gamma_j \pi_j$$

for some coefficients γ_j. As a result,

$$0 = ||u||^2 = ||\pi_{k+1}||^2 + \sum_{j=0}^{k} {\gamma_j}^2 ||\pi_j||^2,$$

which leads $||\pi_{k+1}|| = 0$. Hence, we cannot add π_{k+1} to the sequence of orthogonal polynomials. $\square$

In applications, we mostly deal with *absolutely continuous* measures where $d\mu(t) = w(t)dt$ and w is a non-negative integrable function on $\mathbb{R}$ called the *weight function*. In that case, *supp*(μ) is mainly an interval-finite, half-infinite, infinite-or possibly a finite number of disjoint intervals.

Another type of measure is *discrete measure* whose support consists of a finite or denumerably infinite number of distinct points t_k at which μ has positive jumps w_k. If the number of points in the support is N, the discrete measure will be denoted by $d\mu_N$, and the inner product associated with it is

$$\int_R u(t)v(t)d\mu_N(t) = \sum_{k=1}^{N} w_k u(t_k)v(t_k). \tag{2.10}$$

It is positive definite on $\mathbb{P}_{N-1}$, but not on any $\mathbb{P}_n$ with $n \geq N$. By Theorem 2.9, there exists only N orthogonal polynomials $\pi_0, \pi_1, \ldots, \pi_{N-1}$. These are called *discrete* *orthogonal polynomials* and they satisfy

$$\sum_{k=0}^{N-1} w_k \pi_r(t_k)\pi_s(t_k) = ||\pi_r||^2 \delta_{rs}. \tag{2.11}$$

Theorem 2.10. *Let $\pi_0, \pi_1, \ldots, \pi_{N-1}$ be the monic orthogonal polynomials relative to the discrete measure $d\mu_N$ of* (2.10)*. Then,*

$$\sum_{k=0}^{N-1} \frac{1}{||\pi_k||^2} \pi_k(t_r)\pi_k(t_s) = \frac{1}{w_r}\delta_{rs}. \tag{2.12}$$

Proof. The condition given in (2.11) can be rewritten in matrix form as $Q^T Q = I$, where Q is a matrix in $\mathbb{R}^{N\times N}$ whose entries are $q_{rs} = \pi_s(t_r)\sqrt{w_r}/||\pi_s||$. Then, we have $QQ^T = I$ as well, which gives us (2.12).

□

Throughout the following sections, we will see several properties of orthogonal polynomials, and assume that $d\mu$ is a positive measure with infinite support, and with finite moments of all orders.

Definition 2.11. *An absolutely continuous measure $d\mu(t) = w(t)dt$ is said to be symmetric if its support interval is $[-b, b]$, $0 < b \leq \infty$, and $w(-t) = w(t)$ for all $t \in \mathbb{R}$.*

Theorem 2.12. *If $d\mu$ is symmetric, then*

$$\pi_k(-t) = (-1)^k \pi_k(t), \quad k = 0, 1, \ldots. \tag{2.13}$$

Thus, depending on the parity of k, π_k is an even or an odd polynomial.

Proof. Define $\hat{\pi}_k = (-1)^k \pi_k(t)$. Then, due to the symmetry assumption

$$(\hat{\pi}_k, \hat{\pi}_l) = (-1)^{k+l}(\pi_k, \pi_l) = 0 \text{ if } k \neq l.$$

Since all $\hat{\pi}_k$ are monic, by Theorem 2.7 we have $\hat{\pi}_k(t) \equiv \pi_k(t)$. □

2.1. Three Term Recurrence Relation

Theorem 2.13. *The monic orthogonal polynomials* $\{\pi_k\}$ *satisfy the recursion formula*

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}, \quad k = 0, 1, \ldots, \tag{2.14}$$

$$\pi_0(t) = 1, \quad \pi_{-1}(t) = 0,$$

where

$$\alpha_k \quad = \tfrac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}, \qquad k = 0, 1, 2, \ldots,$$

$$\beta_k \quad = \tfrac{(\pi_k, \pi_k)}{(\pi_{k-1}, \pi_{k-1})}, \qquad k = 1, 2, \ldots.$$

Proof. We take the inner product of $t\pi_k$ with π_n. By orthogonality, we have

$$(\pi_n, t\pi_k) = (p_{n+1}, \pi_k) = 0$$

for $n \leq k - 2$, where p_n represents any polynomial with degree n.

Thus, in the representation (2.9) of the polynomial $t\pi_k$ only the three coefficients are nonzero; namely c_{k+1}, c_k, c_{k-1}. As done in the proof of Lemma 2.8, the coefficients are find as follows; for c_{k+1} we obtain

$$\begin{aligned} c_{k+1}(\pi_{k+1}, \pi_{k+1}) = (t\pi_k, \pi_{k+1}) \quad &= \quad ((\pi_{k+1} + p_k), \pi_{k+1}) \\ &= \quad (\pi_{k+1}, \pi_{k+1}) + (p_k, \pi_{k+1}) \\ &= \quad (\pi_{k+1}, \pi_{k+1}). \end{aligned}$$

and in a similar manner we find

$$c_{k-1}(\pi_{k-1}, \pi_{k-1}) = (\pi_k, \pi_k).$$

and

$$c_k = \frac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}.$$

Therefore,

$$t\pi_k = \pi_{k+1} + \frac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}\pi_k + \frac{(\pi_k, \pi_k)}{(\pi_{k-1}, \pi_{k-1})}\pi_{k-1}$$

Setting

$$\alpha_k = \tfrac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}, \quad \beta_k = \tfrac{(\pi_k, \pi_k)}{(\pi_{k-1}, \pi_{k-1})}$$

and putting the terms multiplied by π_{k-1} together, we obtain the desired formula. □

Remark The index set in (2.14) may be finite or infinite depending on if the inner product is positive definite on $\mathbb{P}$, or on $\mathbb{P}_d$ but not on $\mathbb{P}_n$ for $n > d$ respectively.

2.2. Zeros

Theorem 2.14. *All zeros of π_n, $n > 1$ are real and simple; if supp$(\mu) \subset [a, b]$, then all zeros of π_n belong to $[a, b]$.*

Proof. Notice that $\int_{\mathbb{R}} \pi_n(t) d\mu(t) = 0$ for $n \geq 1$. Therefore, there exist at least one point in the interior of $[a, b]$ at which π_n changes sign. Let $\tau_1, \tau_2, \ldots, \tau_k$, $k \leq n$, be all such points. If $k < n$ were true, then

$$\int_{\mathbb{R}} \pi_n(t) \prod_{j=1}^{k} (t - \tau_j) d\mu(t) = \int_{[a,b]} \pi_n(t) \prod_{j=1}^{k} (t - \tau_j) d\mu(t) = 0$$

as a consequence of orthogonality. This, however, cannot be the case since the integrand has constant sign. As the sum of the multiplicities of the zeros is n, it follows from this that $k = n$ and that all the zeros τ_j are simple. $\square$

Let us denote the zeros of π_n as

$$\tau_{n,n} < \cdots < \tau_{2,n} < \tau_{1,n}.$$

Theorem 2.15. *The zeros of π_{n+1} alternate with those of π_n, that is,*

$$\tau_{n+1,n+1} < \tau_{n,n} < \tau_{n,n+1} < \tau_{n-1,n} < \cdots < \tau_{1,n} < \tau_{1,n+1}. \tag{2.15}$$

Proof. We firstly show that

$$\text{sign } \pi_{n-1}(\tau_{k,n}) = (-1)^{k+1} \text{ for } n \geq 1 \tag{2.16}$$

holds, by induction. Since $\pi_0 = 1$, (2.16) is satisfied for $n = 1, k = 1$. Now let us assume that (2.16) is fulfilled for an $n \geq 1$ and for every $1 \leq k \leq n$. By Theorem 2.13, we have $\pi_{n+1}(\tau_{k,n}) = -\beta_n \pi_{n-1}(\tau_{k,n})$, $\beta_n > 0$; hence

$$\text{sign } \pi_{n+1}(\tau_{k,n}) = (-1)^k.$$

For sufficiently large values of t the sign of $\pi_n(t)$ is determined by the term t^n, so we have

$$\text{sign } \pi_{n+1}(+\infty) = 1 \quad , \text{sign } \pi_{n+1}(-\infty) = (-1)^{n+1}. \tag{2.17}$$

In each of the following $(n+1)$ intervals

$$(-\infty, \tau_{n,n}), (\tau_{n,n}, \tau_{n-1,n}), \ldots, (\tau_{2,n}, \tau_{1,n}), (\tau_{1,n}, +\infty)$$

there is at least one zero of π_{n+1}. As π_{n+1} has exactly $n+1$ zeros, it follows that

$$\begin{aligned} \tau_{n+1,n+1} < \tau_{n,n} < \tau_{n,n+1} < \cdots &< \tau_{k+1,n+1} < \tau_{k,n} < \tau_{k,n+1} < \cdots < \\ < \tau_{2,n+1} &< \tau_{1,n} < \tau_{1,n+1}. \end{aligned}$$

This in turn yields that in the interval $[\tau_{k,n+1}, \infty)$ the polynomial π_n changes its sign exactly $k-1$ times (namely at the zeros $\tau_{1,n}, \tau_{2,n}, \ldots, \tau_{k-1,n}$), it follows from (2.17) that sign $\pi_n(\tau_{k,n+1}) = (-1)^k$. This proves (2.16), and in the course of the proof we have also seen that (2.15) is a consequence of (2.16).

□

Theorem 2.16. *If $c < d$ and $\mu(c) = \mu(d)$, then $\pi_n(d\mu, t)$ has at most one zero in the interval $[c, d]$.*

Proof. By the way of contradiction assume that there are two zeros $c \leq \tau_{i,n} < \tau_{j,n} \leq d$, and let all the other zeros, within [c,d] or without, be $\tau_{k,n}$. Notice that

$$\pi_n \textstyle\prod_{k\neq i,j}(t-\tau_{k,n}) \geq 0 \quad \text{on} \quad (-\infty, c] \cup [d, \infty).$$

Then, by orthogonality

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \pi_n \prod_{k\neq i,j}(t-\tau_{k,n})d\mu(t) \\ &= \int_{-\infty}^{c} \pi_n \prod_{k\neq i,j}(t-\tau_{k,n})d\mu(t) + \int_{d}^{\infty} \pi_n \prod_{k\neq i,j}(t-\tau_{k,n})d\mu(t) \\ &= \int_{-\infty}^{c} \prod_{k\neq i,j}(t-\tau_{k,n})^2(t-\tau_{i,n})(t-\tau_{j,n})d\mu(t) + \int_{d}^{\infty} \prod_{k\neq i,j}(t-\tau_{k,n})^2(t-\tau_{i,n})(t-\tau_{j,n})d\mu(t). \end{aligned}$$

in contradiction to Proposition 2.3. □

2.3. Extremal Properties

The set of monic polynomials with degree n will be denoted by $\mathbb{P}_n^{\circ}$.

Theorem 2.17. *For any monic polynomial $\pi \in \mathbb{P}_n^{\circ}$ there holds*

$$\int_{\mathbb{R}} \pi^2(t) d\mu(t) \geq \int_{\mathbb{R}} \pi_n^2 d\mu(t), \tag{2.18}$$

with equality if and only if $\pi = \pi_n$. In other words, π_n minimizes the integral on the left over all $\pi \in \mathbb{P}_n^{\circ}$:

$$\min_{\pi \in \mathbb{P}_n^{\circ}} \int_{\mathbb{R}} \pi^2(t) d\mu(t) = \int_{\mathbb{R}} \pi_n^2 d\mu(t). \tag{2.19}$$

Proof. According to Lemma 2.8, the polynomial can be represented in terms of the orthogonal polynomials $\pi_0, \pi_1, \ldots, \pi_n$ as

$$\pi(t) = \pi_n(t) + \sum_{k=0}^{n-1} c_k \pi_k(t). \tag{2.20}$$

Then,

$$\int_{\mathbb{R}} \pi^2(t) d\mu(t) = \int_{\mathbb{R}} \pi_n^2(t) d\mu(t) + \sum_{k=0}^{n-1} \int_{\mathbb{R}} c_k^2 \pi_k^2(t) d\mu(t).$$

This proves inequality (2.18) and equality if and only if $c_0 = c_1 = \cdots = c_{n-1} = 0$, that is, $\pi = \pi_n$.

□

Remark Another way of seeing (2.19) can be given as follows. Consider the left-hand integral as a function $\phi(a_0, a_1, \ldots, a_{n-1})$ of the coefficients in the monic polynomial

$\pi(t)$. Setting the partial derivative with respect to each a_k equal to zero yields

$$\int_{\mathbb{R}} \pi(t)t^k d\mu(t) = 0, \quad k = 0, 1, \ldots, n-1. \tag{2.21}$$

Notice that this is precisely the conditions of orthogonality that $\pi = \pi_n$ must satisfy. Furthermore, the Hessian matrix of ϕ is twice the Hankel matrix $\mathbb{M}_n$ in (2.4), which is positive definite by Theorem 2.4, confirming the minimality of π_n.

2.4. The Gauss-Jacobi quadrature formula

In this section we will deal with orthonormal polynomials

$$\tilde{\pi}_n = \pi_n/||\pi_n||,$$

which are no longer monic, and use them to develop a Gaussian quadrature formula. The question here is that if $p(t)$ is a polynomial of degree $\leq 2n-1$, can we obtain a formula like

$$\int_{\mathbb{R}} p(t)d\mu(t) = \sum_{i=1}^{K} \lambda_n(\xi_i)p(\xi_i), \tag{2.22}$$

for some function $\lambda_n(t)$ and some points $\xi_1, \ldots, \xi_K \in supp(\mu)$? The point of (2.22) is that generally if one fixed n points, using the Lagrange interpolation formula one can only hope to fit polynomials up to degree $n-1$ by adjusting the constants. But here we get a space of almost twice the dimension. Both the formula (2.22) and the weights λ_n deserve special names which will be specified later.

In order to form the Lagrange fundamental polynomials, let us now focus on the expression

$$\psi_n(t,\xi) = \psi_{n,\xi}(t) = \tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t), \tag{2.23}$$

where the parameter ξ ranges in real numbers. It can be observed that the degree of ψ_n, as a function t, depends on whether $\tilde{\pi}_{n-1}(\xi) = 0$ or not. If $\tilde{\pi}_{n-1}(\xi) \neq 0$, it is a polynomial of degree $n-1$, as $\tilde{\pi}_{n-1}(\xi)$ and $\tilde{\pi}_n(\xi)$ cannot be zero at the same time, according to Theorem 2.15. We denote the degree of $\psi_n(t,\xi)$ by n^*. Therefore,

$$n^* = \begin{cases} n & \text{for} \quad \tilde{\pi}_{n-1}(\xi) \neq 0 \\ n-1 & \text{for} \quad \tilde{\pi}_{n-1}(\xi) = 0 \end{cases} \tag{2.24}$$

Theorem 2.18. *All zeros (with respect to the variable t) of the polynomial $\psi_n(t,\xi)$ are real and simple. If supp $(\mu) \subset [a,b]$, then at least $n-1$ zeros lie in (a,b).*

Proof. If $\tilde{\pi}_{n-1}(\xi) = 0$ or $\tilde{\pi}_n(\xi) = 0$, then ψ_n is a multiple of $\tilde{\pi}_n(t)$ or, respectively, of $\tilde{\pi}_{n-1}(t)$, the statement follows from Theorem 2.14. Thus, we assume that $\tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(\xi) \neq 0$, and $n^* = n$. By (2.16) we have

$$\text{sign}\psi_n(\tau_{kn},\xi) = -\text{sign}\tilde{\pi}_n(\xi)\text{sign}\tilde{\pi}_{n-1}(\tau_{kn}) = (-1)^k\text{sign}\tilde{\pi}_n(\xi).$$

We see from this formula that $\psi_n(t,\xi)$ has odd number of zeros, counting multiplicity, in any of the intervals $(\tau_{n,n},\tau_{n-1,n}), (\tau_{n-1,n},\tau_{n-2,n}), \ldots, (\tau_{2,n},\tau_{1,n})$. Combining this with the fact that the total number of zeros is equal to n, we see that there is exactly one zero of ψ_n in each of those intervals. There remains only a single zero η which must be real too, since the complex roots occur in conjugate pairs. Now, we are going to show that η lies outside of $[\tau_{n,n},\tau_{1,n}]$. Firstly, as $\text{sign}\psi_n(\tau_{i,n},\xi) \neq 0$, η cannot coincide with any of the $\tau_{i,n}$. Moreover, as we discussed above, it cannot be in $(\tau_{i+1,n},\tau_{i,n})$. To sum up, there is one zero in each interval $(\tau_{i+1,n},\tau_{i,n})$ and one outside of $[\tau_{n,n},\tau_{1,n}]$, so that every zero is simple. By Theorem 2.14 $[\tau_{n,n},\tau_{1,n}] \subset (a,b)$, whence the second assertion of the theorem follows. $\square$

We will denote the zeros of $\psi_n(t,\xi)$ in a decreasing order by

$$\xi_1 > \xi_2 > \cdots > \xi_{n^*}.$$

Notice that ξ itself is one of these zeros, since $\psi_n(t,\xi)$ vanishes at $t=\xi$. By the Lagrange interpolation formula for $n \geq 2$ an arbitrary polynomial $p_{n^*-1}(t)$ of degree at most equal to n^*-1 can be represented in the form

$$p_{n^*-1}(t) = \sum_{i=1}^{n^*} p_{n^*-1}(\xi_i) l_n(t,\xi_i) \tag{2.25}$$

with

$$l_n(t,\xi_i) = \frac{\psi_n(t,\xi)}{\psi_n'(\xi_i,\xi)(t-\xi_i)} \tag{2.26}$$

where $\psi_n'(t,\xi)$ represents the derivative of ψ_n with respect to t. Since all of its zeros are simple, $\psi_n'(\xi_i,\xi) \neq 0$. The degree of the Lagrange fundamental functions $l_n(t,\xi_i)$ is equal to n^*-1 and,

$$l_n(\xi_k,\xi_i) = \begin{cases} 1 & \text{for} \quad \xi_k = \xi_i, \\ 0 & \text{for} \quad \xi_k \neq \xi_i. \end{cases} \tag{2.27}$$

holds.

Theorem 2.19. *The polynomial $l_n(t,\xi_i)$ is uniquely determined by the following three properties:*

(i) $l_n(t,\xi_i) \in \mathbb{P}_{n^*-1}(t)$

(ii) $l_n(\xi_i,\xi_i) = 1$

(iii) The relation

$$\int_{\mathbb{R}} l_n(t,\xi_i) p_{n-1}(t) d\mu(t) = 0$$

holds for every polynomial $p_{n-1}(t)$ of degree at most equal to n^*-1 and vanishing at the point ξ_i.

Proof. Statements (1) and (2) are clear. In order to prove (3), we set $p_{n-1}(t) =$

$(t-\xi_i)p_{n-2}(t)$, and

$$\begin{aligned}\int_{\mathbb{R}} l_n(t,\xi_i)p_{n-1}(t)d\mu(t) &= \int_{\mathbb{R}} \frac{\psi_n(t,\xi)}{\psi_n'(\xi_i,\xi)(t-\xi_i)}(t-\xi_i)p_{n-2}(t)d\mu(t)\\ &= \frac{1}{\psi_n'(\xi_i,\xi)}\left\{\tilde{\pi}_{n-1}(\xi)\int_{\mathbb{R}}\tilde{\pi}_n(t)p_{n-2}(t)d\mu(t)-\right.\\ &\qquad \left.-\tilde{\pi}_n(\xi)\int_{\mathbb{R}}\tilde{\pi}_{n-1}(t)p_{n-2}(t)d\mu(t)\right\} = 0\end{aligned}$$

the last step is justified by the orthogonality of the polynomials $\tilde{\pi}_{n-1}(t)$ and $\tilde{\pi}_n(t)$.

For the uniqueness, we consider another function, $l_n^*(t,\xi)$, satisfying (1),(2) and (3). By (1) and (2) we have

$$l_n(t,\xi_i) - l_n^*(t,\xi_i) = p_{n-1}(t)$$

is a polynomial of degree at most equal to $n-1$ and vanishing for $t=\xi$. We infer from (3) that

$$\begin{aligned}\int_{\mathbb{R}}[l_n(t,\xi_i)-l_n^*(t,\xi_i)]^2 d\mu(t) &= \int_{\mathbb{R}} l_n(t,\xi_i)p_{n-1}(t)d\mu(t) -\\ &\quad -\int_{\mathbb{R}} l_n^*(t,\xi_i)p_{n-1}(t)d\mu(t) = 0.\end{aligned}$$

Due to the hypothesis on the measure $\mu(t)$ we arrive at the conclusion that

$$l_n^*(t,\xi_i) = l_n(t,\xi_i).$$

Since the expression

$$\frac{\psi_n(t,\xi)}{\psi_n'(\xi_i,\xi)(t-\xi_i)}$$

satisfies the given conditions, it follows that

$$l_n(t,\xi_i) = \frac{\psi_n(t,\xi)}{\psi_n'(\xi_i,\xi)(t-\xi_i)}$$

□

This result is of fundamental importance for the next theorems. To a real number ξ we have adjoined the zeros $\xi_1, \xi_2, \ldots, \xi_{n^*}$ of the polynomial $\psi_n(t,\xi)$, where ξ was one of these numbers. When we constructed the Lagrange fundamental polynomials $l_n(t,\xi_i)$ this way, the further nodes ξ_j $(j \neq i)$, in the formula (2.25) can then be obtained as zeros of $l_n(t,\xi_i)$.

Theorem 2.20. *For an arbitrary polynomial $p_{n'}(t)$ of degree at most equal to $n' = (n+n^*-2)$ the quadrature formula*

$$\int_{\mathbb{R}} p_{n'}(t)d\mu(t) = \sum_{i=1}^{n^*} \lambda_n(\xi_i)p_{n'}(\xi_i) \tag{2.28}$$

holds with

$$\lambda_n(\xi_i) = \int_{\mathbb{R}} [l_n(t,\xi_i)]^2 d\mu(t) > 0$$

for every $i = 1, 2, \ldots, n^*$.

Remark The values $\lambda_n(\xi_i)$ are obtained by substituting the value $\eta = \xi_i$ into the function

$$\lambda_n(\eta) = \int_{\mathbb{R}} [l_n(t,\eta)]^2 d\mu(t) \tag{2.29}$$

defined for every real η. Formula of type (2.28) are called *quadrature formula*, and the coefficients $\lambda_n(\xi_i)$ are named *Christoffel numbers.*

Proof. Let $p_{n'}$ and $q_{n'}$ be two polynomials of degree at most equal to n' for which $p_{n'}(\xi_i) = q_{n'}(\xi_i)$ holds for $i = 1, 2, \ldots, n^*$. Then we have

$$q_{n'}(t) - p_{n'}(t) = \psi_n(t, \xi)p_{n-2}(t)$$

for some polynomial $p_{n-2}(t)$ of degree at most equal to $n-2$ (since the degree of $\psi_n(t, \xi)$ is at most equal to n^*, the degree of $p_{n-2}(t)$ is at most equal to $n' - n^* = n-2$). Hence, $p_{n-2}(t)$ is orthogonal to $\tilde{\pi}_n$, as well as to $\tilde{\pi}_{n-1}$,

$$\begin{aligned} \int_{\mathbb{R}} \psi_n(t, \xi)p_{n-2}d\mu(t) \quad &= \quad \tilde{\pi}_{n-1}(\xi) \int_{\mathbb{R}} \tilde{\pi}_n(t)p_{n-2}(t)d\mu(t) - \\ &\quad -\tilde{\pi}_n(\xi) \int_{\mathbb{R}} \tilde{\pi}_{n-1}(t)p_{n-2}(t)d\mu(t) = 0. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} p_{n'}(t)d\mu(t) = \int_{\mathbb{R}} q_{n'}(t)d\mu(t).$$

For $q_{n'}$, we substitute the uniquely determined polynomial

$$\sum_{k=1}^{n^*} p_{n'}(\xi_k)l_n(t, \xi_k)$$

of degree at most equal to $n^* - 1$, agreeing with $p_{n'}(t)$ for $t = \xi_k$ $(k = 1, 2, \ldots, n^*)$, by the Lagrange interpolation formula. This way we obtain

$$\int_{\mathbb{R}} p_{n'}(t)d\mu(t) = \sum_{k=1}^{n^*} \int_{\mathbb{R}} p_{n'}(\xi_k)l_n(t, \xi_k)d\mu(t).$$

In order to finish the proof, we need to show that

$$\int_{\mathbb{R}} l_n(t, \xi_i)d\mu(t) = \int_{\mathbb{R}} l_n^2(t, \xi_i)d\mu(t) \tag{2.30}$$

for $i = 1, 2, \ldots, n^*$. This can be seen by substituting $p_{n'}(t) = l_n^2(t, \xi_i)$ into the last but one formula, which is possible since the degree of $l_n^2(t,\xi)$ is equal to $2(n^*-1) \leq n+n^*-2$. Taking into account of (2.27),

$$l_n^2(\xi_k,\xi_i) = \begin{cases} 1 & \text{for} \quad \xi_k = \xi_i, \\ 0 & \text{for} \quad \xi_k \neq \xi_i. \end{cases}$$

we obtain (2.30). □

2.5. Consequences of Quadrature Formula

Taking into consideration the relation (2.27) we obtain from the quadrature formula (2.28) for $k \leq n-1$

$$\int_{\mathbb{R}} \tilde{\pi}_k(t) l_n(t,\xi) = \lambda_n(\xi)\tilde{\pi}_k(\xi),$$

hence by the expansion (2.9)

$$l_n(t,\xi) = \lambda_n(\xi)\sum_{k=0}^{n-1}\tilde{\pi}_k(\xi)\tilde{\pi}_k(t) = \lambda_n(\xi)K_n(t,\xi) \tag{2.31}$$

with

$$K_n(t,\xi) = \sum_{k=0}^{n-1}\tilde{\pi}_k(\xi)\tilde{\pi}_t$$

moreover, from (2.26) we have

$$K_n(t,\xi) = \{\lambda_n(\xi)\psi_n'(\xi,\xi)\}^{-1}\frac{\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)}{(t-\xi)}$$

Comparing the coefficients of t^{n-1} on both sides of the formula

$$\tilde{\pi}_{n-1}(\xi)\frac{1}{||\pi_{n-1}||} = \{\lambda_n(\xi)\psi_n'(\xi,\xi)\}^{-1}\tilde{\pi}_{n-1}(\xi)\frac{1}{||\pi_n||}$$

holds true. As this equation must hold identically with respect to ξ, we have

$$\lambda_n(\xi)\psi_n'(\xi,\xi) = \frac{\frac{1}{||\pi_n||}}{\frac{1}{||\pi_{n-1}||}} \tag{2.32}$$

and

$$K_n(t,\xi) = \sum_{k=0}^{n-1} \tilde{\pi}_k(\xi)\tilde{\pi}_k(t) = \frac{||\pi_n||}{||\pi_{n-1}||}\frac{\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)}{(t-\xi)}. \tag{2.33}$$

Formula (2.33) is called the *Christoffel-Darboux summation formula*; and it has a significant role at the treatment of expansions of functions in orthogonal polynomials, since $K_n(t,\xi)$ represents the kernel of the partial sums of the orthogonal expansion. Notice that $K_n(t,\xi)$ has the following symmetry relation:

$$K_n(t,\xi) = K_n(\xi,t). \tag{2.34}$$

The following theorem provides a direct proof of the reproducing kernel relation.

Theorem 2.21. *For an arbitrary polynomial $p_{n-1}(t)$ of degree at most equal to $n-1$ we have*

$$\int_{\mathbb{R}} K_n(t,\xi)p_{n-1}(t)d\mu(t) = p_{n-1}(\xi). \tag{2.35}$$

Proof. For an arbitrary $p_{n-1}(t)$,

$$\frac{p_{n-1}(t) - p_{n-1}(\xi)}{(t-\xi)}$$

is a polynomial of degree at most equal to $n-2$, and therefore it is orthogonal to both $\tilde{\pi}_n(t)$ and $\tilde{\pi}_{n-1}(t)$. Accordingly,

$$\int_{\mathbb{R}} \frac{p_{n-1}(t) - p_{n-1}(\xi)}{(t-\xi)}[\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)]d\mu(t) = 0,$$

which implies that

$$\int_{\mathbb{R}} \frac{\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)}{t-\xi} p_{n-1}(t)d\mu(t)$$
$$= p_{n-1}(\xi) \int_{\mathbb{R}} \frac{\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)}{t-\xi} d\mu(t) = \Lambda_{n-1}(\xi)p_{n-1}(\xi).$$

Substituting $p_{n-1}(t) = \tilde{\pi}_k(t)$ in this formula for $k = 0, 1, \cdots, n-1$, we obtain the coefficients of $\tilde{\pi}_k(t)$ in the orthogonal expansion of $\frac{\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)}{t-\xi}$:

$$\int_{\mathbb{R}} \frac{\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)}{t-\xi} \tilde{\pi}_k(t)d\mu(t) = \Lambda_{n-1}(\xi)\tilde{\pi}_k(\xi),$$

and then

$$\frac{\tilde{\pi}_{n-1}(\xi)\tilde{\pi}_n(t) - \tilde{\pi}_n(\xi)\tilde{\pi}_{n-1}(t)}{t-\xi} = \Lambda_{n-1}(\xi) \sum_{k=0}^{n-1} \tilde{\pi}_k(\xi)\tilde{\pi}_k(t).$$

Comparing the coefficients of t^{n-1} on both sides we obtain the value $\frac{1}{||\pi_n||} / \frac{1}{||\pi_{n-1}||}$ for $\Lambda_{n-1}(\xi)$, independent of ξ, in accordance with (2.35). $\square$

Remark Another important consequence of the quadrature formula follows from (2.31). Using (2.27), we observe that if $j \neq k$

$$0 = l_n(\xi_k, \xi_j) = \lambda_n(\xi_j)K_n(\xi_k, \xi_j). \tag{2.36}$$

Since $\lambda_n(\xi_j) > 0$, $K_n(\xi_k, \xi_j) = 0$ holds for all $j \neq k$.

3. BACKGROUND

Before we get to the proofs of the main results, we give brief introductions into the areas we will use. In Section 3.1, we give an overview of de Branges spaces of entire functions. In Section 3.2, we give some details of the theory of entire functions of exponential type, as it features prominently in this work.

3.1. de Branges Spaces of Entire Functions

In this section the Hilbert space of entire functions associated with a function E is defined and useful properties from this space are cited.

de Branges spaces are built around the Hermite-Biehler class. An entire function E is said to belong to the *Hermite-Biehler class* if it has no zeros in the upper half-plane $\mathbb{C}^+ = \{z : \text{Im } z > 0\}$ and

$$|E(z)| \geq |E(\bar{z})| \text{ for } z \in \mathbb{C}^+. \tag{3.1}$$

We write $E \in \overline{HB}$. Recall that the *Hardy space* $H^2(\mathbb{C}^+)$ is the set of all functions g analytic in the upper-half plane, for which

$$\sup_{y>0} \int_{\mathbb{R}} |g(x+iy)|^2 dx < \infty.$$

Given an entire function g, we let

$$g^*(z) = \overline{g(\bar{z})}. \tag{3.2}$$

One interpretation of de Branges spaces is to see them as weighted versions of Paley Wiener spaces. The reader may refer to [10] and [11]. Here, we recall the Paley-Wiener theorem.

Fix $\sigma > 0$, and let

$$\hat{f}(k) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ikx} dx$$

for $f \in L_2(-\sigma, \sigma)$. The function $\hat{f}$ is called the Fourier transform of f, originally defined as an element of $L_2(\mathbb{R})$, and extends to an entire function. Paley-Wiener space PW_σ is defined as the space of Fourier transforms $\hat{f}$ of functions f from $L_2(\mathbb{R})$. The Paley-Wiener theorem says that

$$PW_\sigma = \{F : \mathbb{C} \to \mathbb{C} : F \text{ entire}, \int_{\mathbb{R}} |F(x)|^2 dx < \infty, |F(z)| \leq C_F e^{\sigma |z|}\}. \tag{3.3}$$

The *de Branges space* is defined in analogy to (3.3). It consists of the entire functions F which are square integrable on the real line with respect to the weight function $|E|^{-2}$ for $E \in \overline{HB}$,

$$\int_{\mathbb{R}} \left| \frac{F(x)}{E(x)} \right|^2 dx < \infty \tag{3.4}$$

and satisfy a growth condition. In the presence of (3.4) there are several ways to state this condition. In the following section, we sum up the results which will be needed in defining the de Branges spaces.

3.1.1. Preliminary Results

We will begin by defining the space $N(\mathbb{C}^+)$, the space of functions of bounded type.

Definition 3.1. *A real valued function $f(x+iy)$ defined on an open subset $\Omega \in \mathbb{C}$ is called harmonic in Ω if f is twice differential with respect to x and y; and the Laplacian $\triangle f$ vanishes, i.e.*

$$\triangle f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} = 0.$$

Definition 3.2. *Let $\Omega \in \mathbb{C}$ be open. A function $f : \Omega \to [-\infty, \infty)$ is called subharmonic on Ω if*

(i) f is upper semi-continuous, i.e. $\{x \in \Omega : f(x) < a\}$ is open for all $a \in \mathbb{R}$,

(ii) for every set A with compact closure $\overline{A} \in \Omega$ and every continuous function $h : \overline{A} \to \mathbb{R}$ whose restriction to A is harmonic, if $f \leq h$ on ∂A, then $f \leq h$ on A.

The following theorem yields some important properties about harmonic and subharmonic functions.

Theorem 3.3. *Let $\Omega \in \mathbb{C}$ be an open set.*

(i) A continuous function $f : \Omega \to \mathbb{R}$ is harmonic if and only if for any closed disk $\{z + re^{i\phi} : \phi \in (0, 2\pi], r \in [0, R]\}$ with center z and radius R contained in Ω

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\phi}) d\phi.$$

(ii) An upper semi-continuous function $f : \Omega \to \mathbb{R}$ is subharmonic if and only if for any closed disk $\{z + re^{i\phi} : \phi \in (0, 2\pi], r \in [0, R]\}$ with center z and radius R contained in Ω

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\phi}) d\phi.$$

In this case,

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\phi}) d\phi \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\phi}) d\phi$$

holds for all $r < R$.

(iii) A function $f : \Omega \to \mathbb{R}$ in $C^2(\Omega)$ is subharmonic if and only if $\triangle u \geq 0$.

(iv) Let f be an analytic function on a region Ω. Then $f, \bar{f}, \mathrm{Re}\, f, \mathrm{Im}\, f$ are harmonic

and

$$log^{+}|f(z)| = max\{log|f(z)|, 0\}$$

is subharmonic on Ω.

Definition 3.4. *A harmonic function* h *on a region* Ω *is called harmonic majorant of a subharmonic function* $f \not\equiv -\infty$, *if* $h \geq f$ *on* Ω.

Theorem 3.5. *Let* Ω *be a simply connected region of* $\mathbb{C}$ *and* f *be an analytic function on* Ω. *Then the following assertions are equivalent.*

(i) There exist analytic and bounded functions g *and* h *on* Ω *such that* $f = \frac{g}{h}$.

(ii) $log^{+}|f(z)|$ *has a harmonic majorant on* Ω.

Definition 3.6. *A function* f *defined and analytic on a simply connected region* Ω *is said to be of bounded type in* Ω, *if it satisfies the equivalent conditions in Theorem* 3.5. *The space of all functions of bounded type on* Ω *is denoted* $N(\mathbb{C}^{+})$.

Before we introduce a subset of $N(\mathbb{C}^{+})$, we will state the Nevanlinna's factorization of functions of bounded type in a half plane. The following results are mainly from the book [12].

Definition 3.7. *An analytic function on* $\mathbb{C}^{+}$ *is called outer if*

$$f(z) = \alpha \exp\left(\frac{1}{\pi i}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)\log K(t)dt\right)$$

where $|\alpha| = 1$, $K(t) > 0$ *and*

$$\int_{\mathbb{R}} \frac{|\log K(t)|}{1+t^2}dt < \infty. \tag{3.5}$$

A function of the form

$$f(z) = \alpha\left(\frac{z-i}{z+i}\right)^{n} \prod_{j\in J} \frac{|z_j^2+1|}{z_j^2+1}\frac{z-z_j}{z-\bar{z}_j}$$

is called a Blascke product where $|\alpha| = 1$, $n \geq 1$, $z_j \in \mathbb{C}^+ \diagdown \{i\}$ *for all* $j \in J$, $J \subset \mathbb{N}$. *An empty product is defined as* 1. *The set* $\{z_j\}$ *forms the zeros of the function* $f(z)$, *and written in the rectangular form* $z_j = x_j + iy_j$, *they satisfy that*

$$\sum_{j\in J} \frac{y_j}{x_j^2 + (y_j+1)^2} < \infty. \tag{3.6}$$

Theorem 3.8. *Let* $f(z) \in N(\mathbb{C}^+)$, $f \not\equiv 0$. *Then*

$$f(z) = e^{i\alpha} e^{-ihz} B(z) F(z) S_1(z)/S_2(z), \tag{3.7}$$

where α, $h \in \mathbb{R}$, $B(z)$ *is a Blaschke product,* $F(z)$ *is an outer function, and* $S_1(z)$ *and* $S_2(z)$ *are functions of the form*

$$S_{1,2}(z) = \exp\left(-\frac{1}{\pi i}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu_{1,2}(t)\right)$$

where $\mu_{1,2}$ *are singular and mutually singular nonnegative Borel measures on the real line satisfying*

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_{1,2}(t) < \infty. \tag{3.8}$$

Except for the choice of α, *the factorization* (3.7) *is unique. Every function of the form* (3.7) *is in* $N(\mathbb{C}^+)$.

The number h in the above theorem which is associated with the function $f(z) \in N(\mathbb{C}^+)$ deserves a special name as it is frequently used in the literature.

Definition 3.9. *The number* h *in the Theorem* 3.8 *is called the mean type of the function* f.

Definition 3.10. *The functions in* $N(\mathbb{C}^+)$ *for which* $h \leq 0$ *form a subclass of* $N(\mathbb{C}^+)$ *called* $N^+(\mathbb{C}^+)$.

Proposition 3.11. *Let* $f(z) \in N(\mathbb{C}^+)$, $f \not\equiv 0$. *Then in* (3.7) *the singular factor* $S_2 \equiv 1$

if and only if $f(z) \in N^+(\mathbb{C}^+)$.

After this introduction of preliminary results we can finally define *de Branges* spaces. Before giving an explicit definition, let us remind that there are various ways to define this space; and the following proposition, proof of which is contained in [13], summarizes the most basic ones.

Proposition 3.12. *Suppose that F is entire and* (3.4) *holds. Then the followings are equivalent:*

(i) $|F(z)/E(z)|, |F^*(z)/E(z)| \leq C_F(\text{Im } z)^{-1/2}$ *for all* $z \in \mathbb{C}^+$.
(ii) $F/E, F^*/E \in N^+(\mathbb{C}^+)$.
(iii) $F/E, F^*/E \in H^2(\mathbb{C}^+)$.

Definition 3.13. *The de Branges space $\mathcal{H}(E)$ corresponding to the entire function $E \in \overline{HB}$, is the set of all entire functions f such that, in addition to* (3.4)*, one of the above conditions are satisfied.*

In de Branges book [10], the condition (ii) is required to define $\mathcal{H}(E)$. Condition (i) is used in [14], while (iii) gives the most famous description of the space $\mathcal{H}(E)$.

The space $\mathcal{H}(E)$ is a vector space over complex numbers, and the next theorem shows that any space $\mathcal{H}(E)$ contains nonzero elements.

Theorem 3.14. $\mathcal{H}(E)$ *endowed with the inner product*

$$(F, G) = \int_{\mathbb{R}} \overline{F(x)} G(x) \frac{dx}{|E(x)|^2}, \tag{3.9}$$

is a Hilbert space. Moreover, for any $\zeta \in \mathbb{C}$, point evaluation is a bounded linear functional. More explicitly, the entire function $K(\zeta, .)$ given by

$$K(\zeta, z) = K_\zeta(z) = \frac{i}{2\pi} \frac{\overline{E(\zeta)}E(z) - E(\bar{\zeta})\overline{E(\bar{z})}}{(z - \bar{\zeta})} \tag{3.10}$$

belongs to $\mathcal{H}(E)$ *for every* $\zeta \in \mathbb{C}$*, and* $(K(\zeta, .), F) = F(\zeta)$ *for all* $F \in \mathcal{H}(E)$*.*

It is possible to give an abstract definition of a de Branges space. One useful alternative involves the reproducing kernel $K(\zeta, z)$ defined in terms of E. Then $\mathcal{H}(E)$ is the set of all entire functions g with

$$||g||_E = \left(\int_{\mathbb{R}} \left|\frac{g}{E}\right|^2\right)^{1/2} < \infty \tag{3.11}$$

and

$$|g(\zeta)| \leq \mathcal{K}(\zeta, \zeta)^{1/2} ||g||_E \text{ for } \text{ all } \zeta \in \mathbb{C}. \tag{3.12}$$

Example $E_\sigma = e^{-i\sigma z}$ is a function from $\overline{HB}$. With this setting, we obtain the classical de Branges space: $\mathcal{H}(E_\sigma) = PW_\sigma$ with the norm

$$||g||_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |g|^2\right)^{1/2}.$$

Note also that the reproducing kernel $K(\zeta, .)$ for $\mathcal{H}(E_\sigma) = PW_\sigma$ is the Dirichlet kernel,

$$K(\zeta, .) = D_\sigma(\bar{\zeta} - z) = \frac{\sin a(\bar{\zeta} - z)}{\bar{\zeta} - z}.$$

Example Another familiar de Branges space is formed in [9]. Let

$$L_n(x, t) = (x - t) K_n(x, t),$$

where $K_n(x, t)$ regarded as a function of t is the reproducing kernel of the space of polynomials whose degree $\leq n - 1$. Then, as shown in [9], $L_n(\bar{a}, .) \in \overline{HB}$ for Im $a > 0$. After normalizing it, we set $E_{n,a} = \sqrt{2\pi} \frac{L_n(\bar{a}, z)}{|L_n(a, \bar{a})|^{1/2}}$. The de Branges space $\mathcal{H}(E_{n,a})$ is the space of polynomials of degree $\leq n - 1$.

Here, and later on, we shall use the generic decomposition of a function $E \in \overline{HB}$ as $E = C - iS$ with

$$C \doteq \tfrac{E+E^*}{2}, \quad S \doteq i\tfrac{E-E^*}{2}.$$

Note that $C(z)$ and $S(z)$ are entire functions which are real for real z.

The notion of a phase function is important in the theory of de Branges spaces. For $E \in \overline{HB}$, there exists a continuous function $\phi(x)$ of real x such that $E(x)e^{i\phi(x)}$ is real for all values of x. The phase function ϕ associated with E is unique up to an additive constant in $\pi\mathbb{Z}$; and it is given by a continuous branch of $-\mathrm{Im}\log E(x)$. Throughout this paper, we may assume that $E(z)$ has no zero in the closed upper half plane. Then, for all $x \in \mathbb{R}$,

$$\begin{aligned}
\phi'(x) &= \frac{d}{dx}\arctan\left(\frac{S(x)}{C(x)}\right) \\
&= \frac{C^2(x)}{|E(x)|^2}\,\frac{S'(x)C(x) - C'(x)S(x)}{C^2(x)} \\
&= \frac{S'(x)C(x) - C'(x)S(x)}{|E(x)|^2}.
\end{aligned}$$

Since $K(x,x) = \frac{1}{\pi}(S'(x)C(x) - C'(x)S(x))$, we obtain that

$$\phi'(x) = \frac{\pi K(x,x)}{|E(x)|^2} > 0. \tag{3.13}$$

Let α be a given real number; and let $\{s_k\}$ denote the increasing sequence such that

$$\phi(s_k) = \alpha + k\pi, \quad k \in \mathbb{Z}. \tag{3.14}$$

Then, the functions $\left\{\frac{K(s_k,z)}{\sqrt{K(s_k,s_k)}}\right\}_k$ form an orthonormal sequence in $\mathcal{H}(E)$, and the only elements of $\mathcal{H}(E)$ which are orthogonal to $\frac{K(s_k,z)}{\sqrt{K(s_k,s_k)}}$ for every k are constant multiples of $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$. If this function does not belong to $\mathcal{H}(E)$, then

$$\int_{\mathbb{R}} \left|\frac{F(t)}{E(t)}dt\right|^2 = \sum_k \frac{\pi|F(s_k)|^2}{\phi'(s_k)|E(s_k)|^2} = \sum_k \frac{|F(s_k)|^2}{K(s_k, s_k)}, \tag{3.15}$$

while for all z,

$$F(z) = \sum_k F(s_k)\frac{K(s_k, z)}{\sqrt{K(s_k, s_k)}}. \tag{3.16}$$

Furthermore, there is at most one real $\alpha \in [0, \pi)$ for which $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ belongs to $\mathcal{H}(E)$ [10, p.55].

3.2. Entire Functions of Exponential Type

Here we review some theory that we shall use about entire functions of exponential type. Most of this results can be found in [15]. To begin with, we shall consider some basic terms which help us understand how fast an entire function can grow. For a general characterization of the growth, the function

$$M_f(r) = \max_{|z|=r} |f(z)|$$

is introduced. By the Maximum Principle, $M_f(r)$ increases monotonically.

Before we define the order and the type of an entire function, let us recall the following notation. An inequality $h(r) < \phi(r)$ which holds for sufficiently large values of r, is called an asymptotic inequality, and write $h(r) <^{as} \phi(r)$. If the same inequality holds for some sequences of values $r_n \to \infty$, then we shall write $h(r) <^{n} \phi(r)$.

Definition 3.15. *An entire function $f(z)$ is called a function of finite order if $M_f(r) <^{as} exp(r^k)$ for some $k > 0$. The order of an entire function f is the greatest lower bound of those values of k for which the given asymptotic inequality is fulfilled. The order of an entire function is denoted by $\rho = \rho_f$.*

It follows from the above definition that

$$e^{r^{\rho-\epsilon}} <^{n} M_f(r) <^{as} e^{r^{\rho+\epsilon}}$$

for all $\epsilon > 0$. By taking the logarithm twice we deduce that

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{3.17}$$

Definition 3.16. *Let ρ be the order of an entire function f. The function is said to have a finite type if for some $A > 0$ the inequality $M_f(r) <^{as} e^{Ar^{\rho}}$ is fulfilled. The greatest lower bound for those values of A for which the latter asymptotic inequality is fulfilled is called the type $\sigma = \sigma_f$ of the function f.*

Using a very similar argument, we obtain that

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho}}. \tag{3.18}$$

If, for a given $\rho > 0$, the type of a function is infinite, then the function is of *maximal type*; for $0 < \sigma_f < \infty$ the type is called *normal* or *mean*; for $\sigma_f = 0$ the type is *minimal.*

Definition 3.17. *Entire functions of order $\rho = 1$ and normal type σ are called entire functions of exponential type σ.*

Let us now consider a function $f(z)$ which is analytic inside a sector $D = \{z = re^{i\theta} : \alpha < \theta < \beta\}$ and satisfies the estimate

$$M_{f_{\alpha,\beta}}(r) <^{as} e^{Ar^{\rho}} \tag{3.19}$$

with $M_{f_{\alpha,\beta}}(r) = \sup_{\{\alpha<\theta<\beta,\ |z|=r\}} |f(z)|$.

Definition 3.18. *The function*

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^\rho} \tag{3.20}$$

is called the indicator function of $f(z)$ with respect to the order ρ.

The indicator function $h_f(\theta)$ describes the growth of the function $f(z)$ along a ray $\{z : \arg z = \theta\}$. There are a couple of frequently used and basic results that the indicator function satisfies.

Proposition 3.19. *Let f and g be functions which are analytic in a sector D and satisfy the estimate* (3.19)*. Then,*

$$h_{fg}(\theta) \leq h_f(\theta) + h_g(\theta),$$

and

$$h_{f+g}(\theta) \leq \max(h_f(\theta), h_g(\theta)).$$

In the theory of the entire functions, functions of exponential type possess a huge importance. According to the Phragmen- Lindelöf theorem, every function f analytic and of exponential type σ in the upper half plane $\mathbb{C}^+$ which is bounded by some constant M on the real axis, satisfies the inequality

$$|f(x+iy)| \leq Me^{\sigma y}, \quad y \geq 0.$$

In this work, entire functions that are bounded on the real axis in a weaker sense are pointed out as well.

Definition 3.20. *By the Cartwright class C we mean the class of all entire functions* *of exponential type satisfying the inequality*

$$\int_{\mathbb{R}} \frac{\log^+ |f(t)|}{1+t^2} dt < \infty.$$

The following theorem of Krein presents a characterization of the class C.

Theorem 3.21. *An entire function f belongs to the class C if and only if f belongs to the classes $N(\mathbb{C}^+)$ and $N(\mathbb{C}^-)$, i.e., $\log|f(z)|$ have positive harmonic majorants in the upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$.*

To study the growth and the distribution of zeros of functions class C, one should look at the book [15]. Here, we concentrate on the representation of $\log|f(z)|$ for $f \in C$.

Definition 3.22. *A set of disks $\{C_j\}_{j=1}^{\infty}$, centered at points z_j of the upper half plane and of radii ρ_j is called a set of finite view if*

$$\sum_{j=1}^{\infty} \frac{\rho_j}{r_j} < \infty,$$

where $r_j = |z_j|$.

Theorem 3.23. *Every function $f(z)$ of class C satisfies the following relations:*

$$\log|f(z)| = \sigma_+ y + o(|z|), \quad y \geq 0,$$
$$\log|f(z)| = \sigma_- y + o(|z|), \quad y \leq 0,$$

for some real numbers σ_+ and σ_-, everywhere outside a system of exceptional disks of finite view.

It follows from Theorem 3.23 that if $f(z)$ belongs to the class C, then the limit

$$\lim_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r} = \begin{cases} \sigma_+ \sin\theta, & 0 \leq \theta \leq \pi \\ \sigma_- |\sin\theta|, & \pi \leq \theta \leq 2\pi \end{cases} \tag{3.21}$$

exists for almost all $\theta \in [0, 2\pi]$. Indeed, for any $\epsilon > 0$, one can choose sufficiently large R_ϵ such that the sum of openings of the angles at which the exceptional disks C_j centered outside the disk $\{z : |z| < R_\epsilon\}$ are viewed, is less than ϵ. Relation (3.21) holds for all θ such that the ray arg$z = \theta$ does not belong to these angles. Since ϵ is an arbitrary small number, the limit in (3.21) exists almost everwhere in $[0, 2\pi]$.

Let $n(f, r)$ denote the number of zeros of a function f in the ball with center 0 and radius r, counting multiplicity. By the famous Jensen formula,

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| d\psi - \int_0^R \frac{n(t)}{t}$$

for functions f that is analytic in the disk $\{z : |z| \leq R\}$ such that $f(0) \neq 0$.

It follows directly from the Jensen formula that

$$\log |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| d\psi.$$

If f is an entire function with $|f(0)| = 1$, then for $r > 0$ we have

$$\log M_f(er) \geq \int_0^{er} \frac{n(t)}{t} dt \geq \int_r^{er} \frac{n(t)}{t} dt \geq n(r),$$

and thus

$$n(r) < \log M_f(er).$$

To study the distribution of zeros of an entire function of class C, we need the following lemma.

Lemma 3.24. *Let $n(t)$ be a nondecreasing function for $t > 0$, let $n(t) = 0$ for $0 \leq t \leq \epsilon$,* *for some positive number ϵ, and let there exists $\alpha > -2$ such that*

$$\psi(R) = \frac{1}{R^{\alpha+2}} \int_0^R t^\alpha n(t)dt \tag{3.22}$$

approaches the limit d as $R \to \infty$. Then the function $\frac{n(R)}{R}$ approaches the limit $(\alpha+2)d$ as $R \to \infty$.

The proof of the lemma is contained in [15]. Here, we use this lemma to show the following result.

Theorem 3.25. *If $f(z)$ belongs to the class C, and it is real valued on the real axis with $f(0) \neq 0$, then*

$$\lim_{r\to\infty} \frac{n_f(R)}{R} = 2\frac{\sigma}{\pi}. \tag{3.23}$$

Proof. According to Theorem 3.23, we have an asymptotic estimate of $\log|f(re^{i\theta})|$ for each $f \in C$. The hypothesis that f being real valued on the real axis, yields that $\sigma_+ = \sigma_- = \sigma$ by Schwarz's lemma where σ is the exponential type of $f(re^{i\theta})$. In other words, $h_f(\theta) = \sigma|\sin\theta|$ exists for almost all $\theta \in [0, 2\pi]$.

Using the Jensen formula, we obtain that

$$\begin{aligned}
\int_0^R \frac{n(t)}{t} &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})|d\psi - \log|f(0)| \\
&= \frac{1}{2\pi}\int_0^{2\pi} R\sigma|\sin\psi|d\psi - \log|f(0)| \\
&= 2R\frac{\sigma}{\pi} - \log|f(0)|.
\end{aligned}$$

outside a set of exceptional disks of finite view. Since the left hand side is a monotonic function of R, the above equality holds for all R.

Therefore,

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \frac{n(t)}{t}dt = 2\frac{\sigma}{\pi}. \tag{3.24}$$

The relation (3.23) is a consequence of the lemma 3.24. $\square$

We now discuss theorems of Phragmen- Lindelöf type. For these results, see [15].

Theorem 3.26. *If $f(z)$, $z = x + iy$, is an analytic function in the half-plane $\mathbb{C}^+$ such that, for all $\epsilon > 0$,*

$$M_f(r) <^{as} e^{(\sigma+\epsilon)r},$$

and $|f(z)| \leq M$ on the real axis, then

$$|f(x + iy)| \leq Me^{\sigma y}. \tag{3.25}$$

Remark The estimate given by (3.25) is sharp. Functions of the type $f(z) = M\gamma e^{-i\sigma z}$, $|\gamma| = 1$ attains the upper bound.

Remark If $f(z)$ is an entire function of exponential type σ, and $|f(x)| \leq M$, $-\infty < x < \infty$, the estimate (3.25) holds in the whole complex plane.

If we require $f \in L_2(\mathbb{R})$ instead of its boundedness along the real axis, we obtain another estimate. For this, we first recall Plancherel- Polya theorem.

Theorem 3.27. *(Plancherel- Polya Theorem) Let $f(z)$ be an analytic function in the upper half-plane $\{y > 0\}$, continuous up to the real axis, and let*

$$|f(z)| <^{as} e^{(\sigma+\epsilon)|z|},$$

for an arbitrary $\epsilon > 0$. *If*

$$\int_{\mathbb{R}} |f(x)|^p dx = M < \infty, \quad p > 0, \tag{3.26}$$

then

$$\int_{\mathbb{R}} |f(x+iy)|^p dx \leq M e^{p\sigma y},$$

for an arbitrary $y > 0$.

When f is an entire function of exponential type, Theorem 3.27 takes the following form.

Remark If $f(z)$ is an entire function of exponential type σ_f such that

$$\int_{\mathbb{R}} |f(x)|^p dx \leq M, \tag{3.27}$$

for some $p > 0$ then Theorem 3.27 yields that

$$\int_{\mathbb{R}} |f(x+iy)|^p dx \leq e^{p\sigma|y|} ||f||^p_{L_p(\mathbb{R})}.$$

Therefore, we have

$$\int_{-1}^{1} \int_{\mathbb{R}} |f(x+i(y+s))|^p dx ds \leq 2e^{p(1+|y|)} ||f||^p_{L_p(\mathbb{R})},$$

for any $y \in \mathbb{R}$. Since $|f|^p$ is a subharmonic function, we obtain

$$|f(x+iy)|^p \leq \frac{2}{\pi} e^{\sigma p(|y|+1)} ||f||^p_{L_p(\mathbb{R})}. \tag{3.28}$$

4. UNIVERSALITY LAWS

In the previous chapters, we have explored several results from various fields. We would like note that all of them will be put in action in this chapter. We now begin to study the main theorems which are presented in the introduction.

4.1. Notation

We will record the notation that will be used throughout this chapter. Independent constants will be denoted by $C, C_1, C_2, \ldots$. We write $C = C(\alpha)$ to denote the dependence on the parameter α. We use $\sim$ in the following sense: given real sequences $\{c_n\}$, $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1, C_2 with

$$C_1 \leq \frac{c_n}{d_n} \leq C_2.$$

Throughout the entire section, μ denotes a finite positive Borel measure with not necessarily compact support on the real line; and J will be the compact set in Theorem 1.1. The corresponding orthonormal polynomials are denoted by $\{p_n\}_{n=0}^{\infty}$, so that

$$\int p_n p_m d\mu = \delta_{mn}.$$

We denote the zeros of p_n by

$$x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n}.$$

The reproducing kernel $K_n(x,t)$ is defined in the following way:

$$K_n(x,t) = \sum_{k=0}^{n-1} p_k(x)p_k(y).$$

and the normalized kernel is

$$\tilde{K}_n(x,y) = \mu'(x)^{1/2}\mu'(y)^{1/2}K_n(x,y).$$

We let

$$\begin{aligned} L_n(x,t) &= (x-t)K_n(x,t) \\ &= \frac{\gamma_{n-1}}{\gamma_n}(p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)). \end{aligned} \tag{4.1}$$

The n^{th} Christoffel function is

$$\lambda_n(x) = 1/K_n(x,x) = \inf_{deg(P)\leq n-1} \frac{\int P^2 d\mu}{P^2(x)}.$$

By the Gauss quadrature formula, whenever P is a polynomial of degree $\leq 2n-1$,

$$\sum_{j=1}^{n} \lambda_n(x_{jn})P(x_{jn}) = \int P d\mu.$$

We shall need another Gauss type of quadrature formula. Given a real number ξ, there are n or $n-1$ points $t_{jn} = t_{jn}(\xi)$, one of which is ξ, such that

$$\sum_j \lambda_n(t_{jn})P(t_{jn}) = \int P d\mu, \tag{4.2}$$

whenever P is a polynomial of degree $\leq 2n-2$. The $\{t_{jn}\}$ are zeros of

$$L_n(\xi,t) = \frac{\gamma_{n-1}}{\gamma_n}(p_n(\xi)p_{n-1}(t) - p_{n-1}(\xi)p_n(t)),$$

regarded as a function of t. Because we consider a sequence $\{\xi_n\}$ of points in J, rather than a fixed ξ, we use the quadrature rule that includes ξ_n, so that

$$t_{jn} = t_{jn}(\xi_n) \text{ for all } j.$$

We set $t_{0n} = \xi_n$, and order the $\{t_{jn}\}$, treated as the origin:

$$\cdots < t_{-2,n} < t_{-1,n} < t_{0n} = \xi_n < t_{1n} < \cdots .$$

The sequence $\{t_{jn}\}$ consists of either n or $n-1$ points, and it is possible that all t_{jn} lie to the left or right of ξ_n. As it was proven in the first section, when $(p_n p_{n-1})(\xi_n) \neq 0$,then one zero of $L_n(\xi_n, t)$ lies in $(x_{jn}, x_{j-1,n})$ for each j, and the remaining zero lies outside (x_{nn}, x_{1n}). For the given sequence $\{\xi_n\}$ in J, we shall define for $n \geq 1$,

$$f_n(a,b) = \frac{K_n(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n,\xi_n)}$$

and

$$\tilde{L}_n(a,b) = (a-b) f_n(a,b). \tag{4.3}$$

The zeros of

$$f_n(0,t) = \frac{K_n(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n,\xi_n)}$$

will be denoted by $\{\rho_{jn}\}_{j\neq 0}$. Since $\{t_{jn}\} = \{t_{jn}(\xi_n)\}$ are the zeros of $L_n(\xi_n, t)$, we have

$$\rho_{jn} = \tilde{K}_n(\xi_n,\xi_n)(t_{jn} - \xi_n).$$

We set

$$\rho_{0n} = 0,$$

corresponding to $t_{0n} = \xi_n$. For an appropriate subsequence $\mathcal{S}$ of integers, we set

$$f(a,b) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(a,b).$$

The zeros of $f(0,.)$ will be denoted by $\{\rho_j\}_{j\neq 0}$ and we set $\rho_0 = 0$. Our ordering of zeros is

$$\cdots \leq \rho_{-2} \leq \rho_{-1} < \rho_0 = 0 < \rho_1 \leq \rho_2 \leq \cdots .$$

4.2. Normality

We begin with a consequence of Bernstein's growth inequality for polynomials in the complex plane. Throughout this section, J is as in Section 4.1, while $\{\xi_n\}$ is a sequence in J. We shall assume the hypotheses of Theorem 1.1, however, shall not assume (1.2).

Lemma 4.1. *Let $[c,d]$ be a real interval and $\mathcal{K}$ be a compact subset of (c,d). Let $A, \eta > 0$ and*

$$\Gamma = \sup_{x\in\mathcal{K}} \frac{1}{\sqrt{(d-x)(x-c)}}.$$

There exists $\eta_0 = \eta_0(A, \mathcal{K}, \eta, \Gamma)$ such that for $\eta \geq \eta_0$, polynomials P of degree n, $x \in \mathcal{K}$ and $|a| \leq A$,

$$\left|P(x + i\frac{a}{n})\right| \leq e^{(1+\eta)\Gamma|a|} ||P||_{L_\infty[c,d]}. \tag{4.4}$$

Proof. We will consider the case $[c,d] = [-1,1]$. The general case follows by a linear transformation. Let $x \in \mathcal{K}$, and $z = x + i\frac{a}{n}$. We will use Bernstein's inequality,

$$|P(z)| \leq \left|z + \sqrt{z^2-1}\right|^n ||P||_{L_\infty[-1,1]}. \tag{4.5}$$

We have

$$\left|x+\sqrt{x^2-1}\right| = \left|x+i\sqrt{1-x^2}\right|$$
$$= \sqrt{x^2+(1-x^2)} = 1.$$

Therefore,

$$\begin{aligned}
\log\left|z+\sqrt{z^2-1}\right| &= \mathrm{Re}\left(\int_x^{x+i\frac{a}{n}}\left(\frac{\frac{d}{du}(u+\sqrt{u^2-1})}{u+\sqrt{u^2-1}}\right)_{|u=x+is} du\right)\\
&= \mathrm{Re}\left(\int_0^{a/n}\frac{d}{du}\log(u+\sqrt{u^2-1})_{|u=x+is} ids\right)\\
&= \mathrm{Re}\left(\int_0^{a/n}\frac{1}{\sqrt{u^2-1}_{|u=x+is}}\,ids\right)\\
&= \frac{1}{\sqrt{1-x^2}}\mathrm{Re}\left(\int_0^{a/n}\frac{ds}{\sqrt{1+\frac{s^2-2ixs}{1-x^2}}}\right)\\
&= \frac{1}{\sqrt{1-x^2}}\left(\frac{a}{n}+O\left(\frac{a}{n}\right)^2/(1-x^2)\right).
\end{aligned}$$

Substituting this result into (4.5), we obtain (4.4). □

Lemma 4.2. *For $n \geq 1$, let*

$$f_n(u,v) = \frac{K_n(\xi_n + \frac{u}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{v}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n,\xi_n)}.$$

(i) $\{f_n(u,v)\}$ is uniformly bounded for u, v in compact subsets of the plane.

(ii) Let $f(u,v)$ denote the locally uniform limit of some subsequence $\{f_n(u,v)\}_{n\in S}$ of $\{f_n(u,v)\}_{n=1}^{\infty}$. Then, for each fixed real number u, $f(u,.)$ is an entire function of exponential type. Moreover, for some C_1 and C_2 independent of u, v, and the subsequence S,

$$|f_n(u,v)| \leq C_1 e^{C_2(|\mathrm{Im}\ u|+|\mathrm{Im}\ v|)}.$$

(iii) For each fixed real number u, $f(u,.)$ has only real zeros.

Proof. (i) According to the assumptions we made in Section 4.1, μ is absolutely continuous in some open set O containing the compact set J, and μ' is bounded above and below there. As J is covered by finitely many open intervals in O, by increasing the size of J, we may assume that J consists of finitely many compact intervals. It then suffices to consider the case where J is just one interval. By the absolute continuity of μ in O and the boundedness of μ' there, we have the following bound

$$K_n(x,x)^{-1} = \lambda_n(x) \sim \frac{1}{n}, \tag{4.6}$$

uniformly in n, and in each compact subset of O. Since we have the freedom of reducing the size of O, we can assume that this holds in O. By Cauchy-Schwarz, we have

$$\frac{1}{n}|K_n(\xi,t)| \leq \left(\frac{1}{n}K_n(\xi,\xi)\right)^{1/2}\left(\frac{1}{n}K_n(t,t)\right)^{1/2} \leq C$$

for $\xi, t \in O$. We apply the Bernstein's growth lemma in the plane (4.4) in each variable separately, and then for each $\xi, t \in O$, $|a|, |b| \leq A$ and $n \geq n_0(A)$, we have

$$\frac{1}{n}\left|K_n\left(\xi + i\frac{a}{n}, t + i\frac{b}{n}\right)\right| \leq Ce^{C_2(|a|+|b|)}. \tag{4.7}$$

(Normally, we have to take a slightly smaller set than O; but we can take care of that problem by relabelling.) In (4.7), the constants C and C_2 are independent of A, ξ, t, a, b. If u, v lie in a bounded subset of the plane, and $\xi \in O$, then for n large enough, we write $\xi + \frac{u}{n} = \xi + \frac{\mathrm{Re}(u)}{n} + i\frac{\mathrm{Im}\ (u)}{n}$, where $\xi + \frac{\mathrm{Re}(u)}{n}$ is contained in a slightly large open set than O. Again, we relabel the open set O. Then, (4.7) takes the form

$$\frac{1}{n}\left|K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right)\right| \leq Ce^{C_2(|\mathrm{Im}\ u|+|\mathrm{Im}\ v|)}.$$

Recalling that $\mu' \sim 1$ in O, we have

$$\tilde{K}_n(\xi_n, \xi_n) = \mu'(\xi_n)K_n(\xi_n, \xi_n) \sim n.$$

Therefore, we see that for $|u|, |v| \leq A$ and $n \geq n_0(A)$

$$|f_n(u,v)| \leq C_1 e^{C_2(|\text{Im } u| + |\text{Im } v|)}, \tag{4.8}$$

where C_1, C_2 are independent of n, u, v, A; which yields the stated uniform boundedness.

(ii) By (4.8) $\{f_n(u,v)\}_{n=1}^{\infty}$ is a normal family of two variables u, v; i.e., the given family contains a subsequence which converges uniformly on compact subset of $\mathbb{C}$ to a continuous function. If $f(u,v)$ is the locally uniform limit through the subsequence $\mathcal{S}$ of integers, then $f(u,v)$ is an entire function in u, v satisfying for all complex u, v,

$$|f(u,v)| \leq C_1 e^{C_2(|\text{Im } u| + |\text{Im } v|)}. \tag{4.9}$$

In particular, $f(u,v)$ is bounded for $u, v \in \mathbb{R}$, and is an entire function of exponential type in each variable. $\square$

Lemma 4.3. *(i) Uniformly for $u \in \mathbb{R}$,*

$$f(u,u) \sim 1. \tag{4.10}$$

(ii) Assume

$$\lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_{\xi_n - \epsilon}^{\xi_n + \epsilon} |w(t) - w(\xi_n)| dt = 0. \tag{4.11}$$

Then for all $u \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} |f(u,s)|^2 ds \leq f(u, \bar{u}). \tag{4.12}$$

(iii) For each $a \in \mathbb{R}$, $f(a,.)$ has infinitely many real zeros.

Proof. (i) By (4.6), we have

$$C_1 \leq \frac{K_n(\xi_n + t, \xi_n + t)}{K_n(\xi_n, \xi_n)} \leq C_2$$

for some $\epsilon > 0$, for $|t| \leq \epsilon$, and for large n. Let us note that $w(\xi_n) \neq 0$ for any $\xi_n \in J$; and w is bounded in J. Then,

$$f(u,u) = \lim_{n\to\infty, n\in\mathcal{S}} \frac{K_n(\xi_n + \frac{u}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{u}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n, \xi_n)}$$

yields (4.10) for any $u \in \mathbb{R}$.

(ii) The identity

$$\begin{aligned} K_n(s,\bar{s}) &= \int_{\mathbb{R}} K_n(s,t)K_n(\bar{s},t)d\mu(t) \\ &= \int_{\mathbb{R}} K_n(s,t)K_n(\bar{s},\bar{t})d\mu(t) \\ &= \int_{\mathbb{R}} |K_n(s,t)|^2 d\mu(t), \end{aligned}$$

is valid for all $s \in \mathbb{C}$. Let $r > 0$. Using the above identity, we write;

$$\begin{aligned} 1 &\geq \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}} \frac{|K_n(s,t)|^2}{K_n(s,\bar{s})} w(t)dt \qquad (4.13) \\ &= w(\xi_n) \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}} \frac{|K_n(s,t)|^2}{K_n(s,\bar{s})} dt + \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n,\xi_n)}} \frac{|K_n(s,t)|^2}{K_n(s,\bar{s})} (w(t) - w(\xi_n))dt \\ &= I_1 + I_2. \end{aligned}$$

Applying Cauchy- Schwarz inequality and the upper bound (4.7),

$$\frac{|K_n(s,t)|^2}{K_n(s,\bar{s})} \leq K_n(t,t) \leq Cn. \qquad (4.14)$$

By (4.14) and (4.6),

$$\begin{aligned}|I_2| &\leq Cn\int_{\xi_n-\frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n+\frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}|w(t)-w(\xi_n)|dt\\&\leq \frac{C\tilde{K}_n(\xi_n,\xi_n)}{r}\int_{\xi_n-\frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n+\frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}|w(t)-w(\xi_n)|dt\\&\to 0,\ n\to\infty,\end{aligned}$$

by (4.11). We now do the following substitutions:

$$s=\xi_n+\tfrac{u}{\tilde{K}_n(\xi_n,\xi_n)},\quad t=\xi_n+\tfrac{y}{\tilde{K}_n(\xi_n,\xi_n)}.$$

Then,

$$I_1=\int_{-r}^{r}\left|\frac{K_n\left(\xi_n+\frac{u}{\tilde{K}_n(\xi_n,\xi_n)},\xi_n+\frac{y}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)}\right|^2\frac{K_n(\xi_n,\xi_n)}{K_n(s,\bar{s})}dy.$$

Since the integrand tends to the limit $\frac{|f(u,u)|^2}{f(u,\bar{u})}$ as $n\to\infty$ through $\mathcal{S}$, we have

$$\liminf_{n\to\infty,n\in\mathcal{S}} I_1\leq\int_{-r}^{r}\frac{|f(u,y)|^2}{f(u,\bar{u})}dy.$$

Substituting into (4.13) yields

$$1\geq\int_{-r}^{r}\frac{|f(u,y)|^2}{f(u,u)}dy.$$

Finally, we may let $r\to\infty$ to obtain the desired result.

(iii)We already showed that $f(a,.)$ has only real zeros in Lemma 4.2. These zeros of $f(a,.)$ will be named in the later results. Here, we write $f(a,.)$ as a product of its zeros.

By being the limit of polynomials with increasing degrees, $f(a,.)$ is non-constant, and not a polynomial. We also know that $f(a,.)$ is real on the real axis. Under the assumption (4.11) we have just shown that it belongs to $L_2(\mathbb{R})$. Moreover, $f(a,a) \neq 0$. By (4.9), we have

$$|f(a,t)| \leq Ce^{C_1(|\text{Im } a|+|\text{Im } t|)} \leq C$$

which yields that $f(a,.)$ belongs to the Cartwright class. We can then write [15, p. 130]

$$f(a,a+z) = f(a,a) \lim_{R\to\infty} \prod_{b:|b|<R \text{ and } f(a,a+b)=0} \left(1 - \frac{z}{b}\right).$$

□

4.3. Proof of Theorems 1.1 and 1.2

In this section we shall be concerned almost exclusively with the properties of the limit function $f(u,v)$. We begin with recalling that $f(a,.)$ is entire function of exponential type for each real a. Let us call it σ_a. Firstly, we show that this exponential type is independent of a; although it could possibly depend on $\{\xi_n\}$ and the subsequence $\mathcal{S}$.

Lemma 4.4. *For $a \in \mathbb{R}$, let $n(f(a,.),r)$ denote the number of zeros of $f(a,.)$ in the ball center 0, radius r, counting multiplicity. Then for any real a, we have as $r \to \infty$,*

$$n(f(a,.),r) - n(f(0,.),r) = O(1). \tag{4.15}$$

Consequently,

$$\sigma_a = \sigma_0 = \sigma, \text{ and }. \tag{4.16}$$

Moreover, for all $a \in \mathbb{R}$, $f(a,.) \in L^2_\sigma$.

Proof. In order to calculate the difference (4.15), we recall the proof of Theorem 2.18 which deals with the zeros of the function

$$\psi_n(\xi,t) = \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} K_n(\xi,t)(\xi - t) = p_n(t)p_{n-1}(\xi) - p_{n-1}(t)p_n(\xi).$$

Given $\xi \in \mathbb{R}$, with $p_{n-1}(\xi)p_n(\xi) \neq 0$, $\psi_n(\xi,t)$ has, as a function of t, simple zeros in each of the intervals

$$(x_{n,n}, x_{n-1,n}), (x_{n-1,n}, x_{n-2,n}), \ldots, (x_{2,n}, x_{1,n}).$$

There is just one zero lying outside $[x_{n,n}, x_{1,n}]$. When $p_{n-1}(\xi)p_n(\xi) = 0$, $\psi_n(\xi,t)$ is a multiple of p_n or p_{n-1}. As zeros of the latter polynomials interlace, then $\psi_n(\xi,t)$ has a simple zero in each of the intervals

$$[x_{n,n}, x_{n-1,n}), [x_{n-1,n}, x_{n-2,n}), \ldots, [x_{2,n}, x_{1,n}).$$

We then deduce that, regardless of ξ, the number j of zeros of $K_n(\xi,t)$ in $[x_{m,n}, x_{k,n}]$ satisfies

$$|j - (m-k)| \leq \begin{cases} 0 & \text{if} \quad p_{n-1}(\xi)p_n(\xi) \neq 0 \\ 1 & \text{if} \quad p_{n-1}(\xi)p_n(\xi) = 0 \end{cases}$$

Consider now

$$f_n(a,t) = K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right) / K_n(\xi_n,\xi_n)$$

and

$$f_n(0,t) = K_n\left(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right) / K_n(\xi_n,\xi_n)$$

as functions of t. In any fixed interval $[-r,r]$, it follows that the difference between the number of zeros of these two functions is at most 2. Letting $n \to \infty$ through $\mathcal{S}$ we obtain (4.15). Then (4.16) follows from (3.23). Finally, since $f(a,.) \in L_2(\mathbb{R})$ under the assumption (4.11), we have $f(a,.) \in L^2_\sigma$.

$\square$

Lemma 4.5. *Assume* (4.11). *Then we have for all* $a \in \mathbb{R}$*,*

$$\int_{\mathbb{R}} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin \sigma(a-s)}{\sigma(a-s)} \right)^2 ds \leq \frac{1}{f(a,a)} - \frac{\pi}{\sigma} \tag{4.17}$$

and

$$\sigma \geq \pi \sup_{a \in \mathbb{R}} f(a,a) \geq \pi. \tag{4.18}$$

Proof. In order to show (4.17), we firstly recall an important identity for L^2_σ that we will use. For any $g \in L^2_\sigma$, we have the following reproducing kernel identity;

$$g(x) = \int_{\mathbb{R}} g(t) \frac{\sin \sigma(x-t)}{\sigma(x-t)} dt, \quad x \in \mathbb{R}. \tag{4.19}$$

In particular, setting $g(t) = \frac{\sin \sigma(x-t)}{\sigma(x-t)}$, we shall also use,

$$\int_{\mathbb{R}} \left(\frac{\sin \sigma(x-s)}{\sigma(x-s)} \right)^2 ds = 1. \tag{4.20}$$

We now expand the square at the left hand side of (4.17), and see that

$$\frac{1}{f(a,a)^2} \int_{\mathbb{R}} f(a,s)^2 ds - \frac{2}{f(a,a)} \int_{\mathbb{R}} f(a,s) \frac{\sin \sigma(a-s)}{\sigma(a-s)} ds + \int_{\mathbb{R}} \left(\frac{\sin \sigma(a-s)}{\sigma(a-s)} \right)^2 ds$$
$$\leq \frac{1}{f(a,a)} - 2\frac{\pi}{\sigma} + \frac{\pi}{\sigma},$$

by (4.12), and the identities (4.19) and (4.20) which are applicable since $f(a,.) \in L^2_\sigma$.

Since the left-hand side of (4.17) is non-negative, we obtain for all real a,

$$\sigma \geq \pi f(a,a).$$

As $f(0,0) = 1$, we then obtain (4.18). □

We are now ready to consider the properties of the zeros of $f(0,z)$. Recall from Section 4.1, The Gauss type quadrature formula, with nodes $\{t_{jn}\}$ including the point $\xi = \xi_n$:

$$\sum_j \lambda_n(t_{jn})P(t_{jn}) = \int P(t)d\mu(t),$$

for all polynomials P of degree $\leq 2n-2$. Recall that we order the nodes as

$$\cdots < t_{-2,n} < t_{-1,n} < t_{0,n} = \xi_n < t_{1,n} < t_{2,n} < \cdots$$

and write

$$t_{jn} = \xi_n + \frac{\rho_{jn}}{\tilde{K}_n(\xi_n,\xi_n)} \tag{4.21}$$

where $\{\rho_{jn}\}_{j\neq 0}$ are the zeros of $f_n(0,t)$.

Lemma 4.6. *(i) For each fixed j, as $n \to \infty$ through $\mathcal{S}$,*

$$\rho_{jn} \to \rho_j,$$

where $\rho_0 = 0$ and

$$\cdots \leq \rho_{-2} \leq \rho_{-1} < 0 < \rho_1 \leq \rho_2 \leq \cdots .$$

(ii) There exists C_1 such that for all integers j,

$$\rho_j - \rho_{j-1} \leq C_1.$$

(iii) The function $f(0,z)$ has (possibly multiple) zeros at ρ_j, $j \neq 0$, and no other zeros.

Proof. (i), (iii) We already know that $f(0,z)$ has infinitely many zeros, and is not identically zero. We also know that $f_n(0,z) = K_n(\xi_n, \xi_n + \frac{z}{\tilde{K}_n(\xi_n,\xi_n)})/K_n(\xi_n,\xi_n)$ has simple zeros at ρ_{jn}, and no other zeros. Since $f_n(0,z)$ converges uniformly to $f(0,z)$ on compacta through the subsequence $\mathcal{S}$, the result then follows by Hurwitz' Theorem. (ii) We will use two results from [5]. By Markov- Stieltjes inequalities [5, p. 33, (5.10)]

$$\int_{t_{j-1,n}}^{t_{j,n}} w \leq \int_{t_{j-1,n}}^{t_{jn}} d\mu \leq \lambda_n(t_{j-1,n}) + \lambda_n(t_{jn})$$

holds true for any j. Fixing j, and using our upper bounds for the Christoffel function, valid in an open set containing J, and the fact that each point of J is a limit point of zeros [5, p. 67], we see that the last right hand side is $O(\frac{1}{n})$. Furthermore, by hypothesis, w is bounded below in an open set containing J. It follows that $n > n_0(j)$,

$$t_{j,n} - t_{j-1,n} \leq \frac{C}{n}.$$

We note that C does not depend on j, since it arises from the upper bound for the Christoffel functions and the lower bound for w. Then from (4.21) and (4.6), for $n \geq n_0(j)$,

$$\rho_{j,n} - \rho_{j-1,n} \leq C.$$

Letting $n \to \infty$ through $\mathcal{S}$ gives the result. $\square$

Lemma 4.7. *Assume* (4.11) *and let*

$$\Lambda = \sup_{x\in\mathbb{R}} f(x,x).$$

For each real a, $f(a,.)$ is entire of exponential type $\sigma = \pi\Lambda$.

Proof. In view of Lemma 4.4, it suffices to show that $f(0,.)$ is entire of exponential type $\sigma = \pi\Lambda$. To this end, we shall consider the zero distribution of $f(0,.)$. Here, we again use the Markov- Stieltjes inequalities: for each $1 \le l \le n$,

$$\sum_{j=1}^{l-1} \lambda_n(t_{j,n}) \le \int_{-\infty}^{t_{l,n}} d\mu \le \sum_{j=1}^{t_{l,n}} \lambda_n(t_{j,n}).$$

Writing the same equation for $k > l$, and subtracting the relevant parts of the inequalities,

$$\sum_{j=l+1}^{k-1} \lambda_n(t_{j,n}) \le \int_{t_{l,n}}^{t_{k,n}} d\mu(t).$$

Assume that $t_{l,n}, t_{k,n} \in O$. Then by the absolute continuity of μ in O, and the substitution $t = \xi_n + \frac{s}{\tilde{K}_n(\xi_n,\xi_n)}$, we obtain

$$\sum_{j=l+1}^{k-1} \frac{K_n(\xi_n,\xi_n)}{K_n(t_{j,n},t_{j,n})} \le \int_{\rho_{l,n}}^{\rho_{k,n}} \frac{w(\xi_n + \frac{s}{\tilde{K}_n(\xi_n,\xi_n)})}{w(\xi_n)} ds.$$

Notice that

$$\int_{\rho_{l,n}}^{\rho_{k,n}} \frac{w(\xi_n + \frac{s}{\tilde{K}_n(\xi_n,\xi_n)})}{w(\xi_n)} = \frac{\tilde{K}_n(\xi_n,\xi_n)}{w(\xi_n)} \int_{\xi_n + \frac{\rho_{l,n}}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{\rho_{k,n}}{\tilde{K}_n(\xi_n,\xi_n)}} w(t)dt \tag{4.22}$$

We now consider the limit

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{1}{w(\xi_n)} \tilde{K}_n(\xi_n,\xi_n) \int_{\xi_n + \frac{\rho_{l,n}}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{\rho_{k,n}}{\tilde{K}_n(\xi_n,\xi_n)}} w(t)dt.$$

Letting $\epsilon = 1/\tilde{K}_n(\xi_n,\xi_n)$ in (4.11), we deduce that

$$\lim_{n\to\infty} \left(\tilde{K}_n(\xi_n,\xi_n) \int_{\xi_n + \frac{\rho_{l,n}}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{\rho_{k,n}}{\tilde{K}_n(\xi_n,\xi_n)}} w(t)dt - \tilde{K}_n(\xi_n,\xi_n) \int_{\xi_n + \frac{\rho_{l,n}}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n + \frac{\rho_{k,n}}{\tilde{K}_n(\xi_n,\xi_n)}} w(\xi_n)dt \right) = 0.$$

So, the right-hand side in (4.22) will be equal to

$$\lim_{n\to\infty}\frac{1}{w(\xi_n)}\tilde{K}_n(\xi_n,\xi_n)\int_{\xi_n+\frac{\rho_{l,n}}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n+\frac{\rho_{k,n}}{\tilde{K}_n(\xi_n,\xi_n)}} w(\xi_n)dt = \lim_{n\to\infty, n\in\mathcal{S}}\tilde{K}_n(\xi_n,\xi_n)\frac{\rho_{k,n}-\rho_{l,n}}{\tilde{K}_n(\xi_n,\xi_n)}$$
$$= \rho_k - \rho_l$$

Next, for each fixed j, as $n\to\infty$ through $\mathcal{S}$,

$$\frac{K_n(t_{j,n},t_{j,n})}{K_n(\xi_n,\xi_n)} = \frac{K_n\left(\xi_n+\frac{\rho_{j,n}}{\tilde{K}_n(\xi_n,\xi_n)},\xi_n+\frac{\rho_{j,n}}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)} = f_n(\rho_{j,n},\rho_{j,n})$$
$$\to f(\rho_j,\rho_j).$$

Thus for each fixed k, l,

$$\sum_{j=l+1}^{k-1}\frac{1}{f(\rho_j,\rho_j)} \le \rho_k - \rho_l.$$

As $f(\rho_j,\rho_j)\le\Lambda$ for all j, we obtain

$$k-l-1 \le \Lambda(\rho_k-\rho_l).$$

In particularly, $\rho_{l+2}-\rho_l \ge C > 0$, so $f(0,.)$ has at most double zeros. Moreover, because $\{\rho_{j,n}\}$ are simple zeros of $f_n(0,.)$, ρ_k can only be a double zero of $f(0,.)$ if it is repeated in the sequence $\{\rho_j\}$. Then, in the interval $[\rho_l,\rho_k]$, the total number of zeros of $f(0,.)$, namely $k-l+1$ or $k-l+2$ or $k-l+3$, if 0 does not belong to $[k,l]$, and $k-l$ or $k-l+1$ or $k-l+2$ if it does, is at most $\Lambda(\rho_l-\rho_k)+4$. We now count the zeros of $f(0,.)$ in the interval $[-r,r]$ whose number is denoted by $n(r)$. Recall that $C_1 \le \rho_{j+2}-\rho_j \le C_2$ and the sequence of zeros $\{\rho_j\}$ is infinite. Therefore, we can pick a ρ_k which is bounded distance from r, and ρ_l a bounded distance from $-r$. We obtain that $n(r)$ is at most the number of zeros in $[\rho_l,\rho_k]$ plus $O(1)$, and hence at most

$\Lambda(\rho_l - \rho_k) + O(1)$. Thus,

$$n(r) \leq 2\Lambda r + O(1).$$

Then, recalling (3.23),

$$\frac{\sigma}{\pi} = \lim_{r\to\infty} \frac{n(r)}{2r} \leq \Lambda.$$

Together with our lower bound $\sigma \geq \pi\Lambda$ from Lemma 4.5, we obtain the result. $\square$

Remark Plugging $\frac{\sigma}{\pi} = \Lambda = \sup_{x\in\mathbb{R}} f(x,x)$ into the Lemma 4.5, we obtain

$$\int_{\mathbb{R}} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin\sigma(a-s)}{\sigma(a-s)} \right)^2 ds \leq \frac{1}{f(a,a)} - \frac{1}{\sup_{x\in\mathbb{R}}}.$$

In particular, if a attains the sup, so that $f(a,a) = \sup_{x\in\mathbb{R}} f(x,x)$, then for all $s \in \mathbb{R}$,

$$\frac{f(a,s)}{f(a,a)} = \frac{\sin\sigma(a-s)}{\sigma(a-s)}.$$

If the supremum is not attained at any finite point, then instead we obtain a sequence $\{a_k\}$ with $\lim_{k\to\infty} |a_k| = \infty$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}} \left(\frac{f(a_k,s)}{f(a_k,a_k)} - \frac{\sin\sigma(a_k-s)}{\sigma(a_k-s)} \right)^2 ds = 0.$$

Note that we have not used the hypothesis (1.2) yet.

4.3.1. Proof of Theorem 1.2

Noting that

$$\lim_{a\to b} \frac{\sin\pi(a-b)}{\pi(a-b)} = 1,$$

it is clear that (ii) implies (i). We now assume (i). Using the uniformity of (1.2) for $\xi \in J$, and the fact that $\tilde{K}_n(\xi_n, \xi_n) \sim n$, our hypothesis (1.2) implies also that

$$\lim_{n\to\infty} f_n(a,a) = \lim_{n\to\infty} \frac{K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)} = 1,$$

and by the uniformity in a,

$$f(a,a) = 1 \text{ for all real a.}$$

Thus,

$$\Lambda = \sup_{x\in\mathbb{R}} f(x,x) = 1.$$

Together with Lemma 4.7 this yields that , for each fixed a, $f(a,.)$ is entire of exponential type $\sigma = \pi$. By (4.17), we then obtain, for each real a,

$$\int_{\mathbb{R}} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin \pi(s-a)}{\pi(s-a)} \right)^2 ds = 0,$$

that is

$$f(a,s) = \frac{\sin \pi(s-a)}{\pi(s-a)}.$$

for all $a, s \in \mathbb{R}$. Using the uniqueness theorem for entire functions, we get

$$f(a,b) = \frac{\sin \pi(a-b)}{\pi(a-b)}$$

for all $a, b \in \mathbb{C}$. All in all,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{K_n(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n,\xi_n)} = \frac{\sin \pi(a-b)}{(a-b)},$$

uniformly for a, b in compact subsets of the plane. Since the limit function is independent of the subsequence $\mathcal{S}$, we obtain

$$\lim_{n\to\infty} \frac{K_n(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n, \xi_n)} = \frac{\sin \pi(a-b)}{(a-b)},$$

uniformly for a, b in compact subsets of the plane. Finally as $\{\xi_n\}$ can be any sequence in J, the conclusion (1.3) holds true uniformly for $\xi \in J$.

4.3.2. Proof of Theorem 1.1

Since w is assumed to be continuous at each point $\xi \in J$, we obtain (1.4) immediately; and the uniformity of (1.4) follows easily from the continuity of w (regarded as a function defined on all of $\operatorname{supp}(\mu)$) at each point of compact J.

4.4. de Branges Spaces of Entire Functions Associated with General Measure μ

In this section, we shall prove four general theorems. Throughout the entire section, we do not assume the hypotheses of Theorem 1.4. In particular, the measure μ may not be absolutely continuous in some open set, nor bounded in its support unless otherwise stated. Therefore, we may need to prove the same results that we have proven in the previous section.

Firstly, recall the notation

$$f_n(a, b) = \frac{K_n(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n, \xi_n)}.$$

Theorem 4.8. *Let μ be a measure with support on the real line, with all power moments $\int x^j d\mu(x)$, $j \geq 0$ finite, and with infinitely many points in its support. Let $\{\xi_n\}$ be a sequence of real numbers. Assume that there is a non-real complex number a, and an* *infinite sequence of integers $\mathcal{S}$, for which there exists*

$$f(a,z) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(a,z), \tag{4.23}$$

uniformly in compact subsets of $\mathbb{C}$, and that

$$f(a,\overline{a}) \neq 0. \tag{4.24}$$

Then

(i) *There exists, for all $z, v \in \mathbb{C}$,*

$$f(z,v) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(z,v),$$

and the limit is uniform for z, v in compact subsets of $\mathbb{C}$.

(ii) *Let*

$$L(z,v) = (z-v)f(z,v).$$

For all complex α, β, z, v,

$$L(z,v)L(\alpha,\beta) = L(\alpha,z)L(\beta,v) - L(\beta,z)L(\alpha,v). \tag{4.25}$$

(iii) *Let* $\operatorname{Im} a > 0$*. Then for* $\operatorname{Im} z > 0$*,*

$$\begin{aligned} |f(\overline{a},z)| &\geq |f(a,z)|; \\ |L(\overline{a},z)| &> |L(a,z)|. \end{aligned} \tag{4.26}$$

In particular, for $\operatorname{Im} z > 0$*,*

$$|L(z,\overline{z})| > 0 \quad \textit{and} \quad f(z,\overline{z}) > 0. \tag{4.27}$$

(iv) If $f(z,v)=0$*, then* $\operatorname{Im} z$ *and* $\operatorname{Im} v$ *have the same sign. In particular,* $\operatorname{Im} z>0 \Rightarrow \operatorname{Im} v>0$*. Consequently, for* $\operatorname{Im} a>0$*,* $L(\overline{a},.)\in\overline{HB}$*.*

Proof. Recall from (4.1) and (4.3) that we set

$$L_n(u,v)=(u-v)K_n(u,v)$$

and

$$\begin{aligned}\tilde{L}_n(a,b)&=(a-b)f_n(a,b)\\&=(a-b)\frac{K_n(\xi_n+\frac{a}{\tilde{K}_n(\xi_n,\xi_n)},\xi_n+\frac{b}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n,\xi_n)}\\&=\frac{(a-b)}{K_n(\xi_n,\xi_n)}K_n(\xi_n+\frac{a}{\tilde{K}_n(\xi_n,\xi_n)},\xi_n+\frac{b}{K_n(\xi_n,\xi_n)})\\&=\mu_n'(\xi_n)L_n(\xi_n+\frac{a}{\tilde{K}_n(\xi_n,\xi_n)},\xi_n+\frac{b}{K_n(\xi_n,\xi_n)}).\end{aligned}$$

By expanding $L_n(z,v)$ using Christoffel- Darboux formula, the following formula is obtained for all z,v,α,β after the usual calculations:

$$L_n(z,v)L_n(\alpha,\beta)=L_n(\alpha,z)L_n(\beta,v)-L_n(\beta,z)L_n(\alpha,v).$$

Clearly, we have the same formula for $\tilde{L}_n(z,v)$; i.e.

$$\tilde{L}_n(z,v)\tilde{L}_n(\alpha,\beta)=\tilde{L}_n(\alpha,z)\tilde{L}_n(\beta,v)-\tilde{L}_n(\beta,z)\tilde{L}_n(\alpha,v)$$

for all z,v,α,β. Putting $\alpha=a$ and $\beta=\bar{a}$, we get

$$\tilde{L}_n(z,v)\tilde{L}_n(a,\bar{a})=\tilde{L}_n(a,z)\tilde{L}_n(\bar{a},v)-\tilde{L}_n(\bar{a},z)\tilde{L}_n(a,v) \tag{4.28}$$

Let us also recall that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ satisfies the conjugate relation $f_n(\bar{a},z)=\overline{f_n(a,\bar{z})}$ and the symmetry $f_n(a,b)=f_n(b,a)$. We now use our hypothesis

(4.23) to obtain

$$\lim_{n\to\infty, n\in\mathcal{S}} \tilde{L}_n(a,z) = (a-z)f(a,z) = L(a,z);$$
$$\lim_{n\to\infty, n\in\mathcal{S}} \tilde{L}_n(\bar{a},z) = (\bar{a}-z)f(\bar{a},z) = L(\bar{a},z);$$
$$\lim_{n\to\infty, n\in\mathcal{S}} \tilde{L}_n(a,\bar{a}) = L(a,\bar{a}).$$

Note that

$$L(a,\bar{a}) = 2i(\mathrm{Im}\ a)f(a,\bar{a}) \neq 0,$$

by our hypothesis (4.24). Then using (4.28), we obtain

$$\lim_{n\to\infty, n\in\mathcal{S}} \tilde{L}_n(z,v) = \frac{1}{L(a,\bar{a})}[L(a,z)L(\bar{a},v) - L(\bar{a},z)L(a,v)]$$

for z, v in compact subsets of $\mathbb{C}$. This then yields that the limit

$$f(z,v) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(z,v) = \frac{L(a,z)L(\bar{a},v) - L(\bar{a},z)L(a,v)}{L(a,\bar{a})(z-v)}, \tag{4.29}$$

exists uniformly for z, v in compact sets with $z \neq v$. We notice that

$$\lim_{z\to v}(z-v)f(z,v) = \lim_{z\to v} \frac{L(a,z)L(\bar{a},v) - L(\bar{a},z)L(a,v)}{L(a,\bar{a})} = 0,$$

and conclude that the singularity of the right hand side of (4.29) is removable. Therefore, the limit in (4.29) is uniform for z, v on compacta.

(ii) This follows from (4.29) and part (i).

To prove (iii) and (iv), we will need the following lemma.

Lemma 4.9. *(i) If* $K_n(z,w) = 0$*, then* $\mathrm{Im}\ z$ *and* $\mathrm{Im}\ w$ *have the same sign. In particular,* $\mathrm{Im}\ z > 0 \Rightarrow \mathrm{Im}\ w > 0$.

(ii) Let $\mathrm{Im}\ a > 0$. *Then for* $\mathrm{Im}\ z \geq 0$,

$$|K_n(\bar{a}, z)| \geq |K_n(a, z)|; \tag{4.30}$$

$$|L_n(\bar{a}, z)| \geq |L_n(a, z)|. \tag{4.31}$$

In particular, $L_n(\bar{a}, .) \in \overline{HB}$.

Proof. When z is real, then all zeros of $K_n(z, .)$ are real [5, p.19]. In this case, Im $z = \mathrm{Im}\ w = 0$. Before the general case, we show the following equality.

$$\begin{aligned} L_n(u, v) &= (u - v)K_n(u, v) \\ &= \frac{\gamma_{n-1}}{\gamma_n} p_n(u)p_n(v)[G_n(v) - G_n(u)] \end{aligned}$$

where

$$G_n(u) = \frac{p_{n-1}(u)}{p_n(u)} = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=1}^{n} \frac{\lambda_n(x_{j,n})p_{n-1}^2(x_{j,n})}{u - x_{j,n}}.$$

Clearly, $G_n(u) = \frac{p_{n-1}(u)}{p_n(u)}$. To prove the second inequality, we apply the Lagrange interpolation to p_{n-1} at the zeros of p_n, namely at the points $\{x_{j,n}\}$. Then,

$$p_{n-1}(z) = \sum_{j=1}^{n} \frac{p_{n-1}(x_{j,n})p_n(z)}{p_n'(x_{j,n})(z - x_{j,n})}. \tag{4.32}$$

By the Christoffel-Darboux formula, we have

$$\lambda_n^{-1}(x) = K_n(x, x) = \frac{\gamma_{n-1}}{\gamma_n}(p_n'(x)p_{n-1}(x) - p_n(x)p_{n-1}'(x)).$$

Setting $x = x_{j,n}$,

$$\lambda_n^{-1}(x_{j,n}) = \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_{j,n})p_{n-1}(x_{j,n}).$$

Substituting this into (4.32), we obtain the formula for $L_n(u,v)$. Now, $K_n(z,w) = 0$ for non-real z yields that

$$G_n(z) = G_n(w),$$

as $p_n p_{n-1}$ has only real zeros. Taking the imaginary parts of the above equation, we get

$$(\text{Im } z) \sum_{j=1}^{n} \frac{\lambda_n(x_{j,n}) p_{n-1}^2(x_{j,n})}{|z - x_{j,n}|^2} = (\text{Im } w) \sum_{j=1}^{n} \frac{\lambda_n(x_{j,n}) p_{n-1}^2(x_{j,n})}{|w - x_{j,n}|^2}.$$

Since both sums are positive, the result follows.

(ii) We will consider the function

$$h(z) := K_n(a,z)/K_n(\bar{a},z)$$

is analytic for z in the closed upper-half plane, and on the real axis

$$|h(x)| = 1.$$

Moreover, the coefficients of the Taylor expansion about 0 of

$$K_n(\bar{a},z) = \sum_{k=0}^{n} p_k(\bar{a}) p_k(z),$$

are the conjugates of

$$K_n(a,z) = \sum_{k=0}^{n} p_k(z).$$

Then, as $z \to \infty$, $|h(z)| \to 1$. Therefore,

$$|h(z)| \leq 1 \quad \text{for} \quad \text{Im } z \geq 0,$$

by the maximum-modulus principle. This shows (4.30); and (4.31) follows from the observation that

$$|\bar{a} - z| \geq |a - z| \quad \text{for Im } z \geq 0.$$

□

(iii), (iv) By taking limits in the previous lemma, we get that

$$|f(\bar{a}, z)| \geq |f(a, z)| \quad \text{and} \quad |L(\bar{a}, z)| \geq |L(a, z)|, \tag{4.33}$$

for Im $z \geq 0$. Before we show the strict inequality in the latter inequality, we show the assertion on zeros. Suppose that Im $v > 0$ and $f(z, v) = 0$. By Hurwitz's Theorem, there exist $\{z_n\}$ with $f_n(z_n, v) = 0$ and

$$\lim_{n \to \infty, n \in \mathcal{S}} z_n = z.$$

Lemma 4.9i then yields that Im $z_n > 0$ which in turn implies that Im $z \geq 0$. We now prove that it is actually positive. Assume Im $z = 0$. By (4.29),

$$\begin{aligned} 0 &= L(a, z)L(\bar{a}, v) - L(\bar{a}, z)L(a, v) \\ &= L(a, z)L(\bar{a}, v) - \overline{L(a, z)}L(a, v). \end{aligned} \tag{4.34}$$

Define

$$h(u) = \frac{L(a, u)}{L(\bar{a}, u)} \text{ for Im } u \geq 0.$$

Notice that h is meromorphic in the upper-half plane which satisfies

$$|h(u)| \leq 1 \text{ for Im } u \geq 0$$

by (4.33) except perhaps at isolated poles which are in fact removable singularities because of the local boundedness of h. Therefore, h is analytic in the upper half plane. Moreover, $|h(x)| = 1$ for real x after we removed any possible isolated singularities on $\mathbb{R}$. Also,

$$|h(v)| = \left|\frac{L(a,v)}{L(\bar{a},v)}\right| = \left|\frac{L(a,z)}{\overline{L(a,z)}}\right| = 1$$

by (4.34). Since Im $v > 0$, the maximum-modulus principle shows that $h = c$ in the upper half plane, for some unimodular constant c. Then (4.29) takes the following form:

$$f(u,v) = \frac{cL(\bar{a},u)L(\bar{a},v) - L(\bar{a},u)cL(\bar{a},v)}{L(a,\bar{a})(u-v)} = 0,$$

for all u, v in the upper half plane. But this contradicts with the fact that $f(0,0) = 1$. So, Im $z > 0$, as desired.

We now show that $|L(\bar{a},z)| > |L(a,z)|$ for Im $z > 0$. Reminding that we have (4.33), suppose that we have equality $|L(\bar{a},z)| = |L(a,z)|$ for some z in the open upper half plane. Since we will follow the previous argument, we form

$$h(u) = \frac{L(a,u)}{L(\bar{a},u)},$$

which is analytic in the upper half plane, and has $|h(u)| \leq 1$ there. According to our assumption on z, $|h(z)| = 1$. Then, by the maximum-modulus principle, $h = c$ for some uni-modular constant c. As above, this yields a contradiction. □

Theorem 4.10. *Assume the hypotheses of Theorem* 4.8. *Fix* a *with* Im $a > 0$, *and let*

$$E_a(z) = \sqrt{2\pi}\frac{L(\bar{a},z)}{|L(a,\bar{a})|^{1/2}}.$$

(i) Then all zeros of E_a lie in the lower half plane, and $E_a \in \overline{HB}$. Moreover,

$$f(z,\bar{\zeta}) = \frac{i}{2\pi}\frac{E_a(z)\overline{E_a(\zeta)} - E_a^*(z)\overline{E_a^*(\zeta)}}{z-\bar{\zeta}}. \tag{4.35}$$

(ii) For all $g \in \mathcal{H}(E_a)$, and all $z \in \mathbb{C}$, we have

$$g(z) = \int_{\mathbb{R}} g(t)\frac{\overline{f(z,t)}}{|E_a(t)|^2}dt. \tag{4.36}$$

Moreover, $f(z,.) \in \mathcal{H}(E_a)$ for all $z \in \mathbb{C}$.

(iii) For any a,b with Im $b > 0$, *$\mathcal{H}(E_a) = \mathcal{H}(E_b)$ and the norms $||.||_{E_a}$ and $||.||_{E_b}$ are equivalent.*

Proof. (Theorem 4.10) (i) By Theorem 4.8iv we know that all zeros of E_a must lie in the open lower half-plane. Besides,

$$|E_a(z)| > |E_a(\bar{z})| \text{ for } \mathrm{Im}\ z > 0,$$

by (4.26). Therefore, $E_a \in \overline{HB}$. We also notice that

$$L(a,\bar{a}) = 2i(\mathrm{Im}\ a)f(a,\bar{a}) = i|L(a,\bar{a})|,$$

so the functional equation (4.25) yields

$$L(z,\bar{\zeta})i|L(a,\bar{a})| = L(a,z)L(\bar{a},\bar{\zeta}) - L(\bar{a},z)L(a,\bar{\zeta});$$

and thus in turn,

$$f(z,\bar{\zeta}) = \frac{i(L(\bar{a},z)\overline{L(\bar{a},\zeta)} - \overline{L(\bar{a},\bar{z})L(\bar{a},\bar{\zeta})})}{(z-\zeta)|L(a,\bar{a})|}.$$

Plugging the definition of E_a, and recalling that $E_a^*(z) = \overline{E_a^*(\bar{z})}$, we obtain

$$f(z,\bar{\zeta}) = \frac{i}{2\pi}\frac{(E_a(z)\overline{E_a(\zeta)} - E_a^*(z)E_a^*(\zeta))}{(z-\zeta)}.$$

(ii) As we showed above, $E_a \in \overline{HB}$; and so $\mathcal{H}(E_a)$ is well-defined. By (1.7) and (4.35) the reproducing kernel $\mathcal{K}$ of the space $\mathcal{H}(E_a)$ is given by the formula

$$\frac{i}{2\pi}\frac{(E_a(z)\overline{E_a(\zeta)} - E_a^*(z)E_a^*(\zeta))}{(z-\zeta)}.$$

Hence, $f(z,\bar{\zeta}) = \mathcal{K}(\zeta, z)$; and (1.8) gives (4.36). As outlined in de Brange's theory $f(z,.) \in \mathcal{H}(E_a)$.

We will need the following lemma for part (iii):

Lemma 4.11. *(i) For* $\mathrm{Im}\ a > 0$, $\mathrm{Im}\ b > 0$ *and* $\mathrm{Im}\ z \geq 0$,

$$\left|\frac{L(z,\bar{b})}{L(z,\bar{a})}\right| \leq 2\left|\frac{L(a,\bar{b})}{L(a,\bar{a})}\right|. \tag{4.37}$$

(ii) For all $u, v \in \mathbb{C}$,

$$|f(u,v)|^2 \leq f(u,\bar{u})f(v,\bar{v}). \tag{4.38}$$

(iii) For all $a, b \in \mathrm{R}$, *with* $L(a,b) \neq 0$, *and all* $z \in \mathbb{C}$,

$$f(z,\bar{z}) \leq \left(\frac{|b-z|}{|\mathrm{Im}\ z|}\frac{|L(a,z)|}{|L(a,b)|}\right)^2 f(b,b). \tag{4.39}$$

Proof. (i) The functional equation (4.25) gives

$$L(z,\bar{b})L(a,\bar{a}) = L(a,z)L(\bar{a},\bar{b}) - L(\bar{a},z)L(a,\bar{b}).$$

As we showed in the Theorem 4.8iii; if Im $z \geq 0$, then $|L(a,z)| \leq |L(\bar{a},z)|$ and

$$\begin{aligned}|L(\bar{a},\bar{b})| &= |(\bar{a}-\bar{b})f(\bar{a},\bar{b})| \\ &= |a-b||\overline{f(a,b)}| \\ &= |L(a,b)| \\ &\leq |L(\bar{a},b)| \\ &= |L(a,\bar{b})|.\end{aligned}$$

Thus,

$$|L(z,\bar{b})L(a,\bar{a})| \leq 2|L(\bar{a},z)L(a,\bar{b})|.$$

(ii) By Cauchy-Schwarz inequality,

$$|K_n(z,w)|^2 \leq K_n(z,\bar{z})K_n(w,\bar{w}).$$

After division by $K_n(\xi_n,\xi_n) > 0$, we obtain

$$|f_n(u,v)|^2 \leq f_n(u,\bar{u})f_n(v,\bar{v})$$

Letting $n \to \infty$ through $\mathcal{S}$, we get the desired result.

(iii) Let $a, b \in \mathbb{R}$. (4.25) gives

$$L(z,\bar{z})L(a,b) = L(a,z)L(b,\bar{z}) - L(b,z)L(a,\bar{z}).$$

We have $a, b \in \mathbb{R}$. By part (ii), we write

$$
\begin{aligned}
L(z,\bar{z})L(a,b) &= 2|\mathrm{Im}\ z||f(z,\bar{z})| \\
&\leq 2|L(a,z)||L(b,z)| \\
&\leq 2|L(a,z)||b-z|f(b,b)^{1/2}f(z,\bar{z})^{1/2}.
\end{aligned}
$$

□

We now prove Theorem (4.10)(*iii*): From (i) of the previous lemma, we see that for all $z \in \mathbb{C}^+$,

$$
\begin{aligned}
\left|\frac{E_b(z)}{E_a(z)}\right| &= \frac{\left|L(\bar{b},z)\right|}{\left|L(b,\bar{b})\right|^{1/2}}\frac{|L(a,\bar{a})|^{1/2}}{|L(\bar{a},z)|} \\
&\leq 2\frac{|L(a,\bar{b})|}{|L(a,\bar{a})|^{1/2}|L(b,\bar{b})|^{1/2}}.
\end{aligned}
\tag{4.40}
$$

Similarly,

$$
\left|\frac{E_a(z)}{E_b(z)}\right| \leq 2\frac{|L(b,\bar{a})|}{|L(b,\bar{b})|^{1/2}|L(a,\bar{a})|^{1/2}}.
\tag{4.41}
$$

Recall that the denominators are positive by (4.27). To show that $\mathcal{H}(E_a) = \mathcal{H}(E_b)$, we pick $g \in \mathcal{H}(E_b)$. Then $g/E_b, g^*/E_b \in H^2(\mathbb{C}^+)$. By the inequality (4.40), $g/E_a, g^*/E_a \in H^2(\mathbb{C}^+)$. So, $\mathcal{H}(E_a) \supseteq \mathcal{H}(E_b)$,. The converse inclusion follows from the other inequality (4.41). These two inequalities also yield the norm equivalence of $\mathcal{H}(E_a)$ and $\mathcal{H}(E_b)$, i.e., for all g, we have

$$
||g||_{E_b} \leq 2\frac{|L(a,\bar{b})|}{|L(a,\bar{a})|^{1/2}|L(b,\bar{b})|^{1/2}}||g||_{E_a}
$$

and the reverse inequality. □

Theorem 4.12. *Assume the hypotheses of Theorem* (4.8)*. Fix a with* $\mathrm{Im}\ a > 0$*.*

(i) Let

$$F(z) = L(z,0) = zf(0,z),$$

and let $\{\rho_j\}$ *be the zeros* ρ *of* F *for which* $f(\rho,\rho) \neq 0$. *These are all real and simple.*

(ii) The set $\left\{ \frac{f(\rho_j,.)}{\sqrt{f(\rho_j,\rho_j)}} \right\}_j$ *is an orthonormal sequence in* $\mathcal{H}(E_a)$ *and for all* $g \in \mathcal{H}(E_a)$,

$$\sum_j \frac{|g(\rho_j)|^2}{f(\rho_j,\rho_j)} \leq \int \left| \frac{g}{E_a} \right|^2, \tag{4.42}$$

while

$$G[g] = \sum_j g(\rho_j) \frac{f(\rho_j,z)}{f(\rho_j,\rho_j)} \in \mathcal{H}(E_a).$$

(iii) Assume that $F \notin \mathcal{H}(E_a)$. *Then for all* $g, h \in \mathcal{H}(E_a)$, *we have*

$$\int_{\mathbb{R}} \frac{g\bar{h}}{|E_a|^2} = \sum_j \frac{(g\bar{h})(\rho_j)}{f((\rho_j,\rho_j))}, \tag{4.43}$$

and

$$G[g] = g.$$

Proof. (i) Since $zf_n(0,z)$ has only real zeros, its uniform limit as $n \to \infty$ through $\mathcal{S}$ cannot have any non-real zeros. Therefore, F has only real zeros. Remember that, for any $E \in \overline{HB}$ we define a phase function on $\mathbb{R}$ by

$$E(x) = |E(x)|e^{-i\phi(x)}.$$

Since $E_a \in \overline{HB}$, we have a corresponding phase function ϕ defined for real x by

$$E_a(x) = |E_a(x)|e^{-i\phi(x)}.$$

Then, by (4.35), for real x,

$$\begin{aligned} F(x) &= xf(x,0) \\ &= \frac{i}{2\pi}(E_a(x)\overline{E_a(0)} - E_a^*(x)\overline{E_a^*(0)}) \\ &= \frac{1}{\pi}|E_a(x)||E_a(0)|\sin(\phi(x) - \phi(0)). \end{aligned} \tag{4.44}$$

Taking the derivative of the last formula,

$$\begin{aligned} F'(x) = \frac{1}{\pi}&\left(\frac{d}{dx}|E_a(x)|\right)|E_a(0)|\sin(\phi(x) - \phi(0)) \\ &+ \frac{1}{\pi}|E_a(x)||E_a(0)|\cos(\phi(x) - \phi(0))\phi'(x). \end{aligned} \tag{4.45}$$

By Theorem 4.10(a), E_a has non-real zeros. It then follows from (4.44) that $F(x) = 0$ if and only if $\sin(\phi(x) - \phi(0)) = 0$.

Let $\alpha = \phi(0)$ and recall that the sequence $\{s_j\}$ were defined at (3.14) by $\phi(s_j) = \alpha + j\pi$, $j \in \mathbb{Z}$. After reordering, we see that the $\{\rho_j\}$ are just $\{s_k\}$. Suppose ρ_j is not simple for some j. Then, it follows from (4.44) and (4.45) that both $\phi(\rho_j) = \alpha + k\pi$ for some k, and $\phi'(\rho_j) = 0$. Using $\mathcal{K}(\zeta, z) = f(z, \bar{\zeta})$ in (3.13) we obtain that

$$f(\rho_j, \rho_j) = \frac{1}{\pi}\phi'(\rho_j)|E_a(\rho_j)|^2 = 0,$$

which contradicts to our hypothesis that $f(\rho_j, \rho_j) \neq 0$. Thus, all zeros $\{\rho_j\}$ are simple of F with $f(\rho_j, \rho_j) \neq 0$ are simple.

(ii) Recall that we set $\alpha = \phi(0)$. Putting this into (4.35),

$$\begin{aligned} F(z) &= zf(z,0) \\ &= \frac{i}{2\pi}(E_a(z)\overline{E_a(0)} - E_a^*(z)\overline{E_a^*(0)}) \\ &= C(e^{i\alpha}E_a(z) - e^{-i\alpha}E_a^*(z)). \end{aligned} \tag{4.46}$$

Since E_a and E_a^* have zeros in opposite half-planes, the constant in (4.46) is non-zero. Here, we do not know that if $e^{i\alpha}E_a(z) - e^{-i\alpha}E_a^*(z)$ belongs to $\mathcal{H}(E_a)$ or not; otherwise we could simply apply de Branges theory. We now turn back to the zeros $\{\rho_{j,n}\}$ of f_n. Here, we will use the following result that we mentioned earlier; if $j' \neq k'$,

$$f_n(\rho_{j',n}, \rho_{k',n}) = \frac{K_n(t_{j',n}, t_{k',n})}{K_n(\xi_n, \xi_n)} = 0. \tag{4.47}$$

Taking limits through $\mathcal{S}$ on some appropriate subsequences $\{j'(n)\}$, $\{k'(n)\}$ whose terms satisfy (4.47) ; and using Hurwitz' Theorem, leads to

$$f(\rho_j, \rho_k) = 0, \qquad j \neq k. \tag{4.48}$$

The reproducing kernel relation (4.36) gives

$$0 = \int_{\mathbb{R}} \frac{f(t,\rho_j)f(t,\rho_k)}{|E_a(t)|^2} dt.$$

Note that $\{\frac{f(\rho_k,.)}{\sqrt{f(\rho_k,\rho_k)}}\}_k$ is an orthonormal sequence in $\mathcal{H}(E_a)$. Let $e_k = \frac{f(\rho_k,.)}{\sqrt{f(\rho_k,\rho_k)}}$. For all $g \in \mathcal{H}(E_a)$, we have the orthonormal expansion

$$\begin{aligned} \sum_j \frac{< g, e_j >}{||e_j||^2} e_j &= \sum_j \frac{g(\rho_j)}{\sqrt{f(\rho_j,\rho_j)}} \frac{f(\rho_j, z)}{\sqrt{f(\rho_j,\rho_j)}}, \\ &= G[g](z). \end{aligned}$$

by (4.36) and the orthonormality of the sequence $\{e_k\}_k$. Using Bessel's inequality,

$$\sum_j | < g, e_j > |^2 = \sum_j \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)}$$
$$\leq ||g||^2_{E_a} = \int_{\mathbb{R}} \left| \frac{g(t)}{E_a(t)} \right|^2 dt. \tag{4.49}$$

Clearly, every partial sum of $G[g]$ lies in $\mathcal{H}(E_a)$. Moreover, (4.49) yields the convergence of the series in the norm of $\mathcal{H}(E_a)$:

$$\begin{aligned}\left\| \sum_{j=1}^{j=N} < g, e_j > e_j - G[g] \right\|^2_{E_a} &= \left\| \sum_{j=N+1}^{\infty} < g, e_j > e_j \right\|^2_{E_a} \\ &= \int_{\mathbb{R}} \frac{| \sum_{j=N+1}^{\infty} < g, e_k > e_k(t)|^2}{|E_a(t)|^2} \\ &\leq \int_{\mathbb{R}} \sum_{j=N+1}^{\infty} | < g, e_j > |^2 \frac{|e_k(t)|^2}{|E_a(t)|^2} \\ &= \sum_{j=N+1}^{\infty} | < g, e_j > |^2 \longrightarrow 0\end{aligned}$$

as N tends to infinity. Since $\mathcal{H}(E_a)$ is a Hilbert space, $G[g] \in \mathcal{H}(E_a)$.

(iii) By hypothesis, $F \notin \mathcal{H}(E_a)$. (4.46) then shows that

$$e^{i\alpha} E_a(z) - e^{-i\alpha} E_a^*(z) \notin \mathcal{H}(E_a).$$

Recalling that $\alpha = \phi(0)$; and that we identified the zeros $\{\rho_j\}$ with $\{s_j\}$, we apply (3.15) and (3.16). □

Theorem 4.13. *Assume, in addition to the hypothesis of Theorem (4.8), that $f(a, .)$ is an entire function of exponential type σ and*

$$f(t, t) \sim 1 \text{ for } t \in \mathbb{R}. \tag{4.50}$$

(i) Then for all complex b, $f(b, .)$ is an entire function of exponential type σ.

(ii) For all $g \in PW_\sigma$,

$$g = G[g] \in \mathcal{H}(E_a).$$

In particular,

$$PW_\sigma \in \mathcal{H}(E_a).$$

(iii) Assume that there exists $C_0 > 0$ such that for a.e. $t \in \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{\mu'(\xi_n + \frac{t}{\tilde{K}_n}(\xi_n, \xi_n))}{\mu'(\xi_n)} \geq C_0, \tag{4.51}$$

or, assume that for each $r > 0$,

$$\lim_{n\to\infty} \int_{-r}^{r} \left| \frac{\mu'(\xi_n + \frac{t}{\tilde{K}_n}(\xi_n, \xi_n))}{\mu'(\xi_n)} - 1 \right| dt = 0 \tag{4.52}$$

Then

$$PW_\sigma = \mathcal{H}(E_a).$$

We do not assume that μ is absolutely continuous in the above result. Recall that

$$L_n(u, v) = (u - v)K_n(u, v)$$

and

$$\begin{aligned}\tilde{L}_n(a, b) &= (a - b)f_n(a, b)\\ &= \mu'(\xi_n)L_n\left(\xi_n + \frac{a}{\tilde{K}_n}(\xi_n, \xi_n), \xi_n + \frac{b}{\tilde{K}_n}(\xi_n, \xi_n)\right).\end{aligned}$$

Proof. (i) Suppose $f(a, z)$ is of exponential type σ for some a, with $\operatorname{Im} a > 0$. Clearly,

$L(a,z) = (z-a)f(a,z)$ is of exponential type σ as well. By conjugate symmetry,

$$f(\bar{a},z) = \overline{f(a,\bar{z})}$$

the same is true for $f(\bar{a},z)$ and $L(\bar{a},z)$. By (4.37), we have

$$|L(z,\bar{b})| \leq 2\frac{|L(a,\bar{b})|}{|L(a,\bar{a})|}|L(z,\bar{a})| \tag{4.53}$$

when Im $b > 0$, Im $z \leq 0$. Also, by Theorem 4.8iii,

$$|L(\bar{z},\bar{b})| = |L(z,b)| \leq |L(z,\bar{b})|. \tag{4.54}$$

We then deduce that the exponential type of $L(\bar{b},.)$ is no greater than that of $L(\bar{a},.)$. The same therefore holds true for $f(\bar{b},.)$ and $f(\bar{a},.)$, and also for the couple $f(b,.)$ and $f(a,.)$. Note that the reverse assertion too holds true since the inequalities (4.53) and (4.54) are symmetric in a and b. Recalling conjugate symmetry once again, the statement generalizes to the cases Im $a < 0$ or Im $b < 0$. Therefore, for any non-real number b, $f(b,.)$ has exponential type σ.

It remains to show that if b is real, $f(b,z)$ is of exponential type σ. Since $L(b,z)$ and $f(b,z)$ are of same type, it suffices to show the result for $L(b,z)$. Let $\alpha, \beta \in \mathbb{C}\setminus\mathbb{R}$, and $L(\alpha,\beta) \neq 0$. From the functional equation (4.25),

$$\begin{aligned} |L(b,z)| &= |L(z,b)| \\ &= \frac{1}{|L(\alpha,\beta)|}|L(\alpha,z)L(\beta,b) - L(\beta,z)L(\alpha,b)|. \end{aligned}$$

As both $L(\alpha,z)$ and $L(\beta,z)$ are of exponential type σ, it follows that $L(b,z)$ is of type at most σ. To show the reverse inequality, let c be a real number with $L(b,c) \neq 0$, and

d be non-real. We will use (4.38) and (4.39):

$$\begin{aligned}|f(d,z)| &\leq f(d,\bar{d})^{1/2} f(z,\bar{z})^{1/2} \\ &\leq f(d,\bar{d})^{1/2} \frac{|c-z|}{|\text{Im } z|} \frac{|L(b,z)|}{|L(b,c)|} f(c,c)^{1/2}\end{aligned}$$

Hence for $|\text{Im } z| \geq 1$, $|f(d,z)|$ grows no faster than $C|z||L(b,z)|$. Since both $f(d,z)$ and $L(b,z)$ are entire functions of type at most σ, we can estimate $f(d,z)$ on the strip $|\text{Im } z| \leq 1$ by Phragmen- Lindelof principle. We then conclude that exponential type of $L(b,z)$ is at least that of $f(c,z)$. Consequently, $L(b,z)$ has exponential type $\geq \sigma$.

For the proof of part (ii), we need the following lemma.

Lemma 4.14. *Assume in addition to the hypotheses of Theorem* 4.8*, that* $f(a,.)$ *is of type* σ *and* (4.50) *holds. Then*

(i) There exists $C > 0$ *such that the zeros* $\{\rho_j\}$ *of* $L(z,0)$ *satisfy for all* j,

$$\rho_{j+1} - \rho_j \geq C.$$

(ii) There exists $C > 0$ *such that for all* $g \in PW_\sigma$,

$$\sum_j |g(\rho_j)|^2 \leq C\|g\|^2_{L_2(\mathbb{R})}. \tag{4.55}$$

(iii) For all $z \in \mathbb{C}$,

$$\sum_{j=1}^{\infty} \frac{|f(\rho_j,z)|^2}{f(\rho_j,\rho_j)} \leq f(z,\bar{z}).$$

Proof. (i) As we mentioned in (4.48)

$$f(\rho_{j+1},\rho_j) = 0.$$

Moreover, (4.50) and (4.38) yields that

$$|f(\rho_{j+1},x)| \le f(\rho_{j+1,j+1})^{1/2} f(x,x)^{1/2} \le C_1.$$

Since $f(\rho_{j+1},.)$ is entire of exponential type σ which is bounded on the real axis, we may apply Bernstein's inequality for entire functions of exponential type [15, p.227]; and then we obtain for all real t,

$$\left|\frac{\partial}{\partial t} f(\rho_{j+1},t)\right| \le C_1\sigma.$$

Then using (4.50) one more time, for some real number ξ between ρ_j and ρ_{j+1},

$$\begin{aligned} C_2 &\le f(\rho_{j+1},\rho_{j+1}) \\ &= f(\rho_{j+1},\rho_{j+1}) - f(\rho_{j+1},\rho_j) \\ &= \left(\frac{\partial}{\partial t} f(\rho_{j+1},\xi)\right)(\rho_{j+1}-\rho_j) \\ &\le C_1\sigma(\rho_{j+1}-\rho_j). \end{aligned}$$

(ii) This follows directly by an estimate on PW_σ [15, p.150].

(iii) Applying Bessel's inequality (4.42) to $g(t) = f(t,z)$, and using the reproducing kernel identity (4.36), we have

$$\sum_j \frac{|f(\rho_j,t)|^2}{f(\rho_j,\rho_j)} \le \int_{\mathbb{R}} \left|\frac{f(t,z)}{E_a(t)} dt\right|^2 = f(\bar{z},z) = f(z,\bar{z}).$$

□

We now turn back to the proof of Theorem 4.13ii. Let $g \in PW_\sigma$ and define

$$G(z) = G[g](z) = \sum_{j-\infty}^{\infty} g(\rho_j)\frac{f(\rho_j,z)}{f(\rho_j,\rho_j)}.$$

By (4.50) and part (ii) of the previous lemma,

$$\sum_j \frac{|g(\rho_j)|^2}{f(\rho_j,\rho_j)} \leq C\|g\|^2_{L_2\mathbb{R}} < \infty.$$

Proceeding as in the proof of Theorem 4.12ii, we see that $G \in \mathcal{H}(E_a)$. We now show that $G = g$. Let

$$\psi(z) = \frac{g(z) - G(z)}{F(z)}.$$

Since $G(\rho_j) = g(\rho_j)$ by (4.48) and F has simple zeros at each ρ_j, ψ is an entire function. As both numerator and denominator are of exponential type, so is ψ. We first show that

$$G(z) = \sum_{j=-\infty}^{j=\infty} g(\rho_j)\frac{F(z)}{F'(\rho_j)(z-\rho_j)}. \tag{4.56}$$

Let $F(\alpha) = L(\alpha, 0) \neq 0$. By (4.25), we have

$$L(z,\rho_j)L(\alpha,0) = L(\alpha,z)L(0,\rho_j) - L(0,z)L(\alpha,\rho_j) = F(z)L(\alpha,\rho_j).$$

Arranging the above equation and using $L(0,\rho_j) = 0$,

$$f(z,\rho_j) = \frac{F(z)L(\alpha,\rho_j)}{F(\alpha)(z-\rho_j)}.$$

Taking limit $z \to \rho_j$, we obtain

$$f(\rho_j,\rho_j) = F'(\rho_j)\frac{L(\alpha,\rho_j)}{F(\alpha)}.$$

We therefore write

$$\frac{f(\rho_j, z)}{f(\rho_j,\rho_j)} = \frac{F(z)}{F'(z)(z-\rho_j)},$$

which gives (4.56). Furthermore,

$$\left|\frac{G(z)}{F(z)}\right| \le \left(\sum_{j=-\infty}^{j=\infty} |g(\rho_j)|^2\right)^{1/2} \left(\sum_{j=-\infty}^{j=\infty} \frac{1}{|F'(\rho_j)(z-\rho_j)|^2}\right)^{1/2}.$$

Let $\epsilon = (0, \pi/2)$, and define $A_\epsilon = \{z : |z| \ge 1 \text{ and } \epsilon \le |\arg z| \le \pi - \epsilon\}$. We claim that

$$\lim_{z\to\infty, z\in A_\epsilon} \left|\frac{G(z)}{F(z)}\right| = 0. \tag{4.57}$$

To this end, we note that there exists C_ϵ such that for all j,

$$|z-\rho_j| \ge C_\epsilon |i-\rho_j|.$$

Moreover,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \frac{1}{|F'(\rho_j)(i-\rho_j)|^2} &= \frac{1}{|F(i)|^2} \sum_{j=-\infty}^{\infty} \frac{|F(i)|^2}{|F'(\rho_j)(i-\rho_j)|^2} \\ &= \frac{1}{|F(i)|^2} \sum_{j=-\infty}^{\infty} \left|\frac{f(\rho_j, i)}{f(\rho_j,\rho_j)}\right|^2 \\ &\le \frac{1}{|F(i)|^2 \inf_{x\in\mathbb{R}} f(x,x)} f(i,-i) < \infty.\end{aligned}$$

Since

$$\limsup_{z\to\infty, z\in A_\epsilon} \sum_{|j|<n} \frac{1}{|z-\rho_j|^2} = 0,$$

for any $n \ge 1$, we see that

$$\limsup_{z\to\infty, z\in A_\epsilon} \left|\frac{G(z)}{F(z)}\right| \le \left(\sum_{j=-\infty}^{\infty} |g(\rho_j)|^2\right)^{1/2} \left(\frac{1}{C_\epsilon^2} \sum_{|j|\ge n} \frac{1}{|F'(\rho_j)(i-\rho_j)|^2}\right)^{1/2}.$$

Since this has limit 0 as $n \to \infty$, we have shown (4.57).

As we discussed earlier $F(z) = zf(0,z)$ is of exponential type σ, has real zeros, and

$$|F(x)| = |xf(0,x)| \leq |x| f(0,0)^{1/2} f(x,x)^{1/2} \leq C|x|$$

by (4.50) and (4.38). Hence, it lies in the Cartwright class. From the proof of Theorem 3.23, for all $\theta \in (-\pi,\pi) \setminus \{0\}$,

$$\lim_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r} = \sigma|\sin\theta|.$$

Let us now assume that g has type $\tau < \sigma$. Since g belongs to $L_2(\mathbb{R})$, it lies in the Cartwright class as well. Similarly, for $\theta \in (-\pi,\pi) \setminus \{0\}$,

$$\limsup_{r\to\infty} \frac{\log|g(re^{i\theta})|}{r} \leq \tau|\sin\theta|.$$

Then, for $\theta \in (-\pi,\pi) \setminus \{0\}$, as $r \to \infty$,

$$\left|\frac{g}{F}\right|(re^{i\theta}) \leq \exp((\tau-\sigma)r|\sin\theta| + o(r)).$$

Therefore, for such θ,

$$\lim_{r\to\infty} \left|\frac{g}{F}\right|(re^{i\theta}) = 0.$$

Combining this with (4.57), we see that

$$\lim_{r\to\infty} |\psi|(re^{i\theta}) = 0$$

Since ψ is an entire function of exponential type, Phragmen-Lindelof principle, applied on sectors of opening angle less than π, shows that it is bounded in the plane, and hence constant. As it has limit 0 at ∞, we have $\psi \equiv 0$, so

$$g = G \in \mathcal{H}(E_a).$$

Finally, if g has type σ, then for $\epsilon \in (0,1)$, $g_\epsilon(z) = g(\epsilon z)$ has type $\epsilon\sigma < \sigma$, so

$$g_\epsilon = G[g_\epsilon].$$

As g_ϵ and $G[g_\epsilon]$ converge to g and $G[g]$ respectively, uniformly on compacta, we let $\epsilon \to 1^-$; and obtain

$$g = G[g].$$

(iii) Let $g \in \mathcal{H}(E_a)$. Then $g/E_a, g*/E_a \in H^2(\mathbb{C}^+)$. Since E_a is of exponential type σ, we claim that g has exponential type at most σ. Recall that $\{t_{jn}\} = \{t_{jn}(\xi_n)\}$ are the quadrature points for μ including ξ_n. Fix $l \geq 1$. By the Gauss quadrature formula (4.2), and the fact that $K_n(t_{jn}, t_{kn}) = 0$ for $j \neq k$,we have

$$\int \left| \sum_{|j| \leq l} g(\rho_j) \frac{K_n(t_{jn}, s)}{K_n(t_{jn}, t_{jn})} \right|^2 d\mu(s) = \sum_{|j| \leq l} \frac{|g(\rho_j)|^2}{K_n(t_{jn}, t_{jn})}.$$

Let us make the substitution

$$s = \xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)} = \xi_n + \frac{t}{K_n(\xi_n, \xi_n)\mu'(\xi_n)}$$

Also, recall that

$$\rho_{jn} = \tilde{K}_n(\xi_n, \xi_n)(t_{jn} - \xi_n),$$

$$f_n(a,b) = \frac{K_n(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)})}{K_n(\xi_n,\xi_n)}.$$

Let $r > 0$. We disregard the singular part of μ, we obtain for large n,

$$\int_{-r}^{r} \left| \sum_{|j|\le l} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \right|^2 \frac{\mu'(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)})}{\mu'(\xi_n)} dt \le \sum_{|j|\le l} \frac{|g(\rho_j)|^2}{f_n(\rho_{jn}, \rho_{jn})}. \tag{4.58}$$

As $n \to \infty$ through S, the right-hand side converges to

$$\sum_{|j|\le l} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} \le \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)}.$$

As we mentioned earlier, uniform convergence of $f_n(0,z)$ to $f(0,z)$ forces the zeros $\{\rho_{jn}\}$ of f_n to converge to those of f. Therefore,

$$\sum_{|j|\le l} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \to \sum_{|j|\le l} g(\rho_j) \frac{f(\rho_j, t)}{f(\rho_j, \rho_j)}.$$

Let $G_l(t)$ denote the right-hand side of the above equation. By Fatou's lemma,

$$\begin{aligned}
&\liminf_{n\to\infty, n\in S} \int_{-r}^{r} \left| \sum_{|j|\le l} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \right|^2 \frac{\mu'(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)})}{\mu'(\xi_n)} dt \\
&\ge \int_{-r}^{r} |G_l(t)|^2 \liminf_{n\to\infty, n\in S} \frac{\mu'(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)})}{\mu'(\xi_n)} dt \\
&\ge C_0 \int_{-r}^{r} |G_l(t)|^2 dt
\end{aligned}$$

under our hypothesis (4.51). If we assume (4.52), we write the left-hand side of (4.58) as

$$\begin{aligned}
&\int_{-r}^{r}\left|\sum_{|j|\le l} g(\rho_j)\frac{f_n(\rho_{jn},t)}{f_n(\rho_{jn},\rho_{jn})}\right|^2 dt\\
&+\int_{-r}^{r}\left|\sum_{|j|\le l} g(\rho_j)\frac{f_n(\rho_{jn},t)}{f_n(\rho_{jn},\rho_{jn})}\right|^2 \left\{\frac{\mu'(\xi_n+\frac{t}{\tilde{K}_n(\xi_n,\xi_n)})}{\mu'(\xi_n)}-1\right\}dt\\
&=\int_{-r}^{r}|G_l(t)|^2dt+o(1)+O\left(\int_{-r}^{r}\left|\frac{\mu'(\xi_n+\frac{t}{\tilde{K}_n(\xi_n,\xi_n)})}{\mu'(\xi_n)}-1\right|dt\right)\\
&=\int_{-r}^{r}|G_l(t)|^2dt+o(1).
\end{aligned}$$

Hence,

$$C_0\int_{-r}^{r}|G_l(t)|^2dt\le\sum_{j=-\infty}^{\infty}\frac{|g(\rho_j)|^2}{f(\rho_j,\rho_j)},\tag{4.59}$$

where $C_0 = 1$ if we have (4.52). Since $g \in \mathcal{H}(E_a)$, $G[g]$ belongs to $\mathcal{H}(E_a)$ as well, by Theorem 4.12ii. In this regard, we can proceed as in the proof of Theorem 4.13ii; and deduce

$$\int_{-r}^{r}|g(t)-G_l(t)|^2dt\to 0 \text{ as } l\to\infty.$$

We then write (4.59) as

$$C_0\int_{-r}^{r}|g(t)|^2dt\le\sum_{j=-\infty}^{\infty}\frac{|g(\rho_j)|^2}{f(\rho_j,\rho_j)}.$$

Letting $r\to\infty$ gives

$$C_0\int_{-\infty}^{\infty}|g|^2\le\sum_{j=-\infty}^{\infty}\frac{|g(\rho_j)|^2}{f(\rho_j,\rho_j)}.\tag{4.60}$$

Therefore, $g \in L_2(\mathbb{R})$, and of exponential type at most σ, so $g \in PW_\sigma$. We have shown that $\mathcal{H}(E_a) \subset PW_\sigma$. Combining this with Theorem 4.13ii gives $\mathcal{H}(E_a) = PW_\sigma$. It remains to prove equivalence of the norms. We first note that $F \notin \mathcal{H}(E_a)$. We could suppose for a contradiction that $F \in \mathcal{H}(E_a)$. Since $F(\rho_j) = 0$ for all j, Theorem 4.13ii shows that

$$F = G[F] = 0,$$

a contradiction. Then (4.43) and (4.60) shows that

$$\|g\|_{E_a}^2 = \int_{\mathbb{R}} \left| \frac{g}{E_a} \right|^2 = \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} \geq C_0 \|g\|_{L_2(\mathbb{R})}^2.$$

In the other direction, as $f(\rho_j, \rho_j) \sim 1$ uniformly in j, (4.55) shows that

$$\|g\|_{E_a}^2 = \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} \leq C_2 \|g\|_{L_2(\mathbb{R})}^2.$$

□

4.4.1. Proof of Theorem 1.4

(i) This follows directly from Lemma 4.2.

(ii) This part was proven in Lemma 4.2 and Theorem 4.13. We only need to justify the hypothesis (4.50) that $f(t, t) \sim 1$ for $t \in \mathbb{R}$. This was shown in Lemma 4.3i.

(iii) This follows from Theorem 4.10.

(iv) This follows from Theorem 4.13. To see that hypothesis (4.51) holds, we will recall that there exists $C > 1$,

$$C^{-1} \leq \mu' \leq C \tag{4.61}$$

in some open set O containing the compact set J where the sequence $\{\xi_n\}$ belongs to. By (4.61), there exists $\epsilon > 0$, for all $|t| \leq \epsilon$,

$$\frac{\mu'(\xi_n + t)}{\mu'(\xi_n)} \geq C_0$$

for some C_0. Since $\tilde{K}_n(\xi_n, \xi_n) = \mu'(\xi_n) K_n(\xi_n, \xi_n) \gtrsim C^{-1} n$, for ant $t \in \mathbb{R}$ we have,

$$\left| \frac{t}{\tilde{K}_n(\xi_n, \xi_n)} \right| < \epsilon$$

for sufficiently large n. Therefore,

$$\liminf_{n \to \infty} \frac{\mu'(\xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)})}{\mu'(\xi_n)} \geq C_0.$$

5. CONCLUSION

The interplay between the universality limits and orthogonal polynomials is a very active and growing field. In this thesis we have only attempted to expose a slice of results that we believe are coherent and could be presented in a self-contained manner. In order to give a taste of other work in this field we would like to cite some related work and theorems without their proofs.

In the last few years, D.S. Lubinsky has produced a number of exciting papers establishing universality limits. In this Master's thesis, the methods presented in [8] and [9] has been taken under review. In [8], he uses ideas from orthogonal polynomials to prove new results about universality limits that do not require regularity of the measure involved. Beginning with a finite positive Borel measure μ with compact support on the real line, we first define orthonormal polynomials p_n with respect to μ. We denote by

$$K_n(x,y)=\sum_{k=0}^{n-1}p_k(x)p_k(y)$$

the corresponding reproducing kernel and by

$$\tilde{K}_n(x,y)=\mu'(x)^{1/2}\mu'(y)^{1/2}K_n(x,y)$$

the normalized kernel, where μ' is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure. Assume that μ is absolutely continuous in a neighborhood of a point x in the support and that μ' is bounded near x, where x is a Lebesgue point of μ'. Then, we have been interested in the limit

$$\lim_{n\to\infty}\frac{K_n(x+\frac{a}{n},x+\frac{a}{n})}{K_n(x,x)}=1, \tag{5.1}$$

where a belongs to the compact subsets of the real line; and, more generally, in the limit

$$\lim_{n\to\infty} \frac{K_n(x+\frac{a}{n}, x+\frac{b}{n})}{K_n(x,x)} = \frac{\sin \pi(a-b)}{\pi(a-b)}, \tag{5.2}$$

where a, b belongs to compact subsets of the complex plane. It has been proven in Theorem 1.1 that the equalities (5.1) and (5.2) are equivalent. More importantly, the above limits are uniform in a compact subset J of the support where μ' is absolutely continuous in a neighborhood of J. In this framework, it was proven that the family of functions $\{f_n(.,.)\}$ with,

$$f_n(a,b) = \frac{K_n(\xi_n + \frac{a}{n}, \xi_n + \frac{b}{n})}{K_n(\xi_n, \xi_n)},$$

where $\{\xi_n\}$ is an arbitrary sequence from J is a normal family of functions over $\mathbb{C}^2$; and any limit of $f(.,.)$ of a subsequence of $f_n(.,.)$ is entire of exponential type σ in both variables for some positive real number σ. This limit function $f(.,.)$ is the so-called universality limit. Theorem 1.4 asserts that, under some mild conditions on μ, the universality limit is the reproducing kernel of de Branges space which is isomorphic to a classical Paley-Wiener space. All of these arguments are present in [8] and [9]. In this context, aforementioned de Branges space is denoted by $\mathcal{H}(E_a)$, where

$$E_a(z) = \sqrt{2\pi}\frac{(\bar{a}-z)f(\bar{a},z)}{|(a-\bar{a})f(a,\bar{a})|^{1/2}}.$$

The isomorphism in Theorem 1.4 means that $\mathcal{H}(E_a)$ and PW_σ are equal as sets, and have equivalent norms. This does not imply isometric isomorphism; therefore the inner products on these two spaces need not be equal. Recalling that the reproducing kernel in PW_σ is

$$\mathcal{K}(\zeta, z) = \frac{\sin \sigma(z-\bar{\zeta})}{\pi(z-\bar{\zeta})},$$

which is called "sinc kernel"; the function $f(.,.)$, which is the universality limit and the reproducing kernel of $\mathcal{H}(E_a)$, need not be a sinc kernel. In [16], Lubinsky addresses this question: given such a universality limit $f(.,.)$, how far is $f(.,.)$ from being a sinc kernel? To this end, the author defines an operator $\mathcal{L} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ as

$$\mathcal{L}[h](t) = \int_{\mathbb{R}} h(t)f(t,x)dt.$$

If f is the sinc kernel of PW_σ, then for $h \in PW_\sigma$ we would have $\mathcal{L}[h] = h$. Thus, to measure how far f is from the sinc kernel of PW_σ one may consider the difference $h - \mathcal{L}[h]$ for all $h \in PW_\sigma$. Theorem 1.5 in [16] addresses this question. The author also gives conditions when the universality limit f is indeed the sinc kernel in Corollary 1.6 in [16].

Aside from these techniques that can be used to establish universality limits in the bulk, Lubinsky applies a wide variety of methods for establishing universality at the hard or soft edge of the spectrum. In [17], the author proved that the universality at the hard edge (endpoints of the support of the measure) for arbitrary parameters a, b is equivalent to universality in the diagonal case ($a = b$). Avila, Last and Simon [6] have shown that these methods can be adapted to prove universality for measures whose support is a Cantor set of positive measure.

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