ANALYSIS OF CONVERGENT INTEGRAL EQUATION METHODS FOR HIGH-FREQUENCY SCATTERING

by

Samet Keserci

B.S., Mathematics, Bogaziçi University, 2009

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2012

ACKNOWLEDGEMENTS

I would like to express my profound respect and gratitude to my thesis advisor Assist. Prof. Fatih Ecevit for his endless patience and invaluable guidance both on an academic and a personal level.

I would like to thank Burak Gürel, Mehmet Burçin Ünlü for participating in my thesis committee and Alp Eden for his support and guidance during my undergraduate and graduate years. I would also like to thank Kürşat Aker for his endless support and priceless friendship.

I would like to thank all faculty members.

I would also like to thank all my officemates for their true friendships.

I gratefully acknowledge support from Bogazici University BAP Projects through the project “Fast and convergent methods for high-frequency acoustic scattering” (project code 5548P).

Most importantly, I am indebted to my family for their sacrifices throughout my education.

ABSTRACT

ANALYSIS OF CONVERGENT INTEGRAL EQUATION METHODS FOR HIGH-FREQUENCY SCATTERING

The main aim of this thesis is to devise numerical methods for the solution of high-frequency scattering problems in 2 dimensional settings by utilizing geometrical optics ansatz and asymptotic properties of solutions for convex obstacles (see [1]). To this end, we formulate the sound soft scattering problem as a well-posed boundary integral equation. Among the numerical methods (Nyström, collocation, Galerkin, two-grid and multi-grid) appropriate for solving integral equations, we focus on the classical but efficacious ones, namely the two- and multi-grid methods. We first portray the defect correction principle for integral equations of the second kind which constitutes a basis for the two- and multi-grid methods, then we define both methods over the defect correction iteration. We also set up these methods to compute the scattering return by the unit circle numerically and compare theoretical and numerical results. By virtue of the geometrical optics ansatz, which expresses the normal derivative of the total field as a highly oscillating complex exponential modulated by a slowly oscillating amplitude, we construct a new Galerkin method well adapted to the slowly oscillating nature of the unknown function which we approximate by polynomials. We hereby eliminate the serious drawbacks arising from high oscillations for approximating the solutions. As our main convergence result will display, our new algorithm entails that it suffices to increase the degrees of freedom proportional to k^{ϵ} (for any $\epsilon > 0$) in order to preserve a given accuracy. In contrast with the previous efforts on the problem, we construct our local approximation spaces with particular emphasis on the transition regions to capture the boundary layers around shadow boundaries and utilize approximation spaces in the deep shadow region to incorporate the effects of grazing rays.

ÖZET

YÜKSEK FREKANSLI SAÇILMALAR İÇİN YAKINSAK İNTEGRAL DENKLEM METODLARININ ANALİZİ

Bu tezin temel amacı, 2. boyutta yüksek-frekanslı dalgaların saçılma problemleri için geometrik optik yaklaşımı vasıtası ile bir nümerik metod tasarlamaktır. Bu maksatla, saçılma problemlerini iyi konulmuş bir sınır integral denklemi haline dönüştürdük. İntegral denklemlerini çözmek için uygun olan nümerik metodlar (Nyström, collocation, Galerkin, iki-grid ve çoklu-grid) arasından, klasik ama etkili bir metod olan iki- ve çoklu-grid metodları üzerine odaklandık. İlk olarak, bu her iki metodun temeli olan hata düzeltme prensibini tasvir edip, 2. tür integral denklemlerinin çözümüne uyarladık. Sonra da bu iki metodu hata düzeltme prensibi üzerinden tanımladık. Aynı zamanda bu nümerik metodları kullanarak birim çemberden saçılan dalgaları hesaplayıp teorik ve nümerik sonuçları karşılaştırdık. Toplam alanın normal türevini yüksek derecede salınım yapan kompleks üstel bir fonksiyon ile düşük salınım yapan bir fonksiyonun çarpımı olarak veren geometrik optik yaklaşım sayesinde, düşük salınımlı fonksiyonun doğasına uygun şekilde yeni bir Galerkin metod dizayn edip polinomlarla bu fonksiyona yaklaştık. Böylece yüksek salınımdan ortaya çıkan ciddi problemleri ortadan kaldırmış olduk. İspatladığımız yakınsak metod kendini gösterdiğinde, yeni algoritmamız, k artıkça, verilen hata payını sabitlemek için serbestlik derecesini k^{ϵ} (keyfi bir ϵ için) ile orantılı bir şekilde artırılmasının yeterli olacağını ortaya koymuştur. Bu problem üzerinde çalışan diğer arastırmacılarin aksine, lokal yakınsama uzaylarını, geçiş bölgelerine özel önem vererek gölge sınır bölgesinde oluşan katmanları ve derin gölge sınır bölgesinde yüzeyi sıyıran dalgaları kavrayacak şekilde inşa ettik.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

$a \lesssim_n b$	$a \leq C_n b$ for some constant $C_n \in \mathbb{R}$ that depends only on n
$C^k(D)$	k-times continuously differentiable functions on a domain D
$C^k(\bar{D})$	k-times continuously differentiable functions whose derivatives of all order $\leq k$ is uniformly continuous functions on a domain D
$C^{0,\alpha}(D)$	Uniformly Hölder continuous functions with exponent α
$C^{1,\alpha}(D)$	Uniformly Hölder continuously differentiable functions with exponent α
d_Λ	Degree of polynomials to approximate $V(s,k)$ on $\Lambda \subset [0,2\pi]$
$\mathfrak{D}$	Double-layer potentials
$\bar{D}$	Closure of a domain D
D_s^n	n^{th} derivative operator with respect to s
$H_n^{(1,2)}$	Hankel functions of the first and second kind of order n
k	Wave number
$L^2(\Lambda)$	Square integrable functions on Λ
$\mathbb{N}$	Natural numbers
$\mathcal{O}$	Big-O notation
o	Little-O notation
$\mathbb{R}$	Real numbers
$\mathfrak{S}$	Single-layer potentials
u^{inc}	Incident field
u^s	Scattered field
u	Total field $u^{inc} + u^s$
Δu	Laplacian of u
∇u	Gradient of u
ν	Unit normal vector
Ω_r	Circle with radius r in $\mathbb{R}^2$
$\Phi(x,y)$	Fundamental solutions of the Helmholtz equation

$\square$	End of proof
$:=$	Equality that includes a definition
$\oplus$	Direct sum
∂D	Boundary of a domain D

LIST OF ACRONYMS/ABBREVIATIONS

2D	Two Dimensional
3D	Three Dimensional
I	Illuminated region
S	Deep Region
SB	Shadow boundary
TI	Transitions in the illuminated region
TS	Transitions in the shadow region

1. INTRODUCTION

In this thesis, we introduce a new method for the computation of scattering returns by smooth convex obstacles in 2-dimensional space for high-frequency scenario. More precisely, we consider the solution of Helmholtz equation $\Delta u + k^2 u = 0$ on the exterior of a bounded domain. Our approach is based upon utilization of a well-posed integral equation formulation of the scattering problem, a non-standard Galerkin approximation space adopted to the known asymptotic expansion of the normal derivative ϑ of the total field u (incident + scattered field) obtained from microlocal analysis [1], and the utilization of geometrical optics ansatz

$$\vartheta(\gamma(s), k) = kV(s, k) \exp(ik\gamma(s) \cdot a) \tag{1.1}$$

where γ is the arc-lenght parametrization of the boundary ∂D of the obstacle D, and $V(s, k)$ is the unknown amplitude varying more gradually than ϑ for large wave numbers k [1].

Solving high frequency scattering problems utilizing their boundary integral equation formulations is a widely used technique. The standard numerical methods for solving highly-oscillatory integral equations force the degrees of freedom to be $\mathcal{O}(k)$ to maintain a given accuracy. Several authors [2–6] have thus used ansatz (1.1) for high-frequency scattering problems. In this connection, Abboud et al. [2] developed a numerical scheme based on boundary integral equation methods, the method of stationary phase and geometrical optics ansatz. They utilized a variational formulation of the problem and showed that, in principle, it requires an $\mathcal{O}(k^{1/3})$ increase in the degrees of freedom in order to fix a given accuracy with increasing k. Bruno et al. [5] presented a method using a combined field approach for solving convex scattering problems in two or three dimensions for which computational complexity of solving the high-frequency problems has been observed to be $\mathcal{O}(1)$ as frequency grows. The ideas underlying their approach are similar to those of [2] as it utilizes ansatz (1.1) and an appropriate boundary integral equation formulation. In addition, they invoked the localized integration technique associated with the method of stationary phase, and they employ a variant of the high-order Nyström method in order to attain a given accuracy. Furthermore, they utilize a change of variables around shadow boundaries to resolve the boundary layers in these regions. In [4], Gilladi and Keller proposed a numerical method based on a formulation of the scattering problem as an integral equation by a boundary element collocation method in which basis functions are asymptotically derived. Since they represent the solution to the scattering problem as a single layer potential, their method deteriorates for some wave numbers k leading to non-uniqueness of solutions. Huybrechs and Vandewalle [6] also presented a numerical method for the scattering problem. Their method grounded on the formulation of the boundary element method with carefully chosen basis functions with effective quadrature rules to combine the asymptotic properties of the solution. Furthermore, they used the numerical steepest descent methods to compute oscillatory integrals. In [7], Graham et al. also devised a numerical method based on the use of combined field integral equations approach, utilization of (1.1) and Galerkin approximation where slowly varying amplitude was approximated locally via polynomials. They cast that, as $k \to \infty$, degrees of freedom has to increase proportional to $k^{1/9}$ to attain a given error prescription. However, they did not carry out an error analysis on the deep shadow region. As in [5, 6], they approximated the solution by zero in the deep shadow region.

Considering the same problem, here we present a novel method for the computation of normal derivative of the total field on the boundary. As a main convergence result, we establish that it requires only a minor increase (k^ϵ for any $\epsilon > 0$) in the number of degrees of freedom to maintain a given accuracy. Additionally, the significant advantage of our method over those in the aforementioned scheme is that our method is fully convergent whereas those in [2, 4–6] are not rigorously analysed, and that in [7] is not convergent for fixed k.

The content of this thesis is concisely described below.

In Chapter 2, we give a brief introduction for 2-dimensional scattering problems. Then we give the basic properties of single-layer and double-layer potentials and their

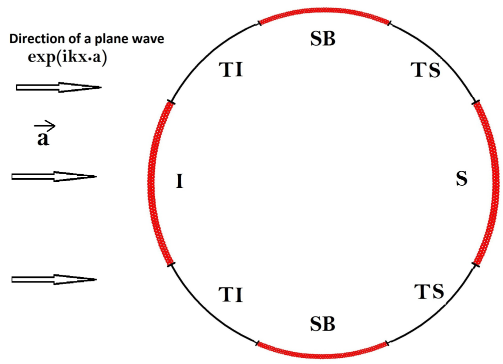


Figure 1.1. Regions.

limiting values on the boundary. Furthermore, we include Green's theorems and identities and prove the existence and uniqueness results for the Helmholtz equation under a physically relevant radiation condition. Towards the end, we formulate our problem as a combined field integral equation. This formulation enable us to determine the scattered field by computing the normal derivative of the total field u on ∂D.

Chapter 3 is devoted to two classical numerical methods, two- and multi-grid methods, adapted to solve the integral equations. Although these methods are first introduced to solve the boundary value problems for the elliptic equations, we see that they can also be applied for integral equations of the second kind. Firstly we begin with the notion of approximate inverse(s) within the framework of defect correction principle which is substructure of the two- and multi-grid iterations adapted upon solving integal equations of the second kind [8, 9]. Following Kress [9], we discuss two variants of the two-grid methods proposed by Brakhage [10] and Atkinson [11] respectively and the multi-grid iterations defined by Hackbusch [12]. In the last section of this chapter, we apply two- and multi-grid methods to the aforementioned scattering problem, and we observe that both methods successfully work for low-frequency problems.

In Chapter 4, we design and analyze our formula for computing the scattering of waves of the form $\exp(ikx \cdot a)$, where a is a vector with norm 1, by an arbitrary smooth convex obstacles in 2-D. First, we review the reformulation of the scattering problem as a boundary integral equation. In the instance of a convex obstacle, we make use of the geometrical optics ansatz (1.1) which enable us to express the normal derivative of the total field by the product of a slowly oscillating amplitude (unknown) and a highly-oscillatory complex exponential (known). Our next step is to approximate the slowly varying function locally by polynomials. To this and, basically we divide the boundary of D into five k-dependent regions as depicted in the Figure 1: Transition regions TI and TS, Illuminated Region I, Deep Shadow Region S and Shadow Boundary SB. Then we construct local Galerkin approximation spaces in each subregion to approximate $V(\cdot, k)$ via polynomials. Next we give the necessary tools for the error analysis of our Galerkin method and state our main convergence result. We also discuss the high frequency asymptotic behavior of $V(s, k)$ (see (1.1)) from [1]. In the last section, we also derive improved approximation errors for $V(s, k)$ over the shadow region by availing of its asymptotic expansion [1]. Finally we carry out the error analysis and prove our main result.

Appendix A contains of the necessary functional analytic tools and Appendix B is dedicated to some auxiliary results.

2. SCATTERING PROBLEM AND ITS INTEGRAL EQUATION FORMULATION

In this chapter, we begin with a brief introduction to wave prorogation. Then we introduce the single- and double- layer potentials and discuss their regularity properties. For later use, we include Green's theorems and identities and prove the existence and uniqueness of solutions to exterior Helmholtz equation under a physically relevant radiation condition. Throughout the manuscript, unless otherwise stated, we always assume that $D \subseteq \mathbb{R}^2$ is an open bounded domain of class C^2.

The propagation of acoustic waves in a homogenous isotropic medium in $\mathbb{R}^2$ is governed by the equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = 0. \tag{2.1}$$

In the case of time-harmonic waves of the form $U(x,t) = Re\{u(x)e^{-i\omega t}\}$ with frequency $\omega > 0$, we see that u satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0,$$

where $k = \omega/c$ is the *wave number* and c is the speed of sound. Let u^{inc} be an incident field impinging on the boundary of an obstacle D and u^s be the scattered field. For a sound-soft obstacle, the total field $u^{inc} + u^s$ must vanish on the boundary. Accordingly, in the case of a sound-soft object, mathematical modelling of the scattering of time-harmonic waves results in Dirichlet problems for the Helmholtz equation [13]. For a more detailed description of the sound-soft scattering problem we refer to Section 2.3. Since we focus on the Helmholtz equation in $\mathbb{R}^2$, we state its fundamental solution

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y. \tag{2.2}$$

In general, $H_n^{(1)} = J_n + iY_n$ is called the *Hankel function* of first kind of order n. Here,

$$J_n(t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!} \left(\frac{t}{2}\right)^{n+2p} \tag{2.3}$$

and

$$\begin{aligned} Y_n(t) = &\frac{2}{\pi}\left\{\ln\frac{t}{2} + C\right\} J_n(t) - \frac{1}{\pi}\sum_{p=0}^{n-1} \frac{(n-1-p)!}{p!}\left(\frac{2}{t}\right)^{n-2p} \\ &- \frac{1}{\pi}\sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p} \{\psi(p+n) + \psi(p)\} \end{aligned} \tag{2.4}$$

are *Bessel functions* where $\psi(0) = 0$,

$$\psi(p) = \sum_{m=1}^{p} \frac{1}{m}, \quad p = 1, 2, ...,$$

and

$$C = \lim_{p\to\infty} \left\{\sum_{m=1}^{p} \frac{1}{m} - \ln p\right\}$$

is the Euler's constant.

For future reference, we note the relations [14]

$$\frac{d}{dr}H_n^{(1)}(r) = \frac{nH_n^{(1)}}{r} - H_{n+1}(r), \quad \text{and} \quad H_{n+1}^{(1)}(r) = \frac{2n}{r}H_n^{(1)}(r) - H_{n-1}^{(1)}(r) \tag{2.5}$$

and the asymptotic behavior [14]

$$H_n^{(1)}(r) = \sqrt{\frac{2}{\pi r}}\exp^{i(r-\frac{n\pi}{2}-\frac{\pi}{4})}\left\{1 + \mathcal{O}\left(\frac{1}{r}\right)\right\}, \qquad r \to \infty. \tag{2.6}$$

Furthermore, from (2.2), (2.3) and (2.4), we conclude that

$$\Phi(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x-y|} + \frac{i}{4} - \frac{1}{2\pi}\ln\frac{k}{2} - \frac{C}{2\pi} + \mathcal{O}\left(|x-y|^2\ln\frac{1}{|x-y|}\right) \tag{2.7}$$

as $|x-y| \to 0$. In the next section, we give the basic properties of single-layer and double-layer potentials and their limiting values on the boundary as we formulate the scattering problem in the form of a combined field integral equation.

2.1. Single- and Double-Layer Potentials

The integrals

$$\mathfrak{S}(x) = \int_{\partial D} \varphi(y)\Phi(x,y)ds(y), \quad x \in \mathbb{R}^2\backslash\partial D \tag{2.8}$$

and

$$\mathfrak{D}(x) = \int_{\partial D} \varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}ds(y), \quad x \in \mathbb{R}^2\backslash\partial D, \tag{2.9}$$

are called single-layer and double-layer potentials for an integrable density function φ. Here ν is the outward unit normal vector to ∂D. They solve the Helmholtz equation inside and outside of the domain D [14].

The next theorem gives the behavior of surface potentials on the boundary for continuous densities. For a proof of the theorem, we refer to Theorems 2.12, 2.13, 2.19 and 2.21 in [13].

Theorem 2.1. [Jump Relations] *Let ∂D be of class C^2 and let φ be continuous. Then the single-layer potential $\mathfrak{S}$ with density φ is continuous throughout $\mathbb{R}^2$ and*

$$||\mathfrak{S}||_{\infty,\mathbb{R}^2} \le C||\varphi||_{\infty,\partial D}$$

for some constant C depending on ∂D. On the boundary we have

$$\mathfrak{S}(x) = \int_{\partial D} \varphi(y)\Phi(x,y)ds(y), \quad x \in \partial D, \tag{2.10}$$

and

$$\frac{\partial \mathfrak{S}_\pm}{\partial \nu}(x) = \int_{\partial D} \varphi(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}ds(y) \mp \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{2.11}$$

where

$$\frac{\partial \mathfrak{S}_\pm}{\partial \nu}(x) := \lim_{h \to 0^+} \nu(x) \cdot \nabla \mathfrak{S}(x \pm h\nu(x))$$

is to be understood in the sense of uniform convergence on ∂D and where the integrals exist as improper integrals. The double-layer potential $\mathfrak{D}$ with density φ can be continuously extended from D to $\overline{D}$ and from $\mathbb{R}^2 \backslash \overline{D}$ to $\mathbb{R}^2 \backslash D$ with limiting values

$$\mathfrak{D}_\pm(x) = \int_{\partial D} \varphi(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds(y) \pm \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{2.12}$$

where

$$\mathfrak{D}_\pm(x) := \lim_{h \to 0^+} \mathfrak{D}(x \pm h\nu(x))$$

and where the integral exists as an improper integral. Furthermore,

$$||\mathfrak{D}||_{\infty,\overline{D}} \leq C||\varphi||_{\infty,\partial D} \quad \textit{and} \quad ||\mathfrak{D}||_{\infty,\mathbb{R}^2\backslash D} \leq C||\varphi||_{\infty,\partial D}$$

for some constant C depending on ∂D and

$$\lim_{h \to 0^+} \left\{ \frac{\partial \mathfrak{D}}{\partial \nu}(x + h\nu(x)) - \frac{\partial \mathfrak{D}}{\partial \nu}(x - h\nu(x)) \right\} = 0, \quad x \in \partial D, \tag{2.13}$$

uniformly on ∂D.

Next, we give the further regularity properties of these surface potentials in the setting of Hölder spaces (see Definition A.8). Observe that, each function φ in $C^{0,\beta}(D)$ also belongs to $C^{0,\alpha}(D)$ for $\alpha < \beta$. Indeed, an appeal to the Arzelà-Ascoli theorem (see Theorem A.6) entails that this embedding is compact as depicted in the next theorem.

Theorem 2.2. [13] *Let $0 < \alpha < \beta \leq 1$ and let D be compact. Then the imbedding operators*

$$I^{\beta} : C^{0,\beta}(D) \to C(D), \quad I^{\alpha,\beta} : C^{0,\beta}(D) \to C^{0,\alpha}(D)$$

are compact.

We can prove the same properties for the Hölder spaces $C^{1,\alpha}(D)$ of uniformly Hölder continuously differentiable functions with norm defined in Definition A.8 [13]. The direct values of acoustic single- and double-layer potentials have more regularity on the boundary as seen in the next theorem. In order to examine the mapping properties of the potentials on the boundary, we first introduce operators

$$(\mathcal{S}\varphi)(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)ds(y), \quad x \in \partial D, \tag{2.14}$$

$$(\mathcal{K}\varphi)(x) := 2\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y), \quad x \in \partial D, \tag{2.15}$$

called as the single- and double-layer operators, and

$$(\mathcal{K}'\varphi)(x) := 2\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)ds(y), \quad x \in \partial D. \tag{2.16}$$

called as the normal derivative operator.

Theorem 2.3. [14] *Let ∂D be of class C^2. Then the operators $\mathcal{S}$, $\mathcal{K}$ and $\mathcal{K}'$ are bounded from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$, and the operators $\mathcal{S}$ and $\mathcal{K}$ are also bounded from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$.*

For a proof, we refer to Theorems 2.12, 2.15, 2.16, 2.22 and 2.23 in [13]. Also, we note that $\mathcal{S}$ is self-adjoint, and $\mathcal{K}$ and $\mathcal{K}'$ are adjoint with respect to the bilinear form

$$\langle \varphi, \psi \rangle := \int_{\partial D} \varphi(y)\psi(y)ds(y).$$

2.2. Green's Representation Theorems and Sommerfeld's Radiation Condition

In this section, we give the basic properties of solutions of Helmholtz equation under Sommerfeld's radiation condition. For any domain $D \subset \mathbb{R}^2$ of class C^2, we define the linear space $\Re(D)$ of all complex valued functions $u \in C^2(D) \cap C(\overline{D})$ for which the normal derivative on the boundary exists in the sense that the limit

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h \to 0^+} \nu(x) \cdot \nabla u(x - h\nu(x)), \quad x \in \partial D,$$

exist uniformly on ∂D. Notice that the assumption $u, v \in \Re(D)$ suffices to guarantee the validity of Green's first theorem

$$\int_D u(y)\Delta v(y)dy = \int_{\partial D} u(y)\frac{\partial v(y)}{\partial \nu(y)}ds(y) - \int_D \nabla u(y) \cdot \nabla v(y)dy \tag{2.17}$$

and Green's second theorem

$$\int_D \{u(y)\Delta v(y) - v(y)\Delta u(y)\}\, dy = \int_{\partial D} \left(u(y)\frac{\partial v(y)}{\partial \nu(y)} - v(y)\frac{\partial u(y)}{\partial \nu(y)} \right) ds(y) \tag{2.18}$$

for a bounded domain D of class C^2. This follows by first integrating over *parallel surfaces* (see Definition A.7) and then passing to the limit as *parallel surfaces* tend to ∂D [13].

As the next theorem shows, any solution of Helmholtz equation can be written as a combination of single- and double-layer potentials.

Theorem 2.4. [13] *Let $u \in \Re(D)$ be a solution to Helmholtz equation*

$$\Delta u + k^2 u = 0 \ \ in \ \ D.$$

Then,

$$\int_{\partial D} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right) ds(y) = \begin{cases} -u(x) & if\ x \in D \\ 0 & if\ x \in \mathbb{R}^2 \backslash \overline{D}. \end{cases}$$

In $\mathbb{R}^2$, the Sommerfeld radiation condition that guarantees the uniqueness of the exterior scattering problem takes on the form

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \qquad r = |x|, \tag{2.19}$$

uniformly for all directions $x/|x|$. The asymptotic expansion

$$H_0^{(1)}(r) = \sqrt{\frac{2}{\pi r}} e^{i(r - \pi/4)} \left(1 + \mathcal{O}\left(\frac{1}{r} \right) \right), \tag{2.20}$$

of the Hankel function $H_0^{(1)}(r)$ implies that (2.2) satisfies (2.19).

As we state in the next theorem, both the single- and double-layer acoustic potentials satisfy the Sommerfeld radiation condition [13].

Theorem 2.5. *Both the single-layer acoustic potential defined by* (2.8) *and the double-layer acoustic potential defined by* (2.9) *satisfy the Sommerfeld radiation condition* (2.19).

The proof is immediate from the next two lemmas.

Lemma 2.6. *For any bounded domain $D \subset \mathbb{R}^2$ of class C^2,*

$$\frac{x}{|x|} \cdot \nabla_x \Phi(x,y) - ik\Phi(x,y) = \mathcal{O}\left(\frac{1}{|x|^{3/2}}\right), \quad |x| \to \infty \tag{2.21}$$

uniformly for all directions $\frac{x}{|x|}$ and uniformly for all y contained in ∂D.

Proof. Recalling the fundamental solution of Helmholtz equation given in (2.2), we have

$$\nabla_x \Phi(x,y) = -\frac{ik}{4} H_1^{(1)}(k|x-y|) \frac{x-y}{|x-y|}. \tag{2.22}$$

Then, from 2.5 and 2.22, we obtain

$$\begin{aligned}
&\frac{x}{|x|} \cdot \nabla_x \Phi(x,y) - ik\Phi(x,y) \\
&= -\frac{ik}{4} H_1^{(1)}(k|x-y|) \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} + \frac{k}{4} H_0^{(1)}(k|x-y|) \\
&= \frac{k}{4}\left(H_0^{(1)}(k|x-y|) - \sqrt{\frac{2}{\pi k|x-y|}} e^{i(k|x-y|-\pi/4)} \right) + \frac{k}{4}\sqrt{\frac{2}{\pi k|x-y|}} e^{i(k|x-y|-\pi/4)} \\
&- \frac{ik}{4} \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \left(H_1^{(1)}(k|x-y|) - \sqrt{\frac{2}{\pi k|x-y|}} e^{i(k|x-y|-3\pi/4)} \right) \\
&- \frac{ik}{4} \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \sqrt{\frac{2}{\pi k|x-y|}} e^{i(k|x-y|-3\pi/4)}.
\end{aligned}$$

Next, take the absolute values of both sides and use the triangle inequality to get

$$\begin{aligned}
&\left|\frac{x}{|x|}\cdot\nabla_x\Phi(x,y)-ik\Phi(x,y)\right|\\
&\leq\frac{k}{4}\left|\left(H_0^{(1)}(k|x-y|)-\sqrt{\frac{2}{\pi k|x-y|}}e^{i(k|x-y|-\pi/4)}\right)\right|\\
&+\frac{k}{4}\left|\left(H_1^{(1)}(k|x-y|)-\sqrt{\frac{2}{\pi k|x-y|}}e^{i(k|x-y|-3\pi/4)}\right)\right|\\
&+\frac{k}{4}\sqrt{\frac{2}{\pi k|x-y|}}\left|e^{i(k|x-y|-\pi/4)}-i\frac{x-y}{|x-y|}\cdot\frac{x}{|x|}e^{i(k|x-y|-3\pi/4)}\right|\\
&=\frac{k}{4}\left|\left(H_0^{(1)}(k|x-y|)-\sqrt{\frac{2}{\pi k|x-y|}}e^{i(k|x-y|-\pi/4)}\right)\right|\\
&+\frac{k}{4}\left|\left(H_1^{(1)}(k|x-y|)-\sqrt{\frac{2}{\pi k|x-y|}}e^{i(k|x-y|-3\pi/4)}\right)\right|\\
&+\frac{k}{4}\sqrt{\frac{2}{\pi k|x-y|}}e^{i(k|x-y|-3\pi/4)}\left|i-i\frac{(x-y)\cdot x}{|x-y||x|}\right|.
\end{aligned}$$

From (2.6), we see that first two terms are $\mathcal{O}\left(\frac{1}{|x|^{3/2}}\right)$ as $|x|\to\infty$. Thus, it is enough to show that the term $\left|i-i\frac{(x-y)\cdot x}{|x-y||x|}\right|$ is $\mathcal{O}\left(\frac{1}{|x|}\right)$. To this end, observe that

$$\begin{aligned}
\left|1-\frac{(x-y)\cdot x}{|x-y||x|}\right|&=\left|\frac{x}{|x|}\cdot\left(\frac{x}{|x|}-\frac{x-y}{|x-y|}\right)\right|\leq\left|\frac{x}{|x|}-\frac{x-y}{|x-y|}\right|\\
&\leq\left|\frac{x}{|x|}-\frac{x}{|x-y|}\right|+\left|\frac{y}{|x-y|}\right|=\frac{|x|}{|x|}\left|\frac{|x-y|-|x|}{|x-y|}\right|+\frac{|y|}{|x-y|}\\
&\leq\frac{|-y|}{|x-y|}+\frac{|y|}{|x-y|}=\frac{2|y|}{|x-y|}=\mathcal{O}\left(\frac{1}{|x|}\right)
\end{aligned}$$

as $|x|\to\infty$. $\square$

The proof of the next lemma is similar.

Lemma 2.7. *For any bounded domain $D\subset\mathbb{R}^2$ of class C^2,*

$$\frac{x}{|x|}\cdot\nabla_x\frac{\partial\Phi(x,y)}{\partial\nu(y)}-ik\frac{\partial\Phi(x,y)}{\partial\nu(y)}=\mathcal{O}\left(\frac{1}{|x|^{3/2}}\right),\quad |x|\to\infty$$

uniformly for all directions $\frac{x}{|x|}$ *and uniformly for all* y *contained in* ∂D.

As in Theorem 2.4 which concerned the solutions of the Helmholtz equation in bounded domains, the next theorem provides an expression for solutions in exterior domains as a combination of single- and double-layer potentials.

Theorem 2.8. [13] *Let* $u \in \Re(\mathbb{R}^2 \backslash \overline{D})$ *be a solution to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \;\; in \;\; \mathbb{R}^2 \backslash \overline{D}$$

satisfying the Sommerfeld radiation condition (2.19). *Then*

$$\int_{\partial D} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right) ds(y) = \begin{cases} 0 & \text{if } x \in D \\ u(x) & \text{if } x \in \mathbb{R}^2 \backslash \overline{D}. \end{cases} \tag{2.23}$$

Proof. First we claim that

$$\int_{|y|=R} |u|^2 ds(y) = \mathcal{O}(1), \quad R \to \infty. \tag{2.24}$$

Indeed, as is readily seen from (2.19), given $\epsilon > 0$ we can choose $R > 0$ so that

$$\begin{aligned} \int_{\Omega_R} \left| \frac{\partial u(y)}{\partial \nu(y)} - iku(y) \right|^2 ds(y) &= \int_{\Omega_R} \left| \nabla u(y) \cdot \frac{y}{|y|} - iku(y) \right|^2 ds(y) \\ &= \int_{\Omega_R} \frac{1}{|y|} \left| |y|^{1/2} \left(\nabla u(y) \cdot \frac{y}{|y|} - iku(y) \right) \right|^2 ds(y) \\ &\leq \frac{1}{R} \epsilon^2 2\pi R \end{aligned}$$

where ν is the outward unit normal to the sphere $\Omega_R := \{y \in \mathbb{R}^2 | |y| = R\}$. This implies

$$\lim_{R \to \infty} \int_{\Omega_R} \left| \frac{\partial u(y)}{\partial \nu(y)} - iku(y) \right|^2 ds(y) = 0$$

so that

$$0 = \lim_{R\to\infty} \int_{\Omega_R} \left| \frac{\partial u(y)}{\partial \nu(y)} - iku(y) \right|^2 ds(y)$$
$$= \lim_{R\to\infty} \int_{\Omega_R} \left(\left| \frac{\partial u(y)}{\partial \nu(y)} \right|^2 + k^2|u|^2 + 2\mathrm{Im}\left(ku(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)} \right) \right) ds(y). \tag{2.25}$$

Next, choose R large enough so that $\Omega_R \subset \mathbb{R}^2 \backslash \overline{D}$ and apply Green's first theorem (2.17) to functions u and $\overline{u}$ in the domain $D_R := \{y \in \mathbb{R}^2 \backslash \overline{D} | |y| < R\}$ to obtain

$$\int_{D_R} u(y)\Delta\overline{u(y)}dy = -\int_{\partial D} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) + \int_{\Omega_R} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) - \int_{D_R} |\nabla u(y)|^2 dy.$$

Since $\Delta\overline{u} = \overline{(-k^2 u)} = -k^2\overline{u}$, this entails

$$-k^2 \int_{D_R} |u(y)|^2 dy = -\int_{\partial D} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) + \int_{\Omega_R} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) - \int_{D_R} |\nabla u(y)|^2 dy$$

so that

$$k\int_{\Omega_R} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) = k\int_{\partial D} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) + k\int_{D_R} |\nabla u(y)|^2 dy - k^3\int_{D_R} |u(y)|^2 dy.$$

We can thus deduce that

$$\mathrm{Im}\left(k\int_{\Omega_R} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) \right) = \mathrm{Im}\left(k\int_{\partial D} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) \right).$$

Accordingly, plugging the last identity into (2.25), we get

$$0 = \lim_{R\to\infty} \int_{\Omega_R} \left| \frac{\partial u(y)}{\partial \nu(y)} \right|^2 + k^2|u(y)|^2 + 2\mathrm{Im}\left(k\int_{\partial D} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)} \right) ds(y)$$

This yields that

$$\lim_{R\to\infty} \int_{\Omega_R} \left| \frac{\partial u(y)}{\partial \nu(y)} \right|^2 + k^2|u(y)|^2 ds(y) = -2\mathrm{Im}\left(k\int_{\partial D} u(y)\frac{\partial \overline{u(y)}}{\partial \nu(y)}ds(y) \right) < C \tag{2.26}$$

for some $C \in \mathbb{R}$. Since the right-hand-side of this identity is finite, we conclude that $\int_{\Omega_R} \left|\frac{\partial u}{\partial \nu}\right|^2$ and $\int_{\Omega_R} k^2|u|^2$ must be bounded as $R \to \infty$. Hence, (2.24) follows. We write the integral in (2.23) as the sum of

$$I_1 := \int_{\Omega_R} u(y) \left(\frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik\Phi(x,y) \right) ds(y)$$

and

$$I_2 := -\int_{\Omega_R} \Phi(x,y) \left(\frac{\partial u(y)}{\partial \nu(y)} - iku(y) \right) ds(y)$$

Then, observe that

$$\begin{aligned}
|I_1|^2 &\leq \left(\int_{\Omega_R} |u(y)| \left| \frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik\Phi(x,y) \right| ds(y) \right)^2 \\
&\leq \left(\int_{\Omega_R} |u(y)|^2 ds(y) \right) \left(\int_{\Omega_R} \left| \frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik\Phi(x,y) \right|^2 ds(y) \right) \\
&\leq M \left(\frac{C_1}{R^3} 2\pi R \right) \to 0, \text{ as } R \to \infty,
\end{aligned}$$

where we have used Cauchy-Schwarz inequality, Lemma 2.6 and (2.24). As to I_2, by (2.19) and the fact that $\Phi(x,y) = \mathcal{O}(\frac{1}{|x|^{1/2}})$ as $|x| \to \infty$, given $\epsilon > 0$ we can choose R sufficiently large so that

$$\begin{aligned}
|I_2|^2 &\leq \left(\int_{\Omega_R} |\Phi(x,y)|^2 ds(y) \right) \left(\int_{\Omega_R} \left| \frac{\partial u(y)}{\partial \nu(y)} - iku(y) \right|^2 ds(y) \right) \\
&\leq \left(\frac{C_2}{R} 2\pi R \right) \left(\int_{\Omega_R} \frac{1}{|y|} \left| |y|^{1/2} \left(\frac{\partial u(y)}{\partial \nu(y)} - iku(y) \right) \right|^2 ds(y) \right) \\
&\leq (C_2 2\pi) \left(\frac{\epsilon^2}{R} 2\pi R \right) = C_2 4\pi^2 \epsilon^2.
\end{aligned}$$

Hence, we get

$$\lim_{R \to \infty} \int_{\Omega_R} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right) ds(y) = 0. \tag{2.27}$$

If $x \in \mathbb{R}^2 \backslash \overline{D}$, choose R large enough and $r > 0$ sufficiently small so that $\Omega_R \subset \mathbb{R}^2 \backslash \overline{D}$ and $\Omega_r \subset D_R$. Then, apply Green's second theorem (2.18) in $D_{R,r} := \{y \in D_R \| x - y| > r\}$ to get

$$\begin{aligned} 0 &= \int_{\partial D_{R,r}} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right) ds(y) \\ &= \int_{\partial D \cup \Omega_R \cup \Omega_r} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right) ds(y). \end{aligned}$$

Here, observe that the integral on Ω_R goes to zero as $R \to \infty$ by (2.27). By Green's second theorem and (2.7), we get that the integral on Ω_r converges to $-u(x)$ as $r \to 0$. Thus, we conclude that

$$\int_{\partial D} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right) ds(y) = u(x)$$

for $x \in \mathbb{R}^2 \backslash \overline{D}$. On the other hand, if $x \in D$, we already have

$$0 = \int_{\partial D} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right) ds(y).$$

□

Remark 2.9. [13] *Observe that any solution u of the Helmholtz equation satisfying the radiation condition also satisfies*

$$u(x) = O\left(\frac{1}{|x|^{1/2}} \right), \quad as\ |x| \to \infty$$

uniformly for all directions $\dfrac{x}{|x|}$.

Since the fundamental solution of Helmholtz equation satisfies the radiation condition with respect to both variables, Theorem 2.4 allows us to conclude that solutions to the Helmholtz equation are analytic as stated in the next theorem.

Theorem 2.10. [13] *Any two times continuously differentiable solution to the Helmholtz equation is analytic.*

Thus, whenever u is a solution to the Helmholtz equation one can always infer that u is twice continuously differentiable, and hence analytic, in the interior of its domain [14].

2.3. Scattering from Sound-Soft Obstacle

The scattering of time harmonic waves by an arbitrary sound-soft scatterer gives rise to the following problem [14].

Direct Acoustic Obstacle Scattering Problem: Let u^{inc} be an incident field and a solution to the Helmholtz equation in $\mathbb{R}^2$. Then, find a total field $u = u^{inc} + u^s$ satisfying the Helmholtz equation

$$\Delta u + k^2 u = 0 \ \text{ in } \ \mathbb{R}^2 \backslash \overline{D} \tag{2.28}$$

so that the scattered field u^s meets the Sommerfeld radiation condition, and the total field u vanishes on the boundary. This direct scattering problem is evidently a special case of the following exterior Dirichlet problem.

Exterior Dirichlet Problem: Given a continuous function f on ∂D, find a radiating solution $u \in C^2(\mathbb{R}^2 \backslash \overline{D}) \cap C(\mathbb{R}^2 \backslash D)$ to the Helmholtz equation which satisfies $u = f$ on ∂D.

The uniqueness of solutions to the exterior Dirichlet problem is based on the following lemma due to Rellich [15].

Lemma 2.11. [13] *Let k be positive and $u \in C^2(\mathbb{R}^2 \backslash \overline{D})$ a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition and*

$$\int_{|x|=R} |u(y)|^2 ds(y) = o(1), \ \ \text{as } R \to \infty. \tag{2.29}$$

Then $u(y) = 0$ in $\mathbb{R}^2 \backslash \overline{D}$.

The preceding Lemma gives rise to the next theorem from which the uniqueness of the exterior Dirichlet problem follows.

Theorem 2.12. [13] *Let* $u \in \Re(\mathbb{R}^2 \backslash \overline{D})$ *be a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition (2.19) and*

$$\operatorname{Im}\left(k \int_{\partial D} u(y) \frac{\partial \overline{u(y)}}{\partial \nu(y)} ds(y)\right) \geq 0. \tag{2.30}$$

Then u(y)=0 in $\mathbb{R}^2 \backslash \overline{D}$.

Proof. Identity (2.26) in the proof of Theorem 2.8 combined with condition (2.30)entails

$$\int_{|x|=R} |u(y)|^2 ds(y) \to 0, \text{ as } R \to \infty. \tag{2.31}$$

Therefore, by Lemma 2.11, u is identically zero in $\mathbb{R}^2 \backslash \overline{D}$. □

The following uniqueness result is now an immediate consequence of the preceding theorem.

Theorem 2.13. [14] *The exterior Dirichlet problem has at most one solution.*

The existence of solutions to the exterior Dirichlet problem can be deduced from the mapping properties of its boundary integral equation. To this end, we seek the solution as a combination of acoustic single-layer and double- layer potentials

$$u(x) = \int_{\partial D} \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta \Phi(x,y) \right\} \varphi(y) ds(y), \quad x \in \mathbb{R}^2 \backslash \partial D, \tag{2.32}$$

with a density $\varphi \in C(\partial D)$ and a real coupling parameter $\eta \neq 0$ [14]. Then, by utilizing Theorem 2.1, observe that the potential u given by (2.32) in $\mathbb{R}^2 \backslash \overline{D}$ solves the exterior Dirichlet problem if the density satisfies the integral equation

$$\varphi + \mathcal{K}\varphi - i\eta \mathcal{S}\varphi = 2f. \tag{2.33}$$

Next, by Theorems 2.2 and 2.3, we deduce the compactness of the operators $\mathcal{S}, \mathcal{K} : C(\partial D) \to C(\partial D)$. Thus, by virtue of Riesz-Fredholm theory (see A.4), the existence of a solution to integral equation (2.33) can be guaranteed [14]. As for the uniqueness, it is enough to show that the homogenous form of the integral equation (2.33) has only zero as a solution. To this end, let $\varphi \in C(\partial D)$ be a solution of the homogenous equation. Then, the potential u given by (2.32) must satisfy $u_+ = 0$ on ∂D. The uniqueness of the exterior dirichlet problem yields that $u = 0$ in $\mathbb{R}^2 \backslash \overline{D}$. Furthermore, since $u = \mathfrak{D} - i\eta\mathfrak{S}$ defined in (2.32) is a difference of double-layer potential $\mathfrak{D}$ and single-layer potential $\mathfrak{S}$, from jump relation (2.12) of the double-layer potential on the boundary and continuity of the single-layer potential (2.10), we have

$$\begin{aligned} u_+ - u_- &= (\mathfrak{D} - i\eta\mathfrak{S})_+ - (\mathfrak{D} - i\eta\mathfrak{S})_- \\ &= \mathfrak{D}_+ - \mathfrak{D}_- - i\eta(\mathfrak{S}_+ - \mathfrak{S}_-) \\ &= \varphi \qquad \text{on } \ \partial D. \end{aligned}$$

Since $u_+ = 0$ on ∂D, we thus get $-u_- = \varphi$. Moreover, again employing jump relations (2.10)-(2.12) for the normal derivatives of layer potentials, we obtain

$$\begin{aligned} \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} &= \frac{\partial \mathfrak{D}_+}{\partial \nu} - \frac{\partial \mathfrak{D}_-}{\partial \nu} - i\eta\left(\frac{\partial \mathfrak{S}_+}{\partial \nu} - \frac{\partial \mathfrak{S}_-}{\partial \nu}\right) \\ &= i\eta\varphi \qquad \text{on } \ \partial D. \end{aligned}$$

By (2.13) and the fact that $u_+ = 0$ implies $\dfrac{\partial u_+}{\partial \nu} = 0$ on ∂D, we have $-\dfrac{\partial u_-}{\partial \nu} = i\eta\varphi$ on ∂D. Now, applying Green's first theorem (2.17) to the function $-\overline{u(y)}_-$ gives

$$\begin{aligned} \int_{\partial D} \overline{u(y)}_- \frac{\partial u(y)_-}{\partial \nu(y)} ds(y) &= \int_D \left\{\overline{u(y)} \Delta u(y) + |\nabla u(y)|^2\right\} dy \\ &= \int_D \left\{-k^2|u(y)|^2 + |\nabla u(y)|^2\right\} dy, \end{aligned}$$

which implies

$$i\eta \int_{\partial D} |\varphi(y)|^2 ds(y) = \int_D \left\{|\nabla u(y)|^2 - k^2|u(y)|^2\right\} dy.$$

Upon taking the imaginary part of this equation and noting that $k > 0$, we get $\varphi = 0$. This proves that $(I + \mathcal{K} - i\eta\mathcal{S}) : C(\partial D) \to C(\partial D)$ is injective. Accordingly, by Riesz-Fredholm Theory (see Theorem A.4), it possess a bounded inverse. Therefore, for any $f \in C(\partial D)$, equation (2.33) is uniquely solvable and the solution depends continuously on f in the maximum norm. This shows the well-posedness of the exterior Dirichlet problem as stated in the next theorem.

Theorem 2.14. [14] *The exterior Dirichlet problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on* $\mathbb{R}^2\backslash D$, *and all its derivatives on closed subsets of* $\mathbb{R}^2\backslash\overline{D}$.

Next, by the imbedding Theorem 2.2, since the operator $I : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is compact, and the operators $\mathcal{S}, \mathcal{K} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ are bounded, we conclude that the operators $\mathcal{S}, \mathcal{K} : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ are compact (see Theorem A.3). Since $(I+\mathcal{K}-i\eta\mathcal{S}) : C(\partial D) \to C(\partial D)$ is injective, the operator $(I+\mathcal{K}-i\eta\mathcal{S}) : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ is also injective. Next, applying Riesz-Fredholm theory (see Theorem A.4), we get that $(I+\mathcal{K}-i\eta\mathcal{S})^{-1} : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ exists and is bounded. This means that for any given $f \in C^{1,\alpha}(\partial D)$, the solution φ of (2.33) belongs to $C^{1,\alpha}(\partial D)$ and depends continuously on f in the $\|\cdot\|_{1,\alpha}$ norm. Furthermore, by (2.32), we find that u is in $C^{1,\alpha}(\partial D)$, and depends continuously on f. Especially, $f \in C^{1,\alpha}(\partial D)$ implies that the normal derivative $\dfrac{\partial u}{\partial \nu} \in C^{0,\alpha}(\partial D)$.

Now, we return to the scattering of waves of the form $u^{inc} = e^{ikx\cdot a}$ by a sound-soft obstacle D. For domains D of class C^2, by previous regularity results and (2.3), u^s belongs to $C^{1,\alpha}(\mathbb{R}^2\backslash D)$. Therefore, apply Green's formula for exterior domains to get

$$\int_{\partial D} \left(u^s(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s(y)}{\partial \nu(y)}\Phi(x,y) \right) ds(y) = u^s(x), \quad x \in \mathbb{R}^2\backslash D. \tag{2.34}$$

Since $u^{inc}(y)$ solves the Helmholtz equation in whole space, invoking Green's second theorem to the pair of functions $u^{inc}(y)$ and $\Phi(x,y)$ gives

$$\begin{aligned}
&\int_{\partial D}\left(u^{inc}(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}-\frac{\partial u^{inc}(y)}{\partial\nu(y)}\Phi(x,y)\right)ds(y)\\
&=\int_D u^{inc}(y)\Delta_y\Phi(x,y)-\Delta u^{inc}(y)\Phi(x,y)dy\\
&=\int_D u^{inc}(y)(k^2)\Phi(x,y)-(k^2)u^{inc}(y)\Phi(x,y)dy\\
&=0.
\end{aligned}\tag{2.35}$$

Next, adding (2.34) to (2.35) yields

$$u^s(x)=\int_{\partial D}\left\{\left(u^{inc}(y)+u^s(y)\right)\frac{\partial\Phi(x,y)}{\partial\nu(y)}-\left(\frac{\partial u^s(y)}{\partial\nu(y)}+\frac{\partial u^{inc}(y)}{\partial\nu(y)}\right)\Phi(x,y)\right\}ds(y)\tag{2.36}$$

for $x\in\mathbb{R}^2\backslash D$. Therefore, imposing the boundary condition $u^{inc}+u^s=0$ yields that the total field must satisfies

$$u(x)=u^{inc}(x)-\int_{\partial D}\frac{\partial u(y)}{\partial\nu(y)}\Phi(x,y)ds(y)\tag{2.37}$$

for $x\in\mathbb{R}^2\backslash D$. Observe that, letting $x\to\partial D$ in (2.37) and using the boundary condition $u^{inc}+u^s=0$, we obtain

$$0=u^{inc}(x)_+-\left\{\int_{\partial D}\frac{\partial u(y)}{\partial\nu(y)}\Phi(x,y)ds(y)\right\}_+.$$

Since the integral on the right-hand side is a single-layer potential with density $\dfrac{\partial u}{\partial\nu}\in C^{0,\alpha}(\partial D)$, we have

$$\begin{aligned}
0&=u^{inc}(x)_+-\left\{\int_{\partial D}\frac{\partial u(y)}{\partial\nu(y)}\Phi(x,y)ds(y)\right\}_+\\
&=u^{inc}(x)-\left\{\int_{\partial D}\frac{\partial u(y)}{\partial\nu(y)}\Phi(x,y)ds(y)\right\}\\
&=u^{inc}(x)-\frac{1}{2}\left(\mathcal{S}\frac{\partial u}{\partial\nu}\right)(x),\quad x\in\partial D,
\end{aligned}\tag{2.38}$$

so that

$$-2i\eta u^{inc}(x) = -i\eta \left(\mathcal{S}\frac{\partial u}{\partial \nu} \right)(x), \quad x \in \partial D. \tag{2.39}$$

Next, taking the normal derivatives in (2.37), we have

$$\frac{\partial u(x)}{\partial \nu(x)} = \frac{\partial u^{inc}(x)}{\partial \nu(x)} - \int_{\partial D} \frac{\partial u(y)}{\partial \nu(y)} \frac{\partial \Phi(x,y)}{\partial \nu(x)} ds(y), \quad x \in \mathbb{R}^2 \backslash D$$

implying that

$$\frac{\partial u^{inc}(x)}{\partial \nu(x)} = \frac{\partial u(x)}{\partial \nu(x)} + \int_{\partial D} \frac{\partial u(y)}{\partial \nu(y)} \frac{\partial \Phi(x,y)}{\partial \nu(x)} ds(y), \quad x \in \mathbb{R}^2 \backslash D.$$

Thus, letting $x \to \partial D$, we find that

$$\left\{ \frac{\partial u^{inc}(x)}{\partial \nu(x)} \right\}_+ = \left\{ \frac{\partial u(x)}{\partial \nu(x)} \right\}_+ + \left\{ \int_{\partial D} \frac{\partial u(y)}{\partial \nu(y)} \frac{\partial \Phi(x,y)}{\partial \nu(x)} ds(y) \right\}_+, \quad x \in \mathbb{R}^2 \backslash D.$$

Since $\dfrac{\partial u^{inc}(x)}{\partial \nu(x)}$ is continuous, we obtain

$$\frac{\partial u^{inc}(x)}{\partial \nu(x)} = \frac{\partial u(x)}{\partial \nu(x)} + \frac{1}{2} \left(\mathcal{K}' \frac{\partial u}{\partial \nu} \right)(x) - \frac{1}{2} \frac{\partial u(x)}{\partial \nu(x)}, \quad x \in \partial D,$$

which gives

$$\frac{\partial u(x)}{\partial \nu(x)} + \left(\mathcal{K}' \frac{\partial u}{\partial \nu} \right)(x) = 2 \frac{\partial u^{inc}(x)}{\partial \nu(x)}, \quad x \in \partial D. \tag{2.40}$$

Now, adding (2.39) to (2.40), we get the following equality

$$\frac{\partial u(x)}{\partial \nu(x)} + \left(\mathcal{K}' \frac{\partial u}{\partial \nu} \right)(x) - i\eta \left(\mathcal{S}\frac{\partial u}{\partial \nu} \right)(x) = 2 \frac{\partial u^{inc}(x)}{\partial \nu(x)} - 2i\eta u^{inc}(x), \quad x \in \partial D.$$

Equivalently, letting $\vartheta(x) = \dfrac{\partial u(x)}{\partial \nu(x)}$, we end up with the following combined field integral equation

$$\vartheta(x) + \int_{\partial D} \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(x)} - i\eta \Phi(x,y) \right\} \vartheta(y) ds(y) = 2 \left(\frac{\partial u^{inc}(x)}{\partial \nu(x)} - i\eta u^{inc}(x) \right) \quad x \in \partial D \tag{2.41}$$

for the unknown $\vartheta(x)$.

In order to guarantee uniqueness of solutions, one must show that the operator $I + \mathcal{K}' - i\eta\mathcal{S} : C(\partial D) \longrightarrow C(\partial D)$ is bijective. To this end, observe that $\mathcal{K}' - i\eta\mathcal{S}$ and $\mathcal{K} - i\eta\mathcal{S}$ are compact operators from $C(\partial D)$ to $C(\partial D)$. Since S is self-adjoint, and $\mathcal{K}$ and $\mathcal{K}'$ are adjoint with respect to the bilinear form $\langle \phi, \varphi \rangle = \int_{\partial D} \phi\varphi ds$ on $C(\partial D)$, the operators $\mathcal{K}' - i\eta\mathcal{S}$ and $\mathcal{K} - i\eta\mathcal{S}$ are also adjoint with respect to this bilinear form. Therefore, by Fredholm's theorem (see Theorem A.5), in the dual system $\langle C(\partial D), C(\partial D) \rangle$, the operators $I + \mathcal{K}' - i\eta\mathcal{S}$ and $I + \mathcal{K} - i\eta\mathcal{S}$ have the same nullity. As we have shown that $I + \mathcal{K} - i\eta\mathcal{S}$ is injective, we can conclude that $I + \mathcal{K}' - i\eta\mathcal{S}$ has trivial null space. Consequently, by the Riesz theorem (see Theorem A.4), we get that the operator $I + \mathcal{K}' - i\eta\mathcal{S}$ is bijective.

3. MULTI-GRID METHOD

In this chapter, following Kress [9] and Hackbusch [12], we discuss two classical iterative numerical methods: two- and multi-grid methods. In the literature, these methods are first introduced for the numerical implementations of general elliptic boundary value problems. We shall see that both methods can be well suited to solve the integral equations as well, and are especially apposite for solving the linear systems originating from the numerical treatment of Fredholm integral equations of the second kind [9]

$$\varphi = A\varphi + f. \tag{3.1}$$

These methods are grounded on the two basic mechanisms: smoothing operation on the present level and defect correction on some coarser levels. We discuss only two version of the two-grid methods due to Brakhage [10] and Atkinson [11] respectively and multi-grid iterations defined by Hackbusch [12]. For these iterative methods, convergence properties are derived using the collectively compact operator theory by Anselone [16] and Atkinson [11]. In the last section we give algorithms and illustrate numerical results for two- and multi-grid methods.

3.1. Preliminaries

In this section, we give the basic functional analysis tools to prove convergence of the two- and multi-grid schemes.

Theorem 3.1. [9] *Let X and Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator with a bounded inverse $A^{-1} : Y \to X$. Assume the sequence $A_n : X \to Y$ of bounded linear operators to be norm convergent $\|A_n - A\| \to 0$, $n \to \infty$. Then for* *sufficiently large n, more precisely for all n with*

$$\|A_{-1}(A_n - A)\| < 1,$$

the inverse operators $A_n^{-1} : Y \to X$ exist and are bounded by

$$\|A_n^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}. \tag{3.2}$$

For the solutions of the equations

$$A\varphi = f \quad \textit{and} \quad A_n\varphi_n = f_n$$

we have the error estimate

$$||\varphi_n - \varphi|| \leq \frac{||A^{-1}||}{1 - ||A^{-1}(A_n - A)||}\{||(A_n - A)\varphi|| + ||f_n - f||\}. \tag{3.3}$$

Proof. Assume that n is sufficiently large so that we have $\|A^{-1}(A_n - A)\| = \|A^{-1}(A - A_n)\| \leq \|A^{-1}\|\|(A - A_n)\| \leq 1$. Then, the Neumann series theorem (see Theorem A.1) entails

$$||[I - A^{-1}(A_n - A)]^{-1}|| \leq \frac{1}{1 - \|A^{-1}(A_n - A)\|}. \tag{3.4}$$

Next, by the identity $A_n^{-1} = (I - A^{-1}(A_n - A))^{-1}A^{-1}$ and the equality above yields

$$||[I - A^{-1}(A_n - A)]^{-1}A^{-1}|| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}.$$

Finally, the estimate (3.3) can be obtained by

$$A_n(\varphi_n - \varphi) = f_n - f + (A - A_n)\varphi.$$

□

Now, we introduce the notion of collectively compact operators.

Definition 3.2. [9] *A set $\mathcal{A} = \{A : X \to Y\}$ of linear operators mapping a normed space X into a normed space Y is called* collectively compact *if for each bounded set $U \subset X$ the image set $\mathcal{A}(U) = \{A\varphi : \varphi \in U, A \in \mathcal{A}\}$ is relatively compact.*

Notice that, any element of a collectively compact set is compact and a set of finitely many compact operators is collectively compact. Moreover, by definition, a sequence $(A_n)_{n\in\mathbb{N}}$ is collectively compact if the set $\{A_n : n \in \mathbb{N}\}$ is. Also note that the limit operator A of a pointwisely convergent sequence $(A_n)_{n\in\mathbb{N}}$ of collectively compact operators is compact [9].

Theorem 3.3. [9] *Let X,Z be normed spaces and Y be a Banach space. Let $\mathcal{A}$ be a collectively compact set of operators mapping X into Y and let $L_n : Y \to Z$ be a pointwise convergent sequence of bounded linear operators with limit operator $L : Y \to Z$. Then*

$$\|(L_n - L)A\| \to 0, \;\; n \to \infty,$$

uniformly for all $A \in \mathcal{A}$, i.e.,

$$\sup_{A\in\mathcal{A}} \|(L_n - L)A\| \to 0, \;\; n \to \infty.$$

The following corollary to Theorem 3.3 is used to prove the convergence of two-grid methods.

Corollary 3.4. [9] *Let X be a Banach space and let $A_n : X \to X$ be a collectively compact and pointwise convergent sequence with limit operator $A : X \to X$. Then*

$$\|(A_n - A)A\| \to 0 \;\; \text{and} \;\; \|(A_n - A)A_n\| \to 0, \quad n \to \infty.$$

From the preceding corollary we can prove the next theorem concerning the error analysis for equations of the second kind due to the Brahkage [10] and Anselone and Moore [17].

Theorem 3.5. [9] *Let $A : X \to X$ be a compact linear operator in a Banach Space X and let $I - A$ be injective. Assume the sequence $A_n : X \to X$ to be collectively compact and pointwise convergent $A_n\varphi \to A\varphi$, $n \to \infty$, for all $\varphi \in X$. Then for sufficiently large n, more precisely for all n with*

$$\|(I - A)^{-1}(A_n - A)A_n\| \leq 1,$$

the inverse operators $(I - A_n)^{-1} : X \to X$ exist and are bounded by

$$\|(I - A_n)^{-1}\| \leq \frac{1 + \|(I - A)^{-1}A_n\|}{1 - \|(I - A)^{-1}(A_n - A)(A_n)\|}. \tag{3.5}$$

For the solutions of the equations

$$\varphi - A\varphi = f \quad and \quad \varphi_n - A_n\varphi_n = f_n$$

we have the error estimate

$$\|\varphi_n - \varphi\| \leq \frac{1 + \|(I - A)^{-1}A_n\|}{1 - \|(I - A)^{-1}(A_n - A)(A_n)\|}\{\|(A_n - A)\varphi\| + \|f_n - f\|\}.$$

Proof. We briefly sketch a proof of the theorem. First observe that, by Riesz Theorem (see Theorem A.4), the inverse of $(I - A)$ exists and satisfies the identity

$$(I - A)^{-1} = I + (I - A)^{-1}A$$

from which we may conclude that $B_n = I + (I - A)^{-1}A_n$ approximate the $(I - A_n)^{-1}$. Also note that

$$B_n(I - A_n) = I - C_n \tag{3.6}$$

where $C_n = (I-A)^{-1}(A_n-A)A_n$. Next Corollary 3.4 entails that norm of C_n decreases to zero as $n \to \infty$. Thus, choosing sufficiently large n so that $\|C_n\| < 1$ enable us to make use of the Neumann series Theorem (see Theorem A.1) to conclude that $(I-C_n)^{-1}$ exists and satisfies

$$\|(I - C_n)^{-1}\| \leq \frac{1}{1 - \|C_n\|}.$$

Furthermore, since A_n is compact, and from (3.6), $I-A_n$ is injective, invoking the Riesz Theorem (see Theorem A.4) yields the existence of the inverse of $I - A_n$. Moreover, again by utilizing (3.6), we obtain $(I - A_n)^{-1} = (I - C_n)^{-1}B_n$ from which (3.5) follows. The error estimate can be established by the identity

$$(I - A_n)(\varphi_n - \varphi) = f_n - f + (A_n - A)\varphi.$$

□

The following corollary is an immediate consequence of the preceding theorem.

Corollary 3.6. [9] *Under the assumptions of Theorem 3.5 we have the error estimate*

$$\|\varphi_n - \varphi\| \leq C\{\|(A_n - A)\varphi\| + \|f_n - f\|\} \tag{3.7}$$

for all sufficiently large n and some constant C.

3.2. Defect Correction

In this section, following [9, 12, 18], we discuss the abstract framework of defect correction principle, which form a basis for the two- and multi-grid, for integral equations of the second kind. It basicly consists of the sequence of operations: *restriction*, *coarse grid solution* and *prolongation*, which we described in details in the last section. For a more detailed description of the defect correction method, we refer to Stetter [8].

Assume that $A : X \to X$ be a bounded linear operator on a Banach space X so that $(I - A)^{-1}$ exists and is bounded . Then, we approximate the solution of the Fredholm integral equations of the second kind

$$\varphi - A\varphi = f \tag{3.8}$$

by solutions of the approximating equations

$$\varphi_n - A_n\varphi_n = f_n \tag{3.9}$$

where A_n approximates A, f_n approximates f, and we expect that φ_n approximates φ. Here we define the approximating operator A_n's by Nyström discretization

$$A_n\vartheta(x) = \sum_{j=0}^{q_n} w_n^j n K(x, x_n^j)\vartheta(x_n^j),$$

where $\{x_n^j : j = 1, 2, ..., q_n, \;\; q_n \in \mathbb{N}\}$ is the set nodal points corresponding to level n. Furthermore, the sequence (A_n) of bounded linear operators $A_n : X \to X$ is collectively compact and pointwise convergent which enable us to utilize Theorem 3.5 to guarantee existence and uniqueness of a solution to the approximate equation (3.9) [16]. In what follows, the index n expresses different levels of Nyström discretization in which the number of nodal points varies as the level changes. For convenience we set $F := (I - A)$ and $F_n := (I - A_n)$.

Suppose that we are given an approximate solution φ_n^{input} of (3.9). Then, in order to reduce the high oscillations in the error $\varphi_n - \varphi_n^{input}$, we smooth the given inital approximation. Among the class of smoothing operations (Gauss-Seidel, Jacobi and Picard's iteration, conjugate gradient methods,...) for general linear and nonlinear system, as is anticipated from the nature of our problem, we shall use *Picard's iteration* to smooth the given approximate input. For more detailed descriptions of class of smoothing operations, we refer to [12] and [19]. Let $\mathcal{P}^j$ denotes the j times Picard's iteration operator. Then, by the smoothing operation

$$\varphi_{n,0} = \mathcal{P}^j[\varphi_{input}],$$

we expect that we now have a better approximation $\varphi_{n,0}$. Next, we elucidate the defect correction iteration.

Observe that if δ_n is the exact correction function of the approximate solution $\varphi_{n,0}$ of $F_n\varphi_n = f_n$, then we must have

$$\begin{aligned} F_n(\varphi_{n,0} + \delta_n) = f_n &\Leftrightarrow F_n(\varphi_{n,0} + \delta_n) = f_n \\ &\Leftrightarrow F_n\varphi_{n,0} + F_n\delta_n = f_n \\ &\Leftrightarrow F_n\delta_n = f_n - F_n\varphi_{n,0} \\ &\Leftrightarrow F_n\delta_n = d_n \end{aligned}$$

where $d_n := f_n - F_n\varphi_{n,0}$ is the defect or residual function. These equivalent statements reveal that δ_n is the exact correction function of $\varphi_{n,0}$ if and only if it satisfies the defect correction equation

$$F_n\delta_n = d_n. \tag{3.10}$$

Next, observe that since $\delta_n = \varphi_n - \varphi_{n,0}$, it is comparatively small with respect to $\varphi_{n,0}$, and thus it is superfluous to solve the defect correction equation exactly. Instead, we write

$$\delta_n = B_n d_n \tag{3.11}$$

where the bounded linear operator B_n approximates the inverse F_n^{-1}. Thus we deduce

$$\begin{aligned}\varphi_{n,1} &:= \varphi_{n,0} + \delta_n \\ &= \varphi_{n,0} + B_n d_n \\ &= \varphi_{n,0} + B_n(f_n - F_n\varphi_{n,0}) \\ &= \varphi_{n,0} + B_n f_n - B_n F_n \varphi_{n,0} \\ &= [I - B_n F_n]\varphi_{n,0} + B_n f_n\end{aligned}$$

as an improved approximate solution to (3.9). Continuing in this way leads to following defect (residual) correction iteration

$$\varphi_{n,i+1} := [I - B_n F_n]\varphi_{n,i} + B_n f_n, \quad i = 0, 1, 2, 3... \tag{3.12}$$

for the solution of (3.9). If $\|I - B_n F_n\| < 1$, or the spectral radius of $I - B_n F_n$ is less than one, then, by contraction mapping principle, iteration (3.12) converges to a solution φ_n of $B_n F_n \varphi_n = B_n f_n$ and the solution is unique. Accordingly, as the unique solution φ_n of (3.9) automatically solves the equation $B_n F_n \varphi_n = B_n f_n$, the iteration (3.12) converges to the unique solution of (3.9) if it happen to converge [9].

3.3. Two-Grid Methods

Following Kress [9], here we discuss two type of the two-grid-method due to Brakhage [10] and Atkinson [11].

Definition 3.7. *The two-grid method is an iterative numerical scheme comprised of two main constituents, namely smoothing operations and defect correction iterations (3.12) with the approximate inverse B_n on the level n given by the correct inverse F_m^{-1} for some $m < n$.*

As is clearly, the name “two-grid” comes from the fact that we use only two levels, namely the present level n and the coarser level $m < n$. Next, observe that, we can write

$$
\begin{aligned}
I - B_n F_n &= I - F_m^{-1} F_n \\
&= I - F_m^{-1} + F_m^{-1} A_n \\
&= F_m F_m^{-1} - F_m^{-1} + F_m^{-1} A_n \\
&= F_m^{-1}(I - A_m - I) + F_m^{-1} A_n \\
&= F_m^{-1}(A_n - A_m). \qquad (3.13)
\end{aligned}
$$

By previous section, recalling that the operators A_n approximate the operator A, the choice of B_n as in preceding definition means to solve the residual correction equation in some coarser level. In this section, we focus on the two extreme cases of the two-grid methods where we use either the preceding level $m = n - 1$ as is done Brakhage [10] or the coarsest level $m = 0$ proposed by Atkinson [11] to solve the residual correction equation.

Thus, taking $B_n^{(1)} = F_{n-1}^{-1}$, the first variant of the two-grid iteration we shall consider reads

$$
\begin{aligned}
\varphi_{n,i+1} &:= [I - B_n^{(1)} F_n]\varphi_{n,i} + B_n^{(1)} f_n, \qquad (3.14) \\
&= F_{n-1}^{-1}(A_n - A_{n-1})\varphi_{n,i} + F_{n-1}^{-1} f_n \quad i = 0, 1, 2, 3... \qquad (3.15)
\end{aligned}
$$

and the second variant reads

$$
\begin{aligned}
\varphi_{n,i+1} &:= [I - B_n^{(2)} F_n]\varphi_{n,i} + B_n^{(2)} f_n, \qquad (3.16) \\
&= F_0^{-1}(A_n - A_0)\varphi_{n,i} + (F_0)^{-1} f_n \quad i = 0, 1, 2, 3... \qquad (3.17)
\end{aligned}
$$

where $B_n^{(2)} = F_0^{-1}$.

The next theorem regards the convergence of the two-grid iterations.

Theorem 3.8. [9] *Assume that the sequence of operators $A_n : X \to X$ is either norm* *convergent or collectively compact and pointwise convergent $A_n\varphi \to A\varphi$, $n \to \infty$, for all $\varphi \in X$. Then two-grid iteration*

$$\varphi_{n,i+1} := F_{n-1}^{-1}\{(A_n - A_{n-1})\varphi_{n,i} + f_n\} \quad i = 0, 1, 2, 3... \tag{3.18}$$

using two consecutive grids converges, provided that n is sufficiently large.

Proof. Let $\|A_n - A\| \to 0$, as $n \to \infty$. Then we immediately get that F_n is norm convergent to F. Accordingly, from estimate (3.2) we get

$$\begin{aligned}\|F_n^{-1}\| &\leq \frac{\|A^{-1}\|}{1 - \|A^{-1}[F_n - F]\|} \\ &= \frac{\|A^{-1}\|}{1 - \|A^{-1}(A - A_n)\|} \\ &\leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|(A - A_n)\|} \leq C\end{aligned}$$

for all sufficiently large $n \in \mathbb{N}$. Thus the result follows from the fact that

$$\|F_{n-1}^{-1}(A_n - A_{n-1})\| \leq C\|A_n - A_{n-1}\| \to 0, \quad as \quad n \to \infty. \tag{3.19}$$

Now we can deduce that for sufficiently large $n \in \mathbb{N}$ so that $\|F_{n-1}^{-1}(A_n - A_{n-1})\| < 1$, the iteration (3.18) converge to the unique solution φ_n of (3.9).

Next assume that the sequence (A_n) is collectively compact and pointwise convergent. Since pointwise convergence $A_n \to A$ of a collectively compact sequence implies the compactness of the limit operator, A is compact and from theorem (3.5) the inverse of F_n's exist and are uniformly bounded for sufficiently large n. Therefore we can deduce that the sequence $\tilde{A}_n := F_{n-1}^{-1}(A_n - A_{n-1})$ is collectively compact.

Then, from the pointwise convergence $A_n\varphi - A_{n-1}\varphi \to 0$, $n \to \infty$ for all $\varphi \in X$, by Theorem (3.3), setting $\mathcal{A} := \left\{\tilde{A}_n : n \in \mathbb{N}\right\}$, $L_n := A_n - A_{n-1} : X \to X$, $L = 0$

operator and $X = Y = Z$, we get

$$\|(A_n - A_{n-1})F_{n-1}^{-1}(A_n - A_{n-1})\| \to 0, \quad n \to \infty.$$

Moreover, by the uniform boundedness of F_{n-1}^{-1} we obtain

$$\begin{aligned} 0 &< \| \left\{ F_{n-1}^{-1}(A_n - A_{n-1}) \right\}^2 \| \\ &\leq \|F_{n-1}^{-1}\| \|(A_n - A_{n-1})F_{n-1}^{-1}(A_n - A_{n-1})\| \to 0, \end{aligned}$$

as $n \to \infty$ which implies

$$\| \left\{ F_{n-1}^{-1}(A_n - A_{n-1}) \right\}^2 \| \to 0,$$

as $n \to \infty$. Hence the result follows. $\square$

The next theorem is about convergence of the two-grid iterations taking the approximate inverse $B_n = F_0^{-1}$ in defect correction principle. We see that this iteration converges whenever A_0 is already a good approximation to A.

Theorem 3.9. [9] *Assume that the sequence of operators $A_n : X \to X$ is either norm convergent or collectively compact and pointwise convergent $A_n\varphi \to A\varphi$, $n \to \infty$, for all $\varphi \in X$. Then the two-grid iteration*

$$\varphi_{n,i+1} := F_0^{-1}\{(A_n - A_0)\varphi_{n,i} + f_n\} \quad i = 0, 1, 2, 3... \tag{3.20}$$

using a fine and coarse grid converges, provided that the approximation A_0 is already sufficiently close to A.

Proof. First assume that the sequence (A_n) of operators are norm convergent. Then, as in the previous theorem, F_n^{-1} are uniformly bounded by a constant C. Thus choose a coarsest grid so that

$$\|A_n - A_0\| \leq \frac{1}{2C} \quad \text{for} \quad n > 0.$$

Accordingly we have,

$$\|F_0^{-1}(A_n - A_0)\| \leq \|F_0^{-1}\|\|(A_n - A_0)\| \leq C\frac{1}{2C} = \frac{1}{2}.$$

which implies that two-grid iteration converges.

Now assume that the sequence (A_n) is collectively compact and pointwise convergent. In order to show the convergence of the iteration (3.20), we can select the coarsest grid so that

$$\|F_0^{-1}(A_m - A_0)F_0^{-1}(A_n - A_0)\| < \frac{1}{2}$$

for all $m, n \geq 0$, which yields

$$\| \left(F_0^{-1}(A_n - A_0)\right)^2 \| < 1.$$

Here we used Theorem 3.3 applied to the collectively compact sequence $\tilde{A}_n := F_0^{-1}(A_n - A_0)$ and $L_m := A_m - A_0$ converging to $L := A - A_0$. Thus the result follows. $\square$

3.4. Multi-Grid Methods

In this section we discuss two different style of the multi-grid method by following Hackbusch [12], Schippers [20] and Kress [9]. The two-grid methods described in Theorem 3.8 and Theorem 3.9 use only two levels whereas the multi-grid methods make use of $n + 1$ levels. The basic idea of multi-grid algorithm is as follows.

Recall that in two-grid algorithm we ended up with the defect correction equation

$$F_{n-1}\delta_{n-1} = d_{n-1}. \tag{3.21}$$

on the level $n-1$ and solved it exactly in that level. Then, instead of solving the correction equation exactly on the level $n-1$, we can go further and apply same defect correction principle for approximating the solution of linear system $F_{n-1}\delta_{n-1} = d_{n-1}$. Thus, as is expected, we now have a new defect correction equation $F_{n-2}\delta'_{n-2} = d'_{n-2}$ on the level $n-2$. Notice that, we have used the three levels $n, n-1$ and $n-2$ up to now. In fact we can also apply same defect correction procedure to approximate the solution of $F_{n-2}\delta'_{n-2} = d'_{n-2}$. and continue in this way until we reach the coarsest level, then we solve the correction equation exactly there. Thus we have used total of $n+1$ levels, namely $\{n, n-1, n-2, ..., 0\}$. In each level, the exact correction function is approximated by applying multi-grid algorithm p times in which the initial approximation is taken to be zero. More compactly, we portray multi-grid algorithm as in the next definition.

Definition 3.10. [9] *The multi-grid method is an iterative numerical scheme based upon composition of two main characteristics: smoothing operations and defect correction iteration of the form (3.12) with the approximate inverse defined recursively by*

$$\begin{aligned} B_0^{(3)} &:= F_0^{-1}, \\ B_n^{(3)} &:= \sum_{m=0}^{p-1} [I - B_{n-1}^{(3)} F_{n-1}]^m B_{n-1}^{(3)}, \quad n = 1, 2, ... \end{aligned} \tag{3.22}$$

for some $p \in \mathbb{N}$.

Next we show that multi-grid method covers the advantages of the two-grid method. First observe that, by induction on p and recursive formula (3.22) we have

$$I - B_n^{(3)} F_{n-1} = [I - B_{n-1}^{(3)} F_{n-1}]^p \tag{3.23}$$

for all $n \in \mathbb{N}$. Let $M_n^{(3)} := I - B_n^{(3)} F_n$ be the multi-grid iteration operator using $n+1$ levels. Then from Definition (3.10) and (3.23), we deduce that

$$\begin{aligned} M_n^{(3)} &= I - B_n^{(3)} F_n = I - B_n^{(3)} I F_n = I - B_n^{(3)} F_{n-1} F_{n-1}^{-1} F_n \\ &= I - \left[I - I + B_n^{(3)} F_{n-1} \right] F_{n-1}^{-1} F_n \\ &= I - F_{n-1}^{-1} F_n + \{ I - B_n^{(3)} F_{n-1} \} F_{n-1}^{-1} F_n \\ &= F_{n-1}^{-1} (A_n - A_{n-1}) + \left\{ M_{n-1}^{(3)} \right\}^p F_{n-1}^{-1} F_n. \end{aligned}$$

This shows that the iteration operators $M_n^{(3)}$ can be defined by the recursive formula

$$\begin{aligned} M_1^{(3)} &= T_1^{(3)}, \\ M_n^{(3)} &= T_n^{(3)} + \left\{ M_{n-1}^{(3)} \right\}^p (I - T_n^{(3)}), \;\; n = 1, 2, ... \end{aligned} \tag{3.24}$$

where

$$T_n^{(3)} := F_{n-1}^{-1} (A_n - A_{n-1}) \tag{3.25}$$

are the two-grid iteration operators.

As is seen in the next theorem, convergence of the multi-grid method depends upon the approximation on each level. In particular, in order to guarantee convergence of the multi-grid method one should set the coarsest level so that approximation on that level is accurate enough.

Theorem 3.11. [9] *Assume that*

$$\| T_n^{(3)} \| \le q^{n-1} C \tag{3.26}$$

for all $n \in \mathbb{N}$ and some constant $q \in (0, 1]$ and $C > 0$ satisfying

$$C \le \frac{1}{2q} (\sqrt{1 + q^2} - 1). \tag{3.27}$$

Then, if $p \geq 2$, *we have*

$$\|M_n^{(3)}\| \leq 2q^{n-1}C < 1, \quad n \in \mathbb{N}.$$

Proof. First observe that $(\sqrt{1+q^2})^2 < (1+q)^2$ for $q > 0$ and that by (3.27) we have

$$\|M_1^{(3)}\| = \|T_1^{(3)}\| \leq q^{1-1}C = C < 2C \leq \frac{1}{q}(\sqrt{1+q^2}-1) < 1.$$

Thus, the assertion holds true for $n = 1$. Now suppose that it is correct for some $n \in \mathbb{N}$. Then, we use the recurrence relation (3.24) to obtain

$$\begin{aligned}
\|M_{n+1}\| &= \|T_{n+1}^{(3)} + \left\{M_n^{(3)}\right\}^p (I - T_{n+1}^{(3)})\| \\
&\leq \|T_{n+1}^{(3)}\| + \|M_n^{(3)}\|^p(\|I\| + \|T_{n+1}^{(3)}\|) \\
&= \|T_{n+1}^{(3)}\| + \|M_n^{(3)}\|^p(1 + \|T_{n+1}^{(3)}\|) \\
&\leq q^nC + (2q^{n-1})^p(1 + q^nC) \\
&\leq q^nC + (2q^{n-1})^2(1 + q^nC) \\
&= q^nC + \frac{4}{q}q^{2n-1}C^2 + \frac{4}{q}q^{2n-1}C^2q^nC \\
&\leq q^nC + \frac{4}{q}q^nC^2 + \frac{4}{q}C^2q^nC \\
&= q^nC\left(1 + \frac{4}{q}C(1+qC)\right) \leq q^nC2.
\end{aligned}$$

The last inequality holds because $4C(1+qC) \leq q$ for all $0 \leq C \leq \frac{\sqrt{1+q^2}-1}{2q}$. This follows from the fact that $f(t) = 4t(1-qt) - q \leq 0$ for all $t \in [0, \frac{\sqrt{1+q^2}-1}{2q}]$. $\square$

Recall that, $\varphi_n \to \varphi$ as $n \to \infty$ and $\varphi_{n,i} \to \varphi_n$ as $i \to \infty$. Since our primary aim is to approximate φ, large number of iterations on each level may be redundant. In the next definition, we describe the full multi-grid scheme or nested iteration which makes iteration error $\|\varphi_{n,i} - \varphi_n\|$ on each level approximately of the same order as the discretization error $\|\varphi_n - \varphi\|$ by fixing the iteration number so that these errors is maintained to be approximatly the same [9].

Definition 3.12. *Starting with $\tilde{\varphi}_0 := F_0^{-1} f_0$, the full multi-grid scheme constructs a sequence $(\tilde{\varphi})$ of approximations by performing k steps of the multi-grid iteration on $n+1$ levels using the preceding $\tilde{\varphi}_{n-1}$ as initial element.*

We note that the computational cost of the multi-grid and full multi-grid methods are $\mathcal{O}(2^{2(n+1)})$ for the case that the grid size on the level n is the half of the grid size on the level $n-1$ [12]. Before we state a convergence theorem for the full multi-grid method, we make some observations.

Let $\tilde{\varphi}_0 := F_0^{-1} f_0$ be the initial element in multi-grid scheme. For the sake of simplicity, set $\varphi_{1,0} = \tilde{\varphi}_0$ and then iterate the following scheme

$$\varphi_{1,i+1} = (I - B_1^{(3)} F_1^{(3)})\varphi_{1,i} + B_1^{(3)} f_1 \quad i : 0, 1, 2, ...$$

k times and set the out put $\varphi_{1,k} =: \tilde{\varphi}_1$ as an input for the iteration in the next level. Then set $\varphi_{2,0} = \varphi_{1,k}$ and perform the following iteration

$$\varphi_{2,i+1} = (I - B_2^{(3)} F_2^{(3)})\varphi_{2,i} + B_2^{(3)} f_2 \quad i : 0, 1, 2, ...$$

k times and set the out put $\varphi_{2,k} =: \tilde{\varphi}_2$ as an input for the iteration in the next level. Then set $\varphi_{3,0} = \varphi_{2,k}$ and perform k steps iteration in the next level. Continuing in this way, we get $\tilde{\varphi}_{n+1} = \varphi_{n+1,k}$ at $n+1$ level. Then,

$$
\begin{aligned}
\tilde{\varphi}_{n+1} &= \varphi_{n+1,k} = (I - B^{(3)}_{n+1}F_{n+1})\varphi_{n+1,k-1} + B^{(3)}_{n+1}f_{n+1} \\
&= [I - B^{(3)}_{n+1}F_{n+1}]\left[(I - B^{(3)}_{n+1}F_{n+1})\varphi_{n+1,k-2} + B^{(3)}_{n+1}f_{n+1}\right] + B^{(3)}_{n+1}f_{n+1} \\
&= \left[I - B^{(3)}_{n+1}F_{n+1}\right]^2 \varphi_{n+1,k-2} + \left[I - B^{(3)}_{n+1}F_{n+1}\right] B^{(3)}_{n+1}f_{n+1} + B^{(3)}_{n+1}f_{n+1} \\
&= \left[I - B^{(3)}_{n+1}F_{n+1}\right]^3 \varphi_{n+1,k-3} + \left[I - B^{(3)}_{n+1}F_{n+1}\right]^2 B^{(3)}_{n+1}f_{n+1} \\
&+ \left[I - B^{(3)}_{n+1}F_{n+1}\right] B^{(3)}_{n+1}f_{n+1} + B^{(3)}_{n+1}f_{n+1} \\
&\dots \\
&= \left[M^{(3)}_{n+1}\right]^k \varphi_{n+1,0} + \sum_{m=0}^{k-1} \left[M^{(3)}_{n+1}\right]^m B^{(3)}_{n+1}f_{n+1} \\
&= \left[M^{(3)}_{n+1}\right]^k \tilde{\varphi}_n + \sum_{m=0}^{k-1} \left[M^{(3)}_{n+1}\right]^m B^{(3)}_{n+1}f_{n+1} \qquad (3.28)
\end{aligned}
$$

We are now ready to prove the following theorem.

Theorem 3.13. [9] *Assume that*

$$\|\varphi_n - \varphi\| \le Cq^n \qquad (3.29)$$

for some constants $0 < q < 1$ *and* $C > 0$, *and that* $t := \sup_{n\in\mathbb{N}} \|M^{(3)}_n\|$ *satisfies*

$$t^k < q.$$

Then for the approximation φ_n *obtained by the full multi-grid method we have that*

$$\|\tilde{\varphi}_n - \varphi_n\| \le C\frac{(q+1)t^k}{q - t^k}q^n, \quad n \in \mathbb{N}. \qquad (3.30)$$

Proof. Let

$$\alpha := \frac{(q+1)}{q - t^k}t^k.$$

Thus we have

$$(\alpha+1+q)t^k = \left(\frac{(q+1)}{q-t^k}t^k + 1 + q\right)t^k = \left(\frac{q(1+q)}{q-t^k}\right)t^k = q\frac{1+q}{q-t^k}t^k = q\alpha.$$

Next observe that (3.30) is trivially satisfied for $n = 0$ since

$$\tilde{\varphi}_0 = F_0^{-1} f_0 = \varphi_0 \Rightarrow \|\tilde{\varphi}_0 - \varphi_0\| = 0.$$

Now suppose that the assertion hold true for some $n \in \mathbb{N}$. Then first note that from $\varphi_{n+1} - A_{n+1}\varphi_{n+1} = f_{n+1}$ we can deduce

$$B_{n+1}^{(3)} f_{n+1} = B_{n+1}^{(3)} F_{n+1}\varphi_{n+1} \tag{3.31}$$

$$= [I - I + B_{n+1}^{(3)} F_{n+1}]\varphi_{n+1} \tag{3.32}$$

$$= \varphi_{n+1} - M_{n+1}^{(3)}\varphi_{n+1}. \tag{3.33}$$

Accordingly, from (3.28) and (3.31) we get

$$\begin{aligned}
\tilde{\varphi}_{n+1} &= \left[M_{n+1}^{(3)}\right]^k \tilde{\varphi}_n + \sum_{m=0}^{k-1} \left[M_{n+1}^{(3)}\right]^m B_{n+1}^{(3)} f_{n+1} \\
&= \left[M_{n+1}^{(3)}\right]^k \tilde{\varphi}_n + \sum_{m=0}^{k-1} \left[M_{n+1}^{(3)}\right]^m (\varphi_{n+1} - M_{n+1}^{(3)}\varphi_{n+1}) \\
&= \left[M_{n+1}^{(3)}\right]^k \tilde{\varphi}_n + \sum_{m=0}^{k-1} \left[M_{n+1}^{(3)}\right]^m \varphi_{n+1} - \left[M_{n+1}^{(3)}\right]^{m+1} \varphi_{n+1}) \\
&= \left[M_{n+1}^{(3)}\right]^k \tilde{\varphi}_n + \varphi_{n+1}) - \left[M_{n+1}^{(3)}\right]^k \varphi_{n+1}).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\|\tilde{\varphi}_{n+1} - \varphi_{n+1}\| &= \| \left[M_{n+1}^{(3)}\right]^k (\tilde{\varphi}_n - \varphi_{n+1})\| \leq t^k \|\tilde{\varphi}_n - \varphi_{n+1}\| \\
&\leq t^k \left(\|\tilde{\varphi}_n - \varphi_n\| + \|\varphi_n - \varphi\| + \|\varphi - \varphi_{n+1}\|\right) \\
&\leq t^k \left(C\alpha q n + Cq + Cq^{n+1}\right) \\
&= Ct^k(\alpha + 1 + q)q^n = C\alpha q^n.
\end{aligned}$$

The proof is completed now. □

3.5. Numerical Results

In this section, we illustrate the theoretical convergence result for the two- and multi-grid methods which are defined by the approximate inverses $B^{(1)}$ (Brakhage's method) and $B^{(3)}$ (Hackbusch's method) in the previous sections. As an example, the integral equation governed by the sound soft scattering problem

$$\vartheta(x) - \int_{\partial D} K(x,y)\vartheta(y)ds(y) = f(x), \quad x \in \partial D \tag{3.34}$$

where

$$K(x,y) = \frac{\partial \Phi(x,y)}{\partial \nu(x)} - ik\Phi(x,y) \text{ and } f(x) = 2\left(\frac{\partial u^{inc}(x)}{\partial \nu(x)} - iku^{inc}(x)\right) \tag{3.35}$$

is solved for various values of the parameter k for the unknown $\vartheta(x)$. We test the two- and multi-grid algorithms for the unit circle D for which, in case $u^{inc}(x) = e^{ikx\cdot a}$ and $a = (1,0)$, the exact solution of (3.34) is [21]

$$\vartheta(x) = -\frac{2i}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{in(x+\pi/2)}}{H_n^{(1)}(k)}, \quad x \in [0, 2\pi].$$

The approximating operators A_n are defined via suitable quadrature rule

$$A_n\vartheta(x) = \sum_{j=0}^{2^{n+1}-1} w_n^j K(x, x_j)\vartheta(x_j),$$

where $\Omega_n = \{x_j = jh_n : j = 1, 2, ..., 2^{n+1}\}$ is the set of nodal points, $h_n = 2\pi(2^{n+1}-1)^{-1}$ is the grid size and the weights $\{w_n^j\}$ are given by $\{h_n/2, h_n, h_n, ...h_n, h_n/2\}$. Referring to Kress [9] and Anselone [16], the approximating operators (A_n) is collectively compact and pointwise convergent which guarantees the convergence of two- and multi-grid iterations for sufficiently large n.

3.5.1. Two-Grid Method: First Variant

As in the previous section in which two- and multi-grid method described in a more abstract framework, assume that we have some approximation $\vartheta_{n,0}$ for the solution of discretised equation

$$\vartheta_n - A_n\vartheta_n = f_n, \quad x \in \partial D. \tag{3.36}$$

Next we shall improve given approximation. To this end, we first smooth the given approximation by a $1-$step Picard's iteration to get an intermediate result

$$\vartheta_{n,1} = A_n\vartheta_{n,0} + f_n. \tag{3.37}$$

Then the error $\delta_n = \vartheta_n - \vartheta_{n,1}$ is smooth compared with $\vartheta_n - \vartheta_{n,0}$ [12]. If δ_n is the exact correction, i.e., $\vartheta_n = \vartheta_{n,1} + \delta_n$, then by pluging $\vartheta_{n,1}$ into the equation $\vartheta_n - A_n\vartheta_n = f_n$, we end up with the defect (residual) $d_n = f_n - \vartheta_{n,1} + A_n\vartheta_{n,1}$. As is seen below, $\vartheta_{n,1}$ exactly solve the (3.36) if and only if $d_n = 0$. Moreover, since

$$(I_n - A_n)\delta_n = (I_n - A_n)\vartheta_{n,1} - (I_n - A_n)\vartheta_n = (I_n - A_n)\vartheta_{n,1} - f_n = d_n, \tag{3.38}$$

the exact correction δ_n is the solution of the defect correction equation $(I_n - A_n)\delta_n = d_n$. Instead of solving the defect correction equation on the present level, we shall approximate δ_n by using some coarser levels. By the fact that approximations of smooth functions by the coarser level can be sufficiently accurate [12], we approximate the solution of $(I_n - A_n)\delta_n = d_n$ by the defect correction equation $(I_m - A_m)\delta_m = d_m$ for some $m < n$. Taking $m = n-1$ to obtain the first variant of the two-grid method (Brakhage), we solve the equation $(I_{n-1} - A_{n-1})\delta_{n-1} = d_{n-1}$ exactly on the level $n-1$ and interpolate to level n in order to approximate the solution of $(I_n - A_n)\delta_n = d_n$. Here note that the matrix $(I_n - A_n)$ is defined for all levels $n \geq 1$. In the previous section, we carried out the convergence analysis of two-grid and multi-grid methods by considering the iteration matrices of the two methods as operators on each level, yet we need some interpolation and restriction operators to serve as a bridge between different levels for numerical implementation. Accordingly, we introduce a linear mapping R_n, called *restriction*, which restricts a function defined on level n to the coarser level $n-1$. we can simply choose R_n to be injection R_n^{inj} which is defined as

$$(R_n^{inj} d_n)(x) = d_n(x) \text{ for all } x \in \Omega_{n-1} \subset \Omega_n.$$

However, since R_n^{inj} omits the values of a function at $x \in \Omega_n \backslash \Omega_{n-1}$, it may result in loss of accuracy. Thus we should also consider components of a function on $\Omega_n \backslash \Omega_{n-1}$. To this end we shall define the restriction operator R_n as

$$R_n d_n(x) = \frac{1}{4}[d_n(x - h_n) + 2d_n(x) + d_n(x + h_n)] \text{ for } x \in \Omega_{n-1}$$

for which the corresponding matrix is then

$$R_n = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}_{(2^n-1)\times(2^{n+1}-1)}$$

Next, letting $d_{n-1} = R_n d_n$, we can now obtain the exact correction $\delta_{n-1} = (I_{n-1} - A_{n-1})^{-1} d_{n-1}$ on the coarser level $n-1$. We expect that we can approximate δ_n by δ_{n-1} which is defined on Ω_{n-1}. Our next aim is to interpolate the exact correction function δ_{n-1} to the present level n. To this end, we introduce the linear interpolation operator (called *prolongation*) denoted by P_n, which approximates the δ_n by interpolating the

δ_{n-1}. We simply choose P_n to be the piecewise linear interpolation defined as

$$(P_n\delta_{n-1})(x) = \begin{cases} \delta_{n-1}(x) & x \in \Omega_{n-1} \\ 0 & x = 0, 2\pi \\ [\delta_{n-1}(x-h_n) + \delta_{n-1}(x+h_n)]/2 & \text{otherwise.} \end{cases} \tag{3.39}$$

Thus P_n can be represented by $P_n = 2[R_n]^\intercal$. Since $\vartheta_n = \vartheta_{n,1} + \delta_n$ is the exact solution and $\widetilde{\delta_n} = P_n\delta_{n-1}$ is an approximation to δ_n, we can improve the value of $\vartheta_{n,1}$ by $\vartheta_{n,1}+\widetilde{\delta_n}$. To summarize, let $\vartheta_{n,0}$ be a given approximation to the solution of $(I_n - A_n)\vartheta_n = f_n$. Then

Step 1: *Smoothing the Input by Picard Iteration*

$\vartheta_{n,1} = A_n\vartheta_{n,0} + f_n$

Step 2: *Calculation of the Defect (Residual)*

$d_n = f_n - (I_n - A_n)\vartheta_{n,1}$

Step 3: *Restriction of the Defect to the Coarse Level*

$d_{n-1} = R_n d_n$

Step 4: *Solution of the Exact Correction on the Coarse Level*

$\delta_{n-1} = (I_{n-1} - A_{n-1})^{-1} d_{n-1}$

Step 5: *Interpolation of the Exact Correction from Coarse to the Present Level*

$\widetilde{\delta_n} = P_n\delta_{n-1}$

Step 6: *Improving the Given Approximate Solution*

$\vartheta_{n,2} = \vartheta_{n,1} + \widetilde{\delta_n}$.

More compactly, after the smoothing operation, one can write

$$\vartheta_{n,2} = \left[I_n - P_n(I_{n-1} - A_{n-1})^{-1}R_n(I_n - A_n)\right]\vartheta_{n,1} - P_n(I_{n-1} - A_{n-1})^{-1}R_n f_n,$$

or, more abstractly

$$\vartheta_{n,2} = \left[I - F_{n-1}^{-1}F_n\right]\vartheta_{n,1} - F_{n-1}^{-1}f_n$$

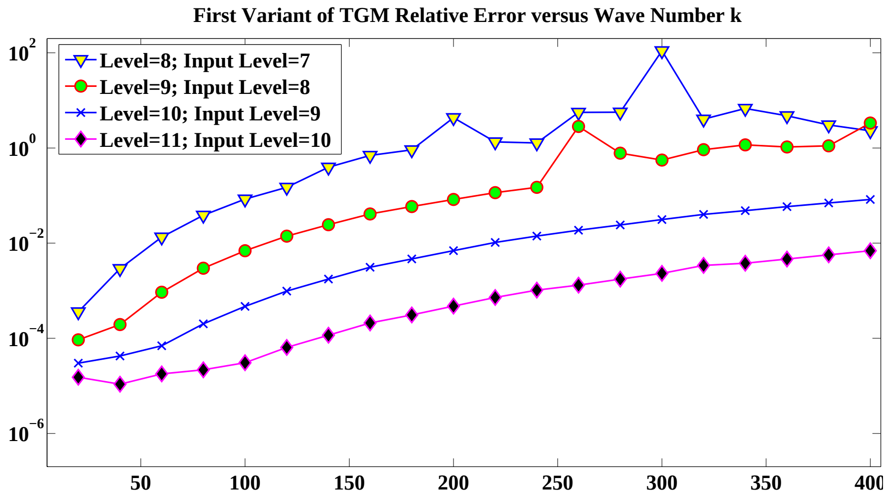
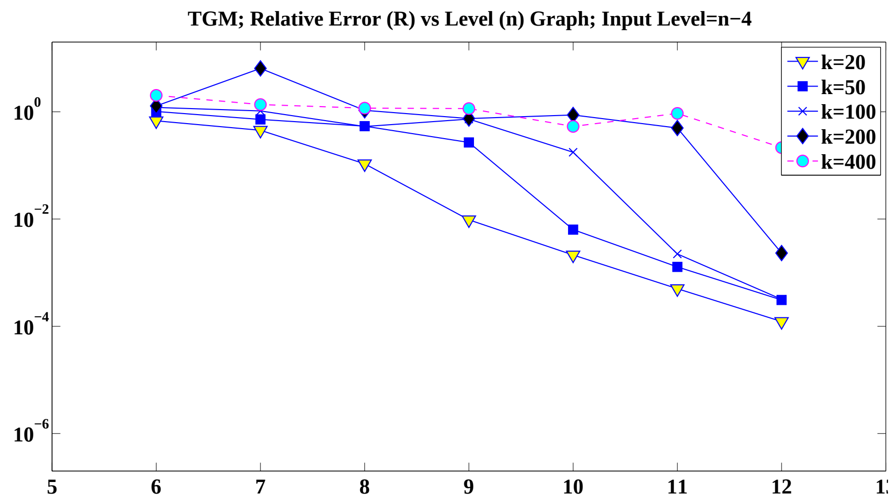


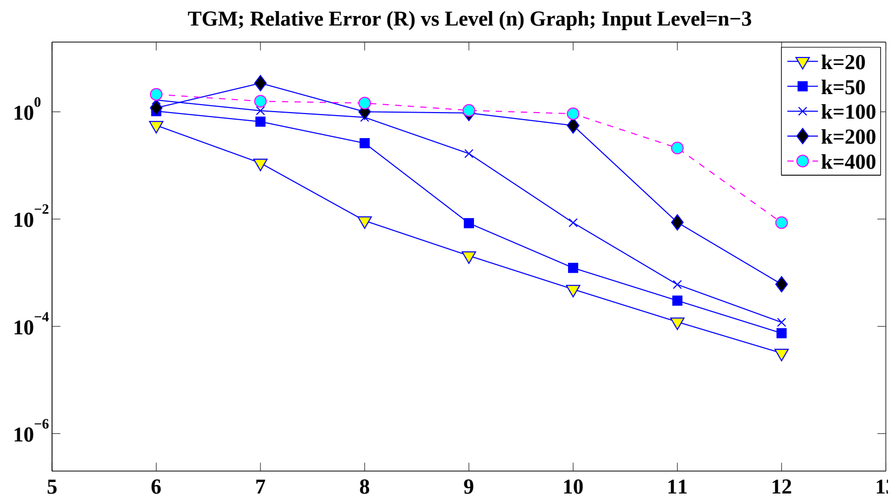
Figure 3.1. Wave Number versus Relative Error.

which is the first variant of the two-grid method proposed by Brakhage [10]. We, of course, choose $n \in \mathbb{N}$ so that Theorem 3.14 guarantees the convergence of two-grid method.

Next, we depict the convergence behavior of the first variant of the two-grid method. In Figure 3.1, we see that relative errors increase as the wave number k increases for fixed level. In each level, we set initial approximations to be Nyström solution of the previous level. As we know that increase in k leads to increase in oscillation of the integrand in 3.34, classical numerical methods naturally loose their robustness for $k \gg 1$. On the other hand, Figures 3.2 and 3.3 are the plots of relative error versus level L for different values of the wave number k. In each plot, initial approximation is the Nyström solution of the integral equation (3.34) on the level $L-4, L-3, L-2$ and $L-1$ where L represents the present level. As is seen clearly, as the level of initial approximation increases which means that the initial function approximates the solution better, the relative error decreases.

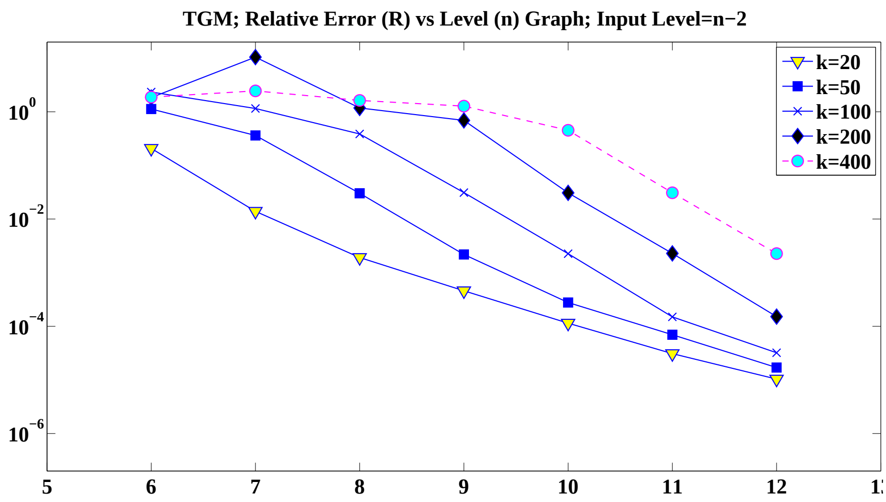


(a) Input Level=Present Level-4

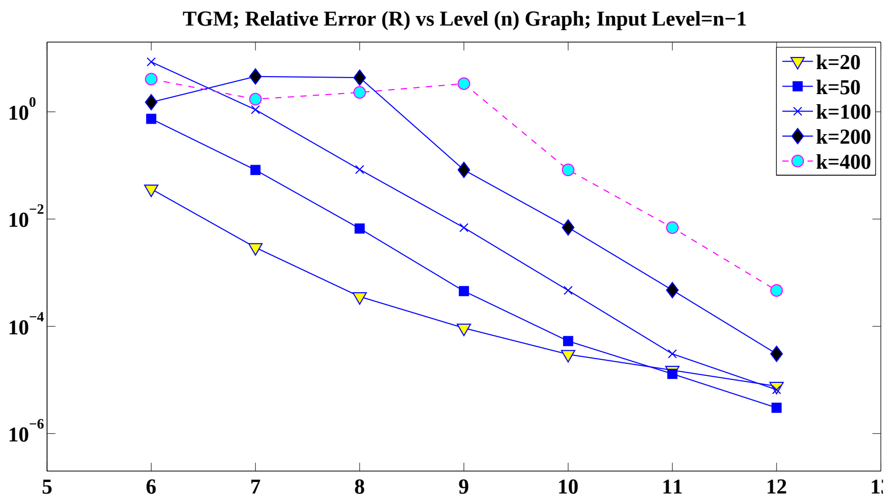


(b) Input Level=Present Level-3

Figure 3.2. Relative Error vs Level for the Two-Grid Method.



(a) Input Level=Present Level-2



(b) Input Level=Present Level-1

Figure 3.3. Relative Error vs Level for the Two-Grid Method.

3.5.2. Multi-Grid: First Variant

In this section, we demonstrate the convergence of the multi-grid algorithm numerically for small wave number k. As in the two-grid method, interpolation and restriction operator also play an important role in the multi-grid method. Following Hackbusch [12], we first observe that one can write $T_n^{(3)}$ as

$$\begin{aligned} T_n^{(3)} &= I_n - P_n(I_{n-1} - A_{n-1})^{-1}R_n(I_n - A_n) \\ &= I_n - P_nR_n + P_n(I_{n-1} - A_{n-1})^{-1}[(I_{n-1} - A_{n-1})R_n - R_n(I_n - A_n)] \\ &= I_n - P_nR_n + P_n(I_{n-1} - A_{n-1})^{-1}[R_nA_n - A_{n-1}R_n] \end{aligned}$$

Hence, from the recursive definition of the multi-grid iteration (3.24), we obtain the recursively defined multi-grid iteration matrices

$$M_1^{(3)} = T_1^{(3)} \tag{3.40}$$

$$M_n^{(3)} = T_n^{(3)} + P_n\big(M_{n-1}^{(3)}\big)^p(I_{n-1} - A_{n-1})^{-1}R_n(I_n - A_n) \tag{3.41}$$

$$= T_n^{(3)} + P_n\big(M_{n-1}^{(3)}\big)^p\big[R_n - (I_{n-1} - A_{n-1})^{-1}(R_nA_n - A_{n-1}R_n)\big], \tag{3.42}$$

where we use the identity

$$(I_{n-1} - A_{n-1})^{-1}R_n(I_n - A_n) = \big[R_n - (I_{n-1} - A_{n-1})^{-1}(R_nA_n - A_{n-1}R_n)\big].$$

For a given approximation $\vartheta_{n,0}$ to the solution of $F_n\vartheta_n = f_n$, the multi-grid algorithm can then be described as follows.

Step1: *Smoothing the Input by Picard's Iteration*

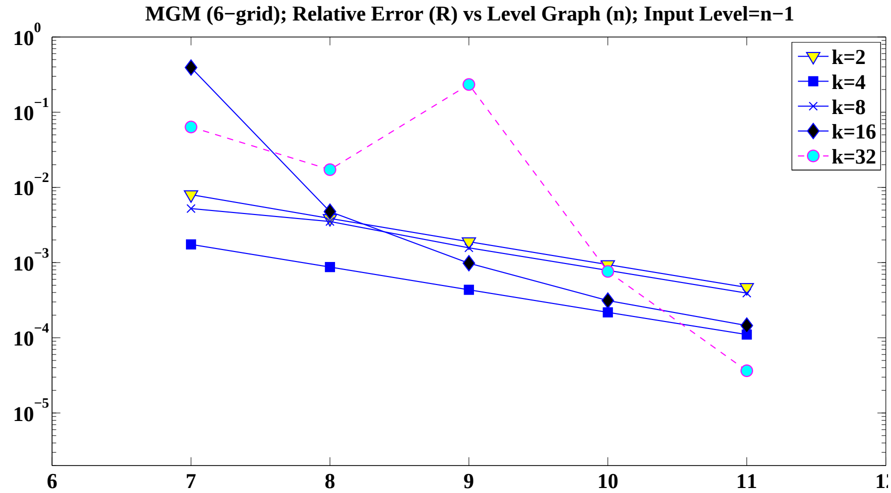
$$\vartheta_{n,1} = A_n\vartheta_{n,0} + f_n$$

Step2: *k Iteration of the Smoothed Approximation with Iteration Matrix* $M_n^{(3)}$

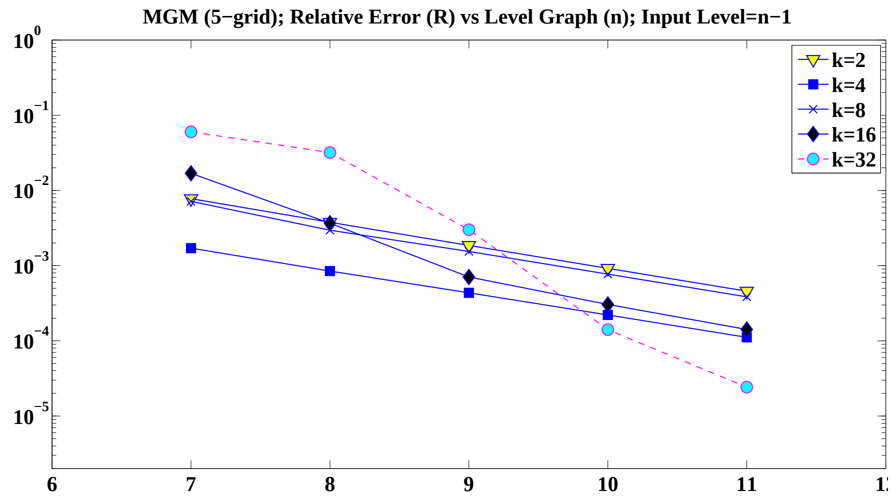
$$\vartheta_{n,j+1} = M_n^{(3)}\vartheta_{n,j} + B_n^{(3)}f_n \;\; j = 1, 2, .., k$$

where $B_n^{(3)}$ are defined recursively in (3.22) and $(M_n^{(3)})$ is defined as above.

In the next two plots (Figures 3.4 and 3.5), multi-grid algorithm is implemented in a way that only initial approximation is smoothed by Picard's iteration, and the improvement of exact correction is done by $2 - step$ (p=2) multi-grid iteration in each step taking the initial correction to be zero.

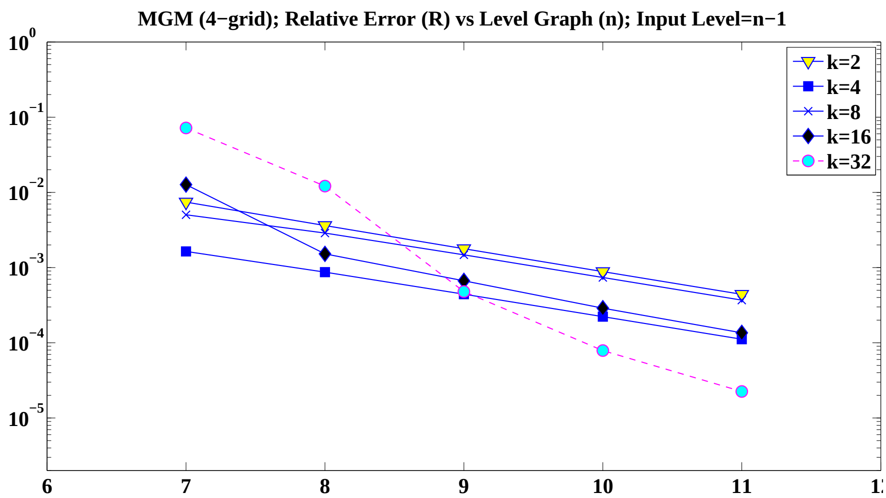


(a) Multi-grid Using 6 Level

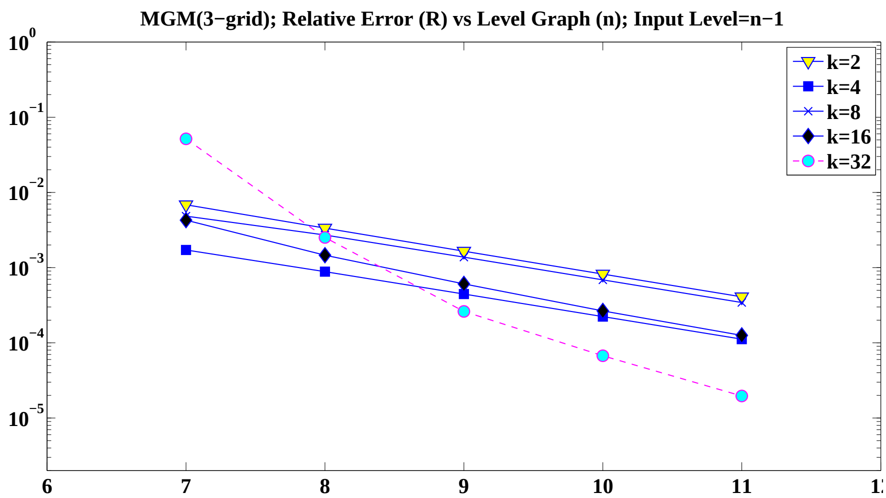


(b) Multi-grid Using 5 Level

Figure 3.4. Relative Error vs Level Graph for Multi-grid Method.



(a) Multi-grid Using 4 Level



(b) Multi-grid Using 3 Level

Figure 3.5. Relative Error vs Level Graph for Multi-grid Method.

4. A CONVERGENT INTEGRAL EQUATION METHOD FOR HIGH FREQUENCY SCATTERING

In this chapter, we devise and analyze a method for computing the scattering returns by any smooth convex barrier in two dimensions. The classical numerical methods developed for the solution of such problems give rise to number of degrees of freedom that (at best) increase linearly with increasing wave number k. Accordingly, they are not suitable for high-frequency ($k \gg 1$) simulations. Our approach is based upon utilization of a well-posed integral equation formulation of the scattering problem, and Galerkin approximations adopted to the known asymptotic properties (boundary layers) of the solution.

4.1. Description of Numerical Method and Main Results

In this section, we describe a numerical method for the sound-soft scattering problems, and we present that it requires only a minor increase (k^ϵ for any $\epsilon > 0$) in the number of degrees of freedom to maintain a fixed accuracy at the end of the section. In chapter 1, we have formulated the sound-soft scattering problem as the boundary integral equation (2.41) which we rewrite as

$$\mathcal{R}_k \vartheta = f_k \tag{4.1}$$

where

$$\mathcal{R}_k = I + \mathcal{K}' - ik\mathcal{S} \text{ and } f = 2\left(\frac{\partial u^{inc}}{\partial \nu}(x) - iku^{inc}(x)\right)$$

and $\mathcal{S}$ and $\mathcal{K}'$ are defined as in (2.14) and (2.16). Here, we have replaced the coupling parameter η with k so as to optimize the condition number of the boundary integral operator $\mathcal{R}_k$ (see Kress [22]).

Now let $\partial D = \{\gamma(s) : s \in [0, 2\pi]\}$ be a 2π-periodic parametrization of ∂D, and choose $t_1, t_2 \in [0, 2\pi]$ so that $\gamma(t_1)$ and $\gamma(t_2)$ correspond to tangency points in accordance with the unit direction vector a of incoming plane wave $u^{inc}(x) = e^{ikx\cdot a}$. Let $\nu(x)$ be the unit normal vector at the point $x \in \partial D$ and $\langle\cdot,\cdot\rangle$ be the usual inner product in $\mathbb{R}^2$. Then I=$\{x \in \partial D | \langle \nu(x), a\rangle < 0\}$ is called the illuminated region and S=$\{x \in \partial D | \langle \nu(x), a\rangle > 0\}$ is called the shadow region. Moreover, we choose parametrization γ so that $\gamma(0)$ is a point in the shadow region. For convenience, assume that $\gamma(s)$ is proportional to arc-length parametrization on ∂D. Hence, with the same symbols, we can rewrite (4.1) as

$$\mathcal{R}_k \vartheta(s) = f_k(s), \text{ for } s \in [0, 2\pi]. \tag{4.2}$$

For any measurable subset Λ of $[0, 2\pi]$, we denote $(\vartheta, w)_{L^2(\Lambda)}$ to be the usual L^2 inner product of complex or real valued functions defined on Λ and $\|\cdot\|_{L^2(\Lambda)}$ to be the induced norm. For simplicity, we write (ϑ, w), $\|\cdot\|$ and L^2 when $\Lambda = [0, 2\pi]$. In the standard Galerkin method, the variational formulation of (4.2) is to find $\vartheta \in L^2$ such that,

$$a_k(\vartheta, w) := (\mathcal{R}_k \vartheta, w) = (f_k, w) \tag{4.3}$$

for all $w \in L^2$. To define our approximation spaces, we first divide ∂D into five parts and construct the approximation spaces in each region in the form of complex exponentials modulated by polynomials. Next, in order to study approximation properties of our algorithm, we study the asymptotic behavior of the function $V(s, k)$, which is appearing in the geometrical optics ansatz $\vartheta(s, k) := \vartheta(\gamma(s), k) = kV(s, k)\exp(ik\gamma(s)\cdot a)$ and oscillates slowly than ϑ, for large k. Then we derive a bound for the derivatives of $V(s, k)$ of all orders.

Next, in order to approximate $V(s, k)$ efficiently, we divide ∂D into subregions as depicted in the next definition.

Definition 4.1. *Let* $0 \le \epsilon_m < \epsilon_{m-1} < \ldots < \epsilon_1 < 1/3$ *and* $\xi_1, \xi_2 > 0$ *be constants.*

Then, for sufficiently large $k > 0$, we define a total of $4m$ subregions as follows:

Illuminated region (I): $\Lambda^I = [t_1 + \xi_1 k^{-1/3} k^{\epsilon_1}, t_2 - \xi_2 k^{-1/3} k^{\epsilon_1}]$

Deep shadow region (S): $\Lambda^S = [t_2 + \xi_2 k^{-1/3} k^{\epsilon_1}, 2\pi + t_1 - \xi_1 k^{-1/3} k^{\epsilon_1}]$

Shadow boundaries (SB_1 and SB_2):

$\Lambda_1^{SB} = [t_1 - \xi_1 k^{-1/3} k^{\epsilon_m}, t_1 + \xi_1 k^{-1/3} k^{\epsilon_m}]$, $\quad \Lambda_2^{SB} = [t_2 - \xi_2 k^{-1/3} k^{\epsilon_m}, t_2 + \xi_2 k^{-1/3} k^{\epsilon_m}]$

Transitions in the illuminated region (TI): *For* $j = 1, 2, .., m-1$,

$\Lambda_j^{I_1} = [t_1 + \xi_1 k^{-1/3} k^{\epsilon_{j+1}}, t_1 + \xi_1 k^{-1/3} k^{\epsilon_j}]$, $\quad \Lambda_j^{I_2} = [t_2 - \xi_2 k^{-1/3} k^{\epsilon_j}, t_2 - \xi_2 k^{-1/3} k^{\epsilon_{j+1}}]$

Transitions in the shadow region (TS): *For* $j = 1, 2, .., m-1$,

$\Lambda_j^{S_1} = [t_1 - \xi_1 k^{-1/3} k^{\epsilon_j}, t_1 - \xi_1 k^{-1/3} k^{\epsilon_{j+1}}]$, $\quad \Lambda_j^{S_2} = [t_2 + \xi_2 k^{-1/3} k^{\epsilon_{j+1}}, t_2 + \xi_2 k^{-1/3} k^{\epsilon_j}]$

Next, we let $\{\chi_j : j = 1, ..., 4m\}$ be the characteristic functions of these subregions, then clearly

$$\chi_j(s)\vartheta(s) = k\chi_j(s) e^{ik\gamma(s)\cdot a} V(s,k), \quad s \in [0, 2\pi]. \tag{4.4}$$

Bearing this in mind, choose integers $d_j \geq 0$ and define the local approximation spaces as

$$\mathcal{L}_k^j = \text{span}\left\{ k\chi_j(s) e^{ik\gamma(s)\cdot a} s^\ell : \ell = 0, 1, 2, ..d_j \right\}. \tag{4.5}$$

Finally, we define our global approximation space as

$$\mathcal{G}_k^{\mathbf{d}} = \oplus_{j=1}^{4m} \mathcal{L}_k^j \tag{4.6}$$

where $|\mathbf{d}| = \sum_{j=1}^{4m}(d_j + 1)$ is the dimension of our approximation space. The Galerkin formulation for (4.4) is thus to find $\tilde{\vartheta} \in \mathcal{G}_k^d$ such that

$$a_k(\tilde{\vartheta}, \tilde{w}) = (f_k, \tilde{w}), \text{ for all } \tilde{w} \in \mathcal{G}_k^d. \tag{4.7}$$

Next, in order to carry out the error analysis, we make use of the following lemma due to Céa (see [23]).

Lemma 4.2. (*Céa's Lemma*) *Suppose a_k satisfies, for all $\vartheta, w \in L^2$, the two assumptions:*

$$\begin{gathered} \textit{continuity } |a_k(\vartheta, w)| \leq B_k \|\vartheta\| \, \|w\|, \;\; B_k > 0, \\ \textit{coercivity } |a_k(\vartheta, \vartheta)| \geq \alpha_k \|\vartheta\|^2, \;\; \alpha_k > 0 \end{gathered}$$

Then both the weak form (4.3) *and its Galerkin approximation* (4.7) *have unique solutions ($\vartheta \in L^2$ and $\tilde{\vartheta} \in \mathcal{G}_k^d$). Moreover,*

$$\|\vartheta - \tilde{\vartheta}\| \leq \frac{B_k}{\alpha_k} \|\vartheta - \tilde{w}\| \tag{4.8}$$

for all $\tilde{w} \in \mathcal{G}_k^d$.

In order to utilize the preceeding lemma in our scattering problem, we use (4.4) to write

$$\vartheta(s) = \sum_{j=1}^{4m} \chi_j(s)\vartheta(s) = k \sum_{j=1}^{4m} \chi_j(s) e^{ik\gamma(s)\cdot a} V(s,k), \tag{4.9}$$

and write $\tilde{w} \in \mathcal{G}_k^d$ as

$$\tilde{w}(s) = k \sum_{j=1}^{4m} \chi_j e^{ik\gamma(s)\cdot a} p_j^{d_j}(s) \tag{4.10}$$

for some $p_j^{d_j} \in \mathbb{P}(d_j)$ for $j = 0, 1, 2...4m$. From (4.8)-(4.10), we have the following corollary [7].

Corollary 4.3.

$$\|\vartheta - \tilde{\vartheta}\| \leq \left(\frac{B_k}{\alpha_k}\right) k \sum_{j=1}^{4m} \left\{ \|e^{ik\gamma(s)\cdot a}\|_{L^\infty} \inf_{p \in \mathbb{P}(d_j)} \|V(\cdot, k) - p\|_{L^2(\Lambda_j)} \right\}. \tag{4.11}$$

In order to derive bounds for the best approximation error, as the preceding corollary shows, on each subregion we need to estimate the best approximation error for the approximation to $V(s,k)$ via polynomials of certain degree. These are given in the next theorem (for a proof, see Section 4.3).

Theorem 4.4. *Let* $0 \leq \epsilon_m < \epsilon_{m-1} < ... < \epsilon_1 < 1/3$*. For* $2 \leq n \leq \min\{d_{IS}, d_{SB}, d_T\}+1$ *and sufficiently large* k*, we have:*

Illuminated and deep shadow regions : *For* $\Omega \in \{\Lambda^I, \Lambda^S\}$

$$\inf_{p\in\mathbb{P}(d_{IS})} \|V(\cdot,k)-p\|_{L^2(\Omega)} \lesssim_n k^{-(1+3\epsilon_1)/2}\left(\frac{k^{(1/3-\epsilon_1)/2}}{d_{IS}}\right)^n$$

Shadow boundaries : *For* $\Omega \in \{\Lambda_1^{SB}, \Lambda_2^{SB}\}$

$$\inf_{p\in\mathbb{P}(d_{SB})} \|V(\cdot,k)-p\|_{L^2(\Omega)} \lesssim_n k^{-1/2}\left(\frac{k^{\epsilon_m}}{d_{SB}}\right)^n$$

Transition regions : *For* $\Omega \in \{\Lambda_j^{I_r}, \Lambda_j^{S_r}\}$ $j : 1,2,..m-1$ *and* $r = 1,2$

$$\inf_{p\in\mathbb{P}(d_T)} \|V(\cdot,k)-p\|_{L^2(\Omega)} \lesssim_n k^{-(1+3\epsilon_{j+1})/2}\left(\frac{k^{(\epsilon_j-\epsilon_{j+1})/2}}{d_T}\right)^n$$

where "$a \lesssim_n b$*" means that* $a \leq C_n b$ *for some constant* C_n *that depends only on* n*, and* d_{IS}, d_{SB} *and* d_T *are degree of polynomials to approximate* $V(s,k)$ *on the regions* Λ^I*,* Λ^S*,* Λ_1^{SB}*,* Λ_2^{SB}*,* $\Lambda_j^{I_r}$ *and* $\Lambda_j^{S_r}$ *respectively, and* $\mathbb{P}(d)$ *is the set of algebraic polynomials of degree* d *or less.*

Since $\|e^{ik\gamma(s)\cdot a}\|_{L^\infty} = 1$, letting $d_T = d_{\Lambda_j^{I_1}} = d_{\Lambda_j^{S_1}} = d_{\Lambda_j^{I_2}} = d_{\Lambda_j^{S_2}}$ for $j = 1,2,..,m-$

1, $d_{IS} = d_{\Lambda^I} = d_{\Lambda^S}$ and $d_{SB} = d_{\Lambda_1^{SB}} = d_{\Lambda_2^{SB}}$, Corollary 4.3 and Theorem 4.4 entail

$$\begin{aligned}
\|\vartheta - \tilde{\vartheta}\| \lesssim_n & \left(\frac{\beta_k}{\alpha_k}\right) k \sum_{j=1}^{m-1} k^{-(1+3\epsilon_{j+1})/2} \left(\frac{k^{(\epsilon_j - \epsilon_{j+1})/2}}{d_T}\right)^n \\
&+ \left(\frac{\beta_k}{\alpha_k}\right) k k^{-(1+3\epsilon_1)/2} \left(\frac{k^{(1/3-\epsilon_1)/2}}{d_{IS}}\right)^n \\
&+ \left(\frac{\beta_k}{\alpha_k}\right) k k^{-1/2} \left(\frac{k^{\epsilon_m}}{d_{SB}}\right)^n .
\end{aligned} \tag{4.12}$$

where α_k and β_k are the coercivity and continuity constants respectively and estimated in [7, sect.4].

Next, in order to obtain an optimum error bound with respect to k, we choose ϵ_j's so that $(\epsilon_j - \epsilon_{j+1})/2 = (1/3 - \epsilon_1)/2 = \epsilon_m$ holds. An easy calculation shows that

$$\epsilon_j = \frac{2m - 2j + 1}{3} \frac{1}{2m+1}$$

for $j = 1, 2, .., m$. With this choice of ϵ_j, inequality (4.12) entails

$$\begin{aligned}
\|\upsilon - \tilde{\upsilon}\| \lesssim_n & \left(\frac{\beta_k}{\alpha_k}\right) k \sum_{j=1}^{m-1} k^{-\frac{2m-j}{2m+1}} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_T}\right)^n \\
&+ \left(\frac{\beta_k}{\alpha_k}\right) k k^{-\frac{2m}{2m+1}} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_{IS}}\right)^n \\
&+ \left(\frac{\beta_k}{\alpha_k}\right) k k^{-1/2} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_{SB}}\right)^n .
\end{aligned}$$

Observe that the first term of the right-hand-side of the inequality above is maximum when $j = m - 1$. Hence, it follows that

$$\begin{aligned}\|\upsilon - \tilde{\upsilon}\| \lesssim_n &\left(\frac{\beta_k}{\alpha_k}\right) k k^{-\frac{m+1}{2m+1}} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_T}\right)^n \\ &+ \left(\frac{\beta_k}{\alpha_k}\right) k k^{-\frac{2m}{2m+1}} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_{IS}}\right)^n \\ &+ \left(\frac{\beta_k}{\alpha_k}\right) k k^{-1/2} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_{SB}}\right)^n .\end{aligned}$$

Now we can summarize the main result in the next theorem.

Theorem 4.5. *Let $\tilde{\upsilon}$ be the solution of the Galerkin approximation* (4.7)*. Assume that polynomials of degree d_T is used in transition regions and d_{IS} in illuminated and deep shadow regions, and d_{SB} in shadow boundaries. Then for all n with $2 \leq n \leq \min\{d_T, d_{IS}, d_{SB}\} + 1$, we have*

$$\frac{\|\upsilon - \tilde{\upsilon}\|}{k} \lesssim_n \left(\frac{\beta_k}{\alpha_k}\right) \left\{ k^{-\frac{m+1}{2m+1}} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_T}\right)^n + k^{-\frac{2m}{2m+1}} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_{IS}}\right)^n + k^{-\frac{1}{2}} \left(\frac{k^{\frac{1}{(6m+3)}}}{d_{SB}}\right)^n \right\}$$

where β_k and α_k are continuity and coercivity constants.

Notice that $k^{-\frac{m+1}{2m+1}} < k^{-1/2}$ and $k^{-\frac{2m}{2m+1}} \leq k^{-2/3}$ for $m \geq 1$. Thus, choosing $d = d_T = d_{IS} = d_{SB}$, we immediately have the following result.

Corollary 4.6. *Let $\tilde{\upsilon}$ be the Galerkin solution as described in* (4.7)*. Also assume that $n \geq 2$ and $d = d_T = d_{IS} = d_{SB} \geq n - 1$. Then we have*

$$\frac{\|\upsilon - \tilde{\upsilon}\|}{k} \lesssim_n \left(\frac{\beta_k}{\alpha_k}\right) k^{-1/2} \left(\frac{k^{\frac{1}{6m+3}}}{d}\right)^n .$$

The preceding corollary implies that in order to fix the accuracy of the method, one should increase the degree of the polynomials d proportional to $k^{\frac{1}{6m+3}}$ where m is the number of subregions in each transition region.

4.2. Asymptotics Expansion of the Normal Derivative of the Total Field

In this section, we give the asymptotic expansions and properties of the slowly oscilating function $V(s,k)$ for the high frequency case ($k \gg 1$). We begin with the following theorem which is proved by from Melrose and Taylor [1].

Theorem 4.7. *There exist $\triangle > 0$ such that $V(s,k)$ has the asymptotic expansion:*

$$V(s,k) \sim \sum_{\ell,m \geqslant 0} k^{-1/3-2\ell/3-m} b_{\ell,m}(s) \Psi^{(\ell)}(k^{1/3} Z(s)) \tag{4.13}$$

valid for $s \in I_\triangle := (t_1 - \triangle, t_1 + \triangle) \cup (t_2 - \triangle, t_2 + \triangle)$, where $\gamma(t_1)$ and $\gamma(t_2)$ are the tangency points. The functions $b_{\ell,m}$, Ψ and Z have the following properties.

- *$b_{\ell,m}$ are C^∞ complex-valued functions on $I_\triangle$.*
- *Z is a C^∞ real valued function on $I_\triangle$, with simple zeros at t_1 and t_2, which is positive valued on $(t_1, t_2) \cap I_\triangle$ and negative valued on $(t_2 - 2\pi, t_1) \cap I_\triangle$.*
- *$\Psi : \mathbb{C} \to \mathbb{C}$ is an entire function specified by*

$$\Psi(\tau) := exp(-i\tau^3/3) \int_c \frac{exp(-iz\tau)}{Ai(e^{2\pi/3} z)} dz \tag{4.14}$$

where Ai is the Airy function [24] and c is an appropriate contour. In particular,

$$\Psi(\tau) = a_0 \tau + a_1 \tau^{-2} + a_2 \tau^{-5} + a_n \tau^{1-3n} + O(\tau^{1-3(n+1)}) \ \ as \ \ \tau \to \infty, \ \ a_0 \neq 0 \tag{4.15}$$

and this expansion remains valid for all derivatives of Ψ by formally differentiating each term on the right- hand-side, including the error term (see [1, Lemma 9.9]). *Moreover, there exist $\beta > 0$ and $c_0 \neq 0$ such that for any $n \in \mathbb{N}$*

$$D^n_\tau \Psi(\tau) = c_0 D^n_\tau \left\{ exp(-i\tau^3/3 - i\tau\alpha_1) \right\} (1 + O(exp(-|\tau|\beta))), \ \ as \ \ \tau \to -\infty \tag{4.16}$$

where $\alpha_1 = exp(-2\pi i/3)\nu_1$ and $0 < \nu_1$ is the right-most root of Ai.

Next, we express $V(s,k)$ as a sum of finitly many explicit $k-$dependent terms and a manageable remainder as stated in the next corollary [7, sect. 5].

Corollary 4.8. *With the same notation as in Theorem 4.7, the functions $b_{\ell,m}$ can be extended to 2π -periodic C^∞ functions such that, for all $L, M \in \mathbb{N}$, the decomposition*

$$V(s,k) = \left[\sum_{\ell,m}^{L,K} k^{-1/3-2\ell/3-m} b_{\ell,m}(s) \Psi^{(\ell)}(k^{1/3} Z(s)) \right] + R_{L,M}(s,k) \tag{4.17}$$

holds for all $s \in [0, 2\pi]$, with remainder term satisfying, for all $n \in \mathbb{N}$

$$|D_s^n R_{L,M}(s,k)| \leq C_{L,M,n}(1+k)^{\mu+n/3}, \tag{4.18}$$

where $\mu := -\min\left\{\frac{2}{3}(L+1), (M+1)\right\}$ and $C_{L,M,n}$ is independent of k.

In the next theorem, we now present bounds on the derivatives of $V(s,k)$ of all order with respect to s, which is utilized to estimate semi-norms and then carry out error analysis.

Theorem 4.9. [7] *For all $n \in \mathbb{N}$ and sufficiently large k, we have*

$$|D_s^n V(s,k)| \lesssim_n \begin{cases} 1 + \sum_{j=2}^n k^{(j-1)/3}(1 + k^{1/3}|\omega(s)|)^{-j-2} & \text{if } n \geq 2 \\ 1 & \text{if } n = 0, 1. \end{cases} \tag{4.19}$$

where $w(s) = (s - t_1)(t_2 - s)$.

Proof. First, note that, by properties of Z given in Theorem 4.7, we have $Z(s) = h(s)\omega(s)$ where h is a smooth function and positive on $[0, 2\pi]$ and $\omega(s) = (s-t_1)(t_2-s)$.

For any $n \in \mathbb{N}$, there exist $L, M \in \mathbb{N}$ so that $-\mu \geq n/3$, where

$$\mu = -\min\{2(L+1)/3, (M+1)\}.$$

Next, applying Corollary 4.8 yields,

$$V(s,k) = A_{L,M}(s,k) + R_{L,M}(s,k)$$

where

$$A_{L,M}(s,k) := k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} B_{l,M}(s) \Psi^{(l)}(k^{1/3} Z(s)). \tag{4.20}$$

Here $B_{l,M}$ is given by the sum $\sum_{m=0}^{M} k^{-m} b_{l,m}(s)$ where $b_{l,m}(s)$ is described as in Theorem 4.7. As $\mu + n/3 \leq 0$, by (4.18), it then follows

$$|D_s^n R_{L,M}(s,k)| \leq C_{n,L,M}(1+k)^{\mu+n/3} \leq C_{n,L,M}$$

for all k. Now, by the Leibniz's rule for the product of functions and Faà Di Bruno's formula [25] for the derivatives of composition of functions and the fact that all derivatives of $B_{l,M}(s)$ are bounded by a constant, we have

$$\begin{aligned}
&|D_s^n A_{L,M}(s,k)| \\
&\leq k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| D_s^n [B_{l,M}(s) \Psi^{(l)}(k^{1/3} Z(s))] \right| \\
&= k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| \sum_{j=0}^{n} \binom{n}{j} B_{l,M}^{(n-j)}(s) (\Psi^{(l)}(k^{1/3} Z(s)))^{(j)} \right| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| \sum_{j=0}^{n} (\Psi^{(l)}(k^{1/3} Z(s)))^{(j)} \right| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| \sum_{j=0}^{n} \sum_{\sum_{y=1}^{j} y m_y = j} (\Psi^{(l+m_1+..+m_j)}(k^{1/3} Z(s))) \prod_{p=1}^{j} (k^{1/3} Z^{(p)}(s))^{(m_p)} \right|.
\end{aligned}$$

Note that $Z(s) = h(s)\omega(s)$ is a smooth function, accordingly all of its derivatives are bounded by a constant independent from k. For convenience, letting $m_1 + m_2 .. + m_j = i$

gives

$$
\begin{aligned}
&|D_s^n A_{L,M}(s,k)| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{j=0}^{n} \sum_{\sum_{y=1}^{j} y m_y = j} k^{(m_1+\ldots+m_j)/3} \left|\Psi^{(l+m_1+\ldots+m_j)}(k^{1/3} Z(s))\right| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{j=0}^{n} \sum_{1\le i\le j} k^{i/3} \left|\Psi^{(l+i)}(k^{1/3} Z(s))\right| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{j=0}^{n} k^{j/3} \left|\Psi^{(l+j)}(k^{1/3} Z(s))\right| \\
&\lesssim_n k^{-1/3} \sum_{j=0}^{n} \sum_{l=0}^{L} k^{(j-2l)/3} \left|\Psi^{(l+j)}(k^{1/3} Z(s))\right|. \qquad (4.21)
\end{aligned}
$$

At this point, we use (4.15) and (4.16) to derive the estimates

$$|\Psi(\tau)| \le C_0(1+|\tau|), \qquad (4.22)$$

$$|\Psi'(\tau)| \le C_1, \qquad (4.23)$$

$$|\Psi^{(l)}(\tau)| \le C_l(1+|\tau|)^{-2-l}, \quad for \ \ l \ge 2. \qquad (4.24)$$

Thus, splitting the outer sum in (4.21) into three parts, we have

$$
\begin{aligned}
&|D_s^n A_{L,M}(s,k)| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left|\Psi^{(l)}(k^{1/3} Z(s))\right| + k^{-1/3} \sum_{l=0}^{L} k^{(1-2l)/3} \left|\Psi^{(l+1)}(k^{1/3} Z(s))\right| \\
&+ k^{-1/3} \sum_{j=2}^{n} \sum_{l=0}^{L} k^{(j-2l)/3} \left|\Psi^{(l+j)}(k^{1/3} Z(s))\right| \\
&\lesssim_n k^{-1/3} \left(1 + k^{1/3}|Z(s)| + k^{-2/3} + \sum_{l=2}^{L} (1+k^{1/3}|Z(s)|)^{-2-l} \right) \\
&+ k^{-1/3} \left(k^{1/3} + \sum_{l=1}^{L} k^{(1-2l)/3} (1+k^{1/3}|Z(s)|)^{-2-l-1} \right) \\
&+ k^{-1/3} \sum_{j=2}^{n} \sum_{l=0}^{L} k^{(j-2l)/3} (1+k^{1/3}|Z(s)|)^{-2-l-j}.
\end{aligned}
$$

Since $Z(s) = h(s)w(s)$ and h is a non-vanishing continuous function, we get

$$
\begin{aligned}
|D_s^n A_{L,M}(s,k)| &\lesssim_n k^{-1/3}\left(1 + k^{1/3}|w(s)| + k^{-2/3} + \sum_{l=2}^{L}(1+k^{1/3}|w(s)|)^{-2-l}\right) \\
&+ k^{-1/3}\left(k^{1/3} + \sum_{l=1}^{L} k^{(1-2l)/3}(1+k^{1/3}|w(s)|)^{-2-l-1}\right) \\
&+ k^{-1/3}\sum_{j=2}^{n}\sum_{l=0}^{L} k^{(j-2l)/3}(1+k^{1/3}|w(s)|)^{-2-l-j} \\
&\lesssim_n 1 + \sum_{j=2}^{n} k^{(j-1)/3}(1+k^{1/3}|\omega(s)|)^{-j-2}\left(1 + \sum_{l=1}^{L} k^{-2l/3}(1+k^{1/3}|\omega(s)|)^{-l}\right) \\
&\lesssim_n 1 + \sum_{j=2}^{n} k^{(j-1)/3}(1+k^{1/3}|\omega(s)|)^{-j-2}.
\end{aligned}
$$

$\square$

4.3. Semi-norm Estimates and Best Approximation Error

In this section, we estimate the semi-norms of $V(s,k)$ on each region described in Definition 4.1 and carry out the error analysis of Galerkin method formulated as in (4.7).

To this end, consider the 2π-periodic function $W(s,k)$ defined on $[0, 2\pi]$ as

$$W(s,k) = k^{-1/3} + |w(s)|$$

where $w(s) = (s - t_1)(t_2 - s)$, and note that

$$
W(s,k) = \begin{cases} -(s-c_1)(d_1 - s) & \text{if } s \in [0,2\pi]\backslash[t_1,t_2], \\ (s-c_2)(d_2-s) & \text{if } s \in [t_1,t_2], \end{cases} \tag{4.25}
$$

for $k \geq 1$ where

$$\begin{aligned}
c_1 &= t_1 + (T - \sqrt{T^2 - k^{-1/3}}) = L - \sqrt{T^2 - k^{-1/3}}, \\
d_1 &= t_2 - (T - \sqrt{T^2 - k^{-1/3}}) = L + \sqrt{T^2 - k^{-1/3}}, \\
c_2 &= t_1 + (T - \sqrt{T^2 + k^{-1/3}}) = L - \sqrt{T^2 + k^{-1/3}}, \\
d_2 &= t_2 - (T - \sqrt{T^2 + k^{-1/3}}) = L + \sqrt{T^2 + k^{-1/3}} \qquad (4.26)
\end{aligned}$$

with $L = \frac{t_2+t_1}{2}$ and $T = \frac{t_2-t_1}{2}$. Next, given an interval $I = (a, b)$, it is well known that there exist $C_n > 0$ such that, for all nonnegative integers n with $n < d + 1$,

$$\inf_{p \in \mathbb{P}(d)} \|f - p\|_{L^2(I)} \leq C_n d^{-n} |f|_{n,I} \qquad (4.27)$$

where

$$|f|_{n,I} := \left[\int_a^b |f^{(n)}(s)|^2 (s-a)^n (b-s)^n ds \right]^{1/2} \qquad (4.28)$$

is the semi-norm defined on I, and $\mathbb{P}(d)$ is the set of univariate polynomials of degree $\leq d$ (see [26, Cor. 3.12]). We employ this to derive bounds for the approximation of $V(s, k)$ via polynomials. We now state and prove a theorem regarding the semi-norm estimates of $V(s, k)$.

Theorem 4.10. *Let $0 \leq \epsilon_m < \epsilon_{m-1} < ... < \epsilon_1 < 1/3$. For $n \geq 2$ and sufficiently large k, we have the following semi-norm estimates on each region given in Definition* 4.1.

Illuminated and Deep Shadow Regions : *For* $\Omega \in \{\Lambda^I, \Lambda^S\}$

$$|V(s, k)|_{n,\Omega} \lesssim_n 1 + k^{-(1+3\epsilon_1)/2} k^{(1/3-\epsilon_1)n/2} \qquad (4.29)$$

Shadow Boundaries : *For* $\Omega \in \{\Lambda_1^{SB}, \Lambda_2^{SB}\}$

$$|V(s, k)|_{n,\Omega} \lesssim_n 1 + k^{-1/2} k^{(\epsilon_m)n}$$

Transition Regions : *For* $\Omega \in \{\Lambda_j^{I_r}, \Lambda_j^{S_r} : j = 1, 2, ..m-1 \text{ and } r = 1, 2\}$.

$$|V(s,k)|_{n,\Omega} \lesssim_n 1 + k^{-(1+3\epsilon_{j+1})/2} k^{(\epsilon_j - \epsilon_{j+1})n/2}.$$

where "$a \lesssim_n b$*" means that* $a \leq C_n b$ *for some constant* C_n *that depends only on* n*.*

Proof. We show only the semi-norm estimate of $V(s,k)$ on the illuminated region. For the sake of simplicity, let $[a,b] := [t_1 + \xi_1 k^{-1/3} k^{\epsilon_1}, t_2 - \xi_2 k^{-1/3} k^{\epsilon_1}] = \Lambda^I$. Observe that, by Theorem 4.9, we have

$$\begin{aligned}
|V(s,k)|^2_{n,\Lambda^I} &= \int_a^b |D_s^n V(s,k)|^2 (s-a)^n (b-s)^n ds \\
&\lesssim_n \int_a^b \left\{ 1 + \sum_{j=2}^{n} \frac{k^{(j-1)/3}}{(1 + k^{1/3}|\omega(s)|)^{j+2}} \right\}^2 (s-a)^n (b-s)^n ds \\
&\lesssim_n \int_a^b \left\{ 1 + k^{-1} \sum_{j=2}^{n} \frac{1}{(k^{-1/3} + |\omega(s)|)^{j+2}} \right\}^2 (s-a)^n (b-s)^n ds \\
&\lesssim_n \int_a^b (s-a)^n (b-s)^n + k^{-2} \sum_{j=2}^{n} \frac{(s-a)^n (b-s)^n}{W(s,k)^{2j+4}} ds \\
&\lesssim_n 1 + k^{-2} \sum_{j=2}^{n} \int_a^b \frac{(s-a)^n (b-s)^n}{W(s,k)^{2j+4}} ds
\end{aligned} \tag{4.30}$$

Next, we calculate the last integral. By Lemma B.2 and noting that $W(s,k) = (s - c_2)(d_2 - s)$ in the given interval $[a,b]$, we get

$$\begin{aligned}
& k^{-2} \sum_{j=2}^{n} \int_a^b \frac{(s-a)^n (b-s)^n}{(s-c_2)^{2j+4} (d_2 - s)^{2j+4}} ds \\
& \lesssim_n k^{-2} \sum_{j=2}^{n} \sum_{i=1}^{2j+4} \sum_{p,q=0}^{n} F(n,p,q,i) \\
& \lesssim_n k^{-2} \sum_{j=2}^{n} \sum_{i=1}^{2j+4} \sum_{p,q=0}^{n} \Big((c_2 - a)^p (c_2 - b)^q (b - c_2)^{2n-(p+q+i)+1} \qquad (4.31) \\
& -(c_2 - a)^p (c_2 - b)^q (a - c_2)^{2n-(p+q+i)+1} \\
& +(a - d_2)^p (b - d_2)^q (d_2 - a)^{2n-(p+q+i)+1} - (a - d_2)^p (b - d_2)^q (d_2 - b)^{2n-(p+q+i)+1} \Big)
\end{aligned}$$

where $F(n,p,q,i)$ is defined as in Lemma B.2. For convenience, letting $\mathcal{T} = \frac{k^{-\frac{1}{3}}}{T+\sqrt{T^2+k^{-\frac{1}{3}}}}$ and recalling (4.26), we have

$$\begin{aligned}
&2T-\xi_2 \le (b-c_2) = T+\sqrt{T^2+k^{-\frac{1}{3}}}-\xi_2 k^{-\frac{1}{3}}k^{\epsilon_1} \le T+\sqrt{T^2+1} \Rightarrow (b-c_2)=O(1)\\
&\xi_1 k^{-\frac{1}{3}+\epsilon_1} \le (a-c_2) = \mathcal{T}+\xi_1 k^{-\frac{1}{3}+\epsilon_1} \le (\frac{1}{2T}+\xi_1)k^{-\frac{1}{3}+\epsilon_1} \Rightarrow (a-c_2)=O(k^{-\frac{1}{3}+\epsilon_1})\\
&\xi_2 k^{-\frac{1}{3}+\epsilon_1} \le (d_2-b) = \mathcal{T}+\xi_2 k^{-\frac{1}{3}+\epsilon_1} \le (\frac{1}{2T}+\xi_2)k^{-\frac{1}{3}+\epsilon_1} \Rightarrow (d_2-b)=O(k^{-\frac{1}{3}+\epsilon_1})\\
&2T-\xi_1 \le (d_2-a) = T+\sqrt{T^2+k^{-\frac{1}{3}}}-\xi_1 k^{-\frac{1}{3}+\epsilon_1} \le T+\sqrt{T^2+1} \Rightarrow (d_2-a)=O(1)
\end{aligned}$$

for sufficiently large k. Plugging these into (4.31) yields

$$\begin{aligned}
&k^{-2}\sum_{j=2}^{n}\int_a^b \frac{(s-a)^n(b-s)^n}{(s-c_2)^{2j+4}(d_2-s)^{2j+4}}ds\\
&\lesssim_n k^{-2}\sum_{j=2}^{n}\sum_{i=1}^{2j+4}\sum_{p,q=0}^{n}[(k^{-1/3+\epsilon_1})^p-(k^{-1/3+\epsilon_1})^p(k^{-1/3+\epsilon_1})^{2n-(p+q+i)+1}\\
&+(k^{-1/3+\epsilon_1})^q-(k^{-1/3+\epsilon_1})^q(k^{-1/3+\epsilon_1})^{2n-(p+q+i)+1}]\\
&\lesssim_n k^{-2}\sum_{j=2}^{n}\sum_{i=1}^{2j+4}\sum_{p,q=0}^{n}\left[(k^{-1/3+\epsilon_1})^p+(k^{-1/3+\epsilon_1})^{2n-(q+i)+1}+(k^{-1/3+\epsilon_1})^q+(k^{-1/3+\epsilon_1})^{2n-(p+i)+1}\right]\\
&\lesssim_n k^{-2}\sum_{j=2}^{n}\sum_{i=1}^{2j+4}\sum_{p,q=0}^{n}\left[1+(k^{-1/3+\epsilon_1})^{2n-(q+i)+1}+(k^{-1/3+\epsilon_1})^{2n-(p+i)+1}\right]\\
&\lesssim_n k^{-2}\left[1+(k^{-1/3+\epsilon_1})^{-n-3}\right]\\
&\lesssim_n \left[k^{-2}+(k^{-1/3+\epsilon_1})^{-n-3}k^{-2}\right] \lesssim_n \left[1+k^{(-1/3+\epsilon_1)(-n-3)-2}\right] \lesssim_n \left[1+k^{(1/3-\epsilon_1)n-(1+3\epsilon_1)}\right]
\end{aligned}$$

Thus, by (4.30), we obtain

$$\begin{aligned}
|V(s,k)|^2_{n,\Lambda^I} &= \int_a^b |D_s^n V(s,k)|^2(s-a)^n(b-s)^n ds\\
&\lesssim_n 1+k^{-2}\sum_{j=2}^{n}\int_a^b \frac{(s-a)^n(b-s)^n}{(s-c_2)^{2j+4}(d_2-s)^{2j+4}}ds\\
&\lesssim_n \left[1+k^{(1/3-\epsilon_1)n-(1+3\epsilon_1)}\right].
\end{aligned}$$

The semi-norm estimates on shadow boundary and transition regions can be established similarly. However, when we estimate the semi-norm on deep shadow region we end up with

$$|V(s,k)|^2_{n,\Lambda^S} \lesssim_n 1 + k^{-2} \sum_{j=2}^{n} \int_a^b \frac{(s-a)^n(b-s)^n}{W(s,k)^{2j+4}} ds$$

where $[a,b] = \Lambda^S = [t_2 + \xi_2 k^{-1/3} k^{\epsilon_1}, 2\pi + t_1 - \xi_1 k^{-1/3} k^{\epsilon_1}]$. Accordingly, we have to estimate the integral $\int_a^b \frac{(s-a)^n(b-s)^n}{W(s,k)^{2j+4}} ds$. To this end, we separate the integral in two parts

$$\int_a^b \frac{(s-a)^n(b-s)^n}{W(s,k)^{2j+4}} ds = \int_a^{2\pi} \frac{(s-a)^n(b-s)^n}{W(s,k)^{2j+4}} ds + \int_{2\pi}^b \frac{(s-a)^n(b-s)^n}{W(s,k)^{2j+4}} ds.$$

The first integral can be estimated in a manner similar to that for the illuminated region, yet for the second integral, we have to make a change of variables $u = s - 2\pi$ by noting the fact that $W(s,k)$ is a 2π-periodic function of s. Thus, we need to estimate

$$\int_0^{b-2\pi} \frac{(u+2\pi-a)^n(b-2\pi-u)^n}{W(u+2\pi,k)^{2j+4}} du = \int_0^{b'} \frac{(u+2\pi-a)^n(b-2\pi-u)^n}{W(u,k)^{2j+4}} du$$

where $b' = b - 2\pi$. The estimation of this integral can now be carried out similar to that for the illuminated region. $\square$

Now, we are ready to derive bounds for the best approximation error. Indeed, recalling (4.27), Theorem 4.4 is now an immediate consequence of the preceding theorem. In more detail, for the first estimate in Theorem 4.4, let $2 \leq n \leq d_I + 1$. By (4.27), we then have

$$\inf_{p \in \mathbb{P}(d_I)} \|V(\cdot,k) - p\|_{L^2(\Lambda^I)} \leq C_n d_I^{-n} |V(s,k)|_{n,\Lambda^I}$$

where $|V(s,k)|_{n,\Lambda^I}$ is the semi-norm defined in (4.28). Next, by Theorem 4.10, it follows that

$$
\begin{aligned}
\inf_{p\in\mathbb{P}(d_I)} \|V(s,k)-p\|_{L^2(\Lambda^I)} &\lesssim_n d_I^{-n}(1+k^{-(1+3\epsilon_1)/2}k^{(1/3-\epsilon_1)n/2}) \\
&\lesssim_n d_I^{-n} + k^{-(1+3\epsilon_1)/2}k^{(1/3-\epsilon_1)n/2}d_I^{-n} \\
&\lesssim_n k^{-(1+3\epsilon_1)/2}k^{(1/3-\epsilon_1)n/2}d_I^{-n} \\
&\lesssim_n k^{-(1+3\epsilon_1)/2}\left(\frac{k^{(1/3-\epsilon_1)/2}}{d_I}\right).
\end{aligned}
$$

The remaining estimates in Theorem 4.4 can be carried out similarly.

4.4. Shadow Region-Revisited

In this section, we derive improved estimates over the shadow region by analyzing the second asymptotic expansion of the function $\Psi(s)$ given in (4.16). First we note the following improvement upon Theorem 4.9 on the shadow region.

Theorem 4.11. *For $s \in [0,t_1)\cup(t_2,2\pi]$ and sufficiently large k, we have,*

$$
|D_s^n V(s,k)| \lesssim_n \begin{cases} 1+\sum_{j=0}^{n} k^{(j-1)/3}e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1} & \text{if } n\geq 2 \\ 1 & \text{if } n=0,1. \end{cases} \tag{4.32}
$$

where $\nu_1<0$ is the right most root of the Airy function $Ai(z)$ [24].

Proof. First note that, by properties of Z (see Theorem 4.7 and (4.13)), we have

$$
Z(s)=h(s)\omega(s)=h(s)(s-t_1)(t_2-s)
$$

where h is a smooth real function and positive on $[0,2\pi]$. Given any $n\in\mathbb{N}$, choose $L,M\in\mathbb{N}$ so that $-\mu\geq n/3$, where μ is defined as

$$
\mu=-\min\{2(L+1)/3,(M+1)\}.
$$

Then, by Corollary 4.8, we get

$$V(s,k) = A_{L,M}(s,k) + R_{L,M}(s,k)$$

where

$$A_{L,M}(s,k) := k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} B_{l,M}(s) \Psi^{(l)}(k^{1/3} Z(s)).$$

Here $B_{l,M}$ is given by the sum $\sum_{m=0}^{M} k^{-m} b_{l,m}(s)$ where $b_{l,m}(s)$ is described as in Theorem 4.7. Since $\mu + n/3 \leq 0$, by (4.18), we then have

$$|D_s^n R_{L,M}(s,k)| \leq C_{n,L,M}(1+k)^{\mu+n/3} \leq C_{n,L,M} \tag{4.33}$$

for all k. Thus, by Leibnitz's rule,

$$\begin{aligned}
&|D_s^n A_{L,M}(s,k)| \\
&\leq k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| D_s^n [B_{l,M}(s) \Psi^{(l)}(k^{1/3} Z(s))] \right| \\
&= k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| \sum_{j=0}^{n} \binom{n}{j} B_{l,M}^{(n-j)}(s) (\Psi^{(l)}(k^{1/3} Z(s)))^{(j)} \right|.
\end{aligned}$$

Next, we use Fa Di Bruno's formula [25] for the derivatives of composed functions. Since all derivatives of $B_{l,M}(s)$ have k-independent bounds, we obtain

$$\begin{aligned}
|D_s^n A_{L,M}(s,k)| &\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| \sum_{j=0}^{n} (\Psi^{(l)}(k^{1/3} Z(s)))^{(j)} \right| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \left| \sum_{j=0}^{n} \sum_{\sum_{y=1}^{j} y m_y = j} (\Psi^{(l+m_1+\ldots+m_j)}(k^{1/3} Z(s))) \prod_{p=1}^{j} (k^{1/3} Z^{(p)}(s))^{(m_p)} \right|.
\end{aligned}$$

The aforementioned properties of the function h in the decomposition $Z(s) = h(s)w(s)$

accordingly yields

$$
\begin{aligned}
&|D_s^n A_{L,M}(s,k)| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{j=0}^{n} \sum_{\sum_{y=1}^{j} y m_y = j} k^{(m_1+\ldots+m_j)/3} \left|\Psi^{(l+m_1+\ldots+m_j)}(k^{1/3}\omega(s))\right| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{j=0}^{n} \sum_{1\le m_1+\ldots+m_j \le j} k^{(m_1+\ldots+m_j)/3} \left|\Psi^{(l+m_1+\ldots+m_j)}(k^{1/3}\omega(s))\right|.
\end{aligned}
$$

Therefore, letting $m_1 + \ldots + m_j = i$ and rearranging sums, we have

$$
\begin{aligned}
|D_s^n A_{L,M}(s,k)| &\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{j=0}^{n} \sum_{1\le i\le j} k^{i/3} \left|\Psi^{(l+i)}(k^{1/3}\omega(s))\right| \\
&\lesssim_n k^{-1/3} \sum_{l=0}^{L} k^{-2l/3} \sum_{j=0}^{n} k^{j/3} \left|\Psi^{(l+j)}(k^{1/3}\omega(s))\right| \\
&\lesssim_n k^{-1/3} \sum_{j=0}^{n} \sum_{l=0}^{L} k^{(j-2l)/3} \left|\Psi^{(l+j)}(k^{1/3}\omega(s))\right|. \qquad (4.34)
\end{aligned}
$$

We now use the asymptotic expansion 4.16 of the derivatives of Ψ for the shadow region. Since the roots of the Airy function Ai(z) are negative [24], when $\tau \to -\infty$, the function Ψ together with its derivatives decay exponentially. If we write (4.16) more explicitly, Faà Di Bruno's formula [25] entails

$$
\begin{aligned}
|D_\tau^n \Psi(\tau)| &= |c_0 D_\tau^n \left[\exp(-i\tau^3/3 - i\tau\alpha_1)\right] || (1 + O(exp(-|\tau|\beta)))| \qquad (4.35) \\
&\lesssim_n \sum_{\sum_{y=1}^{n} y m_y = n} |\exp^{(m_1+\ldots+m_n)}(-i\tau^3/3 - i\tau\alpha_1)|| \prod_{p=1}^{n} (g^{(p)}(\tau))^{m_p}| \qquad (4.36)
\end{aligned}
$$

where $g(\tau) = -i\tau^3/3 - i\tau\alpha_1$. Since $g(\tau)$ vanishes after 3 times differentiation, we get

$$
\begin{aligned}
|D_\tau^n \Psi(\tau)| &\lesssim_n \sum_{\sum_{y=1}^n y m_y = n} |\exp^{(m_1+...+m_n)}||(-i\tau^3/3 - i\tau\alpha_1)||\prod_{p=1}^n (g^{(p)}(\tau))^{m_p}| \\
&\lesssim_n \sum_{\sum_{y=1}^n y m_y = n} |\exp^{(m_1+...+m_n)}||(-i\tau^3/3 - i\tau\alpha_1)||(g'(\tau))^{m_1}(g''(\tau))^{m_2}(g'''(\tau))^{m_3}| \\
&\lesssim_n \sum_{\sum_{y=1}^n y m_y = n} |\exp^{(m_1+...+m_n)}||(-i\tau^3/3 - i\tau\alpha_1)||(-i\tau^2 - i\alpha_1)^{m_1}||(-2i\tau)^{m_2}||(-2i)^{m_3}|.
\end{aligned}
\tag{4.37}
$$

Next, let $\mathfrak{X}(s,k) := e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1}$ and note that

$$
\begin{aligned}
|e^{-ik\omega^3(s)/3 - ik^{1/3}\omega(s)\alpha_1}| &= |e^{-ik\omega^3(s)/3 - ik^{1/3}\omega(s)\{cos(-2\pi/3)+isin(-2\pi/3)\}\nu_1}| \\
&= |e^{-ik\omega^3(s)/3}||e^{-ik^{1/3}\omega(s)cos(2\pi/3)\nu_1 - k^{1/3}\omega(s)sin(2\pi/3)\nu_1}| \\
&= |e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1}| = |\mathfrak{X}(s,k)|.
\end{aligned}
\tag{4.38}
$$

Accordingly, by (4.37) and (4.38), we have

$$
\begin{aligned}
&|\Psi^{(l+j)}(k^{1/3}\omega(s))| \\
&\lesssim_n \sum |e^{-ik\omega^3(s)/3 - ik^{1/3}\omega(s)\alpha_1}(-ik^{2/3}\omega^2(s) - i\alpha_1)^{m_1}(-2ik^{1/3}\omega)^{m_2}(-2i)^{m_3}| \\
&\lesssim_n \sum |e^{-ik\omega^3(s)/3 - ik^{1/3}\omega(s)\alpha_1}||(-ik^{2/3}\omega^2(s) - i\alpha_1)^{m_1}||(-2ik^{1/3}\omega)^{m_2}||(-2i)^{m_3}| \\
&\lesssim_n \sum |e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1}|| - ik^{2/3}\omega^2(s) - i\alpha_1|^{m_1}| - 2k^{1/3}\omega(s)|^{m_2}|2|^{m_3} \\
&\lesssim_n \sum |\mathfrak{X}(s,k)|| - ik^{2/3}\omega^2(s) - sin(2\pi/3)\nu_1 - icos(2\pi/3)\nu_1|^{m_1}| - 2k^{1/3}\omega(s)|^{m_2} \\
&\lesssim_n \sum |\mathfrak{X}(s,k)|| - sin(2\pi/3)\nu_1 - i[k^{2/3}\omega^2(s) + cos(2\pi/3)\nu_1]|^{m_1}| - 2k^{1/3}\omega(s)|^{m_2} \\
&\lesssim_n \sum |\mathfrak{X}(s,k)||\nu_1 + k^{4/3}\omega^4(s)|^{m_1}| - 2k^{1/3}\omega(s)|^{m_2}
\end{aligned}
$$

where all sums above are taken over the set

$$
\left\{(m_1, ..., m_{l+j}) : \sum_{y=1}^{l+j} y m_y = l+j, m_y \in \mathbb{N} \text{ for } 1 \leq y \leq l+j\right\}.
$$

As exponential functions of negative exponents decay faster than any polynomial at infinity, we obtain

$$\begin{aligned}
&|\Psi^{(l+j)}(k^{1/3}\omega(s))| \\
&\lesssim_n \sum_{\sum_{y=1}^{l+j} ym_y = l+j} e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1}(\nu_1 + k^{4/3}\omega^4(s))^{m_1}(-2k^{1/3}\omega(s))^{m_2} \\
&\lesssim_n e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1}
\end{aligned} \tag{4.39}$$

whenever $\omega(s) > 0$. Therefore, using (4.39) in (4.34) gives

$$\begin{aligned}
|D_s^n A_{L,M}(s,k)| &\lesssim_n k^{-1/3}\sum_{j=0}^{n}\sum_{l=0}^{L} k^{(j-2l)/3}|\Psi^{(l+j)}(k^{1/3}Z(s))| \\
&\lesssim_n k^{-1/3}\sum_{j=0}^{n}\sum_{l=0}^{L} k^{(j-2l)/3}e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1} \\
&\lesssim_n \sum_{j=0}^{n} k^{(j-1)/3}e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1}.
\end{aligned}$$

Since $R_{L,M}$ and its derivatives of all order are bounded by (4.18), we finally have

$$|D_s^n V(s,k)| \lesssim_n 1 + \sum_{j=0}^{n} k^{(j-1)/3}e^{-k^{1/3}\omega(s)sin(2\pi/3)\nu_1}. \tag{4.40}$$

$\square$

As we anticipated, in the next section, we estimate the semi-norm of $V(s,k)$ in the shadow region by utilizing the preceding theorem.

Theorem 4.12. [Improved semi-norm estimate in the shadow] *Let $\epsilon > 0$ be given. Then, for any interval $[a,b] \subset [0,t_1)$ or $(t_2, 2\pi]$ and $n \geq 2$ we have*

$$|V(s,k)|^2_{n,(a,b)} \lesssim_n \left(1 + e^{k^{1/3}(\epsilon - 2\bar{\omega} sin(2\pi/3)\nu_1)}\right)$$

for sufficiently large k, where $\bar{\omega} = \max_{s\in[a,b]}\omega(s)$ and $\nu_1 < 0$ is the right most root of the Ai.

Proof. By (4.28), we have

$$
\begin{aligned}
|V(s,k)|^2_{n,(a,b)} &= \int_a^b |D_s^n V(s,k)|^2 (s-a)^n (b-s)^n ds \\
&\lesssim_n \int_a^b \left(1 + \sum_{j=0}^n k^{(j-1)/3} e^{-k^{1/3}\omega(s) sin(2\pi/3)\nu_1}\right)^2 (s-a)^n (b-s)^n ds \\
&\lesssim_n \int_a^b \left(1 + \sum_{j=0}^n k^{(2j-2)/3} e^{-2k^{1/3}\omega(s) sin(2\pi/3)\nu_1}\right) (s-a)^n (b-s)^n ds \\
&\lesssim_n \int_a^b (s-a)^n (b-s)^n ds + \sum_{j=0}^n k^{(2j-2)/3} \int_a^b e^{-2k^{1/3}\omega(s) sin(2\pi/3)\nu_1} (s-a)^n (b-s)^n ds \\
&\lesssim_n 1 + \sum_{j=0}^n k^{(2j-2)/3} \int_a^b e^{-2k^{1/3}\omega(s) sin(2\pi/3)\nu_1} (s-a)^n (b-s)^n ds.
\end{aligned}
$$

Next, since $\nu_1 < 0$, we get

$$
\begin{aligned}
|V(s,k)|^2_{n,(a,b)} &\lesssim_n 1 + \sum_{j=0}^n k^{(2j-2)/3} \int_a^b e^{-2k^{1/3}\omega(s) sin(2\pi/3)\nu_1} (s-a)^n (b-s)^n ds \\
&\lesssim_n 1 + \sum_{j=0}^n k^{(2j-2)/3} e^{-2k^{1/3}\bar{\omega} sin(2\pi/3)\nu_1} (b-a)^{2n+1} \\
&\lesssim_n 1 + k^{(2n-2)/3} (b-a)^{2n+1} e^{-2k^{1/3}\bar{\omega} sin(2\pi/3)\nu_1} \\
&\lesssim_n 1 + e^{\epsilon k^{1/3}} e^{-2k^{1/3}\bar{\omega} sin(2\pi/3)\nu_1}
\end{aligned}
$$

where $\bar{\omega} = \max_{s\in[a,b]} \omega(s)$. □

Next, recalling (4.27), we derive the best approximation error in the deep shadow region as depicted in the next theorem.

Theorem 4.13. [Improved best approximation in the shadow] *Let $\epsilon > 0$ be given. Then, for any interval $\Omega \subset [0, t_1)$ or $(t_2, 2\pi]$ and $2 \le n \le d_\Omega + 1$, we have*

$$
\inf_{p\in\mathbb{P}(d_\Omega)} \|V(\cdot,k) - p\|_{L^2(\Omega)} \le C_n \frac{e^{k^{1/3}(\epsilon - 2\bar{\omega} sin(2\pi/3)\nu_1)}}{d_\Omega^n}
$$

where $\bar{\omega} = \max_{s\in[a,b]} \omega(s)$ and $\nu_1 < 0$ is the right most root of the Ai, and d_Ω is the degree of polynomials to approximate $V(s,k)$.

5. CONCLUSION

The main aim of this thesis was to devise numerical methods for high-frequency scattering problems in 2 dimensional settings by utilizing the geometrical optics ansatz (1.1) for convex obstacles rigorously established by Melrose and Taylor [1]. To this end, we transformed the sound soft scattering problem into a well-posed boundary integral equation and, by virtue of the ansatz, which expresses the normal derivative of the total field as a highly oscillating complex exponential modulated by a slowly oscillating amplitude, we constructed a new Galerkin method to capture the oscillations for large wave number k and approximate the slowly oscillating part of the solution. In essence our method was based on the refinement of transition regions given in Definition 4.1 where asymptotic properties of the solution changes from polynomial to exponential. As a main convergence result, we showed that it suffices to increase the degrees of freedom proportional to k^{ϵ} for any ϵ in order to retain a fixed accuracy.

In the third chapter, we implemented two- and multi-grid methods for integral equations of the second kind. We tested both methods with sound-soft scattering problem for the unit circle for which the exact solution is known. As is seen in numerical results, multi-grid method is not efficient for high-frequency problems if the number of level is large.

Our method can also be applied to multiple scattering problems by a carefully design of a Galerkin approximation spaces in accordance with the given configuration.

APPENDIX A: Functional Analysis

The following theorems and definitions can be found in [9].

Theorem A.1. [Neumann Series] *Let $A : X \to X$ be a bounded linear operator on a Banach space X with $\|A\| < 1$ and let $I : X \to X$ denote the identity operator. Then $I - A$ has a bounded inverse on X that is given by the Neumann series*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and satisfies

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Definition A.2. [Compact Operators] *A linear operator $A : X \to Y$ from a normed space X into a normed space Y is called* compact *if it maps each bounded set in X into a relatively compact set in Y.*

Theorem A.3. *Let X, Y and Z be normed spaces and let $A : X \to Y$ and $B : Y \to Z$ be bounded linear operators. Then the product $BA : X \to Z$ is compact if one of the two operators A or B is compact.*

Theorem A.4. [Riesz Theory for Compact Operators] *Let $A : X \to X$ be a compact linear operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective (and therefore also bijective), then the inverse operator $(I - A)^{-1} : X \to X$ is bounded.*

Theorem A.5. [Fredholm Theorem] *Let $\langle X, Y\rangle$ be a dual system and $A : X \to X$, $B : Y \to Y$ be compact adjoint operators. Then the nullspaces of the operators $I - A$ and $I - B$ have the same finite dimension.*

Theorem A.6. [Arzelà-Ascoli] *A set $U \subset C(D)$ is relatively compact if and only if it* *is bounded and equicontinuous, i.e., if there exist a constant C such that*

$$|\varphi(x)| \leq C$$

for all $x \in D$ and all $\varphi \in U$, and for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(y)| < \epsilon$$

for all $x, y \in D$ with $|x - y| < \delta$ and all $\varphi \in U$.

Definition A.7. [Parallel Surface] *For a bounded domain D of class C^m, $m \geq 1$, we have the notion of the parallel surface described by*

$$\partial D_h := \{z = x + h\nu(x) : x \in \partial D\} \tag{A.1}$$

where ν is the unit normal to the ∂D and h is a real parameter.

Definition A.8. [14] [Hölder Spaces] *A real or complex valued function φ defined on a set $D \subset \mathbb{R}^2$ is called uniformly Hölder continuous with Hölder exponent $0 < \alpha \leq 1$ if there is a constant C such that*

$$|\varphi(x) - \varphi(y)| \leq C|x - y|^\alpha$$

for all $x, y \in D$. We define the Hölder space $C^{0,\alpha}(D)$ to be the linear space of all functions defined on D which are bounded and uniformly Hölder continuous with exponent α. It is a Banach Spaces with the norm

$$\|\varphi\|_{0,\alpha} := \|\varphi\|_{0,\alpha,D} := \sup_{x \in D} |\varphi(x)| + \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}.$$

We also introduce the Hölder space $C^{1,\alpha}(D)$, $0 < \alpha \leq 1$, of uniformly Hölder continuously differentiable functions as the space of differentiable functions for which gradφ

belongs to $C^{0,\alpha}(D)$*. With the norm*

$$\|\varphi\|_{1,\alpha} := \|\varphi\|_{1,\alpha,D} := \|\varphi(x)\|_\infty + \|\mathrm{grad}\varphi\|_{0,\alpha}$$

$C^{1,\alpha}(D)$ *is again a Banach space.*

APPENDIX B: Auxiliary Results

Lemma B.1. *For $c < d$ and $m \in \mathbb{N}$, we have*

$$\frac{1}{(s-c)^m(d-s)^m} = \sum_{j=1}^{m} \binom{2m-j-1}{m-j} \frac{1}{(d-c)^{2m-j}} \left[\frac{1}{(s-c)^j} + \frac{1}{(d-s)^j}\right].$$

Proof. We first compute the partial fraction decomposition

$$\begin{aligned} \frac{1}{(s-c)^m(d-s)^m} &= \frac{A_0}{(s-c)^m} + \frac{A_1}{(s-c)^{m-1}} + ... + \frac{A_{m-1}}{(s-c)} \\ &+ \frac{B_0}{(d-s)^m} + \frac{B_1}{(d-s)^{m-1}} + ... + \frac{B_{m-1}}{(d-s)} \end{aligned} \tag{B.1}$$

where $A_k, B_k \in \mathbb{R}$ for $k = 0, 1, ..., m-1$. To this end, we multiply both sides by $(s-c)^m$ to obtain

$$\begin{aligned} \frac{1}{(d-s)^m} &= A_0 + A_1(s-c) + A_2(s-c)^2 ... + A_{m-1}(s-c)^{m-1} \\ &+ (s-c)^m \left[\frac{B_0}{(d-s)^m} + \frac{B_1}{(d-s)^{m-1}} + ... + \frac{B_{m-1}}{(d-s)}\right] \end{aligned} \tag{B.2}$$

so that setting $s = c$ we get

$$\begin{aligned} \frac{1}{(d-c)^m} &= A_0 + A_1(c-c) + A_2(c-c)^2 ... + A_{m-1}(c-c)^{m-1} \\ &+ (c-c)^m \left[\frac{B_0}{(d-s)^m} + \frac{B_1}{(d-s)^{m-1}} + ... + \frac{B_{m-1}}{(d-s)}\right] \\ &= A_0. \end{aligned}$$

Next, we differentiate $(B.2)$ to obtain

$$\begin{aligned}\frac{m}{(d-s)^{m+1}} &= A_1 + A_2(s-c) + 3A_3(s-c)^2 ... + A_{m-1}(m-1)(s-c)^{m-2} \\ &\quad + m(s-c)^{m-1}\left[\frac{B_0}{(d-s)^m} + \frac{B_1}{(d-s)^{m-1}} + ... + \frac{B_{m-1}}{(d-s)}\right] \\ &\quad + (s-c)^m (d/ds)\left[\frac{B_0}{(d-s)^m} + \frac{B_1}{(d-s)^{m-1}} + ... + \frac{B_{m-1}}{(d-s)}\right]\end{aligned}$$

so that for $s = c$, we get

$$\frac{m}{(d-c)^{m+1}} = A_1.$$

Continuing in this way, we have

$$A_k = \binom{m+k-1}{m-1}\frac{1}{(d-c)^{m+k}} \quad \text{for all } 0 \le k \le m-1.$$

In a similar manner, one can multiply equation $(B.2)$ by $(d-s)^m$ to get

$$B_k = A_k \text{ for all } 0 \le k \le m-1.$$

Next, writing the coefficients A_k and B_k in (B.2) and letting let $m - k = j$, we get

$$\begin{aligned}\frac{1}{(s-c)^m(d-s)^m} &= \sum_{k=0}^{m-1}\binom{m+k-1}{k}\frac{1}{(d-c)^{m+k}}\left[\frac{1}{(s-c)^{m-k}} + \frac{1}{(d-s)^{m-k}}\right] \\ &= \sum_{j=1}^{m}\binom{2m-j-1}{m-j}\frac{1}{(d-c)^{2m-j}}\left[\frac{1}{(s-c)^j} + \frac{1}{(d-s)^j}\right].\end{aligned}$$

$\square$

Lemma B.2. *Let $t_1, t_2 \in [0, 2\pi]$ be so that $\gamma(t_1)$ and $\gamma(t_2)$ correspond to tangency points in accordance with the unit direction vector a where γ is a 2π-periodic parametrization* *of boundary of a bounded convex domain D. Also let us define*

$$\begin{aligned} c_1 &= t_1 + (T - \sqrt{T^2 - k^{-1/3}}) = L - \sqrt{T^2 - k^{-1/3}}, \\ d_1 &= t_2 - (T - \sqrt{T^2 - k^{-1/3}}) = L + \sqrt{T^2 - k^{-1/3}}, \\ c_2 &= t_1 + (T - \sqrt{T^2 + k^{-1/3}}) = L - \sqrt{T^2 + k^{-1/3}}, \\ d_2 &= t_2 - (T - \sqrt{T^2 + k^{-1/3}}) = L + \sqrt{T^2 + k^{-1/3}} \end{aligned} \tag{B.3}$$

where $L = \frac{t_2+t_1}{2}$ and $T = \frac{t_2-t_1}{2}$. Suppose that either $[\alpha, \beta] \subseteq [t_1, t_2]$ and $c = c_I$, $d = d_I$ or $[\alpha, \beta] \cap (t_1, t_2) = \emptyset$ and $c = c_S$, $d = d_S$. Then, for any $a, b \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, there holds

$$\int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m} ds = \sum_{j=1}^{m} \sum_{p,q=0}^{n} \binom{2m-j-1}{m-j} \binom{n}{p} \binom{n}{q} \frac{(-1)^n}{(d-c)^{2m-j}} F(n,p,q,j)$$

where,

$$F(n,p,q,j) = (c-a)^p(c-b)^q \log\left(\frac{\beta-c}{\alpha-c}\right) + (a-d)^p(b-d)^q \log\left(\frac{d-\alpha}{d-\beta}\right)$$

when $2n-p-q-j+1 = 0$,

$$\begin{aligned} F(n,p,q,j) = {} & \frac{(c-a)^p(c-b)^q}{2n-p-q-j+1}\left[(\beta-c)^{2n-p-q-j+1} - (\alpha-c)^{2n-p-q-j+1}\right] \\ & + \frac{(a-d)^p(b-d)^q}{2n-p-q-j+1}\left[(d-\alpha)^{2n-p-q-j+1} - (d-\beta)^{2n-p-q-j+1}\right] \end{aligned}$$

when $2n-p-q-j+1 \neq 0$.

Proof. By Lemma B.1, we have

$$
\begin{aligned}
&\int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m} ds \\
&= \int_\alpha^\beta \sum_{j=1}^m \binom{2m-j-1}{m-j} \frac{(s-a)^n(b-s)^n}{(d-c)^{2m-j}} \left[\frac{1}{(s-c)^j} + \frac{1}{(d-s)^j}\right] ds \\
&= \int_\alpha^\beta \sum_{j=1}^m \binom{2m-j-1}{m-j} \frac{1}{(d-c)^{2m-j}} \left[\frac{(s-a)^n(b-s)^n}{(s-c)^j} + \frac{(s-a)^n(b-s)^n}{(d-s)^j}\right] ds \\
&= \sum_{j=1}^m \binom{2m-j-1}{m-j} \frac{1}{(d-c)^{2m-j}} \left[\int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(s-c)^j} ds + \int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(d-s)^j} ds\right].
\end{aligned}
$$

Next, we make the change of variables $s-c=u$ in the first integral and $d-s=u$ in the second to get

$$
\begin{aligned}
&\int_\alpha^\beta \frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m} ds \\
&= \sum_{j=1}^m \binom{2m-j-1}{m-j} \frac{1}{(d-c)^{2m-j}} \left[\int_{\alpha-c}^{\beta-c} \frac{(u+c-a)^n(b-u-c)^n}{u^j} du \right.\\
&\left. + \int_{d-\alpha}^{d-\beta} \frac{(d-u-a)^n(b-d+u)^n}{u^j} du\right] \\
&= \sum_{j=1}^m \binom{2m-j-1}{m-j} \frac{(-1)^n}{(d-c)^{2m-j}} \left[\int_{\alpha-c}^{\beta-c} \frac{(u+c-a)^n(u+c-b)^n}{u^j} du \right.\\
&\left. + \int_{d-\alpha}^{d-\beta} \frac{(u+a-d)^n(u+b-d)^n}{u^j} du\right].
\end{aligned}
$$

As binomial theorem entails

$$
\frac{(u+c-a)^n(u+c-b)^n}{u^j} = \sum_{p=0}^n \sum_{q=0}^n \binom{n}{p} \binom{n}{q} u^{2n-p-q-j} (c-a)^p (c-b)^q
$$

and

$$
\frac{(u+a-d)^n(u+b-d)^n}{u^j} = \sum_{p=0}^n \sum_{q=0}^n \binom{n}{p} \binom{n}{q} u^{2n-p-q-j} (a-d)^p (b-d)^q,
$$

it follows that

$$
\begin{aligned}
&\int_{\alpha}^{\beta} \frac{(s-a)^n(b-s)^n}{(s-c)^m(d-s)^m} ds \\
&= \sum_{j=1}^{m} \frac{\binom{2m-j-1}{m-j}(-1)^n}{(d-c)^{2m-j}} \sum_{p,q=0}^{n} \binom{n}{p}\binom{n}{q} \left[\int_{\alpha-c}^{\beta-c} u^{2n-p-q-j}(c-a)^p(c-b)^q du \right. \\
&\left. + \int_{d-\alpha}^{d-\beta} u^{2n-p-q-j}(a-d)^p(b-d)^q du \right] \\
&= \sum_{j=1}^{m} \frac{\binom{2m-j-1}{m-j}(-1)^n}{(d-c)^{2m-j}} \sum_{p,q=0}^{n} \binom{n}{p}\binom{n}{q} \left[(c-a)^p(c-b)^q \int_{\alpha-c}^{\beta-c} u^{2n-p-q-j} du \right. \\
&\left. + (a-d)^p(b-d)^q \int_{d-\alpha}^{d-\beta} u^{2n-p-q-j} du \right].
\end{aligned}
$$

Then, the result follows. □

REFERENCES

1. Melrose, R. and M. Taylor, “Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle”, *Adv. Math.*, Vol. 55, pp. 242–315, 1985.

2. Abboud, T., J. C. Nédélec and B. Zhou, “Méthode des équations intégrales pour les hautes fréquencies”, *C.R. Acad. Sci. Paris*, Vol. 318, p. 165–170, 2004.

3. Giladi, E. and J. Keller, “A hybrid numerical asymptotic method for scattering problems”, *Journal of Computational Physics*, Vol. 174, pp. 226–247, 2001.

4. Giladi, E. and J. Keller, “An asymptotically derived boundary element method for the Helmholtz equation”, *Journal of Computational and Applied Mathematics*, Vol. 198, pp. 52–74, 2007.

5. Bruno, O., C. Geuzaine, J. Monroe and F. Reitich, “Prescribed error toler-ances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case”, *Phil. Trans. Roy. Soc. London*, Vol. 362, pp. 629–645, 2004.

6. Huybrechs, D. and S. Vandewalle, “A sparse discretization for integral equation formulations of high frequency scattering problems”, *SIAM J. Sci. Comput.*, Vol. 29, p. 2305–2328, 2007.

7. Dominguez, V., I. G. Graham and V. P. Smyshlyaev, “A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering”, *Numerische Mathematik*, Vol. 106, pp. 471–510, 2004.

8. Stetter, H. J., “The defect correction principle and discretization methods”, *Numer. Math.*, Vol. 29, pp. 425–443, 1978.

9. Colton, D. and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*,

Springer, New York, 1998.

10. Brakhage, H., “Über die numerische Behandlung von Integralgleichungen nach der Quadraturformelmethode”, *Numer. Math.*, Vol. 2, pp. 183–196, 1960.

11. Atkinson, K., “Iterative variants of the Nyström method for the numerical solution of integral equations”, *Numerische Mathematik*, Vol. 22, pp. 17–31, 1974.

12. Hackbusch, W., *Multi-Grid Methods and Applications*, Springer-Verlag, New York, 1985.

13. Colton, D. and R. Kress, *Integral Equation Methods in Scattering Theory*, Krieger Publishing Company, Florida, 1983.

14. Colton, D. and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, New York, 1998.

15. Rellich, F., “Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten”, *Jber. Deutsch. Math. Verein.*, Vol. 53, pp. 57–65, 1943.

16. Anselone, P., *Collectively compact operator approximation theory*, Prentice-Hall, New Jersey, 1971.

17. Anselone, P. M. and R. Moore, “Approximate solutions of integral and operator equations”, *J. Math. Anal. Appl.*, Vol. 9, pp. 268–277, 1964.

18. Atkinson, K. E., *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, United Kingdom, 1997.

19. Bramble, J. H., *Multigrid Methods*, John Wiley & Sons, Inc., New York, 1993.

20. Hemker, P. W. and H. Schippers, “Multiple grid methods for the solution of Fredholm integral equations of the second kind”, *Math. Comput.*, Vol. 36, pp. 215–232, 1981.

21. Çağan Özen, H., *Robus High-Frequency Solvers*, M.S. Thesis, Boğaziçi University, 2012.

22. Kress, R., "Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering", *Q. Jl. Mech. appl. Math.*, Vol. 38, pp. 323–341, 1985.

23. Céa, J., *Approximation variationnelle des problèmes aux limites*, Ph.D. Thesis, Ann. Ins. Fourier, 1964.

24. Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1965.

25. Bruno, F. D., "Sullo sviluppo delle Funzioni", Vol. 6, p. 479–480, 1855.

26. Schwab, C., *p- and hp- Finite Element Methods*, Oxford University Press, Oxford, 1998.