INDICES OF ORBITS IN CONTACT HOMOLOGY

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ABSTRACT

INDICES OF ORBITS IN CONTACT HOMOLOGY

In this thesis, we investigate the basics of the contact homology including the grading of periodic orbits and pseudoholomorphic curves. We also present the alternative approach to the subject by F. Bourgeois [1] and present the computation of the contact homology of Brieskorn manifolds by O. van Koert [2].

ÖZET

TEMAS HOMOLOJİSİNDE YÖRÜNGE İNDİSLERİ

Bu tezde, holomorfik eğriler ve peryodik yörüngelerin derecelendirmesi dahil temas homolojisinin temelleri incelenmiştir. Aynı zamanda, F. Bourgeois [1] tarafından geliştirilen farklı bir yaklaşım incelenmiş ve O. van Koert [2] tarafından yapılmış Brieskorn çokkatlılarının temas homolojilerinin hesabı sunulmuştur.

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LIST OF SYMBOLS

$\mathbb{1}$	The identity matrix of appropriate dimension
$\oplus$	The direct sum of appropriate structures
$\times_S$	The fibered product of two spaces over S
$\otimes$	The tensor product of differential forms
$\equiv_k$	Equivalent modulo k
$(-)^*$	Pullback operator
$(-)_*$	Pushforward operator
$\wedge$	The wedge product of differential forms
$-/-$	The quotient space under a group action
$\lvert - \rvert$	The reduced index of a periodic Reeb orbit
$\lfloor - \rfloor$	The floor function
$\lceil - \rceil$	The ceiling function
$\arg(-)$	The argument of a complex number
$\mathbb{C}$	The field of complex numbers
$c_1(-)$	The first Chern class of a complex vector bundle
$Crit(\mathcal{A})$	Critical points of the action functional $\mathcal{A}$
$\mathrm{d}\alpha$	The exterior derivative of the differential form α
$\gcd(-)$	The greatest common divisor of a set of integers
$H_n(X, G)$	The n^{th} homology group of X with the coefficient group G
$HC_n(M, \xi)$	The n^{th} contact homology group of the contact manifold (M, ξ)
$\operatorname{Hess} f$	The Hessian of the function f
I	The closed interval $[0, 1] \subset \mathbb{R}$
$\Im(-)$	The imaginary part of a complex number
$\mathcal{L}_X$	Lie derivative along the vector field X
$\operatorname{lcm}(-)$	The least common multiple of a set of integers
$m(-)$	The multiplicity of a periodic Reeb orbit or a pseudoholomorphic cylinder

$\mathcal{M}(-;-)$	The moduli space of pseudoholomorphic curves
$\tilde{\mathcal{M}}(-;-)$	The moduli space of unparameterized pseudoholomorphic curves
$\mathcal{P}$	The set of all periodic Reeb orbits of a contact manifold (M, α)
$\mathcal{P}_g$	The set of all good periodic Reeb orbits of a contact manifold (M, α)
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{R}^+$	The positive elements in $\mathbb{R}$
$\mathbb{R}^*$	Non-zero elements in $\mathbb{R}$
sign	Matrix sign function
S^n	The n-dimensional sphere
$SO(n)$	The group of n-by-n special orthonormal matrices
$Sp(2n)$	The group of $2n$-by-$2n$ real symplectic matrices
$Sp(2n)^+$	The set of matrices A in $Sp(2n)$ such that $\det(\mathbb{1} - A) > 0$
$Sp(2n)^-$	The set of matrices A in $Sp(2n)$ such that $\det(\mathbb{1} - A) < 0$
$Sp(2n)^0$	The set of matrices A in $Sp(2n)$ such that $\det(\mathbb{1} - A) = 0$
$Sp(2n)^*$	The set of matrices A in $Sp(2n)$ such that $\det(\mathbb{1} - A) \neq 0$
$\mathrm{span}(-)$	The span of a set of vectors
TM	Tangent bundle of the manifold M
$\mathrm{tr}(-)$	The trace of a matrix
$U(n)$	The unitary group of degree n
$\mathbb{Z}$	The set of integers

∇	The covariant derivative
$\iota(X)\alpha$	Contraction of the differential form α with the vector field X
$\mu(-)$	The Maslov index
$\mu_{CZ}(-)$	The Conley-Zehnder index of a symplectic path or a periodic Reeb orbit
$\sigma(\alpha)$	The set of critical values of the action functional corresponding to α

1. INTRODUCTION

1.1. Contact Geometry

Contact geometry is the study of odd-dimensional manifolds together with a differential structure. It can be seen as an odd-dimensional companion of symplectic geometry. They arise in classical mechanics especially in the study of time-dependent Hamiltonian systems.

Definition 1.1. *A contact manifold* (M^{2n-1}, ξ) *is an odd dimensional manifold where* ξ *is a maximally non-integrable hyperplane distribution, called* the contact structure.

Assuming the contact structure ξ is co-orientable, we can think of the contact manifold as $(M^{2n-1}, \xi = \ker\alpha)$ where α is a global form such that $\alpha \wedge (\mathrm{d}\alpha)^{n-1} > 0$. Here, α is called *the contact form.* Note that the non-integrability condition always provides such a form *locally.* From now on, we assume that ξ is co-orientable. Also, if we multiply the contact form with a sign definite function f, the resulting form $f\alpha$ will be again a contact form.

The most basic example of a contact manifold is $\mathbb{R}^{2n-1} = \{(\mathbf{x}, \mathbf{y}, z) \colon z \in \mathbb{R},\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^{2n-2}\}$ together with the so-called standard contact form $\alpha_{\text{std}} = dz + \mathbf{x}\mathrm{d}\mathbf{y}$. A keen eye would remark the similarity to the standard symplectic structure, simply by calculating $\mathrm{d}\alpha = \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y}$. Similar to the symplectic case, all contact manifolds locally look exactly like standard $\mathbb{R}^{2n-1}$, as the following theorem states.

Theorem 1.2. (Darboux) *Let* $(M^{2n-1}, \xi = \ker\alpha)$ *be a contact manifold. Then for every* $p \in M$ *there exist an open neighborhood* U *of* p *and a diffeomorphism* $\varphi \colon \mathbb{R}^{2n-1} \to U$ *with the local coordinates* $\mathbf{x}, \mathbf{y}, z$ *such that* $\varphi^*\alpha = dz + \mathbf{x}d\mathbf{y}$

1.2. Contact Homology

Contact homology was introduced by Y. Eliashberg and H. Hofer [3] and followed by the Symplectic Field Theory [4]. The main motivation for introducing this theory was that there exist closed manifolds with contact structures which belong to the same formal homotopy class[1] but not contactomorphic[2] . The aim of this theory is to construct a contact-geometric counterpart of the theory of pseudoholomorphic curves in symplectic manifolds à la M. Gromov [5]. We basically count pseudoholomorphic curves in the symplectization of the contact manifold with some further conditions. It is an analogue of the Morse-Floer theories, namely we introduce an action functional and try to construct a homological theory using its critical points which in this case happen to be closed Reeb orbits (see Lemma 2.1). The underlying chain complex of contact homology is a module over these periodic orbits. There is a big difficulty to this theory, namely the symmetric nature of the contact dynamics associated to the contact form should be perturbed in order to get this machinery working. Using the contact homology invariant, I. Ustilovsky showed that there exist infinitely many (non-contactomorphic) contact structures on S^{4m+1} [6]. Alternatively, as presented in [1], one can get rid of this perturbation and calculate the contact homology easier than the perturbed case.

In Chapter 2, we will introduce the contact homology following [7]. In Chapter 3 we will present the non-perturbed version of the theory. Using this last approach, we will calculate the contact homology of Brieskorn manifolds in Chapter 4 following [2].

[1] the formal homotopy class $[\xi]$ of the contact structure is ξ together with an almost complex structure J or equivalently a reduction of the structure group of TM to $U(n-1) \times I$

[2] two contact structures (M_1, ξ_1) and (M_2, ξ_2) are said to be *contactomorphic* if there exist a diffeomorphism $\Phi\colon M_1 \to M_2$ such that $\Phi_* \xi_1 = \xi_2$

2. PRINCIPLES OF CONTACT HOMOLOGY

The contact homology is a tool to differentiate between contact manifolds, which can be viewed as an invariant of the contact structure on a contact manifold [3]. The main ingredients of contact homology are Reeb orbits. We shall introduce the contact homology following [7]. Let $(M, \xi = \ker \alpha)$ be a $(2n-1)$-dimensional closed orientable contact manifold. The *Reeb vector field* R_α associated to the contact form α is determined by the equations

$$\begin{aligned} \iota\left(R_\alpha\right) \mathrm{d}\alpha &= 0 \\ \alpha\left(R_\alpha\right) &= 1 \end{aligned} \tag{2.1}$$

Since $\alpha \wedge \mathrm{d}\alpha^{n-1} \neq 0$, we have for all $x \in M$, $R_\alpha\big|_x \in \ker \mathrm{d}\alpha\big|_x$ where $\ker \mathrm{d}\alpha$ is a line bundle. Thus, with the second property that normalizes R_α, the Reeb vector field is unique. Consider the following action functional

$$\mathcal{A}\colon C^\infty\left(S^1, M\right) \to \mathbb{R} \text{ where } \mathcal{A}\left(\gamma\right) = \int\limits_\gamma \alpha \tag{2.2}$$

The following lemma makes closed Reeb orbits worth examining more in detail.

Lemma 2.1. *$\gamma \in Crit(\mathcal{A})$ if and only if γ is a closed Reeb orbit of period $\mathcal{A}(\gamma)$.*

Proof. Let γ_t, $t \in [0,1]$ be a 1-parameter family of loops in M with $\gamma_0 = \gamma$. We have

$$\begin{aligned} \frac{d}{dt}\mathcal{A}\left(\gamma_t\right)\bigg|_{t=0} &= \frac{d}{dt}\bigg|_{t=0} \int\limits_{S^1} \gamma_t^* \alpha \\ &= \int\limits_\gamma \mathcal{L}_X \alpha \text{ where } X = \frac{d}{dt}\gamma_t\bigg|_{t=0} \\ &= \int\limits_\gamma \iota\left(X\right)\mathrm{d}\alpha \\ &= \int\limits_{S^1} \mathrm{d}\alpha\left(X, \dot{\gamma}\right) dt \end{aligned}$$

Here, we used the definition of the Lie derivative on the second line and the Cartan formula on the third line. Now if γ is a critical point of the functional, then $\iota\left(\dot{\gamma}\right)\mathrm{d}\alpha = 0$, which means that $\dot{\gamma}$ is proportional to R_α. Moreover, if we parametrize γ such that $\dot{\gamma} = R_\alpha$ we have

$$\int_\gamma \alpha = \int_\gamma \underbrace{\alpha\left(R_\alpha\right)}_{=1} = T$$

where T is the period of γ. $\square$

From the point-of-view of Morse theory, we should study the Hessian of $\mathcal{A}$ at the critical points, which corresponds to linearized Reeb flow near a periodic orbit. Moreover, we should impose a non-degeneracy condition on the contact form and provide a perturbation lemma which shows that the non-degeneracy condition is generic.

2.1. Perturbation of The Contact Form

Let γ be a closed orbit of period T and $p \in \gamma$. Let $\varphi_t \colon M \to M$ be the Reeb flow after time t. Since $\mathcal{L}_{R_\alpha}\alpha = \mathrm{d}\iota(R_\alpha)\alpha + \iota(R_\alpha)\mathrm{d}\alpha = 0$, the contact structure ξ is preserved under the Reeb flow. The *linearized return map* $\Psi_\gamma \colon \xi_p \to \xi_p$ is the restriction of the differential of the map φ_T to ξ_p. Note that Ψ_γ is symplectic.

Definition 2.2. *The closed Reeb orbit γ is non-degenerate if the map $\Psi_\gamma \colon \xi_p \to \xi_p$ has no eigenvalue equal to 1.*

We should note here that one can require that this map has no eigenvalue equal to a power of a root of unity [4], since we will be interested in multiple covers of periodic orbits. Also γ is non-degenerate if and only if it is a non-degenerate critical point of $\mathcal{A}$ since the Hessian of the action functional coincides with the linearized return map. Now, we show that non-degeneracy condition is a generic one.

Lemma 2.3. *For any contact structure ξ on M, there exists a contact form α for ξ such that all closed orbits of R_α are non-degenerate.*

Proof. Consider the graph Γ_α of the Reeb flow in $\mathbb{R}^+ \times M \times M$: $(t, x, y) \in \Gamma_\alpha$ if and only if $\varphi_t(x) = y$, $t \geqslant 0$. Fix $T > 0$. The aim is to perturb α to α_T such that the perturbed graph Γ_{α_T} is transverse to $[0,1] \times \Delta_M$, that is the graph of the identity in the compact set $[0,1] \times M \times M$. Let $p \in \Gamma_\alpha \cap ([0,1] \times \Delta_M)$ be a non-transversal intersection point. Define $\alpha_\epsilon = (1 + \epsilon f_p)\,\alpha$ where $\mathrm{d}f_p = 0$, f_p is supported in a small tubular neighborhood of γ and $\epsilon > 0$ is very small -in order to ensure $1 + \epsilon f_p > 0$. Denote the Reeb flow corresponding to α_ϵ by R_{α_ϵ}. Then

$$\iota\left(R_{\alpha_\epsilon}\right) \mathrm{d}\alpha_\epsilon = 0 \text{ and } \alpha_\epsilon\left(R_{\alpha_\epsilon}\right) = 1$$

To ease the analysis, let $X = R_{\alpha_\epsilon} - R_\alpha$. Then a straightforward calculation yields to the following equations for the additional vector field X

$$\begin{aligned} \iota\left(X\right) \mathrm{d}\alpha &= \frac{\epsilon \mathrm{d}f_p}{\left(1+\epsilon f_p\right)^2} - \frac{\epsilon \mathrm{d}f_p\left(R_{\alpha_\epsilon}\right)}{1+\epsilon f_p}\alpha \\ \alpha(X) &= \frac{-\epsilon f_p}{1+\epsilon f_p} \end{aligned}$$

In order to calculate the additional term due to X on the Poincaré return map, one must establish a suitable metric on the tubular neighborhood of γ. So let $g(U,V) = \mathrm{d}\alpha(U, \tilde{J}V) + \alpha(U)\alpha(V)$ where $\tilde{J}(U) = JU$ for $U = U' + \alpha(U)R_\alpha$ and J a compatible almost complex structure. Clearly, g is a metric and the 1-form corresponding to X is

$$\beta = g(X, \cdot) = \frac{\epsilon}{\left(1+\epsilon f_p\right)^2}\,\mathrm{d}f_p \circ \tilde{J} - \frac{\epsilon\,\mathrm{d}f_p\left(R_{\alpha_\epsilon}\right)}{1+\epsilon f_p}\,\alpha \circ \tilde{J} - \frac{\epsilon f_p}{1+\epsilon f_p}\,\alpha$$

The covariant derivative of β is calculated as follows:

$$\begin{aligned} \nabla\beta = &\frac{\epsilon}{\left(1+\epsilon f_p\right)^2} \nabla \mathrm{d}f_p \circ \tilde{J} - \frac{2\epsilon^2}{\left(1+\epsilon f_p\right)^3} \mathrm{d}f_p \otimes \mathrm{d}f_p \circ \tilde{J} \\ &- \epsilon\,\mathrm{d}f_p(X)\left[\frac{1}{1+\epsilon f_p}\,\nabla\alpha \circ \tilde{J} - \frac{\epsilon}{\left(1+\epsilon f_p\right)^2}\,\mathrm{d}f_p \otimes \alpha \circ \tilde{J}\right] \\ &+ \frac{1}{1+\epsilon f_p}\,\nabla\alpha - \frac{\epsilon}{\left(1+\epsilon f_p\right)^2}\,\mathrm{d}f_p \otimes \alpha \end{aligned}$$

Now $\mathrm{d}f_p \equiv 0$ along γ, thus

$$\nabla\beta\Big|_{\mathrm{T}\gamma} = \frac{\epsilon}{(1+\epsilon f_p)^2}\nabla \mathrm{d}f_p \circ \tilde{J} + \frac{1}{1+\epsilon f_p}\nabla\alpha$$

Since the degeneracy of the closed orbit depends on the value of the derivative restricted to ξ, it is enough to check that the covariant derivative of the additional part does not vanish for any vector $U \in \xi$. But for $U \in \xi_p$, $\nabla_U = g(\underbrace{\nabla_U R_\alpha}_{=0}, \cdot) = 0$ implies

$$\nabla_U\beta\Big|_{\mathrm{T}\gamma} = \frac{\epsilon}{(1+\epsilon f_p)^2}\nabla \mathrm{d}f_p \circ \tilde{J} = \frac{\epsilon}{(1+\epsilon f_p)^2} J\,\mathrm{Hess}\,(f_p)$$

where $\mathrm{Hess}\,(f_p)$ is the Hessian of f_p. Hence, with a suitable choice of f_p, the flow of R_{α_ϵ} along γ can be made non-degenerate. Moreover, choosing ϵ small enough, we can guarantee that no periodic orbit of period less than or equal to T will be created since as in Lemma 2.1 the period depends on the integral of the contact form α_ϵ. If we do the same for all degenerate points, we obtain a collection of functions, but since $[0,1]\times\Delta_M$ is compact, we can extract finite number of those points $\{p_1 \ldots, p_N\}$ together with their neighborhoods $\{U_{p_1} \ldots, U_{p_N}\}$ and functions $\{f_{p_1} \ldots, f_{p_N}\}$. Let K be the vector space generated by these functions and consider

$$\Gamma_{\alpha,K} := \Big\{(t,x,y) \in \mathbb{R}^+ \times M \times M \colon \varphi_t^f(x) = y,\ f \in K, \|f\| \text{ small}\Big\}$$

where φ_t^f is the flow of the Reeb vector field of the contact form $(1+f)\,\alpha$. By construction, $\Gamma_{\alpha,K}$ is transverse to $[0,1]\times\Delta_M$. Using the projection map $\pi\colon \Gamma_{\alpha,K} \cap ([0,1]\times\Delta_M) \to K$ and Sard's theorem for π, we can pick a regular value $f_T \in K$. Hence, the closed Reeb orbits of period $\leqslant T$ of the contact form $\alpha_T = (1+f_T)\,\alpha$ are all non-degenerate.

Let $\mathcal{C}_T$ denote the set of contact forms corresponding to the contact structure ξ such that all closed Reeb orbits are non-degenerate. We showed here that $\mathcal{C}_T$ is open and dense in the set of all contact forms of ξ, for any $T \in \mathbb{R}^+$. By Baire's theorem,

$\bigcap_{T\in\mathbb{R}^+} \mathcal{C}_T$ is nonempty. $\square$

In order to define contact homology, we choose a generic contact form α such that all closed Reeb orbits of α are non-degenerate. To continue with the homology, we need to define a grading on closed Reeb orbits in the first place, and for that, we should introduce Conley-Zehnder index of a closed Reeb orbit.

2.2. Conley-Zehnder Index and Chains

2.2.1. Conley-Zehnder Index

The Conley-Zehnder index first appeared on the analysis of periodic orbits of Hamiltonian systems [8,9]. It is a Morse type index for periodic orbits and the definition can also be done through paths in Lagrangian Grassmanian [10]. Since both approaches have their own advantages, we shall use both of them whenever they may be useful.

Let γ be a non-degenerate closed Reeb orbit of period T. We should pass to a symplectic set-up in order to be able to talk about an index theory. Notice that the contact structure $\xi = \ker\alpha$ is a symplectic bundle since $\mathrm{d}\alpha$ is a symplectic structure on ξ_p for every $p \in M$. So using the linearized Reeb flow $\mathrm{d}\varphi_t \colon \xi_p \to \xi_{\varphi_t(t)}$ we obtain a one-parameter family of symplectic transformations.

Definition 2.4. *Let (E, M, π, ω) be a symplectic bundle with total space E, base M, the projection $\pi\colon E \to M$ and ω the symplectic structure on fibers $\pi^{-1}(x)$ for $x \in M$. A* symplectic trivialization *of the symplectic bundle E is a smooth map $M \times \mathbb{R}^{2n} \to E\colon\ (p,\zeta) \mapsto \Phi(p)\zeta$ which pulls back the symplectic form ω to the standard one on $\mathbb{R}^{2n}$, that is, $\Phi^*\omega = \omega_0$.*

We can talk about a symplectic trivialization along γ; that is a symplectic trivialization of the pull-back bundle of $\gamma^*\xi$. Fixing a symplectic trivialization along γ, we obtain a path of symplectic matrices $\Psi_\gamma(t)$, $t \in [0,T]$. Clearly, $\Psi_\gamma(0) = \mathbb{1}$ and -due to non-degeneracy- $\det(\Psi_\gamma(T) - I) \neq 0$. A number $t \in [0,T]$ is called a *crossing* if $\det(\Psi_\gamma(t) - \mathbb{1}) = 0$. Denoting $\ker(\det(\Psi_\gamma(t) - \mathbb{1}))$ by E_t, we define the *crossing form*

$\Gamma(\Psi_\gamma, t)$ at a crossing t as follows

$$\Gamma(\Psi_\gamma, t)\, v = \omega_0 \left(v, \dot{\Psi}_\gamma v\right) \text{ for every } v \in E_t.$$

A crossing t is said to be *regular* if the crossing form $\Gamma(\Psi_\gamma, t)$ is non-degenerate. For the path Ψ_γ having regular crossings, we define the *Conley-Zehnder index* as follows

$$\mu_{CZ}(\gamma) := \frac{1}{2} \operatorname{sign} \Gamma(\Psi_\gamma, 0) + \sum_{\substack{t \neq 0 \\ t \text{ crossing}}} \operatorname{sign} \Gamma(\Psi_\gamma, t)$$

Using the following lemma, this index is well defined for any path of symplectic matrices

Lemma 2.5. ([7, Lemma 2.2], [11]) *Every path of symplectic matrices Ψ is homotopic with fixed end points to another path $\tilde{\Psi}$ having regular crossings. Moreover, $\mu_{CZ}(\Psi) = \mu_{CZ}(\tilde{\Psi})$.*

Definition 2.6. *The* Conley-Zehnder index *of a path of symplectic matrices $\Psi(t)$, $t \in [0, T]$ with $\Psi(0) = \mathbb{1}$ and $\Psi(T)$ non-degenerate, is defined as*

$$\mu_{CZ}(\Psi) := \frac{1}{2} \operatorname{sign} \Gamma(\tilde{\Psi}, 0) + \sum_{\substack{t \neq 0 \\ t \ crossing}} \operatorname{sign} \Gamma(\tilde{\Psi}, t)$$

through $\tilde{\Psi}$ homotopic with fixed endpoints to Ψ having regular crossings.

We must state here the main properties of this index, which is useful for calculations. Following [11], we collect these properties in the following theorem

Theorem 2.7. *Let $\mu_{CZ}(\Psi)$ be the Conley-Zehnder index of a path of symplectic matrices $\Psi\colon [0,1] \to Sp(2n)$. Then we have*

- **(Naturality)** *For any path $\Phi\colon [0,1] \to Sp(2n)$, $\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi)$*
- **(Homotopy)** *The Conley-Zehnder index is constant for fixed end homotopic paths*
- **(Zero)** *If $\Psi(t)$ has no eigenvalue on the unit circle for $t \in (0,1]$ then $\mu_{CZ}(\Psi) = 0$*
- **(Product)** *For $n + m = k$, $\Phi \in Sp(2m)$, identifying $Sp(2n) \oplus Sp(2m)$ with a subgroup of $Sp(2k)$, we have $\mu_{CZ}(\Psi \oplus \Phi) = \mu_{CZ}(\Psi) + \mu_{CZ}(\Phi)$.*

- **(Loop)** *For any loop* $\Phi\colon [0,1] \to Sp(2n)$ *such that* $\Phi(0) = \Phi(1) = I$*, we have*

$$\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\Phi)$$

  where $\mu(\Phi)$ *is the Maslov index of the path as in the sequel of [10].*

- **(Signature)** *If* $S = S^T \in \mathbb{R}^{2n\times 2n}$ *is a symmetric matrix with* $\|S\| < 2\pi$ *and* $\Psi(t) = e^{JSt}$*, then*

$$\mu_{CZ}(\psi) = \frac{1}{2}\,\mathrm{sign}(S)$$

  where sign(S) *is the number of positive minus the number of negative eigenvalues.*

- **(Determinant)** $(-1)^{n-\mu_{CZ}(\Psi)} = \mathrm{sign}\det(\mathbb{1} - \Psi(1))$.
- **(Inverse)** $\mu_{CZ}(\Psi^{-1}) = \mu_{CZ}(\Psi^T) = -\mu_{CZ}(\Psi)$.

Definition 2.8. *The* reduced index *of a periodic orbit* γ *is denoted by* $|\gamma| = \mu_{CZ}(\Psi_\gamma) + n - 3$

2.2.2. Iteration of The Index on Multiple Covers

Before going further into the algebraic construction of the chains, we must also clarify how the index iterates on multiple covers of Reeb orbits. The construction of contact homology will be based upon a special subset of the orbits and their multiple covers. For instance, let $\mathcal{P}$ be the set of all multiple covers of Reeb orbits. The first logical step is now to extend the definition of the non-degeneracy to this set, namely

Definition 2.9. *A Reeb orbit* γ *is* non-degenerate *if all its multiple covers are non-degenerate in the sense of Definition 2.2.*

As in Lemma 2.3, this condition on the contact form is generic and one can assume that all elements in $\mathcal{P}$ are non-degenerate. It is now interesting to see how the index of a multiple cover of a Reeb orbit is related with the index of the original one. Let $m\gamma$ denote the m-cover of $\gamma \in \mathcal{P}$ and $n(\gamma)$ denote the number of real negative eigenvalues -counted with algebraic multiplicities- of the Poincaré return map of the Reeb flow along γ. Since this return map is symplectic, the real eigenvalues come in pairs $\lambda, 1/\lambda$ and $n(\gamma)$ is always even. Let

$$\mathcal{P}_o = \{\gamma \in \mathcal{P} \colon n(\gamma)/2 \text{ is odd}\}\,, \quad \mathcal{P}_b = \{k\gamma \colon \gamma \in \mathcal{P}_o,\, k \text{ is even}\}\,, \quad \mathcal{P}_g = \mathcal{P} - \mathcal{P}_b$$

The next lemma characterizes the behaviour of the parity of the index of a multiple cover.

Lemma 2.10. ([6, Lemma 3.2.4]) *For $\gamma \in \mathcal{P}$, $k \in \mathbb{N}$, $\mu_{CZ}(\gamma) - \mu_{CZ}(k\gamma)$ is odd if and only if $\gamma \in P_o$ and k is even.*

Proof. Let $\Phi \in Sp(2n-2)$ be the Poincaré return map of the Reeb flow along γ. Since this path is non-degenerate, by the determinant rule in Theorem 2.7, we have

$$(-1)^{\mu(\gamma)+n-1} = \operatorname{sign}\det(\mathbb{1} - \Phi)$$

Following [9, Lemma 1.5], the eigenvalues of a generic symplectic matrix comes in groups which are

$$(e^{i\theta}, e^{-i\theta}), \quad (\rho, \rho^{-1}), \quad (e^{i\theta}\rho, e^{i\theta}\rho^{-1}, e^{-i\theta}\rho, e^{-i\theta}\rho^{-1})$$

This analysis is made possible by a choice of generic perturbation of the path, which does not change the index. Denote the number of pairs (ρ, ρ^{-1}) for ρ positive by $p(\Phi)$, and the negative ones by $n(\Phi)$. Notice that by definition $n(\Phi) = n(\gamma)/2$. By these eigenvalues, we can construct the block decomposition of Φ consisting of the following blocks

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix} \quad \begin{bmatrix} \rho R(\theta) & 0 \\ 0 & \rho^{-1}R(\theta) \end{bmatrix}$$

First two blocks are 2-by-2 and the last one is 4-by-4. Since $\operatorname{sign}\det(\mathbb{1} - R(\theta)) = 1$, we have $\operatorname{sign}\det(\mathbb{1} - \Phi) = (-1)^{p(\Phi)}$. Moreover, for every eigenvalue λ of Φ, λ^k is an eigenvalue of Φ^k and therefore

$$p(\Phi^k) \equiv_2 p(\Phi) + (k-1)n(\Phi)$$

So we have $\mu(\gamma) - \mu(k\gamma) \equiv_2 p(\Phi) - p(\Phi^k) \equiv_2 (k-1)n(\Phi)$. Therefore, right-hand-side is 1 if and only if k is even $n(\Phi)$ is odd -that is $\gamma \in P_o$. $\square$

The key observation afterward is that if $\mu_{CZ}(\gamma) - \mu_{CZ}(k\gamma) \equiv_2 (k-1)n(\gamma)$ then $\mu_{CZ}(k\gamma) - \mu_{CZ}(l\gamma) \equiv_2 (k-l)n(\gamma)$. It is now obvious that the parity for even multiples is constant and the parity for odd multiples is also constant. The orbits in $\mathcal{P}_g$ are called *good* and orbits in $\mathcal{P}_b$ *bad*, since all multiples of good orbits have the same parity whereas the parity for even multiples of bad orbits differ from their odd multiples. The motivation for this definition will be clear after introducing the moduli spaces and trying to compute their dimensions in Section 2.3.

2.2.3. Algebraic Construction of Chains

Although Definition 2.6 looks robust for a path of symplectic matrices, one must examine carefully the procedure of the definition to reveal the obstructions in the contact case. We started by choosing a symplectic trivialization to find the path of symplectic matrices. For the moment, let us assume that the Reeb orbit γ is homologically trivial so that we can choose a spanning surface S_γ. Given $A \in H_2(M, \mathbb{Z})$, we have the identity

$$\mu_{CZ}(\gamma;\, S_\gamma \# A) = \mu_{CZ}(\gamma;\, S_\gamma) + 2\langle c_1(\xi), A\rangle$$

where # denotes the connected sum and $c_1(\xi) \in H^2(M, \mathbb{Z})$ is the first Chern class of the contact structure.

The chain complex is constructed via a free module generated by good orbits using a coefficient ring. To remove the obstruction caused by the trivialization, one can either use $\mathbb{Z}_2$ as coefficient ring or alternatively introduce a grading over $H_2(M, \mathbb{Z})$ to remove it. Namely, for $A \in H_2(M, \mathbb{Z})$ let

$$|A| = -2\langle c_1(\xi), A\rangle$$

We quotient out the zero graded sub-module $\mathcal{R}$, and obtain a well-defined grading on $H_2(M, \mathbb{Z})/\mathcal{R}$. So as another candidate for coefficient ring is $\mathbb{Q}\left[H_2(M, \mathbb{Z})/\mathcal{R}\right]$ with the grading above.

We will be mainly using the coefficient ring $\mathbb{Z}_2$ since the calculations are far easier than any other ring. For a more general approach on the algebra, we refer the reader to [3].

2.3. Pseudoholomorphic Curves

This section is dedicated to understand the pseudoholomorphic curves in symplectizations of contact manifolds. These curves introduced by M. Gromov [5] are widely used in the area of symplectic geometry and their moduli spaces are thoroughly examined in [12]. The symplectizations were left apart since the behaviour of near the ends were not known. After H. Hofer [13] used these techniques to prove the Weinstein conjecture on S^3, pseudoholomorphic curves in symplectizations became an active topic of study, especially for 3-manifolds. We start by introducing the natural symplectic extension of contact manifolds.

Definition 2.11. *The symplectization of the contact manifold* (M, α) *is the symplectic manifold* $(\mathbb{R} \times M, d(e^t\alpha))$ *where* t *denotes the coordinate in* $\mathbb{R}$*.*

Consider the set of almost complex structures on the contact structure ξ, that is

$$\mathcal{J} = \left\{J \in \mathrm{End}(\xi)\colon J^2 = -\mathbb{1},\, \mathrm{d}\alpha(J\cdot, J\cdot) = \alpha(\cdot, \cdot),\, \mathrm{d}\alpha(\cdot, J\cdot) > 0\right\}$$

This set is contractible as a subset of endomorphisms of ξ [14, Proposition 2.50]. Given $J \in \mathcal{J}$, we can extend it to an almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ compatible with

$\omega = \mathrm{d}(e^t\alpha)$ as follows:

- $\tilde{J}(\partial/\partial t) = R_\alpha$
- $\tilde{J}\big|_\xi = J$

Abusing the notation we shall use J as the extended complex structure on $\mathbb{R}\times M$. As in the compact symplectic manifold case, we want to examine the pseudoholomorphic (or J-holomorphic) curves, that is the maps $F : (\Sigma, j) \to (\mathbb{R}\times M, J)$ where Σ is a Riemann surface with an almost complex structure j and $\mathrm{d}F \circ j = J \circ \mathrm{d}F$. First we observe that the last condition for being pseudoholomorphic is equivalent to the Cauchy-Riemann equation $\bar{\partial}F = \frac{1}{2}(\mathrm{d}F + J \circ \mathrm{d}F \circ j) = 0$. Moreover, we require that these curves converge asymptotically to Reeb orbits in positive and negative ends. To understand this picture, let us restrict ourselves to the punctured sphere $\Sigma = S^2 - \{x, y_1, \ldots, y_s\}$ and impose the following conditions on $F = (F_\mathbb{R}, F_M) \in \mathbb{R}\times M$ with (ρ_u, θ_u) designating a polar chart around a puncture $u \in \{x, y_1, \ldots, y_s\}$:

- $\lim\limits_{\rho_x\to 0} F_\mathbb{R}(\rho_x, \theta_x) = +\infty$
- $\lim\limits_{\rho_{y_k}\to 0} F_\mathbb{R}(\rho_{y_k}, \theta_{y_k}) = -\infty$ where $k = 1, \ldots, s$
- $\lim\limits_{\rho_x\to 0} F_M(\rho_x, \theta_x) = \gamma\left(-\frac{T}{2\pi}\theta_x\right)$ where $\gamma\colon [0, T] \to M$ is a periodic Reeb orbit.
- $\lim\limits_{\rho_{y_k}\to 0} F_M(\rho_{y_k}, \theta_{y_k}) = \gamma_k\left(\frac{T_k}{2\pi}\theta_{y_k}\right)$ where $\gamma_k\colon [0, T_k] \to M$ are periodic Reeb orbits for $k = 1, \ldots, s$.

Let $\mathcal{Hol}\,(\gamma; \gamma_1, \ldots, \gamma_s)$ denote the set of J-holomorphic curves converging to γ on the positive end and to $\gamma_1, \ldots, \gamma_s$ on the negative end in the sense above. Then, it is natural to truncate this huge set via an equivalence relation, namely let $F_1\colon (S^2 - \{x_1, y_1^1, \ldots, y_s^1\}, j_1) \to \mathbb{R}\times M$ be equivalent to $F_2\colon (S^2 - \{x_2, y_1^2, \ldots, y_s^2\}, j_2) \to \mathbb{R}\times M$ when there exists a bi-holomorphic map $h : (S^2, j_1) \to (S^2, j_2)$ so that $h(x_1) = x_2$, $h(y_k^1) = y_k^2$ for $k = 1, \ldots, s$ and $F_1 = F_2 \circ h$.

Definition 2.12. *The moduli space of J-holomorphic curves $\mathcal{M}\,(\gamma; \gamma_1, \ldots, \gamma_s)$ is the set of equivalence classes in $\mathcal{Hol}\,(\gamma; \gamma_1, \ldots, \gamma_s)$*

Notice that $\mathbb{R}$ acts on $\mathcal{M}\,(\gamma; \gamma_1, \ldots, \gamma_s)$ via translations $t \mapsto t + \Delta t$ in $\mathbb{R}\times M$.

To have a nice geometric structure on this space we have to perturb the Cauchy-Riemann equation as $\bar{\partial} F = \dfrac{1}{2}\left(\mathrm{d}F + J \circ \mathrm{d}F \circ j\right) = \nu(F)$ where $\nu(F)$ is invariant under bi-holomorphisms. After completing this space with such perturbations by the methods which exceeds the scope of this work, we obtain the following structure

Theorem 2.13. ([7, Proposition 1]) $\mathcal{M}\left(\gamma; \gamma_1, \ldots, \gamma_s\right)/\mathbb{R}$ *is a compact branched labeled manifold with corners, i.e. the union of manifolds with corners along a co-dimension 1 branching locus, with each manifold having a rational weight, so that near each branching point, the sum of all entering weights equals the sum of all exiting weights. Each manifold with corner has dimension*

$$(n-3)(1-s) + \mu_{CZ}(\gamma) - \sum_{k=1}^{s} \mu_{CZ}(\gamma_s) + 2c_1^{rel}(\xi, \Sigma) - 1$$

where $c_1^{rel}(\xi, \Sigma)$ *is the relative first Chern class of* ξ *on* Σ*, relative to the fixed trivializations of* ξ *along the closed Reeb orbits at the punctures.*

This statement is very delicate and hard to grasp, especially to picture the phenomena while compactifying this space. To ease this process we will consider the cylindrical case only, i.e. $\Sigma = S^2 - \{x^+, x^-\} \cong \mathbb{R} \times S^1$ together with two periodic Reeb orbits γ^+, γ^- so that the J-holomorphic curve converges to γ^+ and γ^- on the positive and negative end respectively. Here we start exactly following [6].

Theorem 2.14. ([6, Theorem 3.2.2]) *For any generic contact form in the sense of Lemma 2.3, there exists a generic almost complex structure* $J \in \mathcal{J}$ *such that*

$$\mathcal{M}\left(\gamma^+; \gamma^-\right)/\mathbb{R}$$

is a smooth manifold of dimension $|\gamma^+| - |\gamma^-| + 2$.

Notice that the group $G = \mathbb{R} \times S^1$ of conformal transformations of the cylinder $\mathbb{R} \times S^1$ acts by composition to this moduli space. Thus by taking the quotient by both the action of G and $\mathbb{R}$, we obtain $\tilde{\mathcal{M}}\left(\gamma^+; \gamma^-\right)$ which is an orbifold branched over multiply covered cylinders of dimension $|\gamma^+| - |\gamma^-| - 1$. First, notice that if $|\gamma^+| - |\gamma^-| =$

1, this space is nothing but a discrete set of points. When $|\gamma^+| - |\gamma^-| = 2$, a problem emerges while taking multiply covered cylinders: if for some $k \in \mathbb{N}$ $|k\gamma^+| - |k\gamma^-| = 1$, then this space fails to be a 1-manifold. This is in fact where we need to introduce good orbits.

Corollary 2.15. *If γ^+ and γ^- are good orbits then for any $k \in \mathbb{N}$, $|k\gamma^+| - |k\gamma^-| \equiv_2 0$ and thus $\tilde{\mathcal{M}}(\gamma^+; \gamma^-)$ is a 1-dimensional smooth manifold.*

Classical Morse-Floer homology theories assert the compactness behaviour of these spaces by the following theorem

Theorem 2.16. ([6, Theorem 3.3.1]) *Let (α, J) be generic in the sense of Lemma 2.3 and Theorem 2.14. Suppose there exists $k \in \mathbb{N}$ such that there are no $\gamma \in \mathcal{P}$ with $1 < |\gamma| < k$. Let $\gamma^+, \gamma^- \in \mathcal{P}$ such that $|\gamma^+| - |\gamma^-| = k$ and $F_n = (F_{\mathbb{R}n}, F_{Mn})$ be a sequence in $\mathcal{M}(\gamma^+; \gamma^-)/G$. Then there exist a subsequence -again denoted by F_n- together with a collection of orbits $\gamma_0 = \gamma^+, \ldots, \gamma_m = \gamma^- \in \mathcal{P}$ connecting cylinders $u^i \in \mathcal{M}(\gamma_{i-1}; \gamma_i)/G$ for $i = 1, \ldots, m$ and sequences c_n^i, $s_n^i \in \mathbb{R}$, $t_n^i \in S^1$ such that as $n \to \infty$, the map $(F_{\mathbb{R}n} + c_n^i, F_{Mn}) \circ (s + s_n^i, t + t_n^i)$ converges to $u^i(s,t)$uniformly with all derivatives on compact subsets of $\mathbb{R} \times S^1$.*

Alternatively, we say that a sequence of J-holomorphic curves converges to a *broken cylinder* $u^1 \# \cdots \# u^m$. A crucial point of this theorem is to justify this gluing operation of two holomorphic curves sharing an end. This process is similar in the Floer homology, so we now follow [15].

As we pointed out in the beginning of this section, it is possible to see J-holomorphic curves as the solutions of a partial differential equation -namely the Cauchy-Riemann equation, or equivalently as the zero sets of some functions between suitable Banach spaces. The equation to solve can be perceived as a partial differential operator

$$\bar{\partial}_J(u) = \frac{\partial u}{\partial t} + J(u)\frac{\partial u}{\partial s}$$

where $(t, s) \in \mathbb{R} \times S^1$ and $u \colon \mathbb{R} \times S^1 \to \mathbb{R} \times M$. Therefore $\bar{\partial}_J(u)$ is a vector field along u and -fixing a curve $u \in \mathcal{M}(\gamma^+; \gamma^-)$ it is natural to investigate the set

$\mathcal{X}_u \subset C^\infty(\mathbb{R} \times S^1, u^*TM)$ consisting of all vectors η along u which satisfy a suitable exponential decay condition as $t \to \pm\infty$ (see [15, Proposition 1.21] how to derive this condition). We first observe that another function in $\mathcal{M}(\gamma^+; \gamma^-)$ can be expressed uniquely as the flow of a vector field $\eta \in \mathcal{X}_u$ in the form $u' = \exp_u(\eta)$. So we can express the set of solutions as the zero set of a function $\mathcal{F}_u \colon \mathcal{X}_u \to \mathcal{X}_u$. Explicitly, it is given by

$$\mathcal{F}_u(\eta) = \Phi_u^{-1}(\eta)\bar{\partial}_J(\exp_u(\eta))$$

for $\eta \in \mathcal{X}_u$, where $\Phi_u(\eta) \colon T_uM \to T_{\exp_u(\eta)}M$ denotes the parallel transport along the geodesic $\tau \mapsto \exp_u(\tau\eta)$. The differential of $\mathcal{F}_u$ at 0 is a linear differential operator, denoted by $D_u = \mathrm{d}\mathcal{F}_u(0)$ and given by

$$D_u\eta = \nabla_t\eta + J(u)\nabla_s\eta + \nabla_\eta J(U)\partial_s u$$

If we complete $\mathcal{X}_u$ appropriately, $D_u \colon W^{1,p} \to L^p$ becomes a Fredholm operator ([15, Theorem 2.2]) where $L^p = L^p(\mathbb{R}\times S^1, u^*TM)$ and $W^{1,p} = W^{1,p}(\mathbb{R}\times S^1, u^*TM)$ together with respective Sobolev norms

$$\|\eta\|_{L^p} = \left(\int_{-\infty}^{\infty}\int_0^1 |\eta|^p\right)^{1/p} \text{ and } \|\eta\|_{W^{1,p}} = \left(\int_{-\infty}^{\infty}\int_0^1 |\eta|^p + |\nabla_t\eta|^p + |\nabla_s\eta|^p\right)^{1/p}$$

To calculate the Fredholm index, we trivialize the pullback bundle, namely choose a unitary trivialization $u^*TM \to \mathbb{E} \times S^1$. Then we obtain a family of vector space isomorphisms $\Phi(t,s) \colon \mathbb{R}^{2n} \to T_{u(s,t)}M$ which pullback the symplectic structure ω and the complex structure J to the standard ones of $\mathbb{R}^{2n} = \{(\mathbf{q},\mathbf{p}) \colon \mathbf{q},\mathbf{p} \in \mathbb{R}^n\}$, namely $\omega_0 = \mathrm{d}\mathbf{q} \wedge \mathrm{d}\mathbf{p}$ and J_0 respectively. Via these isomorphisms, the linear operator can be expressed as

$$D\eta = \partial_s\eta + J_0\partial_t\eta + S\eta$$

for $\eta \colon \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$. Here the matrices $S(t,s) \in \mathbb{R}^{2n\times 2n}$ are defined by $S = \Phi^{-1}D_u\Phi$.

Since the trivializations are unitary, the limit matrices

$$S^{\pm}(s) = \lim_{t\to\pm\infty} S(t,s) = \Phi^{-1} J \nabla_s \Phi$$

are symmetric and hence up to a compact perturbation of D we can assume S is symmetric for all t and s. We construct a path of symplectic matrices using these symmetric matrices, simply by considering the solution of the differential equation $J_0 \partial_s \Psi + S\Psi = 0$ with the initial condition $\Psi(t,0) = \mathbb{1}$ where $S\colon \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n\times 2n}$ is a symmetric matrix valued function and $\Psi\colon \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n\times 2n}$ is a symplectic matrix valued function. Let $\Psi^{\pm}(s) = \lim_{t\to\pm\infty} \Psi(t,s)$, then the Fredholm index of D is given by $\operatorname{index} D = \mu_{CZ}(\Psi^{+}) - \mu_{CZ}(\Psi^{-})$ [11, Theorem 4.1].

To glue two J-holomorphic cylinder, we use Floer's argument as presented in [15]. Let $v \in \mathcal{M}(\gamma^3;\gamma^2)$ and $u \in \mathcal{M}(\gamma^2;\gamma^1)$ with $|\gamma^3|-1 = |\gamma^2| = |\gamma^1|+1$ and surjective Fredholm operators D_v and D_u respectively. We shall construct an approximate solution $\tilde{w}_R = v \#_R u$ and then conclude by using implicit function theorem ([15, Proposition 3.9]) that there is a solution $w_R \in \mathcal{M}(\gamma^1;\gamma^3)$ near $\tilde{w}_R$. The glued version is given explicitly as follows

$$v \#_R u = \begin{cases} v(t+R,s), & t \leqslant -R/2-1 \\ \exp_{y(s)}(\beta(-t-R/2)\eta_2(t+R,s)), & -R/2-1 \leqslant t \leqslant -R/2 \\ y(s), & -R/2 \leqslant t \leqslant R/2 \\ \exp_{y(s)}(\beta(t-R/2)\eta_1(t-R,s)), & R/2 \leqslant t \leqslant R/2+1 \\ v(t+R,s), & R/2+1 \leqslant t \end{cases}$$

where $\eta_1(t,s), \eta_2(t,s) \in T_{y(s)}M$ are chosen such that $u(t,s) = \exp_{y(s)}(\eta_1(t,s))$ for all s and large negative t and $v(t,s) = \exp_{y(s)}(\eta_2(t,s))$ for all s and large positive t. Here $\beta\colon \mathbb{R} \to [0,1]$ is a cutoff function equal to 1 for $t \geqslant 1$ and equal to 0 for $s \leqslant 0$.

If the original moduli spaces $\tilde{\mathcal{M}}(\gamma^3;\gamma^2)$ and $\tilde{\mathcal{M}}(\gamma^2;\gamma^1)$ are zero dimensional then the glued cylinders describe precisely the ends of the one dimensional moduli space $\tilde{\mathcal{M}}(\gamma^3;\gamma^1)$. In other words, the complement of these cylinders is a compact

1-manifold. Following [6], we can represent the moduli space $\tilde{\mathcal{M}}(\gamma^3;\gamma^1)$ as a graph with labeled edges and with weights. To construct this graph, we must observe some basic properties of non-equivalent (through reparametrization) cylinders. For that, let us assume that the middle orbit γ^2 be of multiplicity $m(y)$ and let $m(\cdot)$ denote the multiplicity of both the orbits and the cylinders. Then for $C_1 \in \tilde{\mathcal{M}}(\gamma^3;\gamma^2)$ and $C_2 \in \tilde{\mathcal{M}}(\gamma^2;\gamma^1)$, we can reparametrize them by $2\pi/m(y)$. By gluing these we get $c(y) := m(y)/\operatorname{lcm}(m(C_1), m(C_2))$ non-equivalent approximate cylinders $C_1 \#_R C_2$, which results $c(y)$ broken cylinders. So the compactification of $\mathcal{M}(\gamma^3;\gamma^1)$ can be seen as a graph Γ with vertices corresponding to broken cylinders and edges corresponding to connected components of $\tilde{\mathcal{M}}(\gamma^3;\gamma^1)$. Each vertex belongs to $c(y)$ edges, each one labeled by $\gcd(m(C_1), m(C_2))$ since this is the multiplicity of these cylinders. All edges in a connected component of Γ have the same label so, let Γ_k be the connected component of the graph with labels equal to k.

2.4. The Boundary Operator

In cylindrical contact homology setup, we consider the free module of good periodic Reeb orbits $\mathcal{P}_g$ with $\mathbb{Z}_2$ as the coefficient ring. Via the reduced Conley-Zehnder index, we obtain a graded module of chains

$$C = \bigoplus_{k\in\mathbb{Z}} C_k$$

where C_k contains good periodic orbits of reduced Conley-Zehnder index $|\cdot| = k$. By Corollary 2.15, for $x, y \in \mathcal{P}_g$ with $|x| - |y| = 1$, the reduced moduli space $\tilde{\mathcal{M}}\,(x;\, y)$ is finite. Set

$$\langle \partial x, y\rangle = \sum_{C\in\tilde{\mathcal{M}}(x;\,y)} \frac{m(y)}{m(C)} \mod 2$$

where $m(C)$ and $m(y)$ denote the multiplicities. Clearly $m(C)|m(y)$ and this expression is well-defined. Now, define the boundary map $\partial = \oplus_{k\in\mathbb{Z}}\partial_k$ where $\partial_k \colon C_k \to C_{k-1}$ is given by

$$\partial_k x = \sum_{\substack{y \in \mathcal{P}_g \\ |y| = k-1}} \langle \partial x, y \rangle y$$

Theorem 2.17. *Assume that $\mathcal{P}$ contains no orbit of reduced index 1. Then $\partial^2 = 0$ and thus (C, ∂) forms a chain complex*

Proof. We follow exactly the same proof in [6]. We need to show that for any $x, z \in \mathcal{P}_g$ with $|x| = |z| + 2$

$$\langle \partial^2 x, z \rangle = \sum_{\substack{y \in \mathcal{P}_g \\ |y| = |x| - 1}} \sum_{\substack{C_1 \in \tilde{\mathcal{M}}(x,y) \\ C_2 \in \tilde{\mathcal{M}}(y,z)}} \frac{m(y)m(z)}{m(C_1)m(C_2)} \equiv_2 0$$

The first equality follows from the definition of the ∂ map. By the observation at the end of Chapter 2.3 we consider the graph of the compactification Γ as the union of its connected components Γ_k. On each connected component Γ_k, by Handshaking Lemma, we have

$$\sum_{\substack{C_1, C_2 \\ \gcd(m(C_1), m(C_2)) = k}} \frac{m(y)}{\operatorname{lcm}(m(C_1), m(C_2))} \equiv_2 0$$

Since k divides $m(z)$, we have

$$\sum_{\substack{C_1, C_2 \\ \gcd(m(C_1), m(C_2)) = k}} \frac{m(y)m(z)}{\operatorname{lcm}(m(C_1), m(C_2))k} \equiv_2 0$$

Summing over k we get

$$\sum_{\substack{y \in \mathcal{P} \\ |y| = |x| - 1}} \sum_{\substack{C_1 \in \tilde{\mathcal{M}}(x,y) \\ C_2 \in \tilde{\mathcal{M}}(y,z)}} \frac{m(y)m(z)}{m(C_1)m(C_2)} \equiv_2 0$$

Notice that the orbits y run through all orbits. So we should show that bad orbits contribute an even term to the sum. So suppose that y is an even multiple of an orbit

$\gamma \in P_o$, say $y = 2^l g\gamma$ where g is odd and $l > 0$. Since $m(C_1)|m(y)$ and $m(C_2)|m(z)$, it is enough to show that 2^l does not divide $m(C_1)$. Assume for a contradiction 2^l divide $m(C_1)$. Then $C_1 = 2^r C$ and -since 2^l divides $m(x)$ and $m(y)$- we have $x = 2^l x'$ and $y = 2^l y'$ where $y' = s\gamma$. Since x is a good orbit, y' is an odd multiple of γ and y is a bad orbit, we have by Lemma 2.10 $|x| - |x'| \equiv_2 0$, $|y'| - |\gamma| \equiv_2 0$ and $|y| - |\gamma| \equiv_2 1$. Therefore

$$|x'| - |y'| = \underbrace{|x'| - |x|}_{\equiv_2 0} + \underbrace{|x| - |y|}_{=1} + \underbrace{|y| - |\gamma|}_{\equiv_2 0} + \underbrace{|\gamma| - |y'|}_{\equiv_2 1} \equiv_2 0$$

This contradicts with the fact that the moduli space $\tilde{\mathcal{M}}(x', y')$ can be embedded in $\tilde{\mathcal{M}}(x, y)$ and -thus by the dimension formula- $|x'| - |y'| = 1$. $\square$

The homology of the complex (C, ∂)

$$HC_*(M, \xi; \alpha, J) = \frac{\ker \partial}{\operatorname{Im} \partial}$$

is called the contact homology of the pair (α, J). Since by definition this homology depends upon the choice of this pair, we shall prove an equivalence result between different pairs.

Theorem 2.18. *Let (α_1, J_1) and (α_2, J_2) be generic in the sense of Lemma 2.3 and Theorem 2.14. Assume there are no periodic orbits in $\mathcal{P}$ with reduced index -1,0 or 1. Then there exist a natural isomorphism between the homology complexes*

$$\varphi^{21} \colon HC_*(M, \xi; \alpha_1, J_1) \to HC_*(M, \xi; \alpha_2, J_2)$$

Notice that in general, the differential map ∂ is hard to compute but we have some cases when it simply vanishes. For example, if all good orbits have even (respectively odd) reduced index, then $C_k = 0$ for k odd (respectively even) and therefore $\partial = 0$. So the number of periodic orbits of each index gives a contact invariant.

3. MORSE-BOTT SETUP

In the very start we mentioned that the contact homology can be seen as a variant of the Morse theory together with the action functional $\mathcal{A}$ as in 2.2. In order to achieve a consistent theory, we required generic contact forms to avoid non-degenerate periodic orbits. But this nondegeneracy condition makes the calculation of the contact homology very hard. If we observes Definition 2.2, we notice that it disallows a continuous collection of periodic orbits; so if we allow -as in Morse-Bott theory- for critical points to be degenerate, we would have harder theoretical results to prove but relatively easier calculations. Thus we introduce a new definition in the contact homology without perturbing the contact form

Definition 3.1. *A contact form α on M is said to be of* Morse-Bott type *if the action spectrum $\sigma(\alpha)$ -i.e. the set of critical values of the action functional corresponding to α- is discrete and if, for every $T \in \sigma(\alpha)$,$N_T = \{p \in M \colon \varphi_T(p) = p\}$ is a closed smooth submanifold of M, such that the rank of $d\alpha\big|_{N_T}$ is locally constant and $T_pN_T =$* $\ker\left(\varphi_{T*} - I\right)_p$

Last two conditions are analogues of the finite dimensional Morse-Bott setup. The Reeb flow on M induces an S^1 action on the submanifolds N_T and we get $S_T = N_T/S^1$. The space S_T is an orbifold with singularity group $\mathbb{Z}_k$ since the action is free except the singularities corresponding to Reeb orbits of period T/k covered k times. We need to extend the grading of orbits to these spaces. First, we observe that the symplectic paths obtained by the trivializations need not to end at a non-degenerate symplectic matrix. Moreover, the endpoint of this path could have 1 as eigenvalue. Therefore, it is imperative to modify the Conley-Zehnder index as follows

$$\mu\left(\Psi_\gamma\right) = \frac{1}{2}\operatorname{sign}\Gamma\left(\tilde{\Psi}, 0\right) + \sum_{\substack{t \neq 0 \\ t \text{ crossing}}} \operatorname{sign}\Gamma\left(\Psi_\gamma, t\right) + \frac{1}{2}\operatorname{sign}\Gamma\left(\tilde{\Psi}, T\right)$$

where $\tilde{\Psi}$ is a small perturbation of Ψ [10]. Notice that this number is not an integer in general; $\mu\left(\Psi_\gamma\right) \in \frac{1}{2}\mathbb{Z}$. Since a non-degenerate path will have a non-zero contribution of $\frac{1}{2}\operatorname{sign}\Gamma\left(\tilde{\Psi}, T\right)$, the number $\mu\left(\Psi_\gamma\right) - \frac{1}{2}\dim S_T$ is an integer. Moreover, by the continuity of the Maslov index [10, Theorem 3.2 (Homotopy)], this number is invariant of the orbit chosen, denoted by $\mu(S_T) = \mu\left(\Psi_\gamma\right) - \frac{1}{2}\dim S_T$.

The Morse theoretical construction is as follows. We start by a Morse function f_0 associated with the orbit space N_{T_0} with the least action $T_0 \in \sigma(\alpha)$. Then to find the Morse function f_T, for each T_k such that $S_{T_k} \subset S_T$, we extend the Morse functions f_{T_k} so that the Hessian is positive definite in the normal directions to S_{T_k} in S_T. This can be achieved non-degenerately following a similar argument of Lemma 2.3, i.e. we can find a generic Morse function f_T supported on a small neighborhood of N_T together with a perturbed contact form such that the perturbed orbits of action $T' \leqslant T$ are all non-degenerate and correspond to the critical points of the action functional. The critical points are graded using the Conley-Zehnder index as follows. For a critical point p, let

$$|p| = \mu(S_T) - \frac{1}{2}\dim S_T + n - 3 + \mathrm{index}_p(f_T)$$

where $\mathrm{index}_p(f_T)$ denotes the Morse index of the critical point p. Note that this provides a shift by the reduced Conley-Zehnder index to the Morse homology of the orbit spaces. Also, the good orbit criterion here is that the parity of

$$\left(\mu(S_T) - \frac{1}{2}\dim S_T\right) - \left(\mu(S_{2T}) - \frac{1}{2}\dim S_{2T}\right)$$

must be odd, since the Morse functions were chosen so that $\mathrm{index}_p(f_T) = \mathrm{index}_p(f_{kT})$ and the parity of the grading thus only depends to the quantity above. The chain complex C^* is generated by the good critical points of the functions f_T for all orbit spaces S_T. Now, the differential is given by counting the broken J-holomorphic curves $C_j = \{F_{1,j}, \ldots, F_{l,j}\}$, $j = 1, \ldots, k$ together with $t_j \in \mathbb{R}^+$ for $j = 1, \ldots, k-1$ such that

- The connecting orbits of each holomorphic curve match, that is, for $j = 2, \ldots, k$ the orbits mapped by the gradient flow of f_T after t_{j-1} in the positive end of C_j coincides with the orbits mapped by the gradient flow of f_T before t_{j-1} in the negative end of C_{j-1}

- The closed orbits at the positive end of C_1 are in the stable manifold $W^s(p^+)$
- The closed orbits at the negative ends of C_k are in the unstable manifolds $W^s(p_1^-)$, $\ldots$,$W^s(p_s^-)$

Proposition 3.2. ([1, Theorem 1.8], [7]) *Assume that α is a contact form of Morse-Bott type for (M, ξ) and that J is an almost complex structure on $(R \times M, d(e^t\alpha))$ invariant under the S^1-action along the submanifolds N_T. Then the homology of the chain complex above is isomorphic to the contact homology $HC_*(M, \xi)$.*

In the cylindrical case, the setup looks easier: we have two critical points p^+, p^- together with two orbit spaces S^+, S^-. Then we consider the -compactified- moduli space of pseudoholomorphic curves $\tilde{\mathcal{M}}(S^+; S^-)$ capped with the stable and unstable manifolds via the fibered product

$$W^u(p^+) \times_{S^+} \tilde{\mathcal{M}}(S^+; S^-) \times_{S^-} W^s(p^-)$$

As before the cylindrical version is isomorphic to the standard one provided that we do not have orbits with indices -1,0 and 1 (see [1, Theorem 1.9]).

4. CONTACT HOMOLOGY OF BRIESKORN MANIFOLDS

For a $(n+1)$-tuple $(a_0, \ldots, a_n)$ of integers greater than 1, the algebraic variety

$$\tilde{\Sigma}(a_0, \ldots, a_n) = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^n : \sum_{k=0}^{n} z_k^{a_k} = 0 \right\}$$

is called the Brieskorn variety. The link of this variety, i.e. $\Sigma(a_0, \ldots, a_n) = S^{2n+1} \cap \tilde{\Sigma}(a_0, \ldots, a_n)$ is called the Brieskorn manifold. It admits a natural contact structure induced by the odd dimensional sphere. But to obtain a more symmetric structure, we will use the contact form

$$\alpha = \frac{i}{8} \sum_{k=0}^{n} a_k (z_k \mathrm{d}\bar{z}_k - \bar{z}_k \mathrm{d} z_k) \tag{4.1}$$

A straightforward calculation shows that the Reeb vector field associated to this contact structure is

$$R = 4i(z_0/a_0, \ldots, z_n/a_n).$$

Consider this vector field on $\mathbb{C}^{n+1}$. Then its flow will be given by

$$\varphi_t(z) = \left(e^{4it/a_0} z_0, \ldots, e^{4it/a_n} z_n\right)$$

Notice that this flow is contained on the unit sphere: if $\sum_{k=0}^{n} |z_k|^2 = 1$, then

$$\sum_{k=0}^{n} |\varphi_t(z)_k|^2 = |z_k|^2 = 1$$

Moreover

$$\sum_{k=0}^{n}(\varphi_t(z))_k^{a_k} = e^{4it}\sum_{k=0}^{n} z_k^{a_k}$$

which vanishes on the variety. Therefore $\varphi_t(z)\big|_{\Sigma(a_0,\ldots,a_n)}$ is the flow of the Reeb vector field. Moreover, all Reeb orbits are closed and degenerate. Thus it is reasonable to use Morse-Bott approach without perturbing the contact form. We follow [2]. Endowing $\mathbb{C}^{n+1}$ with the symplectic structure $\omega = \mathrm{d}\alpha = i/4\sum_{k=0}^{n} a_k \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k$, we seek for the symplectic complement ξ^ω of the contact structure. The aim is to split the tangent space of $\mathbb{C}^{n+1}$ as $\mathbb{C}^{n+1} = \xi \oplus \xi^\omega$ to calculate the Maslov indices using the product property in Theorem 2.7. First, through a straightforward calculation, we get $\xi^\omega = \operatorname{span}(X_1, Y_1, X_2, Y_2)$ where

$$\begin{aligned} X_1 &= (\bar{z}_0^{a_0-1}, \ldots, \bar{z}_n^{a_n-1}) \quad Y_1 = iX_1 \\ X_2 &= -2i(z_0/a_0, \ldots, z_n/a_n) \quad Y_2 = (z_0, \ldots, z_n) \end{aligned}$$

We use the Gram-Schmidt process to have a symplectic orthonormal basis $\{\tilde{X}_1, \tilde{Y}_1, \tilde{X}_2, \tilde{Y}_2\}$ where

$$\tilde{X}_1 = X_1/\sqrt{\omega(X_1, Y_1)}, \quad \tilde{Y}_1 = iX_1, \quad \tilde{X}_2 = X_2$$

and

$$\tilde{Y}_2 = Y_2 - \frac{\omega(X_1, Y_2)Y_1 - \omega(Y_1, Y_2)X_1}{\omega(X_1, Y_1)} = Y_2 - \frac{\sum a_k z_k^{a_k}}{2\omega(X_1, Y_1)}X_1$$

Notice that the key observation is that $\omega(X_1, Y_1) = \frac{1}{2}\sum_k a_k \left|z_k\right|^{2(a_k-1)} > 0$. This basis provides a symplectic trivialization of the bundle ξ^ω thus $c_1(\xi^\omega) = 0$. Since the Chern class is additive for direct sums, we have $c_1(\xi) = 0$.

To calculate the action spectrum of the contact form, let $I = \{0, 1, \ldots, n\}$. Notice that for any $J \subset I$, the Reeb flow will have a periodic orbit of period $\frac{\pi}{2}\operatorname{lcm}_{k\in J} a_k$. So we have $\mathcal{T} = \left\{\frac{\pi}{2}\operatorname{lcm}_{k\in J} a_k \colon J \subset I\right\}$ as the set of minimal periods. This set is finite and we enumerate is elements as $\mathcal{T} = \{T_1, \ldots, T_l\}$. Also, let J_{T_k} denote the largest subset of I such that $\frac{\pi}{2}\operatorname{lcm}_{k\in J_{T_k}} a_k = T_k$ for $k = 1, \ldots, l$. Considering the Reeb vector field as a vector field over $\mathbb{C}^{n+1}$, we obtain the return map

$$T\varphi_t = \begin{bmatrix} e^{4it/a_0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{4it/a_n} \end{bmatrix}$$

for a fixed minimal period $T_k \in \mathcal{T}$. Let $\Phi_{\mathbb{C}^{n+1}}$ denote the path of symplectic matrices for $t \in [0, NT_k]$. Combining the product rule and (A.2), we get

$$\mu(\Phi_{\mathbb{C}^{n+1}}) = 2\sum_{a_j\in J_{T_k}} \frac{2NT_k}{\pi a_j} + 2\sum_{a_j\in I-J_{T_k}} \left\lfloor \frac{2NT_k}{\pi a_j} \right\rfloor + |I - J_{T_k}|$$

On the other hand, flowing the symplectic basis for ξ^ω with $\varphi_t(z)$, we calculate how much the symplectic basis vectors rotate

$$T\varphi_t(\tilde{X}_1) = e^{4it}\tilde{X}_1(\varphi_t), \quad T\varphi_t(\tilde{Y}_1) = e^{4it}\tilde{Y}_1(\varphi_t)$$
$$T\varphi_t(\tilde{X}_2) = \tilde{X}_1(\varphi_t), \quad T\varphi_t(\tilde{Y}_2) = \tilde{Y}_1(\varphi_t)$$

Again using the product rule and (A.2), the contribution of the complement bundle ξ^ω is given as

$$\mu(\Phi_{\xi^\omega}) = 4N\frac{T_i}{\pi}$$

Therefore the Maslov index of the quotient space S_{NT_k} is given by

$$\mu(S_{NT_k}) = 2\sum_{a_j\in J_{T_k}} \frac{2NT_k}{\pi a_j} + 2\sum_{a_j\in I-J_{T_k}} \left\lfloor \frac{2NT_k}{\pi a_j} \right\rfloor + |I - J_{T_k}| - 4N\frac{T_i}{\pi}$$

Here one must check the index positivity/negativity condition, namely we must have

$$\sum_{k=0}^{n} \frac{1}{a_k} > 0 \text{ or } \sum_{k=0}^{n} \frac{1}{a_k} < 0$$

This conditions provides the non-existence of orbit spaces with index -1,0 and 1 and we can use the cylindrical homology calculations, which are considerably easier.

To calculate the dimension of the resulting orbifold, notice that N_{T_k} is nothing but the points in the manifold satisfying $z_l = 0 \, \forall \, l \in I - J_{T_k}$ since the set J_{T_k} is maximal. This is exactly the Brieskorn manifold $\Sigma(a_{l_1}, \ldots, a_{l_s})$ where $\{a_{l_1}, \ldots, a_{l_s}\} = J_{T_k}$. Thus the dimension of $S_{T_k} = N_{T_k} \big/ S^1$ is equal to $2\,|J_{T_k}| - 4$.

Following [2], for $p \in S_T$, the differential in the Morse-Bott setup can be calculated using

$$\mathrm{d}p = \partial p + \sum n_q q$$

where the first term designates the Morse differential for the critical points of f_T and the second term counts the number of the elements in the zero-dimensional part of the fibered product

$$\left(W^u(p) \times_S \tilde{\mathcal{M}}(S; S') \times'_S W^s(q) \right) /\mathbb{R}$$

Since S^1 acts on the cylinders through the Reeb flow, except the vertical cylinders, the fibered product is one dimensional. So the only differential to calculate is the Morse differential with a degree shift of $\mu(S_{NT_k}) + n - 3 - \frac{1}{2}\dim S_{T_k}$. So we need the homology groups of the orbit spaces. It is known that (see [16]) the rational homology group of each orbit space is

$$H_q(S_{T_k}, \mathbb{Q}) = \left\{ \begin{array}{ll} \mathbb{Q}, & q \text{ even}, \, 0 \leqslant q \leqslant \dim S_{T_k} \\ 0, & \text{otherwise} \end{array} \right\} \oplus \left\{ \begin{array}{ll} \mathbb{Q}^{\kappa}, & q = \frac{1}{2}\dim S_{T_k} \\ 0, & \text{otherwise} \end{array} \right\}$$

where κ denotes the rank of the rational homology of N_{T_k} -or more precisely-

$$\kappa = \operatorname{rank} \tilde{H}_{|J_{T_k}|-2}(N_{T_k}) = \sum_{I_s \subset J_{T_k}} (-1)^{J_{T_k}-s} \frac{\prod_{j \in I_s} a_j}{\operatorname{lcm}_{j \in I_s} a_j}$$

The cylindrical contact homology with $\mathbb{Q}$-coefficients of M with induced contact structure is a $\mathbb{Q}$-vector space, where the number of generators in each degree can be determined as follows. For each $T_k \in \mathcal{T}$, we get $\operatorname{rank} \tilde{H}_j(S_{T_k}, \mathbb{Q})$ generators in degree $\mu(S_{NT_k}) + n - 3 - \frac{1}{2} \dim S_{T_k} + j$ for $j = 0, \ldots, \dim S_{T_k}$ and $N \in \mathbb{N}$ such that for $j \neq i$ the multiples N_{T_k} are not divisible by T_j whenever $J_{T_k} \subset J_{T_j}$.

APPENDIX A: IN-DEPTH ANALYSIS OF $Sp(2)$

This appendix is dedicated to a detailed analysis of $Sp(2)$ following [17]. Let $A \in Sp(2)$ be a 2–by–2 symplectic matrix and let $A = PO$ be its polar decomposition, where P is symmetric and positive definite and O is orthogonal. Thus, we have

$$O = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and

$$\begin{aligned} \mathcal{U}\colon Sp(2) &\to S^1 \subset \mathbb{C} \\ PO &\mapsto e^{i\theta} \end{aligned} \tag{A.1}$$

A.1. Topology of $Sp(2)$

First, note that the eigenvalues λ_1, λ_2 of P are real and positive with $\lambda_1\lambda_2 = \det P = 1$ and $\lambda_1 + \lambda_2 = \operatorname{tr} P$. Thus we have $\operatorname{tr} P \geqslant 2$ and we can set $\operatorname{tr} P = 2\cosh\tau$ where $\tau \geqslant 0$. Using this substitution, we get

$$P = \begin{bmatrix} \cosh\tau + a & b \\ b & \cosh\tau - a \end{bmatrix} \Rightarrow \det P = 1 = \cosh^2\tau - a^2 - b^2 \Rightarrow b^2 = \sinh^2\tau - a^2$$

The positivity of the last equations implies that $|a| \leqslant |\sinh\tau|$. So let $a = \cos\sigma\sinh\tau$ to get $b = \sin\sigma\sinh\tau$ for $\sigma \in [0, 2\pi]$. Considering $\tau = |z|$ and $\sigma = \arg z$ for $z \in \mathbb{C}$, the set of 2–by–2 symmetric positive definite matrices is homeomorphic with $\mathbb{C}$ and P is given by

$$P(\tau e^{i\sigma}) = \begin{bmatrix} \cosh\tau + \cos\sigma\sinh\tau & \sin\sigma\sinh\tau \\ \sin\sigma\sinh\tau & \cosh\tau - \cos\sigma\sinh\tau \end{bmatrix}$$

Since the complex plane is homeomorphic to open disk $D = \{z \in \mathbb{C}\colon |z| < 1\}$ via the map $r = \tanh^2 \tau$, $Sp(2)$ is homeomorphic to $S^1 \times D$.

A.2. The Rotation Function on $Sp(2)$

Using the homeomorphism in Section A.1, an element of $Sp(2)$ can be characterized by (θ, r, σ). Now, $SO(2) \subset Sp(2)$ and in fact one can see that $SO(2)$ consists of orthogonal matrices so $P = I$ and thus $SO(2) = \{(\theta, r, \sigma) \in S^1 \times D\colon r = 0\}$. Via this identification, it is clear that $SO(2)$ is a deformation retract of $Sp(2)$. Moreover, following (A.1), $\mathcal{U}(\theta, 0, \sigma) = e^{i\theta}$, so $\mathcal{U}$ restricts to the standard isomorphism from $SO(2)$ onto S^1. Notice that U is not symplecticly invariant, i.e. for $M, A \in Sp(2)$, $\mathcal{U}(M^{-1}AM) \neq \mathcal{U}(A)$ and also $\mathcal{U}(A^2) \neq \mathcal{U}(A)$. We seek for a symplecticly invariant rotation function $\rho\colon Sp(2) \to S^1$ homotopic to $\mathcal{U}$ satisfying $\rho(A^n) = \rho(A)^n \ \forall A \in Sp(2)$ –which will be called *rotation function*.

Let $G\colon = -iJ$ which is Hermitian with respect to the standard inner product $\langle w, z\rangle = |w_1| z_1 + |w_2| z_2$ in $\mathbb{C}^2$. Then, it is clear to see that a real matrix A is symplectic if and only if $A^*GA = G$. Assume that $A \in Sp(2)$ has eigenvalues $\lambda \neq \pm 1$ and $\bar{\lambda}$ on S^1 with $\xi, \bar{\xi}$ corresponding eigenvectors. Then

$$\langle G\xi, \bar{\xi}\rangle = \langle A^*GA\xi, \bar{\xi}\rangle = \langle GA\xi, A\bar{\xi}\rangle = |\lambda|^2 \langle G\xi, \bar{\xi}\rangle$$

Since $\lambda \neq \pm 1$, $\langle G\xi, \bar{\xi}\rangle = 0$. Thus, $\left\{\xi, \bar{\xi}\right\}$ is a G–orthogonal basis and $\langle G\xi, \xi\rangle$ and $\langle G\bar{\xi}, \bar{\xi}\rangle$ are real and non-zero. Thus, following this observation we give the following definition.

Definition A.1. *If $\lambda \in S^1 - \{-1, 1\}$ is an eigenvalue of $A \in Sp(2)$ and ξ is an eigenvector, the* Krein sign of λ *is the sign of $\langle G\xi, \xi\rangle$ and we say that λ is* Krein positive *or* Krein negative *depending on this sign.*

Since G has signature $(1, 1)$, λ is Krein positive implies that $\bar{\lambda}$ is Krein negative. Moreover, $A \in Sp(2)$ has eigenvalues either on the real line or the unit circle S^1. In the light of these observations, we define the rotation function as follows:

$$\rho\colon Sp(2) \to S^1 \quad \rho\left(A\right) = \begin{cases} \lambda & \text{if } \lambda \in S^1 - \{-1,1\} \text{ is Krein positive} \\ 1 & \text{if } \lambda \in \mathbb{R}^+ \\ -1 & \text{if } \lambda \in \mathbb{R}^- \end{cases}$$

Since ρ is defined via eigenvalues, it is symplecticly invariant. Moreover, noting that $\rho\left(A\right) = \dfrac{\lambda}{|\lambda|}$ where $\lambda \in \mathbb{R}^*$ or Krein positive, ρ is continuous. Now, the rotation matrix $R\left(\theta\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$ with $\xi = \begin{bmatrix} i \\ 1 \end{bmatrix}$ corresponding to $e^{i\theta}$. Thus

$$G\xi = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} \Rightarrow \langle G\xi, \xi \rangle = -i^2 + 1 = 2$$

Hence $e^{i\theta}$ is Krein positive and $\rho\left(R\left(\theta\right)\right) = e^{i\theta}$. Therefore ρ and $\mathcal{U}$ agree on $SO(2)$. Using the retraction of $Sp(2)$ to $SO(2)$, we can conclude that ρ is homotopic to $\mathcal{U}$. Moreover, since λ^k is an eigenvalue of A^k for an eigenvalue λ of A, $\rho\left(A^k\right) = \rho\left(A\right)^k$.

The question remains that whether we can express this rotation function in terms of (θ, r, σ). To find the eigenvalues of the symplectic matrix $A\left(\theta, r, \sigma\right) = A$, we solve the characteristic polynomial

$$\det\left(\lambda \mathbb{1} - A\right) = 0 \Rightarrow \lambda^2 - \left(\operatorname{tr} A\right)\lambda + 1 = 0 \Rightarrow \Delta = \left(\operatorname{tr} A\right)^2 - 4 = 4\left(\cosh^2\tau\cos^2\theta - 1\right)$$

Using the identity $\cosh^2\tau = 1/\left(1 - \tanh^2\tau\right) = 1/\left(1 - r\right)$, we observe that $\lambda = \pm 1 \Leftrightarrow r = \sin^2\theta$ and $r = \sin^2\theta$ is the surface of symplectic matrices with double eigenvalues. Since r tends to 1 as θ tends to $\pm\pi/2$, this surface consists of two connected components. Note that $\Delta > 0 \Leftrightarrow r > \sin^2\theta$ and thus $\rho = \pm 1$ so that $\rho = 1$ on the part containing the identity matrix I and $\rho = -1$ on the part containing $-I$. On the other hand, $\Delta < 0 \Leftrightarrow r < \sin^2\theta$ and -as in the previous case- we have two connected components -namely Ω_+ and Ω_- on which $\sin\theta > 0$ and $\sin\theta < 0$ respectively. Since

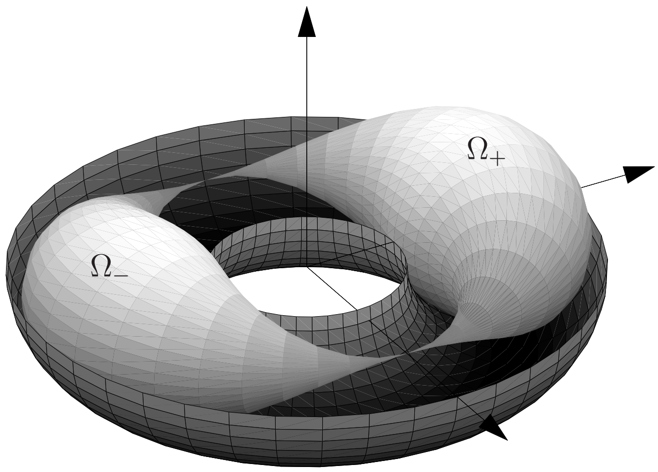

Figure A.1. Domains for the rotation function ρ

$\rho\left(R\left(\theta\right)\right) = e^{i\theta}$, ρ takes values in $S^1_+ = \{z \in S^1 \colon \Im z > 0\}$ and in $S^1_- = \{z \in S^1 \colon \Im z < 0\}$ on Ω_+ and Ω_- respectively.

A.3. Maslov index for non-degenerate paths in $Sp(2)$

One can divide $Sp(2)$ into three subsets

$$Sp(2)^+ = \{A \in Sp(2) \colon \det\left(\mathbb{1} - A\right) > 0\}$$
$$Sp(2)^- = \{A \in Sp(2) \colon \det\left(\mathbb{1} - A\right) < 0\}$$
$$Sp(2)^\circ = \{A \in Sp(2) \colon \det\left(\mathbb{1} - A\right) = 0\}$$

and set $Sp(2)^* = Sp(2)^- \cup Sp(2)^+$. Our aim is to associate an integer to every continuous path $\gamma \colon [0,1] \to Sp(2)$ such that $\gamma(0) = \mathbb{1}$ and $\gamma(1) \in Sp(2)^*$. Such paths are called *non-degenerate*. Note that $-\mathbb{1} \in Sp(2)^+$ with $\rho\left(-\mathbb{1}\right) = -1$ and

$$W = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \in Sp(2)^- \text{ with } \rho\left(W\right) = 1$$

For every continuous path $\alpha \colon [0,1] \to Sp(2)$, we can find a continuous argument function $\delta_\alpha \colon [0,1] \to \mathbb{R}$ such that $\rho\left(\alpha(t)\right) = e^{i\delta_\alpha(t)}$, and set $\Delta_t(\alpha) = \left(\delta_\alpha(t) - \delta_\alpha(0)\right) \big/ \pi$. Now given any matrix $A \in Sp(2)^*$, one can always find a path $\gamma_A \colon [0,1] \to Sp(2)^*$ such that $\gamma_A(0) = A$ and $\gamma(1) \in \{-\mathbb{1}, W\}$ depending whether $A \in Sp(2)^+$ or $A \in Sp(2)^-$. Since $Sp(2)^+$ and $Sp(2)^-$ are both contractible, $\Delta_1(\gamma_A)$ does not depend on the chosen path. We define therefore $\mathcal{R}\colon Sp(2)^* \to \mathbb{R}$ as $\mathcal{R}(A) = \Delta_1(\gamma_A)$. Now we are ready to define the Maslov index:

Definition A.2. *Let* $\gamma\colon [0,1] \to Sp(2)$ *be a continuous path such that* $\gamma(0) = I$ *and* $\gamma(1) \in Sp(2)^*$. *The* Maslov index of γ *is*

$$\mu(\gamma) = \Delta_1(\gamma) + \mathcal{R}(\gamma(1))$$

Example A.3. Let

$$\Phi(t) = e^{it} = R(t),\ t \in [0,T]$$

be a path of symplectic matrices in $\mathbb{R}^2 = C$. Then we have $\rho(R(t)) = e^{it}$ as calculated before. Now, if $T \in 2\pi\mathbb{Z}$, then $\Phi(T) = \mathbb{1} \in Sp(2)^\circ$ and $\mu_{CZ}(\Phi) = T/\pi$. Otherwise, $\Phi(T) \in Sp(2)^+$. Therefore, we have to connect the endpoint of the path to $-\mathbb{1}$. So choose the odd multiple of π nearest to T in clockwise direction, in other words, extend Φ to

$$\tilde{\Phi}(t) = e^{it},\ t \in [0,T'] \text{ where } T' = \left(2\left\lfloor \frac{T}{2\pi} \right\rfloor + 1\right)\pi$$

Thus $\mu_{CZ}(\tilde{\Phi}) = T'$. So altogether, we have

$$\mu_{CZ}(e^{it}\big|_{t\in[0,T]}) = \begin{cases} \frac{T}{\pi} & \text{if } T \in 2\pi\mathbb{Z} \\ 2\left\lfloor \frac{T}{2\pi} \right\rfloor + 1 & \text{otherwise} \end{cases} \tag{A.2}$$

or in a better shape as

$$\mu_{CZ}(e^{it}\big|_{t\in[0,T]}) = \left\lfloor \frac{T}{2\pi} \right\rfloor + \left\lceil \frac{T}{2\pi} \right\rceil$$

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