RADICALS OF INCIDENCE ALGEBRAS

by

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ABSTRACT

RADICALS OF INCIDENCE ALGEBRAS

The incidence algebra of a locally finite partially ordered set X, with the partial ordering "$\leq$", over a ring with identity T is defined as the set of all mappings $f : X \times X \to T$ where $f(x, y) = 0$ for all $x, y \in X$ with $x \not\leq y$ and denoted by $I(X, T)$. The operations on $I(X, T)$ are given by

$$\begin{aligned} (f+g)(x,y) &= f(x,y) + g(x,y) \\ (f \cdot g)(x,y) &= \sum_{x \leq z \leq y} f(x,z) \cdot g(z,y) \\ (r \cdot f)(x,y) &= rf(x,y) \end{aligned}$$

for $f, g \in I(X, T)$, $r \in T$ and $x, y \in X$. When the ring R is commutative, the ring $I(X, R)$ becomes an algebra.

The aim of this study is to investigate some special radicals of incidence algebras and determine the necessary and sufficient conditions characterizing elements of these radicals by using the very definition of the strong product property.

ÖZET

ÇAKIŞMA CEBİRLERİNİN KÖKLERİ

Üzerinde "$\leq$" bağıntısı tanımlanmış yerel sonlu kısmi sıralı bir X kümesinin birimli bir T halkası üzerinde çakışma cebiri "$x \not\leq y$" olacak biçimdeki her $x, y \in X$ için $f(x, y) = 0$ koşulunu sağlayan $f : X \times X \to T$ fonksiyonlarından oluşan ve $I(X, T)$ ile gösterilen kümesi üzerinde aşağıda tanımlanan işlemlerle verilen halkadır: $f, g \in I(X, T)$, $r \in T$ ve $x, y \in X$ olmak üzere

$$\begin{aligned}(f+g)(x,y) &= f(x,y)+g(x,y)\\ (f\cdot g)(x,y) &= \sum_{x\leq z\leq y} f(x,z)\cdot g(z,y)\\ (r\cdot f)(x,y) &= rf(x,y)\end{aligned}$$

R'nin değişmeli halka olması durumunda $I(X, R)$ bir cebir olur.

Bu çalışmanın amacı çakışma cebirlerinin bazı özel radikallerini araştırıp, kuvvetli çarpım özelliğinin tanımından hareketle bu radikallerin elemanlarını belirleyen gerek ve yeter koşullar vermektir.

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LIST OF SYMBOLS/ABBREVIATIONS

$\square$	End of proof
$Ann(t)$	Annihilator of t
C_n	Interval of length n
f_D	Diagonal part of f
f_U	Upper triangular part of f
$I(X,R)$	Incidence algebra of X over R
$I(X,T)$	Incidence algebra of X over T
$\sqrt{J}$	Radical of the ideal J
$\mathfrak{J}(T)$	Jacobson radical of T
${}_TM$	Left T module M
$N_0(T)$	Sum of all the nilpotent ideals of T
$\mathcal{N}_*(T)$	Lower nilradical of T
$\mathcal{N}^*(T)$	Upper nilradical of T
$\mathcal{P}(T)$	Periodic radical of T
(r)	Ideal generated by r
R	Commutative ring
$\mathbb{Z}$	Ring of integers
$\mathbb{R}$	Field of real numbers
T	Noncommutative ring
$\prod_{i\in I} T_i$	Direct product of rings $\{T_i \mid i \in I\}$
X	Locally finite partially ordered set
$Z(I(X,T))$	Set of all strictly upper triangular functions of $I(X,T)$
δ	Identity element of $I(X,T)$
poset	Partially ordered set
spp	Strong product property

1. INTRODUCTION

The aim of this thesis is to study the existing results on radicals of incidence algebras. We will first start with a brief historical outline of the subject:

The various radicals which have been defined by several mathematicians such as Levitzki, Jacobson, Brown-McCoy, and others constitute an important tool in the study of the structure of rings. The purpose of this survey is thus to determine some of these radicals of incidence algebras.

The upper and the lower nilradicals were considered first by M. Baer [1], and are also known as the upper and the lower Baer radicals. Later, the lower nilradical is generalized by Amitsur [2]. In addition, an axiomatic study of radicals can be found in [3], [4] and [5].

In the study of radicals, Köthe [6] suggested the use of nil rings. Yet, the upper nilradical failed to be useful, since the study of rings with no two sided nil ideals still required dealing with one sided nil ideals. This raised the famous Köthe Conjecture which is not readily solved in general.

The theory of incidence algebras goes back to the 60s when it was first introduced by Gian-Carlo Rota and R. P. Stanley. These theorists, however, looked at the issue from a combinatorial point of view. Later on, after a couple of decades, the subject was focused on and analyzed with an algebraic point of view which is also the case in this study. The main topic under consideration will be the incidence algebra of locally finite partially ordered set over a (both commutative and noncommutative) ring with identity. However, there are some researchers who have studied incidence algebras of pre-ordered sets over a field or division ring.

The lower nilradical, or Baer radical of the incidence algebra has been determined when R is a field by Farkas (1974) (see [7]), when R is an integral domain by

Lerous and Sarraillé (1981) (see [8]), when R is a commutative ring by Spiegel (1994) (see [9]), and when R is any ring by Spiegel (2004) (see [10]). The upper nilradical of the incidence algebra is determined where the coefficient ring is noncommutative by Spiegel [11]. In [12] and [13], Bell and Klein showed that periodicity is a radical property in the sense of Kurosh and Amitsur. Guo [14] continued the study of this periodic radical by showing that $\mathcal{P}(T)$ is an intersection of suitable prime ideals and, consequently, that the periodic radical is a special radical (see Divinsky [15] for details). In the case of incidence algebras, a complete description is obtained whenever the coefficient ring is commutative with identity (see [16]).

At this point, we shall sketch the organization of the thesis.

In Chapter 1, introductory explanations are given.

In Chapter 2, basic notations and preliminary results used in the thesis are presented.

In Chapter 3, incidence algebras are examined.

In Chapter 4, the radical property is introduced. Some special radicals such as the upper nilradical, the lower nilradical and the Jacobson radical are presented.

In Chapter 5, the upper nilradical and the lower nilradical of an incidence algebra are examined where the incidence algebra is taken over a commutative ring with unity and taken over a noncommutative ring with unity, respectively.

Finally, *In Chapter* 6, the notion of periodic radical is presented and the periodic radical of an incidence algebra is investigated.

2. PRELIMINARIES

In this chapter, our aim is to present basic definitions and results which will be used in the subsequent chapters of this study. The proofs of all results can be found in any book on abstract algebra.

Throughout the text by a ring we assume an associative ring with or without identity.

Definition The *direct product* $\prod_{i \in I} T_i$ of rings $\{T_i \mid i \in I\}$ is the set of sequences $(t_i)_{i \in I}$ where $t_i \in T_i$ for each $i \in I$ with the operations defined componentwise.

Proposition 2.0.1. *Let* $\{T_i \mid i \in I\}$ *be a family of rings. Then the direct product* $\prod_{i \in I} T_i$ *is a ring.*

Proposition 2.0.2. *Let* t *be an element of a ring with identity* T. *Then*

$$Ann_r(t) = \{x \in T \mid tx = 0\}$$

is a right ideal and

$$Ann_l(t) = \{x \in T \mid xt = 0\}$$

is a left ideal (called respectively the right and the left annihilators of t *in* T*).*

Proposition 2.0.3. *If* T *is a ring with identity, then every ideal of* T *is contained in a maximal ideal.*

Definition A ring is called *simple* if it contains no nontrivial ideals.

Proposition 2.0.4. *For a ring with identity* T, *an ideal* I *maximal implies that* T/I *simple.*

Definition An element e of a ring T is called *idempotent* if $e^2 = e$.

Proposition 2.0.5. *If an element e of a ring with identity T is idempotent, then $T = T(1-e) \oplus Te$.*

If T is a ring with identity, the set of $n \times n$ matrices will be denoted by $M_n(T)$, the set of $n \times n$ upper triangular matrices by $T_n(T)$ and the set of $n \times n$ lower triangular matrices by $L_n(T)$. We also denote by $T_\infty(T)$ and $L_\infty(T)$ the rings of countable upper and lower triangular T-matrices, respectively. Standard matrix multiplication is defined in each of these rings as all sums involve only finitely many non-zero terms.

Definition A *left module* M over a ring T is an abelian group $(M, +)$ with a "multiplication by scalars", that is, a map

$$\begin{aligned} T \times M &\longrightarrow M \\ (r, m) &\longmapsto rm \end{aligned}$$

such that the following are satisfied for all $m_1, m_2, m \in M$, for all $r_1, r_2, r \in T$:

$$\begin{aligned} r(m_1 + m_2) &= rm_1 + rm_2 \\ (r_1 + r_2)m &= r_1 m + r_2 m \\ (r_1 r_2)m &= r_1(r_2 m) \end{aligned}$$

Definition A left T-module M is said to be *unitary* if T has identity 1_T and

$$1_T.m = m$$

for all $m \in M$.

Definition An *algebra* A is a ring which is also a R-module over a commutative ring

R such that the following condition is satisfied:

$$r(ab) = (ra)b = a(rb)$$

for all $r \in R$, $a, b \in A$.

The most natural example of an algebra is $n \times n$ matrices over a commutative ring or $n \times n$ upper or lower triangular matrices over a commutative ring.

Definition Let X be a set and $\leq$ be a binary relation on X. Then, X is called a *pre − ordered* set if the relation $\leq$ is reflexive and transitive. If $\leq$ is reflexive, transitive and antisymmetric, then X is called a *partially ordered set* or simply a *poset*. In this case, the relation $\leq$ is called a partial ordering and a partially ordered set X with the partial ordering $\leq$ is denoted by $(X, \leq)$.

Definition Let $(X, \leq)$ be a partially ordered set.

(i) An element $x \in X$ is called *maximal* if for any $y \in X$, $x \leq y$ implies $y = x$. If, in addition, for this $x \in X$, $y \leq x$ holds for each $x \in X$, then it is called the *maximum* element of X.

(ii) An element $x \in X$ is called *minimal* if for any $y \in X$, $y \leq x$ implies $y = x$ and it is the *minimum* element of X if $x \leq y$ for each $y \in X$.

Definition Let C be a subset of a partially ordered set $(X, \leq)$. Then, C is called a *chain* of X if for all $x, y \in X$, either $x \leq y$ or $y \leq x$. C is called an *antichain* if any distinct pair of elements are not comparable, that is, for any $x, y \in X$ with $x \neq y$ both $x \nleq y$ and $y \nleq x$.

A chain is said to be of *length-n* if it has n elements and a chain of length n is usually denoted by C_n.

Definition Suppose X is a partially ordered set with the partial ordering "$\leq$" and Y is a partially ordered set with the partial ordering "$\preccurlyeq$". Then X and Y are *isomorphic* as *posets* if there is an order preserving bijection between X and Y, that is, if there exists a bijection $\varphi : X \to Y$ with the property that if $x \leq y$ for $x, y \in X$, then $\varphi(x) \preccurlyeq \varphi(y)$ in Y.

Zorn's Lemma If $\mathscr{S}$ is a non-empty partially ordered set such that every chain in $\mathscr{S}$ has an upper bound in $\mathscr{S}$, then $\mathscr{S}$ has a maximal element.

Principle of Transfinite Induction Let $\mathscr{P}(x)$ be a statement involving the symbol x. Let $(A, \leq)$ be a well-ordered set. Suppose

(i) $\mathscr{P}(a)$ is true where a is the smallest element of A

(ii) if a is not the smallest element of A and $\mathscr{P}(b)$ is true whenever $b < a$, then $\mathscr{P}(a)$ is true.

Then $\mathscr{P}(a)$ is true for all $a \in A$.

3. INCIDENCE ALGEBRAS

3.1. Locally Finite Partially Ordered Sets

Definition Let $(X, \leq)$ be a partially ordered set and $x, y \in X$ such that $x \leq y$. An *interval* or *segment* from x to y, denoted $[x, y]$, is defined to be the set

$$[x, y] = \{z \in X \mid x \leq z \leq y \}$$

A partially ordered set X is *locally finite* if every interval of X is finite.

Definition An interval $[x, y]$ in a partially ordered set X is said to have *length-n* if there is a chain of length n in $[x, y]$, and any chain in this interval has length less than or equal to n.

Definition Let X be a partially ordered set. Then, X is said to be *bounded* if there exists a positive integer $n \in \mathbb{Z}^+$ such that every interval $[x, y]$ of X is at most of length n. A partially ordered set X is called *unbounded* if X is not bounded.

Examples of unbounded locally finite partially ordered sets containing an infinite chain include $\mathbb{Z}^+$, the positive integers under the usual ordering, and $\mathbb{Z}^-$, the partially ordered set of negative integers with the usual ordering. If we define the partially ordered set $\bigcup_{n\in\mathbb{N}} C_n$ to be the set $\{x_{11}, x_{21}, x_{22}, x_{31}, x_{32}, x_{33}, x_{41}, \ldots\}$ with the relation that $x_{ij} \leq x_{kl}$ whenever $i = k$ and $j \leq l$, for $x_{ij}, x_{kl} \in \bigcup_{n\in\mathbb{N}} C_n$, then $\bigcup_{n\in\mathbb{N}} C_n$ is an unbounded locally finite partially ordered set with no infinite chain. In fact, if m is a positive integer and

$$A(m) = \{x_{mj} \in \bigcup_{n\in\mathbb{N}} C_n \mid 1 \leq j \leq m \},$$

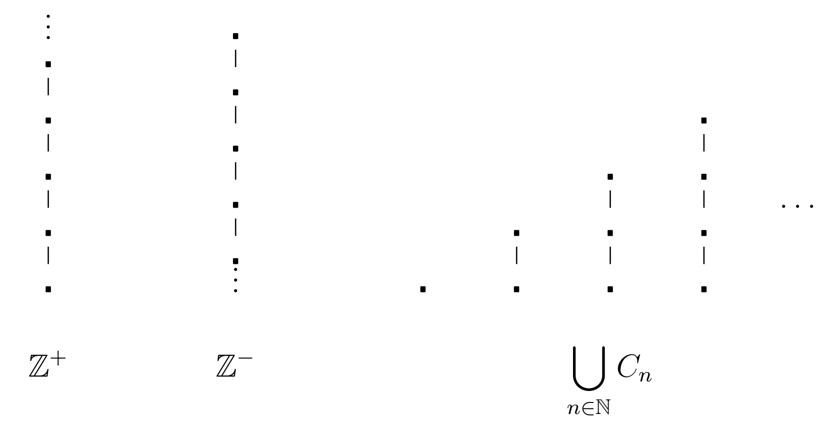

Figure 3.1. The three most basic unbounded posets.

then $A(m)$, as a subpartially ordered set, is a chain of length m. No element of $A(m_1)$ and $A(m_2)$ are related if $m_1 \neq m_2$, and

$$\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{m\in\mathbb{N}} A(m).$$

The Hasse diagrams of $\mathbb{Z}^+, \mathbb{Z}^-$, and $\bigcup C_n$ are given in Figure 3.1.

Theorem 3.1.1. *Let X be an unbounded, partially ordered set. Then X contains a subpartially ordered set isomorphic to $\mathbb{Z}^+, \mathbb{Z}^-$ or $\bigcup_{n\in\mathbb{N}} C_n$.*

Proof. See [17]. □

Lemma 3.1.2. *Suppose X is an unbounded locally finite partially ordered set. Then for each $m, n \in \mathbb{Z}^+$, we can find disjoint intervals of length m and of length n for $m \neq n$.*

Proof. Assume X is an unbounded locally finite partially ordered set. By Theorem 3.1.1, there exists a chain of length n for each $n \in \mathbb{Z}^+$. Let us denote an interval of length n by A_n. Put $B_1 = A_1$ and, inductively, $B_i = A_i \backslash B_{i-1}$ for each i. Then $\bigcup_i A_i = \bigcup_i B_i$ and B_i's are disjoint.

Now we construct disjoint intervals C_i's so that C_i has length i as follows:

Set $C_1 = B_1$. If B_2 contains an interval of length 2, let C_2 be this interval. If not, check B_3. If B_3 contains an interval of length 2, then let C_2 be this interval. If not, check B_4. When we find a chain, say D_{i_1} of length 2, then choose C_2 to be D_{i_1}. Continuing in this manner, we obtained our disjoint intervals C_i's. □

3.2. The Incidence Algebra

Throughout this text, the letter "R" denotes a commutative ring and "T" denotes a ring which is not necessarily commutative. We will define the incidence algebra, $I(X, R)$, of locally finite partially ordered set X over a commutative ring with identity R. Later, we will construct $I(X, T)$ over a ring with identity T which does not form an algebra structure in this case. But, by convention, we will call $I(X, T)$ as an incidence algebra.

Definition The *incidence algebra* $I(X, R)$ of the locally finite partially ordered set X over the commutative ring with identity R is

$$I(X, R) = \{f : X \times X \to R \mid f(x, y) = 0 \text{ if } x \not\leq y\}$$

with the operations given by

$$\begin{aligned} (f+g)(x,y) &= f(x,y) + g(x,y) \\ (f \cdot g)(x,y) &= \sum_{x \leq z \leq y} f(x,z) \cdot g(z,y) \\ (r \cdot f)(x,y) &= rf(x,y) \end{aligned}$$

for $f, g \in I(X, R)$ with $r \in R$ and $x, y \in X$.

Remark Given that if X is locally finite, the above sum is well-defined. We could also write

$$(f \cdot g)(x, y) = \sum_{z \in X} f(x, z) \cdot g(z, y)$$

as $f(x, z) = 0$ if $x \not\leq z$ and $g(z, y) = 0$ if $z \not\leq y$.

It is easy to check that $I(X, R)$ is an R-algebra. However, if we take a noncommutative ring, then $I(X, T)$ is not necessarily an algebra.

Now, we will introduce some special elements of $I(X, T)$.

1. Define

$$\begin{array}{rccc} \delta : & X \times X & \rightarrow & T \\ & (x, y) & \mapsto & \delta(x, y) \end{array}$$

such that

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Here we can clearly see that, for all $f \in I(X, T)$, and for all $x, y \in X$

$$\begin{aligned} (f \cdot \delta)(x, y) &= \sum_{x \leq z \leq y} f(x, z)\delta(z, y) \\ &= f(x, y) \end{aligned}$$

and

$$(\delta \cdot f)(x, y) = f(x, y),$$

that is, $\delta \in I(X, T)$ is the multiplicative identity.

2. Define

$$\begin{array}{rccc} \chi: & X \times X & \rightarrow & T \\ & (x,y) & \mapsto & \chi(x,y) \end{array}$$

where

$$\chi(x,y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise .} \end{cases}$$

3. Define

$$\begin{array}{rccc} \zeta: & X \times X & \rightarrow & T \\ & (x,y) & \mapsto & \zeta(x,y) \end{array}$$

such that

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if otherwise.} \end{cases}$$

namely

$$\zeta(x,y) = \delta(x,y) + \chi(x,y).$$

By definition of multiplication on $I(X,T)$, if an interval $[x,y]$ is of length n, then $\chi^n(x,y) = 0$, giving $\delta - \chi + \chi^2 - ...$ is a finite sum. Then we get $(\delta + \chi)(\delta - \chi + \chi^2 - ...) = \delta$, that is, $\delta + \chi = \zeta \in I(X,T)$ is invertible and its inverse is called the *Möbius function of* $I(X,T)$ and denoted by μ.

Lemma 3.2.1. *Let T be a ring with unity and $s \in T$. If $s \in T$ has both a left and a right inverse, then it is a unit.*

Proof. Let $ls = sr = 1_R$ for $l, r \in I(X,R)$. Then, $l = l(sr) = (ls)r = r$, that is, $l = r$. □

Theorem 3.2.2. *Suppose X is a locally finite partially ordered set and R is a commutative ring with unity. For $f \in I(X, R)$, the followings are equivalent:*

(i) f has a right inverse

(ii) f has a left inverse

(iii) f is a unit

(iv) $f(x, x)$ is a unit in R, for all $x \in X$.

Proof. We show the equivalence of (i) and (iv), the equivalence of (ii) and (iv) can be proven in a similar manner. By Lemma 3.2.1 for a ring with identity R if $s \in R$ has both right and a left inverse, then s is a unit. So (iv) implies both (i) and (ii). Then it follows that (iv) implies (iii). Finally, since (iii) obviously implies both (i) and (ii), the theorem will be proved.

(i)⇒(iv) Suppose that f has a right inverse g. Then, for all $x \in X$, we have

$$(f \cdot g)(x, x) = f(x, x)g(x, x) = \delta(x, x) = 1$$

and therefore $f(x, x)$ is a unit in R.

(iv)⇒(i) Suppose that $f(x, x)$ is a unit for all $x \in X$. We define a right inverse, say g, of f inductively on the length of the intervals of X as follows. If $|[x, y]| = 0$, then $x \not\leq y$ and set $g(x, y) = 0$. If $|[x, y]| = 1$, then $x = y$ and let $g(x, x) = (f(x, x))^{-1}$. Let $n > 1$ and assume that for $x, y \in X$ with $|[x, y]| < n$, $g(x, y)$ is already defined. Let $[x, y]$ be the interval of length n. We want

$$\begin{aligned} 0 = \delta(x, y) = (fg)(x, y) &= \sum_{x \leq z \leq y} f(x, z)g(z, y) \\ &= f(x, x)g(x, y) + \sum_{x < z \leq y} f(x, z)g(z, y) \end{aligned}$$

As $f(x, x)$ is invertible, we can solve this equation for $g(x, y)$. Thus, define

$$g(x, y) = \left[- \sum_{x < z \leq y} f(x, z) \cdot g(z, y) \right] \cdot f(x, x)^{-1}$$

Since the interval $[z, y]$ has length less than n, the function g has been defined for $z, y \in X$ by our induction hypothesis. Therefore, $f \cdot g = \delta$. □

For any cardinal number κ and a ring T, the set of $\kappa \times \kappa$ matrices will be denoted by $M_\kappa(T)$ which forms a T-module structure. A submodule of $M_\kappa(T)$ in which all sums in the formal matrix products of its elements involve only finitely many summands will form a matrix ring contained in $M_\kappa(T)$. By convention, we will refer to such a ring as a subring of $M_\kappa(T)$. Hence, we will show in the next proposition that the multiplication of elements in the incidence algebra and multiplication on matrix rings are closely related.

Proposition 3.2.3. *Let X be a locally finite partially ordered set and T a ring with identity. Then, the incidence algebra $I(X,T)$ is isomorphic to a subring of $M_{|X|}(T)$.*

Proof. Suppose that the elements of X is ordered so that $X = \{x_i \mid i \in I\}$ where I is an indexing set. Consider the entries of an element A in $M_{|X|}(T)$ as indexed by $I \times I$. Define

$$\begin{aligned} \varphi : \ I(X,T) &\longrightarrow M_{|X|}(T) \\ f &\mapsto \varphi(f) \end{aligned}$$

such that

$$\begin{aligned} \varphi(f) : \ I \times I &\longrightarrow T \\ (i_1, i_2) &\mapsto f(x_{i_1}, x_{i_2}) \end{aligned}$$

Now, if $\varphi(f) = 0$, then $(\varphi(f))(i_1, i_2) = f(x_{i_1}, x_{i_1}) = 0$, for all $x_{i_1}, x_{i_2} \in X$, thus φ is injective. On the other hand, by definition of addition and multiplication on $I(X,T)$, for all $f, g \in I(X,T)$ we have $\varphi(f + g) = \varphi(f) + \varphi(g)$ and $\varphi(f.g) = \varphi(f)\varphi(g)$. Hence, φ is a ring isomorphism between $I(X,T)$ and a subring of $M_{|X|}(T)$. □

Now suppose that $X = \{x_i \,|\, i \in I\}$ such that $x_i \leq x_j$ implies $i \leq j$, for all $x_i, x_j \in X$. Then, for any $f \in I(X,T)$, corresponding $\varphi(f)$ is an upper triangular $I \times I$ matrix. Therefore, we can say that $I(X,T)$ is isomorphic to $T_I(T)$, a subring of upper triangular matrices.

If $X = \{x_i \,|\, i \in I\}$ such that $x_i \leq x_j$ implies $j \leq i$, for all $x_i, x_j \in X$, then $\varphi(f)$ becomes a lower triangular $I \times I$ matrix and hence, $I(X,T)$ is isomorphic to $L_I(T)$, a subring of lower triangular matrices.

If in particular $X = \mathbb{Z}^+$, then we have $I(X,T) \cong T_\infty(T)$ and if $X = \mathbb{Z}^-$, then $I(X,T) \cong L_\infty(T)$.

Definition Elements x, y of a partially ordered set X is called *connected* if there exists elements $x_0, x_1, \ldots, x_n$ in X with $x_0 = x$, $x_n = y$ and either $x_i \leq x_{i+1}$ or $x_{i+1} \leq x_i$ for $i = 0, 1, \ldots, n-1$.

Note that, connectedness of elements of a partially ordered set X is an equivalence relation. The equivalence class of an element $x \in X$ is called the *connected component* of x. Then, X can be written as the disjoint union of its connected components. Moreover, if $X = \bigcup\limits_n X_n$, where X_n's are the connected components of X, then $f(x,y)$ will be zero when x and y are not in the same connected component or $x \not\leq y$. Thus, by definition of φ, we can say that $I(X,T)$ is isomorphic to a subring of $\prod\limits_n M_{|X_n|}(T)$.

If, in particular, X is an antichain, then all X_n's will be singletons, hence

$$I(X,T) \cong \prod_{x \in X} T.$$

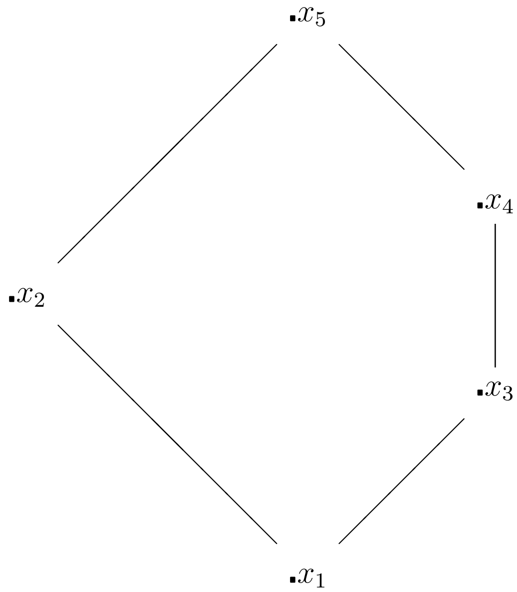

Figure 3.2. The Hasse diagram of N_5.

For $X = \bigcup_{n\in\mathbb{N}} C_n$, if we consider

$$I(C_n, T) \cong T_{|C_n|}(T),$$

for all n, then $I(X,T) \cong \prod_n T_{|C_n|}(T)$ and if $I(C_n,T) \cong L_{|C_n|}(T)$, for all n, then

$$I(X,T) \cong \prod_n L_{|C_n|}(T).$$

Example Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ such that $x_1 \leq x_2 \leq x_5$ and $x_1 \leq x_3 \leq x_4 \leq x_5$. The Hasse diagram of X is given in Figure 3.2.

Consider

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y; \\ 0 & \text{otherwise .} \end{cases}$$

Then,

$$\varphi(\zeta) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where φ is defined as in the proof of the previous proposition.

Proposition 3.2.4. *Let X be a locally finite partially ordered set and R a commutative ring with the unity. If X' is a subpartially ordered set of X, then $I(X', R)$ is a subalgebra of $I(X, R)$.*

Proof. By definition, $I(X', R) = \{f \in I(X, R) : f(x, y) = 0 \text{ if } x \notin X' \text{ or } y \notin X'\}$. Then we can easily verify that it is a subalgebra. □

Proposition 3.2.5. *If S is an ideal of R, then $I(X, S)$ is a subalgebra of $I(X, R)$.*

Proof. Consider similarly $I(X, S) = \{f \in I(X, R) : f(x, y) = 0 \text{ if } f(x, y) \notin S\}$. □

Definition Let X be a locally finite partially ordered set and T be a ring with unity.

(i) An element $f \in I(X, T)$ is called *diagonal* if $f(x, y) = 0$ for any $x, y \in X$ with $x \neq y$ and denoted f_D.

(ii) An element $f \in I(X, T)$ is called *strictly upper triangular* if $f(x, x) = 0$ for any $x \in X$ and denoted f_U.

Remark Given $f \in I(X, T)$, we can write f uniquely as $f = f_D + f_U$, because

$$f_D(x, y) = \begin{cases} f(x, x) & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$f_U(x,y) \quad = \quad \begin{cases} 0 & \text{if } x = y \\ f(x,y) & \text{if } x \neq y \end{cases}$$

and therefore,

$$f(x,y) \quad = \quad f_D(x,y) + f_U(x,y)$$

for all $x, y \in X$.

Remark The set of all strictly upper triangular functions, denoted $Z(I(X,T))$, and the set of all diagonal functions, denoted $D(I(X,T))$, are each subalgebras of $I(X,T)$. In addition,

$$I(X,T) = Z(I(X,T)) \oplus D(I(X,T)).$$

4. RADICAL PROPERTY

Definition Let $\mathscr{S}$ be a certain property that a ring may have. A ring T is called an *$\mathscr{S}$-ring* if it has the property $\mathscr{S}$ and an ideal J of T is called an *$\mathscr{S}$-ideal* if J is an $\mathscr{S}$-ring.

If $\mathscr{S}$ satisfies the following

(1) A homomorphic image of an $\mathscr{S}$-ring is an $\mathscr{S}$-ring,

(2) Every ring contains an $\mathscr{S}$-ideal S which contains every other $\mathscr{S}$-ideal of the ring,

(3) The factor ring T/S does not contain any non-zero $\mathscr{S}$-ideals,

then $\mathscr{S}$ is called a *radical property* and the $\mathscr{S}$-ideal S is called the *$\mathscr{S}$-radical* of T.

Definition An element r of a ring T is *nilpotent* if there exists a positive integer $n \in \mathbb{Z}^+$ such that $r^n = 0$. The smallest such n is called the *index of nilpotency* of r in T. A subring A of the ring T is *nil* if each element of A is nilpotent. The subring A is *nilpotent* if there exists $n \in \mathbb{Z}^+$ such that $a_1 \cdot a_2 \dots a_n = 0$ for every $a_1, a_2, \dots, a_n \in A$, that is, $A^n = 0$.

We now take $\mathscr{S}$ to be the nil property, and a ring T is an $\mathscr{S}$-ring if it is nil. We shall show that $\mathscr{S}$ is a radical property.

Lemma 4.0.6. *(i) If T is a nil ring, so is every subring of T.*

(ii) If T is a nil ring, so is every homomorphic image of T.

(iii) If A is an ideal of T with both A and T/A nil, then T is a nil ring.

Proof. (i) For any subring T' of T, T' consists of nilpotent elements and therefore it is nil.

(ii) If $\varphi : T \to T'$ is an epimorphism, then for all $r \in T$, $\varphi(r)$ is nil and therefore T' is nil.

(iii) Suppose A is an ideal of T with both A and T/A nil. Then, for all $r \in T$, there exists $m \in \mathbb{Z}^+$ with $(r+A)^m = r^m + A = A$, that is, $r^m \in A$. But A is also nil, so, there exists $n \in \mathbb{Z}^+$ with $(r^m)^n = r^{mn} = 0$. Hence r is nilpotent with mn. □

Lemma 4.0.7. *If A and B are two nil ideals of a ring T, so is $A+B$.*

Proof. Since $(A+B)/A \cong B/(A \cap B)$, by the second isomorphism theorem, and the right-hand side is nil as it is a factor ring of a nil ring, $(A+B)/A$ is nil. But then, by the previous lemma, A and $(A+B)/A$ are nil so that $A+B$ is nil. □

Lemma 4.0.8. *The sum of all the nil ideals of a ring T is a nil ideal.*

Proof. Let $\mathcal{N}$ denote the sum of all the nil ideals of T. For $r \in \mathcal{N}$, there are nil ideals $A_1, A_2, \ldots, A_k$ of T such that $r \in A_1 + A_2 + \ldots + A_k$. Since $A_1 + A_2 + \ldots + A_k$ is a nil ideal, r is nilpotent. Therefore, $\mathcal{N}$ is a nil ideal. □

Remark $\mathcal{N}$ is the largest nil ideal of T and $T/\mathcal{N}$ contains no nonzero nil ideals, that is, $T/\mathcal{N}$ contains no nontrivial nilpotent ideals. Thus, we have,

Corollary 4.0.9. *The nil property is a radical property.*

Definition The sum of all of the nil ideals of a ring T is called the *upper nilradical of T* and denoted by $\mathcal{N}^*(T)$.

Next we shall check that the Jacobson radical satisfies the radical property.

Definition Let T be a ring. Then *Jacobson radical* of T, denoted by $\mathcal{J}(T)$, is the intersection of all maximal left ideals of T.

Lemma 4.0.10. *Let T be a ring with unity. Then the following are equivalent:*

(i) $y \in \mathfrak{J}(T)$

(ii) $1 - xy$ *is left invertible, for all* $x \in T$

(iii) $y\ {}_TM = 0$, *for all simple* T*-modules* ${}_TM$.

Proof. *(i)⇒ (ii)* Let $y \in \mathfrak{J}(T)$ and assume $1 - x_0y$ is not left invertible for some $x_0 \in T$. Then $T(1 - x_0y)$ is a proper left ideal of T. Since T has the identity element, $T(1 - x_0y)$ is contained in a maximal left ideal I, say. Also, $y \in I$ implies $x_0y \in I$. Therefore $1 - x_0y + x_0y = 1 \in I$, a contradiction.

(ii)⇒ (iii) Let $1 - xy$ is left invertible for all $x \in T$. Assume there exists a simple T-module ${}_TM$ such that $y\ {}_TM \neq 0$. So, there exists $m \in M$ such that $ym \neq 0$. Since ${}_TM$ is simple, we can express it as ${}_TM = Tym$. It follows that there exists $x \in T$ such that $m = xym$, that is, $(1 - xy)m = 0$. Then $m = 0$ as by assumption $1 - xy$ is left invertible which is a contradiction.

(iii)⇒ (i) Suppose $y_TM = 0$ for all simple T-modules ${}_TM$. Let I be a maximal left ideal of T. Then ${}_T(T/I)$ is simple. So, $y\ {}_T(T/I) = 0$. In particular, $y \cdot \overline{1} = \overline{0}$. Hence, $y + I = I$ giving that $y \in I$. □

Proposition 4.0.11. *The Jacobson radical $\mathfrak{J}(T)$ of a ring T is a right ideal.*

Proof. Take any $y \in \mathfrak{J}(T)$ and $t \in T$. We show $1 - x(yt)$ is left invertible for all $x \in T$. Since $xy \in \mathfrak{J}(T)$, $1 - txy$ is left invertible for all $t \in T$, that is, there exists $v \in T$ such that $v(1 - txy) = 1$, that is, $vtxy = v - 1$ Then,

$$\begin{aligned}(1 + xyvt)(1 - xyt) &= 1 - xyt + xyvt - xyvtxyt \\ &= 1 - xyt + xyvt - xyvt + xyt \\ &= 1\end{aligned}$$

Hence, $yt \in \mathfrak{J}(T)$. □

Let T be a ring with identity. Define an ideal I to be an $\mathscr{S}$-ideal if for any $y \in I$, $1-xy$ is left invertible for each $x \in T$. We show that this property is a radical property and the $\mathscr{S}$-radical of T is precisely the Jacobson radical of T.

Proposition 4.0.12. *Let $\mathscr{S}$ be as above. Then $\mathscr{S}$ satisfies the radical property.*

Proof. *(i)* Let I be an $\mathscr{S}$-ring and $\varphi : I \to B$ be an epimorphism. We check B is an $\mathscr{S}$-ring. Fix $b \in B$. Then there exists an element $a \in I$ such that $\varphi(a) = b$. Take any $y \in B$. Then there exists $x \in I$ with $\varphi(x) = y$. Since I is an $\mathscr{S}$-ring, $1 - xa$ is left invertible with t, say. So, $t(1 - xa) = 1$. It follows that

$$\varphi(t(1 - xa)) = \varphi(t)(1 - \varphi(x)\varphi(a)) = \varphi(t)(1 - yb) = 1$$

This means that $1 - yb$ is left invertible, therefore, B is an $\mathscr{S}$-ring.

(ii) Suppose T is a ring with unity. Obviously, any $\mathscr{S}$-ideal is contained in $\mathfrak{J}(T)$ by Lemma 4.0.10 and thus $\mathfrak{J}(T)$ is the maximal $\mathscr{S}$-ideal of T.

(iii) Let T be a ring with unity. Suppose $B/\mathfrak{J}(T)$ is an $\mathscr{S}$-ideal of $T/\mathfrak{J}(T)$. Fix $b \in B$. Then $\overline{1} - \overline{a}\overline{b}$ is left invertible for all $\overline{a} \in T/\mathfrak{J}(T)$. Let $\overline{t}$ be a left inverse of $\overline{1} - \overline{a}\overline{b}$. Then, $\overline{t}(\overline{1} - \overline{a}\overline{b}) = \overline{1}$, that is, $1 - t(1 - ab) \in \mathfrak{J}(T)$. So, $1 - (1 - t(1 - ab)) = t(1 - ab)$ is left invertible. If t' is a left inverse of $t(1 - ab)$, then we have $t't(1 - ab) = 1$. This means that $1 - ab$ is left invertible for all $a \in T$, that is, $b \in \mathfrak{J}(T)$. □

Next, we consider the nilpotent property and see whether it is also a radical property.

Lemma 4.0.13. *(i) If T is a nilpotent ring, so is every subring of T and so is every factor ring of T.*

(ii) If A is an ideal of T with both A and T/A nilpotent, then T is nilpotent.

Proof. *(i)* is clear.

(ii) Since A is nilpotent there exists $n \in \mathbb{Z}^+$ such that $a_1 \cdot a_2 \cdots a_n = 0$ for every

a_1,a_2, $\cdots a_n \in A$. And since T/A is nilpotent there exists $m \in \mathbb{Z}^+$ such that

$$(r_1 + A) \cdot (r_2 + A) \cdots (r_m + A) = (r_1 \cdot r_2 \cdots r_m) + A = A$$

for every $\overline{r_i} \in T/A$, $1 \leq i \leq m$. Thus for $k = mn$, we have

$$r_{11} \cdot r_{12} \cdots r_{1n} \cdot r_{21} \cdot r_{22} \cdots r_{2n} \cdots r_{m1} \cdot r_{m2} \cdots r_{mn} = 0$$

where $r_1 \cdot r_{i2} \cdots r_{in} \in A$ and $r_{ij} \in T$ with $1 \leq i \leq m$ and $1 \leq j \leq n$. Hence, T is nilpotent with index k. □

Lemma 4.0.14. *If A and B are nilpotent ideals of a ring T, so is $A + B$.*

Proof. Suppose A is nilpotent with n and B is nilpotent with m. We show that $A + B$ is nilpotent with $m + n$. Let $a_i + b_i$ be elements of $A + B$ for $i = 1, 2, \ldots, m + n$. Then $\prod_{i=1}^{m+n} (a_i + b_i) = 0$ because each summand of this expression contains either m many elements of A or n many elements of B as A and B are ideals of T and therefore equals to 0.

□

Lemma 4.0.15. *The sum of all nilpotent ideals of a ring T is nil.*

Proof. Every nilpotent ideal is obviously nil and therefore, by Lemma 4.0.8 the sum of all the nilpotent ideals is a nil ideal. □

The sum of all nilpotent ideals of a ring T is not necessarily a nilpotent ideal as the following example illustrates.

Example Let A be an algebra over a field F with basis $\{x_\alpha\}_{\alpha \in I}$ where

$$I = \{\alpha \in \mathbb{R} \,|\, 0 < \alpha < 1\}$$

with the multiplication of basis elements given by:

$$x_\alpha \cdot x_\beta = \begin{cases} x_{\alpha+\beta} & if\ \alpha+\beta<1 \\ 0 & else \end{cases}$$

A, as a ring, consists of elements of the form $\sum_{finite} f_\alpha x_\alpha$ where f_α's are elements of the field F. We define addition as

$$\begin{array}{ll} f_\alpha x_\alpha + {f_\beta}' x_\beta & \text{just written together if } \alpha \neq \beta \\ f_\alpha x_\alpha + {f_\alpha}' x_\alpha = (f_\alpha + {f_\alpha}')x_\alpha & \text{if } \alpha = \beta \end{array}$$

Now if we choose $n \in \mathbb{Z}^+$ such that $n > \dfrac{1}{\alpha}$ then $(x_\alpha)^n = x_{\alpha n} = 0$ and for $a = \sum f_\alpha x_\alpha$ if β is the smallest subscript in the expression of a then an integer $k \in \mathbb{Z}^+$ such that $k > \dfrac{1}{\beta}$ will give $(a)^k = 0$. Thus, each element of A is nilpotent, that is, A is a nil ring.

However, A is not nilpotent because for all $n \in \mathbb{Z}^+$,

$$x_{\frac{1}{2}} \cdot x_{\frac{1}{4}} \cdot x_{\frac{1}{8}} \cdots x_{\frac{1}{2^n}} \neq 0.$$

Now, take any basis element x_α and consider the ideal (x_α) generated by x_α. Then (x_α) is a nilpotent ideal with an integer t satisfying $t > \frac{1}{\alpha}$. But the union of all the ideals (x_α) is A and, therefore, the union of the nilpotent ideals of A is not a nilpotent ideal.

Corollary 4.0.16. *The nilpotent property is not a radical property.*

Definition The *lower nilradical of* T, denoted $\mathcal{N}_*(T)$, is the smallest nil ideal of T such that $T/\mathcal{N}_*(T)$ contains no non-zero nilpotent ideals.

We shall show the existence of $\mathcal{N}_*(T)$ by Zorn's lemma:
Let T be a ring. Consider

$$\mathscr{S} = \{I \mid I \text{ is a nil ideal of } T \text{ and } T/I \text{ contains no nontrivial nilpotent ideals } \}$$

We have $\mathscr{S} \neq \emptyset$ as $\mathcal{N}^*(T) \in \mathscr{S}$. Now, order $\mathscr{S}$ by $\preccurlyeq$ where $I_1 \preccurlyeq I_2$ means $I_2 \subseteq I_1$. Let $\mathscr{C}$ be a chain in $\mathscr{S}$. Then $S = \bigcap_{I \in \mathscr{C}} I$ is a nil ideal and T/S does not contain any nonzero nilpotent ideals because otherwise if K/S is a nilpotent ideal of T/S then there exists a positive integer N such that $K^N \subseteq S \subseteq I$, for some $I \in \mathscr{C}$ and therefore K/I is a nilpotent ideal of T/I which is not possible. Note that S is an upper-bound for $\mathscr{C}$. Hence, by Zorn's lemma, $\mathscr{S}$ has a maximal element. Now, we check this element is unique. Suppose I_1, I_2 are maximal elements of $\mathscr{S}$. Then $I_1 \cap I_2$ is also a nil ideal. If $A/I_1 \cap I_2$ is a nonzero nilpotent ideal of $T/I_1 \cap I_2$, then $(I_1 + A)/I_1$ is a nonzero nilpotent ideal in $T/I_1 \cap I_2$. So, $I_1 \cap I_2 \in \mathscr{S}$ with $I_1 \cap I_2 \subseteq I_1, I_2$. This contradicts maximality of I_1 and I_2. Hence, $\mathscr{S}$ contains a unique maximal element which is the lower nilradical of T.

Zorn's lemma gives the existence of the lower nilradical of a ring T, however, does not characterize the lower nilradical. In order to have other characterizations of $\mathcal{N}_*(T)$ we determine $\mathcal{N}_*(T)$ in a constructive way.

Construction of $\mathcal{N}_*(R)$:
We use transfinite induction by defining an ideal $A(\alpha)$ of R for each ordinal α.

Induction Bases: Let $A(0) = N_0(T)$ where $N_0(T)$ denotes the sum of all the nilpotent ideals of T.

Induction Hypothesis: Let β be an ordinal such that $A(\alpha)$ has been defined for each $\alpha < \beta$.

Induction Step: If β is a limit ordinal, then define

$$A(\beta) = \sum_{\alpha<\beta} A(\alpha).$$

If β is not a limit ordinal, then there exists an ordinal, say α_0 satisfying $\beta = \alpha_0 + 1$. We set $A(\beta) = B$ such that $N_0\left(\frac{T}{A(\alpha_0)}\right) = \frac{B}{A(\alpha_0)}$. Note that whether β is limit ordinal or not $A(\beta)$ is nil. So, $A(\beta) \subseteq \mathcal{N}^*(T)$, that is, $\mathcal{N}^*(T)$ is an upper bound for $A(\beta)$. This means that there exists an ordinal, say γ, satisfying $A(\gamma) = A(\gamma + 1)$. Then, the lower nilradical of T is $A(\gamma)$.

Note that the above construction of the lower nilradical also shows that the lower nilradical satisfies the radical property. See [15] for details.

Before computing the lower nilradical of a ring, we need to review some ring theoretic results.

Definition An ideal P of a ring with identity T is *prime* if whenever the ideals A, B of T have the property $A \cdot B \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$.

Proposition 4.0.17. *Let T be a ring with identity, P be an ideal of T and $k \geq 2$ a positive integer. Then the followings are equivalent:*

(i) P is a prime ideal.

(ii) If $b_1, b_2, \ldots, b_k \in T$ with $b_1 \cdot T \cdot b_2 \cdot T \cdots T \cdot b_k \subseteq P$, then $b_i \in P$ for some index $1 \leq i \leq k$.

Proof. *(i)⇒(ii)* Suppose that $b_1, b_2, \ldots, b_k$ are elements of T such that

$$b_1 \cdot T \cdot b_2 \cdot T \cdots T \cdot b_k \subseteq P.$$

Then $(T \cdot b_1 \cdot T)(T \cdot b_2 \cdot T) \cdots (T \cdot b_k \cdot T) \subseteq P$. If we set $B_i = T \cdot b_i \cdot T$, then B_i is the ideal generated by b_i and so $B_1 \cdot B_2 \cdots B_k \subseteq P$ gives $B_i \subseteq P$ for some i with $1 \leq i \leq k$, as P is prime. Then $b_i \in P$ for some $1 \leq i \leq k$.

(ii)⇒(i) Suppose that $P, B_1, B_2, \ldots B_k$ are ideals of T such that $B_1 \cdot B_2 \cdots B_k \subseteq P$. Assume that $B_i \not\subseteq P$ for $i = 1, 2, \ldots, k-1$. For $1 \leq i < k$, choose $b_i \in B_i \backslash P$ and let $b \in B_k$. Then

$$b_1 \cdot T \cdot b_2 \cdot T \cdots t \cdot b_{k-1} \cdot T \cdot b \subseteq P,$$

and as $b_i \notin P$ for $1 \leq i < k$, we have $b \in P$ (by assumption). Therefore, $B_k \subseteq P$ and P is prime.

□

Definition An element s of a ring T is *strongly nilpotent* if given a sequence $s_0, s_1, s_2, \ldots$ with $s_0 = s$ and $s_{i+1} \in s_i T s_i$ for $i = 1, 2, \ldots,$ there exists a positive integer $n \in \mathbb{Z}^+$ such that $s_n = 0$.

Proposition 4.0.18. *The intersection of the prime ideals of a ring with identity T is the set of all strongly nilpotent elements.*

Proof. Let

$$A = \{a \in T \,|\, a \text{ is strongly nilpotent}\}$$

and

$$\mathcal{N}(T) = \bigcap P$$

where the intersection is taken over all the prime ideals of T.

(Necessity) We will prove this part by contraposition. If $a \notin \mathcal{N}(T)$, then there is a prime ideal P of T such that $a \notin P$. Then, by the previous proposition, $a \cdot T \cdot a \not\subseteq P$

and so there exists $a_1 \in a \cdot T \cdot a$ such that $a_1 \notin P$. Then again $a_1 \cdot T \cdot a_1 \nsubseteq P$ and so there exists $a_2 \in a_1 \cdot T \cdot a_1$ such that $a_2 \notin P$. Continuing in this manner, we obtain a sequence $a, a_1, a_2, \ldots$ in T such that $a_i \in a_{i-1} \cdot T \cdot a_{i-1}$ and $a_i \notin P$ for each i. Thus, $a_i \neq 0$ for each i and therefore a is not strongly nilpotent, that is $a \notin A$. Hence, $A \subseteq \mathcal{N}(T)$.

(Sufficiency) Conversely, suppose that $a \notin A$. Then there exists a sequence $a_0, a_1, a_2, \ldots$ of nonzero elements of T such that $a_0 = a$ and $a_i \in a_{i-1} \cdot T \cdot a_{i-1}$ for $i = 1, 2, \ldots$. Let $\mathscr{S} = \{a_0, a_1, \cdots\}$. By Zorn's lemma, there exists an ideal P of T such that P is maximally disjoint from $\mathscr{S}$. We claim that P is prime. Suppose not, then there are ideals A and B of T satisfying $AB \subseteq P$ with $A \nsubseteq P$ and $B \nsubseteq P$. Then $A+P$ and $B+P$ are ideals with $P \nsubseteq A+P$ and $P \nsubseteq B+P$. So there are indices i and j such that $a_i \in A+P$ and $a_j \in B+P$. Without loss of generality, assume $i \leq j$. Then

$$a_j \in a_{j-1} \cdot T \cdot a_{j-1} \subseteq a_{j-2} \cdot T \cdot a_{j-2} \cdot T \cdot a_{j-1} \cdot T \cdot a_{j-1} \subseteq \cdots \subseteq T \cdot a_i \cdot T \subseteq A+P$$

Thus,

$$a_{j+1} \in a_j \cdot T \cdot a_j \subseteq (A+P)T(B+P) \subseteq P$$

as

$$(a+p_1)r(b+p_2) = (ar+p_1r)(b+p_2) = arb + arp_2 + p_1rb + p_1rp_2 \in P$$

for all $a \in A$, $b \in B$, $r \in T$ and $p_1, p_2 \in P$. So, $a_{j+1} \in P$ which contradicts our assumption that $P \cap S = \emptyset$. Therefore, P is a prime ideal of T and $a \notin P$. Hence, $\mathcal{N}(T) \subseteq P$. □

Proposition 4.0.19. *The lower nilradical of a ring with identity T is the set of all strongly nilpotent elements.*

Proof. (Necessity) Let a be a nonzero strongly nilpotent element of a ring T and A_0 be the ideal generated by a. Assume that $a \notin \mathcal{N}_*(T)$. Now $(A_0)^2 \nsubseteq \mathcal{N}_*(T)$ because otherwise if $(A_0)^2 \subseteq \mathcal{N}_*(T)$, then

$$(\mathcal{N}_*(T) + x)(\mathcal{N}_*(T) + y) = \mathcal{N}_*(T) + xy = \mathcal{N}_*(T)$$

for all $x, y \in A_0$, that is, $\{\mathcal{N}_*(T) + x \,|\, x \in A_0\}$ is a nilpotent ideal of $T/\mathcal{N}_*(T)$ contradicting the fact that $T/\mathcal{N}_*(T)$ contains no nonzero nilpotent ideals. Therefore, $(A_0)^2 \not\subseteq \mathcal{N}_*(T)$ and there exists $s_1 \in T$ with $a \cdot s_1 \cdot a \notin \mathcal{N}_*(T)$ because otherwise $a \cdot s_1 \cdot a \in \mathcal{N}_*(T)$ implies $(A_0)^2 \subseteq \mathcal{N}_*(T)$. Let $a_1 = a \cdot s_1 \cdot a$ and A_1 be the ideal generated by a_1. Then, again, $(A_1)^2 \not\subseteq \mathcal{N}_*(T)$ and there exists $s_2 \in T$ with $a_1 \cdot s_2 \cdot a_1 \notin \mathcal{N}_*(T)$. Let $a_2 = a_1 \cdot s_2 \cdot a_1$ and continue in this manner to obtain a sequence $a_0 = a, a_1, a_2, \ldots$ with $a_i \in a_{i-1} \cdot T \cdot a_{i-1}$ and $a_i \notin \mathcal{N}_*(T)$ for $i = 1, 2, \ldots$. Then $a_i \neq 0$ for each i contradicting the strongly nilpotent property of a. Thus, the set of strongly nilpotent elements is contained in $\mathcal{N}_*(T)$.

(Sufficiency) To prove the converse, we show $\mathcal{N}_*(T)$ is contained in every prime ideal. Let P be a prime ideal. We check that $A(\alpha) \subseteq P$ for each ordinal α. If $\alpha = 0$, then $A(\alpha)$ is defined as the sum of all the nilpotent ideals of T. If B_i is a nilpotent ideal of T, then there exists $k_i \in \mathbb{Z}^+$ such that ${B_i}^{k_i} = \{0\} \subseteq P$ and so $B_i \subseteq P$ (because P is a prime ideal). It follows that the sum of all the nilpotent ideals $\sum B_i = A(0) \subseteq P$. Now, suppose that β is an ordinal satisfying $A(\alpha) \subseteq P$ for each ordinal $\alpha < \beta$. If β is a limit ordinal, then $A(\beta) = \sum_{\alpha<\beta} A(\alpha)$ giving that $A(\beta) \subseteq P$. If β is not a limit ordinal, then there is a successor of α, say γ, such that $\beta = \gamma + 1$. By definition, $A(\beta) = B$ where $N_0(T/A(\gamma)) = B/A(\gamma)$. Therefore, $A(\beta)$ is the sum of the nilpotent ideals B_i such that $A(\gamma) \subseteq B_i$ and $B_i/A(\gamma)$ is nilpotent. Then there exists a positive integer k such that ${B_i}^k \subseteq A(\gamma) \subseteq P$ where the second inclusion is verified by the assumption. It follows that $B_i \subseteq P$ for each i and therefore $A(\beta) \subseteq P$. Hence, $\mathcal{N}_*(T) \subseteq P$ which completes the proof. $\square$

Proposition 4.0.18 and 4.0.19 show that the lower nilradical coincides with the intersection of the prime ideals of the ring and therefore also known as the prime radical.

Definition Let T be a ring with identity.

(i) An ideal J is said to be a *semi-prime ideal* if, for any ideal A of T, $A^2 \subseteq T$ implies that $A \subseteq T$.

(ii) T is called a *semi-prime ring* if (0) is a semi-prime ideal.

Remark Note that for any ideal P of a ring T, the factor ring T/P is a semi-prime ring if and only if P is a semi-prime ideal. Therefore, T/P is semi-prime if and only if $0_{T/P} \cong P$ is semi-prime.

Proposition 4.0.20. *Suppose T is a ring with identity and J is an ideal in T. Then the followings are equivalent:*

(i) *J is semi-prime*

(ii) *For each $r \in T$, $(r)^2 \in J$ implies that $r \in J$*

(iii) *For each $r \in T$, $rTr \subseteq J$ implies that $r \in J$*

(iv) *For any left ideal A in T, $A^2 \subseteq J$ implies that $A \subseteq J$.*

Proof. *(i)* implies *(ii)*, *(ii)* implies *(iii)* and *(iv)* implies *(i)* follow from the definition of the semi-prime ideal. We check *(iii)* implies *(iv)*. Assume that $A^2 \subseteq P$ for some left ideal A of T, but $A \nsubseteq P$. Take $a \in A \backslash P$. Then, $aTa \subseteq P$. Using *(iii)*, we get $a \in P$, which is a contradiction. □

Definition Let T be a ring with unity and J be an ideal of T. The *radical* of J, denoted by $\sqrt{J}$, is the intersection of prime ideals of T containing J, that is,

$$\sqrt{J} = \bigcap_{P\, prime\ and\ J \subseteq P} P$$

Lemma 4.0.21. *Suppose T is a ring with identity and J is an ideal in T. Then the followings are equivalent:*

(i) *J is a semi-prime ideal*

(ii) *J is an intersection of prime ideals*

(iii) $J = \sqrt{J}$

Proof. *(iii)* $\Rightarrow$ *(ii)* Obvious by the definition of a radical ideal.
(ii) $\Rightarrow$ *(i)* Let A be an ideal such that $A^2 \subseteq J$. By assumption, J is an intersection of prime ideals, therefore, A is contained in each of these ideals. Hence, $A \subseteq J$.
(i) $\Rightarrow$ *(iii)* We show $\sqrt{J} \subseteq J$. Let $a \notin J$. By Proposition 4.0.20,

$$aTa, aTaTaTa, aTaTaTaTaTaTaTa, \ldots \nsubseteq J$$

Choose $t_1 \in T$ with $at_1a \notin J$. Since J is semi-prime and $aTaTaTa \nsubseteq J$, there exists an element $t_2 \in T$ with $at_1at_2at_1a \notin J$. Similarly, there exists $t_3 \in T$ with $at_1at_2at_1at_3at_1at_2at_1a \notin J$. Continuing in this manner we can find $t_i \in T$ for each $i \in \mathbb{Z}^+$. Let S be the set of $a, at_1a, at_1at_2at_1a, at_1at_2at_1at_3at_1at_2at_1a, \ldots$. By Zorn's lemma, there exists an ideal P, say, which is maximally disjoint from S. Since $J \cap S = \emptyset$, we have $J \subseteq P$. We show that P is a prime ideal in T. Suppose $x \notin P$, $y \notin P$ but $(x)(y) \in P$. By maximality of P, there exists $s, s' \in S$ with $s \in P+(x)$ and $s' \in P+(y)$. So, there exists $t \in T$ with $sts' \in S$. Then

$$sts' \in (P + (x))T(P + (y)) \subseteq P + (x)(y) \subseteq P$$

which is a contradiction. Hence, P is prime. It follows that $a \notin \sqrt{J}$. □

Corollary 4.0.22. *Let J be an ideal of a ring T. Then $\sqrt{J}$ is the smallest semi-prime ideal in T satisfying $J = \sqrt{J}$. In particular, $\mathcal{N}_*(T) = \sqrt{(0)}$ is the smallest semi-prime ideal in T.*

Proposition 4.0.23. *Let T be a ring. Then $\mathcal{N}_*(T) = A(\alpha)$ for any ordinal α with $card\, \alpha > card\, T$ where $A(\alpha)$ is defined as in the construction of $\mathcal{N}_*(T)$.*

Proof. Note that $A(\alpha)$'s form an ascending chain of ideals in $\mathcal{N}_*(T)$. Write $B = A(\alpha)$ where α is an ordinal with $card\, \alpha > card\, T$. Then, for any ordinal β with $card\, \beta > card\, T$, we have $B = A(\beta)$. Since $B \subseteq \mathcal{N}_*(T)$, it is sufficient to show that $\mathcal{N}_*(T) \subseteq B$. Now, T/B has no nonzero nilpotent ideals, so it is a semiprime ring. This means that B is a semiprime ideal. Hence $\mathcal{N}_*(T) \subseteq B$ since $\mathcal{N}_*(T)$ is the smallest semiprime ideal of T. □

Next, we will see the relation between the upper nilradical and the Jacobson radical. First, we need to prove the following lemma.

Lemma 4.0.24. *Let T be left artinian ring. Then $\mathfrak{J}(T)$ is the largest nilpotent (left) ideal.*

Proof. Enough to show that there exits $n \in \mathbb{Z}^+$ such that $(\mathfrak{J}(T))^n = 0$. Consider the descending chain

$$\mathfrak{J}(T) \supseteq (\mathfrak{J}(T))^2 \supseteq (\mathfrak{J}(T))^3 \supseteq \dots$$

Since T is artinian, there exists $N \in \mathbb{Z}^+$ such that $(\mathfrak{J}(T))^N = (\mathfrak{J}(T))^{N+1} = \ldots = I$. Hence, we need to see that $I = 0$. Assume not. Then there exists a left ideal J in T such that $IJ \neq 0$. Consider

$$\mathscr{S} = \{J \mid \ J \text{ is an ideal in } T \text{ satisfying } IJ \neq 0 \ \}$$

Since T is left artinian there exists minimal left ideal J_0, say, satisfying $IJ_0 \neq 0$ by Zorn's lemma. So, there exists $a \in J_0$ such that $Ia \neq 0$. Note that $I(Ia) = I^2a = Ia \neq 0$, that is, Ia satisfies this property. We have $a \in J_0$, so, $Ia \subseteq J_0$ and since J_0 was minimal $J_0 \subseteq Ia$. Therefore, $Ia = J_0$, that is, there exists $x \in I$ such that $a = xa$, that is, $(1 - x)a = 0$ where $x \in I \subseteq \mathfrak{J}(T)$. Since $1 - x$ is invertible we have $a = 0$, a contradiction. Hence, $I = 0 = (\mathfrak{J}(T))^N$. □

Corollary 4.0.25. *Suppose T is a (left) artinian ring with identity. Then, any (left) nil ideal of T is also (left) nilpotent.*

Proof. Let I be nil left ideal of T. Then $I \subseteq \mathfrak{J}(T)$ where $\mathfrak{J}(T)$ is nilpotent in this case. Therefore, I is nilpotent. □

Proposition 4.0.26. *Let T be a ring with identity. Then*

$$\mathcal{N}_*(T) \subseteq \mathcal{N}^*(T) \subseteq \mathfrak{J}(T)$$

If T is left artinian, then

$$\mathcal{N}_*(T) = \mathcal{N}^*(T) = \mathcal{J}(T)$$

Proof. $\mathcal{N}_*(T)$ is contained in $\mathcal{N}^*(T)$ as $\mathcal{N}_*(T)$ is a nil ideal. On the other hand $\mathcal{J}(T)$ contains every nil (left) ideal of T as if $y \in I$ for a left ideal I of T, and for each $x \in T$, we have $xy \in I$, therefore, there exists $t \in \mathbb{Z}^+$ such that $(xy)^t = 0$ and

$$(1 + xy + \cdots + (xy)^{t-1})(1 - xy) = 1$$

that is $1 - xy$ is left invertible. Similarly, $1 - xy$ is right invertible and thus $y \in \mathcal{J}(T)$. Assume now T is left artinian. Then $\mathcal{J}(T)$ is the largest nilpotent (right) ideal by Proposition 4.0.24. Since (0) is the unique nilpotent ideal in $T/\mathcal{N}_*(T)$, $T/\mathcal{N}_*(T)$ semiprime, so, if $A^2/\mathcal{N}_*(T) = \mathcal{N}_*(T)/\mathcal{N}_*(T)$, then $A/\mathcal{N}_*(T) = \mathcal{N}_*(T)/\mathcal{N}_*(T)$, that is $A = \mathcal{N}_*(T)$; hence $A/\mathcal{N}_*(T) = \mathcal{N}_*(T)/\mathcal{N}_*(T)$) it follows that $\mathcal{J}(T) \subseteq \mathcal{N}_*(T)$. Hence

$$\mathcal{J}(T) \subseteq \mathcal{N}_*(T) \subseteq \mathcal{N}^*(T) \subseteq \mathcal{J}(T)$$

giving that all three radicals are equal. □

Theorem 4.0.27. *Let T be a ring with unity. Then any ideal J of $M_n(T)$ has the form $M_n(I)$ for a uniquely determined ideal I of T.*

Proof. First note that if I is an ideal of T, then $M_n(I)$ is an ideal of $M_n(T)$. Define

$$\begin{aligned} \varphi \ : \ & A \to B \\ & I \mapsto M_n(I) \end{aligned}$$

where A is the set of ideals of T and B is the set of ideals of $M_n(T)$. We check that φ is a bijection. φ is well-defined and one-to-one as for any ideals I_1, I_2 of T, $I_1 = I_2$ if and only if $M_n(I_1) = M_n(I_2)$. Let J be an ideal of $M_n(T)$. Then form the set I of all the $(1,1)$-entries of matrices in J, that is, if $m = (e_{ij}) \in J$, then put $e_{11} \in I$ and

$I = \{a_{11} \in T \mid (a_{ij}) \in J\ \}$.

Claim 1. I is an ideal of T.

Claim 2. $M_n(I) = J$.

Proof of Claim 1. Take any $x, y \in I$, then there exits $(a_{ij}), (b_{ij}) \in J$ such that $x = a_{11}, y = b_{11}$, then $(a_{ij}) + (b_{ij}) = (c_{ij}) \in J$ and $c_{11} = a_{11} + b_{11} = x + y \in I$. Let $r \in T$, then

$$re_{11}(a_{ij}) \in J \text{ such that } ra_{11} \in I$$

$$(a_{ij})re_{11} \in J \text{ such that } a_{11}r \in I$$

So, I is an ideal of T.

Proof of Claim 2. Let $M = (m_{ij}) \in J$, take any m_{ij} fixed, then $e_{1i}Me_{j1} = m_{ij}e_{11} \in J$ implies $m_{ij} \in I$. So, $M \in M_n(I)$, that is, $J \subseteq M_n(I)$. Conversely, take any $(a_{ij}) \in M_n(I)$. So for any a_{ij}, there exists $M \in J$ such that $a_{ij} = m_{11.}$ Then,

$$a_{ij}e_{ij} = m_{11}e_{ij} = e_{i1}Me_{1j} \in J$$

therefore,

$$\sum_{i,j=1}^{n} a_{ij}e_{ij} = (a_{ij}) \in J$$

that is, $M_n(I) \subseteq J$. □

Proposition 4.0.28. *A ring T is semi-prime if and only if $M_n(T)$ is semi-prime.*

Proof. Assume T is not a semi-prime ring. This implies that (0) is not a semi-prime ideal. So, there exists a non-zero ideal I in T with $I^2 = (0)$. Then $(M_n(I))^2 = (0)$, so $M_n(T)$ is not semi-prime. Conversely, if $M_n(T)$ is not semi-prime, then it has a non-zero ideal $\mathfrak{J}$ such that $(\mathfrak{J})^2 = (0)$. By Theorem 4.0.27, there exists an ideal I in T

with $\mathcal{I} = M_n(I)$. Then $\mathcal{I}^2 = (0)$ implies that $I^2 = (0)$, so T is not semi-prime.

□

Theorem 4.0.29. *For any ring with identity T, we have $\mathcal{N}_*(M_n(T)) = M_n(\mathcal{N}_*(T))$.*

Proof. We have $T/\mathcal{N}_*(T)$ is semi-prime, so $M_n(T/\mathcal{N}_*(T))$ is also semi-prime by Proposition 4.0.28. But then $M_n(T)/M_n(\mathcal{N}_*(T))$ is semi-prime, so $\mathcal{N}_*(M_n(T)) \subseteq M_n(\mathcal{N}_*(T))$ as $\mathcal{N}_*(T)$ is the smallest semi-prime ideal in $M_n(T)$. Using Theorem 4.0.27, write the ideal $\mathcal{N}_*(M_n(T))$ of $M_n(T)$ in the form $M_n(I)$, where I is an ideal in T. Then

$$M_n(T/I) \cong M_n(T)/M_n(I) = M_n(T)/\mathcal{N}_*(M_n(T))$$

is semi-prime, and so is T/I by Proposition 4.0.28. This implies that $\mathcal{N}_*(T) \subseteq I$, so we have

$$M_n(\mathcal{N}_*(T)) \subseteq M_n(I) = \mathcal{N}_*(M_n(T))$$

and the equality holds. □

It is not known if the equation

$$\mathcal{N}^*(M_n(T)) = M_n(\mathcal{N}^*(T))$$

holds for the upper nilradicals. In fact, the above equation for all n and for all rings T is equivalent to the famous Köthe's Conjecture which can be found in [**?**].

Köthe's Conjecture For any ring T, $\mathcal{N}^*(T) = (0)$ implies that T has no non-zero nil one sided ideals.

For several classes of rings, the conjecture has been shown to be true. For example, it can be found in [**?**] that the conjecture holds for the class of right noetherian rings. However, the conjecture is not solved in general yet.

5. UPPER NILRADICAL AND LOWER NILRADICAL OF INCIDENCE ALGEBRAS

In this chapter, our aim is to determine the upper and the lower nilradicals of an incidence algebra. First, we will investigate the upper and the lower nilradicals when the incidence algebra is defined over a commutative ring with unity. Then, we determine necessary and sufficient conditions to characterize the upper and the lower nilradicals of incidence algebras over a noncommutative ring with unity.

5.1. Upper and Lower Nilradicals of $I(X, R)$

Definition Let R be a commutative ring with identity and S be a subset of a locally finite partially ordered set X. A function $f \in I(X, R)$ is *fully-nilpotent of index n on S* if there exists a positive integer n such that given any chain of the form

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \ldots \leq x_n \leq y_n$$

in S, $\prod_{i=1}^{n} f(x_i, y_i) = 0$. A function that is fully-nilpotent on X will simply be called *fully nilpotent*.

Remark If f is fully-nilpotent of index n on $S \subseteq X$, then $(f(x, x))^n = 0$, for all $x \in S$.

Remark When the ring R is an integral domain, then the previous definition is equivalent to the following:

there exists $n \in \mathbb{N}$ such that given any chain of the form

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \ldots \leq x_n \leq y_n$$

in S, $f(x_i, y_i) = 0$ for some $1 \leq i \leq n$.

Proposition 5.1.1. *If T is a ring with identity, then strong nilpotency implies nilpotency.*

Proof. Suppose $s \in T$ is strongly nilpotent. Consider the sequence $s_0, s_1, s_2, \ldots$ where $s_i = s^{2^i}$ for each i. Then $s_0 = s^{2^0} = s$ and $s_{i+1} = s^{2^{i+1}} = s^{2^i 2^i} = s_i s_i$ for each i and therefore $s_n = s^{2^n} = 0$ for some n as s is strongly nilpotent. Hence s is nilpotent with 2^n. □

However, the converse need not be true as the following example illustrates.

Example Consider $\mathbb{Z}^+$ under the usual ordering and let R be a commutative ring with identity. Observe that $\mathbb{Z}^+$ is unbounded. Define $f \in I(\mathbb{Z}^+, R)$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x = 2^k \text{ and } y = 2^k + 1, \text{ for some } k \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

We show that f is nilpotent but not strongly nilpotent. Since for all $x, y \in \mathbb{Z}^+$,

$$f^2(x,y) = \sum_{x \le z \le y} f(x,z)f(z,y) = 0,$$

that is, $f^2 = 0$ giving that f is nilpotent with 2. Define the function $g \in I(\mathbb{Z}^+, R)$ by

$$g(x,y) = \begin{cases} 1 & \text{if } x = 2^k + 1 \text{ and } y = 2^{k+1}, \text{ for some } k \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

We construct a sequence $h_0, h_1, h_2, \ldots$ in $I(\mathbb{Z}^+, R)$ as follows. Put $h_0 = f$ and inductively $h_{i+1} = h_i g h_i$, for $i = 1, 2, \dots$. Then

$$\begin{array}{rcccl} h_1 & = & h_0 g h_0 & = & fgf \\ h_2 & = & h_1 g h_1 & = & fgfgfgf \\ h_3 & = & h_2 g h_2 & = & fgfgfgfgfgfgfgf \\ & & \vdots & & \\ h_k & = & h_{k-1} g h_{k-1} & & \\ & & \vdots & & \end{array}$$

Observe that f occurs 2 times in the expression of h_1, 2^2 times in the expression of h_2, 2^3 times in the expression of h_3. Hence f appears 2^k times in the expression of h_k. Now consider $h_k(2, 2^{2^k} + 1)$, for all $k \in \mathbb{Z}^+$.

$$h_k(2, 2^{2^k} + 1) = f(2, 2^1 + 1) g(2^1 + 1, 2^2) f(2^2, 2^2 + 1) g(2^2 + 1, 2^3) \dots f(2^{2^k}, 2^{2^k} + 1) = 1$$

Hence, f is not strongly nilpotent.

Proposition 5.1.2. *If a ring R with identity is commutative, then nilpotency is equivalent to strong nilpotency.*

Proof. Suppose $r \in R$ is nilpotent with n. We are to show that r is strongly nilpotent. Let $r_0, r_1, \dots$ be a sequence in R with $r_0 = r$ and $r_{i+1} \in r_i R r_i$ for $i = 1, 2, \dots$. Then

$$\begin{array}{rcccccccl} r_1 & = & r t_1 r & = & r^2 t_1 & & & & \text{for some } t_1 \in R \\ r_2 & = & r_1 t_2 r_1 & = & r^2 t_1 t_2 r^2 t_1 & = & r^4 t_1 t_2 t_1 & = & r^4 t_2' \\ & & & & & & & & \text{for some } t_2, t_2' \in R \\ & & \vdots & & & & & & \\ r_k & = & r^{2^k} t & & & & & & \text{for some } t \in R \\ & & \vdots & & & & & & \end{array}$$

Let k be a positive integer such that $2^k \geq n$. Then $r_k = 0$ and thus, r is strongly nilpotent. □

Proposition 5.1.3. *Let X be a locally finite partially ordered set and R a commutative ring with identity. A function $f \in I(X, R)$ is fully-nilpotent if and only if f is strongly nilpotent.*

Proof. (Necessity) Suppose that f is fully-nilpotent. Then there exists $n \in \mathbb{Z}^+$ such that whenever

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_n \leq y_n$$

is a chain on X, we have

$$\prod_{i=1}^{n} f(x_i, y_i) = 0.$$

Let $h_0 = f$ and set $h_{i+1} = h_i g_i h_i$ for $i = 1, 2, \ldots,$ where $g_i \in I(X, R)$. As f appears 2^k times in the expression of h_k, choose k so that $2^k \geq n$. Then

$$h_k(x, y) = \sum f(x_1, y_1) g(y_1, x_2) f(x_2, y_2) \ldots g(y_{2^k - 1}, x_{2^k}) f(x_{2^k}, y_{2^k})$$

where the sum is over all possible chains

$$x = x_1 \leq y_1 \leq \ldots \leq x_{2^k} \leq y_{2^k} = y.$$

But $\prod_{i=1}^{2^k} f(x_i, y_i)$ is a factor of each summand, and as $n \leq 2^k$, it follows that each summand is zero. Hence, $h_k = 0$ and f is strongly nilpotent.

(Sufficiency) Conversely, suppose that f is strongly nilpotent and assume for a contradiction that f is not fully-nilpotent. Then one of the following two possibilities must hold.

(i) For each $n \geq 1$ there exists $x_n \in X$ such that $(f(x_n, x_n))^n \neq 0$.

(ii) For each $n \geq 1$ there exists a chain

$$x_{n,1} \leq y_{n,1} < x_{n,2} \leq y_{n,2} < \cdots < x_{n,n} \leq y_{n,n}$$

in X with $\prod_{i=1}^{n} f(x_{n,i}; y_{n,i}) \neq 0$.

If (i) holds, then the sequence $f_n = f^{2^n}$ is a sequence of nonzero functions, contradicting the strong nilpotency of f.

If (ii) holds, then X is unbounded and, by Lemma 3.1.2, we may assume that the intervals $[x_{m,1}; y_{m,m}]$ and $[x_{n,1}; y_{n,n}]$ are disjoint for $m \neq n$.

Let $k \in \mathbb{N}$ and define the function g_k as follows. For $n \geq 1$ set

$$g_k(y_{n,i}; x_{n,i}) = \begin{cases} 1 & \text{if } i \equiv 2^{k-1} (mod\, 2^k) \text{ and } i \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

We define a sequence of functions $\{f_n\}$ inductively by setting $f_1 = f$ and for $m \leq 1$, set $f_{m+1} = f_m g_m f_m$. Thus, if $r \in \mathbb{N}$, then

$$f_r(x_{2^r,1}; y_{2^r,2^r}) = \prod_{i=1}^{2^r} f(x_{2^r,i}; y_{2^r,i}) \neq 0.$$

This means that the sequence $\{f_n\}$ is not zero for any integer n, and thus, f is not strongly nilpotent which contradicts our assumption. □

Combining this result with Proposition 4.0.19 we conclude the following.

Theorem 5.1.4. *Let X be a locally finite partially ordered set and R be a commutative ring with identity. Then the lower nilradical $\mathcal{N}_*(I(X,R))$ is the set of fully-nilpotent functions of $I(X,R)$.*

Proposition 5.1.5. *Suppose that X is a bounded, locally finite partially ordered set and R a commutative ring with identity. Nilpotent functions of $I(X,R)$ are strongly* *nilpotent.*

Proof. It is enough to show that if f is nilpotent, then f is fully-nilpotent.
As f is nilpotent, there exists $n \in \mathbb{Z}^+$ such that $f^n = 0$. Also, since X is bounded, there exists $k \in \mathbb{N}$ such that when $x_1 < x_2 < \cdots < x_s$ is a chain in X, then $s < k$. Let $N = n(k-1)+1$ and consider a chain in X given by

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_N \leq y_N.$$

There can be at most $k-1$ strict inequalities in the above chain, hence there is a string of n consecutive subscripts, say i through $i+n-1$, such that

$$x_i = y_i = \cdots = x_{i+n-1} = y_{i+n-1}.$$

It follows that $\prod_{j=1}^{N} f(x_j, y_j)$ contains a factor of the form

$$\prod_{j=i}^{i+n-1} f(x_j, y_j) = (f(x_i, y_i))^n = f^n(x_i, x_i) = 0.$$

Hence f is fully-nilpotent. □

Corollary 5.1.6. *If X is a bounded partially ordered set, then $\mathcal{N}_*(I(X,R))$, where R is a commutative ring with identity, is the set of nilpotent functions of $I(X,R)$.*

Proof. If X is bounded, then $f \in I(X,R)$ is strongly nilpotent if and only if f is nilpotent by Proposition 5.1.5. Then the result follows. □

It is obvious that $\mathcal{N}_*(T) \subseteq \mathcal{N}^*(T)$ for a ring T. In general, $\mathcal{N}_*(T) \neq \mathcal{N}^*(T)$. We can also define other nilradicals $\mathcal{M}$ such that $\mathcal{M}$ is a nil ideal of T with $T/\mathcal{M}$ contains no nonzero nilpotent ideals. However, all such radicals lies between upper and lower nilradical as the following proposition states.

Proposition 5.1.7. *If $\mathcal{M}$ is a nilradical of a ring T, then*

$$\mathcal{N}_*(T) \subseteq \mathcal{M} \subseteq \mathcal{N}^*(T)$$

Proof. $\mathcal{M}$ is nil, so, is contained in the sum of all the nil ideals of T, namely $\mathcal{N}^*(T)$. The first inclusion is also clear as $\mathcal{N}_*(T)$ is the smallest nil ideal satisfying $T/\mathcal{N}_*(T)$ contains no nonzero nilpotent ideals. □

Corollary 5.1.8. *If X is bounded, $I(X, R)$ contains unique nilradical where R is a commutative ring with identity.*

Proof. We have $\mathcal{N}_*(T) \subseteq \mathcal{N}^*(T)$ for any ring T. By the previous corollary, $\mathcal{N}_*(I(X, R))$ is the set of nilpotent elements and since $\mathcal{N}^*(T)$ consists of nilpotent elements, we get

$$\mathcal{N}^*(I(X, R)) \subseteq \mathcal{N}_*(I(X, R))$$

which completes the proof. □

This result does not depend on the boundedness of the locally finite partially ordered set as the following theorem states.

Theorem 5.1.9. *Let X be a locally finite partially ordered set and R be a commutative ring with identity. The incidence algebra $I(X, R)$ contains a unique nilradical.*

Proof. If X is bounded, the result follows by Corollary 5.1.8. Suppose now X is unbounded. It is enough to check $\mathcal{N}^*(I(X, R)) \subseteq \mathcal{N}_*(I(X, R))$. Let $f \in \mathcal{N}^*(I(X, R))$. We check f is strongly nilpotent. So it is enough to check f is fully-nilpotent. By way of contradiction, assume f is not fully-nilpotent. Then for each positive integer n, there exists a chain

$$x_{n,1} \leq y_{n,1} \leq x_{n,2} \leq y_{n,2} \leq \cdots \leq x_{n,n} \leq y_{n,n}$$

in X with

$$\prod_{i=1}^{n} f(x_{n,i}; y_{n,i}) \neq 0$$

Since X is unbounded, we may assume intervals $[x_{n,i}; y_{n,i}]$ and $[x_{m,i}; y_{m,i}]$ are disjoint for $m \neq n$. Hence we can define a well-defined function $g \in I(X, R)$ as follows. For any positive integer n,

$$g(u, v) = \begin{cases} 1 & \text{if } u = y_{n,i} \text{ and } v = x_{n,i+1} \text{ for some } n \in \mathbb{Z}^{+} \\ 1 & \text{if } u = v = y_{n,n} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} (fg)^{n}(x_{n,1}; y_{n,n}) &= fgfg \cdots fg(x_{n,1}; y_{n,n}) \\ &= f(x_{n,1}; y_{n,1})g(y_{n,1}; x_{n,2}) \cdots f(x_{n,n}; y_{n,n})g(y_{n,n}; y_{n,n}) \\ &= \prod_{i=1}^{n} f(x_{n,i}; y_{n,i}) \\ &\neq 0 \end{aligned}$$

for each positive integer n and therefore fg is not nilpotent. On the other hand, $fg \in \mathcal{N}^{*}(I(X, R))$ and so nilpotent which is a contradiction. □

5.2. Upper and Lower Nilradicals of $I(X, T)$

In this chapter we will consider incidence algebras over a ring with identity T and investigate the lower and upper nilradicals of them. First, we will extend the definition of fully-nilpotent functions to the strong product property.

Definition Let X be a locally finite partially ordered set and T be a ring with identity. An element $f \in I(X, T)$ has the *strong product property (spp) of index* n if there exists a positive integer n such that, given any chain

$$x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$$

in X, then

$$f(x_1, y_1)Tf(x_2, y_2)T \cdots Tf(x_n, y_n) = 0.$$

Remark Let $s \in T$ where T is a ring with unity. Then $f = s\,\delta \in I(X,T)$ satisfies spp if and only if $s \in T$ is strongly nilpotent.

Proposition 5.2.1. *Any element in the T-submodule generated by*

$$\{e_{x,y} \in I(X,T) \mid x < y\}$$

satisfies spp where

$$e_{x,y}(u,v) = \begin{cases} 1 & \text{if } x = u \text{ and } y = v \\ 0 & \text{otherwise.} \end{cases}$$

for all $x, y \in X$.

Proof. Let f be an element generated by the submodule generated by

$$\{e_{x,y} \in I(X,T) \mid x < y\}$$

Then

$$f = \sum_{i=1}^{n} t_i e_{x_i,y_i} s_i + \sum_{j=1}^{m} n_j e_{x_j,y_j}$$

where for each i, j, with $1 \leq i \leq n$ and $1 \leq j \leq m$, $s_i, t_i \in T$, $n_j \in \mathbb{Z}$ and

$(x_i, y_i), (x_j, y_j) \in X \times X$ with $x_i < y_i$ and $x_j < y_j$. For any $u, v \in X$, we have

$$f(u,v) = \sum_{i=1}^{n} t_i e_{x_i,y_i}(u,v) s_i + \sum_{j=1}^{m} n_j e_{x_j,y_j}(u,v) = 0$$

if $(x_i, y_i), (x_j, y_j) \neq (u, v)$ for each i, j. Now consider any chain of the form

$$u_1 \leq v_1 < u_2 \leq v_2 < \cdots < u_{n+m+1} \leq v_{n+m+1}$$

in X. Then for some k with $1 \leq k \leq n+m+1$, $(u_k, v_k) \neq (x_i, y_i)$ and $(u_k, v_k) \neq (x_j, y_j)$. So, $f(u_k, v_k) = 0$. Therefore

$$f(u_1, v_1) T f(u_2, v_2) T \cdots T f(u_k, v_k) T \cdots T f(u_{n+m+1}; v_{n+m+1}) = 0.$$

It follows that f has spp with $n + m + 1$. □

Proposition 5.2.2. *Let $I(X,T)$ be the incidence algebra of a ring with identity T over a locally finite partially ordered set X. Then*

(i) $(f+g)_D = f_D + g_D$

(ii) $(fg)_D = f_D g_D$

for all $f, g \in I(X,T)$.

Proof. *(i)*

$$\begin{aligned}
(f+g)_D(x,y) &= \begin{cases} (f+g)(x,x) & \text{if } x = y \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} f(x,x) + g(x,x) & \text{if } x = y \\ 0 & \text{else} \end{cases} \\
&= f_D(x,y) + g_D(x,y) \\
&= (f_D + g_D)(x,y)
\end{aligned}$$

(ii) Take any $x, y \in X$. If $x = y$, then

$$(fg)_D(x,x) = (fg)(x,x) = f(x,x)g(x,x) = f_D(x,x)g_D(x,x) = (f_D g_D)(x,x)$$

If $x \neq y$, then $(fg)_D(x,y) = 0$ and

$$(f_D g_D)(x,y) = \sum_{x \leq z \leq y} f_D(x,z)g_D(z,y) = 0$$

as $f_D(x,z) \neq 0$ only if $z = x$ and $g_D(z,y) \neq 0$ only if $z = y$. So, $(fg)_D(x,y) = f_D g_D$. $\square$

Proposition 5.2.3. *Let X be a locally finite partially ordered set, T be a ring with unity and $f = \sum_{finite} t_{xy}e_{xy} \in I(X,T)$. Then $f \in \mathcal{N}^*(I(X,T)) \backslash \mathcal{N}_*(I(X,T))$ if and only if $t_{xx} \in \mathcal{N}^*(T)$ for each $x \in X$ and $t_{xx} \in \mathcal{N}^*(T) \backslash \mathcal{N}_*(T)$ for at least one $x \in X$.*

Proof. *(Sufficiency)* We first check, f_D generates a nil ideal in $\prod_{x \in X} T$. Note that f_D has only finitely many non-zero terms and each of them is nilpotent as t_{xx} is nilpotent in T. Then f_D is nilpotent with the maximum number of the nilpotency in its components. Now, we check that f has spp. By the above example each $t_{xy}e_{xy}$ has spp for $x < y$. Also each $t_{xx}e_{xx}$ has spp for any $x \in X$ because for any chain of the form

$$x_1 \leq y_1 < x_2 \leq y_2$$

in X, either $x_1 \neq x$ or $x_2 \neq x$ giving that

$$t_{xx}e_{xx}(x_1,y_1)Tt_{xx}e_{xx}(x_2,y_2) = 0.$$

Therefore, $t_{xx}e_{xx}$ has spp with 2. Thus, $t_{xy}e_{xy}$ has spp for all $x, y \in X$ say with n_{xy}. Then, $f = \sum_{finite} t_{xy}e_{xy}$ has spp with the sum of n_{xy}'s. Now, we show f_D is not strongly nilpotent in $\prod_{x \in X} T$. Assume the contrary. Since $t_{x_0} \notin \mathcal{N}_*(T)$ for some x_0 in the expression of f, there exist elements $s_0, s_1, s_2, \ldots$ in T such that the sequence $t_0, t_1, t_2, \ldots$ for which $t_0 = t$ and $t_i = t_{i-1}s_{i-1}t_{i-1}$ consists of non-zero elements of T.

Fix any $x \in X$. Consider elements (g_i) in $\prod_{x\in X} T$ defined for each $i = 0, 1, 2, \ldots$ as

$$(g_i)_y = \begin{cases} s_i & \text{if } y = x_0 \\ 0 & \text{else} \end{cases}$$

Construct a sequence $(f_0), (f_1), \ldots$ in $\prod_{x\in X} T$ with $(f_0) = f_D$ and $(f_i) = (f_{i-1})(g_{i-1})(f_{i-1})$ for each $i = 1, 2, \ldots$. Since f_D is assumed to be strongly nilpotent, there exists a positive integer k with $(f_k) = 0$. It follows that

$$\begin{array}{rclcl} (f_0)_{x_0} & = & te_{x_0x_0}(x_0, x_0) & = t & = t_0 \\ (f_1)_{x_0} & = & (f_0)_{x_0}(g_0)_{x_0}(f_0)_{x_0} & = ts_0t & = t_1 \\ (f_2)_{x_0} & = & (f_1)_{x_0}(g_1)_{x_0}(f_1)_{x_0} & = ts_0ts_1ts_0t & = t_2 \\ \vdots & & & & \end{array}$$

is a sequence of non-zero elements which is a contradiction.

(Necessity) Conversely, suppose $f \in \mathcal{N}^*(I(X,T)) \backslash \mathcal{N}^*(I(X,T))$. Since

$$f = \sum_{finite} t_{xy}e_{xy} \in \mathcal{N}^*(I(X,T))$$

we have

$$f_D = \sum_{finite} t_{xx}e_{xx} \in \mathcal{N}^*(\prod_{x\in X} T)$$

It follows that $t_{xx} \in \mathcal{N}^*(T)$ for each $x \in X$. On the other hand, $f_D \notin \mathcal{N}_*(\prod_{x\in X} T)$. So there exists a sequence (f_i) of nonzero elements in $\prod_{x\in X} T$ with $(f_0) = f_D$ and $(f_i) \in (f_{i-1}) \prod_{x\in X} T(f_{i-1})$ for $i = 1, 2, \ldots$. It follows that at least one of the component of the sequence is non-zero, say $(f_n)_{x_0}$, for each $n \in \mathbb{Z}^+$. This means that $t_{x_0x_0} \in T$ is not strongly nilpotent.

□

Proposition 5.2.4. *Let X be a locally finite partially ordered set and T be a ring with identity. If $f \in I(X,T)$ has spp, then so does f_D and f_U.*

Proof. Suppose f has spp with N but f_D does not satisfy spp. Let

$$x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$$

be a chain in X with $n \geq N$. Since f_D does not satisfy spp we can find elements $a_1, a_2, \ldots, a_{n-1} \in T$ with

$$f_D(x_1, y_1)a_1 f_D(x_2, y_2)a_2 \cdots a_{n-1} f_D(x_n, y_n) \neq 0.$$

If $x_i \neq y_i$ for some i, $1 \leq i \leq n$, then $f_D(x_i, y_i) = 0$. Then $x_i = y_i$ for each i, $1 \leq i \leq n$. So, $f_D(x_i, y_i) = f(x_i, y_i)$ for each i, $1 \leq i \leq n$, giving that

$$f(x_1, y_1)a_1 f(x_2, y_2)a_2 \cdots a_{n-1} f(x_n, y_n) \neq 0.$$

This contradicts the fact that f has spp with N. f_U has spp can be proven in a similar way. □

Proposition 5.2.5. *Let X be a locally finite partially ordered set and T be a ring with unity. If $I(X,T)$ contains an element which does not satisfy spp, then X is unbounded.*

Proof. Suppose $f \in I(X,T)$ does not satisfy spp and X is bounded with n. Then for any interval $[x, y]$ in X, $[x, y]$ has length at most n. This means that any chain $[x, y]$ has length at most n. Since there is no chain of length $n+1$ in X, f automatically has spp with $n+1$ which contradicts our assumption. □

Remark If $f \in I(X,T)$ does not satisfy spp, then for each $n \in \mathbb{Z}^+$, there exists a chain

$$x_{n,1} \leq y_{n,1} < x_{n,2} \leq y_{n,2} < \cdots < x_{n,n} \leq y_{n,n}$$

in X and elements $a_{n,1}, a_{n,2}, \ldots, a_{n,n-1} \in T$ such that

$$f(x_{n,1}; y_{n,1})a_{n,1}f(x_{n,2}; y_{n,2})a_{n,2} \ldots a_{n,n-1}f(x_{n,n}; y_{n,n}) \neq 0.$$

Proposition 5.2.6. *Let X be a locally finite partially ordered set and T be a ring with identity. If $f \in \mathcal{N}_*(I(X,T))$, then $f_D \in \mathcal{N}_*(\prod_{x \in X} T)$.*

Proof. Define

$$\begin{array}{rccc} \varphi: & I(X,T) & \rightarrow & \prod_{x \in X} T \\ & f & \mapsto & f_D \end{array}$$

Note that φ is a surjective ring homomorphism. Let $f \in \mathcal{N}_*(I(X,T))$. We check that $f_D \in \mathcal{N}_*(\prod T)$.

Let $(t_1), (t_2), \ldots$ be a sequence in $\prod_{x \in X} T$ with $(t_1) = f_D$ and $(t_{i+1}) = (t_i)(r_i)(t_i)$ for each i and $(r_i) \in \prod_{x \in X} T$. Set $g_i \in I(X,T)$ such that

$$g_i(x,y) = \begin{cases} (r_i)_x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

for each i. Then $f_1, f_2, \ldots$ is a sequence in $I(X,T)$ with $f_1 = f$ and $f_{i+1} = f_i g_i f_i$ for each i. Since $f \in \mathcal{N}_*(I(X,T))$, f is strongly nilpotent, so, there exists a positive integer n with $f_n = 0$. Therefore,

$$\varphi(f_n) = (f_n)_D = (fg_1fg_2 \cdots fg_1f)_D = f_D(g_1)_D f_D \cdots f_D(g_1)_D f_D = 0$$

and since $(f_n)_D = t_n$ we get f_D is strongly nilpotent and contained in $\mathcal{N}_*(\prod_{x\in X} T)$. $\square$

Theorem 5.2.7. *Let X be a locally finite partially ordered set and T be a ring with identity T. Then $f \in \mathcal{N}_*(I(X,T))$ if and only if $f_D \in \mathcal{N}_*(\prod_{x\in X} T)$ and f has spp.*

Proof. (Necessity) Suppose $f \in \mathcal{N}_*(I(X,T))$. By the previous proposition, we have $f_D \in \mathcal{N}_*(\prod_{x\in X} T)$. We check f satisfies spp. Suppose not, then for each $n \in \mathbb{Z}^+$, there exists a chain

$$x_{n,1} \leq y_{n,1} < x_{n,2} \leq y_{n,2} < \cdots < x_{n,n} \leq y_{n,n}$$

in X and elements $a_{n,1}, a_{n,2}, \ldots a_{n,n-1} \in T$ with

$$f(x_{n,1}; y_{n,1})a_{n,1}f(x_{n,2}; y_{n,2})a_{n,2}\ldots a_{n,n-1}f(x_{n,n}; y_{n,n}) \neq 0.$$

Since f does not have spp, by Proposition 5.2.5 , X is unbounded. By Lemma 3.1.2, we may assume the intervals $[x_{n,1}; y_{n,n}]$ and $[x_{m,1;y_{m,m}}]$ are disjoint for $m \neq n$.

Now, define, for each $k \in \mathbb{Z}^+$, $g_k \in I(X,T)$ as follows:
Let $n \geq 1$ and

$$g_k(u,v) = \begin{cases} a_{n,i} & \text{if } u = y_{n,i},\ v = x_{n,i+1} \text{ and } i \equiv 2^{k-1}\ (mod\, 2^k) \text{ for } i \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

for each $u, v \in X$. Now, we construct a sequence $f_1, f_2, \ldots$ in $I(X,R)$ as follows.

Set $f_1 = f$ and $f_{j+1} = f_j g_j f_j$ for each j. Then, for any $r \in \mathbb{Z}^+$

$$f_r(x_{2^r,1}; y_{2^r,2^r}) = f(x_{2^r,1}; y_{2^r,1})a_{2^r,1}f(x_{2^r,2}; y_{2^r,2})a_{2^r,2}\cdots a_{2^r,2^r-1}f(x_{2^r,2^r}; y_{2^r,2^r}) \neq 0$$

Hence, $f_m \neq 0$ for each $m \in \mathbb{Z}^+$. Therefore f is not strongly nilpotent, that is, $f \notin \mathcal{N}_*(I(X,T))$ which is a contradiction.

(Sufficiency) Conversely, suppose $f_D \in \mathcal{N}_*(\prod_{x\in X} T)$ and f has spp. We check f satisfies spp. Let $f_1, f_2, \dots$ be a sequence in $I(X,T)$ with $f_1 = f$ and $f_{i+1} = f_i g_i f_i$ where $g_i \in I(X,T)$ for each i. Then $(f_1)_D, (f_2)_D, \dots$ is a sequence in $\prod_{x\in X} T$ with $(f_1)_D = f_D$ and $(f_{i+1})_D = (f_i)_D(g_i)_D(f_i)_D$ for each i. Since $f_D \in \mathcal{N}_*(\prod_{x\in X} T)$, f_D is strongly nilpotent. So there exists a positive integer t such that $(f_t)_D = 0$.

On the other hand, f has spp, say of index N. This means that for any chain

$$x_1 \le y_1 < x_2 \le y_2 < \cdots < x_n \le y_n$$

in X, with $n \ge N$,

$$f(x_1, y_1)Tf(x_2, y_2)T \cdots Tf(x_n, y_n) = 0.$$

Now consider f_{t+N}. We claim that $f_{t+N} = 0$. Suppose not, then there exists $x, y \in X$ with $f_{t+N}(x, y) \ne 0$. If $x = y$, then

$$f_{t+N}(x, y) = f_{t+N}(x, x) = f_t(x, x)s_1 f_t(x, x)s_2 \cdots s_{2^N-1} f_t(x, x) = 0$$

for some $s_1, s_2, \dots s_{2^N-1} \in T$ as $f_t(x, x) = 0$. So, there exists $x, y \in X$ such that $x \ne y$ and $f_{t+N}(x, y) \ne 0$. Then there exists a chain

$$x = x_1 < y_1 \le x_2 < y_2 \le \cdots \le x_{2^N} < y_{2^N}$$

in X with

$$f_{t+N}(x, y) = f_t(x_1, y_1)t_1 f_t(x_2, y_2)t_2 \cdots t_{2^{N-1}} f_t(x_{2^N}, y_{2^N}) \ne 0$$

for some $t_i \in T$ with $i = 1, 2, \dots, 2^N - 1$. Now let $x_{2m-1} = u_m$ and $y_{2m-1} = v_m$ for

$m = 1, 2, \ldots, 2^{N-1.}$ Then

$$u_1 < v_1 < u_2 < v_n < \cdots < u_{2^{N-1}}$$

is a chain in X with $2^{N-1} \geq N$ and

$$f_{t+N}(x,y) = f(u_1, v_1)\tilde{t}_1 f(u_2, v_2)\tilde{t}_2 \cdots \tilde{t}_{2^{N-1}-1} f(u_{2^{N-1}, v_{2^{N-1}}}) \neq 0$$

which contradicts our assumption that f has spp with N. □

Now, we shall determine the upper nilradical of an incidence algebra where the coefficient ring is noncommutative with identity. First, we need the following results.

Lemma 5.2.8. *Let X be a locally finite partially ordered set and T be a ring with identity. Suppose $f \in Z(I(X,T))$. Then the following are equivalent:*

(i) f satisfies spp,

(ii) The left ideal generated by f is nil,

(iii) The right ideal generated by f is nil,

(iv) The ideal generated by f is nilpotent.

Proof. *(ii)⇔(iii)* Suppose the left ideal generated by f, f_L, is a nil ideal and $g \in I(X,T)$. Then $(gf)^n = 0$ for some $n \in \mathbb{Z}^+$. If we multiply $(gf)^n$ by f on left and by g on right, we get $fgfgf \cdots gfg = (fg)^{n+1} = 0$, that is, the right ideal generated by f, f_R, is a nil ideal. Similarly, if f_R is nil so is f_L.

(iv)⇒(ii) Since every nilpotent ideal is nil, the result follows.

(ii)⇒(i) Suppose the left ideal generated by f, f_L is nil. Assume for a contradiction that f does not satisfy spp. Since $f \in Z(I(X,T))$, for each $n \in \mathbb{Z}^+$, there is a chain

$$x_{n,1} < y_{n,1} < x_{n,2} < y_{n,2} < \cdots < x_{n,n} < y_{n,n}$$

in X, and elements $a_{n,1}, a_{n,2}, \ldots, a_{n,n-1} \in T$ such that

$$f(x_{n,1}; y_{n,1})a_{n,1}f(x_{n,2}; y_{n,2})a_{n,2} \cdots a_{n,n-1}f(x_{n,n}; y_{n,n}) \neq 0$$

Since f does not satisfy spp, by Proposition 5.2.5, X is unbounded. Hence, by Lemma 3.1.2, we may assume the intervals $[x_{n,1}; y_{n,n}]$ and $[x_{m,1}; y_{m,m}]$ are disjoint for $n \neq m$.

Consider an element g of $I(X,T)$ defined as follows:

$$g(u,v) = \begin{cases} 1 & \text{if } u = v = x_{n,1},\ n = 1, 2, \ldots \\ a_{n,i} & \text{if } u = y_{n,i} \text{ and } v = x_{n,i+1} \\ 0 & \text{otherwise} \end{cases}$$

for all $u, v \in X$. Then for each $n \in \mathbb{Z}^+$,

$$\begin{aligned} (gf)^n(x_{n,1}; y_{n,n}) &= (gfgf \cdots gf)(x_{n,1}; y_{n,n}) \\ &= g(x_{n,1}; y_{n,1})f(x_{n,1}; y_{n,1})g(y_{n,1}; x_{n,2}) \cdots f(x_{n,n}; y_{n,n}) \\ &= f(x_{n,1}; y_{n,1})a_{n,1}f(x_{n,2}; y_{n,2})a_{n,2} \cdots a_{n,n-1}f(x_{n,n}; y_{n,n}) \\ &\neq 0 \end{aligned}$$

Therefore, gf, which is an element in the left ideal generated by f, is not a nilpotent element. This contradiction establishes the result.

(i)⇒(iv) Suppose f satisfies spp of index n. Let K be the two sided ideal generated by f. We claim that $K^{2n} = 0$. Assume that $K^{2n} \neq 0$. Then, there are elements $\alpha_1, \alpha_2, \ldots, \alpha_{2n} \in K$ with $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_{2n} \neq 0$. It follows that there are elements $u, v \in X$ with $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_{2n}(u,v) \neq 0$. Note that for each i,

$$\alpha_i = \sum_{j=1}^{m_i} \beta_{i,j} f \gamma_{i,j}$$

where $\beta_{i,j}, \gamma_{i,j} \in I(X,T)$. Therefore,

$$\sum_{j=1}^{m_1} \beta_{1,j} f \gamma_{1,j} \sum_{j=1}^{m_2} \beta_{2,j} f \gamma_{2,j} \cdots \sum_{j=1}^{m_{2n}} \beta_{2n,j} f \gamma_{2n,j}(u,v) \neq 0.$$

so, there exists $\beta_{k_t,j_t}, \gamma_{k_t,j_t} \in I(X,T)$ for $1 \leq t \leq n$, such that

$$\beta_{k_1,j_1} f \gamma_{k_1,j_1} \beta_{k_2,j_2} f \gamma_{k_2,j_2} \cdots \beta_{k_{2n},j_{2n}} f \gamma_{k_{2n},j_{2n}}(u,v) \neq 0.$$

Since $f \in Z(I(X,T))$, there exists a chain

$$u \leq u_1 < v_1 \leq u_2 < v_2 \leq \cdots \leq u_{2n} < v_{2n} \leq v$$

in X with $f(u_i, v_i) \neq 0$ for $i = 1, 2, \ldots, 2n$. Note that for each i we have u_i strictly less than v_i because otherwise $f(u_i, v_i) = 0$ as $f \in Z(I(X,T))$. It follows that for

$$u_1 < v_1 < u_3 < v_3 < \cdots < u_{2n-1} < v_{2n-1}$$

which is a chain of length n in X, we have

$$a_1 f(u_1, v_1) a_2 f(u_2, v_2) a_3 \cdots a_n f(u_n, v_n) a_{n+1} \neq 0$$

for some $a_1, a_2, \ldots, a_{n+1} \in T$ which contradicts the fact that f has spp with n. □

Lemma 5.2.9. *Let X be a locally finite partially ordered set, T be a ring with identity. Suppose $f \in I(X,T)$ has spp. Then, for each $g \in I(X,T)$, fg and gf have spp.*

Proof. Suppose f satisfies spp. Let $g \in I(X,T)$. Assume for contradiction that fg does not satisfy spp. Then for each positive integer n, there exists a chain

$$x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$$

in X and elements $a_1, a_2, \ldots, a_{n-1} \in T$ such that

$$(fg)(x_1, y_1)a_1(fg)(x_2, y_2)a_2 \cdots a_{n-1}(fg)(x_n, y_n) \neq 0.$$

Then for each i with $1 \leq i \leq n$ there exists $u_i \in [x_i, y_i]$ such that

$$f(x_1, u_1)g(u_1, y_1)a_1 f(x_2, y_2)g(u_2, y_2)a_2 \cdots a_{n-1} f(x_n, u_n)g(u_n, y_n) \neq 0$$

where $g(u_i, y_i)a_i \in T$ for each i. This contradicts the assumption that f has spp with n. Hence, fg satisfies spp for each $g \in I(X, T)$. Similarly, gf satisfies spp. □

Lemma 5.2.10. *Let T be a ring with identity, X be a locally finite partially ordered set and $f \in I(X, T)$. If fg satisfies spp for each $g \in Z(I(X, T))$ then f satisfies spp.*

Proof. Suppose that fg satisfies spp for each $g \in Z(I(X, T))$ but f does not satisfy spp. Then, for all $n \in \mathbb{Z}^+$, there exists a chain

$$x_{n,1} \leq y_{n,1} < x_{n,2} \leq y_{n,2} < \cdots < x_{n,n} \leq y_{n,n}$$

in X and elements $a_{n,1}, a_{n,2}, \ldots, a_{n,n-1} \in T$ such that

$$f(x_{n,1}; y_{n,1})a_{n,1} f(x_{n,2}; y_{n,2})a_{n,2} \cdots a_{n,n-1} f(x_{n,n}; y_{n,n}) \neq 0 \tag{5.1}$$

Since f does not satisfy spp, by Proposition 5.2.5, X is unbounded. By Lemma 3.1.2, we may select chains so that $[x_{n,1}; y_{n,n}]$ and $[x_{m,1}; y_{m,m}]$ are disjoint for $n \neq m$.

Consider an element $g \in Z(I(X; T))$ defined as follows:

$$g(u, v) = \begin{cases} a_{n,i} & \text{if } u = y_{n,i}, v = x_{n,i+1}, n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then for each $n \in \mathbb{Z}^+$,

$$\begin{aligned}(fg)(x_{n,i}; x_{n,i+1}) &= f(x_{n,i}; x_{n,i})g(x_{n,i}; x_{n,i+1}) + f(x_{n,i}; y_{n,i})g(y_{n,i}; x_{n,i+1}) \\ &\quad + f(x_{n,i}; x_{n,i+1})g(x_{n,i+1}; x_{n,i+1}) \\ &= f(x_{n,i}; y_{n,i})a_{n,i}\end{aligned}$$

Therefore,

$$\begin{aligned}\prod_{i=1}^{2n-1} (fg)(x_{2n,i}; x_{2n,i+1}) &= f(x_{2n,1}; y_{2n,1})a_{2n,1}f(x_{2n,2}; y_{2n,2})a_{2n,2} \\ &\quad \cdots f(x_{2n,2n-1}; y_{2n,2n-1})a_{2n,2n-1} \\ &\neq 0\end{aligned}$$

by (5.1), which contradicts the fact that fg has spp. □

Proposition 5.2.11. *If* $f \in \mathcal{N}^*(I(X,T))$, *then* $f_D \in \mathcal{N}^*(\prod_{x \in X} T)$.

Proof. Suppose $f \in \mathcal{N}^*(I(X,T))$. Then, there exists a nil ideal A of $I(X,T)$ containing f. Consider

$$A_T = \{g_D \in \prod_{x \in X} T \mid g \in A\}$$

We show that A_T is a nil ideal of $\prod_{x \in X} T$ containing f_D. Let $g_D, h_D \in A_T$. Then there exits $g, h \in A$ with $g_D, h_D \in \prod_{x \in X} T$. Since A is an ideal, $g - h \in A$ and so

$$(g-h)_D = g_D - h_D \in \prod_{x \in X} T.$$

Thus $g_D - h_D \in A_T$. Let $(t) \in \prod_{x \in X} T$ and $g_D \in A_T$. Then there exists $g \in A$ with

$g_D \in \prod_{x \in X} T$ and

$$f(x,y) = \begin{cases} t_x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

is an element of $I(X,T)$. So $(fg_D)_D = f_D g_D \in \prod_{x \in X} T$. Hence $(t)g_D \in A_T$. Similarly $g_D(t) \in A_T$ giving that A_T is an ideal. We now check A_T is nil. Take any $g_D \in A_T$. Then there exists $g \in A$ with $g_D \in \prod_{x \in X} T$. Since A is nil, g is nilpotent. So g_D is nilpotent showing that A_T is a nil ideal. □

Proposition 5.2.12. *Suppose T is a ring with identity and X is a locally finite partially ordered set. If $f \in \mathcal{N}^*(I(X,T))$ and $g \in Z(I(X,T))$, then fg has spp.*

Proof. Suppose $f \in \mathcal{N}^*(I(X,T))$ and $g \in Z(I(X,T))$. Then, for all $x \in X$,

$$(fg)_D(x,x) = f(x,x)g(x,x) = 0$$

as $g(x,x) \in Z(I(X,T))$ giving that $fg \in Z(I(X,T))$. To show fg has spp, it is sufficient to check that the left ideal generated by fg is nil (by Lemma 5.2.8).

Consider the left ideal, A, generated by fg. Pick any $x \in A$. Then $x = \sum_{finite} s_i f g_i$ for some $s_i, g_i \in I(X,T)$. Then $x \in \mathcal{N}^*(I(X,T))$, because $f \in \mathcal{N}^*(I(X,T))$ and $\mathcal{N}^*(I(X,T))$ is an ideal. Therefore, x is a nilpotent element, that is, A is a nil ideal. Thus fg satisfies spp. □

Proposition 5.2.13. *Suppose T is a ring with identity and X is a locally finite partially ordered set. Then*

$$A = \{f \in (I(X,T)) \mid f_D \in \mathcal{N}^*(\prod_{x \in X} T) \text{ and } f \text{ has spp } \}$$

is an ideal of $I(X,T)$.

Proof. Let $f, g \in A$. Then $f_D, g_D \in \mathcal{N}^*(\prod_{x\in X} T)$. So, $(f-g)_D = f_D - g_D \in \mathcal{N}^*(\prod_{x\in X} T)$. Since f, g has spp so does $f-g$ and therefore $f-g \in A$. Let $f \in A, g \in I(X,T)$. Then fg satisfies spp by previous lemma. Since $(fg)_D(x,x) = (f_D g_D)(x,x)$ for all $x \in X$, and $f_D \in \mathcal{N}^*(\prod_{x\in X} T)$, we get $(fg)_D \in \mathcal{N}^*(\prod_{x\in X} T)$. So $fg \in A$, that is, A is a right ideal of $I(X,T.)$ Similarly, A is a left ideal of $I(X,T)$. □

Proposition 5.2.14. *Suppose T is a ring with identity and X is a locally finite partially ordered set. If $f \in I(X,T)$ has spp and $f_D \in \mathcal{N}^*(\prod_{x\in X} T)$, then the ideal generated by f_D is a nil ideal in $I(X,T)$.*

Proof. Suppose f has spp and g is an element of the ideal generated by f_D. Then there are elements $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k \in I(X,T)$ with

$$g = \alpha_1 f_D \beta_1 + \alpha_2 f_D \beta_2 + \cdots + \alpha_k f_D \beta_k.$$

Then

$$g_D = \sum_{i=1}^{k} (\alpha_i)_D f_D (\beta_i)_D \in \mathcal{N}^*(\prod_{x\in X} T).$$

We claim that $g_U \in Z(I(X,T))$ has spp. Assume for contradiction that g_U does not have spp. Then for all $n \in \mathbb{Z}^+$ there exists a chain

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_n < v_n$$

in X, and elements $a_{n,1}, a_{n,2}, \ldots, a_{n,n-1} \in T$ with

$$g_U(u_1,v_1)a_{n,1}g_U(u_2,v_2)a_{n,2}\cdots a_{n,n-1}g_U(u_n,v_n) \neq 0.$$

Then there are i_t's with $1 \leq i_t \leq k$ and $1 \leq t \leq n$ such that

$$(\alpha_{i_1} f_D \beta i_1)(u_1,v_1)a_{n,1}(\alpha_{i_2} f_D \beta i_2)(u_2,v_2)a_{n,2}\cdots a_{n,n-1}(\alpha_{i_n} f_D \beta_{i_n})(u_n,v_n) \neq 0.$$

So, there are $u_i', v_i' \in [u_i, v_i]$ with $1 \leq i \leq n$ such that

$$\alpha_{i_1}(u_1, u_1')f_D(u_1', v_1')\beta_{i_1}(v_1', v_1)a_{n,1} \cdots a_{n,n-1}\alpha_{i_n}(u_n, u_n')f_D(u_n', v_n')\beta_{i_n}(v_n', v_n) \neq 0.$$

So f_D does not satisfy spp. This contradicts Proposition 5.2.4. Hence g_U has spp. Also we have $g_U \in Z(I(X,T))$, so, be Lemma 5.2.8, g_U generates a nilpotent ideal. On the other hand, g_D is nilpotent as $g_D \in \mathcal{N}^*(\prod_{x \in X} T)$. So, there exists $m \in \mathbb{Z}^+$ with $(g_D)^m = 0$. It follows that g^m is an element of the ideal generated by g_U ($g^m = (g_U + g_D)^m = {g_U}^m + \cdots + {g_D}^m$) and is thus nilpotent. Hence, g is nilpotent. Thus f_D generates a nil ideal. □

We can now describe the upper nilradical of the incidence algebra $I(X,T)$ in terms of the upper nilradical of $\prod_{x \in X} T$ and the strong product property.

Theorem 5.2.15. *Let T is a ring with identity and X is a locally finite partially ordered set. Then, $f \in \mathcal{N}^*(I(X,T))$ if and only if $f_D \in \mathcal{N}^*(\prod_{x \in X} T)$ and f has spp.*

Proof. (Necessity) Suppose $f \in \mathcal{N}^*(I(X,T))$. By Proposition 5.2.11, we have $f_D \in \mathcal{N}^*(\prod_{x \in X} T)$. We check f satisfies spp. Assume for a contradiction that f does not have spp. Then, for all $n \in \mathbb{Z}^+$, there exists a chain

$$x_{n,1} \leq y_{n,1} < x_{n,2} \leq y_{n,2} < \cdots < x_{n,n} \leq y_{n,n}$$

in X and elements $a_{n,1}, a_{n,2}, \ldots, a_{n,n-1} \in T$ such that

$$f(x_{n,1}; y_{n,1})a_{n,1}f(x_{n,2}; y_{n,2})a_{n,2} \cdots a_{n,n-1}f(x_{n,n}; y_{n,n}) \neq 0$$

It follows that X is unbounded and since X is locally finite we may select chains so that $[x_{n,1}; y_{n,n}]$ and $[x_{m,1}; y_{m,m}]$ are disjoint for $n \neq m$.

Consider an element $g \in Z(I(X;T))$ defined as follows:

$$g(u,v) = \begin{cases} a_{n,i} & \text{if } u = y_{n,i}, v = x_{n,i+1}, n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then, as in the proof of Lemma 5.2.8, fg does not satisfy spp. But this contradicts Proposition 5.2.12 . Hence, f satisfies spp.

(Sufficiency) We have seen before

$$A = \{f \in I(X,T) \mid f_D \in \mathcal{N}^*(\prod_{x \in X} T) \text{ and } f \text{ has spp}\}$$

is an ideal of $I(X,T)$. By the necessity part of the proof we have $\mathcal{N}^*(I(X,T)) \subseteq A$.

Now, let $f \in A$. We first show $f_U \in \mathcal{N}^*(I(X,T))$. Using Theorem 5.2.7 we check $f_U \in \mathcal{N}_*(I(X,T))$. Since f has spp, f_U has spp by Proposition 5.2.4. For all $x \in X$, $(f_U)_D(x,x) = 0$, so $(f_U)_D \in \mathcal{N}_*(\prod_{x \in X} T)$. Hence $f_U \in \mathcal{N}_*(I(X,T)) \subseteq \mathcal{N}^*(I(X,T))$.

In order to show $f_D \in \mathcal{N}^*(I(X,T))$, it is sufficient to show that f_D generates a nil ideal in $(I(X,T))$. This result follows from Proposition 5.2.14. Therefore $f \in \mathcal{N}^*(I(X,T))$ and $A = \mathcal{N}^*(I(X,T))$. □

Proposition 5.2.16. *Let T be a ring with identity and X be a locally finite partially ordered set. Then $t \in \mathcal{N}^*(T) \backslash \mathcal{N}_*(T)$, if and only if $te_{xx} \in \mathcal{N}^*(I(X,T)) \backslash \mathcal{N}_*(I(X,T))$, for all $x \in X$.*

Proof. (Necessity) First we check $te_{xx} \in \mathcal{N}^*(I(X,T))$. Using Theorem 5.2.15, we show te_{xx} has spp and $(te_{xx})_D \in \mathcal{N}^*(\prod_{x \in X} T)$.

Consider any chain of the form $x_1 \leq y_1 < x_2 \leq y_2$ in X. Then

$$te_{xx}(x_1, y_1) T te_{xx}(x_2, y_2) = 0$$

since either $x \neq x_1$ or $x \neq x_2$. So te_{xx} has spp with 2.

Now consider $(te_{xx})_D \in \prod_{x\in X} T$. We check $(te_{xx})_D$ generates nil ideal in $\prod_{x\in X} T$. Let $(s) = \sum_{finite} (\alpha)(te_{xx})_D(\beta)$ be an element of the ideal generated by $(te_{xx})_D$ where $(\alpha),(\beta) \in \prod_{x\in X} T$. Then for any $y \in X$,

$$\begin{aligned}(s)_y &= \sum_{finite} (\alpha)_y((te_{xx})_D)_y(\beta_y) \\ &= \begin{cases} 0 & \text{if } x \neq y \\ (\alpha)_y t(\beta)_y & \text{else}\end{cases}\end{aligned}$$

But $(\alpha)_y t(\beta)_y \in \mathcal{N}^*(T)$ as $t \in \mathcal{N}^*(T)$ and therefore $(s)_y$ is nilpotent. Hence, $(te_{xx})_D \in \mathcal{N}^*(\prod_{x\in X} T)$.

Now we check $(te_{xx})_D \in \prod_{x\in X} T$ is not strongly nilpotent. Assume the converse. Let $(f_0),(f_1),(f_2),\ldots$ be a sequence in $\in \prod_{x\in X} T$ with $(f_0) = (te_{xx})_D$ and $(f_i) = (f_{i-1})(g_{i-1})(f_{i-1})$ for some $(g_{i-1}) \in I(X,T)$, $i = 1,2,\ldots$. So there exists $m \in \mathbb{Z}^+$ such that $(f_m) = 0$. Consider now

$$\begin{aligned}(f_0)_x &= f_0(x,x) &&= te_{xx}(x,x) &&= t \\ (f_1)_x &= f_1(x,x) &&= f_0(x,x)g_0(x,x)f_0(x,x) &&= t(g_0)_x t \\ (f_2)_x &= f_2(x,x) &&= f_1(x,x)g_1(x,x)f_1(x,x) &&= t(g_1)_x t(g_2)_x t(g_1)_x t \\ &\vdots\end{aligned}$$

This implies that

$$(f_0)_x, (f_1)_x, (f_2)_x, \ldots$$

is a sequence in T with $f_0(x,x) = t$ and

$$(f_i)_x = (f_{i-1})_x(g_{i-1})_x(f_{i-1})_x$$

for $i = 1, 2, \ldots$. Also we have $(f_n)_x = 0$ for each $x \in X$ giving that t is strongly nilpotent, a contradiction.

(Sufficiency) Since $te_{xx}(u,v) = \begin{cases} t & \text{if } u = v = x \\ 0 & \text{otherwise} \end{cases}$, the result easily follows. □

Proposition 5.2.17. *Let T is a ring with identity and X is a locally finite partially ordered set. If $I(X,T)$ has a unique nilradical, so does T.*

Proof. Suppose $I(X,T)$ has a unique nilradical but T does not have a unique nilradical. Then, there exists an element $t \in \mathcal{N}^*(T)$ which is not contained in $\mathcal{N}_*(T)$. Let $x \in X$. Then by Proposition 4.2.13, $te_{xx} \in \mathcal{N}^*(I(X,T))/\mathcal{N}_*(I(X,T))$ which contradicts the fact that $I(X,T)$ has a unique nilradical. □

Proposition 5.2.18. *Suppose X is a finite partially ordered set and T is a ring with unity. Then T has a unique nilradical if and only if $I(X,T)$ has unique nilradical.*

Proof. *(Necessity)* Assume T has unique nilradical. We check

$$\mathcal{N}^*(I(X,T)) \subseteq \mathcal{N}_*(I(X,T)).$$

Pick $f \in \mathcal{N}^*(I(X,T))$. By Theorem 5.2.7, we have $f_D \in \mathcal{N}^*(\prod_{x\in X} T)$ and f has spp. In order to show $f \in \mathcal{N}_*(I(X,T))$ we must check $f_D \in \mathcal{N}_*(\prod_{x\in X} T)$. Since $f_D \in \mathcal{N}^*(\prod_{x\in X} T)$, f_D generates a nil ideal in $\prod_{x\in X} T$. This means that $f_D(x,x) = (f_D)_x$ generates a nil ideal in T, for each $x \in X$. It follows that $(f_D)_x \in \mathcal{N}^*(T) = \mathcal{N}_*(T)$, for each $x \in X$. That is to say $(f_D)_x$ is strongly nilpotent, for each $x \in X$.

Now, we show that f_D is strongly nilpotent. Pick a sequence $(t_1), (t_2), \ldots$ in $\prod_{x\in X} T$ with $(t_1) = f_D$ and $(t_i) \in (t_{i-1}) \prod_{x\in X} T(t_{i-1})$ for $i = 1, 2, \ldots$. Then $(t_1)_x, (t_2)_x, \ldots$ is a sequence in T with $(t_1)_x = (f_D)_x$ and $(t_i)_x \in (t_{i-1})_x T(t_{i-1})_x$, for each $x \in X$. As $(f_D)_x$ is strongly nilpotent, for each $x \in X$, there exists $n_x \in \mathbb{Z}^+$ such that $(t_{n_x})_x = 0$. Set $n = max\,\{n_x \,|\, x \in X\}$. Then $(t_n)_x = 0$, for each $x \in X$, giving that $(t_n) = 0$. It follows that f_D is strongly nilpotent.

(Sufficiency) Follows from Proposition 5.2.17. □

The converse of the Proposition 5.2.17 is still an open problem.

Question Does $I(X,T)$ have a unique nilradical if T has a unique nilradical?

6. THE PERIODIC RADICAL OF INCIDENCE ALGEBRAS

In this chapter, we first introduce the notion of periodic radical. Secondly, we will determine the necessary and sufficient conditions for an element to belong to the periodic radical of an incidence algebra over a commutative ring with unity.

6.1. The Periodic Radical

Definition Let T be a ring. An element x in T is called *periodic* if there exists positive integers m, n with $m \neq n$ such that $x^m = x^n$. A ring consisting of periodic elements is called a *periodic ring.*

Proposition 6.1.1. *Let x belong to a ring T.*

(i) x is periodic if and only if x^n is an idempotent for some positive integer n.

(ii) If x is periodic and T has no nonzero nilpotent elements, then $x^n = x$ for some integer n with $n \geq 2$.

Proof. *(i)* Let x be periodic. Say $x^m = x^n$ with $d = m - n > 0$. Then inductively we have $x^n = x^{n+sd}$ for all $s \geq 1$ because

$$\begin{aligned} x^n &= x^{n+(m-n)} \\ &= x^n x^d \\ &= x^{n+d} x^d \\ &= x^n x^{2d} \\ &\cdots \\ &= x^{n+sd}. \end{aligned}$$

Hence $x^n = x^{2n+r}$ for some $r \geq 0$, in which case x^{n+r} is an idempotent as

$$x^{2(n+r)} = x^{2n+2r} = x^{2n+r}x^r = x^{n+r}.$$

It follows that x^{n+r} is idempotent. Conversely, if x is idempotent, then x is obviously periodic.

(ii) Let x be periodic and T has no nonzero nilpotent elements. So $x^m = x^n$ for some $m, n \in \mathbb{Z}^+$ with $m \neq n$. Say $m > n$. It follows that $(x^{m-n+1} - x)$ is nilpotent as

$$\begin{aligned}
(x^{m-n+1} - x)^{n+1} &= (x^{m-n+1} - x)(x^{m-n+1} - x)\cdots(x^{m-n+1} - x) \\
&= (x^{m-n+1} - x)x^{n-1}(x^{mn-n^2+1} - \cdots + (-1)^n x) \\
&= (x^m - x^n)(x^{mn-n^2+1} - \cdots + (-1)^n x) \\
&= 0
\end{aligned}$$

By assumption T has no non-zero nilpotent elements, therefore, $x^{m-n+1} - x = 0$. This implies that $x^{m-n+1} = x$ where $m - n + 1 \geq 2$. □

Theorem 6.1.2. *Let T be a ring; and suppose that for all $t \in T$, there exists a positive integer $n = n_t$ and a polynomial $p(x) = p_t(x) \in \mathbb{Z}[x]$ such that $t^n = t^{n+1}p(t)$. Then T is periodic.*

Proof. Pick any $t \in T$. We identify the ring $t\mathbb{Z}[t]$ generated by t with K. Choose $n \in \mathbb{Z}^+$ and $\mathrm{p}(x) \in \mathbb{Z}^+[x]$ such that $t^n = t^{n+1}p(t)$. Then $t - t^2\mathrm{p}(t) \in Ann(t^{n-1})$ as

$$(t - t^2\mathrm{p}(t))t^{n-1} = t^n - t^{n+1}\mathrm{p}(t) = 0.$$

Let $\overline{K} = K/Ann(t^{n-1})$ and $\bar{t}$ be the canonical image of t in $\overline{K}$. We have $\bar{t} = \overline{t^2}\mathrm{p}(\bar{t})$ and the element $\bar{e} = \bar{t}\mathrm{p}(\bar{t})$ is an idempotent as

$$\bar{e}^2 = \bar{t}\mathrm{p}(\bar{t})\bar{t}\mathrm{p}(\bar{t}) = \overline{t^2}\mathrm{p}(\bar{t})\mathrm{p}(\bar{t}) = \bar{t}\mathrm{p}(\bar{t}) = \bar{e}.$$

In addition, we have $\bar{t} = \bar{t}\bar{e}$.

Now, if $\overline{e} = \overline{0}$, then $\overline{t} = \overline{0}$, that is, $tt^{n-1} = 0$, that is, $t^n = 0$ giving that t is periodic.

Suppose $\overline{e}$ has infinite additive order in $\overline{K}$. Define

$$\begin{array}{cccc} \varphi : & \mathbb{Z} & \rightarrow & \mathbb{Z}\overline{e} \leq \overline{K} \\ & m & \mapsto & m\overline{e} \end{array}$$

Then φ is obviously onto. In addition, φ is one-to-one as if $m_1\overline{e} = m_2\overline{e}$, then we have $(m_1 - m_2)\overline{e} = \overline{0}$ and since $\overline{e}$ has infinite additive order $m_1 - m_2 = 0$ giving that $m_1 = m_2$. On the other hand for any $m_1, m_2 \in \mathbb{Z}$,

$$\varphi(m_1 + m_2) = (m_1 + m_2)\overline{e} = m_1\overline{e} + m_2\overline{e} = \varphi(m_1) + \varphi(m_2)$$

and

$$\varphi(m_1m_2) = (m_1m_2)\overline{e} = (m_1m_2)\overline{e}^2 = (m_1\overline{e})(m_2\overline{e}) = \varphi(m_1)\varphi(m_2).$$

Hence φ is a ring isomorphism. This implies that $\overline{K}$ contains an isomorphic copy of $\mathbb{Z}$. Note that, $\overline{K}$ satisfies our original hypothesis which yields a contradiction as $\mathbb{Z}$ does not satisfy the hypothesis. Thus, $\overline{e}$ has finite additive order, and so does $\overline{t}$.

Suppose m is the additive order of $\overline{t}$. Then $m\overline{K} = \overline{0}$, as $\overline{K}$ is generated by $\overline{t}$. Let $\overline{N}$ be the set of all nilpotent elements of $\overline{K}$. Then $\overline{N}$ is an ideal of $\overline{K}$. Now consider the factor ring $\tilde{K} = \overline{K}/\overline{N}$. If $\tilde{k} \in \tilde{K}$, then we claim that $\tilde{k}$ is of square-free order. Suppose not. Let n^2 be the additive order of $\tilde{k}$. Then $n^2\tilde{k} = \tilde{0}$ implies $n^2\tilde{k}^2 = \tilde{0}$, that is, $(n\tilde{k})^2 = \tilde{0}$. But since $\tilde{K} = \overline{K}/\overline{N}$ does not contain any non-zero nilpotent elements, we get $n\tilde{k} = \tilde{0}$. Hence, $\tilde{K}$ has all of its elements of square-free order. Moreover, p^2q cannot be order of an element in $\tilde{K}$ because otherwise if $p^2q\tilde{k} = \tilde{0}$, then $qp^2q\tilde{k}\tilde{k} = (pq\tilde{k})^2 = \tilde{0}$ and $pq\tilde{k} = \tilde{0}$ as $\tilde{K}$ contains no nontrivial nilpotent elements. Let $p_1p_2\cdots p_s\tilde{k} = \tilde{0}$ for some primes $p_1, p_2, \ldots, p_s \in \mathbb{Z}^+$, for some $\tilde{k} \in \tilde{K}$. Then $\tilde{k}$ can be written as $\sum_{i=1}^{s} p_1p_2\cdots\hat{p_i}\cdots p_sa_i\tilde{k}$ where $\sum_{i=1}^{s} p_1p_2\cdots\hat{p_i}\cdots p_sa_i = 1$ by Euclidean algorithm as $p_1, p_2, \ldots, p_s$ are primes where $\hat{p_i}$ denotes that p_i is not in the multiplication of p_j's. Note that each $p_1p_2\cdots\hat{p_i}\cdots p_sa_i\tilde{k}$ generates an ideal $\tilde{I_i}$ of characteristic p_i, therefore, $\tilde{I_i}\cong\mathbb{Z}_{p_i}$ for each $1\leq i\leq s$. Also, since we have $m\tilde{K}=\tilde{0}$, there exist only finitely many $\tilde{I_i}$'s. On the other hand, for any $i\neq j$, we have $\tilde{I_i}\cap\tilde{I_j}=\tilde{0}$ as if $\tilde{x}\in\tilde{I_i}\cap\tilde{I_j}$, then $p_i\tilde{x}=p_j\tilde{x}=\tilde{0}$ for some primes p_i,p_j and $1=ap_i+bp_j$ for some $a,b\in\mathbb{Z}$ yields $\tilde{x}=ap_i\tilde{x}+bp_j\tilde{x}=\tilde{0}$. Thus, $\tilde{K}=\tilde{I_1}\oplus\tilde{I_2}\oplus\cdots\oplus\tilde{I_n}$. It follows that $\tilde{t}$ generates a finite ring, so there exist distinct $n_1,n_2\in\mathbb{Z}^+$ satisfying $\tilde{t}^{n_1}=\tilde{t}^{n_2}$, that is, $\bar{t}^{n_1}-\bar{t}^{n_2}\in\overline{N}$. But this forces $\bar{t}$ to be algebraic over $\mathbb{Z}$, so that $\bar{t}$ generates a finite subring of $\overline{K}$. Consequently, there exists $j,k\in\mathbb{Z}^+$ such that $\bar{t}^j=\bar{t}^k$, that is, $t^j-t^k\in Ann(t^{n-1})$ or $t^{j+n-1}=t^{k+n-1}$. Thus t is periodic. □

We shall check now that the periodicity is a radical property.

Lemma 6.1.3. *Let T be a ring and I_1,I_2 be periodic ideals of T. Then I_1+I_2 is periodic.*

Proof. Suppose I_1,I_2 are periodic ideals of a ring T. By the second isomorphism theorem, we have $(I_1+I_2)/I_1\cong I_2/(I_1\cap I_2)$. So $(I_1+I_2)/I_1$ is periodic. Therefore, for all $a\in I_1+I_2$, there exists $m,n\in\mathbb{Z}^+$, $m\neq n$ such that $a^m-a^n\in I_1$. By assumption, I_1 is also periodic, so, there exists $k,j\in\mathbb{Z}^+$, $k\neq j$ such that $(a^n-a^m)^j=(a^n-a^m)^k$. Without lost of generality, suppose $j<k$ and $n<m$. Then

$$(a^n-a^m)^j=(a^n-a^m)^k$$

yields

$$a^{nj}-\cdots+(-1)^ja^{mj} \quad = \quad a^{nk}-\cdots+(-1)^ka^{mk}.$$

Then, it follows that

$$\begin{aligned} a^{nj} &= a^{nk}-\cdots+(-1)^ka^{mk}+\cdots+(-1)^ja^{mj} \\ &= a^{nj+1}p(a) \end{aligned}$$

where $p(x)$ is a polynomial in $\mathbb{Z}[x]$. Thus, $I_1 + I_2$ is periodic by Theorem 6.1.2. □

Corollary 6.1.4. *For a ring T, the sum of all periodic ideals is periodic.*

Proof. Let $P(T)$ be the sum of all periodic ideals of T and $x \in P$. Then

$$x \in I_1 + I_2 + \cdots + I_n$$

for some periodic ideals $I_1, I_2, \ldots, I_n$ and hence x is periodic by the previous lemma. □

Lemma 6.1.5. *Let T be a ring and $P(T)$ be the sum of periodic ideals of T. Then $T/P(T)$ contains no nonzero periodic ideals.*

Proof. If $I/P(T)$ is a nonzero periodic ideal of $T/P(T)$, then $I + P(T)$ is a periodic ideal containing $P(T)$ which is a contradiction as $P(T)$ is the sum of periodic ideals of T. □

Obviously, a homomorphic image of a periodic ring is periodic. Hence, we have

Corollary 6.1.6. *Periodicity is a radical property.*

Definition The *periodic radical* of a ring T, denoted by $\mathcal{P}(T)$, is the sum of all the periodic ideals of T.

The next result describes an important relationship among the periodic radical $\mathcal{P}(T)$, the Jacobson radical $\mathcal{J}(T)$ and the upper nilradical $\mathcal{N}^*(T)$.

Proposition 6.1.7. *For any ring with identity T, we have $\mathcal{P}(T) \cap \mathcal{J}(T) = \mathcal{N}^*(T)$.*

Proof. Suppose $x \in \mathcal{N}^*(T)$. Then there exists a positive integer n such that $x^n = 0$. It follows that $x^n = x^{2n} = 0$ and x is periodic. We now check that $x \in \mathcal{J}(T)$. Take any

$t \in T$. Then $tx \in \mathcal{N}^*(T)$ is nilpotent, say with integer k. Then we get

$$(1 + tx + (tx)^2 + \cdots + (tx)^{k-1})(1 - tx) = 1 - (tx)^k = 1$$

giving that $1 - tx$ is left invertible, that is, $x \in \mathcal{J}(T)$. Hence $\mathcal{N}^*(T) \subseteq \mathcal{P}(T) \cap \mathcal{J}(T)$.

Conversely, suppose $x \in \mathcal{P}(T) \cap \mathcal{J}(T)$. Since $x \in \mathcal{P}(T)$, x^n is an idempotent for some integer n. But then $x^n \in \mathcal{J}(T)$ forces $x^n = 0$. If $x^n \neq 0$, then by Proposition 2.0.5, $T = T(1-x^n) \oplus Tx^n$ as x^n is an idempotent. We have $T(1-x^n)$ is an ideal of T, so, by Proposition 2.0.3, contained in a maximal ideal M, say. Then, $(1-x^n) \in M$. On the other hand, $\mathcal{J}(T)$ is the intersection of all maximal left ideals of T and $x^n \in \mathcal{J}(T)$ yields $x^n \in M$. Hence, we have $x^n, 1 - x^n \in M$, that is, $1 \in M$, a contradiction. Thus, $\mathcal{P}(T) \cap \mathcal{J}(T) \subseteq \mathcal{N}^*(T)$. □

If T is a ring with identity, then the periodic radical of T is an intersection of some suitable prime ideals as the following theorem states.

Theorem 6.1.8. *Let T be a ring with identity. Then $\mathcal{P}(T) = \bigcap_{\alpha} P_\alpha$, where the intersection is taken over the set of prime ideals P_α such that T/P_α contains no nontrivial periodic ideals and such that if an integer z is a non-zero divisor in T, then it is still a non-zero divisor in T/P_α.*

If there are no prime ideals P_α such that T/P_α contains no nontrivial periodic ideals, we say that the intersection is T.

Proof. If $\mathcal{P}(T) = T$ the result is obviously correct. Suppose then that $\mathcal{P}(T) \neq T$. Let P_α be a prime ideal of T such that T/P_α contains no nontrivial periodic ideals and such that if $z \in \mathbb{Z}^+$ is a nonzero divisor in T, then it is still a nonzero divisor in $T/P(T)$. If $\mathcal{P}(T) \not\subseteq P_\alpha$, then

$$\frac{P_\alpha + \mathcal{P}(T)}{P_\alpha} = \{x + P_\alpha \,|\, x \in P_\alpha + \mathcal{P}(T)\} = \{y + P_\alpha \,|\, y \in \mathcal{P}(T)\}$$

is a nontrivial periodic ideal of T/P_α, which is a contradiction. Thus, $\mathcal{P}(T) \subseteq P_\alpha$, and hence $\mathcal{P}(T) \subseteq \bigcap_{\alpha} P_\alpha$.

On the other hand, for any $a \in T - \mathcal{P}(T)$, the ideal (a) generated by a is not periodic. Thus, by the Theorem 6.1.2, there exists an element $b \in (a)$ such that $b^n - b^{n+1}p(b) \neq 0$, for all $n \in \mathbb{Z}^+$ and for all $p(x) \in \mathbb{Z}[x]$. Let

$$H = \{z(b^n - b^{n+1}p(b)) \mid n \in \mathbb{Z}^+,\ p(x) \in \mathbb{Z}[x],\ z \in \mathbb{Z}^+ \text{ non-zero divisor in } T\}$$

and $\mathcal{A}$ be the set of all ideals P in T with $P \cap H = \emptyset$. Then $\mathcal{A} \neq \emptyset$ since $0 \in \mathcal{A}$, so there exists a maximal element P_β in $\mathcal{A}$ by Zorn's lemma.

We claim that P_α is a prime ideal in T. Let A and B be ideals in T such that $A \not\subseteq P_\beta$ and $B \not\subseteq P_\beta$. Then, both $A + P_\beta$ and $B + P_\beta$ intersect with H, say $z_1(b^m - b^{m+1}f(b)) \in A + P_\beta$ and $z_2(b^n - b^{n+1}g(b)) \in B + P_\beta$ for some $m, n \in \mathbb{Z}^+$, $f(x), g(x) \in \mathbb{Z}[x]$ and non-zero divisors z_1, z_2 in T. Then

$$\begin{aligned} z_1z_2(b^{m+n} - b^{m+n+1}h(b)) &= z_1(b^m - b^{m+1}f(b))z_2(b^n - n^{n+1}g(b)) \\ &\in (A + P_\beta)(B + P_\beta) \\ &\subseteq AB + P_\beta \end{aligned}$$

where $h(x) = f(x) + g(x) - xf(x)g(x)$. But $z_1z_2(b^{m+n} - b^{m+n+1}h(b)) \notin P_\beta$, hence $AB \notin P_\beta$ giving that P_β is prime.

Next, we prove that T/P_β contains no nontrivial periodic ideals. Let $I \supset P_\beta$ be an ideal of T and I/P_β be a nontrivial periodic ideal of T/P_β. Then, by the maximality of P_β, there exists an integer $m \in \mathbb{Z}^+$, a polynomial $f(x) \in \mathbb{Z}[x]$ and $z \in \mathbb{Z}^+$, a non-zero divisor in T such that $z(b^m - b^{m+1}f(b)) \in I$, so there exists distinct positive integers s and t with $s < t$ such that

$$(z(b^m - b^{m+1}f(b)) + P_\beta)^s = (z(b^m - b^{m+1}f(b)) + P_\beta)^t$$

and therefore

$$z^s(b^m - b^{m+1}f(b))^s - z^t(b^m - b^{m+1}f(b))^t \in P_\beta$$

which contradicts the choice of P_β since

$$z^s(b^m - b^{m+1}f(b))^s - z^t(b^m - b^{m+1}f(b))^t$$

can be written in the form

$$z^s(b^{ms} - b^{ms+1})h'(b)$$

where $h'(x) \in \mathbb{Z}[x]$ and z^s a non-zero divisor in T. Then T/P_β contains no nontrivial periodic ideals.

We must also check that if z is a non-zero divisor in T, then z is also a non-zero divisor in T/P_β. Suppose z is a non-zero divisor in T. Let $z\bar{t} = \bar{0}$ for some $\bar{t} \in T/P_\beta$. We check $\bar{t} = \bar{0}$. Consider the ideal $(z1_T)$ generated by $z1_T$ and the ideal (t) generated by t. We have $(z1_T)(t) \subseteq P_\beta$ as $zt \in P_\beta$. Since P_β prime, either $(z1_T) \subseteq P_\beta$ or $(t) \subseteq P_\beta$. If $(z1_T) \subseteq P_\beta$, then $z1_T \in P_\beta$. This implies that $(z1_T)(b^n - b^{n+1}p(b)) = z(b^n - b^{n+1}p(b)) \in P_\beta$ for any $p(x) \in \mathbb{Z}[x]$ and $n \in \mathbb{Z}^+$. This yields a contradiction as $H \cap P_\beta = \emptyset$. Hence, $(t) \subseteq P_\beta$ and therefore $t \in P_\beta$. Thus, $\bar{t} = \bar{0}$ and z is a non-zero divisor in T/P_β.

Since $a \notin P_\beta$, we have $a \notin \bigcap P_\alpha$ where the intersection is taken over the set of prime ideals P_α such that T/P_α contains no nontrivial periodic ideals and such that if an integer z is a non-zero divisor in T, then it is still a non-zero divisor in T/P_α. Thus, $\bigcap P_\alpha \subseteq \mathcal{P}(T)$ which completes the proof. $\square$

6.2. The Periodic Radical of $I(X,R)$

Let $x \in \mathcal{P}(R)$. For any $y \in R$, define $e_x(y)$ to be the smallest positive integer such that $(xy)^{e_x(y)}$ is an idempotent. Then define e_x as follows:

$$e_x = \begin{cases} max\{e_x(y) \mid y \in R\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

Proposition 6.2.1. *Assume R is a commutative ring and $A = \prod_{i \in I} R_i$, with $R_i \cong R$, for all $i \in I$. Let $a = (a_i)_{i\in I} \in A$. Then $a \in \mathcal{P}(A)$ if and only if the following conditions all hold:*

(i) $a_i \in \mathcal{P}(R)$, for all $i \in I$

(ii) $|\{i \mid e_{a_i} = \infty\}| < \infty$

(iii) There exists $N \in \mathbb{Z}^+$ such that whenever $e_{a_i} < \infty$, then $e_{a_i} < N$, for all $i \in I$.

Proof. Suppose $a \in \mathcal{P}(A)$. Obviously, $a_i \in \mathcal{P}(R)$ for all $i \in I$. If either *(ii)* or *(iii)* fails to hold, then we can find a subset $\{i_1, i_2, \ldots\}$ of I and elements $b_{i_1}, b_{i_2}, \ldots$ of R such that

$$e_{a_{i_1}}(b_{i_1}) < e_{a_{i_2}}(b_{i_2}) < \cdots$$

Consider $c = (c_i)_{i\in I} \in A$ such that $c_k = b_{i_j}$ if $k = i_j$ for some j and $c_k = 0$ otherwise. Then $ac \in \mathcal{P}(A)$. So, there exists $n \in \mathbb{Z}^+$ such that $(ac)^n = (ac)^{2n}$. This means that $(a_ic_i)^n = (a_ic_i)^{2n}$ for all i which contradicts the fact that $e_{a_{i_j}}(b_{i_j}) > n$ for some j.

Conversely, suppose that *(i)*, *(ii)* and *(iii)* hold. First observe that if x^n is an idempotent, then so is x^{mn}, for all $n \geq 1$ because $x^{2nm} = (x^{2n})^m = (x^n)^m = x^{mn}$. Now let $i_1, \ldots, i_k$ be indices such that $e_{a_{i_j}} = \infty$ for $1 \leq j \leq k$. Let $n_1, n_2, \ldots, n_k$ be positive integers such that ${a_{i_j}}^{n_j}$ is an idempotent. Put $t = n_1 \cdot n_2 \cdots n_k$. Then ${a_{i_j}}^t$ is an idempotent for $1 \leq j \leq k$. Now consider a_i's where $i \in I\backslash\{i_1, \ldots, i_k\}$. By *(iii)*, we have $e_{a_i} < N$ for some $N \in \mathbb{Z}^+$. In particular, for all $m_i \in \mathbb{Z}^+$ satisfying ${a_i}^{m_i}$ is an idempotent, we have $m_i < N$. Therefore, there are at most N many distinct m_i's. Let s be the multiplication of m_i's. Then, ${a_i}^s$ is an idempotent for $i \in I \backslash \{i_1, \ldots, i_k\}$. Hence, a^{st} is an idempotent, that is, $a \in \mathcal{P}(A)$. □

Theorem 6.2.2. *Suppose R is a commutative ring with unity and X is a locally finite partially ordered set. Then the Jacobson radical of the incidence algebra $I(X, R)$, namely $\mathfrak{J}(I(X, R))$, is the set of all functions $f \in I(X, R)$ such that $f(x, x) \in \mathfrak{J}(R)$, for all $x \in X$.*

Proof. By Lemma 4.0.10, $f \in \mathfrak{J}(I(X, R))$ if and only if $\delta - fg$ is (left) invertible for all $g \in I(X, R)$. But, by Theorem 3.2.2, $\delta - fg$ is invertible if and only if $1 - f(x, x) \cdot g(x, x)$ is a unit in R, for all $x \in X$ and $g \in I(X, R)$. But this holds if and only if $f(x, x) \in \mathfrak{J}(R)$ for all $x \in X$.

□

Definition Let R be a commutative ring with identity. An element $f \in I(X, R)$ is called *fully-periodic* if the following conditions are satisfied:

(i) $f_D \in \mathcal{P}(\prod_{x \in X} R)$
(ii) There exists a positive integer n such that if

$$x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$$

in X, then $\prod_{i=1}^{n} f(x_i, y_i) = 0$.

Proposition 6.2.3. *Fully-nilpotent elements are fully-periodic.*

Proof. Suppose $f \in I(X, R)$ is fully-nilpotent. This means that, there exists a positive integer n such that given any chain

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_n \leq y_n$$

in X, $\prod_{i=1}^{n} f(x_i, y_i) = 0$. We check f satisfies conditions of the definition of fully-periodicity. Obviously, *(ii)* holds. We show $f_D \in \mathcal{P}(\prod_{x \in X} R)$ by satisfying conditions of the Theorem 6.2.1.

(i) Since f is fully-nilpotent, say with integer n, we have $f^n(x,x) = 0$, for all $x \in X$, that is, $f^{2n}(x,x) = (f(x,x))^{2n} = (f(x,x))^n = f^n(x,x) = 0$ giving that $f^n(x,x) = (f(x,x))^n$ is an idempotent. Hence, $f(x,x) \in \mathcal{P}(R)$, for all $x \in X$.

(ii) Let $e_{f(x,x)}(r) = n_x$ where n_x is the smallest positive integer so that $(rf(x,x))^{n_x}$ is an idempotent. Since

$$f^{2n}(x,x) = f^n(x,x) = 0$$

we have

$$(rf(x,x))^n = (rf(x,x))^{2n} = 0,$$

so, $n_x \leq n$ for all $x \in X$. Hence $e_{f(x,x)} \neq \infty$ for all $x \in X$, that is,

$$|\{x \mid e_{f(x,x)} = \infty\}| = 0$$

(iii) By above, $e_{f(x,x)} = n_x \leq n$ for all x.

Hence, we conclude that f is fully-periodic. □

Proposition 6.2.4. *If f is fully-periodic, then f_U is fully-nilpotent.*

Proof. Suppose f is fully-periodic. Then by *(ii)* of the definition of fully-periodicity, there exists $n \in \mathbb{Z}^+$ such that given any chain

$$x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$$

in X, $\prod_{i=1}^{n} f(x_i, y_i) = 0$.

We claim that f_U is fully-nilpotent with integer $2n$. Let

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_{2n} \leq y_{2n}$$

be a chain in X. For

$$x_1 = y_1 \leq x_2 = y_2 \leq \cdots \leq x_{2n} = y_{2n}$$

we have

$$\prod_{i=1}^{2n} f_U(x_i, y_i) = 0$$

as $f_U(x, x) = 0$ for all $x \in X$. Hence, it is enough to check when the given chain is of the form

$$x_1 < y_1 \leq x_2 < y_2 \leq \cdots \leq x_{2n} < y_{2n}.$$

We want to show that

$$\prod_{i=1}^{2n} f_U(x_i, y_i) = 0$$

Consider the subchain

$$x_1 < y_1 < x_3 < y_3 < \cdots < x_{2n-1} < y_{2n-1}.$$

If we reindex this chain by $x_{2n-1} = u_n$ and $y_{2n-1} = v_n$, then we have

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_n < v_n$$

and since f is fully-periodic with n, we get,

$$\prod_{i=1}^{n} f_U(u_i, v_i) = \prod_{i=1}^{n} f(u_i, v_i) = 0,$$

that is,

$$\prod_{i=1}^{2n-1} f_U(x_{2i-1}, y_{2i-1}) = 0.$$

Hence,

$$\prod_{i=1}^{2n} f_U(x_i, y_i) = 0$$

as

$$\prod_{i=1}^{2n-1} f_U(x_{2i-1}, y_{2i-1})$$

is a factor of

$$\prod_{i=1}^{2n} f_U(x_i, y_i).$$

□

Theorem 6.2.5. *If R is a commutative ring with identity, then $\mathcal{P}(I(X,R))$ is precisely the set of fully-periodic elements of $I(X,R)$.*

Proof. Let

$$K = \{ f \in I(X, R) \,|\, f \,\text{fully-periodic}\ \}\,.$$

Then K is an ideal of $I(X, R)$:

Let $f, g \in K$. We check $f + g \in K$.

(i) $f_D, g_D \in \mathcal{P}(\prod_{x\in X} R)$ implies $f_D + g_D \in \mathcal{P}(\prod_{x\in X} R)$ as $\mathcal{P}(\prod_{x\in X} R)$ is an ideal of $\prod_{x\in X} R$.

(ii) Suppose f satisfies the condition *(ii)* of the definition of fully-periodicity with n and g satisfies the condition *(ii)* with m. Then $f + g$ satisfies the condition *(ii)* with $m + n$ since if

$$x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$$

is a chain in X, then

$$\prod_{i=1}^{n+m} (f + g)(x_i, y_i) = \prod_{i=1}^{n+m} f(x_i, y_i) + \prod_{i=1}^{n+m} g(x_i, y_i) = 0 + 0 = 0$$

Hence, $f + g \in K$.

Suppose now $f \in K$, $h \in I(X, R)$. We check that $fg \in K$.

(i)$f_D \in \mathcal{P}(\prod_{x\in X} R)$, so $f_D g_D = (fg)_D \in \mathcal{P}(\prod_{x\in X} R)$ as $\mathcal{P}(\prod_{x\in X} R)$ is an ideal of $\prod_{x\in X} R$.

(ii) Suppose $f \in K$ satisfies condition *(ii)* of the definition of fully-periodicity with n. Then fh satisfies condition *(ii)* with n as if

$$x_1 \leq y_1 < x_2 \leq y_2 < \cdots < x_n \leq y_n$$

is a chain in X, then

$$\begin{aligned}\prod_{i=1}^{n}(fh)(x_i,y_i) &= (fg)(x_1,y_1)\cdots(fh)(x_n,y_n)\\ &= \left(\sum_{x_1\le z_1\le y_1} f(x_1,z_1)h(z_1,y_1)\right)\cdots\left(\sum_{x_n\le z_n\le y_n} f(x_n,z_n)h(z_n,y_n)\right)\\ &= \sum f(x_1,z_1)\cdots f(x_n,z_n)h(z_1,y_1)\cdots h(z_n,y_n)\\ &= 0\end{aligned}$$

Therefore $fh \in K$. Similarly $hf \in K$ and K is an ideal of $I(X,R)$.

Let $f \in K$. Then $f_D \in \mathcal{P}(\prod_{x\in X} R)$, therefore, there exists positive integers m,n with $m \neq n$ such that ${f_D}^m = {f_D}^n$. So

$$f^m - f^n = (f^m - f^n)_D + (f^m - f^n)_U = (f^m - f^n)_U.$$

Since $f \in K$ and K is an ideal of $I(X,R)$, we have $f^m - f^n \in K$. This means that $f^m - f^n$ is fully-periodic. By the previous proposition, $(f^m - f^n)_U$ is fully-nilpotent, therefore, $(f^m - f^n)_U \in \mathcal{N}^*(I(X,R)) = \mathcal{N}_*(I(X,R))$ which consists of fully-nilpotent elements of $I(X,R)$. Then

$$\begin{aligned}\overline{K} = K/\mathcal{N}^*(I(X,R)) &= \{f + \mathcal{N}^*(I(X,R)) \mid f \text{ fully-periodic }\}\\ &= \{f_D + \mathcal{N}^*(I(X,R)) \mid f \in I(X,R) \text{ fully-periodic}\}\end{aligned}$$

It follows that

$$\begin{aligned}\overline{K} &\subseteq \{f_D + \mathcal{N}^*(I(X,R)) \mid f \in I(X,R) \text{ and } f_D \in \mathcal{P}(\prod_{x\in X} R)\ \}\\ &\subseteq \{f_D + \mathcal{N}^*(I(X,R)) \mid f \in I(X,R) \text{ and } f_D^n \text{ is an idempotent for some } n \in \mathbb{Z}^+\}\\ &= \{f_D + \mathcal{N}^*(I(X,R)) \mid f \in I(X,R) \text{ and } f_D \text{ is periodic }\}\end{aligned}$$

Therefore,

$$\overline{K} \subseteq \mathcal{P}(I(X,R)/\mathcal{N}^*(I(X,R))) = \{\overline{f} \mid \overline{f^m} = \overline{f^{2m}} \text{ for some } m \in \mathbb{Z}^+ \}$$

Claim. $(\overline{K} \subseteq)\, \mathcal{P}(I(X,R)/\mathcal{N}^*(I(X,R))) = \mathcal{P}(I(X,R))/\mathcal{N}^*(I(X,R))$

Let $\overline{f} = f + \mathcal{N}^*(I(X,R)) \in \mathcal{P}(I(X,R))/\mathcal{N}^*(I(X,R))$. Then there exists a positive integer n such that $f^n = f^{2n}$. So $\overline{f}^n = \overline{f}^{2n}$, that is, $\overline{f} \in \mathcal{P}(I(X,R)/\mathcal{N}^*(I(X,R)))$.

Now, suppose $\overline{f} \in \mathcal{P}(I(X,R)/\mathcal{N}^*(I(X,R)))$. Then there exists a positive integer m such that $\overline{f}^m$ is an idempotent, that is, $f^m - f^{2m} \in \mathcal{N}^*(I(X,R))$. Since $\mathcal{N}^*(I(X,R))$ consists of nilpotent elements, there exists a positive integer t such that $(f^m - f^{2m})^t = 0$. Then

$$f^{mt} - \cdots + (-1)^t f^{2mt} = 0.$$

It follows that $f^{mt} = f^{mt+1}p(f)$ for some polynomial $p(x) \in \mathbb{Z}[x]$. By Theorem 6.1.2, f is periodic. Hence, $\overline{f} \in \mathcal{P}(I(X,R))/\mathcal{N}^*(I(X,R))$.

Hence, we get $K \subseteq \mathcal{P}(I(X,R))$.

Conversely, assume $f \in \mathcal{P}(I(X,R))$ is not fully-periodic. Since the condition *(i)* of definition of fully-periodicity is clearly satisfied, the condition *(ii)* fails to hold, thus, for all $n \in \mathbb{Z}^+$, there exists a chain

$$x_{n,1} \leq y_{n,1} < x_{n,2} \leq y_{n,2} < \cdots < x_{n,n} \leq y_{n,n}$$

such that

$$\prod_{i=1}^{n} f(x_{n,i}; y_{n,i}) \neq 0.$$

Since X is locally finite, using Lemma 3.1.2, we may assume the intervals $[x_{n,1}; y_{n,n}]$ and $[x_{m,1}; y_{m,m}]$ are disjoint for $m \neq n$.

Define an element $h \in I(X, R)$ as follows.

$$\begin{array}{ll} h(y_{n,i}; x_{n,i+1}) = 1 & \text{for } i = 1, 2, \ldots, n-1 \text{ and } n \geq 2 \\ h(x, y) = 0 & \text{in all other cases.} \end{array}$$

Then $(fh)_D = 0$ because $h(x, x) = 0$ for all $x \in X$. So, $(fh)(x, x) = 0 \in \mathcal{J}(R)$ for all $x \in X$ giving that $fh \in \mathcal{J}(I(X, R))$. Since f is chosen from $\mathcal{P}(I(X, R))$, we get $fh \in \mathcal{P}(I(X, R)) \cap \mathcal{J}(I(X, R)) = \mathcal{N}^*(I(X, R))$. This means that fh is fully-nilpotent. Now consider chains

$$x_{n,1} < x_{n,2} \leq x_{n,2} < x_{n,3} \leq x_{n,3} < \cdots < x_{n,n-1} \leq x_{n,n-1} < x_{n,n}$$

for each $n \in \mathbb{Z}^+$. Then

$$\begin{aligned} \prod_{i=1}^{n-1} (fh)(x_{n,i}; x_{n,i+1}) &= f(x_{n,1}; y_{n,1}) h(y_{n,1}; x_{n,2}) \cdots f(x_{n,n-1}; y_{n,n-1}) h(y_{n,n-1}; x_{n,n}) \\ &= f(x_{n,1}; y_{n,1}) \cdots f(x_{n,n-1}; y_{n,n-1}) \\ &\neq 0 \end{aligned}$$

This contradicts the fully-nilpotency of f. Hence, f is fully-periodic. □

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