SOME MEAN VALUE PROBLEMS ABOUT DIRICHLET L-FUNCTIONS AND THE RIEMANN ZETA-FUNCTION

by

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ABSTRACT

SOME MEAN VALUE PROBLEMS ABOUT DIRICHLET L-FUNCTIONS AND THE RIEMANN ZETA-FUNCTION

The average values of higher derivatives of the Riemann zeta-function and Dirichlet L-functions over a set of special points, specifically the set of nontrivial zeta zeros, are important tools to understand the distribution of zeta zeros and the relationships between the Riemann zeta function and Dirichlet L- functions. In this thesis, we study the following sums

$$\sum_{\substack{\rho:\zeta(\rho)=0\\ 0<\Im\rho\leq T}} \zeta^{(j)}(\rho) \qquad \text{and} \qquad \sum_{\substack{\rho:\zeta(\rho)=0\\ 0<\Im\rho\leq T}} L^{(j)}(\rho).$$

ÖZET

DIRICHLET L-FONKSİYONLARI VE RIEMANN ZETA-FONKSİYONU İLE İLGİLİ BAZI ORTALAMA DEĞER PROBLEMLERİ

Riemann zeta fonksiyonunun ve Dirichlet L-fonksiyonlarının türevlerinin bazi özel nokta kümelerinde, spesifik olarak zeta fonksiyonunun sıfırlarında, ortalama degerleri bu fonksiyonları ve aralarindaki ilişkileri anlamak için önemli araçlardir. Bu tezde,

$$\sum_{\substack{\rho:\zeta(\rho)=0\\ 0<\Im\rho\leq T}} \zeta^{(j)}(\rho) \qquad \text{ve} \qquad \sum_{\substack{\rho:\zeta(\rho)=0\\ 0<\Im\rho\leq T}} L^{(j)}(\rho)$$

toplamları incelenmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

$e(\theta)$	$= e^{2\pi i\theta}$
$f^{(j)}(s)$	j^{th} derivative of $f(s)$, $f^{(0)} = f$.
$L(s,\chi)$	A Dirichlet L-function.
β	The real part of a zero of the zeta function or of an L-function.
$\Gamma(s)$	$= \int_0^\infty e^{-x}x^{s-1}dx$ for $\sigma > 0$; called the *Gamma function*.
γ	The imaginary part of a zero of the zeta function or of an L-function.
$\zeta(s)$	The Riemann zeta-function.
$\Lambda(n)$	$= \log p$ if $n = p^k$, $= 0$ otherwise; known as the *von Mangoldt Lambda function*.
$\mu(n)$	$= (-1)^{\omega(n)}$ for square-free n, $n = 0$ otherwise. Known as the *Möbius mu function*.
ρ	$= \beta + i\gamma$; a zero of the zeta function or of an Dirichlet L-function.
τ	$= \lvert t\rvert + 4$.
$\tau(\chi)$	$= \sum_{a=1}^{q} \chi(a)e(a/q)$; known as the *Gauss sum* of χ.
$\phi(n)$	The number a, $1 \le a \le n$, for which $(a,n) = 1$; knowns as *Euler's Totient function*.
$\chi(n)$, $\psi(n)$	A Dirichlet character.
$\chi(s)$	The function defined by the functional equation for $\zeta(s)$.
$\chi(s,\psi)$	The function defined by the functional equation for $L(s,\psi)$.
$\omega(n)$	The number of distinct primes dividing n.
$[\![x]\!]$	The unique integer such that $[\![x]\!] \le x < [\![x]\!] + 1$; called the *integer part* of x.
$\{x\}$	$= x - [\![x]\!]$; called the *fractional part* of x.
$f(x) = O(g(x))$	$\lvert f(x)\rvert \le C\lvert g(x)\rvert$, where C is an absolute constant.
$f(x) \ll g(x)$	$f(x) = O(g(x))$.
$f(x) = O_{\alpha_1,\alpha_2,\ldots}(g(x))$	$\lvert f(x)\rvert \le Cg(x)$, where C is a constant which depends on α_1, $\alpha_2, \ldots$.
$f(x) \ll_{\alpha_1,\alpha_2,\ldots} g(x)$	$f(x) = O_{\alpha_1,\alpha_2,\ldots}(g(x))$.

$f(x) \gg g(x)$	$g(x) = O(f(x))$.
$f(x) \gg_{\alpha_1, \alpha_2, \ldots} g(x)$	$g(x) = O_{\alpha_1, \alpha_2, \ldots}(f(x))$.
$f(x) \asymp g(x)$	$f(x) = O(g(x))$ and $g(x) = O(f(x))$.
$f(x) \sim g(x)$	$\lim_{x \to \infty} f(x)/g(x) = 1$.

1. INTRODUCTION AND STATEMENT OF RESULTS

D. Shanks conjectured that the average value of $\zeta'(\rho)$, over the nontrivial zeros ρ of the Riemann zeta-function $\zeta(s)$, is a positive real number. J. B. Conrey, A. Ghosh and S. M. Gonek [1] proved that this is true asympotically, and that if we average over the zeros with $0 < \Im(\rho) \le T$ then this average is $\sim \frac{1}{2}\log T$. Fujii [2] gave the more precise estimate

$$\sum_{0<\gamma\le T} \zeta'(\rho) = \frac{T}{4\pi}\log^2\frac{T}{2\pi} + (\gamma_0 - 1)\frac{T}{2\pi}\log\frac{T}{2\pi} + (\gamma_1 - \gamma_0)\frac{T}{2\pi} + O\left(T\exp\left(-c\sqrt{\log T}\right)\right) \tag{1.1}$$

for this average, where γ_0 and γ_1 arise from the Laurent expansion of the Riemann zeta function (see (2.11)) and c is some positive constant. Under the Riemann hypothesis (R.H.), Fujii sharpened the above error term to $T^{\frac{1}{2}}(\log T)^{\frac{7}{2}}$. In this thesis, we shall prove the following extension of (1.1).

Theorem 1.1. *Let $j \ge 1$ be a fixed integer. Then, for large T*

$$\sum_{0<\gamma\le T} \zeta^{(j)}(\rho) = \frac{(-1)^{j+1}}{j+1}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j+1} + O_j\left(T\log^j T\right).$$

The average value of $\zeta^{(j)}(\rho)$, over the zeros ρ of $\zeta(\rho) = 0$ with $0 < \Im(\rho) \le T$, i.e.

$$\frac{1}{N(T)}\sum_{0<\gamma\le T} \zeta^{(j)}(\rho),$$

where $N(T)$ is the number of terms in the above sum, by the fact that $N(T) \sim \dfrac{T}{2\pi}\log T$ (see (2.9)), is $\dfrac{(-1)^{j+1}}{j+1}\left(\log\dfrac{T}{2\pi}\right)^j$. So this tells us about the size of $\zeta^{(j)}(s)$ at certain points (namely the nontrivial zeros ρ). We can say that, on R.H., there exist points s

on the critical line with $0 < \Im(s) \leq T$ such that

$$\left|\zeta^{(j)}(s)\right| \gg_j \log^j T.$$

In the second part of our thesis, we shall be concerned with the mean value of $L^{(j)}(\rho)$, over the nontrivial zeros ρ of $\zeta(s)$. We'll obtain the following formulas.

Theorem 1.2. *Let ψ be a primitive character modulo $q \geq 3$. For large T and A an arbitrarily large fixed number, we have*

$$\sum_{0<\gamma\leq T} L(\rho,\psi) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} - \frac{\mu(q)\psi(-1)\tau(\psi)}{\phi(q)}\frac{T}{2\pi}L(1,\overline{\psi}) + \frac{T}{2\pi}\frac{L'}{L}(1,\psi)$$
$$+ O_A\left(T\exp\left(-\mathfrak{u}\sqrt{\log T}\right)\right),$$

and, for $j \geq 1$,

$$\sum_{0<\gamma\leq T} L^{(j)}(\rho,\psi) = -\frac{\mu(q)\psi(-1)\tau(\psi)}{\phi(q)}\frac{T}{2\pi}\sum_{\omega=0}^{j}\frac{j!}{\omega!}\sum_{\nu=\omega}^{j}\frac{(-1)^\nu}{(\nu-\omega)!}L^{(\nu-\omega)}(1,\overline{\psi})\left(\log\frac{qT}{2\pi}\right)^\omega$$
$$+ \frac{T}{2\pi}\left(\frac{L'}{L}\right)^{(j)}(1,\psi) + O_{A,j}\left(T\exp\left(-\mathfrak{u}\sqrt{\log T}\right)\right). \qquad (1.2)$$

In both cases, the asymptotic formulas are valid under the restriction $q \leq \log^A T$ and $\mathfrak{u}$ is a non-effective constant depending on j and A.

As obvious corollaries of Theorem 1.2 for $j = 0$, we can write

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\leq T}\{L(\rho,\psi) - 1\} = -\frac{\mu(q)\psi(-1)\tau(\psi)}{\phi(q)}L(1,\overline{\psi}) + \frac{L'}{L}(1,\psi),$$

which was proved by Fujii (Under the R.H.) [3], and

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{0<\gamma\leq T} L(\rho,\psi) = 1,$$

i.e. the average value of $L(\rho,\psi)$ is asymptotically 1. However, for $j=1$, we have

$$\sum_{0<\gamma\le T} L'(\rho,\psi) = \frac{\mu(q)\psi(-1)\tau(\psi)}{\phi(q)}\frac{T}{2\pi}\left\{L\left(1,\overline{\psi}\right) - L'\left(1,\overline{\psi}\right) - L\left(1,\overline{\psi}\right)\log\frac{qT}{2\pi}\right\}$$
$$+\frac{T}{2\pi}\left(\frac{L'}{L}\right)'(1,\psi) + O_{A,j}\left(T\exp\left(-\mathfrak{u}\sqrt{\log T}\right)\right),$$

and so

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{0<\gamma\le T} L'(\rho,\psi) = \frac{\mu(q)\psi(-1)\tau(\psi)}{\phi(q)}L\left(1,\overline{\psi}\right).$$

By the facts that $\phi(q) \gg \frac{q}{\log q}$, $|\tau(\psi)| = \sqrt{q}$ and $L\left(1,\overline{\psi}\right) \ll \log q$ (see Proposition 4.1), we can say that the average value of $L'(\rho,\psi)$ is asymptotically very close to 0 for large q.

We now determine the exact order of the sum $\sum\limits_{0<\gamma\le T} L^{(j)}(\rho,\psi)$. We first note that by Proposition 4.1 we have $L^{(\kappa)}\left(1,\overline{\psi}\right) \ll_\kappa (\log q)^{\kappa+1}$, $\kappa\in\mathbb{N}$. In addition to this, we need a lower bound for $L\left(1,\overline{\psi}\right)$ and an upper bound for $\left(\frac{L'}{L}\right)^{(\kappa)}(1,\psi)$, $\kappa\in\mathbb{N}$. It easily follows from (2.17), the equation (11.7) in [4], and Proposition 5.2 that

$$\left(\frac{L'}{L}\right)^{(j)}(1,\psi) \ll_{A,j} q^{\frac{j+1}{A}} \quad\text{and}\quad L\left(1,\overline{\psi}\right) \gg_A \frac{1}{q^{\frac{1}{A}}}.$$

So, in the range $q\le \log^A T$, we have

$$\left(\frac{L'}{L}\right)^{(j)}(1,\psi) \ll_{A,j} \log^{j+1} T \quad\text{and}\quad L\left(1,\overline{\psi}\right) \gg_A \frac{1}{\log T}.$$

In the range considered for q using these estimates in Theorem 1.2 gives

$$\sum_{0<\gamma\le T} L(\rho,\psi) = \frac{T}{2\pi}\log\frac{T}{2\pi}\left(1+O_A\left(\frac{\log\log T}{\log T}\right)\right),$$

and, if $\mu(q) \neq 0$, i.e. q is square-free,

$$\sum_{0<\gamma\leq T} L^{(j)}(\rho,\psi) = (-1)^{j+1}\frac{\mu(q)\psi(-1)\tau(\psi)}{\phi(q)} L\left(1,\overline{\psi}\right)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j}\left(1+O_{A,j}\left(\frac{\log\log T}{\log T}\right)\right). \tag{1.3}$$

If q is not square-free, then (1.2) becomes

$$\sum_{0<\gamma\leq T} L^{(j)}(\rho,\psi) = \frac{T}{2\pi}\left(\frac{L'}{L}\right)^{(j)}(1,\psi) + O_{A,j}\left(T\exp\left(-\mathfrak{u}\sqrt{\log T}\right)\right). \tag{1.4}$$

The main handicap of (1.4) is that $\left(\frac{L'}{L}\right)^{(j)}(1,\psi)$ can be too close to 0 for some values of q in the range $q \leq \log^{A} T$ so that the O-term in (1.4) can dominate the main term in (1.4). If this is the case, then we lose the asymptotic fomula. We need a lower bound for $\left|\left(\frac{L'}{L}\right)^{(j)}(1,\psi)\right|$, but we do not know even whether $\left(\frac{L'}{L}\right)^{(j)}(1,\psi)$ is 0 or not. So, in this case, we can only say that

$$\sum_{0<\gamma\leq T} L^{(j)}(\rho,\psi) \ll_{j,A} T \tag{1.5}$$

in the range $q \leq \log^{A} T$.

Finally, we mention two points on the above formulas. First, comparing (1.3) and (1.5) shows the effect of the modulus of the Dirichlet character on mean values. Second, (1.3) expresses a connection of the distribution of ρ with the important arithmetic quantity $L\left(1,\overline{\psi}\right)$.

1.1. Outline for the Proofs of Theorem 1.1 and Theorem 1.2

The basic idea of the proofs of our main theorems is to interpret the sums of $\zeta^{(j)}(\rho)$ and $L^{(j)}(\rho,\psi)$ as a sum of residues. By Cauchy's theorem we have

$$\sum_{0<\gamma\leq T} f(\rho)=\frac{1}{2\pi i}\int_R f(s)\frac{\zeta'}{\zeta}(s)ds, \tag{1.6}$$

where $f(s)$ is $\zeta^{(j)}(s)$ or $L^{(j)}(s,\psi)$ and R is the rectangle joining the points $a+i$, $a+iT$, $1-a+iT$ and $1-a+i$, a is a fixed number >1.

In chapter 2 and 4, we give some estimates of $\zeta^{(j)}(\rho)$, $L^{(j)}(\rho,\psi)$ and $\frac{\zeta'}{\zeta}(s)$. Using these estimates, in both cases of $f(s)$, the horizontal integrals and the vertical integral from $a+i$ to $a+iT$ can be bounded trivially and the contribution of these three integrals are very small and the main terms of the formulas in Theorems 1.1 and 1.2 come from the vertical integral from $1-a+i$ to $1-a+iT$.

The next step is to obtain the main term from the remaining integral. Using certain exponential integral estimates, Lemmas 3.11 and 3.15, we convert this integral to a finite sum, but in this case, summand of this finite sum includes more elementary factors, not like $\zeta^{(j)}(\rho)$ or $L^{(j)}(\rho,\psi)$. Finally, applying standard techniques of analytic number theory, such as Perron's formula, Dirichlet hyperbola method, partial summation formula and so on, we complete the last part in chapters 6 and 7.

2. SOME FORMULAE and ESTIMATES

Before we develop the basic idea of the proof of the Theorem, it will be useful to set down certain formulae and estimates. These are given without proof, the proofs can be found in most classical books on Analytic Number Theory, see, for example, [5], [6], [4] or [7]. Throughout this paper $s = \sigma + it$ denotes a complex variable, $\tau = |t| + 4$ and $\rho = \beta + i\gamma$ denotes nontrivial zeros (i.e. complex zeros) of the Riemann zeta-function.

2.1. χ-Function

The functional equation of the Riemann zeta-function can be expressed in the asymmetric form

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{2.1}$$

where

$$\chi(s) := 2^s \pi^{-1+s} \sin\left(\frac{1}{2} s\pi\right) \Gamma(1-s) \tag{2.2}$$

$$= \pi^{s-\frac{1}{2}} \frac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma(\frac{1}{2}s)}. \tag{2.3}$$

Firstly, we state two well-known asymptotic formulas involving the Γ function:

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right), \tag{2.4}$$

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right). \tag{2.5}$$

These formulas are valid as $|s| \to \infty$, in the angle $-\pi + \delta < \arg s < \pi - \delta$, for any fixed $\delta > 0$. By using the above formulas, it is easy to show that

$$\Gamma(s) = |t|^{\sigma + it - \frac{1}{2}} \, (2\pi)^{\frac{1}{2}} \, \exp\left\{ \operatorname{sgn}(t) \left[\frac{i\pi\sigma}{2} - \frac{t\pi}{2} - \frac{i\pi}{4} \right] - it \right\} \left[1 + O\left(\frac{1}{|t|} \right) \right], \tag{2.6}$$

$$\chi(s) = \left(\frac{|t|}{2\pi} \right)^{\frac{1}{2} - \sigma} \exp\left(-it \log \frac{|t|}{2\pi e} + \frac{i\pi}{4} \operatorname{sgn}(t) \right) \left(1 + O\left(\frac{1}{|t|} \right) \right), \tag{2.7}$$

$$\frac{\chi'}{\chi}(s) = -\log \frac{|t|}{2\pi} + O\left(\frac{1}{|t|} \right), \tag{2.8}$$

uniformly in $\alpha \leq \sigma \leq \beta$ and $|t| \geq 1$, for any fixed real numbers α and β, where

$$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if } \; t > 0 \\ 0 & \text{if } \; t = 0 \\ -1 & \text{if } \; t < 0. \end{cases}$$

2.2. Estimate for $\frac{\zeta'}{\zeta}$

The Riemann-von Mangoldt formula states that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{2.9}$$

where $N(T)$ is the number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$. So, we have $N(T+1) - N(T) \ll \log T$. In other words, there are at most $O(\log n)$ non-trivial zeros $\rho = \beta + i\gamma$ with $n < \gamma \leq n+1$ for $n = 2, 3, ...$ Among the gaps between the ordinates of the zeta zeros there must be a gap of length $\gg (\log n)^{-1}$. Hence, there exists a $T_n \in (n, n+1]$ such that $|\, T_n - \gamma \,| \gg \frac{1}{\log n} \geq \frac{1}{\log T_n}$ for all zeros ρ. Thus, we get a set $\mathscr{F} := \left\{ T \in \mathbb{R} : \exists n \geq 2, \, n < T < n+1, \, |T - \gamma| > \frac{1}{\log T} \; \forall \rho \right\}$ with $\mathscr{F} \cap (n, n+1] \neq \emptyset$ for any $n = 2, 3, \ldots.$ Consider the standard formula

$$\frac{\zeta'}{\zeta}(\sigma + iT) = \sum_{|T - \gamma| \leq 1} \frac{1}{s - \rho} + O(\log T) \; .$$

This estimate is for large T and uniformly for $-1 \le \sigma \le 2$ and the sum is limited to those ρ for which $|T-\gamma| \le 1$ (See [7], §9.6). Since there are at most $O(\log T)$ terms in the sum of the above formula and each term is $\ll \log T$ if $T = \mathscr{F}$, we have the estimate

$$\frac{\zeta'}{\zeta}(\sigma + iT) \ll (\log T)^2, \qquad (\text{for } -1 \le \sigma \le 2). \tag{2.10}$$

2.3. Some Laurent Expansions

The Riemann zeta function is a meromorphic function with a simple pole with residue 1 at $s = 1$. So, it has a Laurent expansion in the neighborhood of $s = 1$

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \gamma_0 + \gamma_1 (s-1) + \gamma_2 (s-1)^2 + \dots \\ &= \frac{1}{s-1} + O(1), \qquad (s \to 1). \end{aligned} \tag{2.11}$$

By differentiation of the above Laurent series, it is easy to see that

$$\zeta^{(j)}(s) = \frac{(-1)^j}{(s-1)^{j+1}} + O_j(1), \qquad (s \to 1), \tag{2.12}$$

and then

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + O(1), \qquad (s \to 1). \tag{2.13}$$

2.4. The Order of $\zeta(s)$ and Its Derivatives

We have, for $k = 0, 1, 2, \ldots$ and $|t| \ge 1$

$$\zeta^{(k)}(\sigma + it) \ll_{\epsilon,k} \begin{cases} |t|^{\frac{1}{2}-\sigma+\epsilon} & \text{if} \quad \sigma \le 0 \\ |t|^{\frac{1}{2}(1-\sigma)+\epsilon} & \text{if} \quad 0 \le \sigma \le 1 \\ |t|^{\epsilon} & \text{if} \quad \sigma \ge 1, \end{cases} \tag{2.14}$$

with an arbitrarily fixed $\epsilon > 0$. (See Gonek [8], section 2)

2.5. Dirichlet's L-Functions

These are the most common zeta functions besides $\zeta(s)$ and are defined by

$$L(s, \psi) = \sum_{n=1}^{\infty} \psi(n) n^{-s} \qquad (\sigma > 1),$$

where for a modulus q (≥ 1), $\psi(n)$ is the arithmetical function known as a Dirichlet's character modulo q [for $q = 1$, taking $\psi \equiv 1$ we retrieve the Riemann zeta-function]. Each character $\psi(n)$ is a totally multiplicative function (i.e. $\psi(nm) = \psi(n)\psi(m)$ for all m and n) which is complex valued and satisfies $|\psi(n)| \leq 1$. It also has the following periodic property: $\psi(n) = \psi(m)$ if $n \equiv m \pmod{q}$, while $\psi(n) = 0$ if $(n, q) > 1$ and $\psi(n) \neq 0$ if $(n, q) = 1$. There exists $\phi(q)$ distinct characters modulo q, and they form an abelian group (under pointwise multiplication) which is isomorphic to the multiplicative group of the reduced system of residues mod q. A special character is the principal character, denoted by $\psi_0(n)$, defined by $\psi_0(n) = 1$ if $(n, q) = 1$ and zero otherwise.

Let $\psi(n)$ be any character to the modulus q other than the principal character. We know that $\psi(n)$ is a periodic function with period q. It is possible, however that for values of n restricted by the condition $(n, q) = 1$, the function $\psi(n)$ may have a period less than q. If so, we say that ψ is imprimitive, and otherwise primitive. We note that if ψ is a primitive character modulo q, then $q \geq 3$.

For any character $\psi(n)$ to the modulus q, the Gaussian sum $\tau(\psi)$ is defined by

$$\tau(\psi) = \sum_{m=1}^{q} \psi(m) e\left(\frac{m}{q}\right),$$

where $e(\alpha) = e^{2\pi i \alpha}$. We have

- For any primitive character ψ modulo q, $|\tau(\psi)| = \sqrt{q}$.
- For any character ψ modulo q, $|\tau(\psi)| \leq \sqrt{q}$.
- $|\tau(\psi_0)| = \mu(q)$.

2.6. The Zero-Free Region for Dirichlet's L-Functions

There is an absolute constant $c_1 > 0$ such that if ψ is a complex Dirichlet character modulo q and $L(\beta + i\gamma, \psi) = 0$ then

$$\beta < 1 - \frac{c_1}{\log q\tau}. \tag{2.15}$$

If ψ is a quadratic character, then (2.15) holds for all zeros of all $L(s,\psi)$, $\psi(\bmod q)$, with at most one exception. The exceptional zero, denoted by β_1, if it exists, is real, simple and may only occur for at most one quadratic character (See [6], §14). Of course, by the Generalized Riemann Hypothesis, all nontrivial zeros of the $L(s,\psi)$ lie on the critical line and no such exceptional zero exists. However, we have two versions of a theorem due to Siegel (See [6], §21), which establish an upper bound for β_1 and a lower bound for $L(1,\psi)$:

For any $\epsilon > 0$, there exist positive numbers $C_1(\epsilon)$, $C_2(\epsilon)$ such that

$$\beta_1 < 1 - \frac{C_1(\epsilon)}{q^\epsilon} \tag{2.16}$$

and

$$L(1,\psi) > \frac{C_2(\epsilon)}{q^\epsilon}. \tag{2.17}$$

The disadvantage of these bounds is that we are unable to compute the value of $C_1(\epsilon)$ and $C_2(\epsilon)$, i.e. these constants are non-effective.

2.7. Functional Equation for Dirichlet's L-Functions

Let $L\,(s,\psi)$ be a Dirichlet L-function, where ψ is a primitive character $\pmod q$, $q \geq 3$. There are two symmetric forms of the functional equation for $L(s,\psi)$(See [6], §9), depending on the value of $\psi(-1)$:

$$\begin{aligned}
&\pi^{-\frac{1}{2}(1-s)}\, q^{\frac{1}{2}(1-s)}\, \Gamma\left[\frac{1}{2}(1-s)\right] L\left(1-s,\,\bar{\psi}\right) \\
&\qquad = \frac{q^{\frac{1}{2}}}{\tau(\psi)}\, \pi^{-\frac{1}{2}s}\, q^{\frac{1}{2}s}\, \Gamma\left(\frac{1}{2}\,s\right) L\,(s,\psi) \qquad\qquad \text{if } \psi(-1)=1; \\
&\pi^{-\frac{1}{2}(2-s)}\, q^{\frac{1}{2}(2-s)}\, \Gamma\left[\frac{1}{2}(2-s)\right] L\left(1-s,\,\bar{\psi}\right) \\
&\qquad = \frac{iq^{\frac{1}{2}}}{\tau(\psi)}\, \pi^{-\frac{1}{2}(s+1)}\, q^{\frac{1}{2}(s+1)}\, \Gamma\left[\frac{1}{2}(s+1)\right] L\,(s,\psi) \qquad \text{if } \psi(-1)=-1.
\end{aligned}$$

Using the following functional relations involving $\Gamma(s)$

$$\begin{aligned}
\Gamma(s+1) &= s\,\Gamma(s), \\
\Gamma(s)\,\Gamma(1-s) &= \frac{\pi}{\sin(\pi s)}, \\
\Gamma(s)\,\Gamma(s+\frac{1}{2}) &= 2^{1-2s}\,\pi^{\frac{1}{2}}\,\Gamma(2s),
\end{aligned}$$

we can convert the above symmetric forms into the unsymmetric forms:

$$\begin{aligned}
L(s,\psi) &= \frac{\tau(\psi)}{q^{\frac{1}{2}}}\pi^{-1+s}\, q^{\frac{1}{2}-s}\, 2^s \sin\left(\frac{\pi s}{2}\right)\Gamma\,(1-s)\, L(1-s,\bar{\psi}) \\
&= \frac{\tau(\psi)}{q^{\frac{1}{2}}}\pi^{-\frac{1}{2}+s}\, q^{\frac{1}{2}-s}\, \frac{\Gamma\left[\frac{1}{2}(1-s)\right]}{\Gamma\left(\frac{1}{2}s\right)} L(1-s,\bar{\psi})
\end{aligned}$$

if $\psi(-1)=1$; and

$$\begin{aligned}
L(s,\psi) &= \frac{\tau(\psi)}{i\,q^{\frac{1}{2}}}\pi^{-1+s}\, q^{\frac{1}{2}-s}\, 2^s \cos\left(\frac{\pi s}{2}\right)\Gamma\,(1-s)\, L(1-s,\bar{\psi}) \\
&= \frac{\tau(\psi)}{i\,q^{\frac{1}{2}}}\pi^{-\frac{1}{2}+s}\, q^{\frac{1}{2}-s}\, \frac{\Gamma\left[\frac{1}{2}(2-s)\right]}{\Gamma\left[\frac{1}{2}(1+s)\right]} L(1-s,\bar{\psi})
\end{aligned}$$

if $\psi(-1) = -1$. It is possible to put together the two unsymmetric forms of the functional equation and we have

$$L\,(s,\psi\,) = \frac{\tau(\psi)}{i^{\mathfrak{a}} q^{\,\frac{1}{2}}}\,\pi^{\,-1+s}\,q^{\,\frac{1}{2}-s}\,2^{s}\,\sin\left[\frac{\pi}{2}(s+a)\right]\,\Gamma\,(1-s)\,L\,(1-s,\bar{\psi}) \tag{2.18}$$

$$= \frac{\tau(\psi)}{i^{\mathfrak{a}}\,q^{\,\frac{1}{2}}}\pi^{\,-\frac{1}{2}+s}\,q^{\,\frac{1}{2}-s}\,\frac{\Gamma\left[\frac{1}{2}(\mathfrak{a}+1-s)\right]}{\Gamma\left[\frac{1}{2}(\mathfrak{a}+s)\right]}L(1-s,\bar{\psi}) \tag{2.19}$$

where $\mathfrak{a}$ is defined by

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \psi(-1) = 1 \\ 1 & \text{if } \psi(-1) = 1. \end{cases}$$

Now, we define a new function

$$\chi\,(s,\psi\,) := \frac{\tau(\psi)}{i^{\,a} q^{\,\frac{1}{2}}}\,\pi^{\,-1+s}\,q^{\,\frac{1}{2}-s}\,2^{s}\,\sin\left[\frac{\pi}{2}(s+a)\right]\,\Gamma\,(1-s) \tag{2.20}$$

$$= \frac{\tau(\psi)}{i^{\mathfrak{a}}\,q^{\,\frac{1}{2}}}\pi^{\,-\frac{1}{2}+s}\,q^{\,\frac{1}{2}-s}\,\frac{\Gamma\left[\frac{1}{2}(\mathfrak{a}+1-s)\right]}{\Gamma\left[\frac{1}{2}(\mathfrak{a}+s)\right]}\,. \tag{2.21}$$

Then, we have

$$L\,(s,\psi\,) = \,\chi\,(s,\psi\,)\,L\,(1-s,\bar{\psi})\,. \tag{2.22}$$

2.8. $\chi(s, \psi)$-Function

Now, we want to get an asymptotic formula for $\chi(s, \psi)$. We insert the equation (2.6) into the equation (2.21) and we get

$$\begin{aligned}
\chi(s, \psi) &= \frac{\tau(\psi)}{i^{\mathfrak{a}} q^{\frac{1}{2}}} \left(\frac{\pi}{q}\right)^{-\frac{1}{2}+s} \left[1 + O\left(\frac{1}{|t|}\right)\right] \\
&\quad \frac{\left(\frac{|t|}{2}\right)^{\frac{1+\mathfrak{a}-\sigma}{2}-\frac{it}{2}-\frac{1}{2}} \exp\left\{-\operatorname{sgn}(t)\left[\frac{i\pi}{2}\left(\frac{1+\mathfrak{a}-\sigma}{2}\right)+\frac{\pi t}{4}-\frac{i\pi}{4}\right]+\frac{it}{2}\right\}}{\left(\frac{|t|}{2}\right)^{\frac{\mathfrak{a}+\sigma}{2}+\frac{it}{2}-\frac{1}{2}} \exp\left\{\operatorname{sgn}(t)\left[\frac{i\pi}{2}\left(\frac{\mathfrak{a}+\sigma}{2}\right)-\frac{\pi t}{4}-\frac{i\pi}{4}\right]-\frac{it}{2}\right\}} \\
&= \frac{\tau(\psi)}{i^{\mathfrak{a}} q^{\frac{1}{2}}} \left(\frac{2\pi}{q|t|}\right)^{\sigma+it-\frac{1}{2}} \exp\left\{\operatorname{sgn}(t)\left(\frac{i\pi}{2}\right)\left(\frac{1}{2}-\mathfrak{a}\right)+it\right\}\left[1+O\left(\frac{1}{|t|}\right)\right] \\
&= \frac{\tau(\psi)}{i^{\mathfrak{a}} q^{\frac{1}{2}}} \left(\frac{2\pi}{q|t|}\right)^{\sigma-\frac{1}{2}} \exp\left\{-it\log\frac{q|t|}{2\pi e}+\operatorname{sgn}(t)\left(\frac{i\pi}{2}\right)\left(\frac{1}{2}-\mathfrak{a}\right)\right\}\left[1+O\left(\frac{1}{|t|}\right)\right],
\end{aligned} \tag{2.23}$$

uniformly in $\alpha \leq \sigma \leq \beta$ and $|t| \geq 1$, for any fixed real numbers α and β. Here the constant of the above O-term depends on α and β.

3. LEMMAS

The essential results in this chapter are Lemmas 3.5, 3.9, 3.15 and 3.17. We'll use these lemmas in the proof of Theorem 1.1 and 1.2 to calculate the integrals involving the zeta, a L-function and their derivatives. In general, proofs of the lemmas are simple, but we'll give all proofs for completeness.

Lemma 3.1. *For $x > 1$ and $k \in \mathbb{Z}^+$,*

$$I_k = \int_1^x \log^k u \, du = x \left\{ \sum_{n=0}^{k} (-1)^n \frac{k!}{(k-n)!} \log^{k-n} x \right\} + (-1)^{k+1} k!.$$

Proof. The proof goes by induction on k. If $k = 1$, then

$$I_1 = \int_1^x \log u \, du = [u \log u - u]_1^x = x \log x - x + 1 = x (\log x - 1) + 1.$$

Assume that the lemma holds for each positive integer $\leq k$, $k > 1$. By partial integration, we have

$$\begin{aligned} I_{k+1} &= \int_1^x \log^{k+1} u \, du = \left[(u \log u - u) \log^k u \right]_1^x - k \int_1^x (u \log u - u) \log^{k-1} u \, \frac{du}{u} \\ &= x \log^{k+1} x - x \log^k x - k \int_1^x \log^k u \, du + k \int_1^x \log^{k-1} u \, du. \end{aligned}$$

Then induction hypothesis gives

$$
\begin{aligned}
I_{k+1} &= x\log^{k+1}x - x\log^k x - k\left[x\left\{\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n}x\right\} + (-1)^{k+1}k!\right] \\
&\quad +k\left[x\left\{\sum_{n=0}^{k-1}(-1)^n\frac{(k-1)!}{(k-1-n)!}\log^{k-1-n}x\right\} + (-1)^k(k-1)!\right] \\
&= x\log^{k+1}x - x\log^k x - k\Bigg[x\log^k x + x\left\{\sum_{n=1}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n}x\right\} + \\
&\quad (-1)^{k+1}k!\Bigg] + k\left[x\left\{\sum_{n=1}^{k}(-1)^{n-1}\frac{(k-1)!}{(k-n)!}\log^{k-n}x\right\} + (-1)^k(k-1)!\right] \\
&= x\log^{k+1}x - (k+1)x\log^k x - kx\left\{\sum_{n=1}^{k}(-1)^n\frac{\log^{k-n}x}{(k-n)!}(k! + (k-1)!)\right\} \\
&\quad -k(-1)^{k+1}k! + k(-1)^k(k-1)! \\
&= x\log^{k+1}x - (k+1)x\log^k x - x\left\{\sum_{n=1}^{k}(-1)^n\frac{(k+1)!}{(k-n)!}\log^{k-n}x\right\} \\
&\quad +(-1)^{k+2}(k+1)! \\
&= x\log^{k+1}x - (k+1)x\log^k x + x\left\{\sum_{n=2}^{k+1}(-1)^n\frac{(k+1)!}{(k+1-n)!}\log^{k+1-n}x\right\} \\
&\quad +(-1)^{k+2}(k+1)! \\
&= x\left\{\sum_{n=0}^{k+1}(-1)^n\frac{(k+1)!}{(k+1-n)!}\log^{k+1-n}x\right\} + (-1)^{k+2}(k+1)!.
\end{aligned}
$$

So mathematical induction completes the proof. $\square$

Lemma 3.2. *For $x > 2$ and $k \in \mathbb{N}$,*

$$
\sum_{n\le x}\log^k n = x\left\{\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n}x\right\} + O_k(\log^k x).
$$

Proof. If $k = 0$ then $\sum_{n\le x}\log^k n = \sum_{n\le x}1 = [\![x]\!] = x - \{x\} = x + O(1)$ since $\{x\} \in [0,1)$ for any $x \in \mathbb{R}$. The case $k = 1$ is a version of the famous Stirling's formula

([6], p. 56). If $k > 1$ then, by partial summation, we have

$$\begin{aligned}
\sum_{n\le x} \log^k n &= [\![x]\!] \log^k x - k \int_1^x [\![u]\!] \log^{k-1} u \, \frac{du}{u} \\
&= (x - \{x\}) \log^k x - k \int_1^x (u - \{u\}) \log^{k-1} u \, \frac{du}{u} \\
&= x \log^k x - k \int_1^x \log^{k-1} u \, du - \{x\} \log^k x + k \int_1^x \{u\} \log^{k-1} u \, \frac{du}{u}.
\end{aligned}$$

Since $\{x\} \in [0, 1)$ for any $x \in \mathbb{R}$, we have

$$\sum_{n\le x} \log^k n = x \log^k x - k \int_1^x \log^{k-1} u \, du - O\left(\log^k x\right) + O\left(k \int_1^x \log^{k-1} u \, \frac{du}{u}\right).$$

In the right-hand side of the above equation the second O-term is

$$\ll \log^k u\Big|_1^x = \log^k x.$$

After this using Lemma 3.1, we can write

$$\begin{aligned}
&\sum_{n\le x} \log^k n \\
&= x \log^k x - k \left[x \left\{ \sum_{n=0}^{k-1} (-1)^n \frac{(k-1)!}{(k-1-n)!} \log^{k-1-n} x \right\} + (-1)^k (k-1)! \right] + O(\log^k x) \\
&= x \log^k x - k \left[x \left\{ \sum_{n=1}^{k} (-1)^{n-1} \frac{(k-1)!}{(k-n)!} \log^{k-n} x \right\} + (-1)^k (k-1)! \right] + O(\log^k x) \\
&= x \log^k x + x \left[\left\{ \sum_{n=1}^{k} (-1)^n \frac{k(k-1)!}{(k-n)!} \log^{k-n} x \right\} \right] - k(-1)^k (k-1)! + O(\log^k x) \\
&= x \log^k x + x \left[\left\{ \sum_{n=1}^{k} (-1)^n \frac{k!}{(k-n)!} \log^{k-n} x \right\} \right] + O_k(\log^k x) \\
&= x \left\{ \sum_{n=0}^{k} (-1)^n \frac{k!}{(k-n)!} \log^{k-n} x \right\} + O_k(\log^k x).
\end{aligned}$$

Hence the proof is completed. $\square$

Lemma 3.3. *For $\Re s > 1$ and $k \in \mathbb{N}$, we have*

$$\int_x^\infty \frac{\log^k u}{u^s}\,du = x^{1-s} \sum_{m=0}^{k} \frac{k!}{(k-m)!(s-1)^{m+1}} \log^{k-m} x. \tag{3.1}$$

Proof. The proof is by induction on k. If $k = 0$, then

$$\int_x^\infty \frac{du}{u^s} = \left[\frac{u^{-s+1}}{-s+1}\right]_x^\infty = \frac{x^{1-s}}{s-1}.$$

Suppose that the lemma is true for some $k \in \mathbb{Z}$, $k \geq 1$, then integration by parts gives

$$\begin{aligned}
\int_x^\infty \frac{\log^{k+1} u}{u^s}\,du &= \left[\frac{\log^{k+1} u}{(-s+1)\,(u^{s-1})}\right]_x^\infty + \frac{k+1}{s-1}\int_x^\infty \frac{\log^k u}{u^s} du \\
&= \frac{\log^{k+1} x}{(s-1)\,(x^{s-1})} + \frac{k+1}{s-1}\left[x^{1-s}\sum_{m=0}^{k} \frac{k!}{(k-m)!(s-1)^{m+1}} \log^{k-m} x\right] \\
&= x^{1-s}\left[\frac{\log^{k+1} x}{s-1} + \frac{k+1}{s-1}\sum_{m=1}^{k+1} \frac{k!}{(k+1-m)!(s-1)^{m}} \log^{k+1-m} x\right] \\
&= x^{1-s}\left[\frac{\log^{k+1} x}{s-1} + \sum_{m=1}^{k+1} \frac{(k+1)!}{(k+1-m)!(s-1)^{m+1}} \log^{k+1-m} x\right] \\
&= x^{1-s}\sum_{m=0}^{k+1} \frac{(k+1)!}{(k+1-m)!(s-1)^{m+1}} \log^{k+1-m} x.
\end{aligned}$$

Thus, induction completes the proof. □

Lemma 3.4. *Let $\Re s > 1$, then for any $k = 0, 1, 2, \ldots$, we have*

$$\begin{aligned}
\sum_{n>x} \frac{\log^k n}{n^s} =& - x^{1-s} \sum_{n=0}^{k} (-1)^n \frac{k!}{(k-n)!}\left[\log^{k-n} x - s\sum_{m=0}^{k-n} \frac{(k-n)!}{(k-n-m)!(s-1)^{m+1}} \log^{k-n-m} x\right] \\
&+ O_k\left(|s|\, x^{-\sigma} \log^k x\right), \quad (x \geq 2).
\end{aligned}$$

Proof. Let $y \in \mathbb{R}$ with $y > x \geq 2$, then by partial summation

$$\sum_{x<n\leq y} \frac{\log^k n}{n^s} = y^{-s}\sum_{n\leq y} \log^k n - x^{-s}\sum_{n\leq x}\log^k n + s\int_x^y \left\{\sum_{n\leq u}\log^k n\right\} u^{-s-1} du.$$

Using Lemma 3.2, we have

$$
\begin{aligned}
\sum_{x<n\le y}\frac{\log^k n}{n^s} =& y^{-s}\left[y\left\{\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n} y\right\}+O_k\left(\log^k y\right)\right]\\
&- x^{-s}\left[x\left\{\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n} x\right\}+O_k\left(\log^k x\right)\right]\\
&+ s\int_x^y\left[u\left\{\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n} u\right\}+O_k\left(\log^k u\right)\right]u^{-s-1}du.
\end{aligned}
$$

Since $\Re s>1$, letting $y\to\infty$ gives

$$
\begin{aligned}
\sum_{n>x}\frac{\log^k n}{n^s} =& -x^{1-s}\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n} x+O_k\left(x^{-\sigma}\log^k x\right)\\
&+\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}s\int_x^\infty\frac{\log^{k-n} u}{u^s}du+O_k\left(|s|\int_x^\infty\frac{\log^k u}{u^{\sigma+1}}\right).
\end{aligned}
$$

We now use Lemma 3.3 and have

$$
\begin{aligned}
\sum_{n>x}\frac{\log^k n}{n^s} =& -x^{1-s}\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\log^{k-n} x+O_k\left(|s|\,x^{-\sigma}\log^k x\right)\\
&+s\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\left\{x^{1-s}\sum_{m=0}^{k-n}\frac{(k-n)!}{(k-n-m)!(s-1)^{m+1}}\log^{k-n-m} x\right\}.
\end{aligned}
$$

Hence, we get the lemma . □

Lemma 3.5. *Let α and β be arbitrarily fixed real numbers with $\alpha<\beta$, then*

$$
\zeta^{(j)}(1-s)=(-1)^j\chi(1-s)\left[1+O\left(\frac{1}{|t|}\right)\right]\sum_{k=0}^{j}\binom{j}{k}\left(\log\frac{|t|}{2\pi}\right)^{j-k}\zeta^{(k)}(s),\quad (|t|\ge 1)
$$

uniformly for $\sigma\in[\alpha,\beta]$.

The proof follows from differentiating the functional equation (2.1) j times and using estimates for $\chi(s)$ and its derivatives. The proof follows the lines of the proof of the formula (3.22). For exact details see the proofs of Lemma 3.7, 3.8 and 3.9. Also this result can be seen in Conrey - Ghosh's paper [9] (See the equation 16).

Lemma 3.6. *Let n be a positive integer then*

$$\left(\frac{d}{ds}\right)^n \frac{\Gamma'}{\Gamma}(s) \ll_{\delta,n} \frac{1}{|s|^n}. \tag{3.2}$$

This is valid for $|s| \geq \frac{1}{2}$ and $|\arg s| < \pi - \delta$, where $\delta > 0$ is arbitrary but fixed.

Proof. Firstly, we have (See [10], p. 47)

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \frac{1}{12s} - \int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{2(s+x)^2}\,dx, \tag{3.3}$$

where B_2 is Bernoulli polynomial defined by

$$B_2(x) = \frac{1}{6} + x^2 - x, \qquad x \in [0,1)\,, \tag{3.4}$$

$$B_2(x) = B_2(x - [\![x]\!]), \qquad \text{for} \quad x \in \mathbb{R}. \tag{3.5}$$

Differentiating the equation (3.3), we have

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} - \frac{1}{12s^2} + \int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{(s+x)^3}\,dx.$$

It is easy to see that

$$\left(\frac{d}{ds}\right)^n \left\{\log s - \frac{1}{2s} - \frac{1}{12s^2}\right\} \ll_n \frac{1}{|s|^n}. \tag{3.6}$$

Now, it is enough to show that

$$\left(\frac{d}{ds}\right)^n \left\{\int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{(s+x)^3}\,dx\right\} \ll_{\delta,n} \frac{1}{|s|^n}, \tag{3.7}$$

to get the result. Firstly, we have

$$\left(\frac{d}{ds}\right)^n \left\{\int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{(s+x)^3}\,dx\right\} = \frac{(-1)^n\,(n+2)!}{2}\int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{(s+x)^{n+3}}\,dx. \tag{3.8}$$

From (3.4) and (3.5), we have

$$|B_2(x)| \le \frac{1}{6} \qquad \forall x \in \mathbb{R}. \tag{3.9}$$

So,

$$\int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{(s+x)^{n+3}}\, dx \;\le\; \frac{1}{6} \int_0^\infty \frac{dx}{|s+x|^{n+3}}. \tag{3.10}$$

From the substitution $x = |s|y$, the inequality (3.10) becomes

$$\int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{(s+x)^{n+3}}\, dx \;\ll\; \frac{1}{|s|^{n+2}} \int_0^\infty \frac{dy}{\left|s|s|^{-1} + y\right|^{n+3}}. \tag{3.11}$$

$\left|s|s|^{-1} + y\right|$ represents the distance from the point $-y$ to a point of the unit circle. Since $|s| \ge \frac{1}{2}$ and $|\arg s| < \pi - \delta$, it is easy to show that geometrically this distance at least $|e^{i(\pi-\delta)} + y|$. So

$$\int_0^\infty \frac{dy}{\left|s|s|^{-1} + y\right|^{n+3}} \ll_{\delta,n} 1 \tag{3.12}$$

Combining (3.8), (3.11) and (3.12), we have

$$\left(\frac{d}{ds}\right)^n \left\{\int_0^\infty \frac{B_2\left(x - [\![x]\!]\right)}{(s+x)^3}\, dx\right\} \ll_{\delta,n} \frac{1}{|s|^{n+2}} \ll_{\delta,n} \frac{1}{|s|^n}. \tag{3.13}$$

Hence, we are done. □

Corollary 3.1. *Let n be a positive integer and $s = \sigma + it$ then, as $t \to \infty$*

$$\left(\frac{d}{ds}\right)^n \frac{\Gamma'}{\Gamma}(s) \ll_{\alpha,\beta,n} \frac{1}{|t|^n}, \tag{3.14}$$

uniformly in $\alpha \le \sigma \le \beta$, for any fixed real numbers α and β.

The result obviously follows from the above lemma.

Lemma 3.7. *Let ψ be primitive character modulo $q \geq 3$, then for large t*

$$\frac{\chi'}{\chi}(s,\psi) = -\log\frac{q|t|}{2\pi} + O_{\alpha,\beta}\left(\frac{1}{|t|}\right) \tag{3.15}$$

uniformly in $\alpha \leq \sigma \leq \beta$, for any fixed real numbers α and β.

Proof. Taking logarithmic derivative of the equation (2.21) gives

$$\frac{\chi'}{\chi}(s,\psi) \;=\; \log\pi \;-\log\, q \;-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left[\frac{1}{2}\left(\mathfrak{a}+1-s\right)\right] - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left[\frac{1}{2}\left(\mathfrak{a}+s\right)\right] \tag{3.16}$$

Using the formula (2.5) in the above expression, we have

$$\frac{\chi'}{\chi}(s,\psi) \;=\; -\log\,\frac{q}{\pi} \;-\frac{1}{2}\log\left(\frac{\mathfrak{a}+1-s}{2}\right) - \frac{1}{2}\log\left(\frac{\mathfrak{a}+s}{2}\right) \;+\, O\left(\frac{1}{|s|}\right), \tag{3.17}$$

where $|t| \geq 1$. In a fixed strip $\alpha \leq \sigma \leq \beta$, as $t \to \infty$, we can neglect the variable σ and (3.17) becomes

$$\frac{\chi'}{\chi}(s,\psi) \;=\; -\log\,\frac{q}{\pi} \;-\frac{1}{2}\log\left(\frac{-it}{2}\right) - \frac{1}{2}\log\left(\frac{it}{2}\right) \;+\, O\left(\frac{1}{|t|}\right). \tag{3.18}$$

Using

$$\log(it) = \log|t| + i\,\mathrm{sgn}(t)\frac{\pi}{2}$$

in (3.18), we get the result. □

Lemma 3.8. *Let ψ be primitive character modulo $q \geq 3$ and $r \in \mathbb{Z}^+$. As $t \to \infty$,*

$$\chi^{(r)}(s,\psi) = \chi(s,\psi)\left(-\log\frac{q|t|}{2\pi}\right)^r\left[1 + O_{\alpha,\beta,r}\left(\frac{1}{|t|\log q\tau}\right)\right] \tag{3.19}$$

uniformly in $\alpha \leq \sigma \leq \beta$, for any fixed real numbers α and β.

Proof. We proceed by induction on r. The case $r = 1$ is true by the above lemma. Assume the statement holds for each $i < r$, $r \in \mathbb{Z}$ with $r > 1$. Taking $(r-1)^{th}$

derivative of the identity

$$\chi'(s,\psi) = \chi(s,\psi)\,\frac{\chi'}{\chi}(s,\psi),$$

we obtain

$$\chi^{(r)}(s,\psi) = \sum_{i=0}^{r-1}\binom{r-1}{i}\chi^{(r-1-i)}(s,\psi)\left(\frac{\chi'}{\chi}\right)^{(i)}(s,\psi).$$

By the induction assumption, we have

$$\begin{aligned}\chi^{(r)}(s,\psi) =& \chi(s,\psi)\left[1+O_{\alpha,\beta,r}\left(\frac{1}{|t|\log q\tau}\right)\right]\\ &\sum_{i=0}^{r-1}\binom{r-1}{i}\left(-\log\frac{q|t|}{2\pi}\right)^{r-1-i}\left(\frac{\chi'}{\chi}\right)^{(i)}(s,\psi).\end{aligned} \tag{3.20}$$

Combining (3.16) and Corollary 3.1, we have

$$\left(\frac{d}{ds}\right)^{i}\frac{\chi'}{\chi}(s,\psi) \ll_{\alpha,\beta,i} \frac{1}{|t|^{i}}, \tag{3.21}$$

for $i \geq 1$. Using Lemma 3.7 and (3.21) in (3.20), the result follows. □

Lemma 3.9. *Let ψ be primitive character modulo $q \geq 3$, then for $j = 0, 1, 2, ...$ and large t we have*

$$\begin{aligned}L^{(j)}(s,\psi) = (-1)^{j}\,\chi(s,\psi)&\left\{\sum_{k=0}^{j}\binom{j}{k}\left(\log\frac{q|t|}{2\pi}\right)^{j-k}L^{(k)}\left(1-s,\overline{\psi}\right)\right\}\\ &\left[1+O_{\alpha,\beta,j}\left(\frac{1}{|t|\log q\tau}\right)\right]\end{aligned} \tag{3.22}$$

uniformly in $\alpha \leq \sigma \leq \beta$, for any fixed real numbers α and β.

Proof. Differentiating j times the equation (2.22), we find

$$L^{(j)}(s,\psi) = \sum_{k=0}^{j}\binom{j}{k}\chi^{(j-k)}(s,\psi)(-1)^{k}L^{(k)}\left(1-s,\overline{\psi}\right).$$

By Lemma 3.8,

$$L^{(j)}(s,\psi) = \sum_{k=0}^{j} \binom{j}{k} \chi(s,\psi) \left(-\log\frac{q|t|}{2\pi}\right)^{j-k} \left[1 + O_{\alpha,\beta,j-k}\left(\frac{1}{|t|\log q\tau}\right)\right]$$
$$(-1)^k L^{(k)}\left(1-s,\overline{\psi}\right).$$

Then the result easily follows. $\square$

Lemma 3.10. *For $m = 0,1,2,...,$ A large, and $A < r \le B \le 2A$,*

$$\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt$$
$$= (2\pi)^{1-a} r^a\, e^{-ir+\frac{\pi i}{4}}\left(\log\frac{r}{2\pi}\right)^m + E(r,A,B)\,(\log A)^m,$$

while for $r \le A$ or $r > B$,

$$\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^m dt = E(r,A,B)\,(\log A)^m,$$

where

$$E(r,A,B) = O\left(A^{a-\frac{1}{2}}\right) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r| + A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-r| + B^{\frac{1}{2}}}\right). \tag{3.23}$$

Lemma 3.11. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that for any $\epsilon > 0$, $b_n \ll_\epsilon n^\epsilon$. Let $a > 1$ and m be a nonnegative integer, then for T sufficiently large,*

$$\frac{1}{2\pi}\int_1^T \chi(1-a-it)\left(\log\frac{t}{2\pi}\right)^m \sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\,dt = \sum_{1\le n\le T/2\pi} b_n \log^m n + O\left(T^{a-\frac{1}{2}}\log^m T\right)$$

Lemmas 3.10 and 3.11 have been proved in Gonek's paper [8]. We want to extend Lemma 3.11 to Dirichlet's L-functions to estimate integrals containing $\chi(s,\psi)$.

Lemma 3.12. *For $m = 0, 1, 2, ...,$ A large, and $A < r \leq B \leq 2A$,*

$$\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$= (2\pi)^{1-a}r^a\, e^{-ir+\frac{\pi i}{4}}\left(\log\frac{qr}{2\pi}\right)^m + E(r,A,B)\,(\log qA)^m,$$

while for $r \leq A$ or $r > B$,

$$\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt = E(r,A,B)\,(\log qA)^m,$$

Proof. From the binomial expansion

$$\left(\log\frac{qt}{2\pi}\right)^m = \sum_{i=0}^m \binom{m}{i}(\log q)^{m-i}\left(\log\frac{t}{2\pi}\right)^i,$$

it follows that

$$\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$= \sum_{i=0}^m \binom{m}{i}(\log q)^{m-i}\int_A^B \exp\left[it\log\frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{t}{2\pi}\right)^i dt,$$

then the result follows from Lemma 3.10. □

Lemma 3.13. *For an arbitrarily fixed number $a > 1$, we have*

$$\sum_{n=1}^\infty \frac{1}{n^a\left(|C-2\pi n| + \sqrt{C}\right)} \ll_a \frac{1}{C}, \quad (C \geq 2). \tag{3.24}$$

Proof. Firstly, we divide the infinite sum in (3.24) into five parts,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^a\left(|C-2\pi n|+\sqrt{C}\right)} &= \sum_{1\le n<\frac{C}{4\pi}}+\sum_{\frac{C}{4\pi}\le n<\frac{C-\sqrt{C}}{2\pi}}+\sum_{\frac{C-\sqrt{C}}{2\pi}\le n<\frac{C+\sqrt{C}}{2\pi}}\\ &\quad+\sum_{\frac{C+\sqrt{C}}{2\pi}\le n<\frac{2C}{2\pi}}+\sum_{n>\frac{2C}{2\pi}}\left\{\frac{1}{n^a\left(|C-2\pi n|+\sqrt{C}\right)}\right\}\\ &= S_1+S_2+S_3+S_4+S_5,\text{say.}\end{aligned}$$

If $1\le n<\frac{C}{4\pi}$, then $|C-2\pi n|+\sqrt{C}\gg C$ and so

$$S_1\ll\frac{1}{C}\sum_{1\le n<\frac{C}{4\pi}}\frac{1}{n^a}.$$

By integral test, the above sum is

$$\le 1+\int_1^{\frac{C}{4\pi}}\frac{dx}{x^a}.$$

Since $a>1$, the above integral is $\ll 1$. Hence,

$$S_1\ll\frac{1}{C}. \tag{3.25}$$

If $\frac{C}{4\pi}\le n<\frac{2C}{4\pi}$, then

$$\frac{1}{n^a}\ll\frac{1}{C^a}. \tag{3.26}$$

So,

$$S_2\ll\frac{1}{C^a}\sum_{\frac{C}{4\pi}\le n<\frac{C-\sqrt{C}}{2\pi}}\frac{1}{|C-2\pi n|+\sqrt{C}}\ll\frac{1}{C^a}\sum_{\frac{C}{4\pi}\le n<\frac{C-\sqrt{C}}{2\pi}}\frac{1}{\frac{C}{2\pi}-n}\ll\frac{1}{C^a}\sum_{\frac{\sqrt{C}}{2\pi}\le m<\frac{C}{4\pi}}\frac{1}{m}.$$

Noting that

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma_0+O\left(\frac{1}{x}\right), \tag{3.27}$$

we can conclude

$$S_2\ll\frac{\log C}{C^a}\ll\frac{1}{C}, \tag{3.28}$$

since $\log C/C^b\to 0$ as $C\to\infty$ for any $b>0$. We shall estimate S_3 and S_4. By (3.26), we have

$$S_3\ll\frac{1}{C^a}\sum_{\frac{C-\sqrt{C}}{2\pi}\leq n<\frac{C+\sqrt{C}}{2\pi}}\frac{1}{|C-2\pi n|+\sqrt{C}}\ll\frac{1}{C^a\sqrt{C}}\sum_{\frac{C-\sqrt{C}}{2\pi}\leq n<\frac{C+\sqrt{C}}{2\pi}}1\ll\frac{1}{C^a}\ll\frac{1}{C}, \tag{3.29}$$

similarly, by (3.26),

$$S_4\ll\frac{1}{C^a}\sum_{\frac{C+\sqrt{C}}{2\pi}\leq n<\frac{2C}{2\pi}}\frac{1}{|C-2\pi n|+\sqrt{C}}\ll\frac{1}{C^a}\sum_{\frac{C+\sqrt{C}}{2\pi}\leq n<\frac{2C}{2\pi}}\frac{1}{n-\frac{C}{2\pi}}\ll\frac{1}{C^a}\sum_{\frac{\sqrt{C}}{2\pi}\leq m<\frac{C}{2\pi}}\frac{1}{m}.$$

By (3.27), the sum in the last statement is $\ll\log C$ and so

$$S_4\ll\frac{\log C}{C^a}\ll\frac{1}{C}. \tag{3.30}$$

For the last case $n\geq\frac{2C}{2\pi}$, we have $|C-2\pi n|+\sqrt{C}\gg C$ and so

$$S_5\ll\frac{1}{C}\sum_{n\geq\frac{2C}{2\pi}}\frac{1}{n^a}\ll\frac{1}{C}\sum_{n=0}^{\infty}\frac{1}{n^a}\ll\frac{1}{C} \tag{3.31}$$

since $a>1$. Finally, combining (3.25), (3.28), (3.29), (3.30) and (3.31), we get the result. □

Lemma 3.14. *Let $E(r,A,B)$ be as in (3.23), where A is large and $A<B\leq 2A$. Assume $\{b_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $b_n\ll_\epsilon n^\epsilon$ for any $\epsilon>0$.*

Then for any fixed $a > 1$,

$$\sum_{n=1}^{\infty} \frac{b_n}{n^a} E\left(\frac{2\pi n}{q}, A, B\right) \ll A^{a-\frac{1}{2}}$$

uniformly for $q \geq 1$, where the implied constant depends on a and ϵ.

Proof. Choose ϵ so that $0 < \epsilon < a - 1$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{b_n}{n^a} E\left(\frac{2\pi n}{q}, A, B\right) \ll & \sum_{n=1}^{\infty} n^{-a+\epsilon} E\left(\frac{2\pi n}{q}, A, B\right) \\ \ll & A^{a-\frac{1}{2}} \sum_{n=1}^{\infty} n^{-a+\epsilon} + A^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\epsilon}\left(\left|A - \frac{2\pi n}{q}\right| + \sqrt{A}\right)} \\ & + B^{a+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{a-\epsilon}\left(\left|B - \frac{2\pi n}{q}\right| + \sqrt{B}\right)}.\end{aligned}$$

The proof of the lemma is completed by noting that

$$\sum_{n=1}^{\infty} n^{-a+\epsilon} \ll 1$$

and, by Lemma 3.13,

$$\sum_{n=1}^{\infty} \frac{1}{n^{a-\epsilon}\left(\left|C - \frac{2\pi n}{q}\right| + \sqrt{C}\right)} \ll \sum_{n=1}^{\infty} \frac{q}{n^{a-\epsilon}\left(\left|Cq - 2\pi n\right| + \sqrt{Cq}\right)} \ll \frac{1}{C}.$$

□

Lemma 3.15. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $b_n \ll_\epsilon n^\epsilon$ for any $\epsilon > 0$ and ψ a primitive character modulo $q \in \mathbb{Z}^+$. Let $a > 1$ and m a non-negative integer. Then for T sufficiently large,*

$$\begin{aligned}\frac{1}{2\pi} \int_1^T \left\{\sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}}\right\} \chi(1 - a - it, \psi) \left(\log \frac{qt}{2\pi}\right)^m dt = \\ \frac{\tau(\psi)}{q} \sum_{1 \leq n \leq \frac{qT}{2\pi}} b_n\, e\left(\frac{-n}{q}\right) \log^m n + O\left((qT)^{a-\frac{1}{2}} (\log qT)^m\right).\end{aligned}$$

Here, the implicit constant of "O" depends on ϵ, a and m.

Proof. By (2.23), we have

$$\frac{1}{2\pi}\int_{\frac{T}{2}}^{T}\left\{\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right\}\chi(1-a-it,\psi)\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$=\frac{\tau(\psi)\exp\left\{\frac{-i\pi}{4}\right\}}{2\pi\, q^{\,1-a}}\int_{\frac{T}{2}}^{T}\left\{\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right\}\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left\{it\log\frac{qt}{2\pi e}\right\}\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$+O\left(\frac{|\tau(\psi)|}{q^{\,1-a}}T^{a-\frac{1}{2}}(\log qT)^m\int_{\frac{T}{2}}^{T}\left\{\sum_{n=1}^{\infty}\frac{|b_n|}{n^a}\right\}\frac{dt}{t}\right). \tag{3.32}$$

Since $b_n \ll_\epsilon n^\epsilon$ and $a > 1$,

$$\sum_{n=1}^{\infty}\frac{|b_n|}{n^a} \ll 1.$$

Thus, the error term in (3.32) is

$$\ll (qT)^{a-\frac{1}{2}}(\log qT)^m$$

by noting that $|\tau(\psi)| = \sqrt{q}$ for any primitive character ψ. Then, the right hand side of (3.32) becomes

$$\frac{\tau(\psi)\exp\left\{\frac{-i\pi}{4}\right\}}{2\pi\, q^{\,1-a}}\sum_{n=1}^{\infty}\frac{b_n}{n^a}\int_{\frac{T}{2}}^{T}\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left\{it\log\frac{qt}{2\pi ne}\right\}\left(\log\frac{qt}{2\pi}\right)^m dt$$
$$+O\left((qT)^{a-\frac{1}{2}}(\log qT)^m\right), \tag{3.33}$$

the inversion of summation and integration being justified by absolute convergence. Now the integral in (3.33) is of the form estimable by Lemma 3.12 with $A = \frac{T}{2}$, $B = T$

and $r = \frac{2\pi n}{q}$. Thus, (3.33) is equal to

$$\frac{\tau(\psi)}{q} \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} b_n \, e\left(\frac{-n}{q}\right) \log^m n + O\left((qT)^{a-\frac{1}{2}} (\log qT)^m\right)$$
$$+ \frac{\tau(\psi) \exp\left\{\frac{-i\pi}{4}\right\}}{2\pi \, q^{1-a}} \left(\log \frac{qT}{2}\right)^m \sum_{\substack{n=1 \\ n \notin \left(\frac{qT}{4\pi}, \frac{qT}{2\pi}\right]}}^{\infty} b_n \, n^{-a} E\left(\frac{2\pi n}{q}, \frac{T}{2}, T\right).$$

By Lemma 3.14, we can conclude that

$$\frac{1}{2\pi} \int_{\frac{T}{2}}^{T} \left\{ \sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}} \right\} \chi(1-a-it, \psi) \left(\log \frac{qt}{2\pi}\right)^m dt$$
$$= \frac{\tau(\psi)}{q} \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} b_n \, e\left(\frac{-n}{q}\right) \log^m n + O\left((qT)^{a-\frac{1}{2}} (\log qT)^m\right) \tag{3.34}$$

for $T \ge T_0$, say. Now let l be the unique integer such that $T_0 \le \frac{T}{2^l} < 2T_0$. Adding the result of (3.34) for the ranges $\left[\frac{T}{2^j}, \frac{T}{2^{j-1}}\right]$ $(j = 1, 2, ..., l)$, we find that

$$\frac{1}{2\pi} \int_{\frac{T}{2^l}}^{T} \left\{ \sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}} \right\} \chi(1-a-it, \psi) \left(\log \frac{qt}{2\pi}\right)^m dt$$
$$= \frac{\tau(\psi)}{q} \sum_{\frac{qT}{2^{l+1}\pi} < n \le \frac{qT}{2\pi}} b_n \, e\left(\frac{-n}{q}\right) \log^m n + O_\epsilon\left((qT)^{a-\frac{1}{2}} (\log qT)^m\right). \tag{3.35}$$

We have

$$\frac{1}{2\pi} \int_{1}^{\frac{T}{2^l}} \left\{ \sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}} \right\} \chi(1-a-it, \psi) \left(\log \frac{qt}{2\pi}\right)^m dt \ll q^{a-\frac{1}{2}} \log^m q, \tag{3.36}$$

since the length of integration is $< T/2^l < 2T_0$, i.e. $\ll 1$, and, by (2.23),

$$|\, \chi(s, \psi) \,| \asymp \left(\frac{q|t|}{2\pi}\right)^{\frac{1}{2}-\sigma}.$$

Also, we have

$$\frac{\tau(\psi)}{q} \sum_{1 \le n \le \frac{qT}{2^{l+1}\pi}} b_n \, e\left(\frac{-n}{q}\right) \log^m n \ll q^{\frac{1}{2}+\epsilon}. \tag{3.37}$$

This follows from trivial calculation and the fact that $|\tau(\psi)| = \sqrt{q}$. Finally, combining (3.35), (3.36) and (3.37) gives the lemma. □

Lemma 3.16. *Let k be a nonnegative integer. For $x \ge 2$, we have*

$$\sum_{n \le x} \frac{\Lambda(n)}{n} \log^k n \; = \; \frac{\log^{k+1} x}{k+1} \; + \, O\left(\log^k x\right),$$

where Λ is the von Mangoldt Lambda function which is defined by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{ for some } k \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case $k = 0$ follows from (3.14.2) of [7]. If $k \ge 1$ then by partial summation formula, we have

$$\sum_{n \le x} \frac{\Lambda(n)}{n} \log^k n \; = \left\{ \sum_{n \le x} \frac{\Lambda(n)}{n} \right\} \log^k x - k \int_1^x \left\{ \sum_{n \le u} \frac{\Lambda(n)}{n} \right\} \log^{k-1} u \, \frac{du}{u}.$$

Using the result for $k = 0$, we can write

$$\sum_{n \le x} \frac{\Lambda(n)}{n} \log^k n \; = \log^{k+1} x + O\left(\log^k x\right) - k \int_1^x \log^k u \frac{du}{u} + O\left(k \int_1^x \log^{k-1} u \frac{du}{u} \right).$$

Then the proof is completed by noting that

$$\int_1^x \log^m u \frac{du}{u} = \left. \frac{\log^{m+1} u}{m+1} \right|_1^x = \frac{\log^{m+1} x}{m+1} \qquad (m = 0, 1, 2...).$$

□

Lemma 3.17. *Let $\zeta^{(k)}(s) \cdot \dfrac{\zeta'}{\zeta}(s) \; = \displaystyle\sum_{n=1}^{\infty} \frac{b_k(n)}{n^s}$ for $\Re s > 1$, where $k \in \mathbb{N}$. Then for*

$x \geq 2$,

$$\sum_{n \leq x} b_k(n) = \frac{(-1)^{k+1}}{k+1} x \log^{k+1} x \,+\, O_{\,k}\left(x \log^k x\right).$$

Proof. Firstly, we note that $\zeta^{(k)}(s) = \sum_{n=1}^{\infty} \frac{(-1)^k \log^k n}{n^s}$ and $\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ for $k = 0, 1, 2, \ldots$ and $\Re s > 1$. Then, by multiplication of Dirichlet series, we can write

$b_k(n) = \sum_{dr=n} (-1)^{k+1} \Lambda(d) \log^k r$. So,

$$\begin{aligned}\sum_{n \leq x} b_k(n) &= \sum_{n \leq x} \sum_{dr=n} (-1)^{k+1} \Lambda(d) \log^k r = (-1)^{k+1} \sum_{dr \leq x} \Lambda(d) \log^k r \\ &= (-1)^{k+1} \sum_{d \leq x} \Lambda(d) \sum_{r \leq \frac{x}{d}} \log^k r.\end{aligned}$$

By Lemma 3.2, we get

$$\sum_{n \leq x} b_k(n) = (-1)^{k+1} x \sum_{d \leq x} \frac{\Lambda(d)}{d} \sum_{n=0}^{k} (-1)^n \frac{k!}{(k-n)!} \left(\log \frac{x}{d}\right)^{k-n} + O_k\left(\sum_{d \leq x} \Lambda(d) \left(\log \frac{x}{d}\right)^k\right)$$

The above error term is

$$\ll_k \log^k x \sum_{d \leq x} \Lambda(d) \ll_k x \log^k x.$$

The last asymptotic inequality follows from the prime number theorem which says that

$$\sum_{d \leq x} \Lambda(d) \sim x.$$

Noting that

$$\left(\log \frac{x}{d}\right)^{k-n} = \sum_{h=0}^{k-n} \binom{k-n}{h} (-1)^h \left(\log x\right)^{k-n-h} \left(\log d\right)^h,$$

we have

$$\sum_{n\leq x} b_k(n) =(-1)^{k+1}x\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\sum_{h=0}^{k-n}(-1)^h\binom{k-n}{h}(\log x)^{k-n-h}\sum_{d\leq x}\frac{\Lambda(d)}{d}\log^h d$$
$$+O_k\left(x\log^k x\right).$$

We now use Lemma 3.16 to obtain

$$\begin{aligned}\sum_{n\leq x} b_k(n) \quad = \quad & (-1)^{k+1}x\sum_{n=0}^{k}(-1)^n\frac{k!}{(k-n)!}\sum_{h=0}^{k-n}(-1)^h\binom{k-n}{h}(\log x)^{k-n-h}\Bigg[\frac{\log^{h+1} x}{h+1}+ \\ & O\left(\log^h x\right)\Bigg] + O_k\left(x\log^k x\right) \\ = \quad & (-1)^{k+1}\,x\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\log^{k+1} x}{h+1} \\ & +(-1)^{k+1}\,x\sum_{n=1}^{k}(-1)^n\frac{k!}{(k-n)!}\sum_{h=0}^{k-n}(-1)^h\binom{k-n}{h}\frac{\log^{k-n+1} x}{h+1} \\ & +O\left(x\sum_{n=0}^{k}\frac{k!}{(k-n)!}\sum_{h=0}^{k-n}\binom{k-n}{h}\log^{k-n} x\right)+O_k\left(x\log^k x\right).\end{aligned}$$

The second and third parts of the right hand side of the above equation is trivially

$$\ll_k x\log^k x,$$

and this gives

$$\sum_{n\leq x} b_k(n) = (-1)^{k+1}\,x\,(\log x)^{k+1}\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{1}{h+1}+O_k\left(x\log^k x\right).$$

We now make some calculations on the coefficient of the main term of the above asymptotic equality;

$$
\begin{aligned}
(-1)^{k+1}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^h}{h+1} &= \frac{(-1)^k}{k+1}\sum_{h=0}^{k}\binom{k+1}{h+1}(-1)^{h+1} \\
&= \frac{(-1)^k}{k+1}\sum_{\nu=1}^{k+1}\binom{k+1}{\nu}(-1)^{\nu} \\
&= \frac{(-1)^k}{k+1}\left\{(1-1)^{k+1}-1\right\} = \frac{(-1)^{k+1}}{k+1},
\end{aligned} \tag{3.38}
$$

and this completes the proof. □

4. THE ORDER OF A DIRICHLET L-FUNCTION AND ITS DERIVATIVES

We need an estimate for $L(s,\psi)$ in the critical strip, $0 \le \sigma \le 1$, where ψ is any primitive Dirichlet character. We first give an integral representation for $L(s,\psi)$, ([6], p. 82),

$$L(s,\psi) = s\int_1^\infty S\left(x,\psi\right)x^{-s-1}dx \quad (\sigma > 0),$$

where $S(x,\psi) := \sum_{n\le x}\psi(n)$. Since $|S(x,\psi)| \le q$, it implies that

$$L(s,\psi) \ll q\tau \qquad \left(\sigma_0 \ge \sigma \ge \frac{1}{2}\right), \tag{4.1}$$

where σ_0 is an arbitrary number $> 1/2$. For $\sigma < \frac{1}{2}$, corresponding results follow from the functional equation (2.22). In any fixed strip $\alpha \le \sigma \le \beta$, by (2.23), we have

$$|\chi\left(s,\psi\right)| \asymp \left(\frac{q|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} \quad (|t| \ge 1). \tag{4.2}$$

Hence

$$L(s,\psi) \ll (q\tau)^{\frac{3}{2}-\sigma} \tag{4.3}$$

for $\sigma_1 \le \sigma < \frac{1}{2}$ and $|t| \ge 1$, where σ_1 is an arbitrary number $< 1/2$.

Now, we'll show that (4.3) is also valid in the bounded region $\sigma_1 \le \sigma < \frac{1}{2}$, $|t| \le 1$. Combining (2.20) with the basic fact $|\tau(\psi)| = \sqrt{q}$, we get

$$\chi(s,\psi) \ll q^{\frac{1}{2}-\sigma}$$

in the region considered. Using this in (2.22), we have

$$L(s,\psi) \ll q^{\frac{1}{2}-\sigma} |L\left(1-s,\bar{\psi}\right)| \tag{4.4}$$

for $\sigma_1 \le \sigma < \frac{1}{2}$, $|t| \le 1$. By (4.1), (4.4) becomes

$$L(s,\psi) \ll q^{\frac{3}{2}-\sigma} \tag{4.5}$$

for $\sigma_1 \le \sigma < \frac{1}{2}$, $|t| \le 1$. It easily follows from (4.3) and (4.5) that

$$L(s,\psi) \ll (q\tau)^{\frac{3}{2}-\sigma} \quad \left(\sigma_1 \le \sigma < \frac{1}{2}\right). \tag{4.6}$$

In addition to (4.1), in any in any half-strip $\sigma \ge 1+\delta$, $\delta > 0$, we have

$$|L(s,\psi)| = \left|\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s}\right| \le \zeta(1+\delta) \ll_\delta 1.$$

Now, we'll prove the following theorem that gives a better estimate near the line $\sigma = 1$.

Proposition 4.1. *Let ψ be a non-principal character modulo $q \ge 3$, then for $j = 0, 1, 2, \ldots$ we have*

$$L^{(j)}(s,\psi) \ll_{j,A} (\log q\tau)^{j+1}, \tag{4.7}$$

uniformly for $1 - \frac{A}{\log q\tau} \le \sigma \le 2$, where A is an arbitrarily large positive constant.

Proof. Let $R(u,\psi) := \sum_{m\le u} \psi(m)\, \log^j m$. Then

$$R(u,\psi) = \sum_{m\le u} [S(m,\psi) - S(m-1,\psi)] \log^j m$$
$$= \sum_{m\le [\![u]\!]-1} S(m,\psi) \left[\log^j m - \log^j (m+1)\right] + S\left([\![u]\!],\psi\right) \log^j \left([\![u]\!]\right).$$

Using the trivial estimate $S(u,\psi) \ll q$, we have

$$R(u,\psi) \ll q \sum_{m\le u} \left|\log^j m - \log^j (m+1)\right| + q \log^j u.$$

Since $\log x$ is a strictly increasing function, we have

$$R(u,\psi) \ll\; q \sum_{m\le u} \left(\log^j (m+1) - \log^j m\right) + q\log^j u \ll\; q\log^j u. \tag{4.8}$$

For $1 - \frac{A}{\log q\tau} \le \sigma \le 2$, we can write

$$\begin{aligned} L^{(j)}(s,\psi) &= (-1)^j \sum_{n=1}^{\infty} \frac{\psi(n)\log^j n}{n^s} \\ &= (-1)^j \sum_{n\le q\tau} \frac{\psi(n)\log^j n}{n^s} + (-1)^j \sum_{n> q\tau} \frac{\psi(n)\log^j n}{n^s} \\ &= O\left(\sum_{n\le q\tau} \frac{\log^j n}{n^{1-\frac{A}{\log q\tau}}}\right) + (-1)^j \sum_{n>q\tau} R(n,\psi) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) \\ &\quad - (-1)^j\, R\left([\![q\tau]\!],\psi\right) \frac{1}{\left([\![q\tau]\!]+1\right)^s}. \end{aligned} \tag{4.9}$$

By (3.27), we have

$$\sum_{n\le q\tau} \frac{\log^j n}{n^{1-\frac{A}{\log q\tau}}} \ll_A \log^j q\tau \sum_{n\le q\tau} \frac{1}{n} \ll_A \log^{j+1} q\tau. \tag{4.10}$$

Using (4.8) and (4.10), (4.9) becomes

$$\begin{aligned} L^{(j)}(s,\psi) =& (-1)^j \sum_{n>q\tau} R(n,\psi)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + O_A\left(\log^{j+1} q\tau\right) \\ =& (-1)^j s \sum_{n>q\tau} R(n,\psi) \int_n^{n+1} \frac{du}{u^{s+1}} + O_A\left(\log^{j+1} q\tau\right). \end{aligned} \tag{4.11}$$

By (4.8),

$$\begin{aligned} L^{(j)}(s,\psi) &= O\left(q\tau \sum_{n>q\tau} \log^j n \int_n^{n+1} \frac{du}{u^{\sigma+1}}\right) + O_A\left(\log^{j+1} q\tau\right) \\ &= O\left(q\tau \sum_{n>q\tau} \frac{\log^j n}{n^{2-\frac{A}{\log q\tau}}}\right) + O_A\left(\log^{j+1} q\tau\right). \end{aligned} \tag{4.12}$$

By Lemma 3.4, the sum in (4.12) is

$$\ll_{j,A} (q\tau)^{-1+\frac{A}{\log q\tau}} \log^j q\tau \ll_{j,A} \frac{\log^j q\tau}{q\tau}. \tag{4.13}$$

Combining (4.12) and (4.13), we get the result. □

We can improve the estimate for $L(s,\psi)$ over the region $\sigma \le 0$. Proposition 4.1 implies that

$$L(s,\psi) \ll_\epsilon (q\tau)^\epsilon \tag{4.14}$$

for $\sigma \ge 1$. Combining this with (4.2) and (2.22), we obtain

$$L(s,\psi) \ll_\epsilon (q\tau)^{\frac{1}{2}-\sigma+\epsilon} \qquad (\sigma \le 0,\ |t| \ge 1). \tag{4.15}$$

By (4.4) and (4.14), the above estimate is valid in the region $\sigma \le 0$, $|t| \le 1$; so we conclude that

$$L(s,\psi) \ll_\epsilon (q\tau)^{\frac{1}{2}-\sigma+\epsilon} \qquad (\sigma \le 0,\ t \in \mathbb{R}). \tag{4.16}$$

Proposition 4.2. *Let ψ be a primitive character modulo $q \geq 3$, then we have*

$$L(s,\psi) \ll_\epsilon q^{\frac{1}{2}(1-\sigma)+\epsilon}, \tag{4.17}$$

uniformly for $0 \leq \sigma \leq 1$, $|t| \leq 1$.

Proof. It follows from (4.7) that

$$L(\sigma+it,\psi) \ll_\epsilon q^\epsilon \tag{4.18}$$

uniformly for $1 - \frac{1}{\log q} \leq \sigma \leq 1$, $|t| \leq 1$. In the right half-plane, we have

$$\begin{aligned}
L(\sigma+it,\psi) &= \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \sum_{n \leq \sqrt{q}\log q} \frac{\psi(n)}{n^s} + \sum_{n > \sqrt{q}\log q} \frac{\psi(n)}{n^s} \\
&= O\left(\sum_{n \leq \sqrt{q}\log q} \frac{1}{n^\sigma}\right) + \sum_{n > \sqrt{q}\log q} S(n,\psi)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) - \frac{S\left([\![\sqrt{q}\log q]\!],\psi\right)}{\left([\![\sqrt{q}\log q]\!]+1\right)^s} \\
&= S_1 + S_2 + S_3, \text{ say.}
\end{aligned}$$

By integral test,

$$S_1 \ll 1 + \int_2^{\sqrt{q}\log q} \frac{dx}{x^\sigma} \ll \left.\frac{x^{1-\sigma}}{1-\sigma}\right|_2^{\sqrt{q}\log q} \ll q^{\frac{1}{2}(1-\sigma)+\epsilon} \tag{4.19}$$

for $1 - \frac{1}{\log q} \geq \sigma \geq \frac{1}{2}$. To treat S_2 we use the Pólya-Vinogradov inequality (See Theorem 9.18 of [4])

$$\sum_{N < n \leq M} \psi(n) \ll \sqrt{q}\log q$$

for any primitive character ψ. (This inequality holds for any non-principal Dirichlet's character.) In the bounded region $\frac{1}{2} \le \sigma \le 1 - \frac{1}{\log q}$, $|t| \le 1$, we have

$$\begin{aligned} S_2 &\ll \sqrt{q}\log q \sum_{n > \sqrt{q}\log q} |s| \int_n^{n+1} \frac{dx}{x^{\sigma+1}} \\ &\ll \sqrt{q}\log q \int_{\sqrt{q}\log q}^{\infty} \frac{dx}{x^{\sigma+1}} \ll q^{\frac{1}{2}(1-\sigma)+\epsilon}. \end{aligned} \tag{4.20}$$

Finally, it follows from the Pólya-Vinogradov inequality that

$$S_3 \ll q^{\frac{1}{2}(1-\sigma)+\epsilon}. \tag{4.21}$$

Combining (4.18), (4.19), (4.20) and (4.21) completes the case $\frac{1}{2} \le \sigma \le 1$, $|t| \le 1$. The other case $0 \le \sigma < \frac{1}{2}$, $|t| \le 1$ easily follows from (4.4). □

Summarizing our results, $|L(s,\psi)|$ has the following upper-bounds:

$$\begin{array}{llll} 1. & L(\sigma+it,\psi) & \ll q\tau & (\sigma \ge \frac{1}{2}, t \in \mathbb{R}) \\ 2. & L(\sigma+it,\psi) & \ll (q\tau)^{\frac{3}{2}-\sigma} & (\sigma < \frac{1}{2}, t \in \mathbb{R}) \\ 3. & L(\sigma+it,\psi) & \ll_\epsilon (q\tau)^{\frac{1}{2}-\sigma+\epsilon} & (\sigma \le 0, t \in \mathbb{R}) \\ 4. & L(\sigma+it,\psi) & \ll_\epsilon (q\tau)^{\epsilon} & (\sigma \ge 1, t \in \mathbb{R}) \\ 5. & L(\sigma+it,\psi) & \ll_\epsilon q^{\frac{1}{2}(1-\sigma)+\epsilon} & (0 \le \sigma \le 1,\ |t| \le 1), \end{array} \tag{4.22}$$

Now, we improve the first and second estimates in the above list. For this, we give a Phragmén-Lindelöf type theorem with slight modifications. (See, for example, Titchmarsh [11], §5.65)

Proposition 4.3. *If $\phi_q(s)$ is regular and $O_\epsilon(e^{\epsilon q\tau})$, $q \in \mathbb{Z}^+$, for every positive ϵ, in the strip $\sigma_1 \le \sigma \le \sigma_2$, and*

$$\phi_q(\sigma_1+it) = O\left((q\tau)^{k_1}\right),\ \phi_q(\sigma_2+it) = O\left((q\tau)^{k_2}\right)\ \ (t \in \mathbb{R}),$$

and

$$\phi_q(\sigma + it) = O(q^{k(\sigma)}),$$

for $\sigma_1 \leq \sigma \leq \sigma_2$, $|t| \leq 1$, $k(\sigma)$ *being the linear function of* σ *which takes the values* k_1, k_2 *at* $\sigma = \sigma_1$, σ_2, *respectively, then*

$$\phi_q(\sigma + it) = O\left((q\tau)^{k(\sigma)}\right)$$

uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$.

Proof. Let

$$\Psi(s) := (-iqs)^{k(s)} = e^{k(s)\log(-iqs)},$$
$$\Phi(s) := \frac{\phi_q(s)}{\Psi(s)},$$

where the logarithm has its principal value. This function is regular for $\sigma_1 \leq \sigma \leq \sigma_2$, $t \geq 1$; also, if $k(s) = as + b$, which takes the values k_1, k_2 for $s = \sigma_1$, σ_2, then

$$\begin{aligned}\Re\left\{k(s)\log(-iqs)\right\} &= \Re\left[\{k(\sigma) + iat\}\log(qt - iq\sigma)\right]\\ &= k(\sigma)\log qt + O(1).\end{aligned}$$

Hence

$$|\Psi(s)| = (q\tau)^{k(\sigma)}e^{O(1)}.$$

So, there exists $M > 0$ such that M does not depend on q and is the upper bound of $\Phi(s)$ on these two lines $\sigma = \sigma_1$, $\sigma = \sigma_2$ and on the line segment between the points $\sigma_1 + i$ and $\sigma_2 + i$. Let $\epsilon > 0$, put

$$g(s) := e^{\epsilon q s i}\Phi(s), \ (\sigma_1 \leq \sigma \leq \sigma_2, \, t \geq 1),$$

then

$$|g(s)| = e^{-\epsilon q t}|\Phi(s)| \leq |\Phi(s)| \leq M$$

for $\sigma = \sigma_1$, $\sigma = \sigma_2$. Also $|g(s)| \to 0$ as $t \to \infty$ for $\sigma_1 \leq \sigma \leq \sigma_2$; and so, if T is large enough, then $|g(s)| \leq M$ on $t = T$, $\sigma_1 \leq \sigma \leq \sigma_2$. Hence $|g(s)| \leq M$ at all points of the rectangle with vertices $\sigma_1 + iT$, $\sigma_2 + iT$, $\sigma_1 + i$, $\sigma_2 + i$. Then the maximum-modulus theorem implies that $|g(s)| \leq M$ at all points in the region $\sigma_1 \leq \sigma \leq \sigma_2$, $t \geq 1$, i.e.

$$|\Phi(s)| \leq e^{\epsilon q t} M \qquad (\sigma_1 \leq \sigma \leq \sigma_2,\, t \geq 1).$$

Making $\epsilon \to 0$, it follows that $|\Phi(s)| \leq M$. So $\phi_q(\sigma + it) = O((q\tau)^{k(\sigma)})$ for $\sigma_1 \leq \sigma \leq \sigma_2$, $t \geq 1$, and similarly for the region $\sigma_1 \leq \sigma \leq \sigma_2$, $t \leq -1$. $\square$

Since $L(s, \psi)$ satisfies the conditions of Proposition 4.3, we can apply it and so

$$L(s, \psi) \ll_\epsilon (q\tau)^{\frac{1}{2}(1-\sigma)+\epsilon}$$

uniformly in $0 \leq \sigma \leq 1$. Summarizing all these results, we have

$$L(\sigma + it, \psi) \ll_\epsilon \begin{cases} (q\tau)^{\frac{1}{2}-\sigma+\epsilon} & \text{if} \quad \sigma \leq 0 \\ (q\tau)^{\frac{1}{2}(1-\sigma)+\epsilon} & \text{if} \quad 0 \leq \sigma \leq 1 \\ (q\tau)^{\epsilon} & \text{if} \quad \sigma \geq 1 \end{cases} \tag{4.23}$$

where ϵ is an arbitrarily small positive number. By Cauchy's integral formula, we have

$$L^{(j)}(s, \psi) = \frac{j!}{2\pi i} \int_C \frac{L(s, \psi)}{(w - s)^{j+1}} dw,$$

where C is an arbitrarily small circle with center s and $j = 1, 2, \ldots$. Using (4.23) in the above integral, it easily follows that

$$L^{(j)}(\sigma + it, \psi) \ll_{j,\epsilon} \begin{cases} (q\tau)^{\frac{1}{2}-\sigma+\epsilon} & \text{if} \quad \sigma \leq 0 \\ (q\tau)^{\frac{1}{2}(1-\sigma)+\epsilon} & \text{if} \quad 0 \leq \sigma \leq 1 \\ (q\tau)^{\epsilon} & \text{if} \quad \sigma \geq 1 \end{cases} \tag{4.24}$$

where ϵ is an arbitrarily small positive number and $j = 0, 1, 2, \ldots$.

5. NOTE ON $L^{(j)}(1,\psi)$ AND $\left(\frac{L'}{L}\right)^{(j)}(1,\psi)$

Proposition 5.1. *Let ψ be any non-principal character modulo $q \geq 3$. Then, for $j = 0, 1, 2, \ldots$, we have*

$$(-1)^j \sum_{n\leq x} \frac{\psi(n)\log^j n}{n} = L^{(j)}(1,\chi) + O_j\left(\frac{\log^j x}{x}\right).$$

Proof. Put $f(u) = \frac{\log^j u}{u}$, $j = 0, 1, ...$, then by partial summation, we have

$$\sum_{x<n\leq y} \frac{\psi(n)\log^j n}{n} = \left\{\sum_{n\leq y}\psi(n)\right\} f(y) - \left\{\sum_{n\leq x}\psi(n)\right\} f(x) - \int_x^y \left\{\sum_{n\leq u}\psi(n)\right\} f'(u)du.$$

We use the Pólya-Vinogradov inequality and make $y \to \infty$ in the above equation to obtain

$$\sum_{n>x} \frac{\psi(n)\log^j n}{n} = O\left(\sqrt{q}\log q\,|f(x)|\right) + O\left(\sqrt{q}\log q \int_x^\infty |f'(u)|du\right).$$

If $j = 0$, then

$$\sum_{n>x} \frac{\psi(n)}{n} = O\left(\frac{\sqrt{q}\log q}{x}\right).$$

It easily follows from the Dirichlet series representation of L-function that

$$\sum_{n\leq x} \frac{\psi(n)}{n} = L(1,\chi) + O\left(\frac{\sqrt{q}\log q}{x}\right).$$

Otherwise, $j \geq 1$, we have

$$\sum_{n>x} \frac{\psi(n)\log^j n}{n} = O\left(\frac{\sqrt{q}\log q \log^j x}{x}\right) + O\left(\sqrt{q}\log q \int_x^\infty \frac{|j\log^{j-1} u - \log^j u|}{u^2}\, du\right).$$

By Lemma 3.3, the integral in the O-term is $\ll_j \dfrac{\log^j x}{x}$. This gives

$$L^{(j)}(1,\psi) - (-1)^j \sum_{n\leq x} \frac{\psi(n)\log^j n}{n} = (-1)^j \sum_{n>x} \frac{\psi(n)\log^j n}{n} \ll_j \frac{\sqrt{q}\log q \log^j x}{x}.$$

Hence, the result follows. □

Proposition 5.2. *Let ψ be a non-principal character modulo $q \geq 3$ and $j = 0, 1, 2, \ldots$, then we have*

$$\left(\frac{L'}{L}\right)^{(j)}(s,\psi) = \frac{(-1)^j j!\, E(\psi)}{(s-\beta_1)^{j+1}} + O_j\left((\log q\tau)^{j+1}\right)$$

uniformly for $\sigma > 1 - c_1/(4\log q\tau)$, $t \in \mathbb{R}$, where

$$E(\psi) = \begin{cases} 1 & \text{if } \beta_1 \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First of all, we quote from Theorem 11.4 of [4]:

Suppose that $\sigma \geq 1 - c_1/(2\log q\tau)$. If $L(s,\psi)$ has no exceptional zero, or if β_1 is an exceptional zero of $L(s,\psi)$ but $|s-\beta_1| \geq 1/\log q$, then

$$\frac{L'}{L}(s,\psi) \ll \log q\tau. \tag{5.1}$$

Alternatively, if β_1 is an exceptional zero of $L(s,\psi)$ and $|s-\beta_1| \leq 1/\log q$, then

$$\frac{L'}{L}(s,\psi) = \frac{1}{s-\beta_1} + O(\log q) \quad (s \neq \beta_1). \tag{5.2}$$

Since β_1, if it exists, is a simple pole of L'/L with residue 1,

$$f(s,\psi) := \frac{L'}{L}(s,\psi) - \frac{E(\psi)}{s-\beta_1} \tag{5.3}$$

is an analytic function in the region

$$\sigma \geq 1 - \frac{c_1}{2\log q\tau}. \tag{5.4}$$

Combining (5.1) and (5.2), we obtain

$$f(s,\psi) \ll \log q\tau \qquad \left(\sigma \geq 1 - \frac{c_1}{2\log q\tau},\ t \in \mathbb{R}\right). \tag{5.5}$$

By Cauchy's integral formula we have

$$f^{(j)}(s,\psi) = \frac{j!}{2\pi i}\int_D \frac{f(w,\psi)}{(w-s)^{j+1}}\,dw, \tag{5.6}$$

where D is a small disc centered at $s = \sigma + it$ and with radius $\asymp 1/\log q\tau$, and $\sigma > 1 - c_1/(4\log q\tau)$. Using (5.5) in (5.6) gives

$$f^{(j)}(s,\psi) \ll_j (\log q\tau)^{j+1} \qquad \left(\sigma \geq 1 - \frac{c_1}{4\log q\tau},\ j = 0,1,2,...\right). \tag{5.7}$$

Differentiating (5.3) j times gives

$$\left(\frac{L'}{L}\right)^{(j)}(s,\psi) = \frac{(-1)^j j! E(\psi)}{(s-\beta_1)^{j+1}} + f^{(j)}(s,\psi). \tag{5.8}$$

Combining (5.7) and (5.8) completes the proof. □

Proposition 5.3. *Let ψ be a non-principal character modulo $q \geq 3$ and A an arbitrarily large fixed number. If $q \leq \log^A x$, then*

$$\sum_{n\leq x}\frac{\psi(n)\Lambda(n)\log^j n}{n} = (-1)^{j+1}\left(\frac{L'}{L}\right)^{(j)}(1,\psi) + O_{A,j}\left(\exp\left(-c_2\sqrt{\log x}\right)\right),\ (j = 0,1,...) \tag{5.9}$$

where c_2 is a non-effective positive constant depending on j and A.

Proof. We first state a general form of Perron's formula (See [12], A.3):

Let $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converge absolutely for $\sigma = \Re s > 1$ and $|a_n| < C\Phi(n)$, where $C > 0$ and for $x \geq x_0$, $\Phi(x)$ is monotonically increasing. Let further

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \ll (\sigma - 1)^{-\alpha}$$

as $\sigma \to 1^+$ for some $\alpha > 0$. If $w = u + iv$ (u, v real) is arbitrary, $b > 0$, $T > 0$, $u + b > 1$, then

$$\sum_{n \leq x} a_n n^{-w} = (2\pi i)^{-1} \int_{b-iT}^{b+iT} A(s+w) x^s s^{-1} ds + O\left(x^b T^{-1} (u+b-1)^{-\alpha}\right) + O\left(T^{-1}\Phi(2x) x^{1-u} \log 2x\right) + O\left(\Phi(2x) x^{-u}\right),$$

and the estimate is uniform in x, T, b, and u provided that b and u are bounded.

Since $|\psi(n)\Lambda(n)\log^j n| \ll (\log n)^{j+1}$ and, by (2.12),

$$\sum_{n=1}^{\infty} \frac{|\psi(n)\Lambda(n)\log^j n|}{n^\sigma} \leq \left| \left(\frac{\zeta'}{\zeta}\right)^{(j)} (\sigma) \right| \ll_j \frac{1}{(\sigma-1)^{j+1}}$$

as $\sigma \to 1^+$, we use Perron's formula to obtain

$$\sum_{n \leq x} \frac{\psi(n)\Lambda(n)\log^j n}{n} = \frac{(-1)^{j+1}}{2\pi i} \int_{\frac{1}{\log x} - iT}^{\frac{1}{\log x} + iT} \left(\frac{L'}{L}\right)^{(j)} (s+1, \psi) \frac{x^s}{s} ds + O\left(T^{-1} \log^{j+2} x\right) + O\left(\frac{\log x}{x}\right) \tag{5.10}$$

and the estimate is uniform in x and T.

The next step is to replace the vertical line of integration in (5.10) by the other sides of the rectangle with vertices

$$\frac{1}{\log x} - iT,\ \frac{1}{\log x} + iT,\ -\delta + iT,\ -\delta - iT. \tag{5.11}$$

We should choose δ so small that there are no zeros of $L(s, \psi)$ inside and on the rectangle above. The zero-free region theorems on Dirichlet L-functions (see §2.6) suggest the choice $\delta = B(\log qT)^{-1}$, B will be determined later. But when ψ is a quadratic character, then the region considered may contain the exceptional zero β_1. For this reason, some condition must be imposed on q in relation to that of T. In the light of Siegel's Theorem (see 2.16) we should guarantee the inequality

$$1 - \frac{C_1(\epsilon)}{q^\epsilon} \leq 1 - \frac{B}{\log qT} \tag{5.12}$$

to make the region considered zero-free. Suppose $q \leq \log^A T$, A is an arbitrarily large fixed positive number, then the above inequality is valid, i.e. there are no zeros inside and on the rectangle determined by (5.11), if we choose $\epsilon = 1/A$ and $B < \min\{c_1/4, C_1(1/A)\}$. Thus $s = 0$ is the only pole of the integrand of the integral in (5.10). Then the theorem of residues gives

$$\begin{aligned}(2\pi i)^{-1} \int_{\frac{1}{\log x} - iT}^{\frac{1}{\log x} + iT} \left(\frac{L'}{L}\right)^{(j)} (s+1, \psi) \frac{x^s}{s} ds &= \left(\frac{L'}{L}\right)^{(j)} (1, \psi) \\ &+ \frac{1}{2\pi i} \left(\int_{\frac{1}{\log x} - iT}^{-\delta - iT} + \int_{-\delta - iT}^{-\delta + iT} + \int_{-\delta + iT}^{\frac{1}{\log x} + iT} \right).\end{aligned} \tag{5.13}$$

It remains to estimate the contribution made by the integrals in (5.13). We first remark that the condition $B < \min\{c_1/4, C_1(1/A)\}$ makes Proposition 5.2 applicable. We start with the horizontal integrals in (5.13). On the line segment between the vertices $1/\log x + iT$ and $-\delta + iT$, we have

$$\left(\frac{L'}{L}\right)^{(j)} (1 + s, \psi) \ll_j (\log qT)^{j+1}.$$

The above estimate follows from Proposition 5.2. So trivial estimation of the horizontal integral from $1/\log x + iT$ to $-\delta + iT$ gives

$$(2\pi i)^{-1} \int_{\frac{1}{\log x}+iT}^{-\delta+iT} \ll_j \frac{(\log qT)^{j+1}}{T}.$$

The second horizontal integral in (5.13) is similarly $\ll_j \dfrac{(\log qT)^{j+1}}{T}$. We now deal with the integral $(2\pi i)^{-1} \displaystyle\int_{-\delta-iT}^{-\delta+iT}$. For any $s \in [-\delta - iT, -\delta + iT]$ we have

$$\begin{aligned}\frac{1}{s+1-\beta_1} &\ll \frac{1}{1 - \frac{B}{\log qT} - \left\{1 - \frac{C_1(\epsilon)}{q^\epsilon}\right\}} \qquad \left(\epsilon = \frac{1}{A},\ B < \min\left\{\frac{c_1}{4}, C_1\left(\frac{1}{A}\right)\right\}\right) \\ &= \frac{q^\epsilon \log qT}{C_1(\epsilon)\log T + (C_1(\epsilon)\log q - Bq^\epsilon)} \\ &\ll_A \log T \end{aligned} \tag{5.14}$$

if $q \le \log^A T$. It follows from Proposition 5.2 and (5.14) that

$$\left(\frac{L'}{L}\right)^{(j)} (s+1, \psi) \ll_{A,j} (\log qT)^{j+1}$$

on the line segment $[-\delta - iT, -\delta + iT]$. So we have

$$\begin{aligned}(2\pi i)^{-1} \int_{-\delta-iT}^{-\delta+iT} &\ll_{A,j} x^{-\frac{B}{\log qT}} (\log qT)^{j+1} \int_{-T}^{T} \frac{dt}{\sqrt{\delta^2 + t^2}} \\ &\ll_{A,j} x^{-\frac{B}{\log qT}} (\log qT)^{j+2}. \end{aligned} \tag{5.15}$$

From the above results, (5.13) becomes

$$\begin{aligned}(2\pi i)^{-1} \int_{\frac{1}{\log x}-iT}^{\frac{1}{\log x}+iT} \left(\frac{L'}{L}\right)^{(j)} (s+1, \psi) \frac{x^s}{s} ds &= \left(\frac{L'}{L}\right)^{(j)} (1, \psi) \\ &+ O_j\left(\frac{(\log qT)^{j+1}}{T}\right) + O_{A,j}\left(x^{-\frac{B}{\log qT}} (\log qT)^{j+2}\right). \end{aligned} \tag{5.16}$$

Inserting (5.16) into (5.10) gives

$$\sum_{n\leq x}\frac{\psi(n)\Lambda(n)\log^j n}{n}=(-1)^{j+1}\left(\frac{L'}{L}\right)^{(j)}(1,\psi)+O_j\left(\frac{(\log qT)^{j+1}}{T}\right)$$
$$+O_{A,j}\left(x^{-\frac{B}{\log qT}}(\log qT)^{j+2}\right)+O\left(T^{-1}\log^{j+2}x\right)+O\left(\frac{\log x}{x}\right). \tag{5.17}$$

Suppose $T=\exp\left(\sqrt{\log x}\right)$. Under the restriction $q\leq\log^A T=\log^{\frac{A}{2}} x$, the error terms in the above equation can be absorbed in

$$O_{j,A}\left(\exp\left(-c_2\sqrt{\log x}\right)\right)$$

where $0<c_2<\min\{1,B\}$ and c_2 is a non-effective constant which depends on j and A. So this completes the proof. $\square$

Proposition 5.4. *Let ψ be any character modulo q and A arbitrarily large fixed poisitive number. If $q\leq\log^A x$, then*

$$\sum_{n\leq x}\psi(n)\Lambda(n)\log^j n$$
$$=\begin{cases}O_{j,A}\left(x\exp\left(-c_2\sqrt{\log x}\right)\right) & \text{if } \psi\neq\psi_0,\\ x\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^{j-\nu}(j-\nu)!\log^{\nu}x+O_{j,A}\left(x\exp\left(-c_5\sqrt{\log x}\right)\right) & \psi=\psi_0,\end{cases}$$

where c_5 is a non-effective positive constant depending on j and A.

Proof. As in the proof of the above theorem, we can apply Perron's formula and we have

$$\sum_{n\leq x}\psi(n)\Lambda(n)\log^j n=\frac{(-1)^{j+1}}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT}\left(\frac{L'}{L}\right)^{(j)}(s,\psi)\frac{x^s}{s}ds$$
$$+O\left(xT^{-1}\log^{j+2}x\right)+O\left(\log^{j+1}x\right) \tag{5.18}$$

and the estimate is uniform in x and T. Let R denote a closed contour that consists of line segments joining the points

$$1+\frac{1}{\log x}+iT,\ 1+\frac{1}{\log x}+iT,\ 1-\delta+iT,\ 1-\delta-iT. \tag{5.19}$$

Case 1. Suppose ψ is a non-principal character. As in the proof of the above theorem, we choose $\delta = B(\log qT)^{-1}$, $0 < B < \min\{c_1/4, C_1(1/A)\}$ so that inside and on R there are no zeros of $L(s,\psi)$ if $q \le \log^A T$. Since ψ is a non-principal character, $L(s,\psi)$ is an entire function, so

$$\left(\frac{L'}{L}\right)^{(j)}(s,\psi)\frac{x^s}{s}$$

has no poles inside and on the contour. By Cauchy's theorem we have

$$(2\pi i)^{-1}\int_R \left(\frac{L'}{L}\right)^{(j)}(s,\psi)\frac{x^s}{s}ds = 0. \tag{5.20}$$

It follows from (5.18) and (5.20)

$$\sum_{n\le x}\psi(n)\Lambda(n)\log^j n = \frac{(-1)^{j+1}}{2\pi i}\left(\int_{1-\delta+iT}^{1+\frac{1}{\log x}+iT}+\int_{1-\delta-iT}^{1-\delta+iT}+\int_{1+\frac{1}{\log x}-iT}^{1-\delta-iT}\right)\left(\frac{L'}{L}\right)^{(j)}(s,\psi)\frac{x^s}{s}ds$$
$$+O\left(xT^{-1}\log^{j+2}x\right)+O\left(\log^{j+1}x\right),$$

if $q \le \log^A T$. As we did in the proof of the previous theorem, the three integrals in the above equation can be bounded by

$$O_{A,j}\left(x^{-\frac{B}{\log qT}}(\log qT)^{j+2}+\frac{(\log qT)^{j+1}}{T}\right)$$

Again we put $T = \exp\left(\sqrt{\log x}\right)$, then under the constraint $q \le \log^A T = \log^{A/2} x$, the proof of Case 1 is completed.

Case 2. Suppose ψ is the principal character modulo q, $\psi = \psi_0$. Firstly, we note two well-known identities involving Dirichlet's $L-$function with the principal character ψ_0:

$$L(s,\psi_0) = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right), \tag{5.21}$$

$$\frac{L'}{L}(s,\psi_0) = \frac{\zeta'}{\zeta}(s) + \sum_{p|q}\frac{\log p}{p^s-1}. \tag{5.22}$$

Equation (5.21) implies that $L(s,\psi_0)$ has a pole at $s=1$ with residue $\phi(q)/q$, it has all the zeros of $\zeta(s)$, and it also has zeros of the form $2\pi ik/\log p$ where k takes integral values and $p|q$. Thus, we can use the zero-free region theorems for $\zeta(s)$; there exists an effective numerical constant c_3 such that $\zeta(s)$, so $L(s,\psi_0)$, has no zeros in the region $\sigma > 1-c_3/\log\tau$. Hence, put $\delta = -c_3/(5\log T)$ in (5.19). However, by differentiating (5.22) j times it is easy to see that the j^{th} derivative of L'/L has a pole at $s=1$ of order $j+1$. By the theorem of residues, (5.18) becomes

$$\begin{aligned}\sum_{n\le x}\psi_0(n)\Lambda(n)\log^j n = {} & (-1)^{j+1}\mathrm{Res}_{s=1}\left\{\left(\frac{L'}{L}\right)^{(j)}(s,\psi_0)\frac{x^s}{s}\right\}\\ & + \frac{(-1)^{j+1}}{2\pi i}\left(\int_{1-\delta+iT}^{1+\frac{1}{\log x}+iT} + \int_{1-\delta-iT}^{1-\delta+iT} + \int_{1+\frac{1}{\log x}-iT}^{1-\delta-iT}\right)\left(\frac{L'}{L}\right)^{(j)}(s,\psi_0)\frac{x^s}{s}ds\\ & + O\left(xT^{-1}\log^{j+2}x\right) + O\left(\log^{j+1}x\right).\end{aligned} \tag{5.23}$$

Next, we calculate the residue at $s=1$;

$$\mathrm{Res}_{s=1}\left\{\left(\frac{L'}{L}\right)^{(j)}(s,\psi_0)\frac{x^s}{s}\right\} = \frac{1}{j!}\frac{d^j}{ds^j}\left\{(s-1)^{j+1}\left(\frac{L'}{L}\right)^{(j)}(s,\psi_0)\frac{x^s}{s}\right\}_{s=1}.$$

By the generalized Leibniz rule, the right side of the above equation is

$$\frac{1}{j!}\sum_{i_1+i_2+i_3=j}\binom{j}{i_1,i_2,i_3}\frac{d^{i_1}}{ds^{i_1}}\left\{\left(\frac{L'}{L}\right)^{(j)}(s,\psi_0)(s-1)^{j+1}\right\}_{s=1}\frac{d^{i_2}}{ds^{i_2}}\left\{x^s\right\}_{s=1}\frac{d^{i_3}}{ds^{i_3}}\left\{\frac{1}{s}\right\}_{s=1}.$$

Since ζ'/ζ has a pole at $s = 1$ with residue -1, we can easily deduce from (5.22) that

$$\left(\frac{L'}{L}\right)^{(j)}(s, \psi_0) = \frac{(-1)^{j+1} j!}{(s-1)^{j+1}} + A_0(j,q) + A_1(j,q)(s-1) + \ldots. \tag{5.24}$$

If $i_1 > 0$ then it easily follows from (5.24) that

$$\frac{d^{i_1}}{ds^{i_1}} \left\{ \left(\frac{L'}{L}\right)^{(j)}(s, \psi_0)(s-1)^{j+1} \right\}_{s=1} = 0.$$

Then we obtain

$$\mathrm{Res}_{s=1} \left\{ \left(\frac{L'}{L}\right)^{(j)}(s, \psi_0) \frac{x^s}{s} \right\} = (-1)^{j+1} x \sum_{i_2+i_3=j} \binom{j}{i_2, i_3} (-1)^{i_3}\, i_3!\, \log^{i_2} x.$$

The last part is to estimate the three integrals in (5.23). For this, we need an estimate for the j^{th} derivative of $\frac{L'}{L}(s, \psi_0)$. From Theorem 6.7 of [4] we deduce that

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + g(s) \quad \left(\sigma > 1 - \frac{c_3}{2\log \tau},\ t \in \mathbb{R}\right), \tag{5.25}$$

where $g(s)$ is an analytic and

$$g(s) \ll \log \tau \tag{5.26}$$

throughout the region $\sigma > 1 - c_3/(2\log \tau)$, $t \in \mathbb{R}$. By Cauchy's integral formula we have

$$g^{(j)}(s) = \frac{j!}{2\pi i} \int_D \frac{g(w)}{(w-s)^{j+1}}\, dw, \tag{5.27}$$

where D is a small disc centered at $s = \sigma + it$ and with radius $\asymp 1/\log \tau$, and $\sigma >$

$1 - c_3/(4\log\tau)$. Then trivial estimation of the above integral gives

$$g^{(j)}(s) \ll_j (\log\tau)^{j+1} \qquad \left(\sigma \geq 1 - \frac{c_3}{4\log\tau},\ t \in \mathbb{R},\ j = 0, 1, 2, ...\right). \tag{5.28}$$

It follows from (5.25) and (5.28) that

$$\left(\frac{\zeta'}{\zeta}\right)^{(j)}(s) = \frac{(-1)^{j+1} j!}{(s-1)^{j+1}} + O_j\left(\log^{j+1}\tau\right) \quad \left(\sigma > 1 - \frac{c_3}{4\log\tau},\ t \in \mathbb{R}\right) \tag{5.29}$$

Differentiating (5.22) j times gives

$$\left(\frac{L'}{L}\right)^{(j)}(s, \psi_0) = \left(\frac{\zeta'}{\zeta}\right)^{(j)}(s) + \frac{d^j}{ds^j}\left\{\sum_{p|q} \frac{\log p}{p^s - 1}\right\}. \tag{5.30}$$

Since $(\log p)/(p^s - 1) \ll 1$ for $\sigma \geq 1/2$, the above sum over p is $\ll \omega(q) \ll \log q$, where $\omega(q)$ denotes the number of distinct prime divisors of q. Then by applying Cauchy's integral formula in an arbitrarily small $\epsilon-$disc centered at $s = \sigma + it$ with $\sigma \geq 1/2 + \epsilon$, we have

$$\frac{d^j}{ds^j}\left\{\sum_{p|q} \frac{\log p}{p^s - 1}\right\} \ll_{j,\epsilon} \log q. \tag{5.31}$$

Combining (5.29), (5.30) and (5.31), we get

$$\left(\frac{L'}{L}\right)^{(j)}(s, \psi_0) = \frac{(-1)^{j+1} j!}{(s-1)^{j+1}} + O_j\left((\log\tau)^{j+1}\right) + O_j(\log q) \tag{5.32}$$

for $\sigma > 1 - c_3/(4\log\tau)$, $t \in \mathbb{R}$.

Now, we return to the estimation of the three integrals in (5.19). Using (5.32), we have

$$\int_{1-\delta+iT}^{1+\frac{1}{\log x}+iT} \left(\frac{L'}{L}\right)^{(j)}(s, \psi_0)\frac{x^s}{s}ds \ll_j xT^{-1}\left\{(\log T)^{j+1} + \log q\right\} \tag{5.33}$$

and, similarly,

$$\int_{1+\frac{1}{\log x}-iT}^{1-\delta-iT} \left(\frac{L'}{L}\right)^{(j)} (s,\psi_0)\frac{x^s}{s}ds \ll_j xT^{-1}\left\{(\log T)^{j+1}+\log q\right\}. \tag{5.34}$$

We have

$$\frac{(-1)^{j+1}j!}{(s-1)^{j+1}} \ll_j \log^{j+1} T$$

for any $s \in [1-\delta-iT, 1-\delta+iT]$. On this line segment, by (5.32), we have

$$\left(\frac{L'}{L}\right)^{(j)} (s,\psi_0) \ll_j \log^{j+1} T + \log q.$$

So

$$\int_{1-\delta+iT}^{1-\delta+iT} \left(\frac{L'}{L}\right)^{(j)} (s,\psi_0)\frac{x^s}{s}ds \ll_j x^{1-c_3/(5\log T)}\left\{(\log T)^{j+1}+\log q\right\}. \tag{5.35}$$

If we choose $T = \exp\left(\sqrt{\log x}\right)$ and $q \leq \log^A T$, A an arbitrarily large fixed positive number, then, by (5.33), (5.34) and (5.35), the sum of the above three integrals can be bounded by

$$O_{j,A}\left(x\exp\left(-c_4\sqrt{\log x}\right)\right), \tag{5.36}$$

where $0 < c_4 < c_3/5$. Combining the above results, (5.23) becomes

$$\sum_{n\leq x}\psi_0(n)\Lambda(n)\log^j n = x\sum_{i_2+i_3=j}\binom{j}{i_2,i_3}(-1)^{i_3}\, i_3!\,\log^{i_2} x + O_{j,A}\left(x\exp\left(-c_5\sqrt{\log x}\right)\right), \tag{5.37}$$

where $0 < c_5 < c_4$. $\square$

6. PROOF OF THEOREM 1.1

Let R denote the closed rectangle in the complex plane with vertices at $a+i$, $a+iT$, $1-a+iT$ and $1-a+i$, where $a=9/8$ and T is a sufficiently large number. Then, consider the integral

$$I = \frac{1}{2\pi i}\int_{\partial R} \zeta^{(j)}(s)\,\frac{\zeta'}{\zeta}(s)\,ds = \sum_{i=1}^{4} I_i,$$

where j is a fixed integer ≥ 1, ∂R is the boundary of R, the integral is taken in the counterclockwise sense and

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{1-a+i}^{a+i} \zeta^{(j)}(s)\,\frac{\zeta'}{\zeta}(s)\,ds,\\
I_2 &= \frac{1}{2\pi i}\int_{a+i}^{a+iT} \zeta^{(j)}(s)\,\frac{\zeta'}{\zeta}(s)\,ds,\\
I_3 &= \frac{1}{2\pi i}\int_{a+iT}^{1-a+iT} \zeta^{(j)}(s)\,\frac{\zeta'}{\zeta}(s)\,ds,\\
I_4 &= \frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i} \zeta^{(j)}(s)\,\frac{\zeta'}{\zeta}(s)\,ds.
\end{aligned}$$

Assume $T \in \mathscr{F}$ (at the end of proof this restriction will be removed), then no zero ρ of $\zeta(s)$ lies on ∂R and all zeros with the condition $0 < \gamma \leq T$ lie in R, since all complex zeros are in the critical strip and the ordinate of the first zero of $\zeta(s)$ above the real axis is > 14. By Cauchy's theorem from complex analysis, we have

$$\sum_{0<\gamma\leq T} \zeta^{(j)}(\rho) = I = \frac{1}{2\pi i}\int_{\partial R} \zeta^{(j)}(s)\,\frac{\zeta'}{\zeta}(s)\,ds. \tag{6.1}$$

Since $\zeta^{(j)}(s)$ and $\frac{\zeta'}{\zeta}(s)$ are analytic functions on the complex plane, except at $s=1$, their modulus are bounded by a positive constant, which does not depend on

T, on the set $\{\sigma + i \in \mathbb{C} : 1 - a \leq \sigma \leq a\}$. Hence, by trivial estimation, we have

$$I_1 \ll_j 1.$$

For $s \in [1 - a + iT, a + iT]$ we have

$$\frac{\zeta'}{\zeta}(s)\zeta^{(j)}(s) \ll \log^2 T \max_{1-a\leq\sigma\leq a} |\, \zeta^{(j)}(\sigma + iT)\, | \ll_{j,\epsilon} T^{a-\frac{1}{2}+\epsilon} \log^2 T \ll_{j,\epsilon} T^{a-\frac{1}{2}+\epsilon}$$

by (2.10) and (2.14). So trivial estimation gives

$$I_3 \ll_{j,\epsilon} T^{a-\frac{1}{2}+\epsilon}.$$

Now, we'll estimate the integral I_2. To do this, we use the Dirichlet series representations of $\zeta^{(j)}(s)$ and $\frac{\zeta'}{\zeta}(s)$:

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i} \int_{a+i}^{a+iT} \zeta^{(j)}(s)\, \frac{\zeta'}{\zeta}(s)\, ds \\
&= \frac{1}{2\pi} \int_1^T \zeta^{(j)}(a+it)\, \frac{\zeta'}{\zeta}(a+it)\, dt \\
&= \frac{1}{2\pi} \int_1^T \left\{ \sum_{n\geq 2} \frac{(-1)^j \log^j n}{n^{a+it}} \right\} \left\{ - \sum_{m\geq 2} \frac{\Lambda(m)}{m^{a+it}} \right\} dt \\
&= \frac{1}{2\pi} \int_1^T \sum_{n,m\geq 2} \frac{(-1)^{j+1}\, \Lambda(m) \log^j n}{(nm)^{a+it}}\ dt \\
&= \frac{1}{2\pi} \sum_{n,m\geq 2} \frac{(-1)^{j+1}\, \Lambda(m) \log^j n}{(nm)^{a}} \int_1^T \frac{dt}{(nm)^{it}},
\end{aligned}$$

where the inversion of the order of summation and integration is justified by absolute convergence. Since $n, m \geq 2$, we have

$$\int_1^T \frac{dt}{(nm)^{it}} = \frac{\exp\left(-it \log nm\right)}{-i \log nm}\Bigg|_1^T \ll 1. \tag{6.2}$$

Thus,

$$I_2 \ll \sum_{n,m \geq 2} \frac{\Lambda(m) \log^j n}{(nm)^a} = \left|\frac{\zeta'}{\zeta}(a)\right| |\zeta^{(j)}(a)| \ll 1,$$

since a is fixed. Hence, we can conclude that

$$I = I_4 + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right). \tag{6.3}$$

Our problem is reduced to estimating I_4. We write directly from the definition of I_4 that

$$I_4 = -\frac{1}{2\pi}\int_1^T \zeta^{(j)}(1-a+it)\,\frac{\zeta'}{\zeta}(1-a+it)\,dt.$$

Taking the complex conjugates of both sides of the above equation we get

$$\begin{aligned}
\overline{I_4} &= \overline{-\frac{1}{2\pi}\int_1^T \zeta^{(j)}(1-a+it)\,\frac{\zeta'}{\zeta}(1-a+it)\,dt} \\
&= \frac{-1}{2\pi}\int_1^T \overline{\zeta^{(j)}(1-a+it)}\,\overline{\left\{\frac{\zeta'}{\zeta}(1-a+it)\right\}}\,dt \\
&= -\frac{1}{2\pi}\int_1^T \zeta^{(j)}\left(\overline{1-a+it}\right)\,\frac{\zeta'}{\zeta}\left(\overline{1-a+it}\right)\,dt \\
&= -\frac{1}{2\pi}\int_1^T \zeta^{(j)}\left(1-a-it\right)\,\frac{\zeta'}{\zeta}\left(1-a-it\right)\,dt
\end{aligned} \tag{6.4}$$

By logarithmic differentitation of the equation (2.1), we have

$$\frac{\zeta'}{\zeta}(1-s) = \frac{\chi'}{\chi}(1-s) - \frac{\zeta'}{\zeta}(s). \tag{6.5}$$

We insert the expression of $\zeta^{(j)}(1-s)$ in Lemma 3.5 and the above expression of $\frac{\zeta'}{\zeta}(1-s)$, with $s = a+it$, into (6.4) and we then get

$$\begin{aligned}
\overline{I_4} &= \frac{(-1)^{j+1}}{2\pi} \int_1^T \chi(1-a-it) \left[1 + O\left(\frac{1}{t}\right)\right] \left\{\sum_{k=0}^{j} \binom{j}{k} \left(\log \frac{t}{2\pi}\right)^{j-k} \zeta^{(k)}(a+it)\right\} \\
&\qquad \left\{\frac{\chi'}{\chi}(1-a-it) - \frac{\zeta'}{\zeta}(a+it)\right\} dt \\
&= \frac{(-1)^{j+1}}{2\pi} \int_1^T \chi(1-a-it) \left\{\sum_{k=0}^{j} \binom{j}{k} \left(\log \frac{t}{2\pi}\right)^{j-k} \zeta^{(k)}(a+it)\right\} \\
&\qquad \left\{\frac{\chi'}{\chi}(1-a-it) - \frac{\zeta'}{\zeta}(a+it)\right\} dt \\
&\quad + O\Bigg(\int_1^T \left|\chi(1-a-it)\right| \left|\sum_{k=0}^{j} \binom{j}{k} \left(\log \frac{t}{2\pi}\right)^{j-k} \zeta^{(k)}(a+it)\right| \\
&\qquad \left|\frac{\chi'}{\chi}(1-a-it) - \frac{\zeta'}{\zeta}(a+it)\right| \frac{dt}{t}\Bigg).
\end{aligned}$$

From the asymptotic formulas (2.7), (2.8) and the facts that

$$\begin{aligned}
|\zeta^{(k)}(a+it)| &\leq |\zeta^{(k)}(a)| \ll 1, \\
\left|\frac{\zeta'}{\zeta}(a+it)\right| &\leq \left|\frac{\zeta'}{\zeta}(a)\right| \ll 1,
\end{aligned}$$

it easily follows that the above error term is

$$\ll_j T^{a-\frac{1}{2}} (\log T)^{j+1} \int_1^T \frac{dt}{t} \ll_{j,\epsilon} T^{a-\frac{1}{2}+\epsilon}.$$

Thus, $\overline{I_4}$ becomes

$$\begin{aligned}
\overline{I_4} &= \frac{(-1)^{j+1}}{2\pi} \sum_{k=0}^{j} \binom{j}{k} \int_1^T \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^{j-k} \zeta^{(k)}(a+it) \frac{\chi'}{\chi}(1-a-it)\, dt \\
&\quad + \frac{(-1)^{j}}{2\pi} \sum_{k=0}^{j} \binom{j}{k} \int_1^T \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^{j-k} \zeta^{(k)}(a+it) \frac{\zeta'}{\zeta}(a+it)\, dt \\
&\quad + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right) \\
&= S_1 + S_2 + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right), \text{ say.}
\end{aligned}$$

Firstly, we deal with S_1. By (2.8), we have

$$S_1 = \frac{(-1)^j}{2\pi} \sum_{k=0}^{j} \binom{j}{k} \int_1^T \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^{j-k+1} \zeta^{(k)}(a+it)\ dt$$
$$+ O\left(\sum_{k=0}^{j} \binom{j}{k} \int_1^T \left| \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^{j-k} \zeta^{(k)}(a+it) \right| \frac{dt}{t} \right).$$

The above error term is, as in the estimation of the error term in I_4,

$$\ll_{j,\epsilon} T^{a-\frac{1}{2}+\epsilon}.$$

Since $\zeta^{(k)}(s) = \sum_{n=1}^{\infty} \frac{(-1)^k \log^k n}{n^s}$ for $\Re s > 1$, we can write

$$S_1 = (-1)^j \sum_{k=0}^{j} \binom{j}{k} \frac{1}{2\pi} \int_1^T \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^{j-k+1} \sum_{n=1}^{\infty} \frac{(-1)^k \log^k n}{n^{\,a+it}}\ dt$$
$$+ O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right).$$

For the integral in the right hand side of the above equation we apply Lemma 3.11 and we get

$$S_1 = (-1)^j \sum_{k=0}^{j} \binom{j}{k} \left[\sum_{n \le \frac{T}{2\pi}} (-1)^k (\log n)^{j+1} + O\left(T^{a-\frac{1}{2}} \log^{j-k+1} T\right) \right] + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right)$$
$$= (-1)^j \sum_{k=0}^{j} \binom{j}{k} (-1)^k \sum_{n \le \frac{T}{2\pi}} \log^{j+1} n + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right).$$

Using Lemma 3.2 in the above equation gives

$$S_1 = (-1)^j \left\{ \sum_{k=0}^{j} \binom{j}{k} (-1)^k \right\} \frac{T}{2\pi} \left\{ \sum_{m=0}^{j+1} \frac{(-1)^m (j+1)!}{(j+1-m)!} \left(\log \frac{T}{2\pi}\right)^{j+1-m} \right\} + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right),$$

but, in the above result the main term disappears since

$$\sum_{k=0}^{j} \binom{j}{k} (-1)^k = (1-1)^k = 0.$$

So, we can conclude that

$$S_1 \ll_{j,\epsilon} T^{a-\frac{1}{2}+\epsilon}.$$

Next, we shall estimate S_2. Firstly, recall from Lemma 3.17 that

$$\zeta^{(k)}(s) \cdot \frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{b_k(n)}{n^s} \; (\Re s > 1),$$

where $k = 0, 1, 2, \ldots$. Then

$$S_2 = (-1)^j \sum_{k=0}^{j} \binom{j}{k} \frac{1}{2\pi} \int_1^T \chi(1-a-it) \left(\log \frac{t}{2\pi}\right)^{j-k} \sum_{n=1}^{\infty} \frac{b_k(n)}{n^{a+it}} \, dt.$$

We now apply Lemma 3.11, and we have

$$\begin{aligned} S_2 &= (-1)^j \sum_{k=0}^{j} \binom{j}{k} \left\{ \sum_{n \le T/2\pi} b_k(n) \log^{j-k} n + O\left(T^{a-\frac{1}{2}} \log^{j-k} T\right) \right\} \\ &= (-1)^j \sum_{k=0}^{j} \binom{j}{k} \sum_{n \le T/2\pi} b_k(n) \log^{j-k} n + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right) \\ &= (-1)^j \sum_{k=0}^{j-1} \binom{j}{k} \sum_{n \le T/2\pi} b_k(n) \log^{j-k} n + (-1)^j \sum_{n \le T/2\pi} b_j(n) + O_{j,\epsilon}\left(T^{a-\frac{1}{2}+\epsilon}\right). \end{aligned}$$

Using Lemma 3.17 gives

$$S_2 = (-1)^j \sum_{k=0}^{j-1} \binom{j}{k} \sum_{n \le T/2\pi} b_k(n) \log^{j-k} n - \frac{1}{j+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi}\right)^{j+1} + O_j\left(T \log^j T\right). \tag{6.6}$$

For $k = 0, 1, .., j-1$, we have by partial summation,

$$\sum_{n \le T/2\pi} b_k(n) \log^{j-k} n = \left\{ \sum_{n \le T/2\pi} b_k(n) \right\} \left(\log \frac{T}{2\pi} \right)^{j-k}$$
$$- (j-k) \int_1^{\frac{T}{2\pi}} \left\{ \sum_{n \le u} b_k(n) \right\} (\log u)^{j-k-1} \frac{du}{u}. \tag{6.7}$$

Lemma 3.17 implies that

$$\sum_{n \le u} b_k(n) \ll_k u \log^{k+1} u.$$

Using this in the above integral and again by Lemma 3.17, (6.7) becomes

$$\begin{aligned}
\sum_{n \le T/2\pi} b_k(n) \log^{j-k} n &= \left\{ \frac{(-1)^{k+1}}{k+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{k+1} + O_k \left(T \log^k T \right) \right\} \left(\log \frac{T}{2\pi} \right)^{j-k} \\
&\quad + O_j \left(\int_1^{\frac{T}{2\pi}} \log^j u \, du \right) \\
&= \frac{(-1)^{k+1}}{k+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right).
\end{aligned}$$

Substituting the last result into (6.6), we have

$$\begin{aligned}
S_2 &= (-1)^j \sum_{k=0}^{j-1} \binom{j}{k} \left\{ \frac{(-1)^{k+1}}{k+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right) \right\} \\
&\quad - \frac{1}{j+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right) \\
&= (-1)^{j+1} \left\{ \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^{k+1}}{k+1} \right\} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right).
\end{aligned}$$

Combining the results of S_1 and S_2 gives

$$\overline{I_4} = (-1)^{j+1} \left\{ \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k}{k+1} \right\} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right).$$

Since the main term of the above statement is real-valued, we have

$$I_4 = (-1)^{j+1} \left\{ \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k}{k+1} \right\} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right).$$

By (3.38) we have

$$I_4 = \frac{(-1)^{j+1}}{j+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right). \tag{6.8}$$

It follows from (6.1), (6.3) and (6.8) that

$$\sum_{0<\gamma\leq T} \zeta^{(j)}(\rho) = \frac{(-1)^{j+1}}{j+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O_j \left(T \log^j T \right). \tag{6.9}$$

We have proven Theorem 1.1 when $T \in \mathscr{F}$. Now, we want to remove this restriction on T. Increasing (or decreasing) T by an amount ≤ 1 means addition (or deletion) of $\ll \log T$ terms to (from) the sum in (6.9) and this produces an error $\ll T^{\frac{1}{2}+\epsilon}$ by (2.14). However, the right hand side of (6.9) changes by $\ll \log^{j+1} T$ as a result of this change on T. But, these errors can be absorbed by the error term in (6.9). So, this means that (6.9) holds for all large T.

7. PROOF OF THEOREM 1.2

Let j be a fixed integer ≥ 0 and ψ a primitive character modulo q. We consider the following integral I around the rectangle R joining the points $a+i, a+iT, 1-a+iT$ and $1-a+i$, where $a = 9/8$, $T \in \mathscr{F}$ and T sufficiently large,

$$\begin{aligned}
I =& \frac{1}{2\pi i}\int_{\partial R} L^{(j)}(s,\psi)\frac{\zeta'}{\zeta}(s)ds \\
=& \frac{1}{2\pi i}\left(\int_{a+i}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+i} + \int_{1-a+i}^{a+i}\right) L^{(j)}(s,\psi)\frac{\zeta'}{\zeta}(s)ds \\
=& I_1 + I_2 + I_3 + I_4, \text{ say.}
\end{aligned}$$

Cauchy's theorem implies that

$$I = \sum_{0<\gamma\leq T} L^{(j)}(\rho,\psi). \tag{7.1}$$

We'll start with the estimation of the integrals I_1, I_2 and I_4. By (4.24) we have

$$I_4 \ll \max_{1-a\leq\sigma\leq a} |L^{(j)}(\sigma+i,\psi)| \ll_{j,\epsilon} q^{a-\frac{1}{2}+\epsilon}.$$

In estimation of I_1, since $a > 1$, the functions in the integrand can be represented by Dirichlet series, so

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{a+i}^{a+iT} L^{(j)}(s,\psi)\,\frac{\zeta'}{\zeta}(s)\,ds \\
&= \frac{1}{2\pi}\int_1^T L^{(j)}(a+it,\psi)\,\frac{\zeta'}{\zeta}(a+it)\,dt \\
&= \frac{1}{2\pi}\int_1^T \left\{\sum_{n\geq 2}\frac{(-1)^j\psi(n)\log^j n}{n^{a+it}}\right\}\left\{-\sum_{m\geq 2}\frac{\Lambda(m)}{m^{a+it}}\right\} dt \\
&= \frac{1}{2\pi}\int_1^T \sum_{n,m\geq 2}\frac{(-1)^{j+1}\,\psi(n)\Lambda(m)\log^j n}{(nm)^{a+it}}\;dt.
\end{aligned}$$

By the absolute convergence of the series in the integrand, we can interchange the order of summation and integration, i.e.

$$I_1 = \frac{1}{2\pi} \sum_{n,m\geq 2} \frac{(-1)^{j+1}\,\Lambda(m)\psi(n)\log^j n}{(nm)^a} \int_1^T \frac{dt}{(nm)^{it}}.$$

By (6.2) we have

$$I_1 \ll \sum_{n,m\geq 2} \frac{\Lambda(m)\log^j n}{(nm)^a} = \left|\,\zeta^{(j)}(a)\right| \left|\frac{\zeta'}{\zeta}(a)\right| \ll_j 1$$

since a is fixed number > 1.

To treat I_2 we write

$$I_2 = \frac{1}{2\pi i} \int_a^{1-a} L^{(j)}(\sigma + iT, \psi)\frac{\zeta'}{\zeta}(\sigma + iT)d\sigma.$$

By (2.10) and (4.24), we have

$$I_2 \ll (\log T)^2 \max_{1-a\leq\sigma\leq a} |L^{(j)}(\sigma + iT, \psi)| \ll_{j,\epsilon} (qT)^{a-\frac{1}{2}+\epsilon}.$$

Collecting the previous results, we obtain

$$I = I_3 + O_{j,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right). \tag{7.2}$$

Now, we 'll estimate I_3,

$$\begin{aligned} I_3 &= -\frac{1}{2\pi i} \int_{1-a+i}^{1-a+iT} L^{(j)}(s, \psi)\frac{\zeta'}{\zeta}(s)ds \\ &= -\frac{1}{2\pi} \int_1^T L^{(j)}(1 - a + it, \psi)\frac{\zeta'}{\zeta}(1 - a + it)dt. \end{aligned}$$

It follows from the facts

$$L^{(j)}(\bar{s},\psi)=\overline{L^{(j)}\left(s,\overline{\psi}\right)} \quad \text{and} \quad \frac{\zeta'}{\zeta}\left(\bar{s}\right)=\overline{\frac{\zeta'}{\zeta}(s)}$$

that

$$\overline{I_3}=-\frac{1}{2\pi}\int_1^T L^{(j)}\left(1-a-it,\overline{\psi}\right)\frac{\zeta'}{\zeta}(1-a-it)dt.$$

By (6.5) and Lemma 3.9, we have

$$\begin{aligned}
\overline{I_3}=&\frac{(-1)^{j+1}}{2\pi}\int_1^T \chi(1-a-it,\overline{\psi})\left\{\sum_{h=0}^{j}\binom{j}{h}\left(\log\frac{qt}{2\pi}\right)^{j-h}L^{(h)}\left(a+it,\psi\right)\right\}\\
&\qquad\left[1+O_j\left(\frac{1}{|t|\log\frac{qT}{2\pi}}\right)\right]\left[\frac{\chi'}{\chi}(1-a-it)-\frac{\zeta'}{\zeta}(a+it)\right]dt\\
=&\frac{(-1)^{j+1}}{2\pi}\sum_{h=0}^{j}\binom{j}{h}\int_1^T\chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^{j-h}L^{(h)}\left(a+it,\psi\right)\frac{\chi'}{\chi}(1-a-it)dt\\
&-\frac{(-1)^{j+1}}{2\pi}\sum_{h=0}^{j}\binom{j}{h}\int_1^T\chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^{j-h}L^{(h)}\left(a+it,\psi\right)\frac{\zeta'}{\zeta}(a+it)dt\\
&+O_j\Bigg(\int_1^T|\chi(1-a-it,\overline{\psi})|\left|\sum_{h=0}^{j}\binom{j}{h}\left(\log\frac{qt}{2\pi}\right)^{j-h}L^{(h)}\left(a+it,\psi\right)\right|\\
&\qquad\left|\frac{\chi'}{\chi}(1-a-it)-\frac{\zeta'}{\zeta}(a+it)\right|\frac{dt}{t\log\frac{qT}{2\pi}}\Bigg).
\end{aligned}$$

Since a is a fixed number >1,

$$L^{(h)}\left(a+it,\psi\right),\frac{\zeta'}{\zeta}(a+it)\ll_h 1.$$

So, by (2.8) and (4.2), the above error term is

$$\ll_j\left(\frac{qT}{2\pi}\right)^{a-\frac{1}{2}}(\log qT)^{j+1}\int_1^T\frac{dt}{t}\ll_{j,\epsilon}(qT)^{a-\frac{1}{2}+\epsilon}.$$

Then

$$\begin{aligned}\overline{I_3} =&\frac{(-1)^{j+1}}{2\pi}\sum_{h=0}^{j}\binom{j}{h}\int_1^T \chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^{j-h} L^{(h)}\left(a+it,\psi\right)\frac{\chi'}{\chi}(1-a-it)dt\\ &-\frac{(-1)^{j+1}}{2\pi}\sum_{h=0}^{j}\binom{j}{h}\int_1^T \chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^{j-h} L^{(h)}\left(a+it,\psi\right)\frac{\zeta'}{\zeta}(a+it)dt\\ &+O_{j,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right). \qquad (7.3)\end{aligned}$$

By (2.8) we can write $-\log\dfrac{qt}{2\pi}+\log q+O\left(\dfrac{1}{t}\right)$ instead of $\dfrac{\chi'}{\chi}(1-a-it)$ in (7.3). Then this equation becomes

$$\begin{aligned}\overline{I_3} =&\frac{(-1)^{j}}{2\pi}\sum_{h=0}^{j}\binom{j}{h}\int_1^T \chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^{j-h+1} L^{(h)}\left(a+it,\psi\right)dt\\ &+\frac{(-1)^{j+1}\log q}{2\pi}\sum_{h=0}^{j}\binom{j}{h}\int_1^T \chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^{j-h} L^{(h)}\left(a+it,\psi\right)dt\\ &+O\left(\sum_{h=0}^{j}\binom{j}{h}\int_1^T |\chi(1-a-it,\overline{\psi})|\left(\log\frac{qt}{2\pi}\right)^{j-h+1} |L^{(h)}\left(a+it,\psi\right)|\frac{dt}{t}\right)\\ &-\frac{(-1)^{j+1}}{2\pi}\sum_{h=0}^{j}\binom{j}{h}\int_1^T \chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^{j-h} L^{(h)}\left(a+it,\psi\right)\frac{\zeta'}{\zeta}(a+it)dt\\ &+O_{j,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right).\end{aligned}$$

The first error term in the right-hand side of the above equation is $\ll_{j,\epsilon}(qT)^{a-\frac{1}{2}+\epsilon}$ similar to the way the error term in (7.3) was obtained. Thus,

$$\begin{aligned}\overline{I_3} = (-1)^j\sum_{h=0}^{j}\binom{j}{h}&\left[A_1(j-h+1,h)-A_1(j-h,h)\log q+A_2(j-h,h)\right]\\ &+O_{j,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right), \qquad (7.4)\end{aligned}$$

where

$$A_1(m,r) := \frac{1}{2\pi}\int_1^T \chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^m L^{(r)}(a+it,\psi)\,dt,$$
$$A_2(m,r) := \frac{1}{2\pi}\int_1^T \chi(1-a-it,\overline{\psi})\left(\log\frac{qt}{2\pi}\right)^m L^{(r)}(a+it,\psi)\,\frac{\zeta'}{\zeta}(a+it)dt$$

and $m, r \in \mathbb{N}$. We note that

$$L^{(r)}(s,\psi) = (-1)^r \sum_{n=1}^{\infty} \frac{\psi(n)\log^r n}{n^s}$$

and we write

$$L^{(r)}(s,\psi)\frac{\zeta'}{\zeta}(s) := \sum_{n=1}^{\infty} \frac{b_r(n)}{n^s}, \tag{7.5}$$

then Lemma 3.15 implies that

$$A_1(m,r) = (-1)^r \frac{\tau\left(\overline{\psi}\right)}{q} \sum_{1\le n\le \frac{qT}{2\pi}} \psi(n)\,e\left(\frac{-n}{q}\right)\log^{m+r} n + O_{m,r,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right) \tag{7.6}$$

and

$$A_2(m,r) = \frac{\tau\left(\overline{\psi}\right)}{q} \sum_{1\le n\le \frac{qT}{2\pi}} b_r(n)\,e\left(\frac{-n}{q}\right)\log^{m} n + O_{m,r,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right).$$

We now calculate $A_1(m,r)$. Firstly, consider the sum

$$\sum_{1\le n\le \frac{qT}{2\pi}} \psi(n)\,e\left(\frac{-n}{q}\right)\log^{m+r} n. \tag{7.7}$$

We convert the exponential factor $e\left(-\frac{n}{q}\right)$ in the above sum into the character sum by the following formula ([6], p.146)

$$e\left(-\frac{n}{q}\right) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(-1)\tau(\chi)\overline{\chi}(n) \tag{7.8}$$

if $(n, q) = 1$. Using this in (7.7), we have

$$\sum_{1 \le n \le \frac{qT}{2\pi}} \psi(n)\, e\left(\frac{-n}{q}\right) \log^{m+r} n = \frac{1}{\phi(q)} \sum_{\chi} \chi(-1)\tau(\chi) \sum_{1 \le n \le \frac{qT}{2\pi}} \left(\overline{\chi}\psi\right)(n)\, \log^{m+r} n. \tag{7.9}$$

We note that (See [12], A.5):

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and $\{b_n\}_{n=1}^{\infty}$ a sequence of real numbers. If $0 \le b_1 \le b_2 \le ...$, then

$$\left| \sum_{M<n\le N} a_n b_n \right| \le 2b_N \max_{M<n\le N} \left| \sum_{M<m\le n} a_m \right|. \tag{7.10}$$

If $\chi \ne \psi$, then the inner sum in (7.9) is

$$\ll \sqrt{q}(\log q) \log^{m+r} qT$$

by the Pólya-Vinogradov inequality and (7.10). Hence,

$$\frac{1}{\phi(q)} \sum_{\chi \ne \psi} \chi(-1)\tau(\chi) \sum_{1 \le n \le \frac{qT}{2\pi}} \left(\overline{\chi}\psi\right)(n)\, \log^{m+r} n \ll q(\log q) \log^{m+r} qT. \tag{7.11}$$

This follows from the facts that $|\tau(\chi)| \le \sqrt{q}$ for any character χ modulo q, and the number of characters modulo q is $\phi(q)$.

Otherwise, if $\chi = \psi$, we consider the following sum

$$\sum_{1 \le n \le x} \chi_0(n) \log^{\mathfrak{e}} n,$$

where χ_0 denotes the principal character modulo q and $\mathfrak{e} \in \mathbb{N}$. Consider the case $\mathfrak{e} = 0$. We have

$$\begin{aligned}\sum_{1 \le n \le x} \chi_0(n) &= \sum_{k=1}^{\left[\!\left[\frac{x}{q}\right]\!\right]} \sum_{n=(k-1)q+1}^{kq} \chi_0(n) + \sum_{q\left[\!\left[\frac{x}{q}\right]\!\right]+1 \le n \le x} \chi_0(n) \\ &= \phi(q) \left[\!\left[\frac{x}{q}\right]\!\right] + O(\phi(q)) \\ &= \frac{\phi(q)}{q} x + O(\phi(q)). \end{aligned} \tag{7.12}$$

Otherwise, if $\mathfrak{e} \ge 1$, by partial summation, we have

$$\sum_{1 \le n \le x} \chi_0(n) \log^{\mathfrak{e}} n = \left\{ \sum_{1 \le n \le x} \chi_0(n) \right\} \log^{\mathfrak{e}} x - \mathfrak{e} \int_1^x \left\{ \sum_{1 \le n \le u} \chi_0(n) \right\} (\log u)^{\mathfrak{e}-1} \frac{du}{u}.$$

Using (7.12) in the above equation gives

$$\begin{aligned}\sum_{1 \le n \le x} \chi_0(n) \log^{\mathfrak{e}} n = &\frac{\phi(q)}{q} x \log^{\mathfrak{e}} x + O\left(\phi(q) \log^{\mathfrak{e}} x\right) \\ &- \mathfrak{e} \int_1^x \left\{ \frac{\phi(q)}{q} u + O\left(\phi(q)\right) \right\} (\log u)^{\mathfrak{e}-1} \frac{du}{u} \\ = &\frac{\phi(q)}{q} x \log^{\mathfrak{e}} x + O\left(\phi(q) \log^{\mathfrak{e}} x\right) - \mathfrak{e} \frac{\phi(q)}{q} \int_1^x (\log u)^{\mathfrak{e}-1} \, du. \end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}\sum_{1 \le n \le x} \chi_0(n) \log^{\mathfrak{e}} n = &\frac{\phi(q)}{q} x \log^{\mathfrak{e}} x + O\left(\phi(q) \log^{\mathfrak{e}} x\right) \\ &- \mathfrak{e} \frac{\phi(q)}{q} x \left\{ \sum_{\nu=0}^{\mathfrak{e}-1} (-1)^{\nu} \frac{(\mathfrak{e}-1)!}{(\mathfrak{e}-1-\nu)!} (\log x)^{\mathfrak{e}-1-\nu} \right\}. \end{aligned} \tag{7.13}$$

Combining (7.12) and (7.13), for $\mathfrak{e} = 0, 1, 2, ...$, we have

$$\sum_{1\le n\le x} \chi_0(n)\log^{\mathfrak{e}} n = \frac{\phi(q)}{q} x\log^{\mathfrak{e}} x + O\left(\phi(q)\log^{\mathfrak{e}} x\right)$$
$$- a_{\mathfrak{e}}\frac{\phi(q)}{q}\, x \left\{\sum_{\nu=0}^{\mathfrak{e}-a_{\mathfrak{e}}}(-1)^{\nu}\frac{\mathfrak{e}!}{(\mathfrak{e}-a_{\mathfrak{e}}-\nu)!}(\log x)^{\mathfrak{e}-a_{\mathfrak{e}}-\nu}\right\}, \tag{7.14}$$

where

$$a_{\mathfrak{e}} := \begin{cases} 0 & \text{if } \mathfrak{e} = 0, \\ 1 & \text{if } \mathfrak{e} \ge 1. \end{cases}$$

Then, it easily follows from (7.9), (7.11), (7.14) and the fact $|\tau(\psi)| = \sqrt{q}$ that

$$\sum_{1\le n\le \frac{qT}{2\pi}} \psi(n)e\left(\frac{-n}{q}\right)\log^{m+r} n = \psi(-1)\tau(\psi)\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^{m+r} + O\left(q\log q\log^{m+r} qT\right)$$
$$- a_{m+r}\psi(-1)\tau(\psi)\frac{T}{2\pi}\left\{\sum_{\nu=0}^{m+r-a_{m+r}}\frac{(-1)^{\nu}(m+r)!}{(m+r-a_{m+r}-\nu)!}\left(\log\frac{qT}{2\pi}\right)^{m+r-a_{m+r}-\nu}\right\}. \tag{7.15}$$

Using the facts $|\tau(\psi)| = \sqrt{q}$, $\tau\left(\overline{\psi}\right) = \psi(-1)\overline{\tau(\psi)}$, and (7.15) in (7.6), we have

$$A_1(m,r) = (-1)^r\frac{T}{2\pi}\Bigg\{\left(\log\frac{qT}{2\pi}\right)^{m+r} - a_{m+r}\Bigg[\sum_{\nu=0}^{m+r-a_{m+r}}\frac{(-1)^{\nu}(m+r)!}{(m+r-a_{m+r}-\nu)!}$$
$$\left(\log\frac{qT}{2\pi}\right)^{m+r-a_{m+r}-\nu}\Bigg]\Bigg\} + O_{m,r,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right). \tag{7.16}$$

From (7.16), it is easy to see that the h dependence of

$$A_1(j-h+1,h) \qquad \text{and} \qquad A_1(j-h,h)$$

is only the factor $(-1)^h$. So the sums

$$\sum_{h=0}^{j}\binom{j}{h}A_1(j-h+1,h) \qquad \text{and} \qquad \sum_{h=0}^{j}\binom{j}{h}A_1(j-h,h)$$

contain the factor $\displaystyle\sum_{h=0}^{j}\binom{j}{h}(-1)^h$, which is 0 since $j \geq 1$. Thus, (7.4) takes form

$$\overline{I_3} = \begin{cases} (-1)^j \sum_{h=0}^{j} \binom{j}{h} A_2(j-h,h) + O_{j,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right) & \text{if } j \geq 1, \\ \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + A_2(0,0) + O_\epsilon\left((qT)^{a-\frac{1}{2}+\epsilon}\right) & \text{if } j = 0. \end{cases} \tag{7.17}$$

Next, we calculate $A_2(j-h,h)$, $h = 0, 1, ..., j$,

$$A_2(j-h,h) = \frac{\tau\left(\overline{\psi}\right)}{q} \sum_{1 \leq n \leq \frac{qT}{2\pi}} b_h(n)\, e\left(\frac{-n}{q}\right) \log^{j-h} n \, + O_{j,h,\epsilon}\left((qT)^{a-\frac{1}{2}+\epsilon}\right). \tag{7.18}$$

Consider the sum

$$\sum_{1 \leq n \leq \frac{qT}{2\pi}} b_h(n)\, e\left(\frac{-n}{q}\right) \log^{j-h} n.$$

From (7.5), we see that

$$b_h(n) = (-1)^{h+1} \sum_{d|n} \psi(d)\,(\log d)^h\, \Lambda\left(\frac{n}{d}\right).$$

We then have

$$
\begin{aligned}
&\sum_{1\le n\le \frac{qT}{2\pi}} b_h(n) e\left(\frac{-n}{q}\right)\log^{j-h} n \\
&\qquad = (-1)^{h+1} \sum_{1\le n\le \frac{qT}{2\pi}} \sum_{d|n} \psi(d)(\log d)^h \Lambda\left(\frac{n}{d}\right) e\left(\frac{-n}{q}\right)\log^{j-h} n \\
&\qquad = (-1)^{h+1} \sum_{dm\le \frac{qT}{2\pi}} \psi(d)(\log d)^h \Lambda(m) e\left(\frac{-dm}{q}\right)\log^{j-h} dm \\
&\qquad = (-1)^{h+1} \sum_{\eta=0}^{j-h} \binom{j-h}{\eta} \sum_{dm\le \frac{qT}{2\pi}} \psi(d)(\log d)^{j-\eta} \Lambda(m) e\left(\frac{-dm}{q}\right)\log^{\eta} m \\
&\qquad = (-1)^{h+1} \sum_{\eta=0}^{j-h} \binom{j-h}{\eta} \Bigg\{ \sum_{d\le\sqrt{\frac{qT}{2\pi}}} \sum_{m\le \frac{qT}{2\pi d}} + \sum_{m\le\sqrt{\frac{qT}{2\pi}}} \sum_{d\le \frac{qT}{2\pi m}} \\
&\qquad\qquad - \sum_{d\le\sqrt{\frac{qT}{2\pi}}} \sum_{m\le\sqrt{\frac{qT}{2\pi}}} \Bigg\} \psi(d)(\log d)^{j-\eta} \Lambda(m) e\left(\frac{-dm}{q}\right)\log^{\eta} m \\
&\qquad = (-1)^{h+1} \sum_{\eta=0}^{j-h} \binom{j-h}{\eta} \left\{ S_1 + S_2 - S_3 \right\}, \text{say.}
\end{aligned}
\tag{7.19}
$$

We write S_1 in the form:

$$
\begin{aligned}
S_1 = \sum_{\substack{d\le\sqrt{\frac{qT}{2\pi}} \\ (d,q)=1}} \psi(d)\,(\log d)^{j-\eta} \;\Bigg\{ &\sum_{\substack{m\le \frac{qT}{2\pi d} \\ (m,q)=1}} \Lambda(m)\, e\left(\frac{-dm}{q}\right)\log^{\eta} m \\
&+ \sum_{\substack{m\le \frac{qT}{2\pi d} \\ (m,q)>1}} \Lambda(m)\, e\left(\frac{-dm}{q}\right)\log^{\eta} m \Bigg\}.
\end{aligned}
$$

Using (7.8), we obtain

$$
\begin{aligned}
S_1 = \frac{1}{\phi(q)} &\sum_{d\le\sqrt{\frac{qT}{2\pi}}} \psi(d)\, \log^{j-\eta} d \sum_{\chi} \chi(-1)\,\tau(\chi)\overline{\chi}(d) \sum_{m\le \frac{qT}{2\pi d}} \Lambda(m)\,\overline{\chi}(m)\log^{\eta} m \\
&+ O\Bigg(\sum_{\substack{d\le\sqrt{\frac{qT}{2\pi}} \\ (d,q)=1}} \log^{j-\eta} d \sum_{\substack{m\le \frac{qT}{2\pi d} \\ (m,q)>1}} \Lambda(m)\log^{\eta} m \Bigg).
\end{aligned}
$$

The error term in the right of the above equation is equal to

$$\sum_{\substack{d \le \sqrt{\frac{qT}{2\pi}} \\ (d,q)=1}} \log^{j-\eta} d \sum_{\substack{p|q \\ p:\text{prime}}} \log p \sum_{\substack{\alpha \\ p^\alpha \le \frac{qT}{2\pi d}}} (\log p^\alpha)^\eta .$$

The number of terms in the innermost sum of the above is

$$\le \frac{\log \frac{qT}{2\pi d}}{\log p} \le \frac{\log qT}{\log p}.$$

So, the error term can be bounded by

$$\log^{\eta+1} qT \sum_{\substack{p|q \\ p:\text{prime}}} 1 \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \log^{j-\eta} d$$

It is easy to see that

$$\sum_{\substack{p|q \\ p:\text{prime}}} 1 \ll \log q.$$

Using this and Lemma 3.2, the error term is

$$\ll_{j,\eta,\epsilon} (qT)^{\frac{1}{2}+\epsilon}.$$

So, we get

$$\begin{aligned}
S_1 =& \frac{1}{\phi(q)} \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \psi(d) \log^{j-\eta} d \sum_{\chi} \chi(-1)\, \tau(\chi) \overline{\chi}(d) \sum_{m \le \frac{qT}{2\pi d}} \Lambda(m)\, \overline{\chi}(m) \log^{\eta} m \\
&+ O_{j,\eta,\epsilon}\left((qT)^{\frac{1}{2}+\epsilon} \right) \\
=& \frac{\tau(\chi_0)}{\phi(q)} \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \psi(d) \log^{j-\eta} d \sum_{m \le \frac{qT}{2\pi d}} \Lambda(m)\, \overline{\chi_0}(m) \log^{\eta} m \\
&+ \frac{1}{\phi(q)} \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \psi(d) \log^{j-\eta} d \sum_{\chi \ne \chi_0} \chi(-1)\, \tau(\chi) \overline{\chi}(d) \sum_{m \le \frac{qT}{2\pi d}} \Lambda(m)\, \overline{\chi}(m) \log^{\eta} m \\
&+ O_{j,\eta,\epsilon}\left((qT)^{\frac{1}{2}+\epsilon} \right).
\end{aligned} \tag{7.20}$$

Using Proposition 5.4 in (7.20), we obtain

$$\begin{aligned}
S_1 =& \frac{\tau(\chi_0) qT}{2\pi\phi(q)} \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \frac{\psi(d) \log^{j-\eta} d}{d} \sum_{\nu=0}^{\eta} \binom{\eta}{\nu} (-1)^{\eta-\nu} (\eta-\nu)! \left(\log \frac{qT}{2\pi d} \right)^{\nu} \\
&+ O_{j,\eta,\epsilon}\left((qT)^{\frac{1}{2}+\epsilon} \right) + O_{A,\eta}\left\{ \frac{qT}{\phi(q)} \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \frac{\log^{j-\eta} d}{d} \sum_{\chi} |\tau(\chi)| \exp\left(-c_6 \sqrt{\log \frac{qT}{2\pi d}} \right) \right\},
\end{aligned} \tag{7.21}$$

where $c_6 = \min(c_2, c_5)$. This holds uniformly with respect to q in the range $q \le \log^A T$. Since the number of characters modulo q is $\phi(q)$, $\tau(\chi_0) = \mu(q)$ and $|\tau(\chi)| \le \sqrt{q}$ for any character χ modulo q, we can conclude that

$$\begin{aligned}
S_1 =& \frac{\mu(q) qT}{2\pi\phi(q)} \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \frac{\psi(d) \log^{j-\eta} d}{d} \sum_{\nu=0}^{\eta} \binom{\eta}{\nu} (-1)^{\eta-\nu} (\eta-\nu)! \left(\log \frac{qT}{2\pi d} \right)^{\nu} \\
&+ O_{j,\eta,\epsilon}\left((qT)^{\frac{1}{2}+\epsilon} \right) + O_{A,\eta}\left(q^{\frac{3}{2}} T \sum_{d \le \sqrt{\frac{qT}{2\pi}}} \frac{\log^{j-\eta} d}{d} \exp\left(-c_6 \sqrt{\log \frac{qT}{2\pi d}} \right) \right).
\end{aligned} \tag{7.22}$$

If $d \le \sqrt{\frac{qT}{2\pi}}$, then we have

$$\exp\left(-c_6\sqrt{\log\frac{qT}{2\pi d}}\right) \le \exp\left(-c_7\sqrt{\log qT}\right),$$

where $0 < c_7 < c_6$. So the error terms in the right of (7.22) is

$$\ll_{A,j,\eta} q^{\frac{3}{2}}T(\log qT)^{j-\eta}\exp\left(-c_7\sqrt{\log qT}\right)\sum_{d\le\sqrt{\frac{qT}{2\pi}}}\frac{1}{d}.$$

By (3.27), since $q \le \log^A T$, the error term can be majorized by

$$C(A,j,\eta)T\exp\left(-c_8\sqrt{\log T}\right),$$

where $C(A,j,\eta)$ is a non-effective constant and $0 < c_8 < c_7$. Thus, (7.22) becomes

$$\begin{aligned} S_1 =& \frac{\mu(q)qT}{2\pi\phi(q)}\sum_{d\le\sqrt{\frac{qT}{2\pi}}}\frac{\psi(d)\log^{j-\eta}d}{d}\sum_{\nu=0}^{\eta}\binom{\eta}{\nu}(-1)^{\eta-\nu}(\eta-\nu)!\left(\log\frac{qT}{2\pi d}\right)^{\nu} \\ &+ O_{A,j,\eta}\left(T\exp\left(-c_8\sqrt{\log T}\right)\right). \end{aligned} \tag{7.23}$$

In (7.23) replacing $\left(\log\frac{qT}{2\pi d}\right)^{\nu}$ by $\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}(-\log d)^{\kappa}\left(\log\frac{qT}{2\pi}\right)^{\nu-\kappa}$ gives

$$\begin{aligned} S_1 =& \frac{\mu(q)qT}{2\pi\phi(q)}\sum_{\nu=0}^{\eta}\sum_{\kappa=0}^{\nu}\binom{\eta}{\nu}\binom{\nu}{\kappa}(-1)^{\eta+\kappa-\nu}(\eta-\nu)!\left(\log\frac{qT}{2\pi}\right)^{\nu-\kappa}\sum_{d\le\sqrt{\frac{qT}{2\pi}}}\frac{\psi(d)\log^{j+\kappa-\eta}d}{d} \\ &+ O_{A,j,\eta}\left(T\exp\left(-c_8\sqrt{\log T}\right)\right) \end{aligned} \tag{7.24}$$

if $q \le \log^A T$. By Proposition 5.1, we have

$$(-1)^{j+\kappa-\eta}\sum_{d\le\sqrt{\frac{qT}{2\pi}}}\frac{\psi(d)\log^{j+\kappa-\eta}d}{d} = L^{(j+\kappa-\eta)}(1,\psi) + O_{j+\kappa-\eta}\left(\frac{\log^{j+\nu-\eta-\kappa}qT}{\sqrt{qT}}\right).$$

Using this in (7.24), we get

$$
\begin{aligned}
S_1 =& \frac{\mu(q)qT}{2\pi\phi(q)} \sum_{\nu=0}^{\eta}\sum_{\kappa=0}^{\nu} \binom{\eta}{\nu}\binom{\nu}{\kappa}(-1)^{j-\nu}(\eta-\nu)! \left(\log\frac{qT}{2\pi}\right)^{\nu-\kappa} L^{(j+\kappa-\eta)}(1,\psi) \\
&+ O_{A,j,\eta}\left(T\exp\left(-c_8\sqrt{\log T}\right)\right) \qquad (7.25)
\end{aligned}
$$

for $q \le \log^A T$.

To treat S_2 we note that

$$
\begin{aligned}
S_2 &= \sum_{m\le\sqrt{\frac{qT}{2\pi}}}\sum_{d\le\frac{qT}{2\pi m}} \psi(d)\,(\log d)^{j-\eta}\,\Lambda(m)\,e\left(\frac{-md}{q}\right)\log^\eta m \\
&= \Bigg\{\sum_{\substack{m\le\sqrt{\frac{qT}{2\pi}}\\(m,q)=1}}\sum_{d\le\frac{qT}{2\pi m}} + \sum_{\substack{m\le\sqrt{\frac{qT}{2\pi}}\\(m,q)>1}}\sum_{d\le\frac{qT}{2\pi m}}\Bigg\}\psi(d)\,(\log d)^{j-\eta}\,\Lambda(m)\,e\left(\frac{-md}{q}\right)\log^\eta m \\
&= S_{21}+S_{22}, \text{say.} \qquad (7.26)
\end{aligned}
$$

We first deal with S_{22}. We rewrite S_{22} in the form

$$
\sum_{\substack{m\le\sqrt{\frac{qT}{2\pi}}\\(m,q)>1}} \Lambda(m)\log^\eta m \sum_{d\le\frac{qT}{2\pi m}} \psi(d)\,(\log d)^{j-\eta}\,e\left(\frac{\frac{-md}{(m,q)}}{\frac{q}{(m,q)}}\right).
$$

In the sum over d each term with $(d,q)>1$ vanishes, since $\psi(d)=0$, so we can assume $(d,q)=1$ and this gives $\left(\frac{md}{(m,q)},\frac{q}{(m,q)}\right)=1$. Thus, we can use (7.8) and have

$$
\begin{aligned}
S_{22} &= \sum_{\substack{m\le\sqrt{\frac{qT}{2\pi}}\\(m,q)>1}} \Lambda(m)\log^\eta m \sum_{d\le\frac{qT}{2\pi m}} \psi(d)(\log d)^{j-\eta}\frac{1}{\phi\left(\frac{q}{(m,q)}\right)}\sum_{\chi\left(\mathrm{mod}\ \frac{q}{(m,q)}\right)} \chi(-1)\tau(\chi)\overline{\chi}\left(\frac{md}{(m,q)}\right) \\
&= \sum_{\substack{m\le\sqrt{\frac{qT}{2\pi}}\\(m,q)>1}} \Lambda(m)(\log m)^\eta\frac{1}{\phi\left(\frac{q}{(m,q)}\right)}\sum_{\chi\left(\mathrm{mod}\ \frac{q}{(m,q)}\right)} \chi(-1)\tau(\chi)\overline{\chi}\left(\frac{m}{(m,q)}\right)\sum_{d\le\frac{qT}{2\pi m}}\left(\psi\overline{\chi}\right)(d)\log^{j-\eta} d.
\end{aligned}
$$

(7.27)

Since ψ is a non-principal character modulo q and χ is a character modulo $q/(m,q)$, $\psi\overline{\chi}$ is a character modulo q. If $\psi\overline{\chi}$ were principal, then $\psi(n)\overline{\chi}(n) = 1$ for any n with $(n,q) = 1$. So $\psi(n) = \chi(n)$, and since the period of χ is equal to $q/(m,q) < q\,,((m,q) > 1)$, this would contradict the primitivity of ψ. Hence, $\psi\overline{\chi}$ is non-principal for any χ modulo $q/(m,q)$. Then the Pólya-Vinogradov inequality and (7.10) implies that

$$\sum_{d \le \frac{qT}{2\pi m}} (\psi\overline{\chi})\,(d)\,\log^{j-\eta} d \ll \sqrt{q}(\log q)\log^{j-\eta}(qT) \tag{7.28}$$

for any χ modulo $q/(m,q)$. Combining (7.27), (7.28) and the following facts that

$$|\tau(\chi)| \le \sqrt{\frac{q}{(m,q)}}$$

for any character χ modulo $q/(m,q)$, and the number of characters modulo $q/(m,q)$ is $\phi\left(q/(m,q)\right)$, we obtain

$$S_{22} \ll \sqrt{q}(\log q)\log^{j-\eta}(qT) \sum_{\substack{m \le \sqrt{\frac{qT}{2\pi}} \\ (m,q)>1}} \Lambda(m)(\log m)^{\eta} \sqrt{\frac{q}{(m,q)}}$$

By the prime number theorem, the last equation becomes

$$S_{22} \ll_{\epsilon,j,A} (T)^{\frac{1}{2}+\epsilon} \tag{7.29}$$

for $q \le \log^{A} T$.

To estimate S_{21}, as we did S_{22}, we use (7.8) to obtain

$$S_{21} = \frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \chi(-1)\tau(\chi) \sum_{m \le \sqrt{\frac{qT}{2\pi}}} \Lambda(m)\,\overline{\chi}(m)\log^{\eta} m \sum_{d \le \frac{qT}{2\pi m}} (\psi\overline{\chi})\,(d)\,\log^{j-\eta} d.$$

We separate the terms with $\chi = \psi$ and put

$$\begin{aligned} S_{211} &:= \frac{1}{\phi(q)}\,\psi(-1)\tau(\psi) \sum_{m \le \sqrt{\frac{qT}{2\pi}}} \Lambda(m)\,\overline{\psi}(m) \log^{\eta} m \sum_{d \le \frac{qT}{2\pi m}} \chi_0(d)\, \log^{j-\eta} d, \\ S_{212} &:= S_{21} - S_{211} \qquad\qquad (7.30) \\ &= \frac{1}{\phi(q)} \sum_{\chi \ne \psi} \chi(-1)\tau(\chi) \sum_{m \le \sqrt{\frac{qT}{2\pi}}} \Lambda(m)\,\overline{\chi}(m) \log^{\eta} m \sum_{d \le \frac{qT}{2\pi m}} (\psi\overline{\chi})\,(d)\, \log^{j-\eta} d. \end{aligned}$$

We first deal with the innermost sum in S_{212}. Since $\chi \ne \psi$, $\psi\overline{\chi}$ is non-principal. Hence, by the Pólya-Vinogradov inequality and (7.10), we can write

$$\sum_{d \le \frac{qT}{2\pi m}} (\psi\overline{\chi})\,(d)\, \log^{j-\eta} d \ll \sqrt{q}(\log q) \left(\log \frac{qT}{2\pi m}\right)^{j-\eta}.$$

Using this and the prime number theorem, we have

$$S_{212} \ll \frac{q\sqrt{T}(\log q) \log^{j}(qT)}{\phi(q)} \sum_{\chi \ne \psi} |\tau(\chi)|.$$

Since the number of characters modulo q is $\phi(q)$ and $|\tau(\chi)| \le \sqrt{q}$ for any non-principal character χ,

$$S_{212} \ll_{A,\epsilon,j} T^{\frac{1}{2}+\epsilon} \tag{7.31}$$

for $q \le \log^{A} T$.

To treat S_{211}, by (7.14), we first note that

$$\sum_{1\le d\le \frac{qT}{2\pi m}} \chi_0(d)\log^{j-\eta} d = \phi(q)\frac{T}{2\pi m}\left(\log\frac{qT}{2\pi m}\right)^{j-\eta} + O\left(\phi(q)\left(\log\frac{qT}{2\pi m}\right)^{j-\eta}\right)$$
$$- a_{j-\eta}\phi(q)\frac{T}{2\pi m}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}(-1)^{\nu}\frac{(j-\eta)!}{(j-\eta-a_{j-\eta}-\nu)!}\left(\log\frac{qT}{2\pi m}\right)^{j-\eta-a_{j-\eta}-\nu}$$
$$= \phi(q)\frac{T}{2\pi m}\sum_{\kappa=0}^{j-\eta}\binom{j-\eta}{\kappa}(-1)^{\kappa}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\log^{\kappa} m$$
$$- a_{j-\eta}\phi(q)\,\frac{T}{2\pi m}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}(-1)^{\nu}\frac{(j-\eta)!}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}(-1)^{\kappa}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\log^{\kappa} m$$
$$+ O\left(\phi(q)\left(\log\frac{qT}{2\pi m}\right)^{j-\eta}\right).$$

Using this result in S_{211}, we have

$$S_{211} = \psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\kappa=0}^{j-\eta}\binom{j-\eta}{\kappa}(-1)^{\kappa}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\sum_{m\le\sqrt{\frac{qT}{2\pi}}}\frac{\Lambda(m)\,\overline{\psi}(m)\log^{\kappa+\eta} m}{m}$$
$$- a_{j-\eta}\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\frac{(-1)^{\nu}(j-\eta)!}{(j-\eta-a_{j-\eta}-\nu)!}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}$$
$$(-1)^{\kappa}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\sum_{m\le\sqrt{\frac{qT}{2\pi}}}\frac{\Lambda(m)\,\overline{\psi}(m)\log^{\kappa+\eta} m}{m}$$
$$+ O\left(|\tau(\psi)|\sum_{m\le\sqrt{\frac{qT}{2\pi}}}\Lambda(m)\,(\log m)^{\eta}\left(\log\frac{qT}{2\pi m}\right)^{j-\eta}\right). \tag{7.32}$$

It easily follows from the prime number theorem and the fact that $|\tau(\psi)| = \sqrt{q}$ that the above error term is

$$\ll_{A,j,\epsilon} T^{\frac{1}{2}+\epsilon} \tag{7.33}$$

in the range $q \le \log^A T$.

Combining (7.30), (7.31), (7.32) and (7.33) gives

$$
\begin{aligned}
S_{21} &= \psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\kappa=0}^{j-\eta}\binom{j-\eta}{\kappa}(-1)^{\kappa}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\sum_{m\le\sqrt{\frac{qT}{2\pi}}}\frac{\Lambda(m)\,\overline{\psi}(m)\log^{\kappa+\eta} m}{m}\\
&- a_{j-\eta}\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\frac{(-1)^{\nu}(j-\eta)!}{(j-\eta-a_{j-\eta}-\nu)!}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}(-1)^{\kappa}\\
&\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\sum_{m\le\sqrt{\frac{qT}{2\pi}}}\frac{\Lambda(m)\,\overline{\psi}(m)\log^{\kappa+\eta} m}{m}+O_{A,\epsilon,j}\left(T^{\frac{1}{2}+\epsilon}\right). \qquad (7.34)
\end{aligned}
$$

if $q \le \log^{A} T$. It follows from Proposition 5.3 that

$$
(-1)^{\kappa+\eta+1}\sum_{m\le\sqrt{\frac{qT}{2\pi}}}\frac{\Lambda(m)\overline{\psi}(m)\log^{\kappa+\eta} m}{m}=\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})+O_{A,\kappa+\eta}\left(\exp\left(-c_2\sqrt{\frac{1}{2}\log\frac{qT}{2\pi}}\right)\right), \qquad (7.35)
$$

under the same condition $q \le \log^{A} T$. Then, using (7.35) and the fact $|\tau(\psi)| = \sqrt{q}$ in (7.34), we have

$$
\begin{aligned}
S_{21} &= -\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\kappa=0}^{j-\eta}\binom{j-\eta}{\kappa}(-1)^{\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})\\
&\quad + a_{j-\eta}\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\frac{(-1)^{\nu+\eta}(j-\eta)!}{(j-\eta-a_{j-\eta}-\nu)!}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}\\
&\quad \left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})+O_{A,j,\eta}\left(T\exp\left(-c_9\sqrt{\log T}\right)\right), \qquad (7.36)
\end{aligned}
$$

where $0 < c_9 < c_2/\sqrt{2}$ and $q \le \log^{A} T$.

Combining (7.26), (7.29) and (7.36), we have

$$
\begin{aligned}
S_2 = &-\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\kappa=0}^{j-\eta}\binom{j-\eta}{\kappa}(-1)^{\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})\\
&+a_{j-\eta}\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\frac{(-1)^{\nu+\eta}(j-\eta)!}{(j-\eta-a_{j-\eta}-\nu)!}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}\\
&\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})+O_{A,j,\eta}\left(T\exp\left(-c_9\sqrt{\log T}\right)\right),
\end{aligned}
\tag{7.37}
$$

for $q \le \log^A T$.

Finally, S_3 can be treated as S_1. In all steps in the estimation of S_1, we should replace the range $1 \le m \le \sqrt{\frac{qT}{2\pi d}}$ with $1 \le m \le \sqrt{\frac{qT}{2\pi}}$ and then, it is easy to verify that for $q \le \log^A T$

$$
\begin{aligned}
S_3 = &\frac{\tau(\chi_0)}{\phi(q)}\sqrt{\frac{qT}{2\pi}}\sum_{\nu=0}^{\eta}\binom{\eta}{\nu}(-1)^{\eta-\nu}(\eta-\nu)!\left(\log\frac{qT}{2\pi}\right)^{\nu}\sum_{d\le\sqrt{\frac{qT}{2\pi}}}\psi(d)\log^{j-\eta} d\\
&+O_{j,\eta,\epsilon}\left((qT)^{\frac{1}{2}+\epsilon}\right)+O_{A,\eta}\left\{\frac{\sqrt{qT}}{\phi(q)}\sum_{d\le\sqrt{\frac{qT}{2\pi}}}\log^{j-\eta} d\sum_{\chi}|\tau(\chi)|\exp\left(-c_6\sqrt{\log\frac{qT}{2\pi}}\right)\right\},
\end{aligned}
\tag{7.38}
$$

which is analogous to (7.21). By Lemma 3.2, it is easy to see that the above error terms can be bounded by $T\exp\left(-c_{10}\sqrt{\log T}\right)$ in the range considered for q. This upper bound is also valid for the first part in the right of (7.38) since

$$
\sum_{d\le\sqrt{\frac{qT}{2\pi}}}\psi(d)\log^{j-\eta} d \ll \sqrt{q}(\log q)\log^{j-\eta}(qT).
$$

Then, we have

$$
S_3 \ll_{A,j,\eta} T\exp\left(-c_{10}\sqrt{\log T}\right). \tag{7.39}
$$

Combining (7.19), (7.25), (7.37) and (7.39), we obtain

$$\sum_{1\le n\le \frac{qT}{2\pi}} b_h(n)e\left(\frac{-n}{q}\right)\log^{j-h} n$$
$$= -\frac{\mu(q)qT}{2\pi\phi(q)}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{\eta}\sum_{\kappa=0}^{\nu}\binom{j-h}{\eta}\binom{\eta}{\nu}\binom{\nu}{\kappa}(-1)^{j+h-\nu}(\eta-\nu)!\left(\log\frac{qT}{2\pi}\right)^{\nu-\kappa}L^{(j+\kappa-\eta)}(1,\psi)$$
$$+\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\eta=0}^{j-h}\sum_{\kappa=0}^{j-\eta}\binom{j-h}{\eta}\binom{j-\eta}{\kappa}(-1)^{h+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})$$
$$-\psi(-1)\tau(\psi)\frac{T}{2\pi}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j-h}{\eta}\frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}(-1)^{h+\nu+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})$$
$$+O_{A,j,h}\left(T\exp\left(-c_{11}\sqrt{\log T}\right)\right), \tag{7.40}$$

where $0 < c_{11} < \min\{c_8, c_9, c_{10}\}$ and $q \le \log^A T$. Since $\overline{\tau(\psi)} = \psi(-1)\tau\left(\overline{\psi}\right)$ and $|\tau(\psi)| = \sqrt{q}$, combining (7.18) and (7.40) gives

$$A_2(j-h,h) = -\frac{\mu(q)\psi(-1)\overline{\tau(\psi)}T}{2\pi\phi(q)}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{\eta}\sum_{\kappa=0}^{\nu}\binom{j-h}{\eta}$$
$$\binom{\eta}{\nu}\binom{\nu}{\kappa}(-1)^{j+h-\nu}(\eta-\nu)!\left(\log\frac{qT}{2\pi}\right)^{\nu-\kappa}L^{(j+\kappa-\eta)}(1,\psi)$$
$$+\frac{T}{2\pi}\sum_{\eta=0}^{j-h}\sum_{\kappa=0}^{j-\eta}\binom{j-h}{\eta}\binom{j-\eta}{\kappa}(-1)^{h+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})$$
$$-\frac{T}{2\pi}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j-h}{\eta}\frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}(-1)^{h+\nu+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi})$$
$$+O_{A,j,h}\left(T\exp\left(-c_{11}\sqrt{\log T}\right)\right). \tag{7.41}$$

So, we have

$$A_2(0,0) = -\frac{\mu(q)\psi(-1)\overline{\tau(\psi)}}{\phi(q)}\frac{T}{2\pi}L(1,\psi) + \frac{T}{2\pi}\left(\frac{L'}{L}\right)(1,\overline{\psi})$$

$$+ O_A\left(T\exp\left(-c_{11}\sqrt{\log T}\right)\right).$$

Using this in (7.17), after this, taking complex conjugate gives

$$I_3 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} - \frac{\mu(q)\psi(-1)\tau(\psi)T}{2\pi\phi(q)}L(1,\overline{\psi}) + \frac{T}{2\pi}\frac{L'}{L}(1,\psi) + O_A\left(T\exp\left(-c_{11}\sqrt{\log T}\right)\right)$$

for $q \leq \log^A T$. Combining this with (7.1) and (7.2) , we get Theorem 1.2 for the case $j = 0$ under the restriction $T \in \mathscr{F}$, but this can be easily removed, as we did in Theorem 1.1.

In the case $j \geq 1$ we can deduce from (7.17) and (7.41) that

$$\begin{aligned}
\overline{I_3} = &-\frac{\mu(q)\psi(-1)\overline{\tau(\psi)}T}{2\pi\phi(q)}\sum_{h=0}^{j}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{\eta}\sum_{\kappa=0}^{\nu}\binom{j}{h}\binom{j-h}{\eta} \\
&\binom{\eta}{\nu}\binom{\nu}{\kappa}(-1)^{h-\nu}(\eta-\nu)!\left(\log\frac{qT}{2\pi}\right)^{\nu-\kappa}L^{(j+\kappa-\eta)}(1,\psi) \\
&+\frac{T}{2\pi}\sum_{h=0}^{j}\sum_{\eta=0}^{j-h}\sum_{\kappa=0}^{j-\eta}\binom{j}{h}\binom{j-h}{\eta}\binom{j-\eta}{\kappa}(-1)^{j+h+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi}) \\
&-\frac{T}{2\pi}\sum_{h=0}^{j}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j}{h}\binom{j-h}{\eta}\frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!} \\
&\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}(-1)^{j+h+\nu+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\overline{\psi}) \\
&+O_{A,j}\left(T\exp\left(-c_{11}\sqrt{\log T}\right)\right).
\end{aligned}$$

Taking complex conjugate of the above equation and combining this result with (7.1) and (7.2), we get

$$\begin{aligned}
\sum_{0<\gamma\le T} L^{(j)}(\rho,\psi) = &-\frac{\mu(q)\psi(-1)\tau(\psi)T}{2\pi\phi(q)}\sum_{h=0}^{j}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{\eta}\sum_{\kappa=0}^{\nu}\binom{j}{h}\binom{j-h}{\eta}\\
&\binom{\eta}{\nu}\binom{\nu}{\kappa}(-1)^{h-\nu}(\eta-\nu)!\left(\log\frac{qT}{2\pi}\right)^{\nu-\kappa} L^{(j+\kappa-\eta)}(1,\overline{\psi})\\
&+\frac{T}{2\pi}\sum_{h=0}^{j}\sum_{\eta=0}^{j-h}\sum_{\kappa=0}^{j-\eta}\binom{j}{h}\binom{j-h}{\eta}\binom{j-\eta}{\kappa}(-1)^{j+h+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\psi)\\
&-\frac{T}{2\pi}\sum_{h=0}^{j}\sum_{\eta=0}^{j-h}\sum_{\nu=0}^{j-\eta-a_{j-\eta}}\sum_{\kappa=0}^{j-\eta-a_{j-\eta}-\nu}\binom{j}{h}\binom{j-h}{\eta}\frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}\\
&\binom{j-\eta-a_{j-\eta}-\nu}{\kappa}(-1)^{j+h+\nu+\eta}\left(\log\frac{qT}{2\pi}\right)^{j-\eta-a_{j-\eta}-\nu-\kappa}\left(\frac{L'}{L}\right)^{(\kappa+\eta)}(1,\psi)\\
&+O_{A,j}\left(T\exp\left(-c_{11}\sqrt{\log T}\right)\right) \qquad (7.42)
\end{aligned}$$

if $j\ge 1$ and $q\le \log^A T$. By the substitutions $\omega=\nu-\kappa$ in the first part, $\omega=\eta+\kappa$ in the second part, and $\omega=\eta+a_{j-\eta}+\nu+\kappa$ in the third part of the right hand side of (7.42), we obtain

$$\begin{aligned}
\sum_{0<\gamma\le T} L^{(j)}(\rho,\psi) = &-\frac{\mu(q)\psi(-1)\tau(\psi)}{\phi(q)}\frac{T}{2\pi}\sum_{\omega=0}^{j}\mathcal{U}_1(\omega,\psi)\left(\log\frac{qT}{2\pi}\right)^{\omega}\\
&+\frac{T}{2\pi}\sum_{\omega=0}^{j}\mathcal{U}_2(\omega)\left(\log\frac{qT}{2\pi}\right)^{j-\omega}\left(\frac{L'}{L}\right)^{(\omega)}(1,\psi)\\
&-\frac{T}{2\pi}\sum_{\omega=0}^{j}\mathcal{U}_3(\omega,\psi)\left(\log\frac{qT}{2\pi}\right)^{j-\omega}\\
&+O_{A,j}\left(T\exp\left(-c_{11}\sqrt{\log T}\right)\right), \qquad (7.43)
\end{aligned}$$

where

$$\mathcal{U}_1(\omega,\psi) := \sum_{h=0}^{j-\omega}\sum_{\eta=\omega}^{j-h}\sum_{\nu=\omega}^{\eta}\binom{j}{h}\binom{j-h}{\eta}\binom{\eta}{\nu}\binom{\nu}{\nu-\omega}(-1)^{h-\nu}(\eta-\nu)!L^{(j+\nu-\omega-\eta)}(1,\overline{\psi}),$$

$$\mathcal{U}_2(\omega) := \sum_{h=0}^{j}\sum_{\eta=0}^{\min(j-h,\omega)}\binom{j}{h}\binom{j-h}{\eta}\binom{j-\eta}{\omega-\eta}(-1)^{j+\eta+h},$$

$$\mathcal{U}_3(\omega,\psi) := \sum_{h=0}^{j}\sum_{\eta=0}^{\min\left\{j-h,\omega-1+[\![\frac{\omega}{j}]\!](1-a_h)\right\}}\sum_{\nu=0}^{\omega-\eta-a_{j-\eta}}\binom{j}{h}\binom{j-h}{\eta}\frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$\binom{j-\eta-a_{j-\eta}-\nu}{\omega-\eta-a_{j-\eta}-\nu}(-1)^{j+h+\nu+\eta}\left(\frac{L'}{L}\right)^{(\omega-a_{j-\eta}-\nu)}(1,\psi)\,.$$

We now make some simplifications on $\mathcal{U}_1(\omega,\psi)$. We have

$$\mathcal{U}_1(\omega,\psi) = \sum_{\eta=\omega}^{j}\sum_{\nu=\omega}^{\eta}\sum_{h=0}^{j-\eta}\binom{j}{\eta}\binom{j-\eta}{h}\binom{\eta}{\nu}\binom{\nu}{\nu-\omega}(-1)^{h-\nu}(\eta-\nu)!L^{(j+\nu-\omega-\eta)}(1,\overline{\psi}).$$

Since $\sum_{h=0}^{j-\eta}\binom{j-\eta}{h}(-1)^h = 0$ unless $j=\eta$. Then, we get

$$\mathcal{U}_1(\omega,\psi) = \frac{j!}{\omega!}\sum_{\nu=\omega}^{j}\frac{(-1)^\nu}{(\nu-\omega)!}L^{(\nu-\omega)}(1,\overline{\psi}).$$

We calculate $\mathcal{U}_2(\omega)$. We have

$$\begin{aligned}\mathcal{U}_2(\omega) &= \sum_{\eta=0}^{\omega}\sum_{h=0}^{j-\eta}\binom{j}{h}\binom{j-h}{\eta}\binom{j-\eta}{\omega-\eta}(-1)^{j+\eta+h}\\ &= \sum_{\eta=0}^{\omega}(-1)^{j+\eta}\binom{j}{\eta}\binom{j-\eta}{\omega-\eta}\sum_{h=0}^{j-\eta}\binom{j-\eta}{h}(-1)^h.\end{aligned}$$

The inner-sum of the last equation is 0 unless $j=\eta$. Thus, $\mathcal{U}_2(\omega)=0$ if $\omega<j$, and $\mathcal{U}_2(j)=1$.

If $\omega < j$, then

$$\mathcal{U}_3(\omega,\psi) = \sum_{h=0}^{j} \sum_{\eta=0}^{\min\{j-h,\omega-1\}} \sum_{\nu=0}^{\omega-\eta-a_{j-\eta}} \binom{j}{h}\binom{j-h}{\eta} \frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$\binom{j-\eta-a_{j-\eta}-\nu}{\omega-\eta-a_{j-\eta}-\nu}(-1)^{j+h+\nu+\eta}\left(\frac{L'}{L}\right)^{(\omega-a_{j-\eta}-\nu)}(1,\psi)$$
$$= \sum_{\eta=0}^{\omega-1} \sum_{\nu=0}^{\omega-\eta-a_{j-\eta}} \sum_{h=0}^{j-\eta} \binom{j}{\eta}\binom{j-\eta}{h} \frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$\binom{j-\eta-a_{j-\eta}-\nu}{\omega-\eta-a_{j-\eta}-\nu}(-1)^{j+h+\nu+\eta}\left(\frac{L'}{L}\right)^{(\omega-a_{j-\eta}-\nu)}(1,\psi)\,.$$

Since $\eta \leq \omega - 1 < j$, the inner-most sum over h vanishes. So $\mathcal{U}_3(\omega,\psi) = 0$ for any $\omega < j$. For the case $\omega = j$, we have

$$\mathcal{U}_3(j,\psi) = \sum_{h=0}^{j} \sum_{\eta=0}^{j-h} \sum_{\nu=0}^{j-\eta-a_{j-\eta}} \binom{j}{h}\binom{j-h}{\eta} \frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$(-1)^{j+h+\nu+\eta}\left(\frac{L'}{L}\right)^{(j-a_{j-\eta}-\nu)}(1,\psi)$$
$$= \sum_{\eta=0}^{j} \sum_{\nu=0}^{j-\eta-a_{j-\eta}} \sum_{h=0}^{j-\eta} \binom{j}{\eta}\binom{j-\eta}{h} \frac{(j-\eta)!a_{j-\eta}}{(j-\eta-a_{j-\eta}-\nu)!}$$
$$(-1)^{j+h+\nu+\eta}\left(\frac{L'}{L}\right)^{(j-a_{j-\eta}-\nu)}(1,\psi)\,.$$

If $\eta \neq j$, then $\displaystyle\sum_{h=0}^{j-\eta}\binom{j-\eta}{h}(-1)^h = 0$; if $\eta = j$, then $a_{j-\eta} = 0$. Thus, $\mathcal{U}_3(j,\psi) = 0$.

Combining these results on $\mathcal{U}_1(\omega,\psi), \mathcal{U}_2(\omega)$ and $\mathcal{U}_3(\omega,\psi)$ with (7.43), and then removing the restriction $T \in \mathscr{F}$, we obtain Theorem 1.2 for $j \geq 1$.

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