O-MINIMAL STRUCTURES

by

Cihan Pehlivan

BS., Financial Mathematics, İstanbul Bilgi University, 2007

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2011

ACKNOWLEDGEMENTS

I would like to express my gratitude to my thesis supervisor Dr. Özlem Beyarslan for her concern, endless support and guidance in preparation of this study.

Also, I would like to thank Assist. Dr. Müge Kanuni and Prof. Oleg Belegradek for participating in my thesis committee. I am thankful to my officemates for their support and tolerance. Lastly, I am indebted to my family and my grandfather for their support and patience through my education.

ABSTRACT

O-MINIMAL STRUCTURES

If $\mathcal{M}$ is a o-minimal expansion of ordered field of reals which contains a definable function which can not dominated by polynomials, then exponential function appears as a definable function. In this study firstly we introduce notions from model theory and o-minimality, then we see definable functions in o-minimal structure forms a Hardy field and we put valuation on Hardy field. By the help of valuation we can show exponential function is definable in a o-minimal expansion of ordered field of reals which is not polynomially bounded.

ÖZET

O-MİNİMAL YAPILAR

Elimizde eğer sıralı gerçel sayılar cisminin bir o-minimal genişlemesi varsa, bu yapı polinomlardan daha hızlı büyüyen bir tanımlı fonksiyon içeriyorsa, yapımız doğal üstsel fonksiyonu tanımlı olarak icerir. Bu çalısmada oncelikle model teori ve o-minimal yapılar hakkında bilgi verilmis, sonrasında o-minimal yapıdaki tanımlı fonksiyonların Hardy cismini olusturdugu incelenmiştir ve Hardy cisminin üzerine valuation konulmuştur. Bu koydugumuz valuation sayesinde sıralı gerçel sayılar cisiminin polinomlarlarla sınırlandırılmayan her o-minimal genişlemesinde doğal üstsel fonksiyonun var oldugu incelenmiştir.

TABLE OF CONTENTS

1. Introduction

Model theory is a branch of mathematics which studies mathematical structures. Mathematical assertions can be true or false depending on which structure we are in, and every statement use mathematical operations such as multiplication and addition. We can interpret mathematical operations differently depending on structure we are using, if we are working on matrix's the meaning of multiplication have to be matrix multiplication. Firstly we need to have a language to compose mathematical statements, then we need a structure to work on, and interpret the language suitable for that structure. If we restrict ourself in language $(+, \cdot, 0, 1)$ where $+$ and $\cdot$ are binary relations and 0 and 1, it is called as a language of rings. Two examples of structures for this language complex numbers $\mathbb{C}$ and matrices over real numbers M_2. For the structure M_2 the constants 0 interpreted as 0_2 and 1 as the identity matrix I, for $\mathbb{C}$ the symbols interpreted trivially. Another important concept in model theory is definable sets which helps to understand structures. Mathematical formulas are true for some elements of structure, the subset which is formed by an elements satisfies certain formula are called definable sets in that structure. In o-minimal structures we keep the definable sets just the finite union of intervals and points.

O-minimality began in the eighties, with the question of Tarski, whether the real field, expanded by the exponential function still has a decidable theory. Decidability has close relationship with definability, so it is important to understand what is definable in $(R, +, \cdot, -, 0, 1, \leq, \exp)$, this is related to the questions if the theory of this structure has quantifier elimination, or model complete. In the begging of eighties L. Dries introduced o-minimality, and A.Pillay and C. Steinhorn proved the basic structure theorems on o-minimal structures and developed their abstract theory. In 1991, A. Wilkie showed that the theory of $(R, +, \cdot, -, 0, 1, \leq, \exp)$ is model complete, and also o-minimal. Further results can be found in his paper [1].

More specifically in this study we concentrate on o-minimal expansion of real closed with emphasizing exponentiation function and polynomial functions. The natural question which we can ask is whether or not exponentiation function appears in a o-minimal expansions of ordered field which is not polynomially bounded. The answer is yes and exponentiation function appears naturally, this proved in Millers paper[2], his result is heavily depends on the article of Rosenlich [3] which studies Hardy fields.

Our main aim is develop the tools we need to understand how exponentiation appears in an o-minimal expansion of ordered field which is not polynomially bounded.

In first chapter we introduce important concepts from Model Theory firstly, then the main theorems and results from o-minimality is given. In second chapter we focuses on the polynomial functions and exponentiation function in o-minimal expansions of ordered fields, and gives the ideas and proofs in Miller's thesis [4] which will be published in his paper [2] we mention previously. Moreover in chapter 2 we explore Hardy fields on the way to the main result.

1.1. Model Theory Background for O-minimal Structures

1.1.1. Basic Definitions

Firstly, we will introduce the basic notion from model theory.

Definition 1.1 (Language). *A language $\mathcal{L}$ is given by the following three ingredients*

(i) A set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$.

(ii) A set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$.

(iv) A set of constant symbols $\mathcal{C}$.

The numbers n_f and n_R tell us that f is a function of n_f variables and n_R is an n_R-ary relation.

Examples 1.2.

(i) The language of rings $\mathcal{L}_r = (0, 1, +, -, \times)$ where $+, -$ and $\times$ function symbols and $0, 1$ are constants.

(ii) The language of ordered rings $\mathcal{L}_{or} = (0, 1, +, -, \times, <)$, where $<$ is a binary relation symbol.

(iv) The language of groups $(e, \cdot, \cdot^{-1})$, e is constant symbol, $\cdot$ is a binary function, $\cdot^{-1}$ is a function.

Definition 1.3 (Structure). *A structure $\mathcal{M}$ is given by the following four ingredients*

(i) A set M which is called the universe, domain or underlying set of $\mathcal{M}$.

(ii) A collection of functions $\{f_i : i \in I_0\}$ where $\{f_i : M^{n_i} \to M\}$ for some $n_i \geq 1$.

(iii) A collection of relations $\{R_i : i \in I_1\}$ where $R_i \subset M^{m_i}$ for some $m_i \geq 1$.

(iv) A collection of distinguished elements $\{c_i : i \in I_2\} \subset M$.

To any structure one can attach a language $\mathcal{L}$ with n_i-ary functions symbol $\tilde{f}_i$

for each f_i, an m_i-ary relation symbol $\tilde{R}_i$ for each R_i and constant symbols $\tilde{c}_i$ for each c_i.

An $\mathcal{L}$-structure is a structure $\mathcal{M}$ where we can interpret all of the symbols of $\mathcal{L}$.

Examples 1.4.

(i) *A graph is a non empty set A with a binary relation P both irreflexive and symmetric. A graph can be viewed as a structure $\mathcal{M}$ in the language L consisting of a unique binary relation symbol R, with $R^{\mathcal{M}} = P$. Also a non empty set A partially ordered by some relation $\leq$ can be regarded as a structure in the same language L; this time, $R^{\mathcal{M}} =\leq$.*

(ii) *A group* G *is a structure of the language $L = \{1, \cdot, \cdot^{-1}\}$ where 1 is constant, $\cdot$ and $\cdot^{-1}$ are the operation symbols of arity 2 and 1 respectively. 1^{G} represents the identity element in* G*, while $\cdot^{\mathrm{G}}$ and $\cdot^{-1^{\mathrm{G}}}$ denote the product and the inverse operation in* G*. We can enrich L with a binary operation symbol $[\ ,\ ]$ corresponds to the commutator operation in* G*.*

(iv) *A field K is a structure of the language $L = \{0, 1, +, -, \cdot\}$ where 0 an 1 are constant, and $+, -$ and $\cdot$ are function symbols each have an usual interpretation in $\mathcal{K}$.*

(v) *An ordered field is a structure in the language $L = \{+, -, \cdot, 0, 1, \leq\}$ obtained by a new relation symbol $\leq$ and it is interpretation is just the order relation in the field.*

If f is a function of arity n we will denote it as f^n

Examples 1.5. *Let $\mathcal{L} = \{f^2, g^2, h^2, c, d\}$ be a language then $(\mathbb{Z}, +, \cdot, <, 0, 1)$, $(\mathbb{R}, +, \cdot, <, 0, 1)$ are $\mathcal{L}$-structures with usual interpretation.*

Definition 1.6. *Let $\mathcal{L}$, $\mathcal{L}'$ be two languages such that $\mathcal{L} \subseteq \mathcal{L}'$. The structure $\mathcal{K}'$ is called extension of $\mathcal{K}$ if $K = K'$, and for any $Q \in \mathcal{L}$ $Q^{\mathcal{K}} = Q^{\mathcal{K}'}$, i.e. the symbols in $\mathcal{L}$ has same interpretation in $\mathcal{L}'$.*

Definition 1.7. *Let M, N be two $\mathcal{L}$-structures $\alpha : M \to N$ is called $L-$monomorphism* *if α is injective and*

(i) *α preserves all $\mathcal{L}$-operations $\alpha(f^{\mathcal{M}}(a_1,..,a_n)) = f^{\mathcal{N}}(\alpha(a_1),...\alpha(a_n))$ for all $a_1,...a_n \in M$, where n is arity of f.*

(ii) *α preserves all $\mathcal{L}$-relations for all $P \in \mathcal{R}$, $(a_1,...,a_n) \in P^{\mathcal{M}} \Leftrightarrow (\alpha(a_1),...\alpha(a_n)) \in P^{\mathcal{N}}$ for all $a_1,...a_n \in M$, where $n = ar(P)$.*

(ii) *α preserves all $\mathcal{L}$-constants, for all $c \in \mathcal{C}$, $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$.*

Definition 1.8. *Bijective monomorphism called an* isomorphism.

Remarks 1.9.

(i) *If $\alpha : \mathcal{M} \to \mathcal{N}$ and $\beta : \mathcal{N} \to \mathcal{K}$ are monomorphisms then their composition is also a monomorphism.*

(ii) *Composition of two isomorphism is an isomorphism.*

(iii) *Inverse of an isomorphism is an isomorphism.*

Notation: $(\mathcal{M} \simeq \mathcal{N}) \Leftrightarrow$ if there exists an isomorphism $(\alpha : \ \mathcal{M} \to \mathcal{N})$.

Definition 1.10. *If $\mathcal{M}$, $\mathcal{N}$ be $L-$structures such that $\mathcal{M} \subseteq \mathcal{N}$ and $id_{MN} : M \to N$ is a monomorphism. Then we say that $\mathcal{M}$ is a substructure of $\mathcal{N}$. That is $(f^{\mathcal{M}}(a_1,..,a_n)) = f^{\mathcal{N}}(a_1,...a_n), c^{\mathcal{M}} = c^{\mathcal{N}}$, $(a_1,...,a_n) \in P^{\mathcal{M}} \Leftrightarrow (a_1,...,a_n) \in P^{\mathcal{N}}$ for any $a_1,...a_n \in \mathcal{M}$.*

If N is a substructure of M, one can need stronger condition about substructures, this leads following definition.

Definition 1.11. *Let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{L}$-structures, $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is called an* elementary structure *of $\mathcal{N}$ if for any $\mathcal{L}$-formula $\phi(x_1,..,x_n)$ and any $\bar{a} \in \mathcal{M}^n$ $\mathcal{M} \models \phi(\bar{a}) \leftrightarrow \mathcal{N} \models \phi(\bar{a})$. We denote elementary structure as $\mathcal{M} \preceq \mathcal{N}$.*

Example 1.12. $(2\mathbb{Z}, <)$ *is a substructure of* $(\mathbb{Z}, <)$.

Example 1.13. *Let* $\phi(x) : \exists y \ (y + y) = x$ *then* $2\mathbb{Z} \nvDash \phi(2)$ *but* $\mathbb{Z} \models \phi(2)$*, so* $(2\mathbb{Z}, +) \npreceq (\mathbb{Z}, +)$*.*

Definition 1.14. *Let* $\mathcal{M}$*,* $\mathcal{N}$ *are* $\mathcal{L}$*-structures,* $\alpha : \mathcal{M} \to \mathcal{N}$*,* α *is called elementary embedding* $\mathcal{M}$ *to* $\mathcal{N}$ *if for any* $\mathcal{L}$*-formula* $\phi(x_1, .., x_n)$*, any* $\bar{a} \in \mathcal{M}^n$ *,* $M \models \phi(\bar{a}) \leftrightarrow N \models \phi(\alpha(\bar{a}))$*.*

Remark 1.15.

(i) $\mathcal{M} \preceq \mathcal{N}$ *if and only if* id_{MN} *is an elementary embedding.*
(ii) $\alpha : \ \mathcal{M} \to \mathcal{N}$ *is an elementary embedding* $\Rightarrow \alpha(\mathcal{M}) \preceq \mathcal{N}$*.*

We can use language $\mathcal{L}$ to create formulas describing properties of $\mathcal{L}$-structures. Formulas build up from atomic terms and atomic terms built up from terms. *Terms* in the language $\mathcal{L}$ are built up by finitely many applications of function symbols to the appropriate number of variables and constant symbols. We allow ourselves to use parentheses as necessary for readability of terms. Formal definition of term is given inductively.

Definition 1.16. *The set of* $\mathcal{L}$*-terms is the smallest set* $\mathcal{T}$ *such that*

(i) $c \in \mathcal{T}$ *for each constant symbol* $\in C$*,*
(ii) each variable symbol $v_i \in \mathcal{T}$ *for* $i = 1, 2...$ *and*
(iii) if $t_1, .., t_{nf} \in T$ *and* $f \in F$*, then* $f(t_1, ...t_{nf}) \in T$*.*

Example 1.17. $\cdot(v_1 - (v_2, 1))$*,* $\cdot(+(v_1, v_2), +(v_3, 1))$ *and* $+(1, +(1, +(1, 1)))$*, are a terms in* $\mathcal{L}_r$*. We can simply denote this terms as* $v_1 \cdot (v_2 - 1)$ *,* $(v1 + v2) \cdot (v3 + 1)$*,* $1 + (1 + (1 + 1))$*.*

An *atomic formula* is an application of a relation symbol or equality to terms.

Example 1.18. $(x \cdot y + 1) \cdot z = 0$ *is an atomic formula in the language* $\mathcal{L}_r$ *and* $(x \cdot y + 1) \cdot z < 0$ *is an atomic formula in* $\mathcal{L}_{or}$*.*

Formulae are formed by finitely many applications of connectives and quantifiers to atomic formulae. Let's give formal definition of formulae.

Definition 1.19. *The set of L-formulas is the smallest set F containing the atomic formulas such that*

(i) if is ϕ is in F, then $\neg\phi$ in F,

(ii) if ϕ and θ and are in F, then $(\phi \wedge \theta)$ and $(\phi \vee \theta)$ are in F.

(iii) If ϕ is in F, then $\exists v_i \phi$ and $\forall v_i \phi$ in F.

Example 1.20. $\Psi(x, y) = (\neg(x = 0) \rightarrow x \cdot y = 1)$ *and* $\theta(x) = (\forall x\ \exists y(x = y \cdot y))$ *are formulae in* $\mathcal{L}_r$

We say that a variable v *occurs freely* in a formula θ if it is not inside a $\forall v$ or $\exists v$ quantifier; otherwise, we say that it is *bound*. For example v_1 is free in the first two formulas and bound in the third, whereas v_2 is bound in both formulas. We call a formula a *sentence* if it has no free variables. Let M be an L-structure. We will see that each L-sentence is either true or false in M. On the other hand, if is a formula with free variables $v_1, ..., v_n$, we will think of as expressing a property of elements of M_n. We often write $(v_1, ..., v_n)$ to make explicit the free variables in . We must define what it means for $(v_1, ..., v_n)$ to hold of $(a_1, ..., a_n) \in M^n$.

Definition 1.21. *Let θ be a formula with free variables from $\bar{v} = (v_{i_1}, .., v_{i_m})$, and let $\bar{a} = (a_{i_1}, ..., a_{i_m}) \in M^m$. We inductively define $M \models \theta(\bar{a})$ as follows.*

(i) If θ is $t_1 = t_2$, then $M \models \theta(\bar{a})$ if $t_1^M(\bar{a}) = t_2^M(\bar{a})$.

(ii) If θ is $R(t_1, ..., t_{n_R})$, then $M \models \theta(\bar{a})$ if $(t_1^M(\bar{a}), ..., t_{n_R}^M(\bar{a})) \in R^{\mathcal{M}}$.

(iii) If θ is $\neg\varphi$ then $M \models \theta(\bar{a})$ if $M \not\models \varphi(\bar{a})$.

(iv) If θ is $\psi \vee \phi$ then $M \models \theta(\bar{a})$ if $M \models \varphi(\bar{a})$ or $M \models \phi(\bar{a})$.

(v) If θ is $\psi \wedge \phi$ then $M \models \theta(\bar{a})$ if $M \models \varphi(\bar{a})$ and $M \models \phi(\bar{a})$.

(vi) If θ is $\exists v_j \psi(\bar{v}, v_j)$ then $M \models \theta(\bar{a})$ if there exists $b \in M$ such that $M \models \psi(\bar{a}, b)$.

(vi) If θ is $\forall v_j\, \psi(\bar{v}, v_j)$ then $M \models \theta(\bar{a})$ if $M \models \psi(\bar{a}, b)$ for all $b \in M$ and $M \models \phi(\bar{a})$.

The elements of structure that satisfy some certain formulas has a important role in Model Theory. We understand structures via these sets.

Definition 1.22 (Definable set)**.** *A set $X \subset M^n$ is definable in the $\mathcal{L}$-structure $\mathcal{M}$ if there is a formula $\phi(x_1, \ldots, x_{n+m})$ and elements $b_1, \ldots, b_m \in M$ such that $X = \{(a_1, \ldots, a_n) : \mathcal{M} \models \phi(a_1, \ldots, a_n, b_1, \ldots, b_m)\}$ X is called A-definable or definable over A, where $A \subset M$, if we can choose that $b_1, \ldots, b_m \in A$. If $m = 0$ we say that X is $\emptyset$-definable.*

Examples 1.23.

(i) $\{x : x > \sqrt{\pi}\}$ *is definable over* $\mathbb{R}$ *but not* $\emptyset$*-definable, while* $\{x : x > \sqrt{2}\}$ *is* $\emptyset$*-definable by the formula* $(xx > 1 + 1) \wedge (x > 0)$.

(ii) *Let* $\mathcal{M} = (R, +, -, \cdot, 0, 1)$ *be the field of real numbers. Let* $\phi(x, y)$ *be the formula* $\exists z(z \neq 0 \wedge y = x + z^2)$. *Because* $a < b$ *if and only if* $\mathcal{M} \models \phi(a, b)$*, the ordering is* $\emptyset$*-definable in* $\mathcal{M}$.

(iii) *Let* F *be a field and* $\mathcal{M} = (F[X], +, -, \cdot, 0, 1)$ *be the ring of polynomials over* F*. Then* F *is definable in* $\mathcal{M}$*. Indeed,* F *is the set of units of* $F[X]$ *and is defined by the formula* $(x = 0) \vee (\exists y : xy = 1)$.

Lemma 1.24. *Let $\mathcal{L}_{or}$ be the language of ordered rings and $(R, +, -, \cdot, , <, 0, 1)$ be the ordered field of real numbers. Suppose that $X \subseteq \mathbb{R}^n$ is A-definable. Then, the topological closure of X is also A-definable.*

Proof: Let $\phi(v_1, ..., v_n, \bar{a})$ define X. Let $\psi(v_1, ..., v_n, \bar{w})$ be the formula $\forall \varepsilon\ [\ \varepsilon > 0 \rightarrow \exists y_1, ..., y_n(\phi(\bar{y}, \bar{w}) \wedge \sum_{i=1}^{n}(v_i - y_i)^2 < \varepsilon)]$. Then, $\bar{b}$ is in the closure of X if and only if $\mathcal{M} \models \psi(\bar{b}, \bar{a})$. □

Definition 1.25 (Elementary Equivalence)**.** *We say that $\mathcal{L}$-structures are elementarily equivalent $\mathcal{M} \equiv \mathcal{N}$ if $\mathcal{M} \models \Phi$ if and only if $\mathcal{N} \models \Phi$ for all $\mathcal{L}$-sentences Φ and it is denoted by $\mathcal{M} \equiv \mathcal{N}$.*

Definition 1.26. *We let $Th(M)$, the full theory of M, be the set of L-sentences Φ such that $\mathcal{M} \models \Phi$.*

It is easy to see that $\mathcal{M} \equiv \mathcal{N}$ if and only if $Th(\mathcal{M}) = Th(\mathcal{N})$.

Theorem 1.27. *Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then, $\mathcal{M} \equiv \mathcal{N}$.*

Proof: We show by induction on formulas that $\mathcal{M} \models \phi(a_1, ..., a_n)$ if and only if $\mathcal{M} \models \phi(j(a_1), ..., j(a_n))$ for all formulas ϕ.

Claim : Suppose that t is a term and the free variables in t are from $\bar{v} = (v_1, ..., v_n)$. For $a = (a_1, ..., a_n) \in M$, we let $j(\bar{a})$ denote $(j(a_1), ..., j(a_n))$. Then $j(t^{\mathcal{M}}(\bar{a})) = t^{\mathcal{N}}(j(\bar{a}))$. We prove this by induction on terms.

(i) If $t = c$, then
$j(t^{\mathcal{M}}(\bar{a})) = j(c^{\mathcal{M}}) = c^{\mathcal{N}} = t^{\mathcal{N}}(j(\bar{a}))$.

(ii) If $t = v_i$, then
$j(t^{\mathcal{M}}(\bar{a})) = j(a_i) = t^{\mathcal{N}}(j(a_i))$.

(iii) If $t = f(t_1, ..., t_m)$,
$j(t^{\mathcal{M}}(\bar{a})) = j(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), ..., t_m^{\mathcal{M}}(\bar{a})) = f^{\mathcal{N}}(j(t_1^{\mathcal{M}}(\bar{a})), ..., j(t_m^{\mathcal{M}}(\bar{a})));$
$= f^{\mathcal{N}}((t_1^{\mathcal{N}}(j(\bar{a}))), ..., (t_m^{\mathcal{N}}(j(\bar{a}))))$
$= t^{\mathcal{N}}(j(\bar{a}))$

We proceed by induction on formulas.

(i) If $\phi(v)$ is $t_1 = t_2$,
then $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$,
$\Leftrightarrow j(t_1^{\mathcal{M}}(\bar{a})) = j(t_2^{\mathcal{M}}(\bar{a}))$ because j is injective
$\Leftrightarrow t_1^{\mathcal{N}}(j(\bar{a})) = t_2^{\mathcal{N}}(j(\bar{a}))$
$\Leftrightarrow \mathcal{N} \models \phi(j(\bar{a}))$.

(ii) If $\phi(v)$ is $R(t_1, ..., t_n)$, then
$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow (t_1^{\mathcal{M}}(\bar{a}), ..., t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$
$\Leftrightarrow (j(t_1^{\mathcal{M}}(\bar{a})), ..., j(t_n^{\mathcal{M}}(\bar{a}))) \in R^{\mathcal{N}}$
$\Leftrightarrow (t_1^{\mathcal{N}}(j(\bar{a})), ..., t_n^{\mathcal{N}}(j(\bar{a}))) \in R^{\mathcal{N}}$

$\mathcal{N} \models \phi(j(\bar{a}))$.

(iii) If ϕ is $\neg\psi$ then by induction $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{M} \models \neg\psi(\bar{a}) \Leftrightarrow \mathcal{N} \models \neg\psi(j(\bar{a}))$ $\Leftrightarrow \mathcal{N} \models \phi(j(\bar{a}))$.

(iv) If ϕ is $\phi \wedge \varphi$ then by induction $\mathcal{M} \models \phi(\bar{a})$and $\mathcal{M} \models \varphi(\bar{a})$ $\mathcal{M} \models \phi(j(\bar{a}))$ and $\mathcal{M} \models \varphi(j(\bar{a})) \Leftrightarrow \mathcal{N} \models \phi(j(\bar{a}))$.

(v) If $\phi(\bar{v})$ is $\exists w\theta(\bar{v}, w)$ then $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \theta(\bar{a}, b)$ for some $b \in M$ $\Leftrightarrow \mathcal{N} \models \theta(j(\bar{a}), c)$ for some $c \in N$ because j is onto $\Leftrightarrow \mathcal{N} \models \phi(j(\bar{a}))$. $\square$

Proposition 1.28. *Let $\mathcal{M} = (M, ., ., .)$ be an $\mathcal{L}-$ structure. If $X \subseteq M^n$ is A-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise fixes X setwise (that is, if is an σ automorphism of $\mathcal{M}$ and $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$).*

Proof: Let $\phi(v, a)$ be the L-formula defining X where $a \in A$. Let σ be an automorphism of $\mathcal{M}$ with $\sigma(a) = a$, and let $b \in M^n$. We showed that if $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. In our proposition $\mathcal{M} = \mathcal{N}$, $\mathcal{M} \models \phi(a)$ if and only if $\mathcal{N} \models \phi(j(a))$. Thus $\mathcal{M} \models \phi(\bar{b}, \bar{a}) \to \mathcal{M} \models \phi(\sigma(\bar{b}), \sigma(\bar{a})) \to \mathcal{M} \models \phi(\sigma(\bar{b}), \bar{a})$. So $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$. $\square$

Proposition 1.29. *The set of real numbers is not definable in the field of complex numbers.*

Proof: If $\mathbb{R}$ were definable in $\mathbb{C}$, then it would be definable over a finite set $A \subseteq \mathbb{C}$. Let $r, s \in \mathbb{C}$ be algebraically independent over A with $r \in \mathbb{R}$ and $s \notin \mathbb{R}$. There is an automorphism σ of $\mathbb{C}$ such that $\sigma|_A$ is the identity and $\sigma(r) = s$. Thus, $\sigma(\mathbb{R}) \neq \mathbb{R}$ so $\mathbb{R}$ is not definable over A. $\square$

1.1.2. Quantifier Elimination

In model theory it is important to understand definable sets, it helps to understand the structures. The study of definable sets with quantifiers is more complicated. It is easy to understand definable sets with quantifier-free formulas. In the structure $(\mathbb{N}, +, -, \cdot, <, 0, 1)$ the quantifier definable sets are polynomial equations and inequalities. Let $\mathcal{L}$ be the language. Two different $\mathcal{L}$-formula $\Psi(\bar{x})$ and $\Phi(\bar{x})$ could the same meaning in a structure $\mathcal{M}$ of $\mathcal{L}$.

Definition 1.30. *A theory $\mathcal{T}$ has quantifier elimination if for every formula θ there is a quantifier-free formula ψ such that $(\mathcal{T} \models \theta) \Leftrightarrow (\mathcal{T} \models \psi)$.*

Examples 1.31. *Let $\theta(a,b,c)$ be the formula $\exists x\, ax^2 + bx + c = 0$. By the quadratic formula, $\mathbb{R} \models \theta(a,b,c) \Leftrightarrow [(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \vee (b \neq 0 \wedge c = 0))]$, whereas in the complex numbers $\mathbb{C} \models \theta(a,b,c) \Leftrightarrow (a \neq 0 \vee b \neq 0 \vee c = 0)$*

Examples 1.32. *Let $\phi(a,b,c,d)$ be the formula $\exists x\, \exists y\, \exists u\, \exists v\, (xa + yc = 1 \wedge xb + yd = 0 \wedge ua + vc = 0 \wedge ub + vd = 1)$.Actually the formula $\phi(a,b,c,d)$ asserts that the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. By the determinant test, $F \models \phi(a,b,c,d) \Leftrightarrow ad - bc \neq 0$ for any field F.*

1.1.3. Complete and Model Complete Theories

Let $\mathcal{L}$ be a language. An $\mathcal{L}$-theory $\mathcal{T}$ is simply a set of consistent $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a model of $\mathcal{T}$ and write $\mathcal{M} \models \mathcal{T}$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in \mathcal{T}$. A theory is satisfiable if it has a model.

Definition 1.33. *A satisfiable theory $\mathcal{T}$ is complete if $\mathcal{T} \models \phi$ or $T \models \neg\phi$ for all $\mathcal{L}$-sentences ϕ.*

Examples 1.34. *The theory of groups is not complete since the formula $\forall x, y(x.y = y.x)$ is just true for abelian groups. The theory of fields of characteristic 0 is not complete since the formula $\exists x(x\ x = 1 + 1)$ is true in $\mathbb{R}$, false in $\mathbb{Q}$. But the theory of algebraically closed fields of characteristic 0 is complete.*

Proposition 1.35. *Let $\mathcal{M}$, $\mathcal{N}$ be two $\mathcal{L}$-structures then $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{N} \models Th(\mathcal{M})$.*

Proof: If $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent it is clear that $\mathcal{N} \models Th(\mathcal{M})$. If $\mathcal{N} \models Th(\mathcal{M})$ then $\mathcal{M} \models \phi \Rightarrow \mathcal{N} \models \phi$ for all ϕ, and if $\mathcal{N} \models \phi$ then $\mathcal{M} \models \phi$, otherwise $\mathcal{M} \models \neg\phi$ implies $\mathcal{N} \models \neg\phi$. $\square$

Definition 1.36. *An $\mathcal{L}$-theory $\mathcal{T}$ is model-complete $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}$, $\mathcal{N} \models T$.*

Proposition 1.37. *If $\mathcal{T}$ has quantifier elimination, then $\mathcal{T}$ is model-complete.*

Proof: Let $\mathcal{M}$, $\mathcal{N}$ be models of $\mathcal{T}$ and $\mathcal{M} \subseteq \mathcal{N}$. We want to show $\mathcal{M}$ is an elementary sub-model of $\mathcal{N}$. Let $\phi(\bar{v})$ be an $\mathcal{L}$-formula, and let $\bar{a} \in M$. Since the theory has quantifier elimination there is a quantifier-free formula $\theta(\bar{v})$ such that $M \models \forall\bar{v}\ \phi(\bar{v}) \Leftrightarrow \theta(\bar{v}))$. Quantifier-free formulas are preserved under substructures and extensions, $\mathcal{M} \models \theta(\bar{a})$ if and only if $\mathcal{N} \models \theta(\bar{a})$. So

$$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{M} \models \theta(\bar{a}) \Leftrightarrow \mathcal{N} \models \theta(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{a}).$$

$\square$

There are methods to understand whether a theory is model complete or not.

Proposition 1.38. *The following are equivalent*

(i) *$\mathcal{T}$ is model complete.*

(ii) *Whenever $\mathcal{K}, \mathcal{N}, \mathcal{M} \models \mathcal{T}$ and $\mathcal{K} \subseteq \mathcal{N}$,and $\mathcal{K} \subseteq \mathcal{M}$, we have $\mathcal{N} \equiv_K \mathcal{M}$.*

Theorem 1.39. *(Robinson's test for model completeness). Let $\mathcal{T}$ be a first-order theory. Suppose that $\mathcal{T}$ has the following property: For any $\mathcal{M}, \mathcal{N} \models T$ such that $\mathcal{M} \subseteq \mathcal{N}$, and any quantifier-free formula $\phi(\bar{v})$ with parameters in $\mathcal{M}$, if $\mathcal{N} \models \exists\bar{v}\ \phi(\bar{v})$ then $\mathcal{M} \models \exists\ \bar{v}\phi(\bar{v})$. Then T is model-complete.*

Proof of these results can be found in [5].

1.2. O-minimality

Definition 1.40 (O-minimality)**.** *The structure $\mathcal{R} := \langle R, <, ..\rangle$ is said to be o minimal if the only definable subsets of R are finite unions of intervals and points, ie. those sets definable with just the ordering.*

There are infinite expansions $\mathcal{M} = (M, \leq, ..)$ of linear orderings such that the subsets of M definable in $\mathcal{M}$ are the finite unions of singletons and open intervals, possibly with infinite endpoints $\pm\infty$. In topological view, it means the only definable sets are unions of singletons and intervals including half-lines and whole M. The ordered field of reals $(\mathbb{R}, +, \cdot, -, 0, 1, \leq)$, as well as any real closed field, is o-minimal.

If we expand the ordered field of reals by addition, or addition and multiplication together, we get an o-minimal structure. On the other hand there exists expansions of $(\mathbb{R}, \leq)$ which are not o-minimal. For example , if we extend the ordered field of reals by sin function, then $\mathbb{Z}$ becomes definable and we loose o-minimality.

1.2.1. Examples of O-minimal Structures

Examples 1.41.

(i) *Let $\mathcal{R} = (R, <)$ be a dense linear order without endpoints. Since the theory of dense linear orders without endpoints has quantifier elimination R and we have only one relation in our language the definable sets are just finite union of intervals.*

(ii) *Let $\mathcal{R} = (\mathbb{R}, +, -, 1, 0)$. R has quantifier elimination so definable subsets of R are boolean combinations of sets of the form $\{x \in \mathbb{R} : p(x) = 0\}$ and $\{x \in \mathbb{R} : p(x) > 0\}$ where $p \in \mathbb{R}[X]$ is polynomial, consequently $\mathcal{R}$ is o-minimal.*

Theorem 1.42. *Let $\mathcal{R}$ be an expansion of* $(\mathbb{R}, <)$ *and suppose that:*

(i) for all $n \in \mathbb{N}$ every quantifier-free definable subset of $\mathbb{R}^n$ has finitely many connected components,

(ii) the theory of $\mathcal{R}$ is model-complete.

Then our structure $\mathcal{R}$ is o-minimal.

Proof: Let X be a definable subset of $\mathbb{R}$. By the model-completeness of $\mathcal{R}$ X is defined by an existential formula $\exists\phi(\bar{y}, x)$, i.e. $\phi(\bar{y}, x)$ is quantifier-free. Let Y be the set defined by $\phi(\bar{y}, x)$ must have finitely many connected components. X is the image of Y under the projection map $\pi : (\bar{y}, x) \to x$ so must have finitely many connected components also. The only connected subsets of $\mathbb{R}$ are intervals and points so X is of the desired form. $\square$

1.2.2. Monotonocity Theorem

Let $\mathcal{M} = (M, \leq, ...)$ be an o-minimal structure. If we put an order topology on M, for every integer n, M^n is a topological space as well, with respect to product topology. We know the definable sets of M. We are trying to find the definable subsets of M^n this will lead to notion of cells. First we will prove for $\mathbb{R}$ but it can be generalized to all real closed fields.

Lemma 1.43. *[6] If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is a definable function then f is continuous at one point of I.*

Proof: $case1$:

Assume there is an open set $J \subset I$ such that f has finite range on J. Pick an element b in the range of f with $\{x \in J : f(x) = b\}$ is infinite. By o-minimality, there is an open interval $J_0 \subset V$ such that f is constantly b on J_0.

$case2$:

In this case we assume each subinterval of I has infinite range under f. We can build $I = I_0 \subset I_1 \subset I_2$... of open subsets of I such that the closure $\overline{I}_{n+1}$ of I_{n+1} is contained in I_n inductively. Given I_n, let X be the range of f on I_n. Since X is infinite, by o-minimality, X contains an interval (a_n, b_n) of length at most $\frac{1}{n}$ with closure is in I_n. The set $Y = \{x \in I_n : f(x) \in (a_n, b_n)\}$ contains a suitable open interval I_{n+1}. $\mathbb{R}$ is locally compact Hausdorff space, by Baire Category theorem $\bigcap_{j=1}^{\infty} I_j = \bigcap_{j=1}^{\infty} \overline{I}_j \neq \emptyset$ let $x \in \bigcap_{j=1}^{\infty} I_j$. If we take any interval U containing $f(x)$ then by construction of I_n's there exists an I_n such that $(a_n, b_n) \subset U$ then $f(I_{n+1}) \subseteq U$ so f is continuous at x. $\square$

This Lemma is first order, so it is true for all Real Closed Fields, by the completeness of Real Closed fields.

Theorem 1.44. *[6] Let F be a real closed field and $f : F \to F$ is a definable function. Then, we can partition F into $I_1 \cup .... \cup I_m \cup X$, where X is finite and the I_j are pairwise disjoint open intervals with endpoints in $F \cup \{\pm\infty\}$ such that f is continuous on each I_j.*

Proof: Let D be the set of points where f is discontinuous i.e.

$$D = \{x : F \models (\exists \epsilon > 0 \forall \delta > 0 ((\exists y \, |x - y| < \delta) \wedge (|f(x) - f(y)| > \epsilon)))\}.$$

Since D is definable, by o-minimality D is either finite or has a nonempty interior. By Lemma 1.43, D must be finite. Thus,$F \backslash D$ is a finite union of intervals on which f is continuous. $\square$

In fact Theorem 1.44 can be improved as follows.

Theorem 1.45 (Monotonocity). *[6] If $f : F \to F$ is (definable) function, then we can partition F into $I_1 \bigcup .... \bigcup I_2 \bigcup X$, where X is finite and the I_j are pairwise disjoint open intervals with endpoints in $F \bigcup \{\mp\infty\}$ such that f is either constant or monotone and continuous on each I_j.*

Proof can be found in [6].

1.2.3. Cells

Let F denote real closed field. We understand so far definable subsets of F, to understand the definable subsets of F^n the notion of cell constructed as follows.

Definition 1.46 (Cells). *We inductively define the collection of cells as follows. $X \subseteq F^n$ is a 0-cell if it is a single point. $X \subseteq F$ is a 1-cell if it is an interval (a, b), where $a \in F \cup \{-\infty\}, b \in F \cup \{+\infty\}$, and $a < b$. If $X \subseteq F^n$ is an n-cell and $f : X \to F$ is a continuous definable function, then $Y = \{(x, f(x)) : x \in X\}$ is an n-cell. Let $X \subseteq F^n$ be an n-cell. Suppose that f is either a continuous definable function from X to F or f is identically $-\infty$ and g is either a continuous definable function from X to F such that $f(\bar{x}) < g(\bar{x})$ for all $\bar{x} \in X$ or g is identically $+\infty$; then $Y = \{(\bar{x}, y) : \bar{x} \in X, f(\bar{x}) < y < g(\bar{x})\}$ is an $n + 1$-cell.*

Theorem 1.47 (Uniform Bounding). *Let $X \in F^{n+1}$ be semi-algebraic. There is a natural number N such that if $\bar{a} \in F^n$ and $X_{\bar{a}} = \{y : (\bar{a}, y) \in X\}$ is finite, then $|X_{\bar{a}}| < N$.*

Proof: The set $X_{\bar{a}}$ is infinite if and only if it contains an interval $(c, d) \subseteq X_{\bar{a}}$. Thus $\{(\bar{a}, b) \in X : X_{\bar{a}}$ is finite $\}$ is definable. Without loss of generality, we may assume that for all $\bar{a} \in F^n, X_{\bar{a}}$ is finite and we may assume that

$$F \models \forall x\, \forall c\, \forall d\, \neg[c < d \wedge \forall y(c < y < d \to y \in X_{\bar{a}})]$$

Consider the following set of sentences in the language of fields with constants added for each element of F and new constants $c_1, ..., c_n$. Let Γ be

$$RCF + \ DiagF + \ \{\exists y_1, ..., y_m \ [\bigwedge_{i<j} y_i \neq y_j \wedge \bigwedge_{i=1}^{m} y_i \in X_{\bar{c}}] : m \in \omega\}.$$

Suppose that Γ is satisfiable. Then, there is a real closed field $K \supseteq F$ and elements $\bar{c} \in K^n$ such that $X_{\bar{c}}$ is infinite. By model-completeness, $F \prec K$. Therefore

$$K \models \forall \bar{x} \ \forall c \ d \neg [c < d \wedge \forall y (c < y < d \rightarrow y \in X_{\bar{a}})].$$

This contradicts the o-minimality of K. Thus is unsatisfiable and there is an N such that

$$RCF + Diag(F) \models \forall \bar{x} \neg \ (\ \exists y_1, ..., y_N [\bigwedge_{i<j} y_i \neq y_j \wedge \bigwedge_{i=1}^{N} y_i \in X_{\bar{x}}])$$

In particular, for all $a \in F^n, |X_{\bar{a}}| < N$. $\square$

1.2.4. Cell Decomposition

Definition 1.48. *We say that $X \subset F^n$ is semialgebraic if it is a finite Boolean combination of sets of the form $\{\bar{x} : f(\bar{x}) > 0\}$ or $\{\bar{x} : f(\bar{x}) = 0\}$, $f \in F[\bar{X}]$.*

Corollary 1.49. *In a real closed ordered field K, the definable sets are exactly the semialgebraic ones.*

Corollary 1.50. *Let F be a real closed field. If $X \subset F^n$ is a closed and bounded semialgebraic set and $f : X \rightarrow F^n$ is continuous and semialgebraic, then the image of X is closed and bounded.*

Theorem 1.51. *[5] Let $X \subseteq F^m$ be semialgebraic. There are finitely many pairwise disjoint cells $C_1, ..., C_n$ such that $X = C_1 \cup ... \cup C_n$.*

Proof: $(m=2)$ For each $a \in F$, let

$$C_a = \{x : \forall \epsilon > 0 \; \exists y, z \in (x-\epsilon, x+\epsilon)[(a,y) \in X \wedge (a,z) \notin X]\}.$$

C_a called the critical values above a. By o-minimality, there are only finitely many critical values above a. By uniform bounding, there is a natural number N such that for all $a \in F$, $|C_a| < N$. We partition F into $A_0, A_1, ..., A_N$, where $A_n = \{a : |C_a| = n\}$. For each $n \leq N$, we have a definable function $f_n : A_1 \cup ... \cup A_n \to F$ by $f_n(a) = n$th element of C_a. As above, $X_a = \{y : (a,y) \in X\}$. For $n \leq N$ and $a \in A_n$, we define $P_a \in 2^{2n+1}$. If $n = 0$, then $P_a(0) = 1$ if and only if $X_a = F$. Suppose that $n > 0$. $P_a(0) = 1$ if and only if $x \in X_a$ for all $x < f_1(a)$. $P_a(2i-1) = 1$ if and only if $f_i(a) \in X$. For $i < n$, $P_a(2i) = 1$ if and only if $x \in X_a$ for all $x \in (f_i(a), f_{i+1}(a))$. $P(2n) = 1$ if and only if $x \in X_a$ for all $x > f_n(a)$. For each possible pattern $\sigma \in 2^{2n+1}$, let $A_{n,\sigma} = \{a \in A_n : P_a = \sigma\}$. Each $A_{n,\sigma}$ is semialgebraic. For each $A_{n,\sigma}$ we will give a decomposition of $\{(x,y) \in X : x \in A_{n,\sigma}\}$ into disjoint cells. Because the $A_{n,\sigma}$ partition F, this will suffice. Fix one $A_{n,\sigma}$ By Corollary 1.43, we can partition $A_{n,\sigma} = C_1 \cup ... \cup C_l$, where each C_j is either an interval or a singleton and f_i is continuous on C_j for $i \leq n$, $j \leq l$. We can now give a decomposition of $\{(x,y) : x \in A_{n,\sigma}\}$ into cells such that each cell is either contained in X or disjoint from X. For $j \leq l$, let $D_{j,0} = \{(x,y) : x \in \; C_j$ and $y < f(1)\}$. For $j \leq l$ and $1 \leq i \leq n$, let $D_{j,2i-1} = \{(x, f_i(x)) : x \in C_j\}$. For $j \leq l$ and $1 \leq i < n$, let $D_{j,2i} = \{(x,y) : x \in C_j, f_i(x) < y < f_{i+1(x)}\}$. For $j \leq l$, let $D_{j,2n} = \{(x,y) : x \in C_j, y > f_n(x)\}$. Clearly, each $D_{j,i}$ is a cell, $\bigcup D_{j,i} = \{(x,y) : x \in A_{n,\sigma}\}$, and each $D_{j,i}$ is either contained in X or disjoint from X. Thus, taking the $D_{j,i}$ that are contained in X, we get a partition of $\{(x,y) \in X : x \in A_{n,\sigma}\}$ into disjoint cells. $\square$

1.2.5. Some Theorems on O-minimal Structures

Lemma 1.52. *[7] Suppose I is an interval, $a \in I$, and $f : I \to \mathbb{R}$ is a definable function. Then both $f'(a^-) = lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$ and $f'(a^+)$ exists in $\mathbb{R} \cup \{\pm\infty\}$.*

Proof: It is enough to prove for $f'(a^+)$, we can prove smilarly for $f'(a^-)$. Let $K = lim_{h\to 0^+} \inf \frac{f(a+h)-f(a)}{h}$ and $L = lim_{h\to 0^+} \sup \frac{f(a+h)-f(a)}{h}$, suppose that $K < L$. Let r be a rational number such that $K < r < L$. The set of points $h > 0$ such that $f(a+h) - f(a) < rh$ definable and it contains a sequence approaching to 0 then by o-minimality this set contains $(0, t)$ where $t > 0$, also the set of points $h > 0$ such that $f(a+h) - f(a) > rh$ is definable and it contains a sequence approaching to 0, then by o-minimality interval containing $(0, k)$ where $t > 0$, contradiction. □

Lemma 1.53. *[7] Suppose I is an interval and $f : I \to \mathbb{R}$ is continuous and $f'(a^+)$ is defined in $\mathbb{R}\cup\{\pm\infty\}$ and greater than 0 for all $a \in I$. Then f is strictly increasing and, its inverse f^{-1} defined on the interval $f(I)$ has the property that $(f^{-1})'(b^+)$ is defined and equal to $\frac{1}{f'(a^+)}$ where $f(a) = b$.*

Proof: $f'(a^+) = lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$, then the set of points $h > 0$ such that $f(a+h) - f(a) > 0$ is definable and this set contains a sequence approaching 0, by o-minimality this set should contain an interval $(0, t)$ where $t > 0$, and f is strictly increasing on that interval. Let $f(a) = b$, $(f^{-1})'(b^+) = lim_{h\to 0^+} \frac{f^{-1}(b+h)-f^{-1}(b)}{h}$, $f^{-1}(b+h) = a + k$ where $k > 0$, also we know $f^{-1}(b) = a$ then $h = f(a+k) - f(a)$, and

$$lim_{h\to 0^+} \frac{f^{-1}(b+h) - f^{-1}(b)}{h} = lim_{k\to 0^+} \frac{a+k-a}{f(a+k) - f(k)} = \frac{1}{f'(a^+)}$$

, since f is continuous k goes to 0 iff h goes to 0. □

Lemma 1.54. *[7] Suppose I is an interval and $f : I \to \mathbb{R}$ is continuous and the maps $a \to f'(a^+)$ and $a \to f'(a^-)$ are well defined, real valued, and continuous on all of I. Then f is differentiable and has a continuous derivative on I.*

Proof: If we show $f'(a^+) = f'(a^-)$ we are done. Suppose $f'(a^+) > f'(a^-)$ for some $a \in I$. Since $a \to f'(a^+)$ and $a \to f'(a^-)$ are well defined and continuous there is a $c \in \mathbb{R}$ and an interval $J \subset I$ containing point a and $f'(x^-) < c < f'(x^+)$ for all $x \in J$. Then if we define $g : J \to \mathbb{R}$ with $g(x) = f(x) - c$ it is continuous and

$g'(x^-) < 0$ and $g'(x^+) > 0$ for all x. By the lemma 1.53 g is strictly increasing on J and if we apply the lemma to $-g(x)$ it shows that $g(x)$ strictly decreasing on J, we get contradiction. So f is continuously differentiable on I. $\square$

Lemma 1.55. *[7] Suppose I is an interval and $f : I \to \mathbb{R}$ is a definable function then there are only finitely many $x \in I$ such that $f'(x^+) = \pm\infty$*

Proof: Suppose that $\{x \in I : f'(x^+) = +\infty\}$ is infinite, since this set is definable it contains an interval J. Also by o-minimality we can assume it is continuous also on J. By the lemma 1.53 we can assume f is strictly increasing in that interval. Also that implies $f'(x^-) \geq 0$ on J. Let $B = \{x \in I : f'(x^-) = +\infty\}$ if this set is infinite we can assume on our interval J, $f'(x^-) = +\infty$. Otherwise we can assume $f'(x^-)$ is finite on J and $x \to f'(x^-)$ is continuous on J. In the first situation, f^{-1} satisfies $(f^{-1})'(b^+)\frac{1}{f'(a^-)} = 0$ for all b, means f^{-1} is constant, contradicting the fact that f is a function.
In second situation we are in the same situation as in the proof of the lemma 1.54 we assume $f'(a^+) > f'(a^-)$ and in similar way we get contradiction. So there are only finitely many $f'(x^+) = \pm\infty$. $\square$

Theorem 1.56. *Let $f : I \to \mathbb{R}$ be definable (semi-algebraic) function. Then, we can partition I into $I_1 \cup .... \cup I_m \cup X$, where X is finite and the I_j are pairwise disjoint open intervals with endpoints in $\mathbb{R} \cup \{\pm\infty\}$ such that f is continuously differentiable on each I_j .*

Proof: By Theorem 1.44 we can partition I in intervals such that f is continuous and $f'(x^+)$ and $f'(x^-)$ are finite on each interval I_j then the functions $x \to f'(x^-)$ and $x \to f'(x^+)$ are definable, hence they are same by the lemma 1.54 and continuous in each interval. $\square$

1.2.6. The Theory of Real Closed Fields

A generalization of theory of real numbers is real closed fields, the study of real closed fields has an important role in the development of o-minimality. Here are some theorems proofs can be found in [5].

Definition 1.57. *A field F is said to be formally real if -1 is not a sum of squares. We say F is real closed if it is formally real and has no proper real algebraic extensions.*

If F is a real closed and $a \neq 0$, then exactly one of a and $-a$ is a square. we can define an order on F such that positive elements are exactly the squares.

Theorem 1.58 (Artin-Schrier)**.** *Let $(F, <)$ be an ordered field. Then the followings are equivalent*

(i) *F is real closed.*
(ii) *F(i) is algebraically closed (were $i = \sqrt{-1}$)*
(iii) *If $p(X) \in F[X]$, $a, b \in F$ such that $a < b$ and $p(a) < p(b)$ then there is a $c \in F$ such that $a < c < b$ and $p(c) = 0$*
(iv) *For any $a \in F$ either a or -a is a square and every polynomial of odd degree has a root.*

By (iv), we can axiomatize the theory of real closed fields in the language of rings $\mathcal{L}_r$ by axioms asserting that F is formally real field of characteristic zero where (iv) holds. We call this theory RCF.

The study of the field of real numbers began with the work of Tarski . Unlike the algebraically closed fields, the theory of the real numbers does not have quantifier elimination in $\mathcal{L}_r$, the language of rings. In $\mathcal{R}$ the formula $\psi(x) = (\exists z z^2 = x)$, defines an infinite coinfinite definable set. In fact the ordering is the only obstruction to quantifier elimination, and the ordering $x < y$ is definable in the real field by the formula $\exists z(z \neq 0 \wedge x + z^2 = y)$.

Theorem 1.59. *The theory of real closed fields(RCF) has the elimination of quantifiers in the language* $L = \{+, -, \cdot, \leq, 0, 1\}$.

Theorem 1.60. *The theory of real closed fields (RCF) is model complete.*

Theorem 1.61. *The theory of real closed fields is complete, RCF is the theory of the ordered field of reals.*

Corollary 1.62. *Every real closed field is o-minimal.*

2. Exponentiation, Power Function and Polynomially Bounded Structures

2.1. Polynomially Bounded O-minimal Structures

We need to fix our notation and we need to make clear which structure we are working on. Let R denotes any ordered field, and $\bar{R} = (R, <, +, -, \cdot, 0, 1)$. Pos$(R)$ denotes the positive elements of R. Moreover $\mathcal{R}$ will denote any o-minimal expansion ordered field $\bar{R} = (R, <, +, -, \cdot, 0, 1)$.

The o-minimality of $\mathcal{R}$ allows us in most cases like we are working in reals. The mean value theorem, the intermediate value theorem, l'Hospital's Rule, derivatives tests can be applied to definable functions.

Definition 2.1 (Van den Dries-Miller)**.** *An expansion R of the real field is said to be polynomially bounded if for all definable functions $f : R \to R$ there exists a natural number N such that $|f(x)| \leq x^N$ for all sufficiently large x.*

Definition 2.2. *Let R be an ordered field. A power function on R is an endomorphism of the multiplicative group of positive elements of R.*

Definition 2.3. *$Pos(R)$ will denote the positive elements of R.*

Example 2.4. *For each $r \in \mathbb{R}$ the map $x^r : Pos(R) \to Pos(R)$ is a power function.*

Proposition 2.5. *[8][Definable Subgroups]There are no proper, non-trivial definable subgroups of either $(R, +, 0)$, $(Pos(R), \cdot, 1)$.*

Proof: Let G be a definable subgroup of $(R, +, 0)$. Assume G is not convex then there exists $r \in R \setminus G$ such that $0 < r < g$ and $g \in G$. Then for all $n \in \mathbb{N}$ $ng < r + ng < (n+1)g$ since G is a subgroup then none of the $r + ng$ is an element of G, hence G can not be finite union of intervals and points. So G is convex let

$s := \sup(G)$ then if we take $a \in (0, s)$ $a \in G$ since G is a subgroup $na \in G$ for all $n \in \mathbb{N}$ that implies $s = +\infty$. So G must be R. Similar proof works for $(Pos(R), \cdot, 1)$. □

Corollary 2.6. *[4] Let $f : (Pos(R), \cdot, 1) \to (Pos(R), \cdot, 1)$ be a definable power function then f is identically equal to 1 or f is an automorphism of $(Pos(R), \cdot, 1)$.*

Proof: We know that image and the kernel of f is a definable subgroup of $Pos(R)$. By the previous proposition, image of function f is either $\{1\}$ or $Pos(R)$ and also kernel of f is $\{1\}$ or $Pos(R)$. So f is identically 1 or it is a bijection. □

Lemma 2.7. *[9] (Lemma 2.2.35)*

(i) Let f be a power function then $f'(x) = f'(1)\frac{f(x)}{x}$.

(ii) f is monotonic power function; if $f'(1) > 0$ then f is strictly monotone increasing, if $f'(1) < 0$ then f is strictly monotone decreasing and if $f'(1) = 0$ then f is constantly 1.

Proof:

(i) If f is a power function for $x > 0$ and $h \in \mathbb{R}$ is sufficiently small then

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x)}{x}\frac{f(1 + hx^{-1}) - 1}{hx^{-1}} = \frac{f(x)}{x}f'(1).$$

Consequently f is differentiable at x if and only if f is differentiable at 1. Since f must be differentiable at all but finitely many points by extended monotonicity f is everywhere differentiable.

(ii) Proofs follows by Lemma 1.53. □

Remark 2.8. *If f is a definable power function, then from Lemma 2.6 f is infinitely differentiable.*

Lemma 2.9. *Let f and g be definable power functions in R and suppose that $f'(1) = g'(1)$. Then $f = g$.*

Proof: $\frac{f}{g}$ is a power function with exponent 0. The result follows from lemma. $\square$

2.1.1. Hardy Fields

Let R be an ordered field and K be the ring of functions $f : R \to R$ containing all constant functions and the identity function under pointwise addition and multiplication of functions. We will say that a property $P(x)$ holds ultimately if $P(x)$ holds for all sufficiently large $x \in R$. Let I be the ideal of K consisting of all those $f \in K$ that are ultimately zero. Let $\mathcal{H} = K/I$, $\mathcal{H}$ is a field if and only if for all $f \in K/I$ there exists $g \in K$ such that fg is ultimately equal to 1; in particular we must have that f is ultimately non-zero. Moreover if for all $f \in K$, f has ultimately constant sign then $\mathcal{H}$ becomes an ordered field when we say that $f < g$ if and only if $g - f$ is ultimately positive.

Definition 2.10. *Notation as being above $\mathcal{H}$ is called a Hardy Field if all $f \in K$ are ultimately differentiable and K is closed under differentiation.*

If we take the ideal that contain functions that ultimately zero, after dividing this ideal we make the functions equal if $f = g$ ultimately.

There is another way to seeing this construction if we just work on reals.

Definition 2.11. *Given $a \in \mathbb{R}$ we define an equivalence relation $\sim$ on the set of real valued functions whose domains contains a neighborhood of a by $f \sim g$ if there is neighborhood of V of a, $V \subseteq dom(f) \bigcap dom(g)$, such that $f\ |_V = g\ |_V$. The equivalence classes are called* germs *at a.*

Remark 2.12. *[10] The elements of Hardy fields as we constructed above coincides with the germs at infinity.*

Definition 2.13 (Valuation)**.** *Let R be an ordered field and R^* denotes nonzero elements of ordered field. A* valuation *on R is a surjective map $v : R^* \to G$ onto an ordered abelian group G satisfying the following properties.*

(i) $v(xy) = v(x) + v(y)$

(ii) If $x + y \neq 0$ then $v(x + y) \geq \min\{v(x), v(y)\}$ with equality if $v(x) \neq v(y)$

(iii) $v(x) < 0$, then $|x| > 1$

The group G is called *value group* of valuation. A valuation can be extended to all R by adding ∞ greater than every element of G, and defining $v(0) = \infty$.

Definition 2.14. *Let f be a power function, $f \mapsto f'(1) : \mathcal{A} \to R$ is an ordered field embedding. The image of this map is called the* field of exponents *of R and denoted by K.*

2.1.2. Exponentiation Function and Main Result

Definition 2.15. *Let F be an ordered field. An exponential function on F is a homomorphism from the additive group of F to the multiplicative group of positive elements of F.*

The map $x \to \exp(x) : \mathbb{R} \to \mathbb{R}$ is an exponential function on the ordered field of real numbers. exp is of course definable in the o-minimal structure R_{exp}. As we mentioned previously $\mathcal{R}$ will denote an o-minimal expansion of $(R, +, -, 0, 1)$.

Lemma 2.16. *[9] Let f be a exponential function on R.*

(i) If f is not identically equal to 1 then f is an isomorphism between $(R, +)$ and $(Pos(R), \cdot)$.

(ii) f *is differentiable with* $f'(x) = f'(0)f(x)$.

(iii) f *is monotonic; if* $f'(0) > 0$ *then* f *strictly monotone increasing, if* $f'(0) < 0$ *then* f *is strictly monotone decreasing and if* $f'(0) = 0$ *then* f *is identically equal to* 1.

Proof:

(i) It follows from 2.5.

(ii) $f'(x) = \lim_{h\to\infty} \frac{f(x+h)-f(x)}{h} = \lim_{h\to\infty} \frac{f(x)(f(h)-1)}{h} = \lim_{h\to\infty} f(x)f'(0)$.

(iii) Since $f'(x) = f'(0)f(x)$, $f'(0) > 0$ implies $f'(x) > 0$. Then f is strictly increasing by Lemma 1.53. $\square$

Lemma 2.17. *[9] If* f *is a definable exponential function in the structure* $\mathcal{R}$ *where* $\mathcal{R}$ *is an o-minimal expansion ordered field and* $f'(0) > 0$ *then for all* $r \in R$, $\frac{f(x)}{x^r} \to \infty$ *as* $x \to \infty$.

Proof: Let $r \in R$ and let $g(x) = \frac{f(x)}{x^{r+1}}$. Then

$$g'(x) = \frac{f(x)}{x^{r+2}}(xf'(0) - (r+1)).$$

Since $f'(0) > 0$ then f is positive eventually by monotonocity. So $g'(x)$ and $g(x)$ are eventually positive then we can find $R > 0$ such that $g(x) > R$ for large x. Therefore $\frac{f(x)}{x^r} = xg(x) \to \infty$ as $x \to \infty$. $\square$

Proposition 2.18. *[4]*

(i) If there is a nonconstant definable function $f : R \to R$ *with* $f(x+y) = f(x)f(y)$ *for all* $x, y \in R$ *in then the structure* R *is exponential.*

(ii) If there exist a nonconstant definable function $h : Pos(R) \to R$ *in structure* $\mathcal{R}$ *with* $h(xy) = h(x) + h(y)$ *for all* $x, y > 0$*, then the structure* $\mathcal{R}$ *is exponential.*

(ii) If there exists a definable function $g : R \to R$ in structure $\mathcal{R}$ with $xg'(x) \to 1$ as $x \to \infty$ then $\mathcal{R}$ is exponential.

Proof:

(i) If $f(a) = 0$ for some $a \in R$ then $f(x) = 0$ for all $x \in R$ but $f(x)$ is nonconstant so $f(x) \neq 0$ for all $x \in R$. For any $x \in R$ we have $f(x) = f(x/2 + x/2) = (f(x/2))^2 > 0$, and for $x = 0$ $f(0) = (f(0))^2$, so $f(0) = 1$ and f is always positive. The kernel of f is definable subgroup of $(R, +, 0)$ and the image of f is a definable subgroup of $(Pos(R), \cdot, 1)$ by the lemma 2.5 the image should be $Pos(R)$ and the kernel should be $\{0\}$, so it is a bijection. Also

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ = \lim_{h \to 0} f(x) \frac{f(h) - 1}{h} &= \lim_{h \to 0} f(x) \frac{f(h) - f(0)}{h} \end{aligned}$$

so $f'(x) = f(x)f'(0)$, by generalized monotonicity it is ultimately differentiable. So if it is differentiable at 0, that means it is differentiable everywhere. Considering $x \to f(x/f'(0))$, we can assume $f'(x) = f(x) > 0$ so f is strictly increasing on R so f is a definable ordered group isomorphism from $(R, <, +, 0)$ to $(Pos(R), <, \cdot, 1)$.

(ii) The kernel of h is a definable subgroup $(R, \cdot, 1)$ and the image of h is a definable subgroup of $(Pos(R), +, 0)$, since h is nonconstant, by lemma 2.5 h is a bijection. The inverse of the h satisfies conditions in (i). We can apply one to inverse function of h.

(iii) Let $g : R \to R$ be definable and $xg'(x) \to 1$ as $x \to +\infty$. If we take $s, t > 0$ such that $\lim_{x \to \infty}[g(tx) - g(x)] \in R$ and $\lim_{x \to \infty}[g(sx) - g(x)] \in R$. Since both limit exist we can write

$$\lim_{x\to\infty}[g(tx)-g(x)]+\lim_{x\to\infty}[g(sx)-g(x)] = \lim_{x\to\infty}[g(tsx)-g(sx)]+\lim_{x\to\infty}[g(sx)-g(x)]$$
$$= \lim_{x\to\infty}[g(tsx)-g(x)].$$

Also $\lim_{x\to\infty}[g(x/t)-g(x)] = \lim_{x\to\infty}[g(x)-g(tx)] = -\lim_{x\to\infty}[g(tx)-g(x)] \in R$. So the set $\{t > 0 : \lim_{x\to\infty}[g(tx)-g(x)] \in R\}$ is a definable subgroup of $(Pos(R), \cdot, 1)$. If we can show kernel contains an element different then 1 then kernel has to be all $Pos(R)$. By mean value theorem, ultimately we have $g(2x) - g(x) = xg'((\xi(x)))$ where $x < \xi(x) < 2x$. Also $1/2 < x/\xi(x) < 1$. Thus $1/2 \le \lim_{x\to+\infty} x/\xi(x) \le 1$ and $1/2 \le \lim_{x\to+\infty}[g(2x)-g(x)] = (\lim_{x\to+\infty} x/\xi(x))(\lim_{x\to+\infty} \xi(x)g'(\xi(x))) \le 1$. Since $\lim_{x\to+\infty} \xi(x)g'(\xi(x)) = \lim_{x\to+\infty} xg'(x) = 1$. □

Let $\mathcal{R}$ be o-minimal expansion of $(R, +, -, 0, 1)$, since definable functions are monotone after some point and by Theorem 1.44 they are ultimately differentiable and have constant sign and also closed under differentiation so they form a differentiable ordered field [10]. Moreover we can regard R as a subfield of $\mathcal{H}$ by identifying $r \in R$ with the germ of constant function $f(x) = r$. We will denote the germ of identity function by x. If we denote nonzero elements of $\mathcal{H}$ as $\mathcal{H}^*$ we can put valuation v on $\mathcal{H}^*$ with following properties.

Theorem 2.19. *Let $\mathcal{H}$ be a Hardy field. Then there exists a map v from the set of nonzero element $\mathcal{H}^*$ onto ordered abelian group such that*

(i) if $f, g \in \mathcal{H}^*$, then $v(fg) = v(f) + v(g)$;
(ii) if $f \in \mathcal{H}^*$, then $v(a) \ge 0$ if and only if $\lim_{x\to+\infty} f(x) \in \mathbb{R}$;
(iii) if $f, g \in \mathcal{H}^*$ and $v(f), v(g) \neq 0$,then $v(f) \ge v(g)$ if and only if $v(f') \ge v(g')$;
(iv) if $f, g \in \mathcal{H}^*$ and $v(f) \ge v(g) \neq 0$, then $v(f') > v(g')$

Proof can be found in theorem 4 of [11].

Remark 2.20. *We will use the following properties of valuation in our proofs.*

(i) $v(f) = 0$ *if* $\lim_{x\to\infty} f(x) \in R^*$

(ii) $v(f) < 0$ *if* $\lim_{x\to\infty} |f(x)| = +\infty$

(iii) $v(f) > 0$ *if* $\lim_{x\to\infty} f(x) = 0$

Definition 2.21. $f \in \mathcal{H}^*$ *is* infinitely increasing *if* $\lim_{x\to\infty} |f(x)| = +\infty$ *i.e* $f > 0$, $v(f) < 0$.

Remark 2.22. *If* $f \in \mathcal{H}^*$ *with valuation different then zero then one of* f, $1/f$, $-f$, $-1/f$ *is infinitely increasing, and* $|v(f)| = |v(-f)| = |v(1/f)| = |v(-1/f)|$.

Proof: We know $v(f + g) = v(f) + v(g)$, then $v(f\ (1/f)) = v(f) + v(1/f)$ since $v(1) = 0$, $v(f) = -v(1/f)$, and $|v(f)| = |(1/f)|$. Others follows similarly. $\square$

Remark 2.23. *Let* $f(x), g(x) \in \mathcal{H}^*$ $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ *if and only if* $v(f(x)) > v(g(x))$.

Proof: Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, $v(\frac{f(x)}{g(x)}) > 0$ and $v(\frac{f(x)}{g(x)}) = v(f(x)) + v(1/g(x))$. So $v(f(x)) - v(g(x)) > 0$ which means $v(f(x)) > v(g(x))$. The other way also trivial.$\square$

Remark 2.24. *Let* $f(x), g(x) \in \mathcal{H}^*$ $f(x) > g(x)$ *then* $v(f(x)) \leq v(g(x))$.

Proof: Since $f(x) > g(x)$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} \geq 1$. Then $v(\frac{f(x)}{g(x)}) \leq 0$, but $v(\frac{f(x)}{g(x)}) = v(f(x)) - v(g(x))$, implies $v(f(x)) \leq v(g(x))$. $\square$

Remark 2.25. *It follows from the Remark 2.23 that* $v(f) = v(g)$ *if and only if* $f \sim cg$ *for some* $c \in R^*$.

Proposition 2.26. *[4] Let* $f \in \mathcal{H}^*$ *with* $v(f) = v(x^r)$. *Then* $f \circ (tx) \sim t^r x$ *and* $f \circ (t + x) \sim f$ *for each* $t > 0$. *If* $r \neq 0$, *then* $xf'/f \sim r$.

Proof: Since $v(f) = v(x^r)$ there exists $c \in R^*$ such that $\lim_{x\to\infty} f/x^r = c$. Let $t > 0$. then

$$\frac{f(tx)}{f(x)} = t^r \frac{f(tx)}{(tx)^r} \frac{x^r}{f(x)}$$

and

$$\frac{f(t+x)}{f(x)} = (1 + (t/x))^r \frac{f(t+x)}{(t+x)^r} \frac{x^r}{f(x)}.$$

Then

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)t^r} = \lim_{x\to\infty} \frac{f(tx)}{(tx)^r} \frac{x^r}{f(x)} = (1/c)\ c = 1$$

so $f \circ (tx) \sim t^r x$ the second one follows similarly. For the last claim, if we suppose $r \neq 0$ then by l'Hospital's rule $f' \sim crx^{r-1}$. Then $xf' \sim rcx^r \sim rf$ and $xf'/f \sim r$.□

Proposition 2.27. *[4] Let $f, g \in \mathcal{H}^*$ with $v(f) \geq 0$ and $v(g) \neq 0$. Then $v(f') > v(g'/g)$.*

Proof: We can assume $v(f) = 0$, otherwise we can change f with $f + 1$. Since $v(g) \neq 0$, $\lim_{x\to\infty} g(x)$ is 0 or ∞, in both cases we apply l'Hopital's rules to function fg/g,

$$f = fg/g \sim (fg' + f'g)/g' = f + f'g/g'.$$

So $1 \sim 1 + (f'/f)(g/g')$, that means $\lim_{x\to\infty}(f'/f)(g/g') = 0$, by Remark 2.23 $v((f'(g/g')) > v(f)$, but $v(f) = 0$, i.e $\lim_{x\to\infty} f(x) \in R^*$, then it follows $v((f')(g/g')) > 0$, then again by Remark 2.23 $v(f') > v(g'/g)$. □

Proposition 2.28. *[3] Let $f, g \in \mathcal{H}^*$ with $|v(f)| \geq |v(g)|$. Then $v(f'/f) \leq v(g'/g)$.*

Proof: By Remark 2.22, we can assume f and g are both infinitely increasing, without effecting $|v(f)|$,$|v(g)|$ $|v(f'/f)|$ $|v(g'/g)|$. Then valuation of f and g will be negative and $v(f) \leq v(g) < 0$. If $v(f) = v(g)$, then by l'Hopital's rule, $v(f') = v(g')$. If $v(f) < v(g)$ by Remark 2.23 $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ so g/f is infinitely increasing and $(f/g)' > 0$, $f'g - g'f > 0$ which implies $f'/f > g'/g$. Then by Remark 2.25 $v(f'/f) \leq v(g'/g)$ □

Proposition 2.29. *[11] Let $f \in \mathcal{H}^*$ with $v(f) < v(1/x)$. Then there exists $g \in \mathcal{H}^*$ with $g' \sim f$.*

Proof: Let choose $u \in \mathcal{H}^*$ with

$$v(u) = \min\{|v(x)|, |v(xf)|\} = \min\{v(1/x), v(1/(xf))\}.$$

Then $|v(u)| > 0$ since the valuation of x and xf cannot be zero. In the Proposition 2.27 if we choose $g(x) = x$, then $v(u') > 0$. Also from Proposition 2.28 we have $v(\frac{u'}{u}) \geq v(\frac{1}{x}) > v(f)$, and thus

$$v(\frac{u'}{fu}) \geq v(\frac{1}{xf}) = |v(xf)| \geq |v(u)| > 0.$$

Also

$$v(\frac{u'}{fu}) = |v(\frac{fu}{u'})|.$$

Since $v(\frac{u'}{fu}) > 0$ so $\frac{u'}{fu}$ is not a constant function so $(\frac{u'}{fu})' \neq 0$. $|v(\frac{fu}{u'})| \geq |v(u)| > 0$ if we apply 2.28 again, we get

$$v(\frac{u'}{u}) \geq v(\frac{(fu/u')'}{fu/u'}) = v(\frac{(fu/u')'}{f}) + v(\frac{u'}{u}).$$

Then $v(\frac{f}{(fu/u')'}) \geq 0$ and we can apply Proposition 2.27 and get $v((\frac{f}{(fu/u')'})') > v(\frac{u'}{u})$, thus from Remark 2.23

$$\lim_{x\to\infty} \frac{u(x)}{u'}(\frac{f(x)}{(f(x)u(x)/u'(x))'})' = 0.$$

Let $g = \frac{f^2u/u'}{(fu/u')'}$. Then

$$\frac{g'}{f} = 1 + (\frac{u}{u'})(\frac{f}{(fu/u')'})',$$

so $g' \sim f$. □

Definition 2.30. *Non zero elements α, β of ordered abelian group are called comparable if there are positive integers $m|\alpha| > |\beta|$ and $n|\alpha| < |\beta|$. Non zero f, g of a hardy field $\mathcal{H}$, are called comparable if $v(f)$ and $v(g)$ are comparable in the value group $v(\mathcal{H}^*)$.*

Lemma 2.31. *If a definable function f is not polynomially bounded, then the function f and $g(x) = x$ aren't comparable.*

Proof: The function f is not polynomially bounded i.e $\lim_{x\to\infty} f(x)/x^n = \infty$, so $v(f(x)/x^n) < 0$ for all $n \in \mathbb{N}$. Moreover $v(f(x)) + v(1/x^n) = v(f(x)/x^n) < 0$ and

$$v(f(x)) < -v(1/x^n) = v(x^n) = nv(x) \quad \forall n \in \mathbb{N},$$

so f and g aren't comparable. □

Proposition 2.32. *[3] Let $\mathcal{H}$ be a Hardy field, $f, g \in \mathcal{H}$, $v(f), v(g) \neq 0$ and $v(f'/f) = v(g'/g)$. Then f and g are comparable.*

Proof: Without loss of generality we may suppose f and g are infinitely increasing by Remark 2.22, since f and g are monotone after some large N we can suppose $f \geq g$. Then $\log f$ and $\log g$ are infinitely increasing, $\log f \geq \log g$ and $\lim_{x\to\infty} \log f/\log g =$

$\lim_{x\to\infty}(\log f)'/(\log g)' = \lim_{x\to\infty}(f'/f)/(g'/g)$, since $v(f'/f) = v(g'/g)$ $\lim_{x\to\infty}(f'/f)/(g'/g)$ is a real number. Thus for some positive integer N we have $\log f/\log g < N$, and $f < g^N$ $v(f) > Nv(g)$ so they are comparable. $\square$

Theorem 2.33. *[2] Let $\mathcal{R}$ be o-minimal and not polynomially bounded. Then the exponential function is definable.*

Proof: Since $\mathcal{R}$ is not polynomially bounded there exist a definable function f which is not polynomially bounded. By Lemma 2.31 f and $g(x) = x$ aren't comparable. Then by Proposition 2.32 we can say $v(f'/f) \neq v(1/x)$ if we choose g as x. Furthermore we can replace f by its compositional inverse if it is necessary we can assume $v(f'/f) < v(1/x)$. Then by 2.29 there exists a $h \in \mathcal{H}^*$ such that $h' = f'/f$. Then

$$\begin{aligned}\lim_{x\to\infty}(h\circ f^{-1}(x))' &= \lim_{x\to\infty} h'(f^{-1}(x)) = \lim_{x\to\infty} h'(f^{-1}(x))1/f'(f^{-1}(x))\\ &= \lim_{x\to\infty} f'(f^{-1}(x))/(f(f^{-1}(x)))1/f'(f^{-1}(x)) = \lim_{x\to\infty} 1/x.\end{aligned}$$

Then by 2.18(3) $\mathcal{R}$ is exponential. $\square$

REFERENCES

1. Wilkie A. J., *Model Completeness Results for Expansions of the Ordered Field of Real Numbers by Restricted Pfaffian Functions and the Exponential Function*, Journal of the American Mathematical Society, Vol. 9 Issue 4, pp. 1051-1094, 1996.

2. Miller C., *Exponentiation is Hard to Avoid*, Proceedings of the American Mathematical Society, Vol. 122 Issue 1, pp. 257-259, 1994.

3. Rosenlicht M., *The Rank of a Hardy Field*, Transactions of the American Mathematical Society, Vol. 280, pp. 659-671, 1983.

4. Miller C., *Polynomially Bounded O-minimal Structures*, Ph.D Thesis, University of Illinois at Urbana-Champaign, 1994.

5. Marker D., *Model Theory: An Introduction*, Springer-Verlag, 2002.

6. Dries L., *Tame Topology and O-minimal Structures*, Cambridge University Press, New York, 1998.

7. Dries L., *Remarks on Tarski's problem concerning,* $(R, +, \cdot, \exp)$, Logic Colloquium 1982 (G. Lolli, G. Longo, and A. Marcja, eds.), pp. 97-121, North-Holland, Amsterdam, 1984.

8. Marcja A., C. Toffalori, *A Guide to Classical and Modern Model Theory*, Kluwer Academic Publishers, Boston, 2003.

9. Foster T., *Power Functions and Exponentials in O-minimal Expansions of Fields*, Ph.D Thesis, University of Oxford, 2010.

10. Dries L. and C. Miller, *Extending Tamm's Theorem*, Annales de L I'nstitute Fourier, Vol. 44 Issue 5, pp. 1367-1395, 1994.

11. Rosenlicht M., *Hardy Fields*, Journal of the Mathematical Analysis and Applications Vol. 93 , pp. 297-311, 1983.