MIXED CONVECTION ABOUT A ROTATING SPHERE

by

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ABSTRACT

This report presents a theoretical analysis of flow and heat transfer characteristics of the effects of rotational speed, buoyancy force and the Prandtl number on laminar boundary layer over a rotating sphere in forced flow. Applying the finite difference method, numerical computations are carried out for various values of the above parameters. Both assisting and opposing flows are considered. Although the heating condition of uniform wall temperature is used in the analysis, the case of uniform surface heat flux is also studied in the formulation.

After an introduction to the subject and an examination of the previous works, the theoretical background chapter supplies a general formulation. In the section which follows, the problem is specified. Then the results of the numerical solution are displayed in graphical form. Finally, the results are discussed and conclusions are arrived at. The computer program is also supplied.

KÜRE ÜZERİNDE KONVEKTİF ISI TRANSFERİ

ÖZET

Bu çalışma, bir akışkan içersinde kendi ekseni etrafında dönmekte olan bir küreye dönme hızı, akışkan yoğunluğundaki farklar ve Prandtl sayısının etkilerini akış ve ısı transferi açısından incelemektedir. Sonlu farklar metodu uygulanarak, adı geçen parametreler için farklı değerlerde nümerik hesaplar yapılmıştır. Esas olarak yüzey sıcaklığı sınır koşulu olarak kulllanıldığı halde başka sınır koşullarının tercihi halinde formulasyonun nasıl değişebileceği ayrıca belirtilmiştir.

Konuya giriş kısmından sonra, ilgili alanda geçmişte yapılan çalışmalar üzerinde durulmuştur. Teorik bilgiler kısmında genel bir formulasyon yer almaktadır. Takip eden kısımda problem matematiksel olarak açıklanmıştır. Elde edilen sonuçlardan alınan örnekler grafiksel olarak verilmiştir. En son olarak sonuçlar üzerinde tartışılmış ve neticeye varılmıştır. Ayrıca hazırlanmış olan bilgisayar programı da bu raporda yer almaktadır.

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LIST OF SYMBOLS

Listed below are the most commonly used symbols. Some others are defined ad hoc in the study.

Bp	Buoyancy parameter
C_f	Friction coefficient
Gr	Grashof number
g	Acceleration due to gravity
g_x	Projection of g on the x-axis
k	Thermal conductivity
Nu	Nusselt number
Pr	Prandtl number
q_w	Local surface heat transfer rate per unit area
R	Sphere radius
Re	Reynolds number
Rp	Rotation parameter
r	Radius of sphere circle at x
T	Temperature
U_e	Velocity at outer edge of the boundary layer
u	Velocity component in x-direction
v	Velocity component in y-direction
w	Velocity component in rotating direction
x	Coordinate measured along surface from stagnation point
y	Coordinate measured normal to x
z	Coordinate measured in rotating direction
α	Thermal diffusivity
β	Thermal expansion coefficient
ν	Kinematic viscosity
ρ	Fluid density
ϕ	Angle measured from stagnation point
Ω	Angular velocity

Subscripts

w	Evaluated at the surface
∞	Evaluated at the approach conditions

Circumflex

$\hat{}$	Non-dimensional form

Other symbols

*	For the case of uniform surface heat flux
Δ	A finite increment

I. INTRODUCTION

Heat transfer from rotating bodies is an area which includes challenging problems for scientists and engineers. Applications include rotating machinery, spinning projectiles, re-entry missiles, fibre-coating, etc.

The major difficulties encountered in the investigations regarding rotating bodies are the mathematical difficulties and the lack of a common formulation for the wide range of body shapes. In this investigation, the geometry is taken to be that of a sphere and the mixed free- and forced-convection is studied. Moreover, rotation in comparable magnitude to forced flow is considered. Mixed convection implies that the buoyancy force is not neglected and a uniform flow parallel to the axis of sphere is present. One may also think of it as a sphere moving in a direction parallel to its axis of rotation in a fluid at rest.

In the study, the effects of the variation of the buoyancy force, rotation speed and the Prandtl number on the flow and heat transfer are examined. Especially, the consequences of the Prandtl number variation are treated since it has not been a subject matter before, according to literature.

Because of the consideration of the buoyancy force, rotation and forced flow at the same time, the equations that govern the system are more complex compared to those of the previous works. The finite difference method is used to solve the coupled system of equations. In the application of this method, there is a transformation process. In spite of the apparent ease of transformations using finite differences, the numerical solution of such coupled systems of partial differential equations is not an easy matter. Like almost every other engineering problem, it requires some original thought and modifications. However, once the computer program is developed, it serves the purpose for any choice of the parameters of the system. Then the problem reduces to the examination and discussion of the results in order to arrive at conclusions.

II. LITERATURE SURVEY

In the literature, it is possible to come across investigations of laminar heat transfer from axisymmetric bodies. Lin and Chao [1] have considered the problem of steady, laminar, free-convection boundary-layer flow over axisymmetric bodies of arbitrary contour placed in an infinite ambient fluid. By way of a suitable coordinate transformation, the solution of the governing conservation equations have been obtained in terms of a sequence of universal functions. They depend on the Prandtl number and a configuration function that is given by the body contour and its orientation relative to the body force. It is analogous to the wedge variable in forced flows. Several of the universal functions have been evaluated and tabulated. To examine the usefulness and limitations of the analysis, the results have been applied to various body shapes. Spheres as well as other ellipsoids of revolution have been considered.

Another investigation which has been conducted for the same case (free-convection over a non-rotating sphere) is of Hasan and Mujumdar [2]. It is a problem of combined heat and mass transfer. This study is practically important. Applications include evaporation of fuel droplets, calm-day vaporisation of mist and fog, drying of grains, controlling polymerisation reaction products by injecting suitable molecular weight reactants along the porous wall of the reactor, etc. Numerical results of the local Sherwood number, the local Nusselt number and the local wall shear stress have been given in tabular form and graphically. The cases of aiding and opposing thermal and concentration buoyancy forces have been considered.

An investigation for a rotating body has been performed by Badr and Dennis [3]. They have considered the problem of laminar forced-convection from an isothermal cylinder rotating about its own axis and placed in a uniform stream. Major emphasis has been given to the effect of the speed of rotation on the thermal boundary-layer geometry and also on the Nusselt number distribution.

In the study of Lee, Jeng and De Witt [4], a procedure has been established for the calculation of the momentum and heat transfer rates through laminar boundary layers over rotating axisymmetric bodies in forced flow. They have used appropriate coordinate transformations and Merk's type of series and have numerically integrated the obtained coupled ordinary differential equations for various values of the rotation parameter and the Prandtl number. As a special case from the formulation for the rotating sphere, the flow and heat transfer characteristics for the rotating disk have been obtained.

The afore-mentioned buoyancy force has been neglected in some of the investigations for non-rotating bodies in forced flow and for rotating bodies either in forced flow (as in [3] and [4]) or in the absence of a uniform flow from infinity. However, the neglect of the buoyancy effect may not prove right when the velocity is small and the temperature difference between the surface and the surrounding fluid is large. In such cases, it is certain that this buoyancy force will affect the momentum and heat transfer rates.

Suwono [5] has considered these effects on flow and heat transfer over rotating axisymmetric round-nosed bodies. In that study, the numerical computations have been made for the case of rotating hemispheres for values of the buoyancy parameter ranging from zero to infinity. Using the results for the hemispheres, the buoyancy force effects on flow and heat transfer over a sphere have been examined. The effects of the buoyancy force on flow eruption have also been included. Since a uniform flow from infinity is absent in that study, it is not of mixed-convection type.

The problem of mixed forced- and free-convection about a sphere has received relatively less attention. Chen and Mucoglu [6] have conducted such an investigation for non-rotating spheres maintained at a uniform surface temperature. They have presented the local wall shear and surface heat transfer results for gases having a Prandtl number of 0.7 for both assisting and opposing flows. The entire regime of mixed convection has been considered, ranging from pure forced-convection to pure free-convection. As an extension of their study, Chen and Mucoglu [7] have also considered the boundary condition of prescribed uniform surface heat flux. In both of the studies [6] and [7], the finite difference method has been used to solve the transformed conservation equations.

Rajasekaran and Palekar [8] have considered mixed convection about a rotating sphere under two kinds of heating conditions, uniform wall temperature and uniform surface heat flux. They have applied appropriate coordinate transformations and Merk's method of series. Numerical computations have been carried out for Prandtl numbers of 0.7 and 1.0 and the effects of buoyancy force and rotation on the results have been investigated. The ratio of the Nusselt number at uniform surface heat flux to the Nusselt number at uniform wall temperature for different speeds of rotation has been examined. It is also stated in this article [8] that the effects of variation of the Prandtl number on the flow and heat transfer due to buoyancy, rotation and forced flow has not yet been considered and that this could be a subject matter for further investigation. In this present study, also those effects will be considered.

III. THEORETICAL BACKGROUND

In this chapter, initially, the formulation of the mixed convection problem over a general three-dimensional body will be considered and developed. Then the dimensionless ratios encountered in this study are discussed for the sake of emphasising their significance. Finally, a few comments are made on rotating systems, since the mathematical models constructed and sometimes even the methods used in such systems have things in common with one another.

A. Boundary-layer Equations

The equations of motion for a laminar, constant property, incompressible boundary-layer flow over a general three-dimensional body can be stated as

$$\frac{\partial V_1}{\partial t} + \frac{V_1}{h_1}\frac{\partial V_1}{\partial x_1} + \frac{V_2}{h_2}\frac{\partial V_1}{\partial x_2} + V_3\frac{\partial V_1}{\partial x_3} + \frac{V_1 V_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} - \frac{V_2^{\,2}}{h_1 h_2}\frac{\partial h_2}{\partial x_1} = -\frac{1}{\rho h_1}\frac{\partial p}{\partial x_1} + \nu\frac{\partial^2 V_1}{\partial x_3^{\,2}} \tag{1}$$

$$\frac{\partial V_2}{\partial t} + \frac{V_1}{h_1}\frac{\partial V_2}{\partial x_1} + \frac{V_2}{h_2}\frac{\partial V_2}{\partial x_2} + V_3\frac{\partial V_2}{\partial x_3} - \frac{V_1^{\,2}}{h_1 h_2}\frac{\partial h_1}{\partial x_2} + \frac{V_1 V_2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} = -\frac{1}{\rho h_2}\frac{\partial p}{\partial x_2} + \nu\frac{\partial^2 V_2}{\partial x_3^{\,2}} \tag{2}$$

The variables that appear in the above equations are defined in the derivation in Appendix-A. V_i are the components of the velocity vector and x_i are the corresponding curvilinear coordinates.

Since in this study the geometry is that of a sphere, an appropriate curvilinear coordinate system is chosen. This system is valid for any rotationally symmetric blunt-nosed body. Let x-y-z be the non-rotating orthogonal curvilinear coordinate system, with velocity components u-v-w, respectively. x is the distance along a meridian curve and it is measured along the surface from the stagnation point. y is the coordinate normal to x and it indicates the distance from the surface. z is measured in the rotating direction. Therefore, for the chosen coordinates,

$$x_1 = x , \quad x_2 = z , \quad x_3 = y$$
$$V_1 = u , \quad V_2 = w , \quad V_3 = v \tag{3}$$

For this particular coordinate system, h_1 and h_2 can be evaluated as

$$h_1 = 1$$
$$h_2 = r(x) \tag{4}$$

where r(x) is the radius of revolution at x. Then the equations of motion for a steady, laminar, constant property, incompressible boundary-layer flow in the above choice of coordinates are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{w^2}{r} \frac{dr}{dx} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{5}$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + \frac{uw}{r} \frac{dr}{dx} = \nu \frac{\partial^2 w}{\partial y^2} \tag{6}$$

The partial derivatives with respect to z do not appear in the above equations since there are no variations in that direction (due to symmetry). The pressure can be determined by the flow above the boundary layer. Let $U_e(x)$ be the velocity at the outer edge of the boundary layer. Then with

$$U_e \frac{dU_e}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{7}$$

equation (5) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w^2}{r}\frac{dr}{dx} = U_e\frac{dU_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{8}$$

The equation of continuity (from Appendix-A),

$$\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 V_1) + \frac{\partial}{\partial x_2}(h_1 V_2)\right] + \frac{\partial V_3}{\partial x_3} = 0 \tag{9}$$

can be rewritten, with the above formulation, as

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{10}$$

Under the above conditions and when dissipation is neglected, the energy equation can be shown, in a similar way, to have the form

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{11}$$

where $\alpha(=k/\rho c)$ is the thermal diffusivity of the fluid. With the above form of the energy equation, surface temperature of the body may vary only in the x-direction, but never in the z-direction. One should be aware of this restriction when stating the boundary conditions. It is also important that, when neglecting the dissipation term in the energy equation, one should keep in mind that high values of the Prandtl number (for example, those values corresponding to oils) may not be considered later in the study.

Convection is associated with the motion of the fluid surrounding the body. If this motion is caused by an externally applied pressure difference, it is called forced-convection. If, however, the motion is because of the density changes and the gravity, the term free-convection is used. As the topic of this study suggests, a mixed type of convection is going to be considered here. Hence, in this case, the effects of free-convection are taken into account as well as those of the forced type. This requires the addition of another term, the buoyancy force per unit mass, on the right-hand-side of equation (8). Let the fluid temperature be T_∞ and the corresponding density be ρ_∞. The buoyancy force per unit volume for an element of fluid, at temperature T and density ρ, will be $(\rho_\infty - \rho)g$, where g is the acceleration due to gravity. Then the buoyancy force per unit mass is $(\rho_\infty - \rho)g/\rho$. If β is the coefficient of thermal expansion,

$$\frac{1}{\rho} = \frac{1}{\rho_\infty}\left[1+\beta\,(T-T_\infty)\right] \tag{12}$$

Then

$$\rho_\infty = \rho\left[1+\beta\,(T-T_\infty)\right] \tag{13}$$

Therefore, the buoyancy force per unit mass is $\beta g_x(T-T_\infty)$, for a more general surface, with

$$g_x(x) = g\left[1-\left[\frac{dr}{dx}\right]^2\right]^{1/2} \tag{14}$$

Then, if the flow is opposite to the gravitational field, equation (8) becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{w^2}{r} \frac{dr}{dx} = U_e \frac{dU_e}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \pm g_x \beta (T - T_\infty) \qquad (15)$$

In equation (15), the positive and negative signs are to be taken for assisting and opposing flows, respectively.

The equations (6), (10), (11) and (15) are the boundary-layer equations. Although they are developed with the geometry of a sphere in mind, they are also valid for various shapes of bodies of revolution.

B. Dimensionless Ratios

It is important to realize the physical significance of the dimensionless ratios used in this study, so that they will mean more than just numbers. That will be essential for interpritation of the results. In this section, firstly, those dimensionless ratios that are commenly employed in heat-transfer calculations will be briefly considered. Those include the Reynolds number, the Nusselt number, the Prandtl number, the Grashof number and the friction coefficient. Later two other dimensionless values which are used together with the Prandtl number as parameters in this investigation will be preseted: The rotation parameter and the buoyancy parameter. They will be defined and some comments will be made upon them.

The Reynolds number is a measure of relative magnitude of the inertial forces to the viscous forces occuring in the flow. The higher the Reynolds number the greater will be the contribution of inertia effects. The smaller the Reynolds number the greater will be the relative magnitude of the viscous stresses.

The Nusselt number gives a measure of the ratio of the heat transfer rate to the rate at which heat would be conducted within the fluid under a temperature gradient.

The Prandtl number is the ratio of kinematic viscosity to thermal diffusivity. Diffusivity is the rate at which a particular effect is diffused through a medium. Kinematic viscosity of a fluid is the rate at which momentum diffuses through the fluid due to molecular motion, and thermal diffusivity is the rate of diffusion of heat in the fluid. Gases, in general, correspond to Prandtl numbers between 0.5 and 1.0. While water has the values of the Prandtl number at the orders of 1 to 10, light organic liquids are known to have values between 6 and 60. Oils match to high values of the Prandtl number. However, they will not be considered in this study due to the reasons explained while constructing equation (11). Liquid metals, on the other hand, form the other extreme. They will be represented with Prandtl numbers of less than 0.02. In this study, they will not be considered either. The above Prandtl number spectrum of fluids is taken from Kays [9].

Another dimensionless number is the local friction coefficient, which is defined by

$$c_f = \frac{\mu\ (\partial u/\partial y)_{y=0}}{0.5\rho u_\infty^2} \tag{16}$$

where u_∞ is the free stream velocity.

The rotation parameter is defined according to the geometry to be considered. Therefore, it is sufficient here only to mention that the rotation parameter is the relative magnitude of the rotation speed to the free stream velocity. A more precise definition will be stated later in the report when the geometry is taken into account.

The definition of the buoyancy parameter includes the Grashof number in the numerator and the Reynols number in the denominator. As a combination of these two dimensionless groups, the buoyancy parameter can be interpreted as the degree of free-convection as compared to forced-convection.

C. Rotating Systems

In the previous sections of this chapter, the formulation of the mixed forced- and free-convection is made. However, the geometry and the rotation are other significants aspects of this study. Therefore, a preliminary discussion exploring rotating systems with similar geometries will contribute to this present investigation.

Heat transfer from bodies of revolution spinning about their axes of symmetry is both theoretically and practically important, in particular, when they are placed in a forced flow field. As explained in the previous section of this chapter, the rotation parameter conveys the information about the extent of rotation. According to literature [3], at high values of this parameter, the flow and thermal fields are strongly influenced.

Previous investigations in the field of rotating systems have commonly employed body shapes from a special class. [10] The bodies of this class have shapes which can be described by a power function of the type

$$r(x) = L\left[\frac{x}{L}\right]^{(2m-1)/3} \quad , \; m \geq 2 \tag{17}$$

where x is the distance from the nose measured along a meridian, r(x) is the radius of revolution of the body, L is a characteristic length of the body and m determines the geometry. In the solutions, m is used as a parameter; for example, m=2 is the case of a rotating disk.

However, cylinders and spheres do not fall in the class of the bodies mentioned above. It is not possible to obtain, for example, a sphere using a single m-value. Therefore, it is more convenient to look for other methods for bodies such as cylinders and spheres. However, although cylinders and spheres are to be treated separately from the class mentioned above, they have features in common with those other body shapes. For example, it is possible to obtain the flow and heat transfer characteristics for a rotating disk, as a special case, from the formulation for the rotating sphere. [4] This is why the comments regarding that special class are included in this chapter. Moreover, for a sphere, which is the considered geometry in this present study, the Nusselt number in the vicinity of the poles can be closely approximated by the equations developed for a rotating disk, which is a body shape of that class. [10]

IV. PROBLEM STATEMENT

A. The Governing Equations

The boundary-layer equations for laminar, steady, non-dissipative, constant property (except changes in density which produce buoyancy forces), incompressible boundary-layer flow over a general rotating axisymmetric body were derived in the first section of the previous chapter (equations (6), (10), (11), (15)). In order to serve for the discussions in this section, it may help to re-state the boundary-layer equations, here.

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{10}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w^2}{r}\frac{dr}{dx} = U_e\frac{dU_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \pm g_x\beta(T-T_\infty) \tag{15}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{uw}{r}\frac{dr}{dx} = \nu\frac{\partial^2 w}{\partial y^2} \tag{6}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{11}$$

The aim of this chapter is to adapt those equations to the geometry of a sphere. As it is clear from Fig. I that gives the geometry, $r(x)$ is defined for a sphere by

$$r(x) = R\sin\phi = R\sin(x/R) \tag{18}$$

where R is the radius of the sphere and ϕ is the angle measured from the stagnation point.

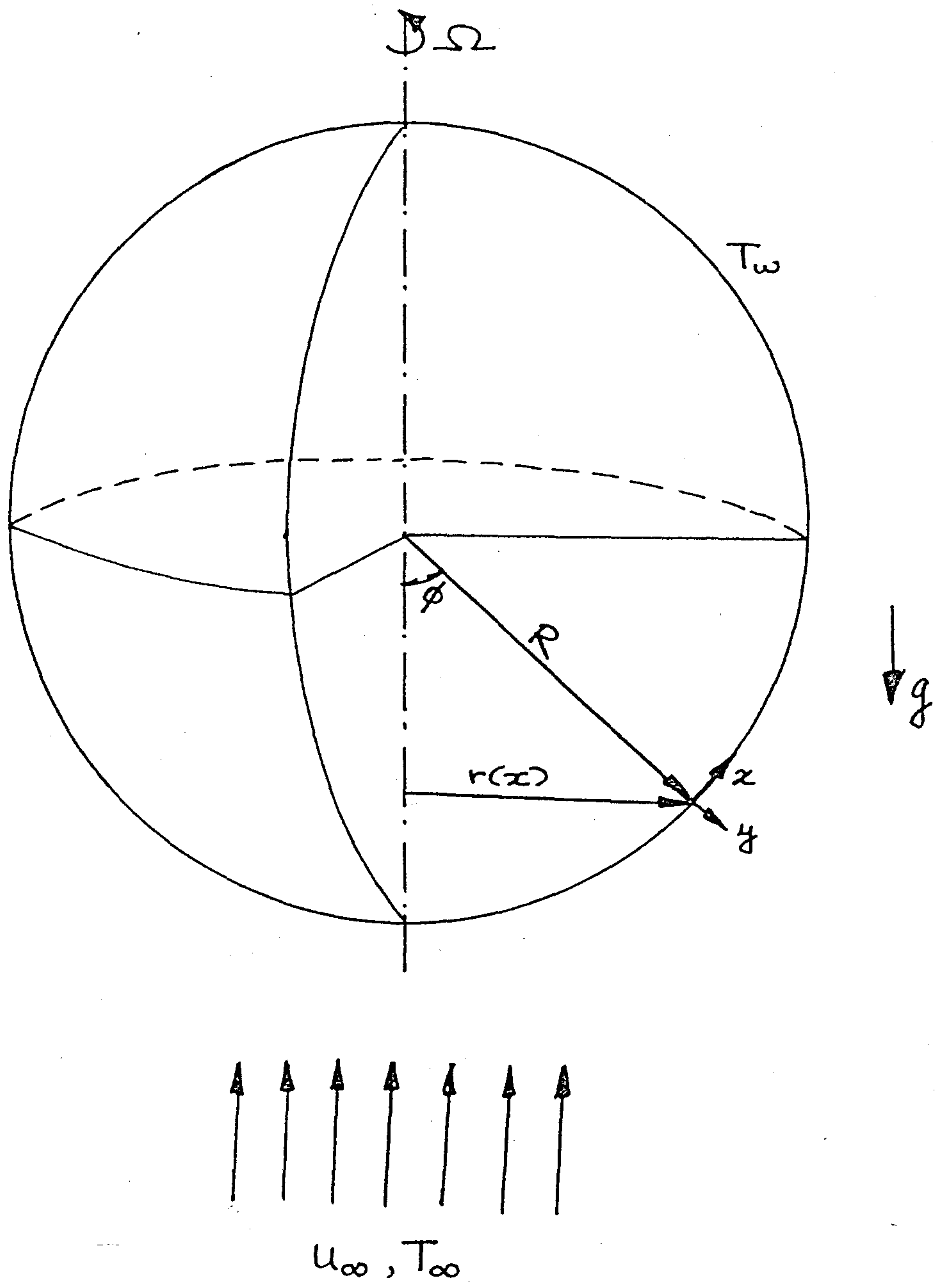

Fig. I - The geometry of the problem.

With the above definition of $r(x)$, g_x in equation (14) becomes

$$g_x(x) = g \sin \phi = g \sin (x/R) \tag{19}$$

As stated earlier, in equation (15), for the sign of g_x, the positive and negative ones are to be taken for assisting and opposing flows, respectively. In the case of assisting flow, $T_w > T_\infty$ and the buoyancy force has a component in the positive x-direction; and in the opposing flow case, $T_w < T_\infty$ and the buoyancy force will have a component in the negative x-direction. This analysis is also valid for downward flow. However, in that case, the x-coordinate is measured from the upper stagnation point. This time, the assisting and opposing flows correspond to $T_w < T_\infty$ and $T_w > T_\infty$, respectively.

U_e, which can be termed as the velocity at the outer edge of the boundary layer or as the local free stream velocity, in general has the expression:

$$\frac{U_e}{u_\infty} = A\,\frac{x}{R} + B\,\frac{x^3}{R^3} + C\,\frac{x^5}{R^5} + D\,\frac{x^7}{R^7} + \cdots \tag{20}$$

where u_∞ is the free stream velocity. [6] The corresponding constants A, B, C, D, etc. for the sphere are given from potential flow solution by

$$A = 3/2\ , \quad B = -\,1/4\ , \quad C = 1/80$$
$$D = -\,1/3360\ , \quad \text{etc.} \tag{21}$$

which forms a sine-series expansion. Therefore, the local free stream velocity for a sphere is

$$U_e(x) = \frac{3}{2}\,u_\infty \sin \phi \tag{22}$$

Also evaluating dr/dx and dU_e/dx, the governing equations of the system may be written as

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{10}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w^2}{r}\cos\phi = \frac{9}{4}\frac{u_\infty^2}{R}\sin\phi\cos\phi + \nu\frac{\partial^2 u}{\partial y^2} \pm g\beta(T-T_\infty)\sin\phi \tag{23}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{uw}{r}\cos\phi = \nu\frac{\partial^2 w}{\partial y^2} \tag{24}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{11}$$

It will be useful for later use to parametrize the above equations. The in-the-previous-chapter-memtioned parameters, the rotation parameter, Rp and the buoyancy parameter, Bp are now defined according to

$$Rp = \left[\frac{2}{3}\frac{\Omega R}{u_\infty}\right]^2 \text{, for sphere} \tag{25}$$

where Ω is the angular velocity of the sphere, and

$$Bp = \frac{Gr}{Re_R^{\,2}} \tag{26}$$

where Re_R is the Reynolds number, Ru_∞/ν.

The Grashof number is defined by

$$Gr = \frac{g\beta(T_w - T_\infty)\ R^3}{\nu^2} \tag{27}$$

The Prandtl number, ν/α, is also used as a parameter in the system. Then the governing equations containing the parameters are:

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{10}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{9}{4}\frac{u_\infty^2}{r}\left[\frac{w}{\Omega R}\right]^2 Rp\cos\phi = \frac{9}{4}\frac{u_\infty^2}{R}\sin\phi\cos\phi$$

$$+\nu\frac{\partial^2 u}{\partial y^2} \pm Bp\frac{u_\infty^2}{R}\frac{(T - T_\infty)}{(T_w - T_\infty)}\sin\phi \tag{28}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{uw}{r}\cos\phi = \nu\frac{\partial^2 w}{\partial y^2} \tag{24}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial y^2} \tag{29}$$

B. The Boundary Conditions

After the derivation of the governing equations, it is now convenient to furnish them with the boundary conditions. As in most of the other problems of interest, it is appropriate to solve the equations under the case of uniform wall temperature.

For the case of uniform wall temperature, the corresponding boundary conditions are

$$
\begin{aligned}
&u = v = 0\ , \quad w = \Omega r\ , \quad T = T_w \qquad && \text{for } y=0 \\
&u = U_e, \quad w = 0\ , \quad T = T_\infty && \text{for } y \to \infty
\end{aligned} \tag{30}
$$

In this study, the above set of boundary conditions is used. However, in order to see what differs in the formulation, it is useful to discuss another possible set, here. In some of the previous investigations, for example in [7], the case of uniform surface heat flux has been employed. For that case of boundary conditions, the following can be written:

$$
\begin{aligned}
&u = v = 0\ , \quad w = \Omega r\ , \quad \frac{\partial T^*}{\partial y} = -\frac{q_w}{k} \qquad && \text{for } y=0 \\
&u = U_e, \quad w = 0\ , \quad T^* = T_\infty && \text{for } y \to \infty
\end{aligned} \tag{31}
$$

For the uniform surface heat flux case, equation (27) of the formulation should be replaced by a new definition of the Grashof number:

$$
Gr^* = \frac{g \beta q_w R^4}{k \nu^2} \tag{32}
$$

In that case, the buoyancy parameter is given by

$$
Bp^* = \frac{Gr^*}{Re_R^{5/2}} \tag{33}
$$

For the boundary conditions defined by equation (31), it is necessary to change equation (28) in view of the new definition of the buoyancy parameter. Later in the report, when the dimensionless system is obtained, the formulation is going to be made such that a single form is attained for both sets of boundary conditions. The equation that takes the place of equation (28) for the second case of boundary conditions is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{9}{4} \frac{u_\infty^2}{r} \left[\frac{w}{\Omega R} \right]^2 Rp \cos \phi = \frac{9}{4} \frac{u_\infty^2}{R} \sin \phi \cos \phi$$

$$+ \nu \frac{\partial^2 u}{\partial y^2} \pm Bp^* \frac{u_\infty^2}{R} \frac{(\overset{*}{T}-T_\infty) \, Re_R^{1/2}}{(q_w \, R/k)} \sin \phi \tag{34}$$

It is clear that, in that case of boundary conditions, the buoyancy force will assist the forced flow for $q_w > 0$, and it will oppose the flow for $q_w < 0$. Therefore, the positive and negative sighs in equation (34) are to be taken accordingly.

V. THE SOLUTION OF THE PROBLEM

In this chapter, initially, the method of solution is decided on. This is achieved by discussing various possible methods, that have been used in previous investigations, with regard to criteria such as convergence and stability. The decision is made on an appropriate method in recognition of its advantages. Later in the chapter, the governing equations and the boundary conditions of the previous chapter are transformed to form the dimensionless system of equations, which are then solved by the chosen method.

A. The Solution Method

In order to obtain solutions for the coupled momentum and energy equations, different procedures are possible. By applying appropriate coordinate transformations and Merk's procedure [11], the governing equations can be reduced to a set of coupled ordinary differential equations. In Merk's method, like in Görtler's method, the series solution is expressed in terms of universal functions. However, Merk's procedure treats the wedge variable as one of the independent coordinates. Merk's procedure for the computation of boundary-layer transfer has been examined in detail by Chao and Fagbanle [11]. The first author had previously discovered the incorrect equations in Merk's procedure. In that study [11], the corrected sequence of the differential equations governing the universal functions associated with the method are provided.

After the system of ordinary differential equations with two-point boundary conditions are obtained using the Merk's method, an approach to the solution is possible by considering a related initial-value problem. A very effective class of numerical methods, which are called initial-value or shooting methods, is based on this notion.

There are two major difficulties associated with the shooting methods. The first one is the problem of convergence, and the second one is that the initial-value problem generated is frequently unstable, i.e. it is very sensitive to perturbations in the initial conditions. Multiple shooting method is developed to overcome those difficulties. In this method, the interval of the problem is devided into many subintervals and for each subinterval a corresponding initial-value problem is generated. Then the problems are solved making sure that the appropriate continuity conditions are satisfied at each of the subdivision points.

Rajasekaran and Palekar [8] have numerically integrated the set of coupled ordinary differential equations, which depend on wedge, rotation and buoyancy parameters, by applying the multiple shooting method. They have used the subroutine DTPTB from IMSL (the International Mathematical and Statistical Library). However, in the instructions given for the usage of this subroutine, it is indicated that the convergence is of vital importance. Therefore, one should take precautions to increase the probability of convergence. It is the best thing to increase the number of shooting points. With many points the program essentially uses a finite difference method, which has less trouble with nonlinearities than shooting methods. In fact, in some of the previous work, for example of [6] and [7], finite difference method is applied.

In this present study, the original partial differential equations are solved by the finite difference method. For the purpose of comparison and with the intension of forming a parallel description to that of multiple shooting method (or shooting methods, in general) as stated above, convergence and stability in the solution of finite difference equations are now examined.

The fundamental concept of the calculus is the interpretation of the derivative as the instantaneous rate of change. For that purpose, a finite increment is used and the limit as that increment approaches zero is examined. In the finite difference method, the inverse of this limit process is used. This is usually termed as "discretization". The discretization of a partial differential equation in a domain of independent variables results in the replacement of this domain by a finite number of preselected, discrete points, referred to as mesh or grid points, and the values at those points are determined.

Let U be the exact solution of a partial differential equation and u be the solution of the difference equations, formed by the discretization process, used to approximate the partial differential equation. Then the finite difference solution is said to be convergent when u tends to U as Δx_i tend to zero, where x_i represent the independent variables. In general, the error (U-u) can be decreased by decreasing Δx_i, but this leads to an increase in the number of equations to be solved, because it means an increase in the number of points. Hence, each additional points adds to the time and labor of calculation. Therefore, this way of improvement is limited by such factors as time, machine storage space, etc.

The discretization error should be considered apart from the round-off errors. If it were possible to carry out all calculations to infinite number of decimal places, the exact solution of the finite difference equations would be obtained. However, calculations are carried out to a finite number of decimal places, which causes round-off errors. A finite difference solution is said to be stable when the total effect of all round-off errors is negligible.

As indicated above, the decision on the method is made in favor of the finite difference method. In the following section, the dimensionless system is obtained to be used in the difference equations.

B. The Dimensionless System of Equations

The computational stage of all numerical methods for solving complex problems generally involves a great deal of arithmetics. It is, therefore, better to arrange the problem such that one solution is sufficient for a variety of different problems. This can be done by expressing all equations in terms of non-dimensional variables. Then all problems with the same non-dimensional mathematical formulation can be dealt with by means of one solution. In this study, the non-dimensional variables are denoted by a circumflex (^) over their original forms. x, y, r can be defined as

$$\hat{x} = x/R \; , \quad \hat{y} = y \; Re_R^{1/2}/R \; , \quad \hat{r} = r/R \tag{35}$$

It is clear that x and o are identical. From equation (18),

$$\hat{r} = \sin \phi = \sin \hat{x} \tag{36}$$

from which it follows that

$$\cos \phi = d\hat{r}/d\hat{x} \tag{37}$$

Accordingly, the non-dimensional forms of the velocities can also be obtained:

$$\hat{u} = \frac{\hat{r}u}{u_\infty} \tag{38}$$

$$\hat{v} = \frac{\hat{r}\, v\, Re_R^{1/2}}{u_\infty} \tag{39}$$

$$\hat{w} = \frac{w}{\Omega R} \tag{40}$$

The nondimensional form of the temperature depends on the choice of the boundary conditions. For the case of uniform wall temperature, the obvious non-dimensional form is

$$\hat{T} = \frac{T - T_\infty}{T_w - T_\infty} \tag{41}$$

whereas if the case of uniform surface heat flux were used, the appropriate non-dimensional form would be

$$\hat{T}^* = \frac{(T - T_\infty)\, Re_R^{1/2}}{q_w R/k} \tag{42}$$

Substituting these non-dimensional forms of equations (35) to (42) in the governing equations ((18), (24), (28), (29)), the dimensionless system of equations can be formed as

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{43}$$

$$\hat{u}\left[\frac{\partial \hat{u}}{\partial \hat{x}} - \frac{\hat{u}}{\hat{r}}\frac{d\hat{r}}{d\hat{x}}\right] + \hat{v}\,\frac{\partial \hat{u}}{\partial \hat{y}} - \frac{9}{4}\,\hat{r}\,Rp\,\hat{w}^2\,\frac{d\hat{r}}{d\hat{x}} = \frac{9}{4}\hat{r}^3\frac{d\hat{r}}{d\hat{x}} + \hat{r}\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \pm Bp\,\hat{r}^3\,\hat{T} \tag{44}$$

$$\hat{u}\,\frac{\partial \hat{w}}{\partial \hat{x}} + \hat{v}\,\frac{\partial \hat{w}}{\partial \hat{y}} + \frac{1}{\hat{r}}\,\hat{u}\,\hat{w}\,\frac{d\hat{r}}{d\hat{x}} = \hat{r}\,\frac{\partial^2 \hat{w}}{\partial \hat{y}^2} \tag{45}$$

$$\hat{u}\,\frac{\partial \hat{T}}{\partial \hat{x}} + \hat{v}\,\frac{\partial \hat{T}}{\partial \hat{y}} = \frac{\hat{r}}{Pr}\,\frac{\partial^2 \hat{T}}{\partial \hat{y}^2} \tag{46}$$

The above equations are also valid for the case of uniform surface heat flux. However, in that case, $\hat{T}$ and Bp would be replaced by $\hat{T}^*$ and Bp^*, respectively.

The boundary conditions should also be written in terms of the non-dimensional variables. Then the set of boundary conditions for the case of uniform wall temperature of equation (30) becomes

$$\begin{aligned} &\hat{u} = \hat{v} = 0\,, \quad \hat{w} = \hat{r}\,, \quad \hat{T} = 1 && \text{for } \hat{y} = 0 \\ &\hat{u} = \frac{3}{2}\,\hat{r}^2\,, \quad \hat{w} = 0\,, \quad \hat{T} = 0 && \text{for } \hat{y} \to \infty \end{aligned} \tag{47}$$

while on the other hand, if the uniform surface heat flux case were to be used, the non-dimensional form of the boundary conditions set would be as

$$\begin{aligned} &\hat{u} = \hat{v} = 0\,, \quad \hat{w} = \hat{r}\,, \quad \frac{\partial \hat{T}^*}{\partial y} = -1 && \text{for } \hat{y} = 0 \\ &\hat{u} = \frac{3}{2}\,\hat{r}^2\,, \quad \hat{w} = 0\,, \quad \hat{T}^* = 0 && \text{for } \hat{y} \to \infty \end{aligned} \tag{48}$$

The local friction coefficient was defined by equation (16). Its form in terms of the non-dimensional variables can be written as

$$\frac{1}{2}\,C_f\,Re_R^{1/2} = \frac{1}{\hat{r}}\,\frac{\partial \hat{u}}{\partial \hat{y}}\bigg|_{\hat{y}=0} \tag{49}$$

The local Nusselt number can be defined as

$$Nu = \frac{hR}{k} = - \frac{R\ (\partial T/\partial y)\big|_{y=0}}{T_w - T_\infty} \tag{50}$$

The nondimensional expression is therefore

$$Nu\ Re_R^{-1/2} = - \frac{\partial \hat{T}}{\partial \hat{y}}\bigg|_{\hat{y}=0} \tag{51}$$

for the uniform wall temperature case. For the case of uniform surface heat flux, the local Nusselt number would be

$$Nu^{*}\ Re_R^{-1/2} = \frac{1}{\hat{T}^{*}}\bigg|_{\hat{y}=0} \tag{52}$$

Now, having established the dimensionless system of equations, it is time to form the difference equations, of the finite difference method, prior to the construction of the computer program.

C. Difference Equations

In the transformation of the differential equations to a form suitable for the finite difference method, certain finite-difference approximations to the derivatives have to be used. After this process of discretisation, which was discussed in the first section of this chapter, the resulting difference equations may be obtained.

In this study, an explicit iteration scheme is preferred and the down-stream direction is taken to be the positive x-direction. Therefore the iteration is performed starting from the stagnation point.

The finite-difference approximations for the derivatives of the system are

$$\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{UD - UU}{\Delta X}, \quad \frac{\partial \hat{u}}{\partial \hat{y}} = \frac{UU(Y+\Delta Y) - UU(Y)}{\Delta Y} \tag{53}$$

$$\frac{\partial \hat{v}}{\partial \hat{y}} = \frac{VU(Y+\Delta Y) - VU(Y)}{\Delta Y} \tag{54}$$

$$\frac{\partial \hat{w}}{\partial \hat{x}} = \frac{WD - WU}{\Delta X}, \quad \frac{\partial \hat{w}}{\partial \hat{y}} = \frac{WU(Y+\Delta Y) - WU(Y)}{\Delta Y} \tag{55}$$

$$\frac{\partial \hat{T}}{\partial \hat{x}} = \frac{TEMPD - TEMPU}{\Delta X}, \quad \frac{\partial \hat{T}}{\partial \hat{y}} = \frac{TEMPU(Y+\Delta Y) - TEMPU(Y)}{\Delta Y} \tag{56}$$

$$\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} = \frac{UU(Y+\Delta Y) - 2\ UU(Y) + UU(Y-\Delta Y)}{(\Delta Y)^2} \tag{57}$$

$$\frac{\partial^2 \hat{w}}{\partial \hat{y}^2} = \frac{WU(Y+\Delta Y) - 2\ WU(Y) + WU(Y-\Delta Y)}{(\Delta Y)^2} \tag{58}$$

$$\frac{\partial^2 \hat{T}}{\partial \hat{y}^2} = \frac{TEMPU(Y+\Delta Y) - 2\ TEMPU(Y) + TEMPU(Y-\Delta Y)}{(\Delta Y)^2} \tag{59}$$

where letters U and D that follow U, V, W and TEMP denote up-stream and down-stream values, respectively. In the new representation of the independent variables, X and Y indicate the values of $\hat{x}$ and $\hat{y}$ at the preselected, discrete points. ΔX and ΔY represent the spacing between those points. U, V, W and TEMP are the solutions of the finite-difference equations. Variations in Y are shown in parentheses. If there is no variation in Y, those parantheses are omitted. The variatimns in X are indicated by U or D, that follows U, V, W and TEMP. The partial derivatives in equations (43) to (46) have now their new representations.

Those finite-difference approximations for the derivatives can now be used together with equations (36), (37), (43), (44), (45) and (46) to form the difference equations, that construct the nucleus of the computer program, which is presented in Appendix-B.

In the following section, some comments will be made on this computer prgram with the intention of simplifying its examination.

D. Some Comments on the Computer Program

At this stage, it may seem that it would be enough only to discuss the symbols in the program. However, there are still some comments about the formulation that are significant to be made. The first one is on the buoyancy parameter. In equation (44), there are plus and minus signs preceding this parameter indicating assisting and opposing flows, respectively. In the computer program, the buoyancy parameter is taken such that it may have either positive or negative values and the preceding sign is therefore chosen plus.

Since the temperature distribution at x=0 is not specified, a new temperature variable is used in the program. That is given by

$$TEMP_i = TEMP \times \sin \phi \qquad (60)$$

where i denotes the intermediate value. As the name indicates, the temperature solutions are converted back into the desired form of formulation later in the program. Therefore, the user does not need to worry about this feature of the program, but should keep in mind that at the points near the stagnation point it is not possible to obtain the temperature distributions and therefore the Nusselt number.

Since the velocity distributions at the stagnation point are known (velocities equal to zero), there is no need for modifications in their formulation. However, the momentum equation for velocity-$\hat{u}$ (equation (44)) should be changed because of the term which includes the temperature variable. In this way, the temperature value is converted back into its earlier form within the calculation in the momentum equation. This process of changing the temperature variable and later converting it back into its original form does not affect the solutions for the velocity distributions since for small values of ϕ, that buoyancy term tends to zero.

The choice of the points, or rather the establishment of the spacing between the points is quite important. For this problem, ΔX is taken much smaller than ΔY, because there is a factor of u before the derivative term and that acts as a divisor in the difference equations. The values of ΔX and ΔY are given in the DATA statements of the computer program together with the other information.

In order to serve as a multi-purpose program, the FORTRAN program of Appendix-B contains a variable NCH, that indicates the work to be done. When NCH equals 1, the velocity and temperature distributions are obtained. The corresponding friction factor and the Nusselt number are also supplied. NCH=3 gives the effect of the Prandtl number on the velocity and temperature profiles, while NCH=4 gives that on the friction factor and the Nusselt number. Actually that is the object of this study. NCH=2 displays both of the results of NCH=3 and NCH=4. When both the effects of the Prandtl number and the velocity and temperature distributions for various angular positions are desired, NCH should be chosen to be zero.

The constants, variables, arrays, parameters, etc. that are present in the computer program are given in Appendix-C.

E. Results

Employing the computer program developed, it is possible to obtain the velocity and temperature distributions, either for assisting or opposing flows, at any angle ϕ (measured from the stagnation point), for any set of values of the rotation parameter, the buoyancy parameter and the Prandtl number. In each case, also the Nusselt number and the friction factor can be evaluated. Although the aim of this study is to observe the effects of the Prandtl number variation, some other results that are attainable all along the study will be also displayed; and some corresponding conclusions will be arrived at. This is done with the intention of keeping the integrity of the subject. Owing to its practical importance, the set of boundary conditions given by equation (30), i.e. the uniform wall temperature case, is considered. Some of the results are displayed in the following graphs. The following DISCUSSION chapter will be based on those figures.

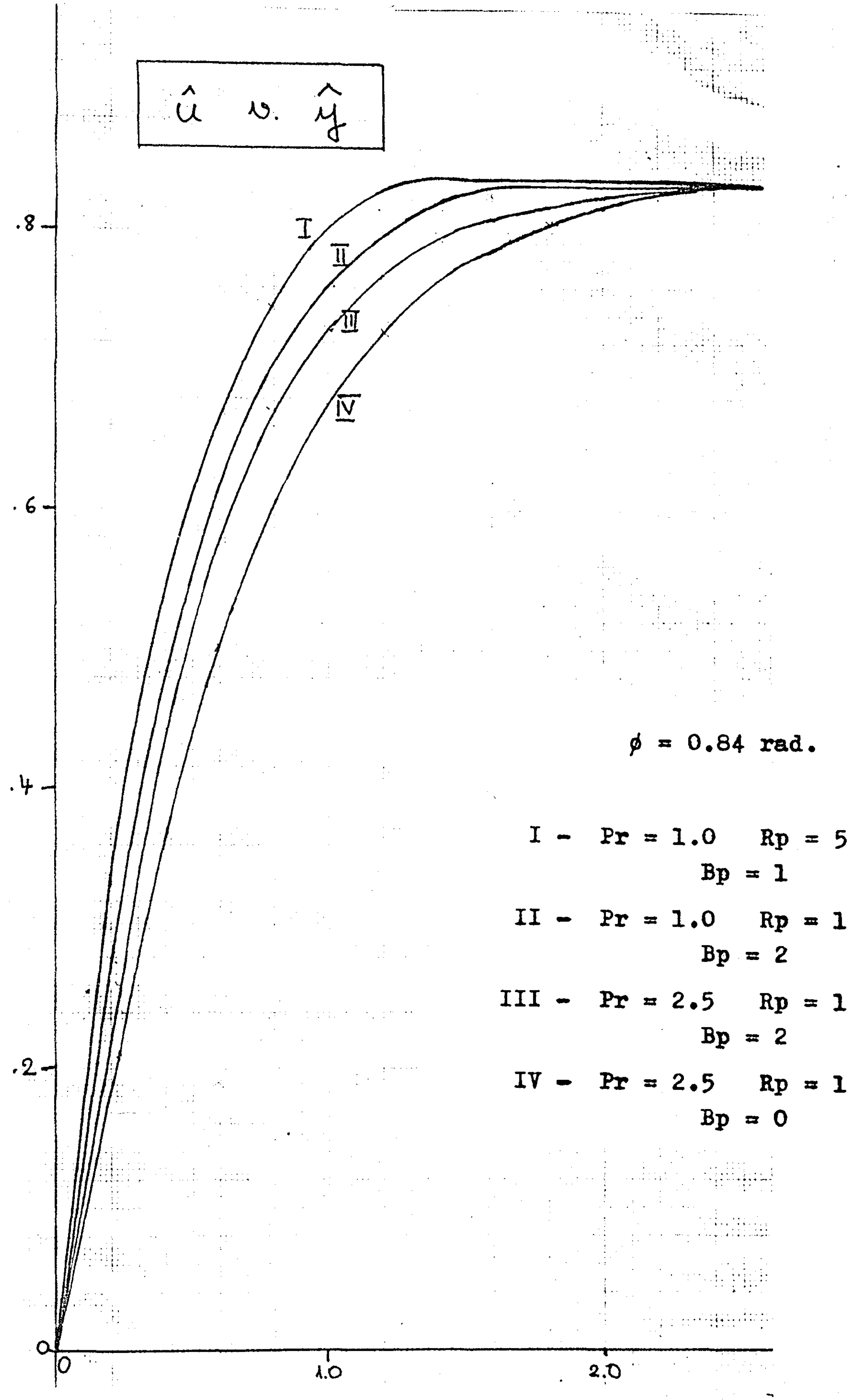

Fig. II - Velocity distributions - Graph 1

Pr = 0.7 , ϕ = 0.84 rad.

I - Rp = 5 Bp = 2

II - Rp = 1 Bp = 2

III - Rp = 1 Bp = 1

IV - Rp = 0 Bp = 1

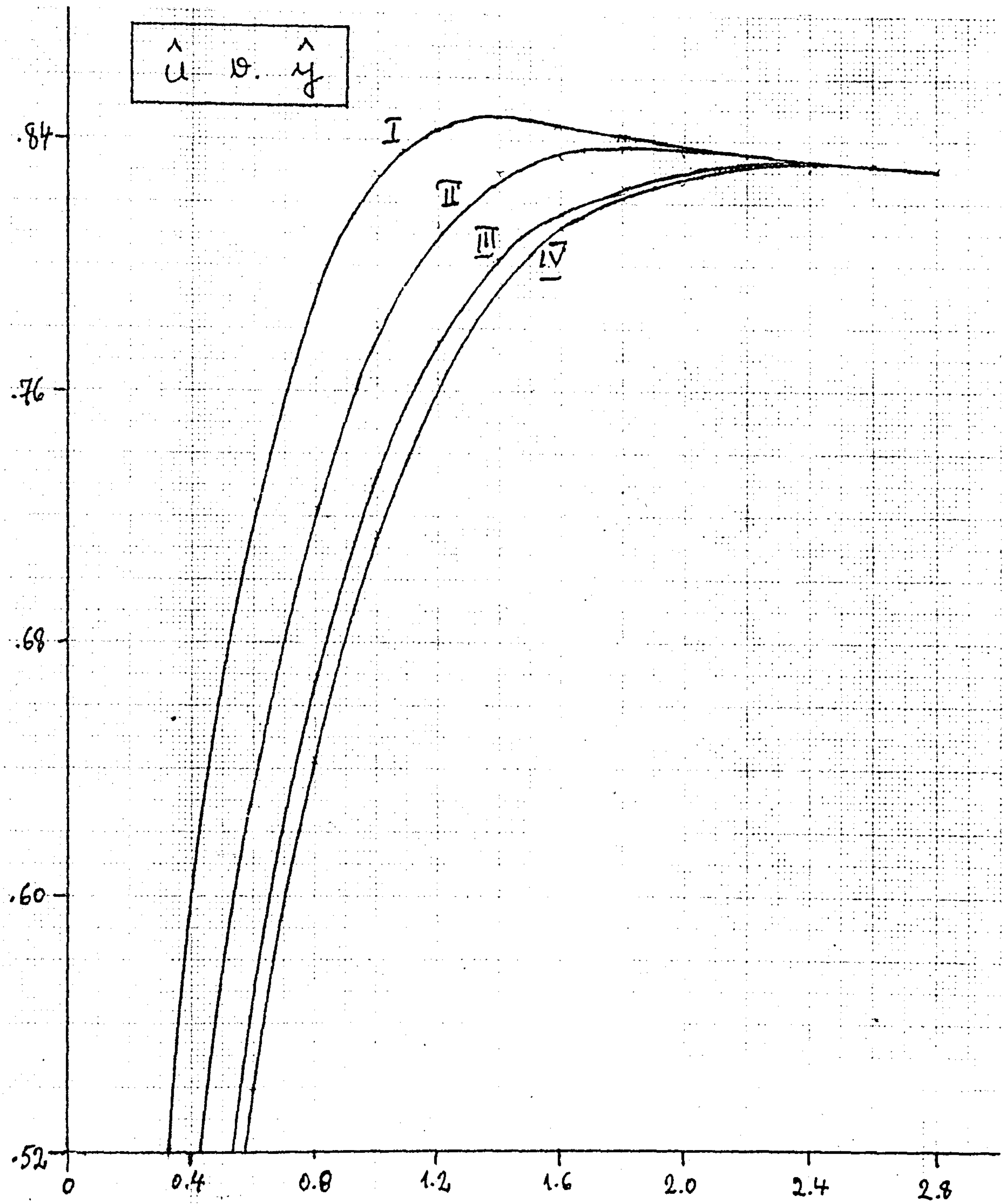

Fig. III - Velocity distributions - Graph 2

Rp = 5.0 ,   Bp = 2.0 , ϕ = 0.84 rad.

I - Pr = 0.7
II - Pr = 1.0
III - Pr = 2.5
IV - Pr = 5.5

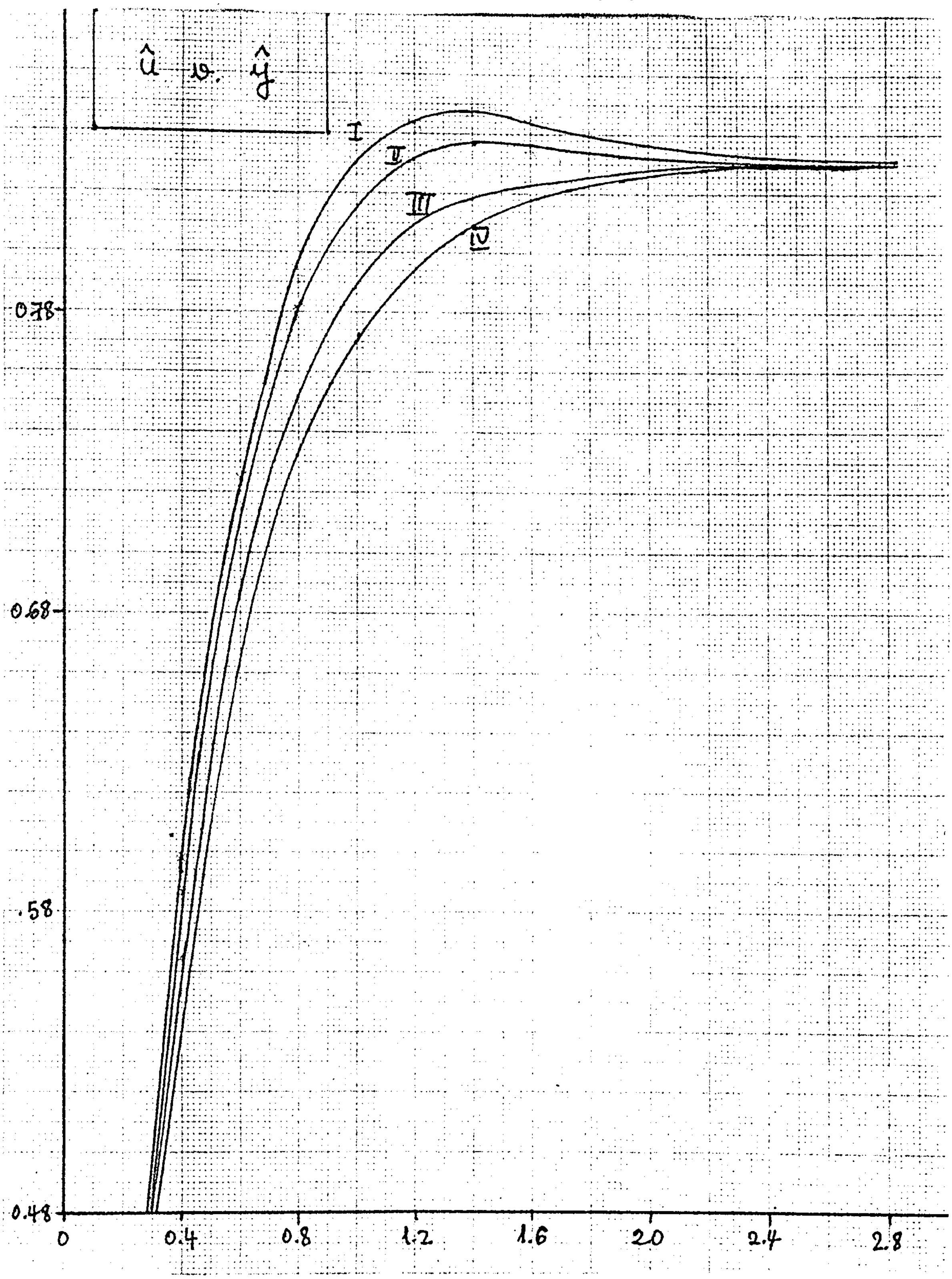

Fig. IV - Velocity distributions - Graph 3

Rp = 1.0 , Bp = -1.0 (opposing flow) , ϕ = 0.84 rad.

I - Pr = 0.7
II - Pr = 1.0
III - Pr = 2.5
IV - Pr = 5.5

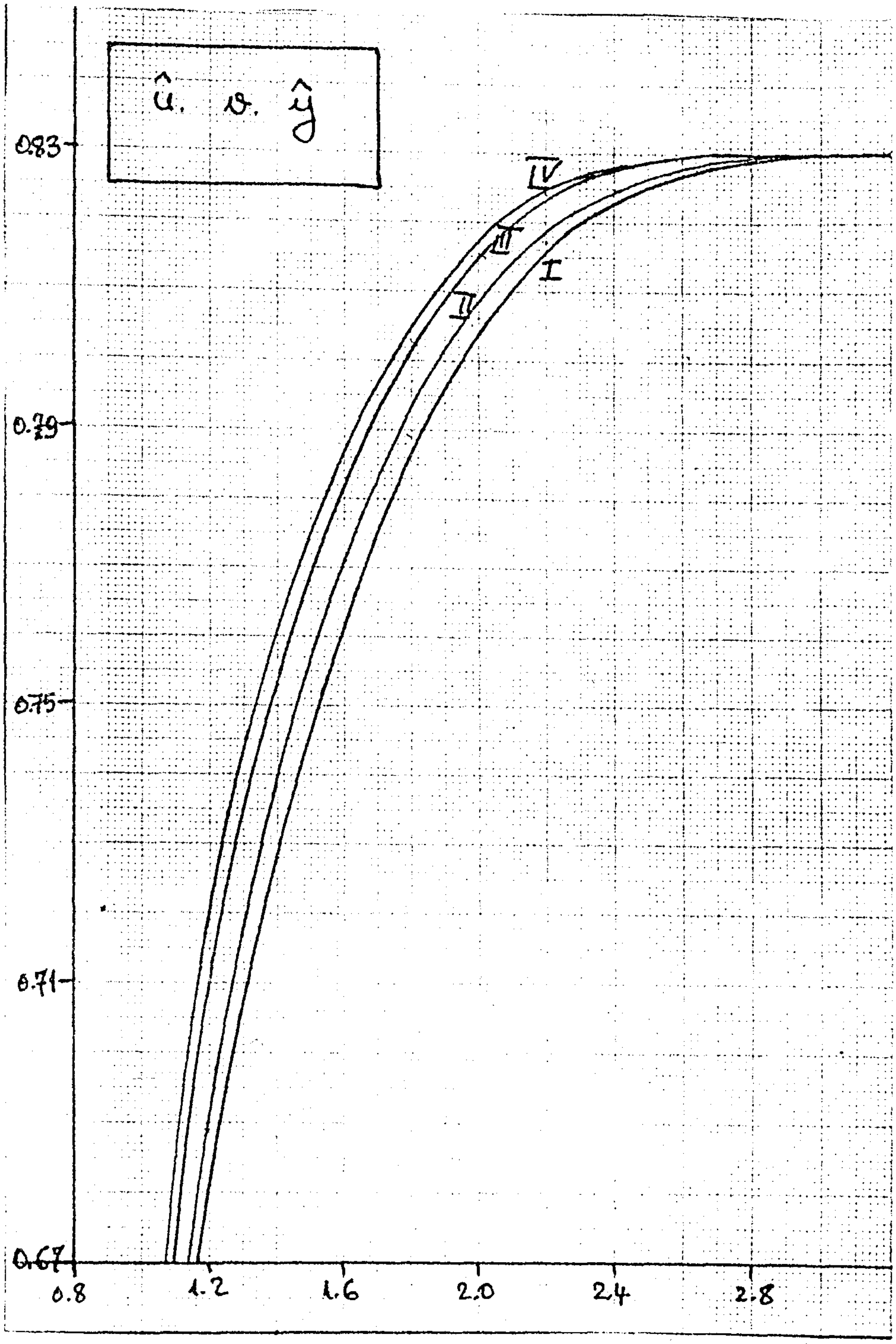

Fig. V - Velocity distributions - Graph 4

Rp = 1.0 , Bp = 1.0 , ϕ = 0.59 rad.

I - Pr = 0.7
II- Pr = 1.0
III- Pr = 5.5

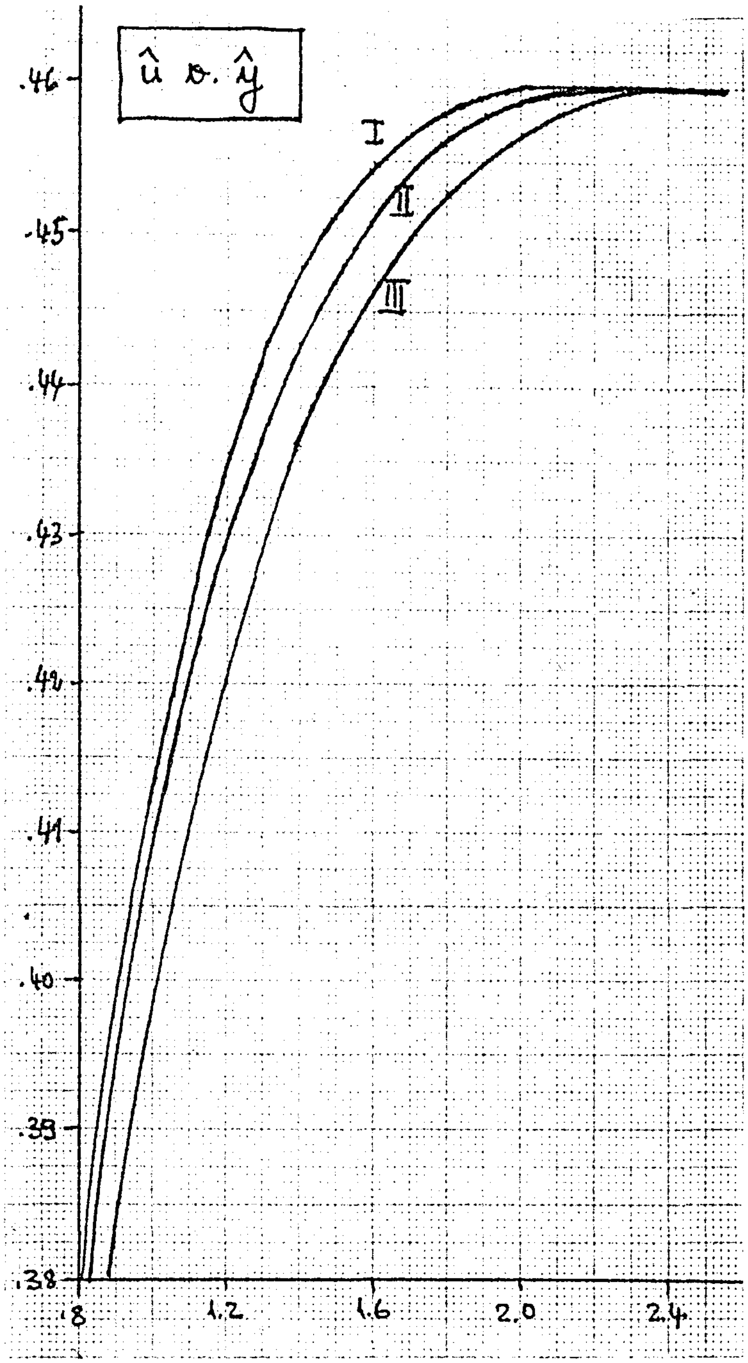

Fig. VI - Velocity distributions - Graph 5

Rp = 1.0 ,   Bp = 1.0 , ϕ = 1.09 rad.

I - Pr = 0.7
II - Pr = 1.0
III - Pr = 2.5
IV - Pr = 5.5

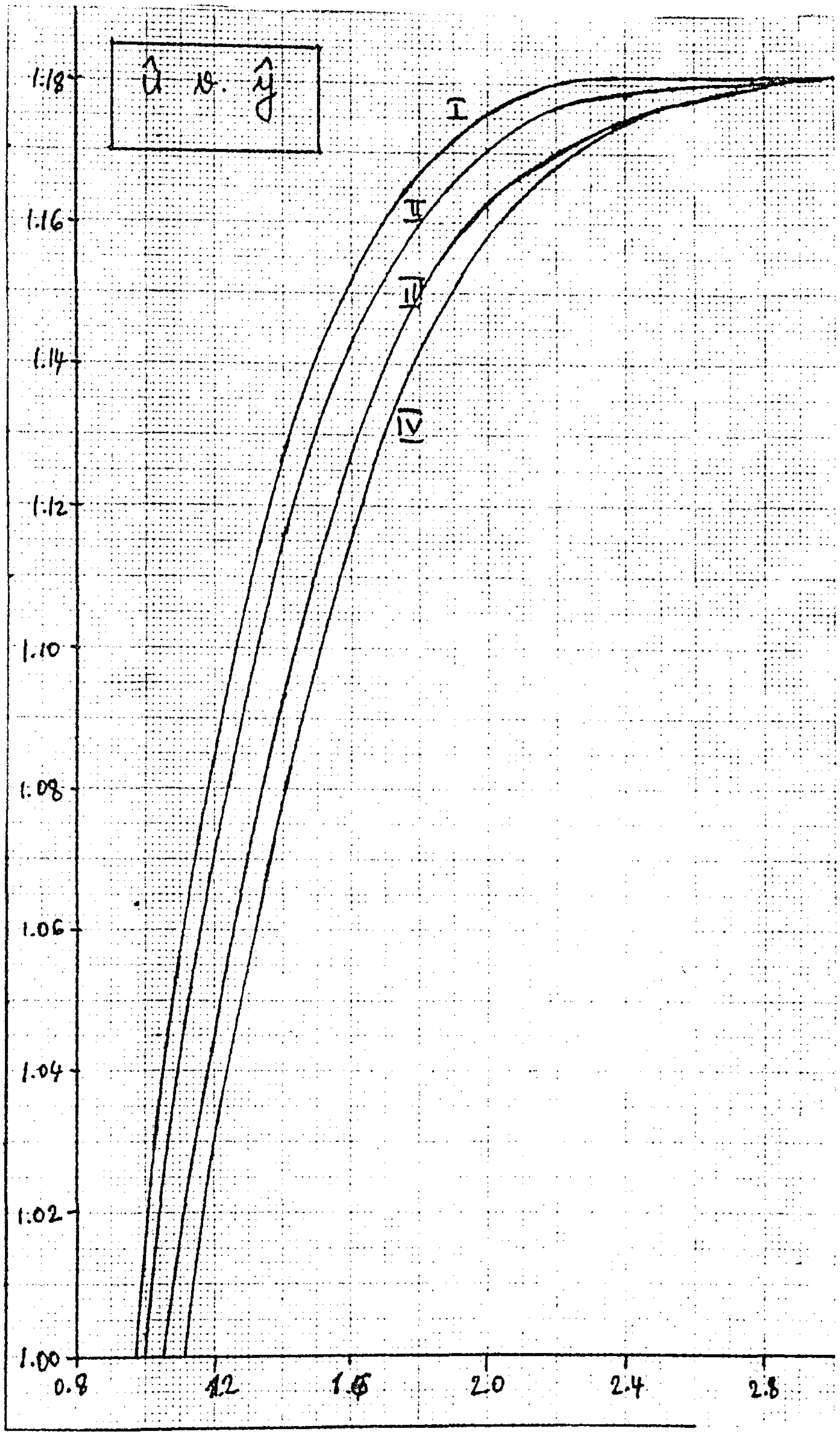

Fig. VII - Velocity distributions - Graph 6

$Bp = 2.0$, $\phi = 0.84$ rad.

I - $Rp = 1$ $Pr = 5.5$

II - $Rp = 5$ $Pr = 5.5$

III - $Rp = 5$ $Pr = 1.0$

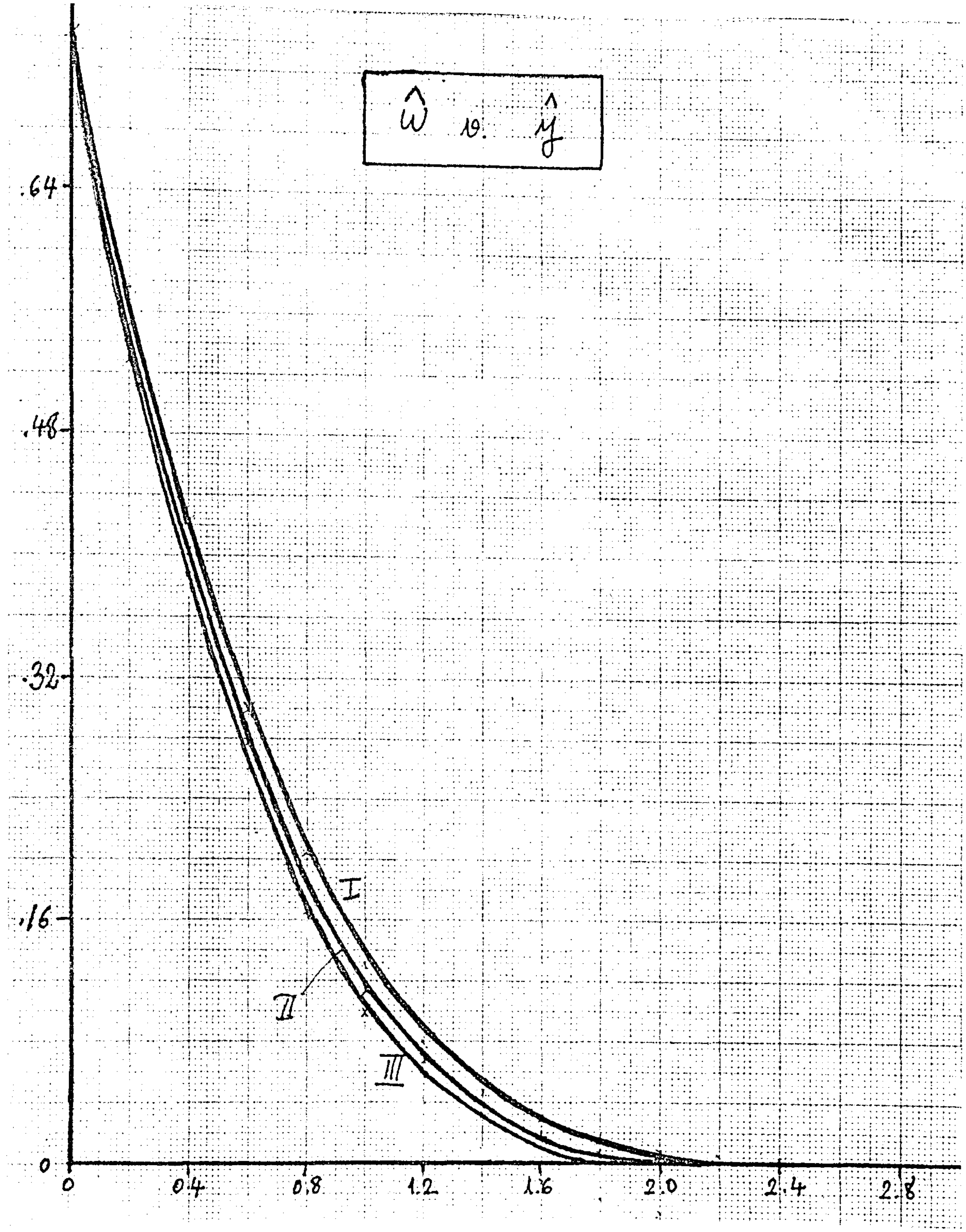

Fig. VIII - Velocity distributions - Graph 7

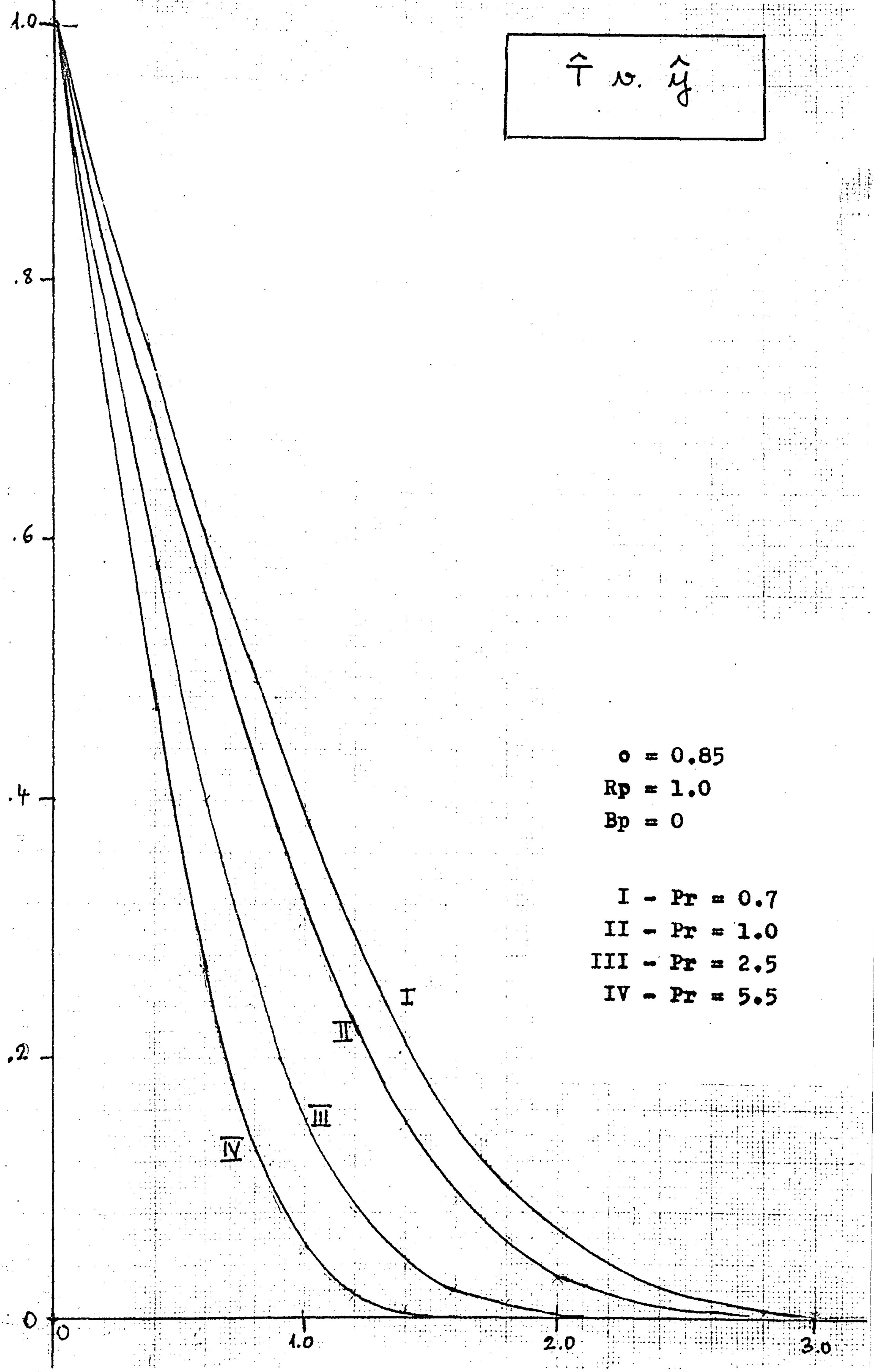

Fig. IX - Temperature distributions

Rp = 1.0 ,   Bp = 1.0

I - Pr = 0.7
II - Pr = 1.0
III - Pr = 2.5
IV - Pr = 5.5

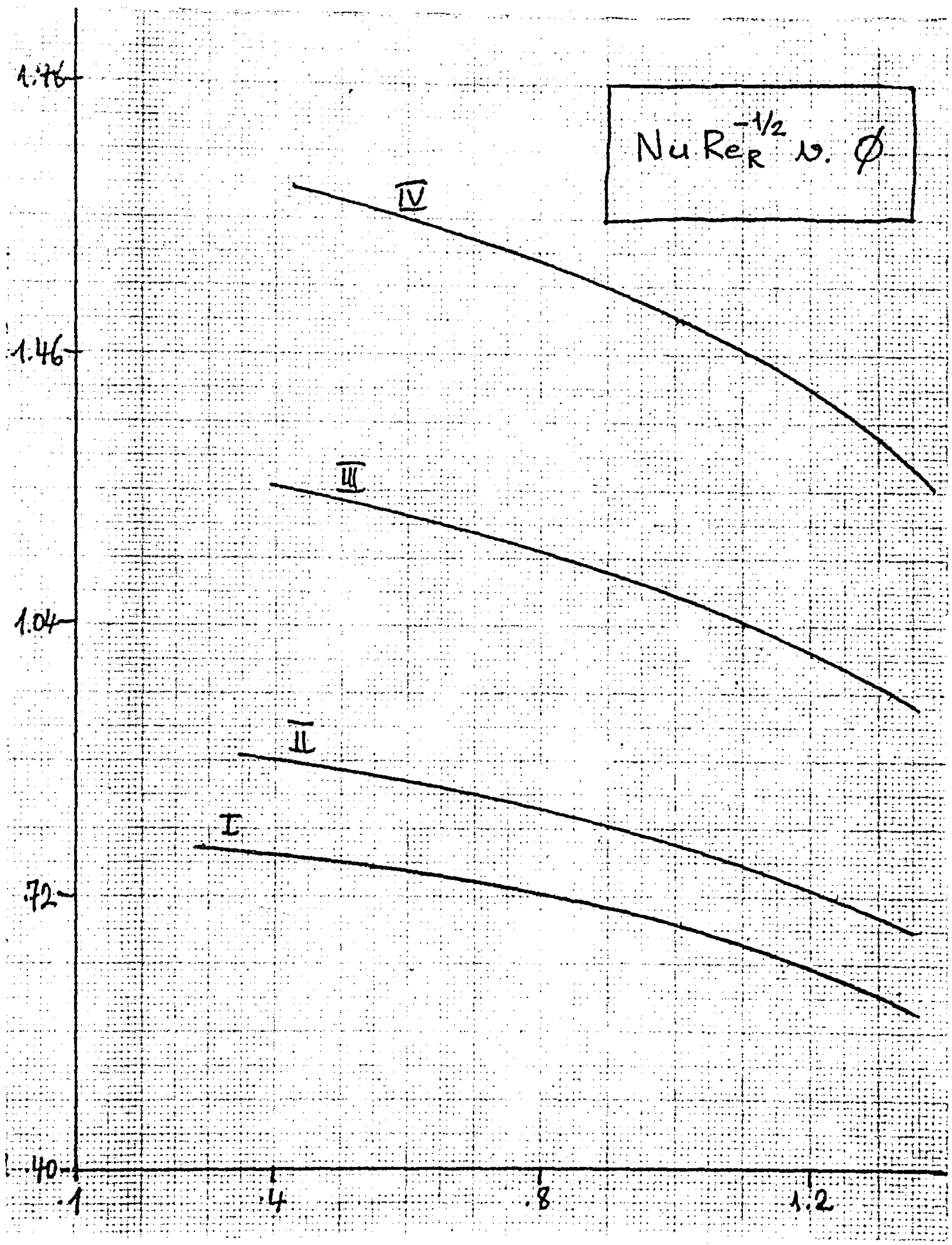

Fig. X - Angular distributions of the local Nusselt number

Bp = 2.0 , Pr = 5.5

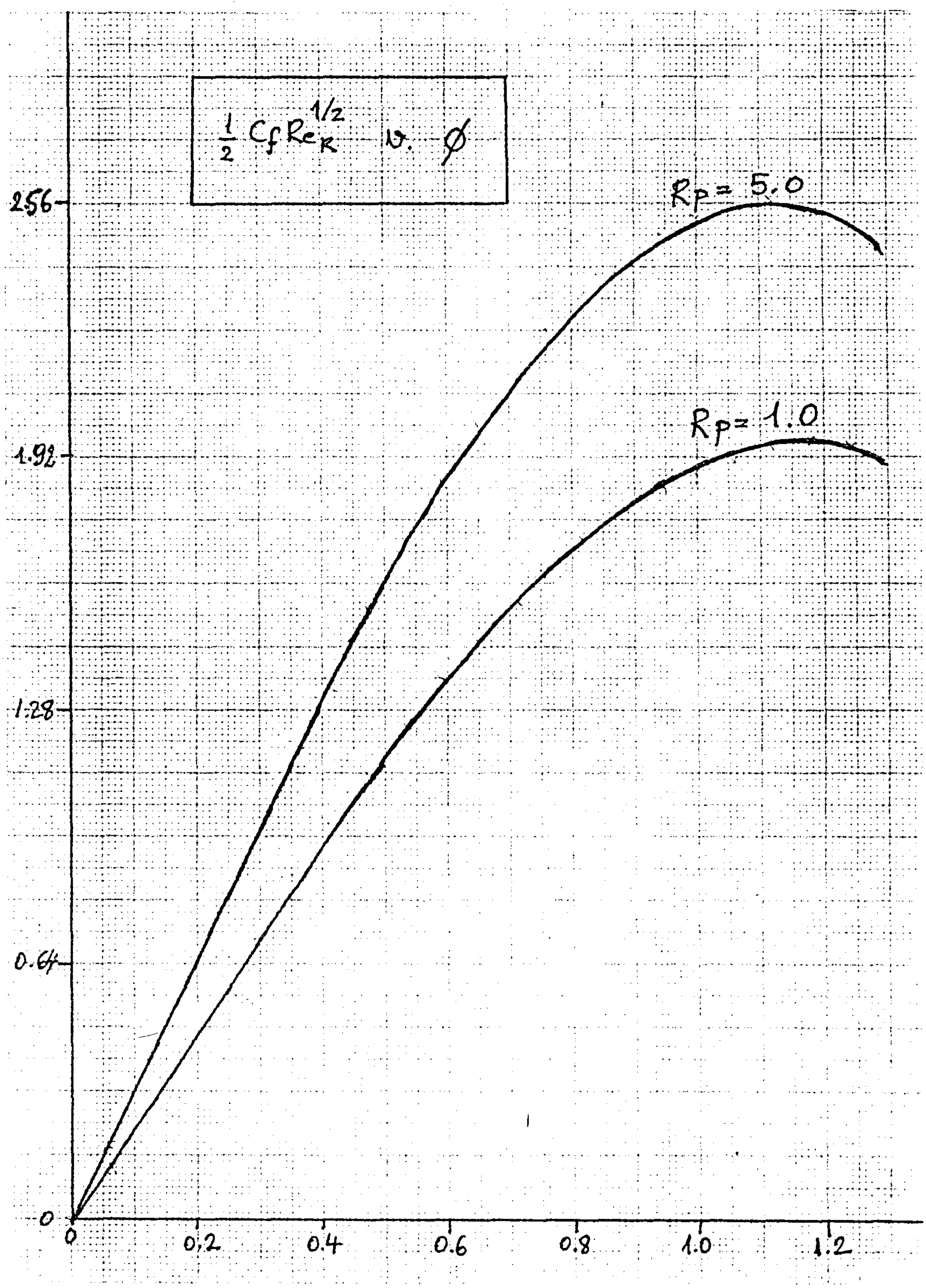

Fig. XI - Angular distributions of the local friction factor - Graph 1

Rp = 1.0 ,   Pr = 2.5

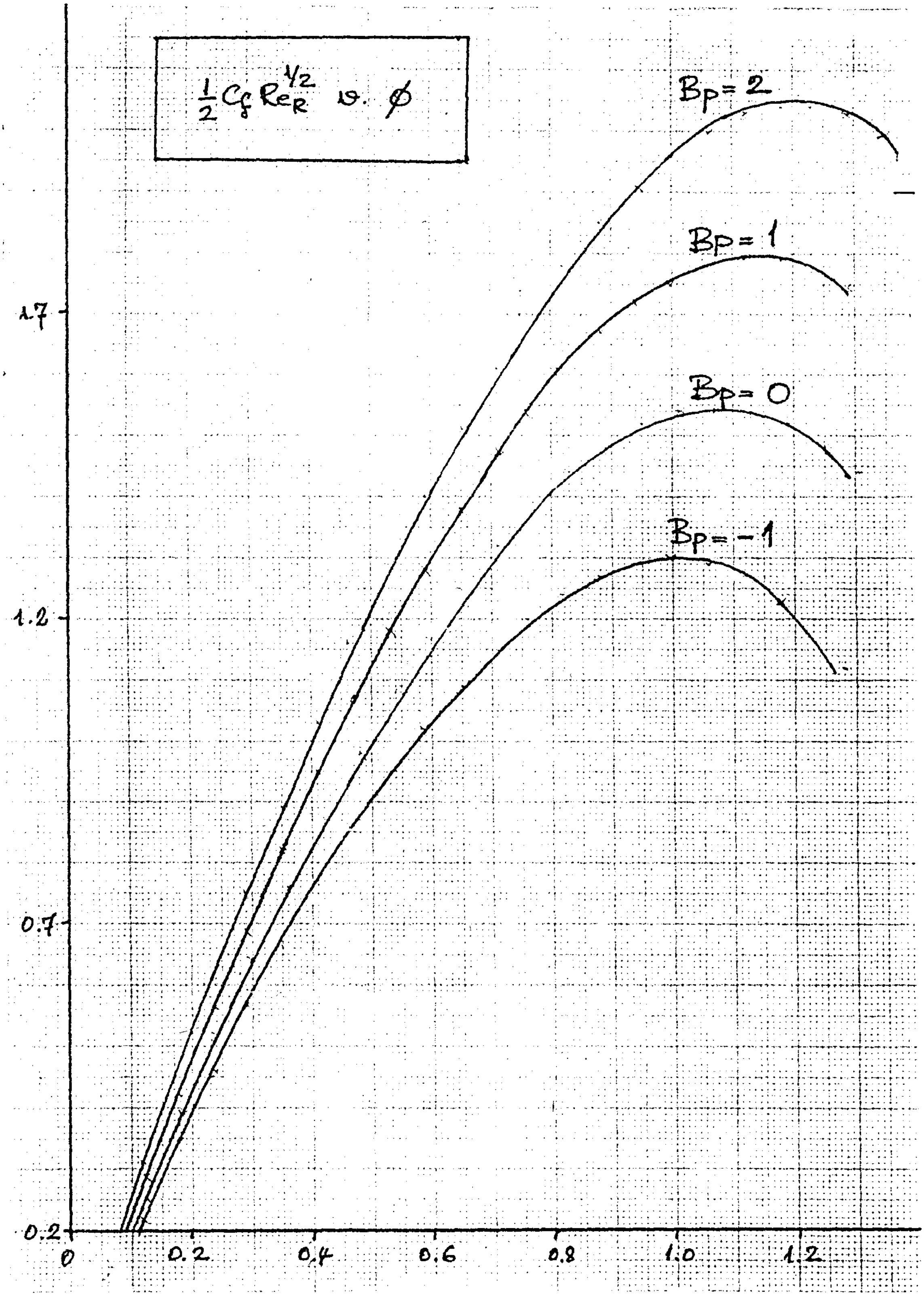

Fig. XII - Angular distributions of the local friction factor - Graph 2

Rp = 1.0 ,   Bp = 2.0

I - Pr = 1.0   III - Pr = 5.5
II - Pr = 2.5   IV - Pr = 15.0

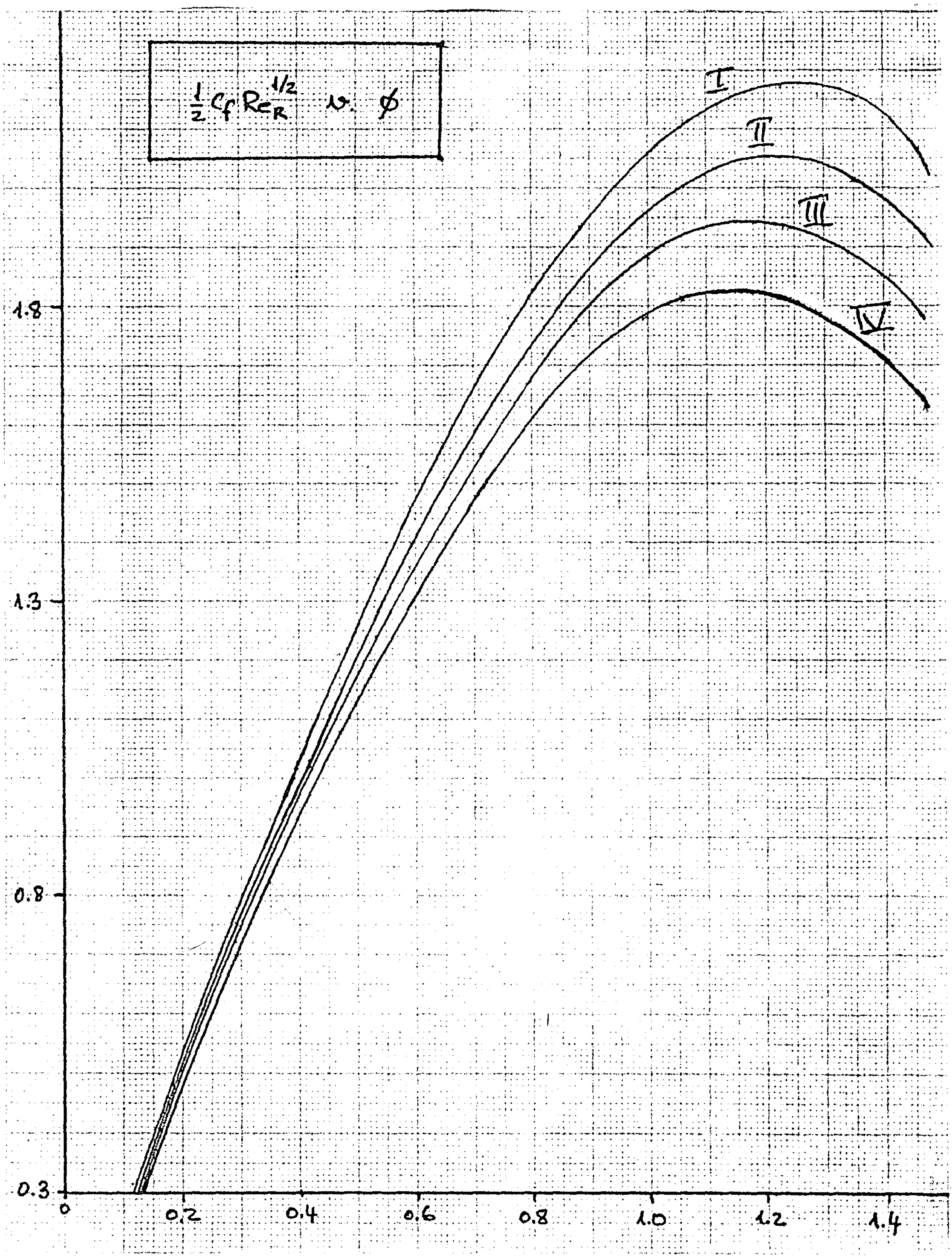

Fig. XIII - Angular distributions of the local friction factor - Graph 3

Rp = 1.0 ,   Bp = -1.0 (opposing flow)

I - Pr = 1.0
II - Pr = 5.5

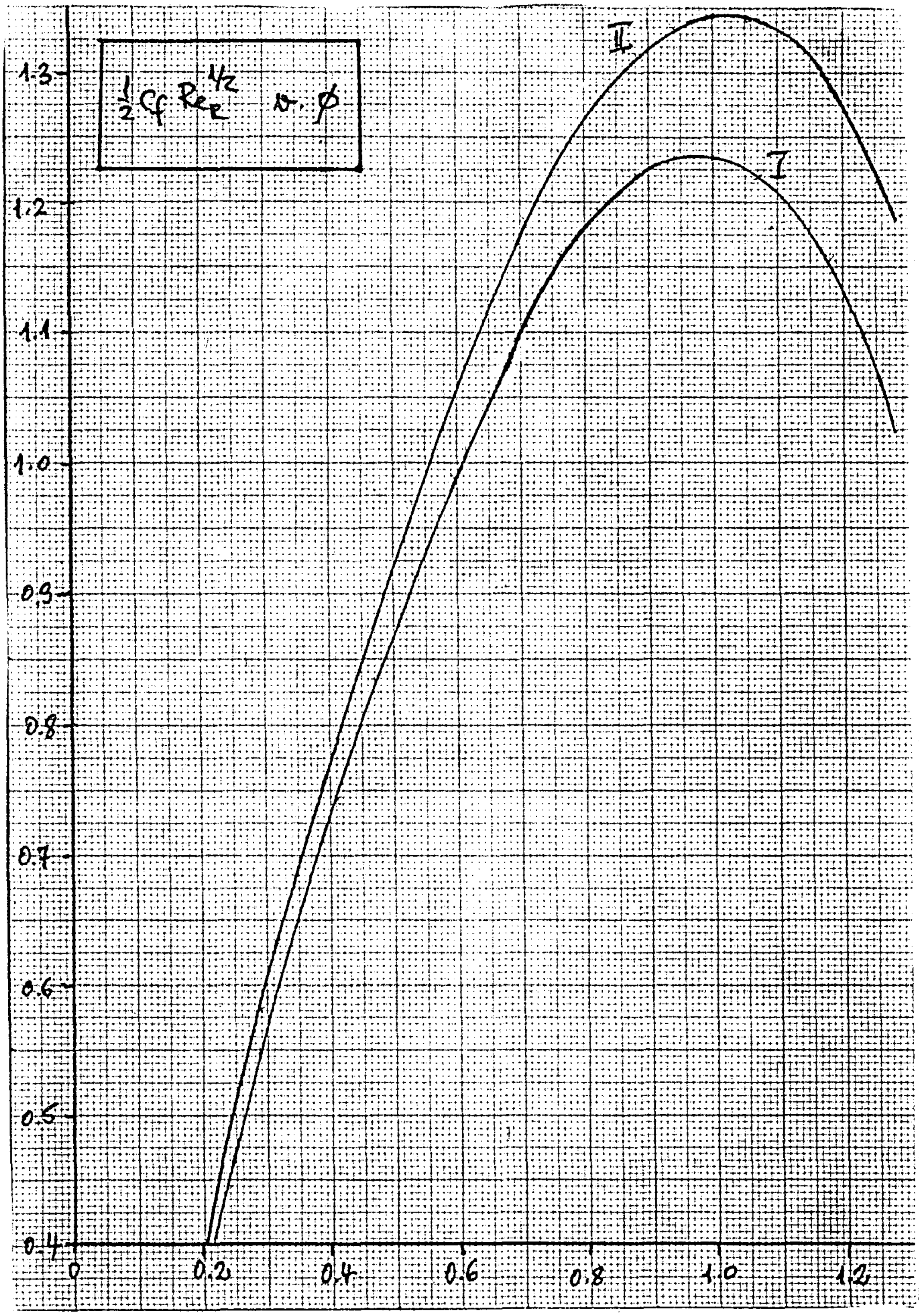

Fig. XIV - Angular distributions of the local friction factor - Graph 4

VI. DISCUSSION

In this chapter, the information of the RESULTS section of Chapter V will be referred to and the effects of the parameters will be examined and discussed. This process will be performed considering each parameter of the system one at a time. Those parameters include the rotation and the buoyancy parameters, the angular position, i.e. the angle measured from the stagnation point, and the Prandtl number. The examination of the effects of the Prandtl number is the actual object in this study, and therefore that will be considered as the last parameter in the chapter.

The effects of the parameters will be examined on the velocity and temperature distributions, the Nusselt number and the friction factor. Some of those are affected directly by any one of the parameters while the values of the others vary indirectly.

A comparison of the results obtained in this study with those of the previous ones is also necessary. However, the formulation differs somewhat in each of the previous investigations. It is not actually the results that are important but rather the conclusions. All those that will be stated for the cases of the rotation parameter effects and the buoyancy parameter effects are in agreement with those of Rajasekaran and Palekar [8]. The effects of the Prandtl number were not previously investigated for a rotaing sphere with mixed type of convection.

A. The Rotation Parameter

The rotation parameter, Rp, was previously defined. However, it is significant to mention here that it is dependent not only on the rotation speed but also on the free stream velocity. Therefore, an increase in the rotation parameter can be interpreted either as a decrease in the free stream velocity, u_∞, or as an increase in the rotation speed, Ω. The smallest value of the rotation parameter is zero, which corresponds to a non-rotating sphere. Very high values of the rotation parameter will not be considered in this study, because in such a case, a degeneration occurs in the formulation. Then the effects of the free stream velocity are diminished and the system looks like the case of the absence of flow. However, the non-dimensional forms are obtained for a non-zero set of values of the free stream velocity.

The effects of the rotation parameter will now be considered after the above establishment of its limits, within which the system will be examined. The velocity distributions and therefore the friction factor are those that are directly affected by the rotation parameter. An increase in that paramater results in a corresponding increase in the velocity gradient at the wall, and therefore a decrease in the velocity boundary-layer thickness. The overshooting of the velocity profiles beyond the local free stream velocity, observed at high values of the buoyancy parameter, takes place earlier as the rotation parameter is increased. This is due to the coupling between the buoyancy and rotation. The above-mentioned increase in the velocity gradiants at the wall reflects differently to velocities $\hat{u}$ and $\hat{w}$. Since the maximum value of the velocity $\hat{u}$, except the region where overshooting is observed, is the local free stream velocity, the velocity profile of $\hat{u}$ increases as the velocity gradiant increases. Overshooting also acts in the same direction. However, velocity $\hat{w}$ decreases with the rotation parameter, since the maximum velocity of rotation is at the surface of the sphere and increasing velocity gradient shifts the $\hat{w}$-profile downwards. However, such a comment is misleading. Although one may talk about a decrease in $\hat{w}$, there is actually an increase in the velocity in the rotating direction, since the rotation parameter increases, i.e. the rotation speed increases. However, this increase is suppressed when the non-dimensional form $\hat{w}$ is obtained in Section B, Chapter IV. As the rotation parameter increases, the friction factor also increases, as expected, due to the increase in the velocity gradient. The effects of the rotation parameter on the velocities and the local friction factor can be observed in figures II, III, VIII and XI. The overshooting is clear in figures III and IV.

The temperature distribution and the Nusselt number are indirectly affected by an increase in the rotation parameter. Because of the resulting increase in the velocity, a decrease in the temperature profile can be observed, if the wall temperature is higher than the surrounding fluid temperature. That, of course, implies an increase in the Nusselt number.

B. The Buoyancy Parameter

As it is pointed out earlier in the report, the buoyancy parameter can be interpreted as the degree of free-convection as compared to forced-convection. This implies that at Bp=0, the problem reduces to pure forced-convection. While on the other hand, as the buoyancy parameter takes higher values, the effect of the temperature on velocities is amplified. It is clear that there is no need for imposing restrictions on the buoyancy parameter as done in the case of the rotation parameter. However, it can be stated that pure forced-convection is possible while pure free-convection is not attainable in this study. This requires different formulations as done in the studies [6] and [7] of Chen and Mucoglu. According to the formulation of Chapter V, the buoyancy parameter may take either positive or negative values. They correspond to assisting and opposing flows, respectively. Explanation, regarding under what conditions the flow may be named as assisting and opposing, is present in Section A, Chapter IV. There is no need to discuss that once more. However, it is necessary to point out that the direction of the gravitational force is significant, since it is the gravitational field which causes the buoyancy force.

As in the case of the rotation parameter, the velocity distributions and therefore the friction factor are those that are directly influenced by the buoyancy parameter. For the assisting flow case, as the name suggests, the velocity increases with increasing values of the buoyancy parameter. In a way, the buoyancy parameter, when it takes positive values, aids the flow. That can be observed in figures III and II. High values of the buoyancy parameter causes overshooting of the velocity profiles beyond the local free stream velocity. The coupling between the rotation and the buoyancy increases the amount of overshooting, as mentioned previously. As in the case of the rotation paramater, the friction factor also increases with increasing buoyancy parameter as seen in figure XII. However when they are compared with each other, rotation has a more pronounced effect.

Also by the buoyancy parameter, the temperature distribution and the Nusselt number are indirectly affected. If the wall temperature is higher than the surrounding fluid temperature, a decrease in the temperature profile and therefore an increase in the Nusselt number is observed in consequence of the increase in velocity, when the buoyancy parameter had positive values (assisting flow) and is increased.

C. The Angular Position

Although the angular position is discussed as a parameter, it is quite different in nature when compared with the other parameters. It determines the point on the surface that will be considered as one of the boundaries. Due to symmetry, the angular position is enough as a parameter for this purpose. According to the formulation of Chapter V, the value of $\hat{x}$ gives directly the angular position. The minimum value is zero and corresponds to the stagnation point.

As mentioned above, the angular position is not a parameter in the sense of the parameters discussed earlier. The local free stream velocity, U_e, defined by equation (22), depends on the angular position. Since it acts as the velocity at the outer edge of the boundary layer and therefore as one of the boundary conditions of velocity $\hat{u}$, it is obvious that a different velocity profile will be obtained for different angular positions. That is due to the geometry of sphere. The same conclusion can be arrived at for velocity $\hat{w}$. Since points at different angular positions on the sphere rotate at different speeds, the corresponding velocity profiles vary accordingly. Therefore, the angular position can be discussed as a parameter only for temperature, the Nusselt number and the friction factor. The local Nusselt number and the local friction factor are represented in graphs as angular distributions. (Figs. X to XIV) The other variables can also be evaluated for various values of the angle ϕ, not for the sake of comparison, but for investigating the distributions at those regions of the sphere. Different angles are considered also in the graphs.

D. The Prandtl Number

As explained in Section B of Chapter III, the Prandtl number, Pr, is the ratio of the diffusivities. The kinematic viscosity is the rate at which momentum diffuses through the fluid due to molecular motion while the thermal diffusivity is the rate of diffusion of heat in the fluid. The range of the Prandtl numbers was also discussed in that section of the report.

The Prandtl number is an important parameter in this study, and its effects are examined for the first time for a rotating sphere in forced flow with buoyancy effects also considered. That is indicated in the study of Rajasekaran and Palekar [8], who have considered the effects of the other parameters and have indicated that the effects of the Prandtl number variation would be a subject matter for further investigations.

A change in the Prandtl number naturally affects the temperature distributions and the Nusselt number. The existence of the buoyancy parameter makes the Prandtl number possible to influence also the velocity distributions and the friction factor. When the buoyancy parameter takes larger values, those effects are easier to observe.

With increasing Prandtl number, the temperature distribution graph shifts downwards, as displayed in Fig. IX, because higher values of the Prandtl number imply much smaller values of the thermal diffusivity, when compared to those of the kinematic viscosity. As a consequence, heat is not diffused at a high rate. That explains why the temperature profiles become steeper. Those steeper profiles imply increased temperature gradient at the wall and therefore decreased thermal boundary layer thickness. As a result, the local Nusselt number increases as observed in Fig. X.

The change in the Prandtl number is reflected to the velocity distributions by way of the buoyancy parameter. As seen in figures II, IV, VI and VII, for assisting flow, an increase in the Prandtl number decreases the velocity profiles. Less steep curves imply increased diffusion of momentum. It may be observed in Fig. IV that the overshooting beyond the local free stream velocity is prevented as the Prandtl number increases. Then the rotation parameter have to take larger values, i.e. the spher has to rotate at a larger speed, before overshooting is detected. For opposing flow, on the other hand, velocity profiles take larger values as Pr increases. That is clear when one takes into account what is meant by opposing flow. That was explained in detail previously in the report. Such a problem with opposing flow is the case when the sphere surface temperature is less than than the fluid temperature. As it is stated above, parallel to the discussion in the previous sections, a decrease in $\hat{u}$ is followed with an increase in $\hat{w}$. That is due to the shape of the $\hat{w}$-profile. When Fig. VIII is observed, it is seen clearly that such an increase means less steeper profiles, as in the case of decreasing $\hat{u}$-curves.

Finally, the effects of the Prandtl number on the local friction faster are discussed. As Pr is increased, the friction factor takes smaller values because of the decreased velocity gradients, for the case of assisting flow. For the case of opposing flow, just the opposite is observed. The corresponding results are displayed in figures XIII and XIV, respectively.

VII. CONCLUSION

In this study of mixed convection about a rotating sphere, the velocity and temperature distributions, the local Nusselt number and the local friction factor are examined for varying parameters. Some of those parameters , the buoyancy and rotation parameters, are also used in the previous studies. The effects of those parameters on the flow and heat transfer are observed to agree well with those of the previous investigations. The Prandtl number variation is considered here , for the first time, according to literature. The results are displayed in graphs and the the effects of this parameter on the flow and heat transfer are examined and the reasons for such effects are analysed.

In previous studies usually the shooting methods were preferred and subroutines from the program libraries were used. In this study, however, in solving the governing equations the finite difference method is directly applied to the problem. The computer program developed here is applicable to various boundary conditions and it can be used with a modification for a wide range of body shapes.

APPENDIX A

The Derivation of the Equations of Motion for Boundary-layer Flow

The following discussion is adapted from that of Rosenhead [12]. The boundary-layer equations of motion together with the equation of continuity are derived for a general three-dimensional body in space.

Let $\vec{V}$ be the velocity vector in the fluid with the components V_1, V_2, V_3 corresponding to the curvilinear coordinates x_1, x_2, x_3. If $\vec{\nabla}$ denotes the gradient operator, the equations of motion of a viscous incompressible fluid can be expressed in the form

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\, \vec{V} = -\frac{1}{\rho} \vec{\nabla} p + \nu \nabla^2 \vec{V} \tag{61}$$

where p is the pressure. Let the surface of the given body be denoted by S. Then the position of a point in space is described by means of its distance x_3 measured along the unit normal $\vec{n}$ to S and the position vector $\vec{a}$ on S. Therefore, the position vector of such a point is

$$\vec{A} = \vec{a}(x_1, x_2) + x_3\, \vec{n}(x_1, x_2) \tag{62}$$

The gradient operator $\vec{\nabla}_S$ for the surface S is

$$\vec{\nabla}_S = \frac{\vec{a}_1}{h_1} \frac{\partial}{\partial x_1} + \frac{\vec{a}_2}{h_2} \frac{\partial}{\partial x_2} \tag{63}$$

where

$$h_1 = \left| \frac{\partial \vec{a}}{\partial x_1} \right|, \quad h_2 = \left| \frac{\partial \vec{a}}{\partial x_2} \right| \tag{64}$$

and

$$\vec{a}_1 = \frac{(\partial \vec{a}/\partial x_1)}{h_1}, \quad \vec{a}_2 = \frac{(\partial \vec{a}/\partial x_2)}{h_2} \tag{65}$$

where $\vec{a}_1$ and $\vec{a}_2$ are unit vectors on S, so that $\vec{a}_1$, $\vec{a}_2$, $\vec{n}$ form an orthogonal triad of unit vectors. The surfaces of x_3=constant make up the system of surfaces parallel to S. Let M denote a member of this system. Then

$$\vec{\nabla}_M = \vec{\nabla}_S + \vec{0}(x_3) \tag{66}$$

where $\vec{0}(x_3)$ denotes operators with coefficients of order x_3. Then the gradient operator $\vec{\nabla}$ for the space is

$$\vec{\nabla} = \vec{\nabla}_M + \vec{n}\frac{\partial}{\partial x_3} \tag{67}$$

Since the velocity vector $\vec{V}$ is in the form

$$\vec{V} = V_1\vec{a}_1 + V_2\vec{a}_2 + V_3\vec{n} = \vec{u} + V_3\vec{n} \tag{68}$$

it can be found that

$$\vec{V}\cdot\vec{\nabla} = (\vec{u} + V_3\vec{n})\left(\vec{\nabla}_M + \vec{n}\frac{\partial}{\partial x_3}\right) = \vec{u}\cdot\vec{\nabla}_M + V_3\frac{\partial}{\partial x_3} \tag{69}$$

while

$$\nabla^2 = \nabla_M^2 - J_M\frac{\partial}{\partial x_3} + \frac{\partial^2}{\partial x_3^2} \tag{70}$$

where J_M is the first curvature of the surface M, defined as

$$J_M = \vec{\nabla}_M\cdot\vec{n} \tag{71}$$

If there is a boundary layer on S, then x_3 and V_3 are small, and the derivatives with respect to x_3 are large compared with those with respect to x_1 and x_2. Therefore, $\vec{\nabla}_M$ can be replaced by $\vec{\nabla}_S$. Then equation (61) becomes

$$\frac{\partial \vec{u}}{\partial t} + \vec{n}\frac{\partial V_3}{\partial t} + (\vec{u}\cdot\vec{\nabla}_S + V_3\frac{\partial}{\partial x_3})(\vec{u} + V_3\vec{n}) = -\frac{1}{\rho}(\vec{\nabla}_S p + \vec{n}\frac{\partial p}{\partial x_3})$$

$$+\nu(\nabla_S^2 - J_S\frac{\partial}{\partial x_3} + \frac{\partial^2}{\partial x_3^2})(\vec{u} + V_3\vec{n}) \qquad (72)$$

After performing the necessary calculations, the boundary-layer equations (1) and (2) stated in Chapter XII can be obtained. The equation of continuity is

$$\vec{\nabla}\cdot\vec{V} = 0 \qquad (73)$$

After the same reasoning, equation (9) is formed.

For mathematical details, one may consult the study in Rosenhead [12].

APPENDIX B

The Computer Program

```
PROGRAM GEPR(INPUT,OUT,OUTPUT=OUT)
DIMENSION U(16,500),V(16,500),W(16,500),TEMP(16,500)
DIMENSION UU(16),UD(16),VU(16)
DIMENSION WU(16),WD(16),TEMPU(16),TEMPD(16)
DIMENSION UF(5,16),WF(5,16),TEMPF(5,16),KNG(5)
DIMENSION VNUS(500,4),VFRI(500,4),YVA(16),PRN(4)
DATA DELX,DELY,IMAX,NI,KMAX,NK/0.00084,0.20,16,1,1800,10/
DATA U1,U2,UIN,V1,VIN/0.0,1.5,0.0,0.0,0.0/
DATA W1,W2,WIN,T1,T2,TIN/1.0,0.0,0.0,1.0,0.0,0.0/
DATA ROTP,BUOP,KMF,NCH/1.0,2.0,1000,2/
DATA AU,AW,AT/1HU,1HW,4HTEMP/
DATA NPR,PRN,KRP/4,1.0,2.5,5.5,15.0,1/
IX=IMAX-1
UU(1)=U1
VU(1)=V1
WU(IMAX)=W2
TEMPU(IMAX)=T2
DO 27 I=1,IMAX,NI
YVA(I)=(I-1)*DELY
27 CONTINUE
DEY2=DELY**2
IF (NCH.EQ.1) PRN(1)=PRN(KRP)
IF (NCH.EQ.1) KRP=1
IF (NCH.EQ.1) NPR=1
IF (NCH.EQ.3) KMAX=KMF+1
DO 28 NP=1,NPR
DO 29 I=2,IX
UU(I)=UIN
VU(I)=VIN
WU(I)=WIN
TEMPU(I)=TIN
29 CONTINUE
NKV=1
DO 37 K=1,KMAX-1
NKV=NKV-1
AX=(K-1)*DELX
SAX=SIN(AX)
SAX2=SAX**2
UU(IMAX)=U2*SAX2
WU(1)=W1*SAX
TEMPU(1)=T1*SAX
VR=0.0
IF (SAX.GT.0.0001) VR=1.0/SAX
COT=COS(AX)*VR
SAX4=SAX**4
DO 30 I=2,IX
DEXU=0.0
IF (UU(I+1).GT.0.0008) DEXU=DELX/UU(I+1)
BUOY=BUOP*TEMPU(I)*SAX2
UD(I)=COT*(UU(I)**2+2.25*(SAX4+SAX2*ROTP*WU(I)**2))
UD(I)=UD(I)+SAX*(UU(I+1)-2.0*UU(I)+UU(I-1))/DEY2
```

```
UD(I)=UD(I)-VU(I)*(UU(I+1)-UU(I))/DELY
UD(I)=(UD(I)+BUOY)*DEXU+UU(I)
VU(I+1)=VU(I)-DELY*(UD(I)-UU(I))/DELX
IF (ROTP.EQ.0.0) GO TO 91
WD(I)=SAX*(WU(I+1)-2.0*WU(I)+WU(I-1))/DEY2
WD(I)=WD(I)-COT*UU(I)*WU(I)
WD(I)=(WD(I)-VU(I)*(WU(I+1)-WU(I))/DELY)*DEXU+WU(I)
91 TEMPD(I)=(TEMPU(I+1)-2.0*TEMPU(I)+TEMPU(I-1))*SAX
TEMPD(I)=TEMPD(I)/PRN(NP)/DEY2+UU(I)*TEMPU(I)*COT
TEMPD(I)=TEMPD(I)-VU(I)*(TEMPU(I+1)-TEMPU(I))/DELY
TEMPD(I)=TEMPD(I)*DEXU+TEMPU(I)
30 CONTINUE
IF (NKV.NE.0) GO TO 83
KN=(K+9)/10
NKV=NK
IF (NCH.GE.2) GO TO 84
IF (NCH.EQ.0.AND.NP.NE.KRP) GO TO 86
DO 34 I=1,IMAX
U(I,KN)=UU(I)
V(I,KN)=VU(I)
W(I,KN)=WU(I)
TEMP(I,KN)=TEMPU(I)*VR
34 CONTINUE
84 IF (NCH.EQ.3) GO TO 83
86 VFRI(KN,NP)=(UU(2)-UU(1))/DELY*VR
VNUS(KN,NP)=(TEMPU(1)-TEMPU(2))/DELY*VR
83 IF (K.NE.KMF.OR.NCH.EQ.1.OR.NCH.EQ.4) GO TO 87
DO 31 I=1,IMAX
UF(NP,I)=UU(I)
WF(NP,I)=WU(I)
TEMPF(NP,I)=TEMPU(I)/SAX
31 CONTINUE
87 DO 38 I=2,IX
UU(I)=UD(I)
WU(I)=WD(I)
TEMPU(I)=TEMPD(I)
38 CONTINUE
37 CONTINUE
28 CONTINUE
DO 32 NP=1,NPR
K=KN
VDB=VNUS(K-1,NP)-VNUS(K,NP)
81 K=K-1
VDA=VDB
VDB=VNUS(K-1,NP)-VNUS(K,NP)
IF (VDA.GE.VDB.AND.VDB.GE.0.0) GO TO 81
KNG(NP)=K
32 CONTINUE
IF (BUOP.GT.0.0) PRINT 25
IF (BUOP.LT.0.0) PRINT 26
PRINT2,ROTP
PRINT3,BUOP
IF (NCH.GE.2) GO TO 80
PRINT1,PRN(KRP)
1 FORMAT(/,10X,'PRANDTL NUMBER : ',F5.2)
PRINT6
```

```
6 FORMAT(///,20X,'U(X,Y)',/)
PRINT4,(YVA(I),I=1,IMAX,NI)
DO 35 K=1,KN,7
XVA=(K-1)*DELX*NK
PRINT5,XVA,(U(I,K),I=1,IMAX,NI),VFRI(K,KRP)
35 CONTINUE
PRINT7
7 FORMAT(///,20X,'V(X,Y)',/)
PRINT4,(YVA(I),I=1,IMAX,NI)
DO 40 K=1,KN,14
XVA=(K-1)*DELX*NK
PRINT9,XVA,(V(I,K),I=1,IMAX,NI)
40 CONTINUE
IF (ROTP.EQ.0.0) GO TO 92
PRINT8
8 FORMAT(///,20X,'W(X,Y)',/)
PRINT4,(YVA(I),I=1,IMAX,NI)
DO 50 K=1,KN,8
XVA=(K-1)*DELX*NK
PRINT10,XVA,(W(I,K),I=1,IMAX,NI)
50 CONTINUE
92 PRINT11
11 FORMAT(///,20X,'TEMP(X,Y)',/)
PRINT4,(YVA(I),I=1,IMAX,NI)
DO 60 K=KNG(KRP),KN,6
XVA=(K-1)*DELX*NK
PRINT5,XVA,(TEMP(I,K),I=1,IMAX,NI),VNUS(K,KRP)
60 CONTINUE
IF (NCH.EQ.1) GO TO 90
80 IF (NCH.EQ.4) GO TO 85
AXF=(KMF-1)*DELX
PRINT12,AXF
12 FORMAT(///,10X,'VARIATION OF PRANDTL NUMBER (X=',F4.2,)
PRINT13,(YVA(I),I=1,IMAX,NI)
PRINT14,(AU,PRN(NP),(UF(NP,I),I=1,IMAX,NI),NP=1,NPR)
IF (ROTP.EQ.0.0) GO TO 93
PRINT15
PRINT14,(AW,PRN(NP),(WF(NP,I),I=1,IMAX,NI),NP=1,NPR)
PRINT15
93 PRINT14,(AT,PRN(NP),(TEMPF(NP,I),I=1,IMAX,NI),NP=1,NPR)
IF (NCH.EQ.3) GO TO 90
85 PRINT 17
DO 36 K=1,KN
V(1,K)=(K-1)*DELX*NK
36 CONTINUE
PRINT19,(PRN(NP),NP=1,NPR)
PRINT16,(V(1,K),(VFRI(K,NP),NP=1,NPR),K=1,KN,7)
PRINT18
PRINT19,(PRN(NP),NP=1,NPR)
DO 33 K=KNG(1),KN,5
IF (K.LT.KNG(4)) GO TO 82
PRINT16,V(1,K),(VNUS(K,NP),NP=1,NPR)
GO TO 33
82 IF (K.GE.KNG(3)) PRINT 20,V(1,K),(VNUS(K,NP),NP=1,3)
IF (K.GE.KNG(3)) GO TO 33
IF (K.GE.KNG(2)) PRINT 21,V(1,K),(VNUS(K,NP),NP=1,2)
IF (K.LT.KNG(2)) PRINT 22,V(1,K),VNUS(K,1)
```

```
33 CONTINUE
90 STOP
25 FORMAT(///,10X,'ASSISTING FLOW')
26 FORMAT(///,10X,'OPPOSING FLOW')
2 FORMAT(//,10X,'ROTATION PARAMETER : ',F5.1)
3 FORMAT(/,10X,'BUOYANCY PARAMETER : ',F5.1)
4 FORMAT(/,4X,'X',16(2X,'Y=',F3.1),/)
13 FORMAT(/,8X,'PN',16(2X,'Y=',F3.1),/)
14 FORMAT(1X,A4,F5.1,16F7.3)
5 FORMAT(1X,F4.2,16F7.3,F8.4)
9 FORMAT(1X,F4.2,16F7.2)
10 FORMAT(1X,F4.2,16F7.3)
15 FORMAT(/)
16 FORMAT(7X,F4.2,4F11.4)
17 FORMAT(///,10X,'FRICTION FACTOR',//)
18 FORMAT(///,10X,'NUSSELT NUMBER',//)
19 FORMAT(10X,'X',4(3X,'PRN=',F4.1))
20 FORMAT(7X,F4.2,3F11.4)
21 FORMAT(7X,F4.2,2F11.4)
22 FORMAT(7X,F4.2,F11.4)
END
```

APPENDIX C

Symbols in the Computer Program

I	:	Variation in the y-direction
K	:	Variation in the x-direction
U(I,K)	:	Velocity $\hat{u}$
V(I,K)	:	Velocity $\hat{v}$
W(I,K)	:	Velocity $\hat{w}$
TEMP(I,K)	:	Temperature $\hat{T}$
UU,VU,WU,TEMPU	:	Upstream values
UD,WD,TEMPD	:	Downstream values
NPR	:	Variation in the Prandtl number
UF(NPR,I)	:	Velocity $\hat{u}$ for various Pr
WF(NPR,I)	:	Velocity $\hat{w}$ for various Pr
TEMPF(NPR,I)	:	Temperature $\hat{T}$ for various Pr
KNG(NPR)	:	Starting point for graphs of Nu for various Pr
VNUS(K,NPR)	:	The local Nu for various Pr
VFRI(K,NPR)	:	The local C_f for various Pr
YVA(I)	:	Value of $\hat{y}$ at I'th point
PRN(NPR)	:	The Prandtl number
DELX	:	$\Delta\hat{x}$
DELY	:	$\Delta\hat{y}$
IMAX	:	Maximum value of I
NI	:	Every NI of I is considered
KMAX	:	Maximum value of K
NK	:	Every NK of K is considered
U1,V1,W1,T1	:	The values of the variables on the surface
U2,W2,T2	:	The values of the variables away from the surface
UIN,VIN,WIN,TIN	:	The values at the stagnation point
ROTP	:	The Rotation parameter
BUOP	:	The Buoyancy parameter
KMF	:	The value of K for the desired angle
NCH	:	The choice of work to be done
KRP	:	PRN(KRP) is the Pr value for which the distributions ase calculated.

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