CLASSICAL AND MODERN TREATMENTS OF RIEMANN ZETA FUNCTION

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ABSTRACT

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Tate's doctoral thesis, *Fourier Analysis in Number Fields and Hecke's Zeta-Functions*, on the analytic properties of the class of L-functions introduced by Erich Hecke, is one of the relatively few such dissertations that have become a byword. In it the methods, novel for that time, of Fourier analysis on groups of adeles, were worked out to recover Hecke's results. In this M.S. thesis, after a brief chapter on the classical treatment of Riemann zeta function, we discuss the local theory, restricted direct products, and the global theory following Tate's thesis. We compute prime divisors of quadratic fields and quasi-characters of p-adic fields.

ÖZET

KLASİK VE MODERN YÖNTEMLERLE RIEMANN ZETA FONKSİYONUNUN İNCELENMESİ

Tate'in doktora tezi, *Sayı Cisimlerinde Fourier Analizi ve Hecke'nin Zeta Fonksiyonları*, adı deyimleşmiş ender tezlerden biridir. Erich Hecke'nin L-fonksiyonlarının analitik özelliklerini konu edinen bu tezde, adeller grubu üzerinde Fourier analizi yapılarak Hecke'nin sonuçları yeniden elde edilmiştir. Bu yüksek lisans tezinde, Riemann zeta fonksiyonunun klasik yöntemle ele alınması üzerine kısa bir bölümün ardından, Tate'in tezi ışığında yerel teori, sınırlı direkt çarpım ve global teori incelenmiştir. Ayrıca kuadratik cisimlerin asal bölenleri ve p-adik cisimlerin kuasi-karakterleri bulunmuştur.

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LIST OF SYMBOLS

$\lvert\ \rvert$	Valuation on the completion k *or*
	Absolute value on $\mathbb{I}$
$[A]$	Characteristic function of A
$\mathbb{A}$	Adele group of the number field k
$\mathbb{A}^{\infty}$	Infinite part of $\mathbb{A}$
$\mathcal{B}(k^{+})$	Borel σ-algebra on k^{+}
$\mathcal{C}(G)$	Set of all continuous functions on G
$\mathcal{C}_c(k^{\times})$	Set of all continuous functions on $k^{\times}$ with compact support
d	Power of $\mathfrak{p}$ which is equal to $\mathfrak{d}$ *or*
	Discriminant of the number field k
D	Additive fundamental domain
$\mathfrak{d}$	Different of k
E	Multiplicative fundamental domain
$\widehat{f}$	Fourier transform of f
$H^{\perp}$	Annihilator of H
$\mathcal{H}(G)$	Set of all holomorphic functions on G
$\mathbb{I}$	Idele group of the number field k
J	Group of ideles of absolute value 1
k	Completion of a number field at a fixed prime divisor *or*
	Number field
k^{+}	Additive group of k
$\widehat{k^{+}}$	Pontryagin dual (character group) of k
$k^{\times}$	Multiplicative group of k
$N(\mathfrak{a})$	Index of the ideal $\mathfrak{a}$ in $\mathfrak{o}$
$\mathfrak{o}$	Ring of integers of k
$[P]$	1 or 0 according as P is true or false
$\mathfrak{p}$	Unique prime ideal of $\mathfrak{o}$ *or*
	Generic prime divisor of the number field k

$\mathcal{Q}$	Set of all quasi-characters of $k^\times$ *or*
	Set of all quasi-characters of $\mathbb{I}$
$\mathcal{Q}_S$	Set of all quasi-characters of $k^\times$ with exponent in S *or*
	Set of all quasi-characters of $\mathbb{I}$ with exponent in S
$\mathfrak{S}(k^+)$	Set of all continuous, integrable maps on k^+ with continuous, integrable Fourier transform
Γ	Gamma function
κ	Volume of the multiplicative fundamental domain
μ	Haar measure on k^+ *or*
	Haar measure on the restricted direct product G *or*
	Haar measure on $\mathbb{A}$
ν	Haar measure on $k^\times$ *or*
	Haar measure on $\mathbb{I}$
ν_J	Haar measure on J

1. CLASSICAL TREATMENT OF RIEMANN ZETA FUNCTION

We begin this chapter with the definition of Riemann zeta function ζ. We continue with the expression of ζ as an infinite product. Finally, we give the meromorphic continuation of ζ to the complex plane, and prove its functional equation.

1.1. Definition of Riemann Zeta Function

In real analysis, real powers of positive real numbers are defined. With the knowledge of complex exponential function this quickly extends to complex powers: for each $t \in (0, \infty)$ and $z \in \mathbb{C}$, we have $t^z = e^{z \log t}$. In particular, n^{-z} is a complex number for all $n \in \mathbb{N}$, $z \in \mathbb{C}$. If $t \in (0, \infty)$ and $z \in \mathbb{C}$, then $|t^z| = |e^{z \log t}| = e^{\operatorname{Re}(z \log t)} = e^{(\operatorname{Re} z) \log t} = t^{\operatorname{Re} z}$.

Theorem 1.1.1. *For each $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$, the complex series $\sum_{n=1}^{\infty} n^{-z}$ is absolutely convergent.*

Proof. Let $z \in \mathbb{C}$ be arbitrary with $\operatorname{Re} z > 1$. For all $n \in \mathbb{N}$, we have $|n^{-z}| = n^{\operatorname{Re}(-z)} = n^{-\operatorname{Re} z} = \frac{1}{n^{\operatorname{Re} z}}$. Thus $\sum_{n=1}^{k} |n^{-z}| = \sum_{n=1}^{k} \frac{1}{n^{\operatorname{Re} z}}$ for all $k \in \mathbb{N}$. By introductory real analysis, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is known to converge for $p > 1$. Consequently, the series $\sum_{n=1}^{\infty} n^{-z}$ is absolutely convergent. Q.E.D.

Absolutely convergent complex series are convergent, so the series $\sum_{n=1}^{\infty} n^{-z}$ converges to a complex number for each $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$.

Definition 1.1.2. *Riemann zeta function ζ* is the complex-valued map defined on the set $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$ by $\zeta(z) := \sum_{n=1}^{\infty} n^{-z}$.

1.2. Expression as an Infinite Product

Infinite product generated by a complex sequence $(z_n)_{n\in\mathbb{N}}$ is defined, in analogy with the definition of infinite sums, as the sequence $\left(\prod_{n=1}^{k} z_n\right)_{k\in\mathbb{N}}$ of partial products. If the limit of $\left(\prod_{n=1}^{k} z_n\right)_{k\in\mathbb{N}}$ exists, then the infinite product of $(z_n)_{n\in\mathbb{N}}$ is said to exist, and is defined as $\lim_{k\to\infty}\prod_{n=1}^{k} z_n$. We shall denote, as we do with infinite series, both the sequence $\left(\prod_{n=1}^{k} z_n\right)_{k\in\mathbb{N}}$ and its limit, if it exists, by $\prod_{n=1}^{\infty} z_n$. The next result is due to Euler (Theorem VII.8.17 of Conway [1], p. 193).

Theorem 1.2.1. *Let (p_n) be the sequence $(2,3,5,\ldots)$ of prime numbers. For each $z\in\mathbb{C}$ with $\operatorname{Re} z>1$, the equality*

$$\zeta(z)=\prod_{n=1}^{\infty}\frac{1}{1-p_n^{-z}}$$

holds.

Proof. Let $z\in\mathbb{C}$ be arbitrary with $\operatorname{Re} z>1$. We are to show that the infinite product on the right-hand side exists, and is equal to $\zeta(z)$. For each prime p, we have $|p^{-z}|=\frac{1}{p^{\operatorname{Re} z}}<\frac{1}{p}<1$ so that the complex number $(1-p^{-z})^{-1}$ is the sum of the geometric series $\sum_{n=0}^{\infty}(p^{-z})^n$. If p and q are distinct primes, then the Cauchy product of the series $\sum_{n=0}^{\infty}(p^{-z})^n$ and $\sum_{n=0}^{\infty}(q^{-z})^n$ is absolutely convergent since these two series are so, and we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}(p^{-z})^m(q^{-z})^{n-m}=\left(\frac{1}{1-p^{-z}}\right)\left(\frac{1}{1-q^{-z}}\right).$$

Consider now the series $\sum_{i,j\geq 0}(p^iq^j)^{-z}$. This series is absolutely convergent since each term of $\sum_{i,j\geq 0}|(p^iq^j)^{-z}|$ appear exactly once as a term of $\sum_{n=1}^{\infty}|n^{-z}|$. Thus the series $\sum_{i,j\geq 0}(p^iq^j)^{-z}$ converges to one and only one complex number irrespective of the enumeration of its terms. If we choose the enumeration $1,p^{-z},q^{-z},(p^2)^{-z},(pq)^{-z},(q^2)^{-z},\ldots$ then $\left(\sum_{n=0}^{k}\sum_{m=0}^{n}(p^m)^{-z}(q^{n-m})^{-z}\right)_{k\in\mathbb{N}\cup\{0\}}$ is a subsequence of the sequence of partial sums of $\sum_{i,j\geq 0}(p^iq^j)^{-z}$, and we get $\sum_{i,j\geq 0}(p^iq^j)^{-z}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(p^m)^{-z}(q^{n-m})^{-z}=$

$\sum_{n=0}^{\infty}\sum_{m=0}^{n}(p^{-z})^m(q^{-z})^{n-m} = \left(\frac{1}{1-p^{-z}}\right)\left(\frac{1}{1-q^{-z}}\right)$. Inductively, this result can be generalized to the case of a finite number of primes so that

$$\sum_{i_1,\ldots,i_k\geq 0}(p_1^{i_1}\cdots p_k^{i_k})^{-z} = \left(\frac{1}{1-p_1^{-z}}\right)\cdots\left(\frac{1}{1-p_k^{-z}}\right)$$

for all $k \in \mathbb{N}$.

Let $\pi_k := \prod_{n=1}^{k}(1-p_n^{-z})^{-1}$ and $\sigma_k := \sum_{n=1}^{k} n^{-z}$ for all $k \in \mathbb{N}$. Take an arbitrary $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} n^{-z}$ is absolutely convergent, there exists $K \in \mathbb{N}$ such that $\sum_{n=k}^{\infty}|n^{-z}| < \varepsilon$ for all $k \geq K$. Given $k \in \mathbb{N}$, each summand of $\sum_{n=1}^{k} n^{-z}$ appears exactly once as a term of $\sum_{i_1,\ldots,i_k\geq 0}(p_1^{i_1}\cdots p_k^{i_k})^{-z}$ and each term of $\sum_{i_1,\ldots,i_k\geq 0}(p_1^{i_1}\cdots p_k^{i_k})^{-z}$ appears exactly once as a term of $\sum_{n=1}^{\infty} n^{-z}$. Hence, for any $k \geq K$, we have

$$\begin{aligned}
|\pi_k - \sigma_k| &= \left|\prod_{n=1}^{k}(1-p_n^{-z})^{-1} - \sum_{n=1}^{k} n^{-z}\right| \\
&= \left|\sum_{i_1,\ldots,i_k\geq 0}(p_1^{i_1}\cdots p_k^{i_k})^{-z} - \sum_{n=1}^{k} n^{-z}\right| \\
&= \left|\sum_{p_1^{i_1}\cdots p_k^{i_k} > k}(p_1^{i_1}\cdots p_k^{i_k})^{-z}\right| \\
&\leq \sum_{p_1^{i_1}\cdots p_k^{i_k} > k}|(p_1^{i_1}\cdots p_k^{i_k})^{-z}| \\
&\leq \sum_{n=k+1}^{\infty}|n^{-z}| \\
&< \varepsilon.
\end{aligned}$$

Thus we have proved $\lim_{k\to\infty}(\pi_k - \sigma_k) = 0$, which implies

$$\prod_{n=1}^{\infty}\frac{1}{1-p_n^{-z}} = \lim_{k\to\infty}\pi_k = \lim_{k\to\infty}\sigma_k = \zeta(z).$$

Q.E.D.

1.3. Meromorphic Continuation and Functional Equation

We begin this section by proving that Riemann zeta function is holomorphic on its domain of definition. For an open subset G of $\mathbb{C}$, we denote by $\mathcal{C}(G)$ the set of all complex-valued continuous functions on G, and by $\mathcal{H}(G)$ the set of all holomorphic functions on G. We endow $\mathcal{C}(G)$ with the metric of uniform convergence on compact subsets.

Theorem 1.3.1. *Riemann zeta function is holomorphic on* $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$.

Proof. Let $G := \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$. For all $n \in \mathbb{N}$, the map $u_n : G \to \mathbb{C}$ defined by $u_n(z) := n^{-z}$ is holomorphic. Then $f_k := \sum_{n=1}^{k} u_n$ belongs to $\mathcal{H}(G)$ for all $k \in \mathbb{N}$. By definition of Riemann zeta function, the sequence (f_k) has pointwise limit ζ. For $\varepsilon > 0$, let $G_\varepsilon := \{z \in \mathbb{C} : \operatorname{Re} z \geq 1 + \varepsilon\}$. If $M_n := n^{-(1+\varepsilon)}$ for all $n \in \mathbb{N}$, then $|u_n(z)| = |n^{-z}| = n^{-\operatorname{Re} z} \leq n^{-(1+\varepsilon)} = M_n$ for all $n \in \mathbb{N}$, $z \in G_\varepsilon$. Since $\sum_{n=1}^{\infty} M_n < \infty$, the sequence (f_k) is uniformly convergent on G_ε by Weierstrass M-test. If K is a compact subset of G, then K is sequentially compact and closed so that $K \subseteq G_\varepsilon$ for some $\varepsilon > 0$. Hence, (f_k) converges to ζ uniformly on compact subsets of G. Thus $\zeta \in \mathcal{C}(G)$ and (f_k) converges to ζ in the metric space $\mathcal{C}(G)$. Since $f_k \in \mathcal{H}(G)$ for all $k \in \mathbb{N}$ and $\mathcal{H}(G)$ is closed in $\mathcal{C}(G)$ (Theorem VII.2.1 of Conway [1], p. 151), we get $\zeta \in \mathcal{H}(G)$. Q.E.D.

Meromorphic continuation of Riemann zeta function can be carried out in several ways. We present below the one which makes use of Poisson summation formula. In this respect, the proof to be given shows parallelism with the proof of the meromorphic continuation of the global zeta function in Tate's thesis [2]. When compared to the style of exposition adopted until now, we shall be rather sketchy.

Let $f : \mathbb{R} \to [0, \infty)$ be a continuous function such that the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists as an improper Riemann integral. If f is increasing on $(-\infty, 0]$ and decreasing on $[0,\infty)$, then we have

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-2\pi int}\,\mathrm{d}t,$$

each series being absolutely convergent (Theorem 11.24 of Apostol [3], p. 332). This is a particular way of expressing Poisson summation formula. It is used to derive for $x > 0$ the transformation equation

$$\theta(x) = x^{-\frac{1}{2}}\theta(x^{-1})$$

for the theta function $\theta : (0,\infty) \to (0,\infty)$ which is defined by $\theta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$ (Apostol [3], p. 334). We shall use this equation to obtain the meromorphic continuation of Riemann zeta function to whole complex plane at one stroke. We follow Whittaker and Watson [4].

Fix an arbitrary complex number s with $\operatorname{Re} s > 2$. Let us denote $\operatorname{Re} s$ by σ for simplicity. By a change of variables, we get from $\Gamma\bigl(\frac{1}{2}s\bigr) = \int_0^\infty e^{-x}x^{\frac{1}{2}s-1}\,\mathrm{d}x$ (Theorem VII.7.15 of Conway [1], p. 180) the equality

$$n^{-s}\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} = \int_0^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}\,\mathrm{d}x.$$

Summing up over all positive integers n, we have

$$\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} = \lim_{k\to\infty}\int_0^\infty \sum_{n=1}^{k} e^{-n^2\pi x}x^{\frac{1}{2}s-1}\,\mathrm{d}x.$$

If we define $\vartheta : (0,\infty) \to (0,\infty)$ by $\vartheta(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x}$ then $\theta(x) = 1 + 2\vartheta(x)$ for all $x > 0$, and the transformation equation for theta yields $1 + 2\vartheta(x) = x^{-\frac{1}{2}}\bigl(1 + 2\vartheta(x^{-1})\bigr)$. Thus we see $\lim_{x\to 0} x^{\frac{1}{2}}\vartheta(x) = \frac{1}{2}$ whence the convergence of $\int_0^\infty \vartheta(x)x^{\frac{1}{2}s-1}\,\mathrm{d}x$.

Consequently, we have

$$\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} = \lim_{k\to\infty}\left(\int_0^\infty \vartheta(x)x^{\frac{1}{2}s-1}\,\mathrm{d}x - \int_0^\infty \sum_{n=k+1}^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}\,\mathrm{d}x\right).$$

The modulus of the last integral tends to zero since

$$\begin{aligned}
\int_0^\infty \sum_{n=k+1}^\infty e^{-n(k+1)\pi x}x^{\frac{1}{2}\sigma-1}\,\mathrm{d}x &= \int_0^\infty \frac{e^{-(k+1)^2\pi x}x^{\frac{1}{2}\sigma-1}}{1-e^{-(k+1)\pi x}}\,\mathrm{d}x \\
&< \frac{1}{\pi(k+1)}\int_0^\infty e^{-(k^2+2k)\pi x}x^{\frac{1}{2}\sigma-2}\,\mathrm{d}x \\
&= \frac{1}{\pi(k+1)}\big((k^2+2k)\pi\big)^{1-\frac{1}{2}\sigma}\Gamma\left(\frac{1}{2}\sigma-1\right)
\end{aligned}$$

for all $k\in\mathbb{N}$. Thus

$$\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} = \int_0^\infty \vartheta(x)x^{\frac{1}{2}s-1}\,\mathrm{d}x.$$

We have

$$\begin{aligned}
\int_0^1 \vartheta(x)x^{\frac{1}{2}s-1}\,\mathrm{d}x &= \int_0^1 \left(-\frac{1}{2}+\frac{1}{2}x^{-\frac{1}{2}}+x^{-\frac{1}{2}}\vartheta(x^{-1})\right)x^{\frac{1}{2}s-1}\,\mathrm{d}x \\
&= -\frac{1}{s}+\frac{1}{s-1}+\int_0^1 x^{-\frac{1}{2}}\vartheta(x^{-1})x^{\frac{1}{2}s-1}\,\mathrm{d}x \\
&= \frac{1}{s(s-1)}+\int_\infty^1 x^{\frac{1}{2}}\vartheta(x)x^{-\frac{1}{2}s+1}\left(\frac{-1}{x^2}\right)\mathrm{d}x \\
&= \frac{1}{s(s-1)}+\int_1^\infty \vartheta(x)x^{\frac{1}{2}(1-s)}x^{-1}\,\mathrm{d}x
\end{aligned}$$

so that

$$\begin{aligned}
\int_0^\infty \vartheta(x)x^{\frac{1}{2}s-1}\,\mathrm{d}x &= \int_0^1 \vartheta(x)x^{\frac{1}{2}s-1}\,\mathrm{d}x+\int_1^\infty \vartheta(x)x^{\frac{1}{2}s-1}\,\mathrm{d}x \\
&= \frac{1}{s(s-1)}+\int_1^\infty \vartheta(x)x^{\frac{1}{2}(1-s)}x^{-1}\,\mathrm{d}x+\int_1^\infty \vartheta(x)x^{\frac{1}{2}s}x^{-1}\,\mathrm{d}x \\
&= \frac{1}{s(s-1)}+\int_1^\infty \big(x^{\frac{1}{2}(1-s)}+x^{\frac{1}{2}s}\big)x^{-1}\vartheta(x)\,\mathrm{d}x.
\end{aligned}$$

Hence, we have proved

$$\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s} - \frac{1}{s(s-1)} = \int_1^\infty \left(x^{\frac{1}{2}(1-s)} + x^{\frac{1}{2}s}\right)x^{-1}\vartheta(x)\,\mathrm{d}x$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 2$. The integral on the right represents a holomorphic function of s since $\vartheta(x) < \sum_{n=1}^{\infty} e^{-n\pi x} = e^{-\pi x}\frac{1}{1-e^{-\pi x}} \leq e^{-\pi x}\frac{1}{1-e^{-\pi}}$ for all $x \in [1,\infty)$. Moreover, it remains unchanged when we replace s by $1-s$. We proved the following (Theorems VII.8.13 and VII.8.14 of Conway [1], pp. 192-193).

Theorem 1.3.2. *Riemann zeta function ζ has a meromorphic continuation to whole complex plane. It has a simple pole at $s = 1$ with residue 1, and it satisfies the functional equation $\xi(s) = \xi(1-s)$ where ξ is the entire function defined by $\xi(s) := s(s-1)\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}$.*

2. TATE'S THESIS

In this chapter, we give a detailed treatment of some parts of Tate's thesis [2], including the local theory, restricted direct products, and most of the global theory.

2.1. The Local Theory

2.1.1. Introduction

Let k_0 be a number field. Two valuations $|\,|_1$ and $|\,|_2$ on k_0 are equivalent if $|\,|_1 = |\,|_2^t$ for some $t > 0$. Equivalence classes of nontrivial valuations are referred to as *prime divisors* of k_0. Every prime divisor $\mathfrak{p}$ determines a metrizable topology on k_0, the metric completion of which is called as the completion of k_0 at $\mathfrak{p}$.

Let k denote the completion of k_0 at a prime divisor $\mathfrak{p}$. If $|\,|$ is a representative of $\mathfrak{p}$, then its restriction $|\,|_{\mathbb{Q}}$ to $\mathbb{Q}$ is a valuation on $\mathbb{Q}$, and is equivalent to, by Ostrowski's theorem (Cassels and Fröhlich [5], p. 45), either the ordinary absolute value $|\,|_\infty$ or the p-adic valuation $|\,|_p$ for some rational prime p. In the first case, $\mathfrak{p}$ is said to be *archimedean*; in the second case, $\mathfrak{p}$ is said to be *discrete*. The field k is a finite extension of the completion of $\mathbb{Q}$ with respect to $|\,|_{\mathbb{Q}}$. This completion is $\mathbb{R}$ or the p-adic field $\mathbb{Q}_p$ according as $|\,|_{\mathbb{Q}}$ is equivalent to $|\,|_\infty$ or $|\,|_p$. Consequently, k is $\mathbb{R}$ or $\mathbb{C}$ if $\mathfrak{p}$ is archimedean, and k is a finite extension of $\mathbb{Q}_p$ for some p if $\mathfrak{p}$ is discrete. In the latter case, we say that k is $\mathfrak{p}$-adic.

Any valuation belonging to $\mathfrak{p}$ naturally extends to a valuation on k, and any two such extensions are equivalent. From the collection of these equivalent valuations, we select $|\,|$ to be the ordinary absolute value if $k = \mathbb{R}$, and the square of the ordinary absolute value if $k = \mathbb{C}$. If $\mathfrak{p}$ is discrete and $|\,|$ is a valuation belonging to $\mathfrak{p}$, then $\mathfrak{o} := \{\alpha \in k : |\alpha| \leq 1\}$ is called as the ring of integers of k. This ring contains a unique prime ideal $\{\alpha \in k : |\alpha| < 1\}$ which we denote by $\mathfrak{p}$, too. With this denotation, the symbol $\mathfrak{p}$ becomes ambiguous; nevertheless, the meaning attached to $\mathfrak{p}$ hereafter shall be clear from the context. The quotient ring $\mathfrak{o}/\mathfrak{p}$ is a finite field. We denote the index of an ideal $\mathfrak{a}$ in $\mathfrak{o}$ by $N(\mathfrak{a})$. In particular, the order of $\mathfrak{o}/\mathfrak{p}$ is $N(\mathfrak{p})$. The ring $\mathfrak{o}$ is a Dedekind ring so that every proper ideal in $\mathfrak{o}$ can be written as a positive power of $\mathfrak{p}$. We define the ordinal number $\operatorname{ord}(\alpha)$ of $\alpha \in k^\times$ as 0 if $|\alpha| = 1$, as n if $|\alpha| < 1$ and $\alpha\mathfrak{o} = \mathfrak{p}^n$, and as $-n$ if $|\alpha| > 1$ and $\alpha^{-1}\mathfrak{o} = \mathfrak{p}^n$. When k is $\mathfrak{p}$-adic, we select $|\,|$ to be the valuation defined by $|\alpha| := N(\mathfrak{p})^{-\operatorname{ord}(\alpha)}$ if $\alpha \neq 0$, and $|\alpha| := 0$ if $\alpha = 0$.

In any case, k is a locally compact field, and a subset of k has compact closure if and only if it is bounded in absolute value.

2.1.2. Additive Characters and Measure

If G is a locally compact abelian group and H is a subgroup of G, then there is an isomorphism $\widehat{G/H} \cong H^\perp$ of topological groups, where $H^\perp$ denotes the annihilator of H in $\widehat{G}$. This result from the theory of topological groups will be used several times in the sequel. One instance is encountered in the proof of the lemma below, by which we shall set up a topological group isomorphism between the additive group k^+ of k and its Pontryagin dual (character group) $\widehat{k^+}$.

Lemma 2.1.1. *If $\psi \in \widehat{k^+}$ is nontrivial, then $\psi_\eta : k^+ \to S^1$ defined by $\psi_\eta(\xi) := \psi(\eta\xi)$ belongs to $\widehat{k^+}$ for each $\eta \in k^+$ and the mapping from k^+ to $\widehat{k^+}$ sending each η to the character ψ_η is an isomorphism of topological groups.*

Proof. Let $\psi \in \widehat{k^+}$ be nontrivial. Each ψ_η, being the composition of the continuous group homomorphism $\xi \mapsto \eta\xi$ with ψ, is in $\widehat{k^+}$. Let $\beta_\psi : k^+ \to \widehat{k^+}$ be defined by $\beta_\psi(\eta) := \psi_\eta$. This map is an algebraic homomorphism since $\psi_{\eta_1+\eta_2}(\xi) = \psi(\eta_1\xi+\eta_2\xi) = \psi(\eta_1\xi)\psi(\eta_2\xi) = \psi_{\eta_1}(\xi)\psi_{\eta_2}(\xi)$ for all $\eta_1, \eta_2, \xi \in k^+$. For any $\eta \in \operatorname{Ker}\beta_\psi$, we have $\psi(\eta\xi) = \psi_\eta(\xi) = 1$ for all $\xi \in k^+$, implying $\eta = 0$ since ψ is nontrivial by hypothesis. Hence, β_ψ is injective.

As β_ψ is an algebraic homomorphism of topological groups, it is enough to prove the continuity at 0 for the continuity of β_ψ. Take an arbitrary open neighborhood $W(K,U)$ of the identity character. The set K, being compact, is bounded in absolute value, say by $M>0$. Since $V:=\psi^{\leftarrow}(U)$ is an open neighborhood of 0, there exists $\varepsilon>0$ such that $B(0,\varepsilon)\subseteq V$. If $V':=B\big(0,\frac{\varepsilon}{M}\big)$, then for any $\eta\in V'$, we have $\psi_\eta\in W(K,U)$ because $\psi_\eta(\xi)=\psi(\eta\xi)\in\psi\big(B(0,\varepsilon)\big)\subseteq U$ for all $\xi\in K$. Hence $\beta_\psi(V')\subseteq W(K,U)$, as desired.

In order to prove $\beta_\psi: k^+\to\operatorname{Im}\beta_\psi$ is open, it is sufficient to show that $\beta_\psi\big(B(0,\varepsilon)\big)$ is open in $\operatorname{Im}\beta_\psi$ for all $\varepsilon>0$, which in turn is proved if, for each $\varepsilon>0$, we can find a $W(K,U)$ such that $W(K,U)\cap\operatorname{Im}\beta_\psi\subseteq\beta_\psi\big(B(0,\varepsilon)\big)$. Let $\varepsilon>0$ be given. Say $\psi(\xi_0)\neq 1$, and let $r:=|\xi_0|>0$. Choose $M>0$ so large that $M\varepsilon\geq r$, and put $K:=B[0,M]$. The set K is closed and, being bounded in absolute value, has compact closure; so it is compact. If U is an open neighborhood of 1 in S^1 such that $\psi(\xi_0)\notin U$, then the inclusion stated must hold. Indeed, if ψ_ξ belongs to the complement of $\beta_\psi\big(B(0,\varepsilon)\big)$ in $\operatorname{Im}\beta_\psi$, then $|\xi|\geq\varepsilon$, implying $\psi(\xi_0)\in\psi\big(B[0,r]\big)\subseteq\psi\big(\{\eta:|\eta|\leq M\varepsilon\}\big)\subseteq\psi_\xi(K)$. By definition of U, $\psi(\xi_0)\notin U$, so $\psi_\xi(K)\nsubseteq U$, which gives $\psi_\xi\notin W(K,U)$.

It remains to prove that β_ψ is onto $\widehat{k^+}$. The image of β_ψ is locally compact since $\beta_\psi: k^+\to\operatorname{Im}\beta_\psi$ is continuous, open and surjective. Consequently, $\operatorname{Im}\beta_\psi$ is closed in $\widehat{k^+}$ because locally compact subgroups of the Hausdorff topological groups are closed. Hence, it is enough to show that $\operatorname{Im}\beta_\psi$ is dense in $\widehat{k^+}$. Let H be the closure of $\operatorname{Im}\beta_\psi$ in $\widehat{k^+}$. The dual of $\widehat{k^+}/H$ is isomorphic to $H^\perp$. Pontryagin duality theorem allows us to see $H^\perp$ as a subgroup of k^+. Since ψ is nontrivial, $H^\perp$ is the trivial subgroup in k^+. As a consequence, the dual of $\widehat{k^+}/H$ is trivial. Then the double dual of $\widehat{k^+}/H$ is also trivial, whence the triviality of $\widehat{k^+}/H$ itself by Pontryagin duality again. Thus we get $\widehat{k^+}=H$, which is to say that $\operatorname{Im}\beta_\psi$ is dense in $\widehat{k^+}$. Q.E.D.

Now we shall construct a special nontrivial character of k^+. Let R denote the completion of $\mathbb{Q}$ with respect to the restriction of any valuation in $\mathfrak{p}$. Thus, R is equal to $\mathbb{R}$ or the p-adic field $\mathbb{Q}_p$ for some p according as $\mathfrak{p}$ is archimedean or discrete. We define a map $\lambda: R\to\mathbb{R}/\mathbb{Z}$ as follows: if $R=\mathbb{R}$, then $\lambda(x):=-x+\mathbb{Z}$; if $R=\mathbb{Q}_p$

for some p, then $\lambda(x) := r + \mathbb{Z}$ where r is a rational number such that (1) $p^n r \in \mathbb{Z}$ for some $n \geq 0$, and (2) $x - r \in \mathbb{Z}_p$. In the latter case, such an $r \in \mathbb{Q}$ indeed exists, for if $x \in \mathbb{Q}_p$, then $x = a_{-n}p^{-n} + \ldots + a_{-1}p^{-1} + a_0 + a_1 p + \ldots$ for some $n \geq 0$ with $0 \leq a_j \leq p-1$ for all $j \geq -n$. If we let r to be 0 or $a_{-n}p^{-n} + \ldots + a_{-1}p^{-1}$ according as $n = 0$ or $n > 0$, then $p^n r \in \mathbb{Z}$ and $x - r = a_0 + a_1 p + \ldots \in \mathbb{Z}_p$. Moreover, if r and s are two rational numbers satisfying (1) and (2), then $r - s = (r - x) + (x - s)$ is a p-adic integer which can be written as m/p^n for some $m \in \mathbb{Z}$ and $n \geq 0$, so we necessarily have $r - s \in \mathbb{Z}$; that is, $r + \mathbb{Z} = s + \mathbb{Z}$.

Lemma 2.1.2. *The map $\lambda : R \to \mathbb{R}/\mathbb{Z}$ defined above is a nontrivial continuous group homomorphism.*

Proof. First, the case $R = \mathbb{R}$. The natural homomorphism from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$ is continuous by definition of the quotient topology on $\mathbb{R}/\mathbb{Z}$, and the map $x \mapsto -x$ is a continuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$, so their composition λ is a continuous homomorphism, which is nontrivial since $\lambda(\frac{1}{2}) = -\frac{1}{2} + \mathbb{Z} \neq 0 + \mathbb{Z} = 0_{\mathbb{R}/\mathbb{Z}}$. Second, the case $R = \mathbb{Q}_p$ for some p. If $\lambda(x) = 0_{\mathbb{R}/\mathbb{Z}}$, then $x \in \mathbb{Z}_p$, so λ is nontrivial since $\mathbb{Z}_p$ is a proper subset of $\mathbb{Q}_p$. If $x, y \in \mathbb{Q}_p$, then $\lambda(x) + \lambda(y)$ is a rational number satisfying the properties (1) and (2) defining $\lambda(x+y)$, so $\lambda(x+y) = \lambda(x) + \lambda(y)$. This proves that λ is a homomorphism. For any sequence (x_n) in $\mathbb{Q}_p$ with $x_n \to 0$, we have $|x_n|_p \to 0$, so (x_n) is eventually in $\mathbb{Z}_p$. Since $\lambda(x) = 0_{\mathbb{R}/\mathbb{Z}}$ for all $x \in \mathbb{Z}_p$, all but a finite number of terms of the sequence $\big(\lambda(x_n)\big)$ is equal to $0_{\mathbb{R}/\mathbb{Z}}$ so that $\lambda(x_n) \to \lambda(0)$. Thus λ is continuous at 0. This implies the continuity of λ because λ is an algebraic homomorphism of topological groups $\mathbb{Q}_p$ and $\mathbb{R}/\mathbb{Z}$. Q.E.D.

The field extension k/R is finite and separable, so we have a map $T_{k/R} : k \to R$ sending each $\xi \in k$ to its trace $T_{k/R}(\xi) \in R$. This map is an additive surjection. It is moreover continuous since k is isomorphic to R^n as a topological vector space if n is the degree of the extension k/R (Proposition 4-13 of Ramakrishnan and Valenza [6], p. 140). We let $\Lambda := \lambda \circ T_{k/R}$. The composition of Λ with the continuous group homomorphism $t + \mathbb{Z} \mapsto e^{2\pi i t}$ is a nontrivial character of k^+. Lemma 2.1.1 yields the next theorem.

Theorem 2.1.3. *The mapping from k^+ to $\widehat{k^+}$ sending each η to the character $\xi \mapsto e^{2\pi i\Lambda(\eta\xi)}$ is an isomorphism of topological groups.*

Let μ be a Haar measure for k^+. Fix $\alpha \in k^\times$ arbitrarily. Then $\xi \mapsto \alpha\xi$ is a topological automorphism of k^+ and it follows that the function $\mu_\alpha : \mathcal{B}(k^+) \to [0, \infty]$ defined by $\mu_\alpha(M) := \mu(\alpha M)$ is a Haar measure on k^+. Since Haar measure on a locally compact group is unique up to a positive constant (Theorem 9.2.3 of Cohn [7], p. 309), there exists $c_\alpha > 0$ such that $\mu_\alpha = c_\alpha\mu$. This constant does not depend on the particular choice of μ. The next lemma renders one reason for why we selected $|\,|$ in the Introduction as such.

Lemma 2.1.4. $c_\alpha = |\alpha|$ *for all* $\alpha \in k^\times$.

Proof. Let $\alpha \in k^\times$. For the cases $k = \mathbb{R}$ and $k = \mathbb{C}$, the equality $c_\alpha = |\alpha|$ is an easy consequence of the definition of $|\,|$. Suppose now k is $\mathfrak{p}$-adic. If $|\alpha| = 1$, then $\alpha\mathfrak{o} = \mathfrak{o}$, so $\mu(\alpha\mathfrak{o}) = |\alpha|\mu(\mathfrak{o})$. If $|\alpha| < 1$, then $\alpha\mathfrak{o} = \mathfrak{p}^n$ for some $n \in \mathbb{N}$. On writing $\mathfrak{o}$ as a disjoint union of cosets of $\mathfrak{p}^n$ and using the translation-invariance of μ, we get $N(\mathfrak{p}^n)\mu(\mathfrak{p}^n) = \mu(\mathfrak{o})$. By definition, $|\alpha| = N(\mathfrak{p})^{-n}$ so that $\mu(\alpha\mathfrak{o}) = N(\mathfrak{p}^n)^{-1}\mu(\mathfrak{o}) = N(\mathfrak{p})^{-n}\mu(\mathfrak{o}) = |\alpha|\mu(\mathfrak{o})$. If $|\alpha| > 1$, then $\mu\big(\frac{1}{\alpha}\mathfrak{o}\big) = \big|\frac{1}{\alpha}\big|\mu(\mathfrak{o})$ by the preceding case, so $|\alpha|\mu\big(\frac{1}{\alpha}\mathfrak{o}\big) = \mu\Big(\alpha\big(\frac{1}{\alpha}\mathfrak{o}\big)\Big)$. As a result, $c_\alpha = |\alpha|$ in any case. Q.E.D.

By linearity of the integral and monotone convergence theorem, it follows by the lemma above that $\int_{k^+} f\,\mathrm{d}\mu = |\alpha| \int_{k^+} f(\alpha\xi)\mu(\mathrm{d}\xi)$ for every integrable map $f : k^+ \to \mathbb{C}$ and $\alpha \in k^\times$.

Now we select a fixed Haar measure for k^+. In the rest of the Local Theory, let μ denote the ordinary Lebesgue measure on $\mathbb{R}$ if $k = \mathbb{R}$, twice ordinary Lebesgue measure on $\mathbb{C}$ if $k = \mathbb{C}$, and the Haar measure on k^+ for which $\mathfrak{o}$ gets measure $N(\mathfrak{d})^{-\frac{1}{2}}$ if k is

$\mathfrak{p}$-adic. This μ induces a measure $\widehat{\mu}$ on $\widehat{k^+}$ via the isomorphism given in Theorem 2.1.3. It turns out that $\widehat{\mu}$ is the dual measure of μ; in other words, $\widehat{\mu}$ satisfies the Fourier inversion formula. This fact is the content of the next theorem. We denote by $\mathfrak{S}(k^+)$ the collection of all continuous, integrable maps f on k^+ with continuous, integrable Fourier transform $\widehat{f}$. Fourier inversion formula holds for such functions (see "Prerequisites" of Tate's thesis [2]).

Theorem 2.1.5. *The measure $\widehat{\mu}$ is the dual measure of μ.*

Proof. It is enough to establish the inversion formula $f(\xi) = \int_{\widehat{k^+}} \widehat{f}(\psi)\psi(\xi)\widehat{\mu}(\mathrm{d}\psi)$ for particular $f \in \mathfrak{S}(k^+)$ and $\xi \in k^+$ with $f(\xi) \neq 0$. We shall choose f as $\xi \mapsto e^{-\pi\xi^2}$ for $k = \mathbb{R}$, as $\xi \mapsto e^{-2\pi|\xi|}$ for $k = \mathbb{C}$, and as the characteristic function of $\mathfrak{o}$ for $\mathfrak{p}$-adic k, and take $\xi = 0$ in all cases. Then $\psi(\xi) = 1$ for all $\psi \in \widehat{k^+}$ and $f(\xi) = 1$ in all cases, so the equality to be proved simplifies as $1 = \int_{\widehat{k^+}} \widehat{f}(\psi)\widehat{\mu}(\mathrm{d}\psi)$, which can equivalently be written as $1 = \int_{k^+} \widehat{f}(\psi_\eta)\mu(\mathrm{d}\eta)$ by definition of $\widehat{\mu}$. This equality will be an easy consequence of the computations to be made in the final subsection devoted to the computation of ρ by special zeta functions. Q.E.D.

2.1.3. Multiplicative Characters and Measure

The mapping $\alpha \mapsto |\alpha|$ from $k^\times$ into $(0, \infty)$ is continuous and multiplicative. Its kernel, which we denote by u, is closed and bounded, hence compact. If k is $\mathfrak{p}$-adic, then u is moreover open since $\mathfrak{o}$ is open, $\mathfrak{p}$ is closed and $u = \mathfrak{o} \setminus \mathfrak{p}$. We shall call a continuous multiplicative map $\chi : k^\times \to \mathbb{C}^\times$ a *quasi-character* of $k^\times$ and reserve the term character for the case $\operatorname{Im}\chi \subseteq S^1$. We say that a quasi-character is *unramified* if $\chi|_u \equiv 1$. The next proposition is stated without proof.

Proposition 2.1.6. *(1) If $\gamma : (0, \infty) \to \mathbb{C}^\times$ is a continuous multiplicative map, then there exists a unique $s \in \mathbb{C}$ such that $\gamma(x) = x^s$ for all $x \in (0, \infty)$.*
(2) Let $\langle N(\mathfrak{p})\rangle$ be the subgroup of $(0, \infty)$ generated by $N(\mathfrak{p})$. If $\gamma : \langle N(\mathfrak{p})\rangle \to \mathbb{C}^\times$ is a

continuous multiplicative map, then there exists an $s \in \mathbb{C}$, *unique modulo* $2\pi i/\log N(\mathfrak{p})$, *such that* $\gamma(x) = x^s$ *for all* $x \in \langle N(\mathfrak{p})\rangle$.

Lemma 2.1.7. *A map* $\chi : k^\times \to \mathbb{C}^\times$ *is an unramified quasi-character of* $k^\times$ *if and only if there exists* $s \in \mathbb{C}$ *such that* $\chi(\alpha) = |\alpha|^s$ *for all* $\alpha \in k^\times$. *If* $\mathfrak{p}$ *is archimedean, then* s *is unique; if* $\mathfrak{p}$ *is discrete, then* s *is unique modulo* $2\pi i/\log N(\mathfrak{p})$.

Proof. Let $\chi : k^\times \to \mathbb{C}^\times$ be an unramified quasi-character. Let $v(k^\times) := \operatorname{Im} |\,|$. We define $\chi_1 : v(k^\times) \to \mathbb{C}^\times$ by $\chi_1\big(|\alpha|\big) := \chi(\alpha)$. This is a well-defined multiplicative map. The diagram

$$\begin{tikzcd} v(k^\times) \arrow[dr, "\chi_1"] & \\ k^\times \arrow[u, "|\,|"] \arrow[r, "\chi"'] & \mathbb{C}^\times \end{tikzcd}$$

commutes and $|\,|$ is an open mapping, so χ_1 is moreover continuous. The conclusion follows by the foregoing proposition. Q.E.D.

If $\mathfrak{p}$ is archimedean, then every $\alpha \in k^\times$ can be written uniquely in the form $\widetilde{\alpha}\rho$ with $\widetilde{\alpha} \in u$ and $\rho > 0$. If $\mathfrak{p}$ is discrete and π is a fixed element of ordinal number 1, then every $\alpha \in k^\times$ can be written uniquely as $\widetilde{\alpha}\rho$ with $\widetilde{\alpha} \in u$ and ρ a power of π. The mapping $\alpha \mapsto \widetilde{\alpha}$ is a continuous homomorphism from $k^\times$ onto u which is identity on u.

Theorem 2.1.8. *A map* $\chi : k^\times \to \mathbb{C}^\times$ *is a quasi-character of* $k^\times$ *if and only if there exists a character* $\widetilde{\chi}$ *of* u *and an* $s \in \mathbb{C}$ *such that* $\chi(\alpha) = \widetilde{\chi}(\widetilde{\alpha})|\alpha|^s$ *for all* $\alpha \in k^\times$. *The map* $\widetilde{\chi}$ *is uniquely determined by* χ, *and* s *is determined as in Lemma 2.1.7.*

Proof. Let $\chi : k^\times \to \mathbb{C}^\times$ be a quasi-character of $k^\times$. Then $\widetilde{\chi} := \chi|_u$ is a continuous homomorphism. Since u is compact, $\widetilde{\chi}(u)$ must be compact, hence bounded in $\mathbb{C}$. This forces $\operatorname{Im} \widetilde{\chi} \subseteq S^1$. So $\widetilde{\chi}$ is a character of u. The mapping $\alpha \mapsto \chi(\alpha)/\widetilde{\chi}(\widetilde{\alpha})$ is an unramified quasi-character of $k^\times$. The result follows by Lemma 2.1.7. Q.E.D.

Now we shall describe the characters of u. For $k = \mathbb{R}$, we have $u = \{-1, 1\}$, so there are two characters of u, the mappings $\alpha \mapsto 1$ and $\alpha \mapsto \alpha$. For $k = \mathbb{C}$, we have $u = S^1$. The characters of S^1 are precisely those maps of the form $\alpha \mapsto \alpha^m$ where $m \in \mathbb{Z}$. Finally, we consider the $\mathfrak{p}$-adic case. For each $n \in \mathbb{N}$, the subgroup $1 + \mathfrak{p}^n$ of u is an open neighborhood of 1. Furthermore, sets of the form $1 + \mathfrak{p}^n$ make up a neighborhood basis at 1. This implies that for any character $\widetilde{\chi}$ of u, there is an $n \in \mathbb{N}$ such that $\widetilde{\chi}(1 + \mathfrak{p}^n) = \{1\}$. If $\widetilde{\chi}$ is trivial, put $n := 0$; otherwise, let n be the minimum natural number satisfying $\widetilde{\chi}(1 + \mathfrak{p}^n) = \{1\}$. We call the ideal $\mathfrak{p}^n =: \mathfrak{f}$ as the *conductor* of $\widetilde{\chi}$. Characters of u are in one-to-one correspondence with characters of the factor group $u/(1 + \mathfrak{f})$. This is a finite group since open subgroups of compact groups have finite index. Consequently, $u/(1 + \mathfrak{f})$ has finitely many characters. Hence, characters of u in the $\mathfrak{p}$-adic case may be described by a finite table of data.

We define the *exponent* of a quasi-character $\chi = \widetilde{\chi}|\ |^s$ to be the real part of s, which is uniquely determined by χ. A quasi-character is a character if and only if its exponent is zero.

We will be able to select a Haar measure on the multiplicative group $k^\times$ in relation with μ. The function $\nu_0 : \mathcal{B}(k^\times) \to [0, \infty]$ defined by $\nu_0(A) := \int_A |\xi|^{-1} \mu(\mathrm{d}\xi)$ is meaningful since $\mathcal{B}(k^\times) \subseteq \mathcal{B}(k^+)$. It follows by monotone convergence theorem that ν_0 is a Borel measure on $k^\times$. Since $k^\times$ is second countable and ν_0 is finite on compact sets, we deduce that ν_0 is regular (Proposition 7.2.3 of Cohn [7], p. 206). Moreover, it is nonzero and, as a consequence of the identity $\int f \,\mathrm{d}\mu = |\alpha| \int f(\alpha\xi)\mu(\mathrm{d}\xi)$, translation-invariant. Therefore, ν_0 is a Haar measure on $k^\times$. By using linearity of the integral and monotone convergence theorem, we obtain the lemma below.

Lemma 2.1.9. *A complex-valued map g defined on $k^\times$ is ν_0-integrable if and only if $g|\ |^{-1}$ is μ-integrable on $k^+ \setminus \{0\}$. In this case, $\int_{k^\times} g(\alpha)\nu_0(\mathrm{d}\alpha) = \int_{k^+ \setminus \{0\}} g(\xi)|\xi|^{-1} \mu(\mathrm{d}\xi)$.*

We choose our Haar measure ν on $k^\times$ as ν_0 if $\mathfrak{p}$ is archimedean, and as $\frac{N(\mathfrak{p})}{N(\mathfrak{p})-1}\nu_0$ if $\mathfrak{p}$ is discrete. In the latter case, we have $\nu(u) = N(\mathfrak{d})^{-\frac{1}{2}}$. Indeed, $\nu_0(u) = \int_u \mathrm{d}\nu_0 =$

$\int_u |\xi|^{-1}\mu(\mathrm{d}\xi) = \int_u \mathrm{d}\mu = \mu(u) = \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\mu(\mathfrak{o})$. Thus $\nu(u) = \mu(\mathfrak{o}) = N(\mathfrak{d})^{-\frac{1}{2}}$ by definition of μ.

2.1.4. The Local Zeta Function and Functional Equation

From this point on, we shall see Fourier transforms $\widehat{f}$ of functions f in $L^1(k^+)$ as if they have domain k^+ by means of the isomorphism $\widehat{k^+} \cong k^+$. We define a collection $\mathcal{Z}$ of functions as

$$\mathcal{Z} = \{f \in \mathfrak{S}(k^+) :\ f|_{k^\times}|\,|^\sigma \text{ and } \widehat{f}|_{k^\times}|\,|^\sigma \text{ belong to } L^1(k^\times) \text{ for } \sigma > 0\}.$$

It will be convenient to denote by $\mathcal{Q}$ the set of all quasi-characters of $k^\times$ and by $\mathcal{Q}_S$ the set of all quasi-characters with exponents in $S \subseteq \mathbb{R}$.

Definition 2.1.10. Corresponding to each $f \in \mathcal{Z}$, the map $\zeta_f : \mathcal{Q}_{(0,\infty)} \to \mathbb{C}$ defined by $\zeta_f(\chi) := \int_{k^\times} f(\alpha)\chi(\alpha)\nu(\mathrm{d}\alpha)$ is called a *zeta function* of k.

The condition $f|_{k^\times}|\,|^\sigma \in L^1(k^\times)$ guarantees that ζ_f has a meaningful definition. We have $\widehat{f} \in \mathcal{Z}$ whenever $f \in \mathcal{Z}$ since the identity $f(\xi) = \widehat{\widehat{f}}(-\xi)$ holds for all $\xi \in k$. In particular, $\zeta_{\widehat{f}}$ is also a zeta function of k for $f \in \mathcal{Z}$.

We define an equivalence relation on $\mathcal{Q}$ by declaring two quasi-characters to be equivalent if their quotient is an unramified quasi-character; that is, if they agree on u. Let us denote by C the equivalence class of χ so that $C = \{\chi|\,|^s : s \in \mathbb{C}\}$. If $\mathfrak{p}$ is archimedean, then the quasi-characters $\chi|\,|^s$ in C are distinct for each $s \in \mathbb{C}$; if $\mathfrak{p}$ is discrete, then they are distinct for $s \in \mathbb{C}$ modulo $2\pi i/\log N(\mathfrak{p})$. Therefore, we may view C as a complex plane in the archimedean case and as a cylinder $\mathbb{C}/\langle 2\pi i/\log N(\mathfrak{p})\rangle$ (a complex plane in which points differing by an integral multiple of $2\pi i/\log N(\mathfrak{p})$ are identified) in the discrete case. Thus we may see a function of quasi-characters as a function defined on a collection of planes or cylinders. Such a function will be said to be holomorphic at a point of its domain if its restriction to the corresponding plane or cylinder is holomorphic at that point. Being holomorphic on a subset is defined accordingly.

Lemma 2.1.11. *Zeta functions of k are holomorphic on $\mathcal{Q}_{(0,\infty)}$.*

Proof. Let $f \in \mathcal{Z}$ and let χ be an arbitrary quasi-character with exponent 0. The complex-valued function $s \mapsto \int_{k^\times} f(\alpha)\chi(\alpha)|\alpha|^s \nu(\mathrm{d}\alpha)$ is holomorphic on the domain $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$. In fact, it can be differentiated under the integral sign: its derivative at s is equal to $\int_{k^\times} f(\alpha)\chi(\alpha)\log|\alpha||\alpha|^s\nu(\mathrm{d}\alpha)$. Hence, the restriction of ζ_f to $C \cap \mathcal{Q}_{(0,\infty)}$ is holomorphic, where C denotes the equivalence class of χ. Since χ is arbitrary, this shows that ζ_f is holomorphic on $\mathcal{Q}_{(0,\infty)}$. Q.E.D.

For each quasi-character χ of $k^\times$ we define $\widehat{\chi}$ to be the quasi-character $\chi^{-1}|\,|$. If χ has exponent σ, then $\widehat{\chi}$ has exponent $1-\sigma$.

Lemma 2.1.12. *For all $\chi \in \mathcal{Q}_{(0,1)}$ and $f, g \in \mathcal{Z}$, we have $\zeta_f(\chi)\zeta_{\widehat{g}}(\widehat{\chi}) = \zeta_{\widehat{f}}(\widehat{\chi})\zeta_g(\chi)$.*

Proof. Fix $\chi \in \mathcal{Q}_{(0,1)}$ arbitrarily. Let $f, g \in \mathcal{Z}$. Then

$$\begin{aligned}
\zeta_f(\chi)\zeta_{\widehat{g}}(\widehat{\chi}) &= \int_{k^\times} f(\alpha)\chi(\alpha)\nu(\mathrm{d}\alpha)\int_{k^\times} \widehat{g}(\beta)\widehat{\chi}(\beta)\nu(\mathrm{d}\beta)\\
&= \int_{k^\times}\int_{k^\times} f(\alpha)\chi(\alpha)\widehat{g}(\beta)\widehat{\chi}(\beta)\nu(\mathrm{d}\alpha)\nu(\mathrm{d}\beta)\\
&= \int_{k^\times}\int_{k^\times} f(\alpha)\widehat{g}(\beta)\chi(\alpha\beta^{-1})|\beta|\nu(\mathrm{d}\alpha)\nu(\mathrm{d}\beta)\\
&= \int_{k^\times\times k^\times} f(\alpha)\widehat{g}(\beta)\chi(\alpha\beta^{-1})|\beta|(\nu\times\nu)\big(\mathrm{d}(\alpha,\beta)\big)
\end{aligned}$$

where the last equality is a consequence of Fubini's theorem (Theorem 5.2.2 of Cohn [7], p. 159). For any $\nu \times \nu$-integrable complex-valued map h on $k^\times \times k^\times$ it follows again by Fubini's theorem and the translation-invariance of ν that

$$\int_{k^\times\times k^\times} h(\alpha,\beta)(\nu\times\nu)\big(\mathrm{d}(\alpha,\beta)\big) = \int_{k^\times\times k^\times} h(\alpha,\alpha\beta)(\nu\times\nu)\big(\mathrm{d}(\alpha,\beta)\big).$$

So the integral

$$\int_{k^\times \times k^\times} f(\alpha)\widehat{g}(\beta)\chi(\alpha\beta^{-1})|\beta|(\nu \times \nu)\big(\mathrm{d}(\alpha,\beta)\big)$$

equals

$$\int_{k^\times \times k^\times} f(\alpha)\widehat{g}(\alpha\beta)\chi(\beta^{-1})|\alpha\beta|(\nu \times \nu)\big(\mathrm{d}(\alpha,\beta)\big)$$

which in turn is equal to

$$\int_{k^\times}\left(\int_{k^\times} f(\alpha)\widehat{g}(\alpha\beta)|\alpha|\nu(\mathrm{d}\alpha)\right)\chi(\beta^{-1})|\beta|\nu(\mathrm{d}\beta).$$

Thus we have shown

$$\zeta_f(\chi)\zeta_{\widehat{g}}(\widehat{\chi}) = \int_{k^\times}\left(\int_{k^\times} f(\alpha)\widehat{g}(\alpha\beta)|\alpha|\nu(\mathrm{d}\alpha)\right)\chi(\beta^{-1})|\beta|\nu(\mathrm{d}\beta).$$

Hence, it suffices to show that the integral in parentheses is symmetric in f and g. Pick an arbitrary $\beta \in k^\times$. Let c be the constant defined by $\nu = c\nu_0$. By Lemma **??**, we have

$$\begin{aligned}
\int_{k^\times} f(\alpha)\widehat{g}(\alpha\beta)|\alpha|\nu(\mathrm{d}\alpha) &= c\int_{k^+\setminus\{0\}} f(\xi)\widehat{g}(\xi\beta)\mu(\mathrm{d}\xi) \\
&= c\int_{k^+} f(\xi)\widehat{g}(\xi\beta)\mu(\mathrm{d}\xi) \\
&= c\int_{k^+} f(\xi)\left(\int_{k^+} g(\eta)e^{-2\pi i\Lambda(\xi\beta\eta)}\mu(\mathrm{d}\eta)\right)\mu(\mathrm{d}\xi) \\
&= c\int_{k^+\times k^+} f(\xi)g(\eta)e^{-2\pi i\Lambda(\xi\beta\eta)}(\mu\times\mu)\big(\mathrm{d}(\xi,\eta)\big).
\end{aligned}$$

The result follows since the last integral above is symmetric in f and g. Q.E.D.

Theorem 2.1.13. *There is a meromorphic function ρ on $\mathcal{Q}$, with domain a superset of $\mathcal{Q}_{(0,1)}$, such that $\zeta_f(\chi) = \rho(\chi)\zeta_{\widehat{f}}(\widehat{\chi})$ for all $f \in \mathcal{Z}$ and $\chi \in \mathcal{Q}_{(0,1)}$. By this functional equation, each zeta function of k is defined to be a meromorphic function on $\mathcal{Q}$.*

Proof. In the next subsection, we will exhibit for each equivalence class C of quasi-characters an explicit function $f_C \in \mathcal{Z}$ such that $\chi \mapsto \zeta_{f_C}(\chi)/\zeta_{\widehat{f_C}}(\widehat{\chi})$ defines a function ρ_C on $C \cap \mathcal{Q}_{(0,1)}$, which will turn out to be a familiar meromorphic function of the parameter s with which we describe the plane or cylinder C. Thus ρ is defined to be a meromorphic function, namely as ρ_C, on all of C for every C; this is exactly what we mean by saying that ρ is meromorphic on $\mathcal{Q}$. Now let $f \in \mathcal{Z}$ and $\chi \in \mathcal{Q}_{(0,1)}$. Call the equivalence class of χ as C. By Lemma 2.1.12, the equality $\zeta_f(\chi)\zeta_{\widehat{f_C}}(\widehat{\chi}) = \zeta_{\widehat{f}}(\widehat{\chi})\zeta_{f_C}(\chi)$ holds. Hence, we have $\zeta_f(\chi) = \rho(\chi)\zeta_{\widehat{f}}(\widehat{\chi})$. Q.E.D.

2.1.5. Computation of ρ by Special Zeta Functions

In this subsection, treating each case $k = \mathbb{R}$, $k = \mathbb{C}$ and $k = \mathfrak{p}$-adic separately, we exhibit for each equivalence class C of quasi-characters an explicit function $f_C \in \mathcal{Z}$, as promised in the proof of the last theorem of the previous subsection, and then compute $\zeta_{f_C}(\chi)$ as well as $\zeta_{\widehat{f_C}}(\widehat{\chi})$ for $\chi \in C \cap \mathcal{Q}_{(0,1)}$ to determine ρ_C.

If φ is a character of u, then $\alpha \mapsto \varphi(\widetilde{\alpha})$ defines a character of $k^\times$. In this way, we may see characters of u as characters of $k^\times$. Every quasi-character of $k^\times$ is equivalent to a character of u by Theorem 2.1.8, and two distinct characters of u are not equivalent. Hence, characters of u make up a complete set of representatives for the set of equivalence classes of quasi-characters.

2.1.5.1. $k = \mathbb{R}$. Let us denote the equivalence class of $\alpha \mapsto 1$ by C, and the equivalence class of $\alpha \mapsto \alpha$ by C_{sgn}. Thus $C = \{|\,|^s : s \in \mathbb{C}\}$ and $C_{\text{sgn}} = \{\text{sgn}|\,|^s : s \in \mathbb{C}\}$. We define f and f_{sgn} by $f(\xi) := e^{-\pi\xi^2}$ and $f_{\text{sgn}}(\xi) := \xi e^{-\pi\xi^2}$ for all $\xi \in \mathbb{R}$ (we use the notations f and f_{sgn} instead of f_C and $f_{C_{\text{sgn}}}$ for simplicity). These maps are continuous and integrable. Now we compute their Fourier transforms. We notice that $\Lambda = \lambda \circ T_{\mathbb{R}/\mathbb{R}} = \lambda$. Thus, for all $\eta \in \mathbb{R}$, we have

$$\widehat{f}(\eta) = \int_{\mathbb{R}} e^{-\pi\xi^2} e^{-2\pi i\Lambda(\eta\xi)} \mu(\mathrm{d}\xi) = \int_{-\infty}^{\infty} e^{-\pi\xi^2 + 2\pi i\eta\xi}\,\mathrm{d}\xi = e^{-\pi\eta^2}$$

where the last equality can be proved by using elementary complex analysis. On the other hand, we have

$$\begin{aligned}
\widehat{f_{\mathrm{sgn}}}(\eta) &= \int_{\mathbb{R}} \xi e^{-\pi\xi^2} e^{-2\pi i \Lambda(\eta\xi)} \mu(\mathrm{d}\xi) \\
&= \int_{-\infty}^{\infty} \xi e^{-\pi\xi^2 + 2\pi i \eta\xi} \, \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\eta} \big(e^{-\pi\xi^2 + 2\pi i \eta\xi} \big) \, \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\int_{-\infty}^{\infty} e^{-\pi\xi^2 + 2\pi i \eta\xi} \, \mathrm{d}\xi \right) \\
&= \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\eta} \big(e^{-\pi\eta^2} \big) \\
&= i\eta e^{-\pi\eta^2}
\end{aligned}$$

for all $\eta \in \mathbb{R}$. Hence, Fourier transforms of f and f_{sgn} are also continuous and integrable so that these maps belong to $\mathfrak{S}(\mathbb{R}^+)$. Moreover, it follows from the integral representation of the gamma function that we have $f \in \mathcal{Z}$ and $f_{\mathrm{sgn}} \in \mathcal{Z}$. We proceed to compute the zeta functions. Let $s \in \mathbb{C}$ be arbitrary with $\operatorname{Re} s > 0$. Then

$$\begin{aligned}
\zeta_f\big(|\,|^s\big) &= \int_{\mathbb{R}^\times} f(\alpha)|\alpha|^s \nu(\mathrm{d}\alpha) \\
&= \int_{\mathbb{R}^+\setminus\{0\}} e^{-\pi\xi^2} |\xi|^{s-1} \mu(\mathrm{d}\xi) \\
&= 2\int_0^{\infty} e^{-\pi\xi^2} \xi^{s-1} \, \mathrm{d}\xi \\
&= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\zeta_{f_{\mathrm{sgn}}}\big(\mathrm{sgn}|\,|^s\big) &= \int_{\mathbb{R}^\times} f_{\mathrm{sgn}}(\alpha)\mathrm{sgn}(\alpha)|\alpha|^s \nu(\mathrm{d}\alpha) \\
&= \int_{\mathbb{R}^+\setminus\{0\}} \xi e^{-\pi\xi^2} \mathrm{sgn}(\xi)|\xi|^{s-1} \mu(\mathrm{d}\xi) \\
&= 2\int_0^{\infty} e^{-\pi\xi^2} \xi^{s} \, \mathrm{d}\xi \\
&= \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right).
\end{aligned}$$

If we further assume $\operatorname{Re} s < 1$, then

$$\zeta_{\widehat{f}}\big(\widehat{|\,|^s}\big) = \zeta_f\big(|\,|^{1-s}\big) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)$$

and

$$\zeta_{\widehat{f_{\mathrm{sgn}}}}\left(\widehat{\mathrm{sgn}|\,|^s}\right) = i\zeta_{f_{\mathrm{sgn}}}\big(\mathrm{sgn}|\,|^{1-s}\big) = i\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right).$$

Therefore, by the identities $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$ and $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\big(z+\frac{1}{2}\big)$, we obtain

$$\rho_C\big(|\,|^s\big) = \zeta_f\big(|\,|^s\big)/\zeta_{\widehat{f}}\big(\widehat{|\,|^s}\big) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)$$

and

$$\rho_{C_{\mathrm{sgn}}}\big(\mathrm{sgn}|\,|^s\big) = \zeta_{f_{\mathrm{sgn}}}\big(\mathrm{sgn}|\,|^s\big)/\zeta_{\widehat{f_{\mathrm{sgn}}}}\left(\widehat{\mathrm{sgn}|\,|^s}\right) = -i2^{1-s}\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s).$$

<u>2.1.5.2. $k = \mathbb{C}$.</u> Let us denote the equivalence class of $\alpha \mapsto \alpha^m$ by C_m for each $m \in \mathbb{Z}$. Then $C_m = \big\{\chi_m|\,|^s : s \in \mathbb{C}\big\}$ where $\chi_m : \mathbb{C}^\times \to S^1$ is defined by $\chi_m(re^{i\theta}) := e^{in\theta}$. We define $f_m : \mathbb{C} \to \mathbb{C}$ (short for f_{C_m}) by

$$f_m(\xi) := \begin{cases} \overline{\xi}^m e^{-2\pi|\xi|}, & \text{if } m \geq 0; \\ \xi^{-m} e^{-2\pi|\xi|}, & \text{if } m < 0. \end{cases}$$

Thus, for all $m \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, we have

$$f_m(x+iy) = \begin{cases} (x-iy)^n e^{-2\pi(x^2+y^2)}, & \text{if } m \geq 0; \\ (x+iy)^{-m} e^{-2\pi(x^2+y^2)}, & \text{if } m < 0. \end{cases}$$

The maps f_m are continuous and integrable. Now we compute their Fourier transforms. We notice $\Lambda = \lambda \circ T_{\mathbb{C}/\mathbb{R}}$ so that $\Lambda(\xi) = \lambda\big(T_{\mathbb{C}/\mathbb{R}}(\xi)\big) = \lambda(2\operatorname{Re}\xi)$ for all $\xi \in \mathbb{C}$. We claim that the equality $\widehat{f_m} = i^{|m|} f_{-m}$ holds for every $m \in \mathbb{Z}$. We shall prove this for $m = 0$, and then for $m > 0$ via induction, and finally for $m < 0$. Let μ_0 denote the Lebesgue measure on $\mathbb{R}$. For all $\eta = u + iv \in \mathbb{C}$, we have

$$\begin{aligned}
\widehat{f_0}(\eta) &= \int_{\mathbb{C}} e^{-2\pi|\xi|} e^{-2\pi i \Lambda(\eta\xi)} \mu(\mathrm{d}\xi) \\
&= 2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi(x^2+y^2)} e^{4\pi i(ux - vy)} \mu_0(\mathrm{d}x) \mu_0(\mathrm{d}y) \\
&= 2 \Big(\int_{-\infty}^{\infty} e^{-2\pi x^2} e^{4\pi i u x} \, \mathrm{d}\xi \Big) \Big(\int_{-\infty}^{\infty} e^{-2\pi y^2} e^{-4\pi i v y} \, \mathrm{d}\xi \Big) \\
&= e^{-2\pi(u^2+v^2)} \\
&= f_0(\eta).
\end{aligned}$$

This proves the claim for $m = 0$. Now suppose the claim is true for arbitrary $m \geq 0$. This means that the equality

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - iy)^m e^{-2\pi(x^2+y^2)} e^{4\pi i(ux-vy)} 2 \, \mathrm{d}x \, \mathrm{d}y = i^m (u + iv)^m e^{-2\pi(u^2+v^2)}$$

holds for all $u, v \in \mathbb{R}$. We apply the operator $D := \frac{1}{4\pi i}\big(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v}\big)$ to both sides:

$$\begin{aligned}
D i^m (u + iv)^m e^{-2\pi(u^2+v^2)} &= i^m (u + iv)^m D e^{-2\pi(u^2+v^2)} \\
&= i^{m+1} (u + iv)^{m+1} e^{-2\pi(u^2+v^2)}.
\end{aligned}$$

On the other hand, the function

$$D \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - iy)^m e^{-2\pi(x^2+y^2)} e^{4\pi i(ux-vy)} 2 \, \mathrm{d}x \, \mathrm{d}y$$

is equal to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D (x - iy)^m e^{-2\pi(x^2+y^2)} e^{4\pi i(ux-vy)} 2 \, \mathrm{d}x \, \mathrm{d}y$$

which in turn equals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-iy)^{m+1}e^{-2\pi(x^2+y^2)}e^{4\pi i(ux-vy)}2\,\mathrm{d}x\,\mathrm{d}y.$$

Hence, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-iy)^{m+1}e^{-2\pi(x^2+y^2)}e^{4\pi i(ux-vy)}2\,\mathrm{d}x\,\mathrm{d}y = i^{m+1}(u+iv)^{m+1}e^{-2\pi(u^2+v^2)}.$$

Thus, our claim is proved for $m+1$. Consequently, the equality $\widehat{f_m} = i^{|m|}f_{-m}$ holds for all $m \geq 0$ by induction. Finally, let $m < 0$ be arbitrary. We know $\widehat{f_{-m}} = i^{|m|}f_m = i^{-m}f_m$ so that $f_m = i^m\widehat{f_{-m}}$. Taking Fourier transforms of both sides, we get

$$\begin{aligned}
\widehat{f_m}(\eta) &= i^m\widehat{\widehat{f_{-m}}}(\eta)\\
&= i^n f_{-m}(-\eta)\\
&= i^m(-1)^{-m}f_{-m}(\eta)\\
&= i^{|m|}f_{-m}(\eta)
\end{aligned}$$

for all $\eta \in \mathbb{C}$. Therefore, the equality $\widehat{f_m} = i^{|m|}f_{-m}$ holds for all $m \in \mathbb{Z}$. As a consequence, we deduce that the maps f_m belong to $\mathfrak{S}(\mathbb{C}^+)$. Moreover, it follows from the integral representation of the gamma function that $f_m \in \mathcal{Z}$ for all $m \in \mathbb{Z}$. We proceed to compute the zeta functions. Let $s \in \mathbb{C}$ be arbitrary with $\operatorname{Re} s > 0$. Then

$$\begin{aligned}
\zeta_{f_m}\big(\chi_m|\,|^s\big) &= \int_{\mathbb{C}^\times} f_m(\alpha)\chi_m(\alpha)|\alpha|^s\nu(\mathrm{d}\alpha)\\
&= \int_0^\infty\int_0^{2\pi} r^{|m|}e^{-im\theta}e^{-2\pi r^2}e^{im\theta}r^{2s}\frac{2r\,\mathrm{d}\theta\,\mathrm{d}r}{r^2}\\
&= 2\pi\int_0^\infty (r^2)^{(s-1)+\frac{|m|}{2}}e^{-2\pi r^2}2r\,\mathrm{d}r\\
&= 2\pi\int_0^\infty t^{(s-1)+\frac{|m|}{2}}e^{-2\pi t}\,\mathrm{d}t\\
&= (2\pi)^{(1-s)-\frac{|m|}{2}}\int_0^\infty t^{s+\frac{|m|}{2}-1}e^{-t}\,\mathrm{d}t\\
&= (2\pi)^{(1-s)-\frac{|m|}{2}}\Gamma\left(s+\frac{|m|}{2}\right).
\end{aligned}$$

If we further assume $\operatorname{Re} s < 1$, then

$$\begin{aligned}
\zeta_{\widehat{f_m}}\left(\widehat{\chi_m|\,|^s}\right) &= \zeta_{i^{|m|}f_{-m}}\left(\chi_{-m}|\,|^{1-s}\right)\\
&= i^{|m|}\zeta_{f_{-m}}\left(\chi_{-m}|\,|^{1-s}\right)\\
&= i^{|m|}(2\pi)^{s-\frac{|m|}{2}}\Gamma\left((1-s)+\frac{|m|}{2}\right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\rho_{C_m}\left(\chi_m|\,|^s\right) &= \zeta_{f_m}\left(\chi_m|\,|^s\right)/\zeta_{\widehat{f_m}}\left(\widehat{\chi_m|\,|^s}\right)\\
&= \frac{(2\pi)^{(1-s)-\frac{|m|}{2}}\Gamma\left(s+\frac{|m|}{2}\right)}{i^{|m|}(2\pi)^{s-\frac{|m|}{2}}\Gamma\left((1-s)+\frac{|m|}{2}\right)}\\
&= (-i)^{|m|}\frac{(2\pi)^{1-s}\Gamma\left(s+\frac{|m|}{2}\right)}{(2\pi)^{s}\Gamma\left((1-s)+\frac{|m|}{2}\right)}.
\end{aligned}$$

2.1.5.3. $k = \mathfrak{p}$-adic. When k is $\mathfrak{p}$-adic, recall that each character of u has a conductor $\mathfrak{p}^n$ for some $n \geq 0$, and there are finitely many characters of u having the same conductor. Unlike the previous cases, we shall not explicitly state each equivalence class of characters of u, but consider arbitrary characters of u with specified conductors to denote any one of those finitely many characters having that conductor. So let, for each $n \geq 0$, χ_n denote any character of u with conductor $\mathfrak{p}^n$ and let C_n denote the equivalence class of χ_n. Thus $C_n = \left\{\chi_n|\,|^s : s \in \mathbb{C}\right\}$ for all $n \geq 0$ where χ_n denotes the map $\alpha \mapsto \chi_n(\widetilde{\alpha})$ on $k^\times$. For any element π of ordinal number 1, we have $\chi_n(\pi) = 1$. Before we define for $n \geq 0$ the corresponding functions f_n (short for f_{C_n}), we shall record some facts.

For all $m \in \mathbb{Z}$, we have $\mathfrak{p}^m = \{\xi \in k : |\xi| \leq N(\mathfrak{p})^{-m}\}$ since $\mathfrak{p}^m = \pi^m\mathfrak{o}$. As a consequence of the ultrametric inequality, $\mathfrak{p}^m$ is a subgroup of k^+. This subgroup, being closed and bounded, is compact. We have $\mu(\mathfrak{p}^m) = \mu(\pi^m\mathfrak{o}) = |\pi^m|\mu(\mathfrak{o}) = N(\mathfrak{p})^{-m}N(\mathfrak{d})^{-\frac{1}{2}}$. Let $A_m := \{\xi \in k : |\xi| = N(\mathfrak{p})^{-m}\}$ for all $m \in \mathbb{Z}$. The equations $A_m = \mathfrak{p}^m \setminus \mathfrak{p}^{m+1}$ and $A_m = \pi^m u$ give alternative definitions. Each A_m is compact, and we have $\mu(A_m) = \mu(\mathfrak{p}^m \setminus \mathfrak{p}^{m+1}) = N(\mathfrak{p})^{-(m+1)}\big(N(\mathfrak{p}) - 1\big)N(\mathfrak{d})^{-\frac{1}{2}}$. The different $\mathfrak{d}$ of k is a nontrivial proper ideal of $\mathfrak{o}$, so $\mathfrak{d} = \mathfrak{p}^d$ for some $d \in \mathbb{N}$. For every $\eta \in k$, we have $\eta \in \mathfrak{d}^{-1}$ if and only if $\Lambda(\eta\mathfrak{o}) = 0$. The following lemma, whose proof we omit, will be frequently used in the sequel.

Lemma 2.1.14. *Let G be a locally compact group, let μ be a Haar measure on G, let H be a subgroup of G, and let $f : G \to \mathbb{C}^\times$ be μ-integrable on H. If f is a nontrivial homomorphism of abstract groups, then $\int_H f \, \mathrm{d}\mu = 0$.*

For each $n \geq 0$, we define $f_n : k \to \mathbb{C}$ as

$$f_n(\xi) := \begin{cases} e^{2\pi i \Lambda(\xi)}, & \text{if } \xi \in \mathfrak{d}^{-1}\mathfrak{p}^{-n}; \\ 0, & \text{if } \xi \notin \mathfrak{d}^{-1}\mathfrak{p}^{-n}. \end{cases}$$

It will be convenient to denote the characteristic function of a set A by $[A]$. Thus, if ψ_η denotes the character of k^+ corresponding to η by Theorem 2.1.3, then $f_n = \psi_1[\mathfrak{d}^{-1}\mathfrak{p}^{-n}]$ for all $n \geq 0$. Each f_n is continuous and integrable; we just notice $\mathfrak{d}^{-1}\mathfrak{p}^{-n} = \mathfrak{p}^{-(d+n)}$. For all $\eta \in k$, we have

$$\widehat{f_n}(\eta) = \int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{-2\pi i \Lambda((\eta-1)\xi)} \mu(\mathrm{d}\xi).$$

If $\eta \equiv 1 \pmod{\mathfrak{p}^n}$, which is to say $\eta - 1 \in \mathfrak{p}^n$, then $(\eta - 1)\xi \in \mathfrak{d}^{-1}$. This implies that $\Lambda\big((\eta - 1)\xi\big) = 0$. Consequently, $\widehat{f_n}(\eta) = \mu(\mathfrak{d}^{-1}\mathfrak{p}^{-n}) = N(\mathfrak{p})^n N(\mathfrak{d})^{\frac{1}{2}}$. If $\eta \not\equiv 1 \pmod{\mathfrak{p}^n}$, then the homomorphism $\xi \mapsto e^{-2\pi i \Lambda((\eta-1)\xi)}$ is nontrivial on the subgroup $\mathfrak{d}^{-1}\mathfrak{p}^{-n}$ whence $\int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{-2\pi i \Lambda((\eta-1)\xi)} \mu(\mathrm{d}\xi) = 0$. Hence, for all $\eta \in k$, we proved

$$\widehat{f_n}(\eta) = \begin{cases} N(\mathfrak{p})^n N(\mathfrak{d})^{\frac{1}{2}}, & \text{if } \eta \equiv 1 \pmod{\mathfrak{p}^n}; \\ 0, & \text{if } \eta \not\equiv 1 \pmod{\mathfrak{p}^n}. \end{cases}$$

Thus $\widehat{f_n} = N(\mathfrak{p})^n N(\mathfrak{d})^{\frac{1}{2}}[1 + \mathfrak{p}^n]$ for all $n \geq 0$. Each $\widehat{f_n}$ is continuous and integrable. Therefore, $f_n \in \mathfrak{S}(k^+)$ for all $n \geq 0$. In fact, each f_n belongs to $\mathcal{Z}$ (see the computations below). We proceed to compute the zeta functions. Let $s \in \mathbb{C}$ be arbitrary with $\operatorname{Re} s > 0$. First, we treat the unramified case $n = 0$. Since χ_0 is the trivial character and $f_0 = [\mathfrak{d}^{-1}]$, we have

$$\begin{aligned}
\zeta_{f_0}\big(\chi_0|\,|^s\big) &= \int_{k^\times} f_0(\alpha)|\alpha|^s\nu(\mathrm{d}\alpha) \\
&= \int_{\mathfrak{d}^{-1}} |\alpha|^s\nu(\mathrm{d}\alpha) \\
&= \int_{\dot{\bigcup}_{m\geq -d} A_m} |\alpha|^s\nu(\mathrm{d}\alpha) \\
&= \sum_{m=-d}^{\infty} \int_{A_m} |\alpha|^s\nu(\mathrm{d}\alpha) \\
&= \sum_{m=-d}^{\infty} N(\mathfrak{p})^{-ms} \int_{A_m} \mathrm{d}\nu \\
&= \sum_{m=-d}^{\infty} N(\mathfrak{p})^{-ms} \frac{N(\mathfrak{p})}{N(\mathfrak{p})-1} \int_{A_m} \mathrm{d}\nu_0 \\
&= \sum_{m=-d}^{\infty} N(\mathfrak{p})^{-ms} \frac{N(\mathfrak{p})}{N(\mathfrak{p})-1} N(\mathfrak{p})^m \mu(A_m) \\
&= \frac{N(\mathfrak{d})^{s-\frac{1}{2}}}{1-N(\mathfrak{p})^{-s}}.
\end{aligned}$$

Now suppose further that $\operatorname{Re} s < 1$. We have $\widehat{f_0} = N(\mathfrak{d})^{\frac{1}{2}}[\mathfrak{o}]$, so

$$\zeta_{\widehat{f_0}}\left(\widehat{\chi_0|\,|^s}\right) = N(\mathfrak{d})^{\frac{1}{2}} \int_{\mathfrak{o}} |\alpha|^{1-s}\nu(\mathrm{d}\alpha) = \frac{1}{1-N(\mathfrak{p})^{s-1}},$$

the latter equality being proved as above. Next, we treat the ramified case $n > 0$. We have

$$\begin{aligned}
\zeta_{f_n}\big(\chi_n|\,|^s\big) &= \int_{k^\times} f_n(\alpha)\chi_n(\alpha)|\alpha|^s\nu(\mathrm{d}\alpha) \\
&= \int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)|\alpha|^s\nu(\mathrm{d}\alpha) \\
&= \sum_{m=-(d+n)}^{\infty} N(\mathfrak{p})^{-ms} \int_{A_m} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha).
\end{aligned}$$

We claim that $\int_{A_m} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha) = 0$ for all $m > -(d+n)$. If $m \geq -d$, then

$A_m \subseteq \mathfrak{d}^{-1}$ so that

$$\begin{aligned}\int_{A_m} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha) &= \int_{A_m} \chi_n(\alpha)\nu(\mathrm{d}\alpha)\\ &= \int_u \chi_n(\pi^m\alpha)\nu(\mathrm{d}\alpha)\\ &= \int_u \chi_n(\alpha)\nu(\mathrm{d}\alpha).\end{aligned}$$

The last integral is 0 because χ_n is nontrivial on u. If $-d > m > -d-n$, then we write A_m as a finite disjoint union: $A_m = \dot{\bigcup}_{\alpha_0\in A_m}(\alpha_0 + \mathfrak{d}^{-1})$. So we need only show $\int_{\alpha_0+\mathfrak{d}^{-1}} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha) = 0$ for each $\alpha_0 \in A_m$. Fix such an α_0. For all $\alpha \in \mathfrak{d}^{-1}$ we have $\Lambda(\alpha_0 + \alpha) = \Lambda(\alpha_0)$. Thus

$$\int_{\alpha_0+\mathfrak{d}^{-1}} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha) = e^{2\pi i\Lambda(\alpha_0)}\int_{\alpha_0+\mathfrak{d}^{-1}} \chi_n(\alpha)\nu(\mathrm{d}\alpha).$$

We compute

$$\begin{aligned}\int_{\alpha_0+\mathfrak{d}^{-1}} \chi_n(\alpha)\nu(\mathrm{d}\alpha) &= \int_{\alpha_0(1+\mathfrak{p}^{-d-m})} \chi_n(\alpha)\nu(\mathrm{d}\alpha)\\ &= \int_{1+\mathfrak{p}^{-d-m}} \chi_n(\alpha_0\alpha)\nu(\mathrm{d}\alpha)\\ &= \chi_n(\alpha_0)\int_{1+\mathfrak{p}^{-d-m}} \chi_n(\alpha)\nu(\mathrm{d}\alpha).\end{aligned}$$

The integral $\int_{1+\mathfrak{p}^{-d-m}} \chi_n(\alpha)\nu(\mathrm{d}\alpha)$ is 0 since χ_n is nontrivial on $1+\mathfrak{p}^{-d-m}$. Thus we have proved $\int_{A_m} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha) = 0$ for all $m > -(d+n)$. As a result, we get

$$\zeta_{f_n}\big(\chi_n|\,|^s\big) = N(\mathfrak{p})^{(d+n)s}\int_{A_{-(d+n)}} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha).$$

We shall rewrite the integral above in a better form. Suppose $\varepsilon_1,\ldots,\varepsilon_r$ make up a complete set of representatives of u modulo $1+\mathfrak{p}^n$. Then A_{-d-n} is a disjoint union of the sets $\varepsilon_j\pi^{-d-n}(1+\mathfrak{p}^n) = \varepsilon_j\pi^{-d-n}+\mathfrak{d}^{-1}$. Fix j arbitrarily. Both Λ and χ_n are constant on $\varepsilon_j\pi^{-d-n}+\mathfrak{d}^{-1}$ since for all $\xi\in\mathfrak{d}^{-1}$ we have $\Lambda(\varepsilon_j\pi^{-d-n}+\xi) = \Lambda(\varepsilon_j\pi^{-d-n})$, and for all $\alpha \in 1+\mathfrak{p}^n$ we have $\chi_n(\varepsilon_j \pi^{-d-n}\alpha) = \chi_n(\varepsilon_j)$. Hence

$$\begin{aligned}
\zeta_{f_n}\big(\chi_n|\,|^s\big) &= N(\mathfrak{p})^{(d+n)s}\sum_{j=1}^{r}\int_{\varepsilon_j\pi^{-d-n}+\mathfrak{d}^{-1}} e^{2\pi i\Lambda(\alpha)}\chi_n(\alpha)\nu(\mathrm{d}\alpha)\\
&= N(\mathfrak{p})^{(d+n)s}\sum_{j=1}^{r} e^{2\pi i\Lambda(\varepsilon_j\pi^{-d-n})}\chi_n(\varepsilon_j)\int_{\varepsilon_j\pi^{-d-n}+\mathfrak{d}^{-1}}\mathrm{d}\nu\\
&= N(\mathfrak{p})^{(d+n)s}\sum_{j=1}^{r} e^{2\pi i\Lambda(\varepsilon_j\pi^{-d-n})}\chi_n(\varepsilon_j)\int_{1+\mathfrak{p}^n}\mathrm{d}\nu\\
&= N(\mathfrak{p})^{(d+n)s}\nu(1+\mathfrak{p}^n)\sum_{j=1}^{r} e^{2\pi i\Lambda(\varepsilon_j\pi^{-d-n})}\chi_n(\varepsilon_j)
\end{aligned}$$

for all $n>0$ (note that r depends on n here). Assume $\operatorname{Re} s<1$ further. Then

$$\zeta_{\widehat{f_n}}\left(\widehat{\chi_n|\,|^s}\right) = N(\mathfrak{p})^n N(\mathfrak{d})^{\frac{1}{2}}\int_{1+\mathfrak{p}^n}\chi_n^{-1}(\alpha)|\alpha|^{1-s}\nu(\mathrm{d}\alpha) = N(\mathfrak{p})^n N(\mathfrak{d})^{\frac{1}{2}}\nu(1+\mathfrak{p}^n).$$

Therefore, for all $s\in\mathbb{C}$ with $0<\operatorname{Re} s<1$, we have

$$\begin{aligned}
\rho_{C_0}\big(\chi_0|\,|^s\big) &= \zeta_{f_0}\big(\chi_0|\,|^s\big)/\zeta_{\widehat{f_0}}\left(\widehat{\chi_0|\,|^s}\right)\\
&= \frac{N(\mathfrak{d})^{s-\frac{1}{2}}}{1-N(\mathfrak{p})^{-s}} \div \frac{1}{1-N(\mathfrak{p})^{s-1}}\\
&= N(\mathfrak{d})^{s-\frac{1}{2}}\frac{1-N(\mathfrak{p})^{s-1}}{1-N(\mathfrak{p})^{-s}}.
\end{aligned}$$

For all $n>0$ and $s\in\mathbb{C}$ with $0<\operatorname{Re} s<1$, we have

$$\begin{aligned}
\rho_{C_n}\big(\chi_n|\,|^s\big) &= \zeta_{f_n}\big(\chi_n|\,|^s\big)/\zeta_{\widehat{f_n}}\left(\widehat{\chi_n|\,|^s}\right)\\
&= \frac{N(\mathfrak{p})^{(d+n)s}\nu(1+\mathfrak{p}^n)\sum_j e^{2\pi i\Lambda(\varepsilon_j\pi^{-d-n})}\chi_n(\varepsilon_j)}{N(\mathfrak{p})^n N(\mathfrak{d})^{\frac{1}{2}}\nu(1+\mathfrak{p}^n)}\\
&= N(\mathfrak{d}\mathfrak{p}^n)^{s-\frac{1}{2}}N(\mathfrak{p}^n)^{-\frac{1}{2}}\sum_j e^{2\pi i\Lambda(\varepsilon_j\pi^{-d-n})}\chi_n(\varepsilon_j)\\
&= N(\mathfrak{d}\mathfrak{f})^{s-\frac{1}{2}}\widetilde{\rho_{C_n}}(\chi_n)
\end{aligned}$$

where $\mathfrak{f}$ is the conductor of χ_n and $\widetilde{\rho_{C_n}}(\chi_n) := N(\mathfrak{f})^{-\frac{1}{2}}\sum_j \chi_n(\varepsilon_j)e^{2\pi i\Lambda(\varepsilon_j/\pi^{\operatorname{ord}(\mathfrak{d}\mathfrak{f})})}$ with $\operatorname{ord}(\mathfrak{p}^n)$ defined to be n for all $n\in\mathbb{N}$.

2.2. Restricted Direct Products

2.2.1. Introduction

Let I be an index set, and let I_∞ be a fixed finite subset of I. Let G_i be a locally compact group for each $i \in I$, and let H_i be a compact open subgroup of G_i for each $i \in I \setminus I_\infty$. We define $\mathcal{S}$ to be the collection of all finite subsets of I containing I_∞. In the sequel, "... for almost all i" means "there is an $S \in \mathcal{S}$ such that ... for all $i \notin S$."

Definition 2.2.1. *Restricted direct product* of the groups G_i relative to the subgroups H_i is defined to be the set of all I-tuples $(x_i)_{i\in I}$ such that $x_i \in H_i$ for almost all i. It is denoted by $\prod'_{i\in I} G_i$.

Restricted direct product $\prod'_{i\in I} G_i$ is a subgroup of the direct product $\prod_{i\in I} G_i$ and it becomes a topological group on stipulating that sets of the form $\prod_{i\in I} N_i$, where N_i is an open neighborhood of the identity in G_i for all i and $N_i = H_i$ for almost all i, make up a neighborhood basis of open sets at the identity 1 of $\prod'_{i\in I} G_i$ (see Section 4 in Chapter II of Higgins [8]). We write G to abbreviate $\prod'_{i\in I} G_i$. We shall most often omit indices from notation whenever a product is over the whole index set I.

For $S \in \mathcal{S}$, let $G_S := \prod_{i\in S} G_i \times \prod_{i\notin S} H_i$. Each G_S is a subgroup of G, and the subspace topology ς on G_S coincides with the product topology ϖ on G_S: the sets $G_S \cap \prod N_i$ are open in the product topology, this implies $\varsigma \subseteq \varpi$; conversely, every basic neighborhood of 1 in the product topology is of the form $\prod N_i$, implying $\varpi \subseteq \varsigma$.

Theorem 2.2.2. *G is locally compact.*

Proof. For any two elements of G, there is a G_S containing both of them. Every G_S is Hausdorff and, being of the form $\prod N_i$, open in G. Consequently, G is Hausdorff. For local compactness, fix any $S \in \mathcal{S}$. Since G_S is locally compact, there is a compact neighborhood of 1 in G_S, hence in G. Thus G is locally compact. Q.E.D.

For each $i \in I$, let π_i denote the projection from G onto G_i. These projections are continuous since their restrictions to the open subsets G_S are continuous.

Theorem 2.2.3. *A subset Y of G has compact closure if and only if $Y \subseteq \prod K_i$ for some compact sets $K_i \subseteq G_i$ with $K_i = H_i$ for almost all i.*

Proof. Let $Y \subseteq G$ have compact closure. The collection $\{G_S : S \in \mathcal{S}\}$ is an open cover for $\overline{Y}$, so $\overline{Y}$ is contained in finitely many of the G_S, hence in one of them, say in G_{S_0}. We define K_i to be $\pi_i(\overline{Y})$ or H_i according as $i \in S_0$ or $i \notin S_0$. Then $Y \subseteq \overline{Y} \subseteq \prod K_i$ where K_i is compact for all i by the continuity of projections. The converse follows from Tychonov's theorem. Q.E.D.

2.2.2. Characters

It will be convenient to abbreviate $\pi_i(y)$ by y_i for $i \in I$, $y \in G$. We denote the subgroup $\prod_{i\in S}\{1_i\} \times \prod_{i\notin S} H_i$ by G^S for $S \in \mathcal{S}$. We identify each G_i with its image under the embedding which sends $x \in G_i$ to the tuple having x in the ith coordinate and identity elements in the other coordinates. In accordance with our previous usage of the term, we mean by a quasi-character a continuous homomorphism into $\mathbb{C}^\times$.

Lemma 2.2.4. *If χ is a quasi-character of G, then χ is trivial on H_i for almost all i, and $\chi(y) = \prod_{i\in I} \chi(y_i)$ for all $y \in G$ (this is a finite product).*

Proof. Let U be an open neighborhood of 1 in $\mathbb{C}^\times$ containing no multiplicative subgroup of $\mathbb{C}^\times$ other than the trivial subgroup; for example, let U be the open ball around 1 of radius $\frac{1}{2}$. There is an open neighborhood $\prod N_i$ of 1 in G with $\chi(\prod N_i) \subseteq U$. Select an $S \in \mathcal{S}$ such that $N_i = H_i$ for all $i \notin S$. Then $G^S \subseteq \prod N_i$, so $\chi(G^S) \subseteq \chi(\prod N_i) \subseteq U$, implying $\chi(G^S) = 1$. Hence, χ is trivial on H_i for all $i \notin S$. Now let $y \in G$. Enlarge S (if needed) so that $y \in G_S$. Write $y = y^S \prod_{i\in S} y_i$ with $y^S \in G^S$. Then $\chi(y) = \chi(y^S) \prod_{i\in S} \chi(y_i) = \prod_{i\in S} \chi(y_i) = \prod_{i\in I} \chi(y_i)$. Q.E.D.

Lemma 2.2.5. *For each i, let χ_i be a quasi-character of G_i. If χ_i is trivial on H_i for almost all i, then $\prod_{i\in I}\chi_i$ is a quasi-character of G.*

Proof. Let $\chi := \prod_{i\in I}\chi_i$ for ease of notation. We show the continuity of χ only. Let U be an open neighborhood of 1 in $\mathbb{C}^\times$. Select an $S \in \mathcal{S}$ such that χ_i is trivial on H_i for all $i \notin S$, and let $n := |S|$. Choose an open neighborhood V of 1 in $\mathbb{C}^\times$ such that $V^n \subseteq U$. For $i \in S$, let N_i satisfy $\chi_i(N_i) \subseteq V$, and for $i \notin S$, let $N_i := H_i$. Then $\chi(\prod N_i) \subseteq V^n \subseteq U$, proving the continuity of χ. Q.E.D.

Suppose that the G_i are abelian in the rest of this subsection. Fix an arbitrary $i \notin I_\infty$. For U as in the proof of Lemma 2.2.4, we have $H_i^\perp = W(H_i, U)$ so that $H_i^\perp$ is an open subgroup of $\widehat{G_i}$. On the other hand, H_i is open in G_i, implying that G_i/H_i is discrete, which in turn gives that $\widehat{G_i/H_i}$ is compact. Consequently, $H_i^\perp$ is compact since $\widehat{G_i/H_i}$ and $H_i^\perp$ are isomorphic as topological groups. Therefore, $H_i^\perp$ is a compact open subgroup of $\widehat{G_i}$ for all $i \notin I_\infty$, and the restricted direct product $\prod'\widehat{G_i}$ of the groups $\widehat{G_i}$ relative to the subgroups $H_i^\perp$ makes sense.

Theorem 2.2.6. *$\widehat{G} \cong \prod'\widehat{G_i}$.*

Proof. It follows from Lemmas 2.2.4 and 2.2.5 that the map $\varphi : \widehat{G} \to \prod'\widehat{G_i}$ defined by $\chi \mapsto (\chi_i)$ is an algebraic isomorphism of groups. We will show φ is moreover a homeomorphism. Take an open neighborhood $\prod O_i$ of the identity in $\prod'\widehat{G_i}$ with $O_i = H_i^\perp$ for almost all i, and $O_i = W(K_i, U_i)$ for the remaining i. Select an $S \in \mathcal{S}$ such that $O_i = H_i^\perp$ for all $i \notin S$. Put $K := \prod_{i\in S} K_i \times \prod_{i\notin S} H_i$ and $U := \bigcap_{i\in S} U_i$. We may assume that U contains no multiplicative subgroup of $\mathbb{C}^\times$ other than the trivial one. Then $\varphi\big(W(K,U)\big) \subseteq \prod O_i$. This proves φ is continuous. Next, take an open neighborhood $W(K,U)$ of the identity in $\widehat{G}$. By Theorem 2.2.3, we have $K \subseteq \prod K_i$ for some compact sets $K_i \subseteq G_i$ with $K_i = H_i$ for almost all i. Select an $S \in \mathcal{S}$ such that $K_i = H_i$ for all $i \notin S$, and let $n := |S|$. Choose an open neighborhood V of 1 in S^1 such that $V^n \subseteq U$. Then $\prod_{i\in S} W(K_i, V) \times \prod_{i\notin S} H_i^\perp$ is a subset of $\varphi\big(W(\prod K_i, U)\big)$, hence of $\varphi\big(W(K,U)\big)$. This proves φ is open. Q.E.D.

2.2.3. Measures

Each G_i admits a left Haar measure μ_i. We assume $\mu_i(H_i) = 1$ for almost all i. For each $S \in \mathcal{S}$, let μ^S denote the Haar measure on G^S such that $\mu^S(G^S) = \prod_{i \notin S} \mu_i(H_i)$, and let μ_S denote Haar measure on G_S which is the transfer of the product measure $\left(\prod_{i \in S} \mu_i\right) \times \mu^S$ on $\left(\prod_{i \in S} G_i\right) \times G^S$ to G_S by means of the isomorphism $G_S \cong \left(\prod_{i \in S} G_i\right) \times G^S$.

Theorem 2.2.7. *There exists a Haar measure μ on G such that $\mu|_{\mathcal{B}(G_S)} = \mu_S$ for all $S \in \mathcal{S}$.*

Proof. For each $S \in \mathcal{S}$, we can choose a Haar measure $\mu(S)$ on G such that $\mu(S)|_{\mathcal{B}(G_S)} = \mu_S$. We will show this choice is independent of S. Let $S, T \in \mathcal{S}$ be arbitrary. We may assume $S \subseteq T$ so that $G_S \subseteq G_T$. It is enough to find a compact $K \subseteq G_S$ of nonzero measure such that $\mu_S(K) = \mu_T(K)$, implying $\mu_T|_{\mathcal{B}(G_S)} = \mu_S$, whence we get $\mu(S) = \mu(T)$. Take compact neighborhoods K_i of 1_i in G_i for $i \in S$, and let $K := \prod_{i \in S} K_i \times \prod_{i \notin S} H_i$. Then

$$\begin{aligned}
\mu_T(K) &= \prod_{i \in S} \mu_i(K_i) \times \prod_{i \in T \setminus S} \mu_i(H_i) \times \mu^T(G^T) \\
&= \prod_{i \in S} \mu_i(K_i) \times \prod_{i \in T \setminus S} \mu_i(H_i) \times \prod_{i \notin T} \mu_i(H_i) \\
&= \prod_{i \in S} \mu_i(K_i) \times \prod_{i \notin S} \mu_i(H_i) \\
&= \prod_{i \in S} \mu_i(K_i) \times \mu^S(G^S) \\
&= \mu_S(K).
\end{aligned}$$

Q.E.D.

We shall denote the measure μ of the preceding theorem by $\prod \mu_i$. From this point on, we assume that G_i is second countable for all $i \in I$ and the index set I is countable. As a consequence, G is second countable. We shall make use of this fact in the next theorem.

Theorem 2.2.8. *Let $f : G \to \mathbb{C}$ be measurable. If $f \geq 0$ or $f \in L^1(G)$, then $\int f \,\mathrm{d}\mu = \lim_{S \in \mathcal{S}} \int_{G_S} f \,\mathrm{d}\mu$.*

Proof. Let $\mathcal{K}$ denote the set of all compact subsets of G. Assume $f \geq 0$. Since G is second countable, we have $\int f \,\mathrm{d}\mu = \lim_{K \in \mathcal{K}} \int_K f \,\mathrm{d}\mu$. This implies the equality above since every $K \in \mathcal{K}$ is contained in some G_S. For the case $f \in L^1(G)$, consider the positive and negative parts of f. Q.E.D.

Let us denote by $\mathcal{F}$ the collection of all finite subsets of I, and by $\| \, \|_{1,i}$ the L^1-norm on G_i for each $i \in I$.

Theorem 2.2.9. *For each $i \in I$, let $f_i : G_i \to \mathbb{C}$ be continuous and integrable. Suppose $f_i|_{H_i} \equiv 1$ for almost all i. If $f : G \to \mathbb{C}$ is defined by $f(y) := \prod f_i(y_i)$ then the following hold.*

(1) f is continuous.

(2) For all $S \in \mathcal{S}$ such that $f_i|_{H_i} \equiv 1$ and $\mu_i(H_i) = 1$ for $i \notin S$, the equality $\int_{G_S} f \,\mathrm{d}\mu = \prod_{i \in S} \big(\int_{G_i} f_i \,\mathrm{d}\mu_i \big)$ holds.

(3) If $\lim_{F \in \mathcal{F}} \prod_{i \in F} \|f_i\|_{1,i}$ is finite, then $f \in L^1(G)$ and the equality $\int f \,\mathrm{d}\mu = \lim_{F \in \mathcal{F}} \prod_{i \in F} \big(\int_{G_i} f_i \,\mathrm{d}\mu_i \big)$ holds.

Proof. (1) It is enough to show the continuity of the restrictions $f|_{G_S}$. Let $S \in \mathcal{S}$ be arbitrary. Select $T \in \mathcal{S}$ such that $f_i|_{H_i} \equiv 1$ for all $i \notin T$. Then, for every $y \in G_S$, we have $f(y) = \prod f_i(y_i) = \prod_{i \in S \cup T} f_i(y_i)$. Since the f_i are continuous by hypothesis, the restriction $f|_{G_S}$ is continuous.

(2) Let S be as stated. Then $f(y) = \prod_{i\in S} f_i(y_i)$ for all $y \in G_S$. Thus

$$\begin{aligned}
\int_{G_S} f \,\mathrm{d}\mu &= \int_{G_S} f|_{G_S} \,\mathrm{d}\mu_S \\
&= \int_{G^S} \mathrm{d}\mu^S \cdot \prod_{i\in S} \left(\int_{G_i} f_i \,\mathrm{d}\mu_i \right) \\
&= \prod_{i\in S} \left(\int_{G_i} f_i \,\mathrm{d}\mu_i \right)
\end{aligned}$$

because $\int_{G^S} \mathrm{d}\mu^S = \mu^S(G^S) = \prod_{i\notin S} \mu_i(H_i) = 1$.

(3) By Theorem 2.2.8, the equality $\int |f| \,\mathrm{d}\mu = \lim_{S\in\mathcal{S}} \int_{G_S} |f| \,\mathrm{d}\mu$ holds (here $|\,|$ stands for the ordinary absolute value of complex numbers). We observe that on replacing each f_i above by $|f_i|$ we construct $|f|$ instead of f. So $\int_{G_S} |f| \,\mathrm{d}\mu = \prod_{i\in S} \left(\int_{G_i} |f_i| \,\mathrm{d}\mu_i \right)$ for all $S \in \mathcal{S}$ such that $f_i|_{H_i} \equiv 1$ and $\mu_i(H_i) = 1$ for $i \notin S$. Fix such an $S_0 \in \mathcal{S}$. Then

$$\begin{aligned}
\int |f| \,\mathrm{d}\mu &= \lim_{S\in\mathcal{S}} \int_{G_S} |f| \,\mathrm{d}\mu \\
&= \lim_{S\supseteq S_0} \int_{G_S} |f| \,\mathrm{d}\mu \\
&= \lim_{S\supseteq S_0} \prod_{i\in S} \left(\int_{G_i} |f_i| \,\mathrm{d}\mu_i \right) \\
&= \lim_{S\supseteq S_0} \prod_{i\in S} \|f_i\|_{1,i} \\
&= \lim_{F\in\mathcal{F}} \prod_{i\in F} \|f_i\|_{1,i} \\
&< \infty
\end{aligned}$$

so that $f \in L^1(G)$. Following the same lines of argument by using f in place of $|f|$ one proves $\int f \,\mathrm{d}\mu = \lim_{F\in\mathcal{F}} \prod_{i\in F} \left(\int_{G_i} f_i \,\mathrm{d}\mu_i \right)$. Q.E.D.

Theorem 2.2.10. *For each $i \in I$, let $f_i : G_i \to \mathbb{C}$ be continuous and integrable. Suppose $f_i = [H_i]$ for almost all i. If $f : G \to \mathbb{C}$ is defined by $f(y) := \prod f_i(y_i)$ then the following hold.*

(1) f is integrable.

(2) If the G_i are abelian, then $\widehat{f}(\chi) = \prod \widehat{f}_i(\chi_i)$ for all $\chi \in \widehat{G}$.

Proof. (1) By hypothesis, $\|f_i\|_{1,i} = \mu_i(H_i)$ for almost all i, so $\|f_i\|_{1,i} = 1$ for almost all i. Hence, by Theorem 2.2.9(3), we have $f \in L^1(G)$.

(2) First of all, note that the product is finite since, given any $i \in I \setminus I_\infty$, we have $\widehat{f_i} = \mu_i(H_i)[H_i^\perp]$ whenever $f_i = [H_i]$. Now let $\chi \in \widehat{G}$. We use Theorem 2.2.9 with the f_i replaced by $f_i\overline{\chi_i}$. Then

$$\begin{aligned}
\widehat{f}(\chi) &= \int f\overline{\chi}\,\mathrm{d}\mu \\
&= \lim_{F\in\mathcal{F}} \prod_{i\in F} \left(\int_{G_i} f_i\overline{\chi_i}\,\mathrm{d}\mu_i \right) \\
&= \lim_{F\in\mathcal{F}} \prod_{i\in F} \widehat{f}_i(\chi_i) \\
&= \prod \widehat{f}_i(\chi_i).
\end{aligned}$$

Q.E.D.

Suppose once again that the G_i are abelian. If $f_i = [H_i]$, then it follows from the equality $\widehat{f_i} = \mu_i(H_i)[H_i^\perp]$ and Fourier inversion formula that $1 = \mu_i(H_i)\widehat{\mu_i}(H_i^\perp)$. Hence, we have $\widehat{\mu_i}(H_i^\perp) = 1$ for almost all i since $\mu_i(H_i) = 1$ for almost all i by assumption. So we can define a measure $\prod \widehat{\mu_i}$ on the restricted direct product $\prod' \widehat{G_i}$. Consequently, by the isomorphism given in Theorem 2.2.6, we obtain a measure on $\widehat{G}$, which we also denote by $\prod \widehat{\mu_i}$. The next theorem presumes the existence of nontrivial functions in $\mathfrak{S}(G_i)$ for each $i \in I_\infty$. One shall see later that this is indeed the case when G is the adele group of a number field.

Theorem 2.2.11. *The measure $\prod \widehat{\mu_i}$ on $\widehat{G}$ is the dual measure of μ.*

Proof. Let $\widehat{\mu}$ denote the dual measure of μ. We know $f(y) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(y)\widehat{\mu}(\mathrm{d}\chi)$ for all $f \in \mathfrak{S}(G)$, $y \in G$. In order to prove $\prod \widehat{\mu_i} = \widehat{\mu}$, we shall view the integrand above as a map defined on $\prod' \widehat{G_i}$ and show that the integral of this map over $\prod' \widehat{G_i}$ with respect to $\prod \widehat{\mu_i}$ is equal to $f(y)$. It is sufficient to do this for particular $f \in \mathfrak{S}(G)$ and $y \in G$ with $f(y) \neq 0$. We define f_i to be $[H_i]$ for $i \notin I_\infty$ and to be a map in $\mathfrak{S}(G_i)$ with $f_i(1_i) = 1$ for $i \in I_\infty$. Let $f : G \to \mathbb{C}$ be defined by $f(y) := \prod f_i(y_i)$. For all $(\chi_i) \in \prod' \widehat{G_i}$, we have $\widehat{f}(\prod \chi_i) = \prod \widehat{f_i}(\chi_i)$ as a consequence of Theorem 2.2.10(2). On replacing the f_i by $\widehat{f_i}$ in Theorem 2.2.9, the integral of $\widehat{f}$ over $\prod' \widehat{G_i}$ with respect to $\prod \widehat{\mu_i}$ is seen to be $\lim_{F \in \mathcal{F}} \prod_{i \in F} \big(\int_{\widehat{G_i}} \widehat{f_i}(\chi_i) \widehat{\mu_i}(\mathrm{d}\chi_i) \big)$, which in turn is equal to $\lim_{F \in \mathcal{F}} \prod_{i \in F} f_i(1_i)$ by using Fourier inversion formula at each coordinate. Since $\lim_{F \in \mathcal{F}} \prod_{i \in F} f_i(1_i) = 1$, the integral of $\widehat{f}$ over $\prod' \widehat{G_i}$ is $f(1)$. Q.E.D.

2.3. The Global Theory

In this section, k denotes a number field. A generic prime divisor of k will be denoted by $\mathfrak{p}$, and the completion of k with respect to $\mathfrak{p}$ by $k_\mathfrak{p}$. All the symbols $|\,|, \mathfrak{o}, N, \mathrm{ord}, \Lambda, u, \mathfrak{d}$ defined for this local field $k_\mathfrak{p}$ also receive the subscript $\mathfrak{p}$: $|\,|_\mathfrak{p}, \mathfrak{o}_\mathfrak{p}, N_\mathfrak{p}, \mathrm{ord}_\mathfrak{p}, \Lambda_\mathfrak{p}, u_\mathfrak{p}, \mathfrak{d}_\mathfrak{p}$.

2.3.1. Additive Theory

Each prime divisor of $\mathbb{Q}$ has a finite number of prolongations to k (Corollary 4-32 of Ramakrishnan and Valenza [6], p. 162), so all but a finite number of primes of k are discrete. Hence, we can define the restricted direct sum of the groups $k_\mathfrak{p}$ relative to the subgroups $\mathfrak{o}_\mathfrak{p}$.

Definition 2.3.1. The restricted direct sum $\bigoplus'_\mathfrak{p} k_\mathfrak{p}$ of the groups $k_\mathfrak{p}$ relative to the subgroups $\mathfrak{o}_\mathfrak{p}$ is called the *adele group* of k. It is denoted by $\mathbb{A}_k$.

Since k is fixed, we write $\mathbb{A}$ instead of $\mathbb{A}_k$ for simplicity. We have the following isomorphisms of topological groups: $\widehat{\mathbb{A}} \cong \prod' \widehat{k_\mathfrak{p}} \cong \bigoplus' k_\mathfrak{p}$ where the restricted direct product is relative to the subgroups $\mathfrak{o}_\mathfrak{p}^\perp$ and the restricted direct sum is relative to the subgroups $\mathfrak{d}_\mathfrak{p}^{-1}$. The first isomorphism is a consequence of the theory of restricted direct products. The second isomorphism follows from the fact that $\widehat{k_\mathfrak{p}} \cong k_\mathfrak{p}$ for all $\mathfrak{p}$, and that $\mathfrak{o}_\mathfrak{p}^\perp$ is mapped onto $\mathfrak{d}_\mathfrak{p}^{-1}$ under this isomorphism for discrete $\mathfrak{p}$. Finally, $\bigoplus' k_\mathfrak{p}$ (relative to $\mathfrak{d}_\mathfrak{p}^{-1}$) is equal to $\mathbb{A}$ since $\mathfrak{d}_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}$ for almost all $\mathfrak{p}$. Hence, we obtain $\widehat{\mathbb{A}} \cong \mathbb{A}$. In order to write this isomorphism explicitly, let multiplication on $\mathbb{A}$ be defined coordinate-wise and let $\Lambda(x)$ be defined as the finite sum $\sum_\mathfrak{p} \Lambda_\mathfrak{p}(x_\mathfrak{p})$ for $x \in \mathbb{A}$.

Theorem 2.3.2. *The mapping from $\mathbb{A}$ to $\widehat{\mathbb{A}}$ sending each y to the character $x \mapsto e^{2\pi i\Lambda(yx)}$ is an isomorphism of topological groups.*

Let $\mu_\mathfrak{p}$ denote the measure on $k_\mathfrak{p}$. For almost all $\mathfrak{p}$, the equality $\mathfrak{d}_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}$ holds so that $\mu_\mathfrak{p}(\mathfrak{o}_\mathfrak{p}) = N_\mathfrak{p}(\mathfrak{d}_\mathfrak{p})^{-\frac{1}{2}} = N_\mathfrak{p}(\mathfrak{o}_\mathfrak{p})^{-\frac{1}{2}} = 1$. We denote by μ the measure $\prod \mu_\mathfrak{p}$ on the restricted direct product $\mathbb{A}$. Let $\widehat{\mu} := \prod \widehat{\mu}_\mathfrak{p}$. By Theorem 2.2.11, the measure $\widehat{\mu}$ on $\widehat{\mathbb{A}}$ is dual to μ, so $f(x) = \int_{\widehat{\mathbb{A}}} \widehat{f}(\psi)\psi(x)\widehat{\mu}(\mathrm{d}\psi)$ for all $f \in \mathfrak{S}(\mathbb{A}), x \in \mathbb{A}$. On writing this integral over $\mathbb{A}$ in view of the isomorphism $\widehat{\mathbb{A}} \cong \mathbb{A}$, we get the next theorem.

Theorem 2.3.3. *The inversion formula $f(x) = \int_\mathbb{A} \widehat{f}(y)e^{2\pi i\Lambda(yx)}\mu(\mathrm{d}y)$ holds for all $f \in \mathfrak{S}(\mathbb{A})$, $x \in \mathbb{A}$.*

For each $a \in \mathbb{A}$, the map $x \mapsto ax$ is a continuous homomorphism from $\mathbb{A}$ to $\mathbb{A}$. We shall see in the next lemma when this map has a continuous inverse, but first we must introduce the idele group of k.

Definition 2.3.4. The restricted direct product $\prod'_\mathfrak{p} k_\mathfrak{p}^\times$ of the groups $k_\mathfrak{p}^\times$ relative to the subgroups $u_\mathfrak{p}$ is called the *idele group* of k. It is denoted by $\mathbb{I}_k$.

As in the case of adele group, we write $\mathbb{I}$ for $\mathbb{I}_k$. Under coordinate-wise multiplication, $\mathbb{A}$ becomes a ring. Then $\mathbb{I}$ is exactly the multiplicative group of units of $\mathbb{A}$.

Lemma 2.3.5. *For all $a \in \mathbb{A}$, the map $x \mapsto ax$ is a topological automorphism of $\mathbb{A}$ if and only if $a \in \mathbb{I}$.*

Proof. If $x \mapsto ax$ is a topological automorphism of $\mathbb{A}$, then it must be surjective so that $ab = 1$ for some $b \in \mathbb{A}$. Thus $a \in \mathbb{A}^\times = \mathbb{I}$. Conversely, if $a \in \mathbb{I}$ then $ab = 1$ for some $b \in \mathbb{A}$. Consequently, the composition of $x \mapsto ax$ and $x \mapsto bx$ in either order is the identity map on $\mathbb{A}$, implying that $x \mapsto ax$ is a topological automorphism of $\mathbb{A}$. Q.E.D.

For all $a \in \mathbb{I}$, the function $\mu_a : \mathcal{B}(\mathbb{A}) \to [0, \infty]$ defined by $\mu_a(M) := \mu(aM)$ is a Haar measure on $\mathbb{A}$ so that $\mu_a = c_a\mu$ for some $c_a > 0$. This c_a turns out to be the finite product $\prod_{\mathfrak{p}} |a_{\mathfrak{p}}|_{\mathfrak{p}}$. We define the absolute value $|a|$ of an idele a as $\prod_{\mathfrak{p}} |a_{\mathfrak{p}}|_{\mathfrak{p}}$.

Lemma 2.3.6. $c_a = |a|$ *for all* $a \in \mathbb{I}$.

Proof. Let $a \in \mathbb{I}$ be arbitrary. It is enough to show $\mu(aK) = |a|\mu(K)$ for a compact $K \subseteq \mathbb{A}$ of nonzero measure. Select a finite set of primes S including the archimedean ones such that $\mu_p(\mathfrak{o}_{\mathfrak{p}}) = 1$ and $a_{\mathfrak{p}} \in u_{\mathfrak{p}}$ for all $\mathfrak{p} \notin S$. We put $K := \prod_{\mathfrak{p} \in S} K_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} \mathfrak{o}_{\mathfrak{p}}$ where $K_{\mathfrak{p}}$ is a compact neighborhood of 0 in $k_{\mathfrak{p}}$ for each $\mathfrak{p} \in S$. Then $\mu(K) = \mu_S(K) = \prod_{\mathfrak{p} \in S} \mu_{\mathfrak{p}}(K_{\mathfrak{p}})$. On the other hand, $\mu(aK) = \mu_S\big(\prod_{\mathfrak{p} \in S} a_{\mathfrak{p}} K_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} \mathfrak{o}_{\mathfrak{p}}\big) = \prod_{\mathfrak{p} \in S} \mu_{\mathfrak{p}}(a_{\mathfrak{p}} K_{\mathfrak{p}}) = \prod_{\mathfrak{p} \in S} |a_{\mathfrak{p}}|_{\mathfrak{p}} \mu_{\mathfrak{p}}(K_{\mathfrak{p}}) = |a| \prod_{\mathfrak{p} \in S} \mu_{\mathfrak{p}}(K_{\mathfrak{p}})$. Therefore, we have $\mu(aK) = |a|\mu(K)$ as desired. Q.E.D.

The field k is canonically embedded in its adele group $\mathbb{A}$ by $\xi \mapsto (\xi, \xi, \ldots)$. Notice that the infinite part of the image of ξ under this embedding consists of the conjugates of ξ relative to $\mathbb{Q}$. We denote the set of archimedean primes of k by S_∞ and the ring of integers of k by $\mathfrak{o}$.

Lemma 2.3.7. $k \cap \mathbb{A}_{S_\infty} = \mathfrak{o}$ *and* $k + \mathbb{A}_{S_\infty} = \mathbb{A}$.

Proof. The first equality is a restatement of the fact that an element of k is an integer if and only if it is an integer with respect to each discrete prime of k. To prove the second equality, let $x \in \mathbb{A}$. Since $x_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$, we can find an integer m such that $mx \in \mathbb{A}_{S_\infty}$. Suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are those primes with respect to which x is not an integer. Let $n_1, \ldots, n_r$ be the exponents of these primes in the factorization of the ideal (m). Since $\mathfrak{o}_\mathfrak{p}$ is dense in $\mathfrak{o}$ for every $\mathfrak{p}$, there exists $\eta_i \in \mathfrak{o}$ with $|mx_i - \eta_i|_i \le N_i(\mathfrak{p}_i)^{-n_i}$ for each $i \in \{1, \ldots, r\}$ (we replaced the subscripts $\mathfrak{p}_i$ by i for ease of notation). Applying Chinese remainder theorem to the ideals $\mathfrak{p}_i^{n_i}$ and integers η_i, we find a $\eta \in \mathfrak{o}$ such that $mx_i \equiv \eta \pmod{\mathfrak{p}_i^{n_i}}$ for all i. Then $|mx_i - \eta|_i \le N_i(\mathfrak{p}_i)^{-n_i}$ for all i. We put $\xi := \frac{\eta}{m}$ so that $x - \xi = \frac{1}{m}(mx - \eta)$ belongs to $\mathbb{A}_{S_\infty}$. This proves $k + \mathbb{A}_{S_\infty} = \mathbb{A}$. Q.E.D.

We put $\mathbb{A}^\infty := \prod_{\mathfrak{p} \in S_\infty} k_\mathfrak{p}$. If $k = \mathbb{Q}(\theta)$ and if the minimal polynomial of θ over $\mathbb{Q}$ has r_1 real roots and r_2 pairs of complex roots, then $\mathbb{A}^\infty$ is the product of r_1 real lines and r_2 complex planes. So $\mathbb{A}^\infty$ is a real vector space of dimension $r_1 + r_2 = n$ where $n := [k : \mathbb{Q}]$. For each $x \in \mathbb{A}$, we denote $(x_\mathfrak{p})_{\mathfrak{p} \in S_\infty}$ by x^∞.

Lemma 2.3.8. *If $\{\omega_1, \ldots, \omega_n\}$ is a minimal basis for the ring of integers $\mathfrak{o}$ of k over the rational integers, then $\{\omega_1^\infty, \ldots, \omega_n^\infty\}$ is a basis for the vector space $\mathbb{A}^\infty$ over $\mathbb{R}$. The parallelotope $D^\infty := \left\{ \sum_{i=1}^n c_i \omega_i^\infty \in \mathbb{A}^\infty : 0 \le c_i < 1 \text{ for all } i \right\}$ spanned by this basis has the volume $\sqrt{|d|}$ where d is the discriminant of k.*

Proof. For the vectors $\omega_1^\infty, \ldots, \omega_n^\infty$ to make up a basis of $\mathbb{A}^\infty$ it is necessary and sufficient that they are linearly independent. This is true since their determinant in ordinary absolute value equals $2^{-r_2}\sqrt{|d|} \neq 0$ (cf. the proof of Theorem 95 of Hecke [9]). We have chosen as our measure on $\mathbb{C}$ twice the Lebesgue measure, so the volume of D^∞ is $2^{-r_2}\sqrt{|d|}$ multiplied by 2^{r_2}. Q.E.D.

Definition 2.3.9. *Additive fundamental domain D* is defined as $D := D^\infty \times \prod_{\mathfrak{p} \notin S_\infty} \mathfrak{o}_\mathfrak{p}$.

Theorem 2.3.10. *(1) $\mathbb{A} = \dot{\bigcup}_{\xi \in k} (\xi + D)$.*

(2) $\mu(D) = 1$.

Proof. (1) Let $x \in \mathbb{A}$ be arbitrary. Since $k + \mathbb{A}_{S_\infty} = \mathbb{A}$, there is $\xi \in k$ such that $x - \xi \in \mathbb{A}_{S_\infty}$. This ξ is unique modulo $\mathfrak{o}$. Indeed, if $\xi' \in k$ is such that $x - \xi' \in \mathbb{A}_{S_\infty}$

then $\xi-\xi' \in k$ as well as $\xi-\xi' = (x-\xi')-(x-\xi) \in \mathbb{A}_{S_\infty}$ so that $\xi-\xi' \in k \cap \mathbb{A}_{S_\infty} = \mathfrak{o}$. Now it is enough to find $\eta \in \mathfrak{o}$ with $(x-\xi)+\eta \in D$, since then $x = (\xi-\eta)+\big((x-\xi)+\eta\big) \in (\xi-\eta)+D$, as desired. If we add any element of $\mathfrak{o}$ to $x-\xi$, we are still in $\mathbb{A}_{S_\infty}$, so we need only consider the archimedean coordinates. By definition of D, there is a unique $\eta \in \mathfrak{o}$ such that $(x-\xi)+\eta \in D$. Thus we have shown $\mathbb{A} = \bigcup_{\xi\in k}(\xi+D)$. To prove that the union is disjoint, suppose $x \in \mathbb{A}$ belongs to both $\xi+D$ and $\xi'+D$ where $\xi, \xi' \in k$. Consider $x-\xi'$. We have just seen that there is a unique $\eta \in \mathfrak{o}$ with $(x-\xi')+\eta \in D$. However, we can take η to be 0 as well as $\xi-\xi'$. Therefore, $\xi = \xi'$ and the union is disjoint.

(2) By the theory of algebraic numbers, we know $N(\mathfrak{d}) = |d|$. Hence, we have $\mu(D) = \sqrt{|d|}\prod_{\mathfrak{p}\notin S_\infty} \mu_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}}) = N(\mathfrak{d})^{\frac{1}{2}} \prod_{\mathfrak{p}\notin S_\infty} N_{\mathfrak{p}}(\mathfrak{d}_{\mathfrak{p}})^{-\frac{1}{2}} = 1$ since $N(\mathfrak{d}) = \prod_{\mathfrak{p}\notin S_\infty} N_{\mathfrak{p}}(\mathfrak{d}_{\mathfrak{p}})$ (see Hasse [10], p. 442). Q.E.D.

Corollary 2.3.11. *k is a discrete subgroup of $\mathbb{A}$. The factor group $\mathbb{A}/k$ is compact.*

Proof. One can obtain, by shifting the infinite part of D, a neighborhood of 0 in $\mathbb{A}$ whose intersection with k is the singleton $\{0\}$. This shows that k is discrete in $\mathbb{A}$. The preceding theorem implies $\mathbb{A} = k + \overline{D}$, where $\overline{D}$ denotes the closure of D in $\mathbb{A}$. As a result, the factor group $\mathbb{A}/k$ is compact since $\overline{D}$ is compact. Q.E.D.

Lemma 2.3.12. $\Lambda(\xi) = 0$ *for all* $\xi \in k$.

Proof. For all $\xi \in k$, we have $\Lambda(\xi) = \sum_p \sum_{\mathfrak{p}|p} \lambda_p\big(T_{k_{\mathfrak{p}}/\mathbb{Q}_p}(\xi)\big) = \sum_p \lambda_p\big(\sum_{\mathfrak{p}|p} T_{k_{\mathfrak{p}}/\mathbb{Q}_p}(\xi)\big) = \sum_p \lambda_p\big(T_{k/\mathbb{Q}}(\xi)\big)$ since the trace is the sum of local traces (see Hasse [10], p. 303). It suffices then to show that $\sum_p \lambda_p(r) = 0$ for $r \in \mathbb{Q}$ because $T_{k/\mathbb{Q}}(\xi) \in \mathbb{Q}$. Let $r \in \mathbb{Q}$ be given. Say $\sum_p \lambda_p(r) = c + \mathbb{Z}$. By definition of the λ_p we must have $c \in \mathbb{Q}$. For any rational prime q, the equalities

$$c+\mathbb{Z} = \lambda_q(r) + \lambda_\infty(r) + \sum_{p\notin\{q,\infty\}} \lambda_p(r) = \big(\lambda_q(r) + (-r+\mathbb{Z})\big) + \sum_{p\notin\{q,\infty\}} \lambda_p(r)$$

hold. Since both $\lambda_q(r)+(-r+\mathbb{Z})$ and the $\lambda_p(r)$ belong to $\mathbb{Z}_q/\mathbb{Z}$, we get $c+\mathbb{Z} \in \mathbb{Z}_q/\mathbb{Z}$ so that $c \in \mathbb{Z}_q$. This is true for all q, implying $c \in \mathbb{Q} \cap \bigcap_q \mathbb{Z}_q = \mathbb{Z}$. Therefore, $\sum_p \lambda_p(r) = 0$ so that $\Lambda(\xi) = 0$. Q.E.D.

Theorem 2.3.13. *For all $x \in \mathbb{A}$, we have $x \in k$ if and only if $\Lambda(x\xi) = 0$ for all $\xi \in k$.*

Proof. Under the isomorphism $\widehat{\mathbb{A}} \cong \mathbb{A}$, we may regard $k^\perp$ as a subgroup of $\mathbb{A}$. Then $x \in k^\perp$ if and only if $\Lambda(x\xi) = 0$ for all $\xi \in k$, so we are to show $k^\perp = k$. The preceding lemma gives $k \subseteq k^\perp$. We prove the converse by showing $[k^\perp : k] = 1$. We have $k^\perp \cong \widehat{\mathbb{A}/k}$, so the compactness of $\mathbb{A}/k$ implies that $k^\perp$ is discrete. Consequently, $k^\perp/k$ is discrete. Discrete subgroups of Hausdorff topological groups are closed (Proposition 1-6 of Ramakrishnan and Valenza [6], p. 8), so $k^\perp/k$ is closed in $\mathbb{A}/k$. Hence $k^\perp/k$ is compact. The factor group $k^\perp/k$, being compact and discrete, is finite. This is possible only if $[k^\perp : k] = 1$ since k is infinite. Therefore, we have $k^\perp = k$. Q.E.D.

2.3.2. Multiplicative Theory

Let us denote by $\mathfrak{M}$ the group of all fractional ideals of $\mathfrak{o}$. We make $\mathfrak{M}$ into a topological group by endowing it with the discrete topology. The map $\varphi : \mathbb{I} \to \mathfrak{M}$ defined by $\varphi(a) := \prod_{\mathfrak{p} \notin S_\infty} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(a_\mathfrak{p})}$ is a homomorphism since $\mathrm{ord}_\mathfrak{p}(a_\mathfrak{p} b_\mathfrak{p}) = \mathrm{ord}_\mathfrak{p}(a_\mathfrak{p}) + \mathrm{ord}_\mathfrak{p}(b_\mathfrak{p})$ for all $a, b \in \mathbb{I}$ and $\mathfrak{p} \notin S_\infty$. It is continuous since each of its restrictions $\varphi|_{\mathbb{I}_S}$ is so. We have $\mathrm{Ker}\,\varphi = \mathbb{I}_{S_\infty}$ and $\mathrm{Im}\,\varphi = \mathfrak{M}$.

The map $a \mapsto |a|$ is a continuous surjective homomorphism from $\mathbb{I}$ onto $(0, \infty)$. We shall denote its kernel by J. Thus $J = \{a \in \mathbb{I} : |a| = 1\}$. This is a closed subgroup of $\mathbb{I}$. We canonically embed $k^\times$ into $\mathbb{I}$ by $\alpha \mapsto (\alpha, \alpha, \ldots)$. Artin's product formula says $k^\times \subseteq J$. We fix an arbitrary archimedean prime $\mathfrak{p}_0$ of k, and let

$$T := \{a \in \mathbb{I} : a_{\mathfrak{p}_0} > 0 \text{ and } a_\mathfrak{p} = 1 \text{ for all } \mathfrak{p} \neq \mathfrak{p}_0\}.$$

The mapping $a \mapsto |a|$ is an isomorphism of topological groups T and $(0, \infty)$. We identify these two groups so that $t \in (0,\infty)$ stands for the idele $(t,1,1,\dots)$ or $(\sqrt{t},1,1,\dots)$ according as $k_{\mathfrak{p}_0} = \mathbb{R}$ or $k_{\mathfrak{p}_0} = \mathbb{C}$ if the $\mathfrak{p}_0$-component is written first. We have an isomorphism of topological groups: $\mathbb{I} \cong T \times J$. Indeed, the assignment $(t,b) \mapsto tb$ defines a topological isomorphism from $T \times J$ onto $\mathbb{I}$. The idele group $\mathbb{I}$ admits a Haar measure as a restricted direct product. Let us call this measure ν. By the isomorphism above, ν induces a measure on $T \times J$, which we also denote by ν. Let ν_T denote the restriction to $(0,\infty)$ of the Haar measure on $\mathbb{R}^\times$ defined in the Local Theory. We select such a Haar measure ν_J on J that the equality $\nu = \nu_T \times \nu_J$ holds. Thus, for any ν-integrable function $f : \mathbb{I} \to \mathbb{C}$, we have

$$\int_{\mathbb{I}} f \,\mathrm{d}\nu = \int_T \int_J f(t,b)\nu_J(\mathrm{d}b)\nu_T(\mathrm{d}t) = \int_J \int_T f(t,b)\nu_T(\mathrm{d}t)\nu_J(\mathrm{d}b)$$

by Fubini's theorem.

We put $J_{S_\infty} := J \cap \mathbb{I}_{S_\infty}$. Equivalently, J is the subgroup of all ideles of norm 1 whose discrete coordinates are all of absolute value 1. Let $S'_\infty := S_\infty \setminus \{\mathfrak{p}_0\}$ and $r := |S'_\infty|$. We note that $r = r_1 + r_2 - 1$. The map $l : J_{S_\infty} \to \mathbb{R}^r$ defined by $l(b) := (\log |b_\mathfrak{p}|_\mathfrak{p})_{\mathfrak{p} \in S'_\infty}$ is a continuous homomorphism. Moreover, l is surjective since we can adjust the $\mathfrak{p}_0$-component of $b \in J_{S_\infty}$ freely. Since $k^\times \subseteq J$ by the product formula, we have $k^\times \cap J_{S_\infty} = k^\times \cap \mathbb{I}_{S_\infty}$. So $k^\times \cap J_{S_\infty}$ consists exactly of the units of the ring of integers $\mathfrak{o}$ of k because α is a unit of $\mathfrak{o}$ if and only if α is a unit of $\mathfrak{o}_\mathfrak{p}$ for all discrete $\mathfrak{p}$. In symbols, $k^\times \cap J_{S_\infty} = \mathfrak{o}^\times$.

We put $W := \mathfrak{o}^\times \cap \operatorname{Ker} l$. It turns out that W is the group of all roots of unity in k. Hence W is a finite cyclic group. We denote its order by w. As a consequence of Dirichlet's unit theorem, we see that the factor group $\mathfrak{o}^\times/W$ is a free abelian group on r generators; that is to say, there exist $\overline{\varepsilon_1},\dots,\overline{\varepsilon_r} \in \mathfrak{o}^\times/W$ such that every $\overline{\varepsilon} \in \mathfrak{o}^\times/W$ can be written uniquely in the form $\overline{\varepsilon} = \overline{\varepsilon_1}^{m_1} \cdots \overline{\varepsilon_r}^{m_r}$ with $m_1,\dots,m_r \in \mathbb{Z}$. Then the r-tuples $l(\varepsilon_1),\dots,l(\varepsilon_r)$ make up a basis for the real vector space $\mathbb{R}^r$. For proofs of the facts mentioned in this paragraph, see Hecke [9], p. 108ff.

For each $b \in J_{S_\infty}$, there are unique scalars $c_1,\dots,c_r \in \mathbb{R}$ such that $l(b) =$

$\sum_{i=1}^{r} c_i l(\varepsilon_i)$. Denote by P the parallelotope in $\mathbb{R}^r$ spanned by the vectors $l(\varepsilon_1), \ldots, l(\varepsilon_r)$. Thus $P = \left\{ \sum_{i=1}^{r} c_i l(\varepsilon_i) \in \mathbb{R}^r : 0 \le c_i < 1 \text{ for all } i \right\}$. Let R be the regulator $|\det(\log |\varepsilon_i|_{\mathfrak{p}})|$ of the field k. The volume of $l^{\leftarrow}(P)$ in J is known to be given by the formula below.

Lemma 2.3.14. $\nu_J(l^{\leftarrow}(P)) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} R.$

Let h denote the class number of k. Choose h ideles $b^{(1)}, \ldots, b^{(h)}$ such that the ideals $\varphi\big(b^{(1)}\big), \ldots, \varphi\big(b^{(h)}\big)$ represent distinct ideal classes. We may assume $b^{(1)}, \ldots, b^{(h)} \in J$. Let $E_0 := \left\{ b \in l^{\leftarrow}(P) : 0 \le \arg(b_{\mathfrak{p}_0}) < \frac{2\pi}{w} \right\}$.

Definition 2.3.15. We define the *multiplicative fundamental domain* for $J/k^{\times}$ to be $E := E_0 b^{(1)} \cup \cdots \cup E_0 b^{(h)}$.

The union above is disjoint. In fact, for each $v \in \{1, \ldots, h\}$, the ideles in $E_0 b^{(v)}$ correspond to $\varphi\big(b^{(v)}\big)$ because $E_0 \subseteq \operatorname{Ker} \varphi$.

Theorem 2.3.16. *(1)* $J = \dot{\bigcup}_{\alpha \in k^{\times}} (\alpha E)$.

(2) $\nu_J(E) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|d|}\, w}$.

Proof. (1) Let $b \in J$. For one and only one v, the ideal $\varphi(b)$ belongs to the class represented by $\varphi\big(b^{(v)}\big)$. Then $\varphi(b) = (\alpha)\varphi\big(b^{(v)}\big)$ for some $\alpha \in k^{\times}$ where (α) denotes the principal ideal generated by α. This implies $\varphi\big(\alpha^{-1}\frac{b}{b^{(v)}}\big) = \mathfrak{o}$ because $\varphi(\alpha) = (\alpha)$. Then $\alpha^{-1}\frac{b}{b^{(v)}} \in \operatorname{Ker}\varphi$, which gives $\alpha^{-1}\frac{b}{b^{(v)}} \in J_{S_\infty}$. There exist unique $c_1, \ldots, c_r \in \mathbb{R}$ such that $l\big(\alpha^{-1}\frac{b}{b^{(v)}}\big) = \sum_{i=1}^{r} c_i l(\varepsilon_i)$. Pick integers $m_1, \ldots, m_r$ to satisfy $0 \le c_i + m_i < 1$ for every i. Then $\prod \varepsilon_i^{m_i} \alpha^{-1} \frac{b}{b^{(v)}} \in l^{\leftarrow}(P)$. We can find a root of unity ζ which multiplies this idele to make the argument of the product lie in the interval $[0, \frac{2\pi}{w})$. Thus $\zeta \prod \varepsilon_i^{m_i} \alpha^{-1} \frac{b}{b^{(v)}} \in E_0$ so that $b \in \beta E_0 b^{(v)} \subseteq \beta E$ where $\beta = \zeta^{-1} \prod \varepsilon_i^{-m_i} \alpha$.

(2) We have $\nu_J(E) = \sum_{v=1}^{h} \nu_J(E_0 b^{(v)}) = h\nu_J(E_0) = \frac{h}{w}\nu_J(l^{\leftarrow}(P)) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|d|}\, w}$.

Q.E.D.

Corollary 2.3.17. *$k^{\times}$ is a discrete subgroup of J. The factor group $J/k^{\times}$ is compact.*

Proof. The idele topology on $\mathbb{I}$ is stronger than the subspace topology on $\mathbb{I}$ inherited from $\mathbb{A}$. The first assertion then follows from the discreteness of k in $\mathbb{A}$. The second one is a consequence of the fact that $l^{\leftarrow}(P)$ is contained in a compact set. Q.E.D.

By a quasi-character of $\mathbb{I}$ we shall mean a continuous homomorphism from $\mathbb{I}$ into $\mathbb{C}^{\times}$ which is trivial on $k^{\times}$. So every quasi-character of $\mathbb{I}$ naturally induces a character of $J/k^{\times}$ since $J/k^{\times}$ is compact. Hence, quasi-characters of $\mathbb{I}$ (when restricted to J) are characters of J. If χ is a quasi-character of $\mathbb{I}$ which is trivial on J, then $\chi_1 : (0,\infty) \to \mathbb{C}^{\times}$ defined by $\chi_1(t) := \chi(t)$ is continuous and multiplicative, so there exists a unique $s \in \mathbb{C}$ such that $\chi_1(t) = t^s$ for all $t \in (0,\infty)$. Consequently, $\chi(a) = \chi\big(|a|\big) = |a|^s$ for all $a \in \mathbb{I}$. Therefore, for each quasi-character χ of $\mathbb{I}$ which is trivial on J, there exists a unique $s \in \mathbb{C}$ such that $\chi(a) = |a|^s$ for all $a \in \mathbb{I}$.

Let χ be any quasi-character of $\mathbb{I}$. Then $|\chi|$ is a quasi-character of $\mathbb{I}$ which is trivial on J, so there is a unique $s \in \mathbb{C}$ such that $|\chi(a)| = |a|^s$ for all $a \in \mathbb{I}$. In particular, $|a|^s > 0$ for all $a \in \mathbb{I}$. This would imply, if the imaginary part of s were nonzero, that $\operatorname{Im}|\,|$ lay in a discrete subgroup of $(0,\infty)$, but we know $\operatorname{Im}|\,| = (0,\infty)$, so s must be real. We call this number as the exponent of χ. A quasi-character of $\mathbb{I}$ is a character if and only if its exponent is zero.

As in the Local Theory, it will be convenient to denote by $\mathcal{Q}$ the set of all quasi-characters of $\mathbb{I}$, and by $\mathcal{Q}_S$ the set of all quasi-characters of $\mathbb{I}$ with exponents in $S \subseteq \mathbb{R}$. For each quasi-character χ of $k^{\times}$ we define as before $\widehat{\chi}$ to be the quasi-character $\chi^{-1}|\,|$. If χ has exponent σ, then $\widehat{\chi}$ has exponent $1-\sigma$.

2.3.3. The Zeta Functions and Functional Equation

By "convergence" of a series $\sum_{\xi \in k} z_\xi$ with terms in $\mathbb{C}$, we mean absolute convergence. Let $\mathcal{Z}$ be the collection of all functions $f : \mathbb{A} \to \mathbb{C}$ satisfying the conditions:

(1) f belongs to $\mathfrak{S}(\mathbb{A})$.

(2) For all $a \in \mathbb{I}, x \in \mathbb{A}$, the series $\sum_{\xi\in k} f(a(x+\xi))$ and $\sum_{\xi\in k} \widehat{f}(a(x+\xi))$ are convergent, the convergence being uniform in the pair (a, x) for x ranging over D and a ranging over any compact subset of $\mathbb{I}$.

(3) $f|_{\mathbb{I}} \cdot |\,|^{\sigma}$ and $\widehat{f}|_{\mathbb{I}} \cdot |\,|^{\sigma}$ belong to $L^1(\mathbb{I})$ for all $\sigma > 1$.

Theorem 2.3.18. *(Poisson summation formula) Let $f : \mathbb{A} \to \mathbb{C}$ be a function satisfying the conditions:*

(1) f is continuous and integrable.

(2) For all $x \in \mathbb{A}$, the series $\sum_{\xi\in k} f(x+\xi)$ is convergent, and the convergence is uniform on D.

(3) The series $\sum_{\xi\in k} \widehat{f}(\xi)$ is convergent.

Then we have

$$\sum_{\xi\in k} \widehat{f}(\xi) = \sum_{\xi\in k} f(\xi).$$

The essential step in the proof of the main theorem of this subsection consists in using the number-theoretic analogue of the Riemann-Roch theorem, which is a corollary of the Poisson summation formula. We state this result below. Hypotheses to be listed are apparently fulfilled for the functions in $\mathcal{Z}$.

Theorem 2.3.19. *Let $f : \mathbb{A} \to \mathbb{C}$ be a function satisfying the conditions:*

(1) f is continuous and integrable.

(2) For all $a \in \mathbb{I}, x \in \mathbb{A}$, the series $\sum_{\xi\in k} f(a(x+\xi))$ is convergent, the convergence being uniform for x ranging over D.

(3) For all $a \in \mathbb{I}$, the series $\sum_{\xi\in k} \widehat{f}(a\xi)$ is convergent.

Then, for all $a \in \mathbb{I}$, we have

$$\frac{1}{|a|}\sum_{\xi\in k} \widehat{f}\left(\frac{\xi}{a}\right) = \sum_{\xi\in k} f(a\xi).$$

Definition 2.3.20. Corresponding to each $f \in \mathcal{Z}$, the map $\zeta_f : \mathcal{Q}_{(1,\infty)} \to \mathbb{C}$ defined by $\zeta_f(\chi) := \int_{\mathbb{I}} f(a)\chi(a)\nu(\mathrm{d}a)$ is called a zeta function of k.

The condition $f|_{\mathbb{I}} \cdot |\,|^{\sigma} \in L^1(\mathbb{I})$ guarantees that ζ_f has a meaningful definition. We have $\widehat{f} \in \mathcal{Z}$ whenever $f \in \mathcal{Z}$ since the identity $f(x) = \widehat{\widehat{f}}(-x)$ holds for all $x \in \mathbb{A}$. In particular, $\zeta_{\widehat{f}}$ is also a zeta function of k for $f \in \mathcal{Z}$.

We define an equivalence relation on $\mathcal{Q}$ by declaring two quasi-characters to be equivalent if they agree on J. As in Local Theory, let us denote by C the equivalence class of χ so that $C = \{\chi|\,|^s : s \in \mathbb{C}\}$. Each $s \in \mathbb{C}$ determines a distinct quasi-character $\chi|\,|^s$ in C. Therefore, we may view C as a complex plane. Thus we may see a function of quasi-characters as a function defined on a collection of complex planes. Such a function will be said to be holomorphic at a point of its domain if its restriction to the corresponding plane is holomorphic at that point. Being holomorphic on a subset is defined accordingly.

One can show, mimicking the corresponding proof given in the Local Theory, that zeta functions of k are holomorphic on their domain of definition.

We shall need two lemmas for the proof of the main theorem. We abbreviate the volume $\frac{2^{r_1}(2\pi)^{r_2}hR}{\sqrt{|d|}w}$ of the multiplicative fundamental domain by κ. Remember that if χ is a quasi-character which is trivial on J, then there is an $s \in \mathbb{C}$ such that $\chi = |\,|^s$.

Lemma 2.3.21. *For all $\chi \in \mathcal{Q}, t \in (0,\infty)$, we have*

$$\int_E \chi(tb)\nu_J(\mathrm{d}b) = \begin{cases} \kappa t^s, & \text{if } \chi = |\,|^s \text{ for some } s \in \mathbb{C}; \\ 0, & \text{if } \chi \text{ is nontrivial on } J. \end{cases}$$

Proof. Let $\chi \in \mathcal{Q}, t \in (0,\infty)$. First of all, we note that the integral $\int_E \chi(tb)\nu_J(\mathrm{d}b)$ exists since the integrand is continuous and E has compact closure. Since χ is multiplicative, we have

$$\int_E \chi(tb)\nu_J(\mathrm{d}b) = \chi(t)\int_E \chi(b)\nu_J(\mathrm{d}b).$$

If $\chi = |\,|^s$ for some $s \in \mathbb{C}$, then χ is trivial on J, hence on E, so that $\int_E \chi(tb)\nu_J(\mathrm{d}b) = |t|^s\nu_J(E) = \kappa t^s$. If χ is nontrivial on J, then the quasi-character on $J/k^\times$ induced by χ is also nontrivial, implying that its integral over $J/k^\times$ is 0. This means $\int_E \chi(b)\nu_J(\mathrm{d}b) = 0$. Therefore, $\int_E \chi(tb)\nu_J(\mathrm{d}b) = 0$. Q.E.D.

Fix a function $f \in \mathcal{Z}$ arbitrarily. Pick one $\sigma_0 > 1$. Since $f \in \mathcal{Z}$, we have $|f_{\mathbb{I}}| \cdot |\,|^{\sigma_0} \in L^1(\mathbb{I})$ (here $|\,|$ stands for the ordinary absolute value of complex numbers). Consequently, by Fubini's theorem, the integral $\int_J |f(tb)||tb|^{\sigma_0}\nu_J(\mathrm{d}b)$ exists for almost all $t \in (0,\infty)$; that is, $\int_J |f(tb)|\nu_J(\mathrm{d}b)$ exists for almost all $t \in (0,\infty)$. Now let $\chi \in \mathcal{Q}$ be arbitrary with exponent σ. For all $t \in (0,\infty), b \in J$, we have $|f(tb)\chi(tb)| = |f(tb)||tb|^\sigma = |f(tb)|t^\sigma$. Therefore, the integral

$$\zeta_f(\chi, t) := \int_J f(tb)\chi(tb)\nu_J(\mathrm{d}b)$$

exists for almost all $t \in (0,\infty)$.

Lemma 2.3.22. *For all $\chi \in \mathcal{Q}$, for almost all $t \in (0,\infty)$, we have*

$$\zeta_f(\chi, t) + f(0)\int_E \chi(tb)\nu_J(\mathrm{d}b) = \zeta_{\widehat{f}}\Big(\widehat{\chi}, \frac{1}{t}\Big) + \widehat{f}(0)\int_E \widehat{\chi}\Big(\frac{1}{t}b\Big)\nu_J(\mathrm{d}b).$$

Proof. Let $\chi \in \mathcal{Q}$ be arbitrary. The above equality makes sense for almost all $t \in (0,\infty)$. Take such a $t \in (0,\infty)$. From $J = \dot{\bigcup}_{\alpha\in k^\times}(\alpha E)$, translation-invariance of ν_J, and $\chi|_k \equiv 1$, we get

$$\begin{aligned}
\zeta_f(\chi,t) &= \int_J f(tb)\chi(tb)\nu_J(\mathrm{d}b)\\
&= \sum_{\alpha\in k^\times}\int_{\alpha E} f(tb)\chi(tb)\nu_J(\mathrm{d}b)\\
&= \sum_{\alpha\in k^\times}\int_{E} f(\alpha tb)\chi(\alpha tb)\nu_J(\mathrm{d}b)\\
&= \sum_{\alpha\in k^\times}\int_{E} f(\alpha tb)\chi(tb)\nu_J(\mathrm{d}b).
\end{aligned}$$

Since the sum $\sum_{\alpha\in k^\times} f(\alpha tb)$ is, by hypothesis, uniformly convergent for b in the relatively compact set E, we have

$$\sum_{\alpha\in k^\times}\int_{E} f(\alpha tb)\chi(tb)\nu_J(\mathrm{d}b) = \int_E\left(\sum_{\alpha\in k^\times} f(\alpha tb)\right)\chi(tb)\nu_J(\mathrm{d}b).$$

Hence

$$\zeta_f(\chi,t) + f(0)\int_E \chi(tb)\nu_J(\mathrm{d}b) = \int_E\left(\sum_{\xi\in k} f(\xi tb)\right)\chi(tb)\nu_J(\mathrm{d}b).$$

By the Riemann-Roch theorem, we have $\sum_{\xi\in k} f(\xi tb) = \frac{1}{|tb|}\sum_{\xi\in k}\widehat{f}\left(\frac{\xi}{tb}\right)$, so

$$\begin{aligned}
\zeta_f(\chi,t) + f(0)\int_E \chi(tb)\nu_J(\mathrm{d}b) &= \int_E\left(\frac{1}{|tb|}\sum_{\xi\in k}\widehat{f}\left(\frac{\xi}{tb}\right)\right)\chi(tb)\nu_J(\mathrm{d}b)\\
&= \int_E\left(\sum_{\xi\in k}\widehat{f}\left(\xi\frac{1}{t}b^{-1}\right)\right)\frac{\chi(tb)}{t}\nu_J(\mathrm{d}b)\\
&= \int_E\left(\sum_{\xi\in k}\widehat{f}\left(\xi\frac{1}{t}b\right)\right)\frac{\chi(tb^{-1})}{t}\nu_J(\mathrm{d}b)\\
&= \int_E\left(\sum_{\xi\in k}\widehat{f}\left(\xi\frac{1}{t}b\right)\right)\widehat{\chi}\left(\frac{1}{t}b\right)\nu_J(\mathrm{d}b).
\end{aligned}$$

Once we make the same computations with the right-hand of the equality lemma asserts, we obtain the last expression above. Q.E.D.

By Fubini's theorem and the definition of ν_T, we have

$$\zeta_f(\chi) = \int_0^\infty \zeta_f(\chi, t)\frac{\mathrm{d}t}{t} = \int_0^1 \zeta_f(\chi, t)\frac{\mathrm{d}t}{t} + \int_1^\infty \zeta_f(\chi, t)\frac{\mathrm{d}t}{t}$$

for all $\chi \in \mathcal{Q}_{(1,\infty)}$. The second integral $\int_1^\infty \zeta_f(\chi, t)\frac{\mathrm{d}t}{t}$ converges in fact for all $\chi \in \mathcal{Q}$. Indeed, we have

$$\int_1^\infty \zeta_f(\chi, t)\frac{\mathrm{d}t}{t} = \int_{\{a\in\mathbb{I}:|a|\geq 1\}} f(a)\chi(a)\nu(\mathrm{d}a)$$

since $[1,\infty)\times J$ corresponds to $\{a \in \mathbb{I} : |a| \geq 1\}$ under the isomorphism $\mathbb{I} \cong T \times J$. The integrability of $a \mapsto f(a)\chi(a)$ on $\{a \in \mathbb{I} : |a| \geq 1\}$ for a quasi-character χ of exponent σ_0 implies the integrability of the same map on the same set for every χ of exponent less than or equal to σ_0. Since the integrability is known for all $\sigma_0 > 1$, it follows that $\int_1^\infty \zeta_f(\chi, t)\frac{\mathrm{d}t}{t}$ converges for each $\chi \in \mathcal{Q}$.

Now we consider the first integral $\int_0^1 \zeta_f(\chi, t)\frac{\mathrm{d}t}{t}$. By Lemma 2.3.22, this integral equals

$$\int_0^1 \left(\zeta_{\widehat{f}}\left(\widehat{\chi}, \frac{1}{t}\right) + \widehat{f}(0)\int_E \widehat{\chi}\left(\frac{1}{t}b\right)\nu_J(\mathrm{d}b) - f(0)\int_E \chi(tb)\nu_J(\mathrm{d}b)\right)\frac{\mathrm{d}t}{t}.$$

We note that, in the collection of planes representing the domain $\mathcal{Q}$ of all quasi-characters of $\mathbb{I}$, one and only one plane consists of those quasi-characters which are trivial on J. For this plane, we identify the quasi-character $|\,|^s$ with the point s. In order to express the above expression succinctly, we introduce this notation: for a mathematical proposition P, we define $[P]$ to be 1 or 0 according as P is true or false. In particular, $\left[\chi = |\,|^s\right]$ is 1 if $\chi = |\,|^s$ for some $s \in \mathbb{C}$, and 0 otherwise (in which case χ is understood to be a quasi-character which is nontrivial on J). Then the above expression can be rewritten as

$$\int_0^1 \left(\zeta_{\widehat{f}}\left(\widehat{\chi}, \frac{1}{t}\right) + \left[\chi = |\,|^s\right]\widehat{f}(0)\kappa\left(\frac{1}{t}\right)^{1-s} - \left[\chi = |\,|^s\right]f(0)\kappa t^s\right)\frac{\mathrm{d}t}{t}$$

by using Lemma 2.3.21. Writing this as a sum of three integrals and computing the last two of them yields

$$\int_0^1 \zeta_{\widehat{f}}\left(\widehat{\chi}, \frac{1}{t}\right)\frac{\mathrm{d}t}{t} + [\chi = |\,|^s]\kappa\left(\frac{\widehat{f}(0)}{s-1} - \frac{f(0)}{s}\right).$$

Finally, as we make the substitution $t \mapsto \frac{1}{t}$ in the integral above, we get

$$\int_1^\infty \zeta_{\widehat{f}}(\widehat{\chi}, t)\frac{\mathrm{d}t}{t} + [\chi = |\,|^s]\kappa\left(\frac{\widehat{f}(0)}{s-1} - \frac{f(0)}{s}\right).$$

Thus we have proved that the equality

$$\zeta_f(\chi) = \int_1^\infty \zeta_f(\chi, t)\frac{\mathrm{d}t}{t} + \int_1^\infty \zeta_{\widehat{f}}(\widehat{\chi}, t)\frac{\mathrm{d}t}{t} + [\chi = |\,|^s]\kappa\left(\frac{\widehat{f}(0)}{s-1} - \frac{f(0)}{s}\right)$$

holds for every $\chi \in \mathcal{Q}_{(1,\infty)}$. The two integrals above exist for all $\chi \in \mathcal{Q}$. In fact, they are holomorphic functions of χ. By this equation, we enlarge the domain of ζ_f to $\mathcal{Q}$ except the two quasi-characters $|\,|^0$ and $|\,|^1$ (belonging to the plane of quasi-characters which are trivial on J). Thus ζ_f is a meromorphic function on $\mathcal{Q}$ having simple poles at $|\,|^0$ and $|\,|^1$ with residues $-\kappa f(0)$ and $\kappa\widehat{f}(0)$, respectively. Moreover, the form of the expression above is unchanged when we replace f and χ by $\widehat{f}$ and $\widehat{\chi}$. Thus we have proved the main theorem of this subsection.

Theorem 2.3.23. *Each zeta function ζ_f of k is meromorphically continued to the domain $\mathcal{Q}$ of all quasi-characters except at $|\,|^0$ and $|\,|^1$ where it has simple poles with residues $-\kappa f(0)$ and $\kappa\widehat{f}(0)$, respectively. Moreover, ζ_f satisfies the functional equation $\zeta_f(\chi) = \zeta_{\widehat{f}}(\widehat{\chi})$.*

3. SOME COMPUTATIONS

3.1. Prime Divisors of Quadratic Fields

Ostrowski's theorem determines the prime divisors of $\mathbb{Q}$. In order to do this for a quadratic field $\mathbb{Q}(\sqrt{m})$, where m is a nonzero square-free rational integer other than 1, we need only find out the prolongations of the prime divisors of $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{m})$.

The degree of the extension $\mathbb{Q}(\sqrt{m})/\mathbb{Q}$ is 2. Hence, for each p, the prime divisor $|\,|_p$ of $\mathbb{Q}$ has two prolongations or one prolongation to $\mathbb{Q}(\sqrt{m})$ according as the minimal polynomial of $\sqrt{m}$ over $\mathbb{Q}$ splits in $\mathbb{Q}_p$ or not; equivalently, $\sqrt{m} \in \mathbb{Q}_p$ or not. In the former case, the two prolongations are defined by $a + b\sqrt{m} \mapsto \left|a + b\sqrt{m}\right|_p$ and $a + b\sqrt{m} \mapsto \left|a - b\sqrt{m}\right|_p$. In the latter case, the unique prolongation is defined by $a + b\sqrt{m} \mapsto \sqrt{|a^2 - mb^2|_p}$ (see Proposition 4-31 of Ramakrishnan and Valenza [6], p. 161).

If $p = \infty$, then $\mathbb{Q}_p = \mathbb{R}$ so that $\sqrt{m} \in \mathbb{Q}_p$ if and only if $m > 0$. Assume $p \neq \infty$ now. We have

$$
\begin{aligned}
\sqrt{m} \in \mathbb{Q}_p \quad &\Longleftrightarrow \quad x^2 = m \text{ for some } x \in \mathbb{Q}_p \\
&\Longleftrightarrow \quad x^2 = m \text{ for some } x \in \mathbb{Z}_p \\
&\Longleftrightarrow \quad x^2 \equiv m \pmod{p^n} \text{ is solvable for all } n \in \mathbb{N}.
\end{aligned}
$$

If $p \mid m$, then $m = pm_1$ for some $m_1 \in \mathbb{Z} \setminus \{0\}$ with $p \nmid m_1$, so the congruence $x^2 \equiv m \pmod{p^2}$ is not solvable because we would otherwise have $x^2 = pm_1 + p^2 l = p(m_1 + pl)$ for some $l \in \mathbb{Z}$, which is impossible since the number of p's in the prime factorizations of x^2 and $p(m_1 + pl)$ cannot be equal. Thus we have $\sqrt{m} \notin \mathbb{Q}_p$ for those m which are divisible by p. If $p \nmid m$, then Hensel's lemma (Theorem 2.23 of Niven et al [11], p. 87) is applicable to simplify the last condition stated above.

Theorem 3.1.1. *(Hensel's lemma) Let $f \in \mathbb{Z}[x], a \in \mathbb{Z}, j \in \mathbb{N}$. If $f(a) \equiv 0 \pmod{p^j}$ and $f'(a) \not\equiv 0 \pmod{p}$, then there exists $t \in \mathbb{Z}$, unique modulo p, such that $f(a+tp^j) \equiv 0 \pmod{p^{j+1}}$.*

We shall put $f(x) := x^2 - m$. Let us consider the case $p \neq 2$, first. If $f(a) \equiv 0 \pmod{p}$ for some $a \in \mathbb{Z}$, then $f'(a) = 2a \not\equiv 0 \pmod{p}$; otherwise, we would have $p \mid a$, implying $p \mid m$, contrary to our assumption. By Hensel's lemma, $f(a+tp) \equiv 0 \pmod{p^2}$ for some $t \in \mathbb{Z}$. Since $f'(a+tp) = 2(a+tp) \equiv 2a \not\equiv 0 \pmod{p}$, there exists, by Hensel's lemma again, $u \in \mathbb{Z}$ such that $f(a+tp+up^2) \equiv 0 \pmod{p^3}$. Proceeding in this manner, it can be shown that $f(x) \equiv 0 \pmod{p^n}$ is solvable for all $n \in \mathbb{N}$. Hence, $f(x) \equiv 0 \pmod{p^n}$ is solvable for all $n \in \mathbb{N}$ if and only if the congruence $f(x) \equiv 0 \pmod{p}$ is solvable. Equivalently, $\sqrt{m} \in \mathbb{Q}_p$ if and only if m is a quadratic residue modulo p.

Second, we consider the case $p = 2$, whence m is odd. From Theorem 47 of Hecke [9], we learn that $x^2 \equiv m \pmod{2^n}$ is solvable for $n \geq 3$ if and only if $x^2 \equiv m \pmod{8}$ is solvable, which is to say that $m \equiv 1 \pmod{8}$; $x^2 \equiv m \pmod{4}$ is solvable if and only if $m \equiv 1 \pmod{4}$; and $x^2 \equiv m \pmod{2}$ is always solvable. Hence, $\sqrt{m} \in \mathbb{Q}_2$ if and only if $m \equiv 1 \pmod{8}$. Thus we have proved the theorem below.

Theorem 3.1.2. *Let m be a nonzero square-free rational integer other than 1. The prime divisor $|\,|_\infty$ has two prolongations to $\mathbb{Q}(\sqrt{m})$ if and only if $m > 0$. If $p \mid m$, then $|\,|_p$ has a unique prolongation. If $p \nmid m$ and $p \neq 2$, then $|\,|_p$ has two prolongations if and only if m is a quadratic residue modulo p; if $2 \nmid m$, then $|\,|_2$ has two prolongations if and only if $m \equiv 1 \pmod{8}$.*

For example, the prime divisors $|\,|_\infty$ and $|\,|_2$ have unique prolongations to $\mathbb{Q}(\sqrt{2})$, and for $p \neq 2$, the prime divisor $|\,|_p$ has two prolongations to $\mathbb{Q}(\sqrt{2})$ if and only if $\left(\frac{2}{p}\right) = 1$. Since $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$ for an odd rational prime p, we have $\left(\frac{2}{p}\right) = 1$ if and only if $p \equiv \pm 1 \pmod{8}$. Consequently, $|\,|_p$ has two prolongations to $\mathbb{Q}(\sqrt{2})$ if and only if $p \equiv \pm 1 \pmod{8}$.

3.2. Quasi-Characters of $\mathbb{Q}_p^\times$

In Local Theory, we have seen that for a local field k, the quasi-characters of $k^\times$ are completely determined once the characters of u (the subgroup of $k^\times$ consisting of elements of absolute value 1) are known. We described then the characters of u explicitly for $k = \mathbb{R}$ and $k = \mathbb{C}$, but we were content with a classification based on conductors in case k is a finite extension of some p-adic field $\mathbb{Q}_p$. In this section, we investigate more closely the characters of u when $k = \mathbb{Q}_p$.

Fix a rational prime p. Then

$$u = \left\{a_0 + a_1p + a_2p^2 + \ldots : 0 \leq a_j < p \text{ for all } j \geq 0, a_0 \neq 0\right\}$$

and

$$1 + \mathfrak{p}^n = \left\{1 + a_np^n + a_{n+1}p^{n+1} + \ldots : 0 \leq a_j < p \text{ for all } j \geq n\right\}$$

for all $n \in \mathbb{N}$, where $\mathfrak{p} = \mathbb{Z}_p \setminus \mathbb{Z}_p^\times$ as usual. We know that we need only determine the characters of each factor group $u/(1 + \mathfrak{p}^n)$. These groups are discrete since $1 + \mathfrak{p}^n$ is open in u for every n, so in fact the question is that of finding the homomorphisms from $u/(1 + \mathfrak{p}^n)$ into S^1.

Let us start with the simplest cases and then generalize. We have

$$\frac{u}{1 + \mathfrak{p}} = \{\overline{a_0} : 0 < a_0 < p\}$$

where $\overline{a_0}$ denotes the coset of a_0. Indeed, if $\overline{a_0 + a_1p + a_2p^2 + \ldots} = \overline{b_0 + b_1p + b_2p^2 + \ldots}$ then $(a_0 + a_1p + a_2p^2 + \ldots)(b_0 + b_1p + b_2p^2 + \ldots)^{-1} \in 1 + \mathfrak{p}$. Say $c_0 + c_1p + c_2p^2 + \ldots = (b_0 + b_1p + b_2p^2 + \ldots)^{-1}$. Then $a_0c_0 \equiv 1$ and $c_0b_0 \equiv 1 \pmod p$, implying $a_0 \equiv b_0 \pmod p$, whence $a_0 = b_0$. Conversely, the equality $\overline{a_0 + a_1p + a_2p^2 + \ldots} = \overline{b_0 + b_1p + b_2p^2 + \ldots}$

is implied by $a_0 = b_0$. Hence, we have $\frac{u}{1+\mathfrak{p}} = \{\overline{a_0} : 0 < a_0 < p\}$. The map

$$\begin{aligned} \frac{u}{1+\mathfrak{p}} &\to \mathbb{Z}_p^\times \\ \overline{a_0} &\mapsto [a_0] \end{aligned}$$

is an isomorphism of groups, where $[a_0]$ denotes the equivalence class of a_0 in $\mathbb{Z}_p^\times$.

Similarly, we have

$$\frac{u}{1+\mathfrak{p}^2} = \left\{\overline{a_0 + a_1 p} : 0 < a_0 < p, 0 \leq a_1 < p\right\}.$$

Indeed, if $\overline{a_0 + a_1p + a_2p^2 + \ldots} = \overline{b_0 + b_1p + b_2p^2 + \ldots}$ and $c_0 + c_1p + c_2p^2 + \ldots = (b_0 + b_1p + b_2p^2 + \ldots)^{-1}$ as above, then $a_0c_0 \equiv 1$, $a_0c_1 + a_1c_0 + \frac{a_0c_0-1}{p} \equiv 0$, $c_0b_0 \equiv 1$, $c_0b_1 + c_1b_0 + \frac{c_0b_0-1}{p} \equiv 0 \pmod p$, implying $a_0 \equiv b_0$ and as a consequence $a_1 \equiv b_1 \pmod p$, whence $a_0 = b_0$ and $a_1 = b_1$. Conversely, the equality $\overline{a_0 + a_1p + a_2p^2 + \ldots} = \overline{b_0 + b_1p + b_2p^2 + \ldots}$ is implied by $a_0 = b_0$ and $a_1 = b_1$. Hence, we have $\frac{u}{1+\mathfrak{p}^2} = \{\overline{a_0 + a_1p} : 0 < a_0 < p, 0 \leq a_1 < p\}$. The map

$$\begin{aligned} \frac{u}{1+\mathfrak{p}^2} &\to \mathbb{Z}_{p^2}^\times \\ \overline{a_0 + a_1p} &\mapsto [a_0 + a_1p] \end{aligned}$$

is an isomorphism of groups. In general, for each $n \in \mathbb{N}$, we have

$$\frac{u}{1+\mathfrak{p}^n} = \left\{\overline{a_0 + a_1p + \ldots + a_{n-1}p^{n-1}} : 0 \leq a_j < p \text{ for all } j, a_0 \neq 0\right\}$$

and the map

$$\begin{aligned} \frac{u}{1+\mathfrak{p}^n} &\to \mathbb{Z}_{p^n}^\times \\ \overline{a_0 + a_1p + \ldots + a_{n-1}p^{n-1}} &\mapsto [a_0 + a_1p + \ldots + a_{n-1}p^{n-1}] \end{aligned}$$

is an isomorphism of groups. Therefore, we are to find the homomorphisms from $\mathbb{Z}_{p^n}^\times$ into S^1. These homomorphisms form a group under pointwise multiplication. We denote this group by $\mathrm{Hom}(\mathbb{Z}_{p^n}^\times, S^1)$. From the theory of finite abelian groups (see §10 in Hecke [9]), we know $\mathrm{Hom}(\mathbb{Z}_{p^n}^\times, S^1)$ is isomorphic to $\mathbb{Z}_{p^n}^\times$ itself.

Let C_m be the cyclic group of order m. The group $\mathbb{Z}_2^\times$ is trivial, we have $\mathbb{Z}_4^\times \cong C_2$, and $\mathbb{Z}_{2^n}^\times \cong C_2 \times C_{2^{n-2}}$ for $n \geq 3$ (Theorem 45 of Hecke [9]). If p is odd, then $\mathbb{Z}_{p^n}^\times \cong C_{(p-1)p^{n-1}}$ (Theorem 44 of Hecke [9]). Thus $\mathrm{Hom}(\mathbb{Z}_2^\times, S^1)$ is trivial, $\mathrm{Hom}(\mathbb{Z}_4^\times, S^1)$ is generated by the homomorphism defined by $g \mapsto -1$ where $\mathbb{Z}_4^\times = \langle g \rangle$, and $\mathrm{Hom}(\mathbb{Z}_{2^n}^\times, S^1)$ is generated by the two homomorphisms defined by $(g, h) \mapsto -1$ and $(g, h) \mapsto \exp\left(\frac{2\pi i}{2^{n-2}}\right)$ where $\mathbb{Z}_{2^n}^\times = \langle g \rangle \times \langle h \rangle$. If p is odd, then $\mathrm{Hom}(\mathbb{Z}_{p^n}^\times, S^1)$ is generated by the homomorphism defined by $g \mapsto \exp\left(\frac{2\pi i}{(p-1)p^{n-1}}\right)$ where $\mathbb{Z}_{p^n}^\times = \langle g \rangle$.

4. CONCLUSIONS

Much information on rational primes is encoded in the Riemann zeta function ζ which is defined by the absolutely convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$$

for complex numbers s such that $\operatorname{Re} s > 1$. This function admits a meromorphic continuation to whole complex plane, except for a simple pole at $s = 1$, and satisfies the functional equation

$$\xi(s) = \xi(1-s)$$

where ξ is the entire function defined by

$$\xi(s) := s(s-1)\zeta(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}.$$

One establishes this meromorphic continuation and the functional equation by making use of the transformation equation

$$\theta(x) = x^{-\frac{1}{2}}\theta(x^{-1})$$

for the theta function θ which is defined by

$$\theta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$$

for $x > 0$. The zeta function could be greatly generalized with the introduction of the Dirichlet series, important examples of which are furthermore generalized simultaneously as the L-function $L(s, \chi)$ associated with a continuous character χ of the idele class group of any number field k. A substantial achievement of Erich Hecke was to establish the meromorphic continuation and the functional equation of $L(s, \chi)$ for any idele class character χ by an enormously complicated application of generalized theta functions and the higher analogues of the equation $\theta(x) = x^{-\frac{1}{2}}\theta(x^{-1})$, which are now understood to be consequences of the Poisson summation formula. One thing that Hecke's method could not describe satisfactorily was the nature of the global constant appearing in the functional equation of $L(s, \chi)$. Then, circa 1950, following a suggestion of his erstwhile thesis advisor Emil Artin, John Tate made use of Fourier analysis on adele groups to reprove both the meromorphic continuation and the functional equation of $L(s, \chi)$, giving in the process a satisfactory description of this global constant.

After a brief chapter on Riemann zeta function, we have discussed in this M.S. thesis the local theory, restricted direct products, and most of the global theory in detail, following Tate's thesis.

We have computed prime divisors of quadratic fields: we proved that if m is a nonzero square-free rational integer other than 1, then the prime divisor $|\,|_\infty$ has two prolongations to $\mathbb{Q}(\sqrt{m})$ if and only if $m > 0$. If $p \mid m$, then $|\,|_p$ has a unique prolongation. If $p \nmid m$ and $p \neq 2$, then $|\,|_p$ has two prolongations if and only if m is a quadratic residue modulo p; if $2 \nmid m$, then $|\,|_2$ has two prolongations if and only if $m \equiv 1 \pmod 8$.

Finally, we gave an explicit description of the quasi-characters of p-adic fields: $\mathrm{Hom}(\mathbb{Z}_2^\times, S^1)$ is trivial, $\mathrm{Hom}(\mathbb{Z}_4^\times, S^1)$ is generated by the homomorphism defined by $g \mapsto -1$ where $\mathbb{Z}_4^\times = \langle g \rangle$, and $\mathrm{Hom}(\mathbb{Z}_{2^n}^\times, S^1)$ is generated by the two homomorphisms defined by $(g,h) \mapsto -1$ and $(g,h) \mapsto \exp\left(\frac{2\pi i}{2^{n-2}}\right)$ where $\mathbb{Z}_{2^n}^\times = \langle g \rangle \times \langle h \rangle$. If p is odd, then $\mathrm{Hom}(\mathbb{Z}_{p^n}^\times, S^1)$ is generated by the homomorphism defined by $g \mapsto \exp\left(\frac{2\pi i}{(p-1)p^{n-1}}\right)$ where $\mathbb{Z}_{p^n}^\times = \langle g \rangle$.

APPENDIX A: ERRORS IN THE LITERATURE

We shall point out in this appendix three errors from the book *Fourier Analysis on Number Fields* [6].

A.1. First Error

If there is a complex Banach space V such that $\mathrm{Aut}(V) \neq \mathrm{Aut}_{\mathrm{top}}(V)$, then Corollary 2-2 is false: take $T \in \mathrm{Aut}(V) \setminus \mathrm{Aut}_{\mathrm{top}}(V)$ and put $G := \mathbb{Z}$. The map

$$\begin{aligned} \rho : G &\to \mathrm{Aut}(V) \\ m &\mapsto T^m \end{aligned}$$

is an abstract representation of G, and for each $x \in V$, the map $m \mapsto T^m(x)$ is continuous from G to V since its domain has the discrete topology. However, ρ cannot be a topological representation; otherwise, the image of G under ρ lies in $\mathrm{Aut}_{\mathrm{top}}(V)$, contrary to $T \notin \mathrm{Aut}_{\mathrm{top}}(V)$.

The additional assumption $\mathrm{Im}\,\rho \subseteq \mathrm{Aut}_{\mathrm{top}}(V)$ corrects Corollary 2-2. This assumption is fulfilled for the reference made to Corollary 2-2 in the proof of Proposition 3-3.

A.2. Second Error

In the proof of Proposition 2-16, the containment $\gamma(T) \in \mathrm{sp}(T)$ lacks justification; nevertheless, we have $\gamma(T) \in \mathrm{sp}_A(T)$, where $\mathrm{sp}_A(T)$ denotes the spectrum of T in A. On the other hand, the inclusion $\mathrm{sp}_A(T) \subseteq \mathbb{R}$ does not follow from Proposition 2-15. The generalization of Proposition 2-15 below proves $\mathrm{sp}_A(T) \subseteq \mathbb{R}$.

Proposition A.2.1. *Let A be a self-adjoint, closed, unital subalgebra of* $\mathrm{End}(H)$. *If $T \in A$ is unitary, then* $\mathrm{sp}_A(T) \subseteq S^1$; *if $T \in A$ is self-adjoint, then* $\mathrm{sp}_A(T) \subseteq \mathbb{R}$.

Proof. For all $T \in A$, $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} T \in A^{\times} &\iff T^{*} \in A^{\times} \\ \lambda \in \mathrm{sp}_{A}(T) &\iff \overline{\lambda} \in \mathrm{sp}_{A}(T^{*}) \end{aligned}$$

and for all $T \in A^{\times}$, $\lambda \in \mathbb{C}^{\times}$, we have

$$\lambda \in \mathrm{sp}_{A}(T) \iff \lambda^{-1} \in \mathrm{sp}_{A}(T^{-1}).$$

In view of these biconditionals, the conclusions are derived as in the proof of Proposition 2-15; just note that the spectral radii of $T \in A$ in $\mathrm{End}(H)$ and in A are the same as a consequence of Theorem 2-6. Q.E.D.

A.3. Third Error

Proposition 4-10, which characterizes multiplicative functions F satisfying the inequality $F(m+n) \leq A \cdot \sup\{F(m), F(n)\}$ for some constant A, has a flawed proof. The auxiliary function f must have codomain $\mathbb{R}$, whence the inequalities thereafter lack justification. Nevertheless, the proposition is proved under the stronger hypothesis that F satisfies the triangle inequality: the argument given in the proof of Ostrowski's theorem in Cassels and Fröhlich [5] works with almost no change. Consequently, we obtain the below corollary in view of the fact that every absolute value is equivalent to one that satisfies the triangle inequality.

Corollary A.3.1. *If k is a nondiscrete locally compact field, then either*
(1) $\mathrm{mod}_k(n) \leq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$*, or*
(2) there is $\lambda > 0$ *such that* $\mathrm{mod}_k(n) = n^{\lambda}$ *for all* $n \in \mathbb{N} \cup \{0\}$.

The authors essentially utilize the corollary above, not Proposition 4-10, in the preliminary analysis for the classification theorem of nondiscrete locally compact fields having zero characteristic.

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