# RESPONSE OF AN ELASTIC HALF-SPACE DUE TO A FINITE SIZED LINE SOURCE

by

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# ABSTRACT

In this thesis, the transient response of an isotropic, homogenous and elastic half-space due to a buried finite sized line source was analyzed. The receiver was always taken on the surface. The solutions corresponding different orientations of line source and receiver were obtained. Numerical results were illustrated in graphs. The time dependence of source functions was taken in the form of Delta-Dirac function.

The results obtained in this work can be used to explain the effect of the size of the transducers used in Nondestructive Testing of materials.

# ÖZET


Bu tezde, homojen, elastik ve isotropik bir yarı-uzayın, gömülü bir sonlu çizgi kuvvet kaynağı nedeniyle oluşan zamana bağlı tepkisi incelenmiştir. Alıcı sürekli olarak yüzeyde tutulmuştur. Farklı kuvvet kaynağı ve alıcı yerleştirmelerine göre farklı çözümler elde edilmiştir. Sayısal sonuçlar grafikler şeklinde verilmiştir. Kuvvet kaynağının zaman fonksiyonu Delta-Dirac fonksiyonu olarak alınmıştır.

Bu çalışmada elde edilen sonuçlar, tahribatsız malzeme kontrollerinde kullanılan çevireçlerin (transducer) yüzey alanlarının kontrol üzerindeki etkisini saptamakta kullanılabilir.

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# I. INTRODUCTION

The quality and integrity of a structural material is greatly affected by the number and size of the defects such as cracks, volds, etc., contained in the material. Under loading or other service conditions, these defects may cause uneven distribution of stresses which in turn may produce more defects and then cause failures. Hence, it is vitally important to detect these defects before any catastrophic failure occurs. The technology used in detecting the defects without changing the original form of the structural material is called Nondestructive Testing (NOT) of materials.

Whenever forces are applied on the material, the plastic deformations or local failures, occured at the defective points, create transient elastic waves due to the rapid release of energy at these failure or yielding points. That wave emission is acoustic. So, the technique of detecting and recording transducers planted at the surface of the material, is called acoustic emission technique in the field of (NDT). The recorded signals are then related to the location and physical characteristics of the local deformation of failure.

For this purpose, the transducers to be used must be calibrated. The transducers are calibrated as a source and as a receiver; through a comparison with a standard source and a standard receiver respectively. A transducer of known characteristics and a transfer media of known theoretical solution is used for calibration. To obtain a known theoretical solution for a chosen media, i.e., half-space, the scientists model the source as equivalent body forces inside the media, distributed over a line or the surface of the defect. The equivalent body force is defined as the body force which should be applied in the absence of the defect to produce the same radiation as a given defect. These idealized sources can be a center of compression, a single point source, a single couple, a double force, a double couple with or without moment, a finite sized line source and a finite sized areal source or any combination of these.

The purpose of this thesis is to study the response of a homogeneous isotropic and elastic half-space due to a finite sized line source. The basis for such an idealized source mechanism is the single point source [5]. That is, the response equations will be obtained by integrating the point source expressions over the finite line of the source. Hence, the numerical results of the surface response can be easily obtained by employing the numerical methods.

The basis for the mathematical analysis of the problem will be the theory of generalized rays. A well documented study on this theory is in the work of Achenbach [2]. In this theory, the response of the media is sorted out into individual rays originating from the same source location, but travelling along different paths before reaching to the receiver. The expression for each ray is in terms of complicated integrals including a source function describing the source, a receiver function depending on the quantity to be calculated, and, the reflection coefficients describing the path upon reflections from the surface. The transient response is then obtained by taking the inverse Laplace transform of the expressions for each ray using the Cagniard's method.

A brief summary of the basic equations of elasticity are given, and the particular solutions for the displacements due to a point force are found in Chapter II.

In Chapter III, a brief history of the method of generalized rays is given. Also in this chapter, the reflection coefficients for a free surface and the ray solution for a half-space are discussed. The expressions for receiver functions and the expressions for source functions for a line source are also given. In Chapter IV, application of the Cagniard's method and the inversion of Laplace transforms are discussed.

Finally, in Chapter V, numerical works and results are presented and discussed. Surface displacements of a half-space due to a line source inside the media are given for different locations of the receiver and for different sizes of the source.

# II. EQUATIONS OF ELASTICITY AND SOLUTION FOR A POINT SOURCE IN AN UNBOUNDED MEDIUM

The basic equations of dynamic elasticity and the particular solution for a single point source in an unbounded, isotropic, homogenous and elastic medium will be presented through this chapter. The linearized equations of motion and solutions of them for a point force in infinite media can be found in the classical book by Love [11] and Achenbach [2]. We recast these solutions in terms of Laplace transforms so that one can modify them for the half-space problems as well.

## 2.1 DYNAMIC EQUATIONS OF ELASTICITY

When forces are applied on a solid body, the body deforms, i.e., the distance between any two points changes. In this thesis, all the strains are taken to be very small, so that the linear equations of the theory of elasticity are applicable.

In the linear theory of elasticity, the displacement field $\underline{u}(\underline{r},t)$ for a homogeneous, isotropic and elastic body satisfies the equation of motion, [2],

$$\mu \nabla^2 \underline{u} + (\lambda + 2\mu)\, \underline{\nabla}\, (\underline{\nabla} \cdot \underline{u}) + \rho\, \underline{F} = \rho\, \underline{\ddot{u}} \qquad (2.1)$$

where $\rho$ is the mass density, $\lambda$ and $\mu$ Lame constants of the material, $\underline{F}$ is the body force per unit mass. A "dot" denotes the partial differentiation with respect to time, t. In the above equation $\nabla^2$ is the Laplacian operator and, $\underline{\nabla}\cdot$ and $\underline{\nabla}$ are the divergence and gradient operators respectively.

The stress-displacement relations for an isotropic elastic material are given by,

$$\underline{\underline{\sigma}} = \lambda\ (\underline{\nabla}\cdot\underline{u})\ \underline{\underline{I}} + \mu(\underline{\nabla}\ \underline{u} + (\underline{\nabla}\ \underline{u})^{T}) \qquad (2.2)$$

where $\underline{\underline{\sigma}}$ and $\underline{\underline{I}}$ are the Cauchy stress and the unit tensors respectively.

Equations (2.1) and (2.2) must be satisfied at every interior point of a body occupying a volume V in space bounded by a surface S. To obtain a solution for the problem, the boundary conditions on S and the initial conditions on the displacement and velocity fields throughout the body at initial time, i.e., t = 0, must be specified. The boundary conditions involving tractions are of the form

$$\underline{T} = \underline{\underline{\sigma}} \cdot \underline{n} \qquad \text{on } S \qquad (2.3)$$

where $\underline{n}$ is the unit outward normal to the surface S. The boundary conditions must be replaced by the radiation conditions in the case of an unbounded medium. That is, all components of the displacement vector must vanish at infinity. The initial conditions to be specified are in general of the form

$$\begin{aligned} \underline{u}\ (\underline{r}, 0) &= \underline{u}_0\ (\underline{r}) \\ \dot{\underline{u}}\ (\underline{r}, 0) &= \dot{\underline{u}}_0\ (\underline{r}) \end{aligned} \qquad (2.4)$$

where $\underline{u}\ \ (\underline{r})$ and $\dot{\underline{u}}_0\ (\underline{r})$ are the initial displacement and velocity fields respectively.

Our approach to solve Eq.(2.1) will be first to reduce it to wave equations, using the Helmholtz decomposition theorem. According to Helmholtz theorem a single valued vector field can be expressed as the sum of the gradient of a scalar field and the curl of a vector field, [5],

$$\underline{u} = \underline{\nabla}\,\phi + \underline{\nabla} \times \Psi_1 \quad , \quad \nabla \cdot \Psi_1 = 0$$
$$\underline{F} = \underline{\nabla}\,G + \underline{\nabla} \times \underline{H} \quad , \quad \nabla \cdot \underline{H} = 0 \tag{2.5}$$

where $\phi$, G, and, $\Psi_1$, H are called the scalar and vector potentials respectively. Substitution of the above equations in Eq.(2.1) to wave equations

$$c^2\, \nabla^2\phi + G = \ddot{\phi} \quad , \quad c^2 = (\lambda + 2\mu)\,/\,\rho$$
$$C^2\, \nabla^2\underline{\Psi}_1 + \underline{H} = \ddot{\underline{\Psi}}_1 \quad , \quad C^2 = \mu/\rho \tag{2.6}$$

The potentials $\phi$ and $\Psi_1$ give rise to waves known as longitudinal waves (P-waves) and shear waves (S-waves) respectively. The longitudinal waves travel with the speed c and shear waves travel with C. For plane waves the particle motion of a P-wave is in the direction of $\underline{n}$, the normal to the wave front, and that of S-wave is in a plane perpendicular to $\underline{n}$.

In curvilinear coordinate systems, the equations in terms of the components of the vector potential are coupled, hence, it is difficult to obtain a solution. However, in the study of propogation of elastic waves, the particle displacement due to S-waves is further decomposed into two orthogonal components. The one that is parallel to a given direction is called the SH-component and the other is called the SV-component. Waves associated with these displacements are called the SH-(horizontally polarized) and SV-(vertically polarized) waves. Thus, a decomposition of the form,

$$\Psi_1 = \chi\, \underline{e}_z \pm \underline{\nabla} \times (\Psi\, \underline{e}_z)$$
$$\underline{H} = H_1\, \underline{e}_z \pm \underline{\nabla} \times (H_2\, \underline{e}_z) \tag{2.7}$$

is possible [20]. In these equations, $\underline{e}_z$ is a unit vector perpendicular to the surface of the half-space. With these equations Eq. (2.6) reduce to,

$$
\begin{aligned}
&c^2 \nabla^2 \phi + G = \ddot{\phi} \\
&C^2 \nabla^2 \Psi + H_2 = \ddot{\Psi} \qquad (2.8) \\
&C^2 \nabla^2 \chi + H_1 = \ddot{\chi}
\end{aligned}
$$

with the initial conditions,

$$
\begin{aligned}
&\phi(\underline{r}, 0) = \phi_0(\underline{r}) \quad , \quad \dot{\phi}(\underline{r}, 0) = \dot{\phi}(\underline{r}) \\
&\Psi(\underline{r}, 0) = \Psi_0(\underline{r}) \quad , \quad \dot{\Psi}(\underline{r}, 0) = \dot{\Psi}(\underline{r}) \qquad (2.9) \\
&\chi(\underline{r}, 0) = \chi_0(\underline{r}) \quad , \quad \dot{\chi}(\underline{r}, 0) = \dot{\chi}_0(\underline{r})
\end{aligned}
$$

It is understood that $\Psi$ gives rise to SV-wave and $\chi$ to SH-wave.

For analyzing the waves in a half space, it is convenient to introduce the following non-dimensional quantities,

$$
\begin{aligned}
&\underline{u} = r_0 \, \underline{\hat{u}} \quad , \quad t = r_0 \, \hat{t}/c \quad , \quad \underline{\nabla} = r_0^{-1} \, \underline{\hat{\nabla}} \\
&\phi = r_0^2 \, \hat{\phi} \quad , \quad \Psi = r_0^3 \, \hat{\Psi} \quad , \quad \chi = r_0^2 \, \hat{\chi} \qquad (2.10) \\
&G = c^2 \, \hat{G} \quad , \quad H_2 = r_0 \, c^2 \, \hat{H}_2 \quad , \quad H_1 = c^2 \, \hat{H}_1 \\
&\underline{\underline{\sigma}} = (\lambda + 2\mu) \, \hat{\underline{\underline{\sigma}}} \quad , \quad \kappa = c/C
\end{aligned}
$$

where "^" denotes the non-dimensional quantities and $r_0$ is the distance between the source and the receiver in r-direction. Employing these non-dimensional quantities, Eq. (2.8) becomes,

$$\hat{\nabla}^2 \hat{\phi} + \hat{G} = \ddot{\hat{\phi}}$$

$$\hat{\nabla}^2 \hat{\Psi} + \kappa^2 \hat{H}_2 = \kappa^2 \ddot{\hat{\Psi}} \qquad (2.11)$$

$$\hat{\nabla}^2 \hat{\chi} + \kappa^2 \hat{H}_1 = \kappa^2 \ddot{\hat{\chi}}$$

The "^" sign, from here on, will be dropped with the understanding that all the quantities used are dimensionless.

## 2.2 PARTICULAR SOLUTION FOR A SINGLE POINT FORCE

In this section, the general solutions for $\phi$, $\Psi$, and $\chi$ in Eq.(2.11) satisfying the boundary and initial conditions will be obtained using the transform techniques. Laplace transform of a function $f(\underline{r},t)$, denoted as $\bar{f}(\underline{r},s)$ is defined as,

$$\bar{F}(\underline{r},s) = \int_0^\infty F(\underline{r},t)\, e^{-st}\, dt$$

$$F(\underline{r},t) = \frac{1}{2\pi i} \int_{Br} \bar{F}(\underline{r},s)\, e^{st}\, dt \qquad (2.12)$$

where s is the transform variable and $B_r$ is the Bromwich contour in the complex s-plane, which is a line parallel to the imaginary axis and to the right of all singularities of $\bar{f}(\underline{r},s)$. Note that the second equation defines the inverse transform.

Consider a concentrated force with a time function f(t) applied at a point $(0,0,z_0)$ and acting in the direction of a unit vector $\underline{a}$, defined as,

$$\underline{a} = a_1\, \underline{e}_1 + a_2\, \underline{e}_2 + a_3\, \underline{e}_3$$

$$= a_r\, \underline{e}_r + a_\theta\, \underline{e}_\theta + a_z\, \underline{e}_z \qquad (2.13)$$

where $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ and $\underline{e}_r$, $\underline{e}_\theta$, $\underline{e}_z$ are the unit vectors in cartesian and cylindrical conditions respectively.

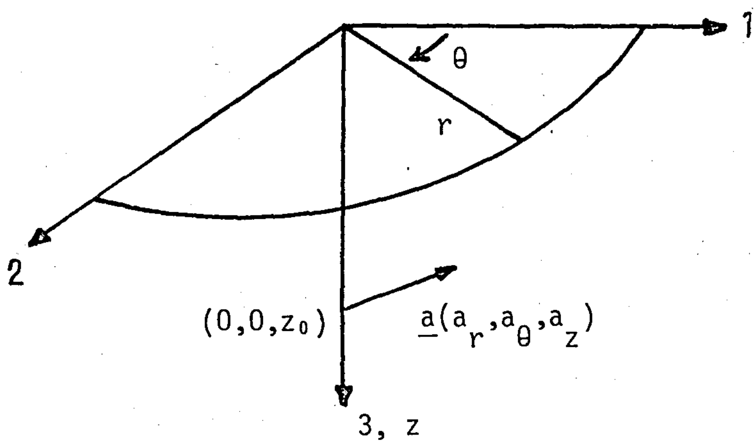


Figure 2.1 Geometry for a point force

Components of the vector $\underline{a}$ in the cylindrical coordinates are related to the components in cartesian coordinates through the relations

$$
\begin{aligned}
a_r &= a_1 \cos\theta + a_2 \sin\theta \\
a_\theta &= -a_1 \sin\theta + a_2 \cos\theta \qquad (2.14) \\
a_z &= a_3
\end{aligned}
$$

The concentrated force, $\underline{F}$ is represented by

$$\underline{F} = \underline{a}\, F_0 F(t)\ \delta(z - z_0)\ \delta(r)/2\pi r \qquad (2.15)$$

and a particular solutions for potentials $\phi$, $\chi$ and $\Psi$ are (see Appendix A),

$$\bar{\phi}(\underline{r},\underline{s},\underline{a}) = a_z \; s^2\bar{F}(s) \int_0^\infty S_p \; e^{-s\eta|z-z_0|} J_0(s\,\xi\, r)\xi \; d\xi$$

$$+ \; a_r \; s^2\bar{F}(s) \int_0^\infty S'_p \; e^{-s\eta|z-z_0|} J_1(s\,\xi\, r)\xi \; d\xi \qquad (2.16a)$$

$$\bar{\chi}(\underline{r},\underline{s},\underline{a}) = -a_\theta \; s^2\bar{F}(s) \int_0^\infty S_H \; e^{-s\zeta|z-z_0|} J_1(s\,\xi\, r) \; d\xi \qquad (2.16b)$$

$$\bar{\Psi}(\underline{r},\underline{s},\underline{a}) = -a_z \; s\;\bar{F}(s) \int_0^\infty S_v \; e^{-s\zeta|z-z_0|} J_0(s\,\xi\, r) \; d\xi$$

$$-a_r \; s\;\bar{F}(s) \int_0^\infty S'_v \; e^{-s\zeta|z-z_0|} J_1(s\,\xi\, r) \; d\xi \qquad (2.16c)$$

where,

$$\bar{F}(s) = F_0 \; \bar{f}(s)/4\pi\,\kappa^2\, s^2\, \mu\, r_0^2 \quad , \quad S_p = -\varepsilon \quad , \quad S'_p = -\varepsilon/\eta$$

$$S_H = \kappa^2/\zeta \quad , \quad S_v = \xi/\zeta \quad , \quad S'_v = \varepsilon \qquad (2.17)$$

$$\zeta = (\xi^2 + \kappa^2)^{1\;2} \qquad \eta = (\xi^2 + 1)^{1\;2} \qquad \varepsilon = \mathrm{sgn}(z - z_0)$$

$\eta$ and $\zeta$ are the slowness in the z direction of the P and S waves respectively, and  is the directivity constant and $S_j$, $S'_j$ are the source functions with the subscript j indicating the type. Note that, Eq. (2.16) completely agrees with Ceranoğlu [5].

## 2.3 DISPLACEMENTS DUE TO A SINGLE POINT FORCE

In studying the response of a half-space due to a point force, it is more convenient to use the cylindrical coordinates (r,θ,z). In this case, the Laplacian operator takes the form,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (2.18)$$

Using Eq.(2.5) and Eq.(2.7), the relations between the displacement and the potentials can be written,

$$u_r = \frac{\partial\phi}{\partial r} + \frac{\partial^2\Psi}{\partial r\partial z} + \frac{\partial\chi}{\partial\theta}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{1}{r}\frac{\partial^2\Psi}{\partial\theta\partial z} - \frac{\partial\chi}{\partial r} \tag{2.19}$$

$$u_z = \frac{\partial\phi}{\partial z} + \frac{\partial^2\Psi}{\partial z^2} - \kappa^2\frac{\partial^2\Psi}{\partial t^2}$$

Substituting the potentials from Eq.(2.19) into Eq.(2.16) we obtain the displacements as,

$$\begin{aligned}\bar{u}_z(\underline{r},\underline{s},\underline{a}) = s^3\,\bar{F}(s)\,\{&a_z \int_0^\infty S_pD_{zp}\,e^{-s\eta|z-z_0|}\,J_0(s\xi r)\xi\,d\xi \\ &+ a_z \int_0^\infty S_vD_{zv}\,e^{-s\zeta|z-z_0|}\,J_0(s\xi r)\xi\,d\xi \\ &+ a_r \int_0^\infty S'_pD_{zp}\,e^{-s\eta|z-z_0|}\,J_1(s\xi r)\xi\,d\xi \\ &+ a_r \int_0^\infty S'_vD_{zv}\,e^{-s\zeta|z-z_0|}\,J_1(s\xi r)\xi\,d\xi\,\}\end{aligned} \tag{2.20a}$$

$$\begin{aligned}\bar{u}_r(\underline{r},\underline{s},\underline{a}) = s^3\,\bar{F}(s)\,\{&a_z \int_0^\infty S_pD_{rp}\,e^{-s\eta|z-z_0|}\,J_1(s\xi r)\xi\,d\xi \\ &+ a_z \int_0^\infty S_vD_{rv}\,e^{-s\zeta|z-z_0|}\,J_1(s\xi r)\xi\,d\xi \\ &- a_r \int_0^\infty S'_pD_{rp}\,e^{-s\eta|z-z_0|}\,J_0(s\xi r)\xi\,d\xi \\ &- a_r \int_0^\infty S'_vD_{rv}\,e^{-s\zeta|z-z_0|}\,J_0(s\xi r)\xi\,d\xi\,\}\end{aligned} \tag{2.20b}$$

$$+ a_r \frac{s^2}{r} \bar{F}(s) \{ \int_0^\infty S'_p D_{rp} e^{-s\eta|z-z_0|} J_1(s\xi r) d\xi$$

$$+ \int_0^\infty S'_v D_{rv} e^{-s\zeta|z-z_0|} J_1(s\xi r) d\xi$$

$$+ \int_0^\infty S_H D_{rH} e^{-s\zeta|z-z_0|} J_1(s\xi r) d\xi \}$$

$$\bar{u}_\theta(\underline{r},s,\underline{a}) = -a_\theta \frac{s^2}{r} \bar{F}(s) \{ \int_0^\infty S'_p D_p e^{-s\eta|z-z_0|} J_1(s\xi r) d\xi$$

$$+ \int_0^\infty S_v D_H e^{-s\zeta|z-z_0|} J_1(s\xi r) d\xi \qquad (2.20c)$$

$$+ \int_0^\infty S_H D_{\theta H} e^{-s\zeta|z-z_0|} J_1(s\xi r) d\xi \}$$

$$+ a_\theta s^3 \bar{F}(s) \int_0^\infty S_H D_{\theta H} e^{-s\zeta|z-z_0|} J_0(s\xi r)\xi d\xi$$

where $J_0$ and $J_1$ are the zeroth and first order Bessel functions and,

$$D_{zp} = -\varepsilon\eta \quad , \quad D_{zv} = -\xi \quad , \quad D_{rp} = D_{\theta p} = -\xi$$
$$D_{rv} = D_{\theta v} = -\varepsilon\zeta \quad , \quad D_{rH} = D_{\theta H} = 1 \qquad (2.21)$$

Note that $D_{ij}$'s denotes the receiver functions with subscript i indicating the direction and subscript j indicating the type of wave. For simplicity, Eq.(2.20) can be written using a different notation as,

$$\bar{u}_{zj}(\underline{r},s,\underline{a}) = s^3 \bar{F}(s) \{ a_z \int_0^\infty S_j D_{zj} e^{-sh_j(z,\xi)} J_0(s\xi r)\xi d\xi$$

$$+ a_r \int_0^\infty S'_j D_{zj} e^{-sh_j(z,\xi)} J_1(s\xi r)\xi d\xi \} \qquad (2.22a)$$

$$\bar{u}_{rj}(\underline{r},s,\underline{a}) = s^3\, \bar{F}(s)\ \{ a_z \int_0^\infty S_j\, D_{rj}\, e^{-sh_j(z,\xi)}\, J_1(s\,\xi\, r)\xi\, d\xi$$

$$- a_r \int_0^\infty S_j'\, D_{rj}\, e^{-sh_j(z,\xi)}\, J_0(s\,\xi\, r)\xi\, d\xi\ \}$$

$$+ a_r \frac{s^2}{r} \bar{F}(s)\ \{ \int_0^\infty S_j'\, D_{rj}\, e^{-sh_j(z,\xi)}\, J_1(s\,\xi\, r)\, d\xi$$

$$+ \int_0^\infty S_{Hj}\, D_{rHj}\, e^{-sh_j(z,\xi)}\, J_1(s\,\xi\, r)\, d\xi\ \} \quad (2.22b)$$

$$\bar{u}_{\theta j} = (\underline{r},s,\underline{a}) = -a_\theta \frac{s^2}{r}\ \bar{F}(s)\ \{ \int_0^\infty S_j'\, D_{\theta j}\, \bar{e}^{sh_j(z,\xi)}\, J_1(s\,\xi\, r)\, d\xi$$

$$+ \int_0^\infty S_{Hj}\, D_{\theta Hj}\, e^{-sh_j(z,\xi)}\, J_1(s\,\xi\, r)\, d\xi \quad (2.22c)$$

$$+ a_\theta\, s^3\, \bar{F}(s) \int_0^\infty S_{Hj}\, D_{\theta Hj}\, e^{-sh_j(z,\xi)}\, J_0(s\,\xi\, r)\xi\, d\xi$$

where,

$$h_j(z,\xi) = \begin{cases} \eta|z - z_0| & \text{for P waves} \\ \zeta|z - z_0| & \text{for SH and SV-waves} \end{cases} \quad (2.23)$$

The transient response of the unbounded medium can be simply obtained by takina the inverse Laplace transform of the Eq's. (2.22a,b,c) [5].

# III. METHOD OF GENERALIZED RAYS AND SOLUTION FOR A HALF-SPACE

In a bounded medium, waves radiated from the source travel among the different paths. Some of the waves originated from the source will reach the observation point after several reflections at the boundaries. Method of generalized rays is based on expressing the solution for such a bounded medium in terms of individual rays which propogate along different paths. Each ray is identified with a source and a receiver function together with a specific combination of the coefficients. The complete solution is then obtained by summing up the responses for all possible rays.

## 3.1. METHOD OF GENERALIZED RAYS, A BRIEF HISTORY

The classical approach in studying the response of an elastic medium under any kind of excitation is to use the theory of normal modes. There, the complementary solutions of the potentials $\phi$, $\Psi$ and $\chi$ of Eq.(2.8) are found with two unknown coefficients for each. For axisymmetric loadings, i.e., $\chi = 0$, the number of unknowns reduce to four. By applying the boundary conditions to the solutions, the unknowns are all determined. Taking the inverse transform, the solution is then obtained in terms of a summation of individual modes. Hence, the accuracy of results are limited with the number of terms taken in the summation. An alternative method is knowns as the generalized ray theory, where the displacement field or the stress field is expressed in terms of the contributions due to different rays travelling along different paths between the source and the receiver. Summing all the particular solutions, one can obtain the final solution.

Generalized wave theory is well known in the field of optics, where the individual rays along which light propogates are obtained by using the high frequency limit of the solution, (Born and Wolf). Aside from this high frequency limit analysis, one can sort out the solution into individual rays by expanding the denominator of constant unknowns of potentials into a series of products of the reflection and the transmission (which is in the case of the two media in contact) coefficients using the Bromwhich expansion.

The generalized ray theory was applied to the propogation of elastic waves by Cagniard [4], when he studied the transient waves in two half-spaces welded together. Through a series of contour deformations and change of integration variables, he was able to find the inverse transform of the expressions for each ray.

Lamb [9] solved the buried force problem in a half-space when he studied the propogation of earth tremors over the surface of the earth. He completed the inversion of Fourier transform in the time domain by changing the integration variables in a manner very similar to Cagniard's. Later on, Pekeris and Lifson [17] solved the buried and surface source problems in a half-space. Lapwood [10] and Garwin [8] formulated the buried line source problem using the generalized ray theory and Cagniard's method. Tangential surface load over a half-space was studied by Chao [7]. Norwood [13] studied the case of rectangular load and proposed a method to remove the singularities that are along the integration path which made the analytical solution for the case of loads applied over finite regions. The generalized ray theory and the Cagniard's method was first applied to the plate problems by Mencher [12], Sherwood [18], Spencer [19] and recently Ceranoğlu [5] are the other contributors of the plate problem. Pao and Gajewski [16] carried the method to a layered media. All the works cited above were the mediums with plane boundaries. Recently, the generalized ray theory was extended to the study of wave propagation media such as hollow spheres and cylinders, Chen [6], Pao and Ceranoğlu [15].

Until 1960's, the individual ray integrals in the generalized ray theory were found by using the Bromwich expansion of the exact solution. This procedure gets tedious for a layered media problem. Spencer [19] showed that the integral solution for each of the multiply reflected and transmitted rays in a layered media can be found by a ray grouping technique. In that technique, the rays are grouped effectively disregarding their mode conversion history. Wu [21] and Norwood [14] used this technique together with Cagniard's method in solving the problem of the propogation of transient waves in an elastic thick plate under an arbitrary load.

## 3.2 REDUCED BOUNDARY CONDITIONS FOR A TRACTION FREE PLANE SURFACE

At a traction free plane surface, the boundary conditions of Eq.(2.3) becomes,

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \tag{3.1}$$

where $\underline{n}$ is the normal vector to the surface. Employing Eq.(2.19) in Eq.(2.2), we obtain

$$\sigma_{zz} = \frac{\kappa^2-2}{\kappa^2}\frac{\partial^2\phi}{\partial t^2} + \frac{2}{\kappa^2}\frac{\partial}{\partial z}\left[\frac{\partial\phi}{\partial z} + \frac{\partial^2\Psi}{\partial z^2} - \kappa^2\frac{\partial^2\Psi}{\partial t^2}\right]$$

$$\sigma_{rr} = \frac{\kappa^2-2}{\kappa^2}\frac{\partial^2\phi}{\partial t^2} + \frac{2}{\kappa^2}\frac{\partial}{\partial r}\left[\frac{\partial\phi}{\partial r} + \frac{\partial^2\Psi}{\partial r\partial z} + \frac{1}{r}\frac{\partial\chi}{\partial\theta}\right]$$

$$\sigma_{\theta\theta} = \frac{\kappa^2-2}{\kappa^2}\frac{\partial^2\phi}{\partial t^2} + \frac{2}{r\kappa^2}\left[\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\chi}{\partial\theta}\right.$$

$$\left. - \frac{\partial^2\chi}{\partial r\partial\theta} + \frac{\partial^2\Psi}{\partial r\partial z} \quad \frac{1}{r}\frac{\partial^2\Psi}{\partial\theta^2\partial z}\right]$$

$$\sigma_{r\theta} = \frac{2}{\kappa^2 r}\frac{\partial}{\partial\theta}\left[\frac{\partial\phi}{\partial r} - \frac{\phi}{r} \quad \frac{\partial^2\Psi}{\partial r\partial z} - \frac{1}{r}\frac{\partial\Psi}{\partial z}\right]$$

$$+ \frac{1}{\kappa^2}\left[\frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2} - \frac{\partial^2\chi}{\partial r^2}\right] \tag{3.2}$$

$$\sigma_{rz} = \frac{1}{\kappa^2}\frac{\partial}{\partial r}\left[2\frac{\partial\phi}{\partial z} + 2\frac{\partial^2\Psi}{\partial z^2} - \kappa^2\frac{\partial^2\Psi}{\partial t^2}\right] + \frac{1}{\kappa^2 r}\frac{\partial}{\partial\theta}\left[\frac{\partial X}{\partial z}\right]$$

$$\sigma_{z\theta} = \frac{1}{\kappa^2 r}\frac{\partial}{\partial\theta}\left[2\frac{\partial\phi}{\partial z} + 2\frac{\partial^2\Psi}{\partial z^2} - \kappa^2\frac{\partial^2\Psi}{\partial t^2}\right] - \frac{1}{\kappa^2 r}\frac{\partial}{\partial r}\left[\frac{\partial X}{\partial z}\right]$$

If the equation of the surface is $z = L$, where $L$ is a constant, then the $z$ components of the stress tensor must disappear

$$\sigma_{zz} = \sigma_{rz} = \sigma_{z\theta} = 0 \quad , \quad \text{at } z = L \tag{3.3}$$

From Eq.(3.2), we then have,

$$(\kappa^2-2)\ddot{\phi} + 2\frac{\partial}{\partial z}\left[\frac{\partial\phi}{\partial z} + \frac{\partial^2\Psi}{\partial z^2} - \kappa^2\ddot{\Psi}\right] = 0 \quad , \quad \text{at } z = L \tag{3.4a}$$

$$\frac{\partial}{\partial r}\left[2\frac{\partial\phi}{\partial z} + 2\frac{\partial^2\Psi}{\partial z^2} - \kappa^2\ddot{\Psi}\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial X}{\partial z}\right) = 0 \quad , \quad \text{at } z = L \tag{3.4b}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left[2\frac{\partial\phi}{\partial z} + 2\frac{\partial^2\Psi}{\partial z^2} - \kappa^2\ddot{\Psi}\right] - \frac{\partial}{\partial r}\left(\frac{\partial X}{\partial z}\right) = 0 \quad , \quad \text{at } z = L \tag{3.4c}$$

Let's consider a function $\sigma(r,\theta)$ such that,

$$\left(2\frac{\partial\phi}{\partial z} + 2\frac{\partial^2\Psi}{\partial z^2} - \kappa^2\ddot{\Psi}\right)\Bigg|_{z=L} = \frac{\partial\sigma}{\partial\theta} \tag{3.5a}$$

$$\left(\frac{\partial X}{\partial z}\right)\Bigg|_{z=L} = -r\frac{\partial\sigma}{\partial r} \tag{3.5b}$$

then Eq.(3.4b) is identically satisfied and Eq.(3.4c) becomes,

$$\frac{\partial^2\sigma}{\partial r^2} + \frac{1}{r}\frac{\partial\sigma}{\partial r} + \frac{1}{r^2}\frac{\partial^2\sigma}{\partial\theta^2} = 0 \qquad (3.6)$$

Note that, the above expression is the Laplacian of $\sigma(r,\theta)$ in the plane $z = L$. Hence $\sigma$ is a harmonic function. Since the solution has to be bounded at infinity, Liouville's theorem states that $\sigma$ is constant. Therefore, the boundary conditions given by Eq.(3.4) reduce to,

$$\left\{ (\kappa^2 - 2)\ddot{\phi} + 2\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z} + \frac{\partial^2\Psi}{\partial z^2} - \kappa^2\ddot{\Psi}\right) \right\}_{z=L} = 0 \qquad (3.7a)$$

$$\Sigma\Big|_{z=L} = \left(2\frac{\partial\phi}{\partial z} + 2\frac{\partial^2\Psi}{\partial z^2} - \kappa^2\ddot{\Psi}\right)\Big|_{z=L} = 0 \qquad (3.7b)$$

$$\Sigma_H\Big|_{z=L} = \left(\frac{\partial X}{\partial z}\right)\Big|_{z=L} = 0 \qquad (3.7c)$$

where $\Sigma$ shows the component of the stresses $\sigma_{rz}$ and $\sigma_{\theta z}$ due to P-wave and SV-wave, and $\Sigma_H$ due to SH-wave. Note that, these are the same boundary conditions used by Ceranoğlu [5] and Başakar [3] . By employing these conditions, it will be possible to obtain the reflection coefficients.

## 3.3. REFLECTION COEFFICIENTS AT A FREE PLANE SURFACE

Some of the waves radiated from a buried source in a half-space travel to the observation point directly. These waves behave just like those which travel in an infinite medium. However, some of the waves reach to the observation point after transmissions or reflections.

The reflection coefficients at a free surface will be derived considering the case of a concentrated point force inside a half-space. Before deriving the stress equations belonging to the reflected waves, we are going to derive the equations belonging to the incident waves. Substitution of potentials Eq.(2.16) into Eq.(3.2) after taking Laplace transform yields,

$$\sigma_{zz}^{(inc)} = s^4 \bar{F}(s) \{ a_z \int_0^\infty S_p D_{zz}^{(p)} e^{-s\eta|z-z_0|} J_0 \xi \, d\xi$$

$$+ a_z \int_0^\infty S_v D_{zz}^{(v)} e^{-s\zeta|z-z_0|} J_0 \xi \, d\xi$$

$$+ a_r \int_0^\infty S'_p D_{zz}^{(p)} e^{-s\eta|z-z_0|} J_1 \xi \, d\xi$$

$$+ a_r \int_0^\infty S'_v D_{zz}^{(v)} e^{-s\zeta|z-z_0|} J_1 \xi \, d\xi \} \qquad (3.8)$$

where,

$$D_{zz}^{(p)} = (\xi^2 + \zeta^2)/\kappa^2 \quad , \quad D_{zz}^{(v)} = 2\,\varepsilon\zeta\xi/\kappa^2 \qquad (3.9)$$

Similar expressions can be obtained for $\bar{\Sigma}^{(inc)}$ and $\bar{\Sigma}_H^{(inc)}$ by using potentials Eq.(2.16) in Eq.(3.7b) and Eq.(3.7c)

$$\bar{\Sigma}^{(inc)} = -s^3 \bar{F}(s) \{ a_z \int_0^\infty S_p D_z^{(p)} e^{-s\eta|z-z_0|} J_0 \, d\xi$$

$$+ a_z \int_0^\infty S_v D_\Sigma^{(v)} e^{-s\zeta|z-z_0|} J_0 \, d\xi$$

$$+ a_r \int_0^\infty S'_p D_\Sigma^{(p)} e^{-s\eta|z-z_0|} J_1 \, d\xi$$

$$+ a_r \int_0^\infty S'_v D_\Sigma^{(v)} e^{-s\zeta|z-z_0|} J_1 \, d\xi \} \qquad (3.10)$$

$$\bar{\Sigma}_H^{(inc)} = -s^3 \bar{F}(s)\, a_\theta \int_0^\infty S_H\, D_\Sigma^{(H)}\, e^{-s\zeta|z-z_0|}\, J_1\, d\xi \qquad (3.11)$$

where,

$$D_\Sigma^{(p)} = -2\varepsilon\eta\xi \quad , \quad D_\Sigma^{(v)} = -(\xi^2 + \zeta^2) \; , \quad D_\Sigma^{(H)} = -\varepsilon\zeta \qquad (3.12)$$

It is well known that elastic waves of either mode, P- or SV-, when incident on a plane surface, will be reflected as two waves, one in each mode. For example, a P-wave will be reflected as a P-wave and a SV-wave, the latter is known as mode conversion. However, a SH-wave will be reflected only as a SH-wave; i.e., there is no mode conversion for SH-wave.

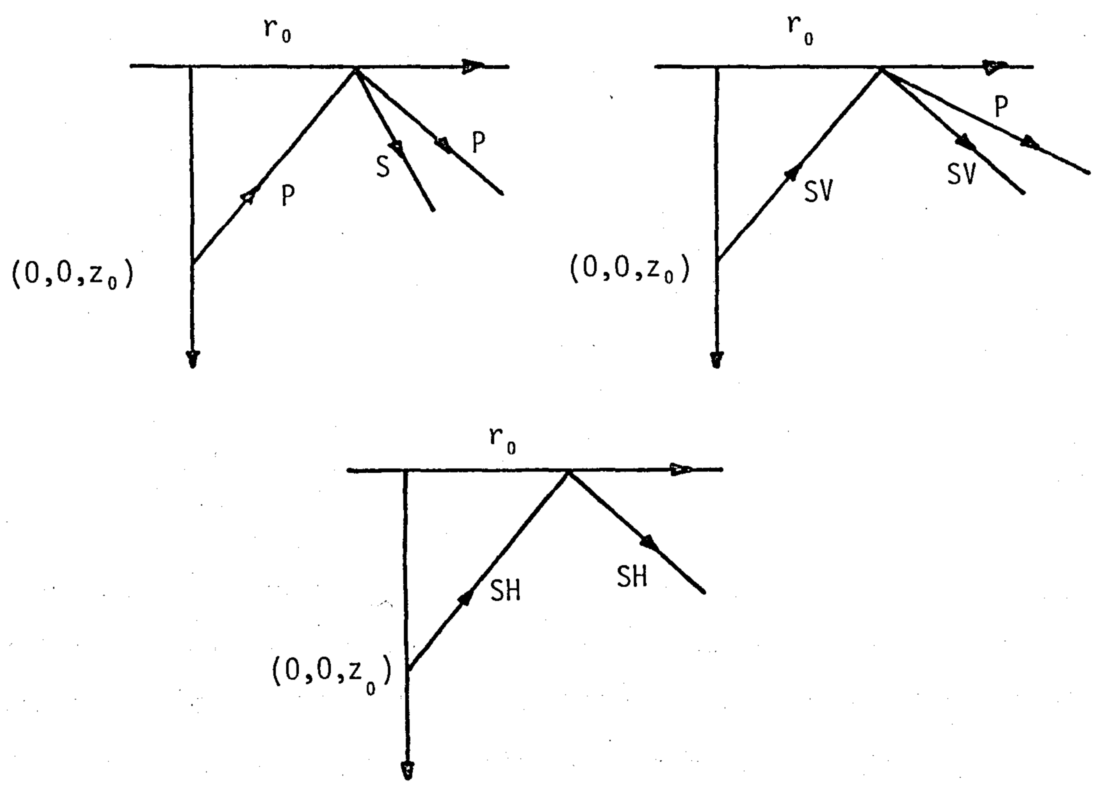


Figure 3.1 Reflection and mode conversion

In half space, $z \geq 0$, the waves which are generated at $z = z_0$ will be reflected when they reach to the surface $z = 0$. We will assume that the ratio, R, of the amplitudes of the reflected waves to the incident waves is some function of the slowness. By this assumption and taking care of the mode conversion, we can express the corresponding of Eq.(3.8), Eq.(3.10) and Eq.(3.11) for reflected waves as,

$$
\begin{aligned}
\bar{\sigma}_{zz}^{(ref)} = s^4 \bar{F}(s) \{ & a_z \int_0^\infty S_p R^{pp} D_{zz}^{(p)} e^{-s\eta(z+z_0)} J_0 \xi \, d\xi \\
& + a_z \int_0^\infty S_p R^{ps} D_{zz}^{(v)} e^{-s(\eta z_0+\zeta z)} J_0 \xi \, d\xi \\
& + a_z \int_0^\infty S_v R^{ss} D_{zz}^{(v)} e^{-s\zeta(z_0+z)} J_0 \xi \, d\xi \\
& + a_z \int_0^\infty S_v R^{sp} D_{zz}^{(p)} e^{-s(\zeta z_0+\eta z)} J_0 \xi \, d\xi \\
& + a_r \int_0^\infty S_p' R^{pp} D_{zz}^{(p)} e^{-s\eta(z+z_0)} J_1 \xi \, d\xi \\
& + a_r \int_0^\infty S_p' R^{ps} D_{zz}^{(v)} e^{-s(\eta z_0+\zeta z)} J_1 \xi \, d\xi \\
& + a_r \int_0^\infty S_v' R^{ss} D_{zz}^{(v)} e^{-s\zeta(z+z_0)} J_1 \xi \, d\xi \\
& + a_r \int_0^\infty S_v' R^{sp} D_{zz}^{(p)} e^{-s(\zeta z_0 + \eta z)} J_1 \xi \, d\xi \} \qquad (3.13a)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\Sigma}^{(ref)} = -s^3 \bar{F}(s) \{ & a_z \int_0^\infty S_p R^{pp} D^{(p)} e^{-s\eta(z_0+z)} J_0 \, d\xi \\
& + a_z \int_0^\infty S_p R^{ps} D^{(v)} e^{-s(\eta z_0+\zeta z)} J_0 \, d\xi \\
& + a_z \int_0^\infty S_v R^{ss} D^{(v)} e^{-s\zeta(z_0+z)} J_0 \, d\xi \\
& + a_z \int_0^\infty S_v R^{sp} D^{(p)} e^{-s(\zeta z_0+\eta z)} J_0 \, d\xi
\end{aligned}
$$

$$+ \; a_r \int_0^\infty S_p' \, R^{pp} \, D^{(p)} \, e^{-s\eta(z_0+z)} \, J_1 \, d\xi$$

$$+ \; a_r \int_0^\infty S_p' \, R^{ps} \, D^{(v)} \, e^{-s(\eta z_0+\eta z)} \, J_1 \, d\xi$$

$$+ \; a_r \int_0^\infty S_v' \, R^{ss} \, D^{(v)} \, e^{-s\zeta(z_0+z)} \, J_1 \, d\xi$$

$$+ \; a_r \int_0^\infty S_v' \, R^{sp} \, D^{(H)} \, e^{-s(\zeta z_0+\eta z)} \, J_1 \, d\xi \; \} \qquad (3.13b)$$

$$\bar{\Sigma}_H^{(ref)} = -s^3 \, \bar{F}(s) \, a_\theta \int_0^\infty S_H \, R^H \, D^{(H)} \, e^{-s\zeta(z_0+z)} \, J_1 \, d\xi \qquad (3.13c)$$

The total wave field, which equals the sum of incident waves and the reflected waves must satisfy the boundary conditions. Hence, by adding the corresponding equations of incidence and reflected waves, and forcing the result to satisfy the traction free boundary condition at z = 0, we have,

$$D_{zz}^{(p)} + R^{pp} \, D_{zz}^{(p)} + R^{ps} \, D_{zz}^{(v)} = 0$$

$$D_{\Sigma}^{(p)} + R^{pp} \, D_{\Sigma}^{(p)} + R^{ps} \, D_{\Sigma}^{(v)} = 0$$

$$D_{zz}^{(v)} + R^{ss} \, D_{zz}^{(v)} + R^{sp} \, D_{zz}^{(p)} = 0 \qquad (3.14)$$

$$D_{\Sigma}^{(v)} + R^{ss} \, D_{\Sigma}^{(v)} + R^{sp} \, D_{\Sigma}^{(p)} = 0$$

$$D_{\Sigma}^{(H)} + R^{H} \, D_{\Sigma}^{(H)} = 0$$

Substituting the values of receiver function, i.e., D's, from Eq.(3.9) and Eq.(3.12) yields,

$$R^{pp} = R^{ss} = |4\eta\zeta\xi^2 + (\xi^2 + \sigma^2)^2| \,/\, \Delta r$$

$$R^{ps} = -4\eta\xi \;(\xi^2 + \zeta^2)/\Delta r$$

$$R^{sp} = (\frac{\zeta}{\eta}) \; R^{ps} \qquad (3.15)$$

$$R^{H} = 1$$

$$\Delta r = 4\eta\zeta\xi^2 - (\xi^2 + \zeta^2)^2$$

The above equations are called the generalized reflection coefficients for the surface $z = 0$.

## 3.4. RAYS SOLUTIONS FOR A HALF SPACE

The complete solution of the displacements due to transient waves in a media is obtained by combining the particular solutions of each ray. These rays may include all types of waves such as longitudinal waves (P-waves), shear waves (S-waves),head waves, Rayleigh waves and Stoneley waves. The last wave type only exists when two media are in contact.

Consider a concentrated force acting at a point $(0,0,z_0)$ and a receiver at a point $Q(r,\theta,z)$

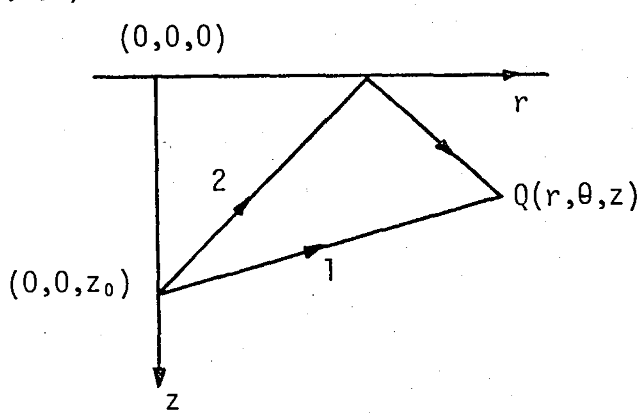


Figure 3.2 Rays in a half-space

It is clear that, there are only two paths of waves. First one is the path 1 where a direct P-wave or a direct S-wave travel along. As there is no reflection in path 1, these waves act just like those in an unbounded medium. Eq.(2.20a) gives the vertical displacement due to the waves travelling along path 1,

$$\begin{aligned}\bar{u}_z^1(\underline{r},s,\underline{a}) = s^3 \bar{F}(s) \{ & a_z \int_0^\infty S_p D_{zp} e^{-s\eta|z-z_0|} J_0(s\xi r)\xi \, d\xi \\ & + a_z \int_0^\infty S_v D_{zv} e^{-s\zeta|z-z_0|} J_0(s\xi r)\xi \, d\xi \\ & + a_r \int_0^\infty S'_p D_{zp} e^{-s\eta|z-z_0|} J_1(s\xi r)\xi \, d\xi \\ & + a_r \int_0^\infty S'_v D_{zv} e^{-s\zeta|z-z_0|} J_1(s\xi r)\xi \, d\xi \} \end{aligned} \qquad (2.20a)$$

The waves that are reflected at the surface will reach the receiver by travelling along the path 2. Again employing Eq.(2.16) in Eq.(2.19) by taking care of the reflection, we get

$$\begin{aligned}\bar{u}_z^2(\underline{r},s,\underline{a}) = s^3 F(s) \{ & [a_z \int_0^\infty S_p R^{pp} D_{zp} e^{-s\eta(z+z_0)} J_0\xi \, d\xi \\ & + a_r \int_0^\infty S'_p R^{pp} D_{zp} e^{-s\eta(z+z_0)} J_1\xi \, d\xi ] \\ & + [a_z \int_0^\infty S_p R^{ps} D_{zv} e^{-s(\eta z_0+\zeta z)} J_0\xi \, d\xi \\ & + a_r \int_0^\infty S'_p R^{ps} D_{zv} e^{-s(\eta z_0+\zeta z)} J_0\xi \, d\xi ] \\ & + [a_z \int_0^\infty S_v R^{ss} D_{zv} e^{-s\zeta(z+z_0)} J_0\xi \, d\xi \\ & + a_r \int_0^\infty S'_v R^{ss} D_{zv} e^{-s\ (z+z_0)} J_1\xi \, d\xi ] \end{aligned}$$

$$+ \left[ a_z \int_0^\infty S_v R^{sp} D_{zp} e^{-s(\zeta z_0+\eta z)} J_0\xi \, d\xi \right.$$

$$\left. + a_r \int_0^\infty S_v' R^{sp} D_{zp} e^{-s(\zeta z_0+\eta z)} J_1\xi \, d\xi \right]\} \qquad (3.16)$$

Similarly we can obtain the displacement equations for $\bar{u}_r^2$ and $\bar{u}_\theta^2$. And total displacement equations for a half-space is written by simply summing up $\bar{u}_z^1$ , $\bar{u}_r^1$ , $\bar{u}_\theta^1$ with $\bar{u}_z^2$ , $\bar{u}_r^2$ , $\bar{u}_\theta^3$ respectively. The phase of each ray is given by the argument of the exponential term. In general we can write the phase function as

$$-sh = -s(\eta z_p + \zeta z_s) \qquad (3.17)$$

where $z_p$ and $z_s$ are the total vertical components of the segments in a particular ray, travelling in the P and S modes respectively. As will be seen in the next chapter every one of these rays have a unique arrival time. Therefore, only the ones that arrive prior to the time of interest are to be considered.

Note that, the terms in the brackets of Eq.(3.16) are all similar in nature, hence, one can write the contribution of the jth ray to the displacements, as

$$\bar{u}_z^j = s^3 \bar{F}(s) \{ a_z \int_0^\infty S_j \Pi_j D_{zj} e^{-sh_j} J_0\xi \, d\xi$$

$$+ a_r \int_0^\infty S_j' \Pi_j D_{zj} e^{-sh_j} J_1\xi \, d\xi \} \qquad (3.18a)$$

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$$\bar{u}_r^j = s^3\bar{F}(s)\{a_z \int_0^\infty S_j\, \Pi_j\, D_{rj}\, e^{-sh_j}\, J_1 \xi\, d\xi$$

$$- a_r \int_0^\infty S'_j\, \Pi_j\, D_{rj}\, e^{-sh_j}\, J_0 \xi\, d\xi \}$$

$$+ \frac{s^2}{r}\; \bar{F}(s)\, a_r \{ \int_0^\infty S_j\, \Pi_j\, D_{rj}\, e^{-sh_j}\, J_1\, d\xi$$

$$+ \int_0^\infty S_{Hj}\, \Pi_{Hj}\, D_{rHj}\, e^{-sh_j}\, J_1\, d\xi \} \qquad (3.18b)$$

$$\bar{u}_\theta^j = -a_\theta \frac{s^2}{r} \bar{F}(s) \int_0^\infty S'_j\, \Pi_j\, D_{\theta j}\, e^{-sh_j}\, J_1\, d\xi$$

$$+ \{ \alpha_\theta\, s^3\, \bar{F}(s) \int_0^\infty S_{Hj}\, \Pi_{Hj}\, D_{\theta Hj}\, e^{-sh_{Hj}}\, J_0 \xi\, d\xi$$

$$- a_\theta\, \frac{s^2}{r}\, \bar{F}(s) \int_0^\infty S_{Hj}\, \Pi_{Hj}\, D_{\theta Hj}\, e^{-sh_{Hj}}\, J_1\, d\xi \} \quad (3.18c)$$

where $\Pi$'s are the reflection coefficients for the jth ray. Along path 1, $\Pi_j = \Pi_{Hj} = 1$ as there is no reflection. Along path 2, $\Pi_j = R^{pp}, R^{ss}, R^{ps}, R^{sp}$ for the mentioned reflected rays in P and S modes and $\Pi_{Hj} = 1$ for the reflected SH-waves.

## 3.5. EXPRESSIONS FOR THE SURFACE RECEIVER FUNCTIONS

Until now, we discussed the case in which both the source and the receiver were buried in a half-space. Now, we are going to discuss the case in which the receiver is at the surface.

To get the surface receiver function, we again consider the buried source and receiver case. Due to reflection; PP, PS, SP, SS waves will also reach to the receiver in addition to direct P and S waves.

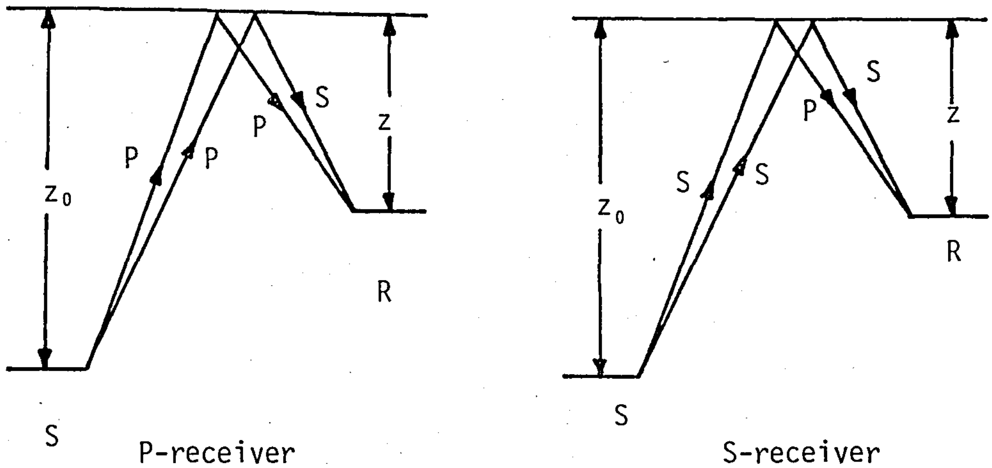



Fig. 3.3 Rays used in derivation of surface receiver functions

From Eq.(3.18), as z approaches zero, the P, PP and PS wave integrals combine and result in a single integral with a new receiver function, that is P surface receiver function, as,

$$D^{*}_{\alpha p} = D_{\alpha p} + D_{\alpha p} R^{pp} + D_{\alpha v} R^{ps} \qquad \alpha = r, \theta, z \qquad (3.19)$$

Note that, subscript denotes the direction. And while z approaches zero, also the S, SS and SP ray integrals combine and result in surface function of S-waves.

$$D^{*}_{\alpha v} = D_{\alpha v} + D_{\alpha v} R^{ss} + D_{\alpha p} R^{sp} \qquad \alpha = r, \theta, z \qquad (3.20)$$

Accordingly, for SH-waves, we have,

$$D^{*}_{\alpha H} = D_{\alpha H} + D_{\alpha H} R^{H} = 2D_{\alpha H} \qquad \alpha = r, \theta, z \qquad (3.21)$$

These functions are all included at Appendix C.

## 3.6. MODIFIED SOURCE FUNCTIONS DUE TO A LINE SOURCE

Until now, we were only concerned with the case of a concentrated single force source. In this section, the modifications due to the case of a vertical line source will be discussed. Consider a line force along the z-axis located between the points $z_1$ and $z_2$. There will be three different cases according to the replacement of the receiver point application. In the first case, the receiver point is above the line force, that is $z < z_1 < z_2$.

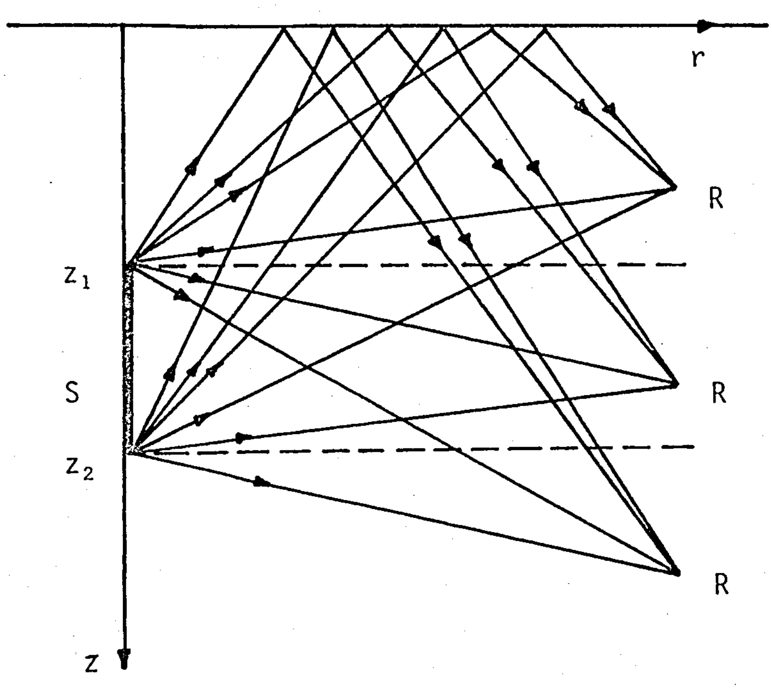



Fig. 3.4 Replacement of the receiver

The second one is the case where a receiver is replaced between $z_1$ to $z_2$, i.e., $z_1 < z < z_2$. Accordingly the third case is $z_1 < z_2 < z$. To find the new expressions for the line force source, we must integrate the displacement equations in $z_0$ from $z_1$ to $z_2$. For simplicity, this procedure will be illustrated using the expression for the first ray, i.e., the incident ray; because the integrands of the second ray, the reflected one, involve the reflection coefficients which are not functions of the parameter $z_0$. Hence, considering only vertical displacement, $\bar{u}_z$ , and integrating Eq.(2.20a) for three different replacement of receiver, we get the new source functions.

$$z < z_1 < z_2 \quad , \qquad |z - z_0| = -(z - z_0)$$

$$\begin{aligned}\int \bar{u}_z^1(\underline{r},s,\underline{a}) = s^3 \bar{F}(s) \{ & a_z \int_{z_1}^{z_2}\int_0^\infty S_p D_{zp} e^{-s\eta(z_0-z)} J_0\xi d\xi\, dz_0 \\ & + a_z \int_{z_1}^{z_2}\int_0^\infty S_v D_{zv} e^{-s\zeta(z_0-z)} J_0\xi d\xi\, dz_0 \\ & + a_r \int_{z_1}^{z_2}\int_0^\infty S'_p D_{zp} e^{-s\eta(z_0-z)} J_1\xi d\xi\, dz_0 \\ & + a_r \int_{z_1}^{z_2}\int_0^\infty S'_v D_{zv} e^{-s\zeta(z_0-z)} J_1\xi d\xi\, dz_0 \}\end{aligned} \qquad (3.22)$$

substituting the values of $S_p$, $S'_p$, $S_v$ and $S'_v$ from Eq.(2.17), then,

$$\begin{aligned}\int \bar{u}_z^1(\underline{r},s,\underline{a}) = s^2\bar{F}(s) \{ & a_z \int_0^\infty (-\frac{1}{\eta}) D_{zp} [ e^{-s\eta(z_2-z)} - e^{-s\eta(z_1-z)} ] J_0\xi\, d\xi \\ & + a_z \int_0^\infty (-\frac{\xi}{\zeta^2}) D_{zv} [ e^{-s\zeta(z_2-z)} - e^{-s\zeta(z_1-z)} ] J_0\xi\, d\xi \\ & + a_r \int_0^\infty (\frac{\xi}{\eta^2}) D_{zp} [ e^{-s\eta(z_2-z)} - e^{-s\eta(z_1-z)} ] J_1\xi\, d\xi \\ & + a_r \int_0^\infty (\frac{1}{\zeta}) D_{zv} [ e^{-s\zeta(z_2-z)} - e^{-s\zeta(z_1-z)} ] J_1\xi\, d\xi\end{aligned} \qquad (3.23)$$

In the same manner, $\bar{u}_z$ is integrated for the other two cases,

$z_1 < z < z_2 \quad , \qquad z_1 \text{ to } z \rightarrow |z-z_0| = (z-z_0)$

$\qquad\qquad\qquad\qquad\quad z \text{ to } z_2 \rightarrow |z-z_0| = -(z-z_0)$

$$\int \bar{u}_z^1(\underline{r},s,\underline{a}) = s^2 \bar{F}(s) \{ a_z \int_0^\infty (-\frac{1}{\eta}) D_{zp} [e^{-s\eta(z_2-z)} - e^{-s\eta(z-z_1)}] J_0\xi \, d\xi$$

$$+ a_z \int_0^\infty (-\frac{\xi}{\zeta^2}) D_{zv} [e^{-s\zeta(z_2-z)} + e^{-s\zeta(z-z_1)} - 2] J_0\xi \, d\xi$$

$$+ a_r \int_0^\infty (\frac{\xi}{\eta^2}) D_{zp} [e^{-s\eta(z_2-z)} + e^{-s\eta(z-z_1)} - 2] J_1\xi \, d\xi$$

$$+ a_r \int_0^\infty (\frac{1}{\zeta}) D_{zv} [e^{-s\zeta(z_2-z)} - e^{-s\zeta(z-z_1)}] J_1\xi \, d\xi \}$$

(3.24)

$z_1 < z_2 < z \quad , \qquad |z-z_0| = -(z-z_0)$

$$\int \bar{u}_z^1(\underline{r},s,\underline{a}) = s^2 \bar{F}(s) \{ a_z \int_0^\infty (-\frac{1}{\eta}) D_{zp} [e^{-s\eta(z-z_2)} - e^{-s\ (z-z_1)}] J_0\xi \, d\xi$$

$$+ a_z \int_0^\infty (-\frac{\xi}{\zeta^2}) D_{zv} [e^{-s\zeta(z-z_2)} - e^{-s\zeta(z-z_1)}] J_0\xi \, d\xi$$

$$+ a_r \int_0^\infty (-\frac{\xi}{\eta^2}) D_{zp} [e^{-s\eta(z-z_2)} - e^{-s\eta(z-z_1)}] J_1\xi \, d\xi$$

$$+ a_r \int_0^\infty (\frac{1}{\zeta}) D_{zv} [e^{-s\zeta(z-z_2)} - e^{-s\zeta(z-z_1)}] J_1 \xi \, d\xi \}$$

(3.25)

Through the numerical applications in this thesis, we are going to concern only the case of surface receiver. This means, only the first one of the receiver applications, that is $z < z_1 < z_2$ , will be considered. Hence, from Eq.(3.23) the modified source functions come out as,

$$\underline{S}_p = \left(-\frac{1}{\eta}\right) \quad , \quad \underline{S}_V = \left(-\frac{\xi}{\zeta^2}\right)$$

$$\underline{S}_p' = \left(\frac{\xi}{\eta^2}\right) \quad , \quad \underline{S}_V' = \left(\frac{1}{\zeta}\right) \qquad (3.26)$$

By going through a similar procedure, one can obtain the expressions for $\bar{u}_\theta^1$ and $\bar{u}_r^1$. Through these expressions, the modified SH-wave source function is

$$\underline{S}_H = -\frac{\kappa^2}{\zeta^2} \qquad (3.27)$$

These modified functions are also listed in Appendix C.

# IV. INVERSION OF LAPLACE TRANSFORM AND CAGNIARD'S METHOD

General expressions for the displacements due to a single concentrated force was given by Eq.(3.18). With the help of sections 3.5 and 3.6 we can rewrite the equations of the total displacements (both incident and reflected) due to a line force lying from $z_1$ to $z_2$ as,

$$\begin{aligned}\bar{u}_z(\underline{r},s,\underline{a}) = s^2\,\bar{F}(s)\,\{\, & a_z \int_0^\infty \underline{S}_p\, D^*_{zp}\, e^{-sh}\, J_0\xi\, d\xi \\ & + a_r \int_0^\infty \underline{S}'_p\, D^*_{zp}\, e^{-sh}\, J_1\xi\, d\xi \\ & + a_z \int_0^\infty \underline{S}_v\, D^*_{zv}\, e^{-sh}\, J_0\xi\, d\xi \\ & + a_r \int_0^\infty \underline{S}'_v\, D^*_{zv}\, e^{-sh}\, J_1\xi\, d\xi \,\}\Big|_{z_1}^{z_2}\end{aligned} \qquad (4.1a)$$

$$\begin{aligned}\bar{u}_r(\underline{r},s,\underline{a}) = s^2\,\bar{F}(s)\,\{\, & a_z \int_0^\infty \underline{S}_p\, D^*_{rp}\, e^{-sh}\, J_1\xi\, d\xi \\ & + a_z \int_0^\infty \underline{S}_v\, D^*_{rv}\, e^{-sh}\, J_1\xi\, d\xi \\ & - a_r \int_0^\infty \underline{S}'_p\, D^*_{rp}\, e^{-sh}\, J_0\xi\, d\xi \\ & - a_r \int_0^\infty \underline{S}'_v\, D^*_{rv}\, e^{-sh}\, J_0\xi\, d\xi \,\}\Big|_{z_1}^{z_2} \\ + \frac{s}{r}\,\bar{F}(s)\, a_r \,\{\, & \int_0^\infty \underline{S}'_p\, D^*_{rp}\, e^{-sh}\, J_1\, d\xi\end{aligned}$$

$$+ \int_0^\infty \underline{S}'_V \, D^*_{rV} \, e^{-sh} \, J_1 \, d\xi$$

$$+ \int_0^\infty \underline{S}_H \, D^*_{rH} \, e^{-sh} \, J_1 \, d\xi \} \Bigg|_{z_1}^{z_2} \qquad (4.1b)$$

$$\bar{u}_\theta(\underline{r},s,\underline{a}) = -a_\theta \frac{s}{r} \; \bar{F}(s) \{ \int_0^\infty \underline{S}'_p \, D^*_{\theta p} \, e^{-sh} \, J_1 \xi \, d\xi$$

$$+ \int_0^\infty \underline{S}'_V \, D^*_{\theta V} \, e^{-sh} \, J_1 \xi \, d\xi \} \Bigg|_{z_1}^{z_2}$$

$$+ \{ a_\theta \, s \, \bar{F}(s) \int_0^\infty \underline{S}_H \, D^*_{\theta H} \, e^{-sh} \, J_0 \xi \, d\xi$$

$$- a_\theta \frac{s}{r} \bar{F}(s) \int_0^\infty \underline{S}_H \, D^*_{\theta H} \, e^{-sh} \, J_1 \, d\xi \} \Bigg|_{z_1}^{z_2} \qquad (4.1c)$$

where,

$$h = z_p \eta + z_s \zeta \quad , \quad \text{for P-waves} \quad z_s = 0 \qquad (4.2)$$

$$\text{for S-waves} \quad z_p = 0$$

All of the integrals appearing in the above equations are of the two kind.

$$\bar{I}_0(\underline{r},s,\underline{a}) = \int_0^\infty E_2(\xi)\,\xi \;\; J_0(s\,\xi\,r) \; e^{-s(z_p\eta+z_s\zeta)} \; d\xi$$

$$\bar{I}_1(\underline{r},s,\underline{a}) = \int_0^\infty E_2(\xi)\,\xi^2 \;\; J_1(s\,\xi\,r) \; e^{-s(z_p\eta+z_s\zeta)} \; d\zeta \qquad (4.3)$$

where $E_1$ and $E_2$ are even functions of $\xi$ involving the source and the receiver functions and the reflection coefficients. The coefficients of these integrals are of the form $s^n\bar{F}(s)$. Therefore after finding the inverse transforms of $I_0$ and $I_1$ , the final solution can be obtained through convolution.

## 4.1. CAGNIARD'S METHOD

### 4.1.1. INTEGRAL REPRESENTATIONS OF $J_0$ AND $J_1$

In the Cagniard's method, the key point is to use the integral representations of Bessel functions $J_0$ and $J_1$. These are, (Abromovitz and Stegun [1]),

$$J_0(z) = \frac{2}{\pi} \text{ Re} \int_0^{\pi/2} e^{izcos\omega} d\omega$$
$$J_1(z) = \frac{2}{\pi} \text{ Im} \int_0^{\pi/2} e^{izcos\omega} \cos\omega \, d\omega \tag{4.4}$$

Substituting Eq.(4.4) in Eq.(4.3) and interchanging the order of integration, since the integrals in $\xi$ are uniformly convergent for all values of $\omega$ between 0 and $\pi/2$,

$$\bar{I}_0(\underline{r},s,\underline{a}) = \frac{2}{\pi} \text{ Re} \int_0^{\pi/2} d\omega \int_0^{\infty} E_1(\xi) \, e^{-s(i\xi r\cos\omega + z_p\eta + z_s\zeta)} \, \xi \, d\xi$$
$$\bar{I}_1(\underline{r},s,\underline{a}) = \frac{2}{\pi} \text{ Im} \int_0^{\pi/2} \cos\omega d\omega \int_0^{\infty} \xi \, E_2(\xi) \, e^{-s(i\xi r\cos\omega + z_p\eta + z_s\zeta)} \, \xi \, d\xi \tag{4.5}$$

By following Cagniard's [4] original approach, we make the following transformations.

### 4.1.2. TRANSFORMATION OF VARIABLE $\xi$ TO t

The second step in Cagniard's method is to make the following transformation,

$$t = -i\,\xi\, r\, \cos\omega + z_p\, \eta + z_s \zeta = g(r,z;\xi) \tag{4.6}$$

Through above transformation the exponential terms of $\bar{I}_0$ and $\bar{I}_1$ take the form of $e^{-st}$. Where, t is a complex quantity. Note that the inverse transform could be obtained rather easily, if we could make the quantity t a real variable.

The function $g(r,z;\xi)$ is a multivalued function with branch points at $\xi = \pm i$ due to the second term and $\xi = \pm i\kappa$ due to the third term. The branch cuts are chosen such that if $\xi$ is real and positive, the radicals are positive.

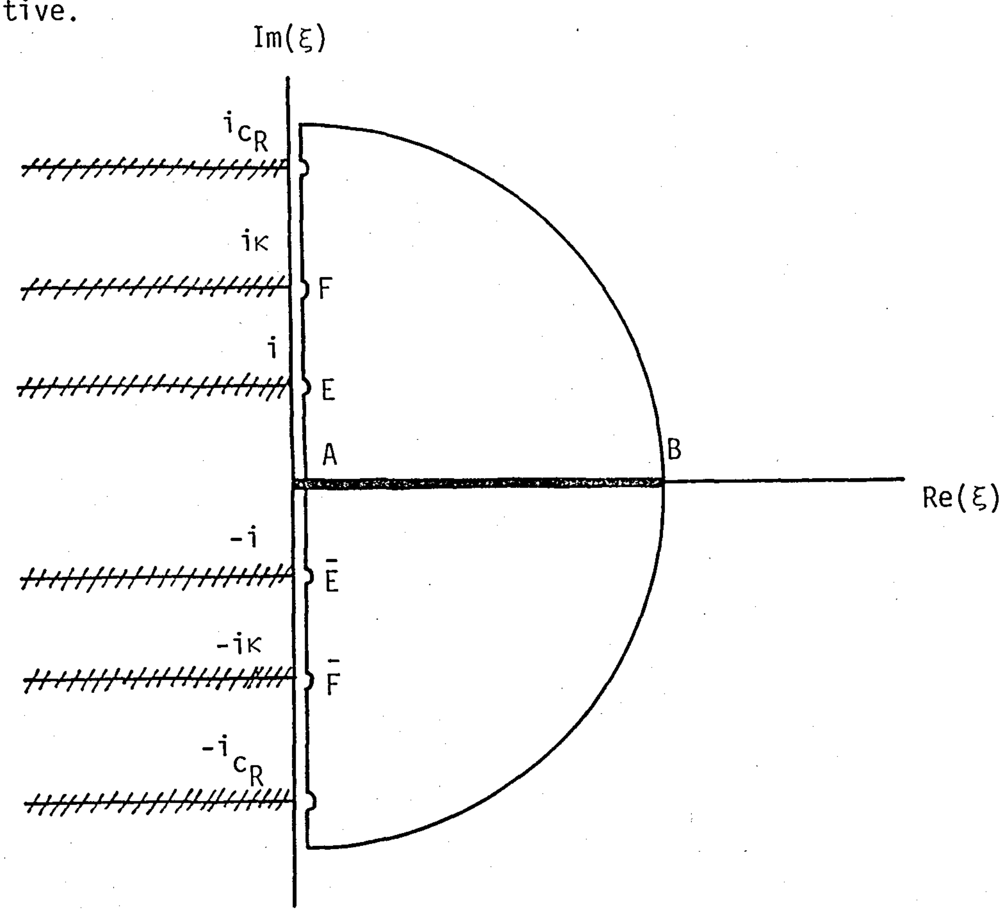



Fig. 4.1 $\xi$ plane, and branch cuts

Note that, the transformation given by Eq.(4.6) transforms the whole complex $\xi$-plane on a complex t-plane.

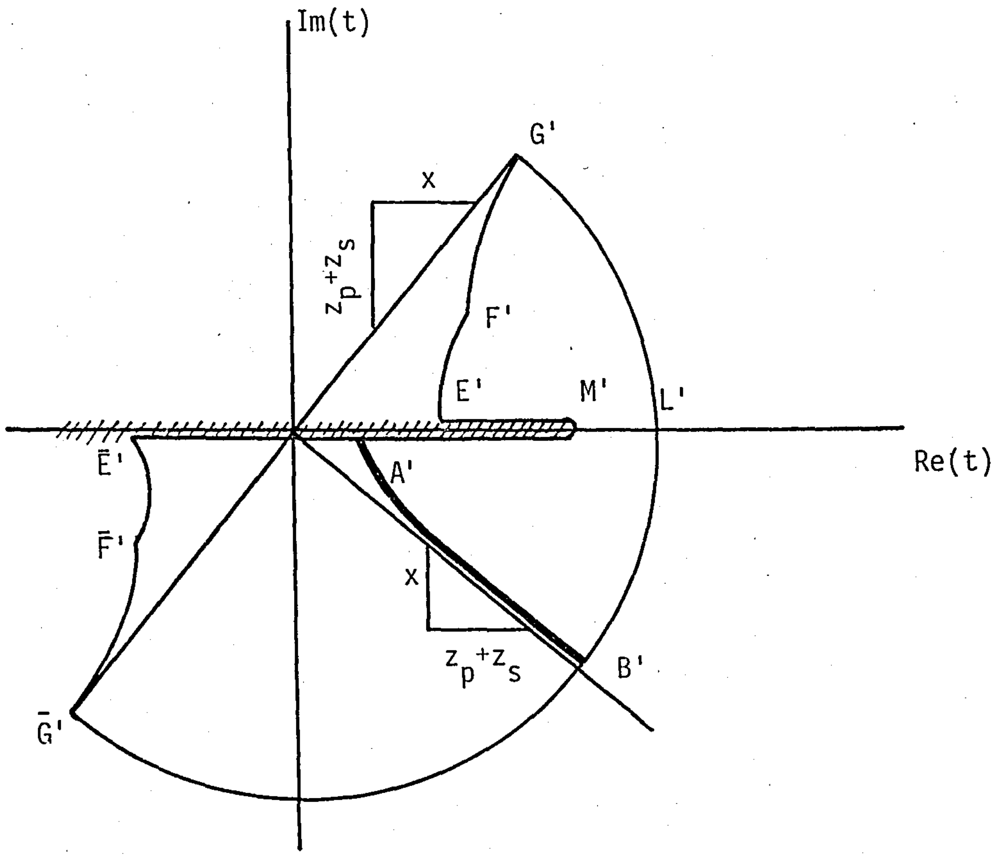



Fig. 4.2. Map of the $\xi$-plane in the t-plane

The original line of integration, that is, the real $\xi$-axis, is mapped into the curve A'B' in the t-plane. The origin, A', of this curve corresponds to the value of $t = t_A$ where, from Eq.(4.6),

$$t_A = z_p + z_s \kappa \tag{4.7}$$

By letting $\xi = \ell$, where $\ell$ is a real variable, one can show by letting $\ell \to +\infty$ that the curve A'B' has an asymptote of the form

$$t = \frac{-x}{z_p + z_s} , \qquad x = r\cos\omega \tag{4.8}$$

On the other hand, by substituting $\xi = i\ell$, one can make the transformation of the imaginary $\xi$-axis.

$$t = r\cos\omega\ell + z_p(1-\ell^2)^{1/2} + z_s\,(\kappa^2-\ell^2)^{1/2} \qquad (4.9)$$

It can be shown that as $\ell \to \pm\infty$, t approaches an asymptote given by

$$t = \frac{z_p + z_s}{x} \quad , \qquad x = r\cos\omega \qquad (4.10)$$

Eq.(4.8) shows that the points on the imaginary axis with $|\ell| < 1$, and if $z_p = 0$ but $z_s \neq 0$, the points with $|\ell| < \kappa$ will lie on the real t-axis. Note that, The mapping of the points for $|\ell|<1$ or $|\ell| < \kappa$ would be double valued if there was a stationary point M. This point must then be an extremum point for $g(r,z,\xi)$ and must satisfy the relation,

$$\left(\frac{\partial t}{\partial \xi}\right)_\omega = -ix + \frac{z_p\,\xi}{\eta} + \frac{z_s\xi}{\zeta} = 0 \qquad (4.11)$$

The above equation has only one root, $\xi = i\,\ell_m$. In our case, where we study the surface response of a half-space (receiver at the surface), we either have $z_p$ or $z_s$ equal to zero, thus,

$$\ell_M = \frac{\alpha\, x}{(z_0^2 + x^2)^{1/2}} \qquad (4.12)$$

where $z_0 = z_p$ and $= 1$ for P-waves, and, $z_0 = z_s$ , $\alpha = \kappa$ for S-waves. To render the single valuedness of the mapping, a branch cut is introduced along the real t-axis starting at the point M', corresponding $t_M$. Thus, the segment AME of the positive imaginary $\xi$-axis is mapped into A'M'E' in the t-plane where A'M' lies below the branch line and M'E' above the branch line. And the point M' is given by,

$$t_M = x\,\ell_M + z_0(\alpha^2 - \ell_M^2)^{1/2} \tag{4.13}$$

### 4.1.3. CHANGE IN THE PATH OF INTEGRATION AND INVERSION OF LAPLACE TRANSFORMATION

According to the transformation Eq.(4.5), the expressions for $\bar{I}_0$ and $\bar{I}_1$ become,

$$\begin{aligned}\bar{I}_0(r,z,s) &= \frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2} d\omega \int_{A'B'} E_1[\xi(t)]\ \xi(t)\left(\frac{d\xi}{dt}\right)_\omega e^{-st}\,dt\\ \bar{I}_1(r,z,s) &= \frac{2}{\pi}\,\mathrm{Im}\int_0^{\pi/2} \cos\omega d\omega \int_{A'B'} E_2[\xi(t)]\ \xi^2(t)\left(\frac{d\xi}{dt}\right)_\omega e^{-st}\,dt\end{aligned} \tag{4.14}$$

where the path of integration is along A'B' in t-plane. Consider the contour A'B'L'M'A'; since there are no singularities inside this contour, the integral along this closed contour, from Cauchy's principle, is zero. Also B'L' is moved to the infinity, the integrands of Eq.(4.11) disappears in this portion of contour. Therefore A'M'L' can be taken as the new path of integration along the real t-axis, instead of A'B'. Then, the integrals of Eq.(4.11) become,

$$\begin{aligned}\bar{I}_0(r,z,s) &= \frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2} d\omega \int_{t_A}^{\infty} E_1[\xi(t)]\ \xi(t)\left(\frac{d\xi}{dt}\right)_\omega e^{-st}\,dt\\ \bar{I}_1(r,z,s) &= \frac{2}{\pi}\,\mathrm{Im}\int_0^{\pi/2} \cos\omega d\omega \int_{t_A}^{\infty} E_2[\xi(t)]\ \xi^2(t)\left(\frac{d\xi}{dt}\right)_\omega e^{-st}\,dt\end{aligned} \tag{4.15}$$

The above analysis is true when $\omega \neq \pi/2$. For the case of $\omega = \pi/2$, the transformation equation of Eq.(4.6) becomes,

$$t = z_p\,\eta + z_s\,\zeta \tag{4.16}$$

Above equation means that the real $\xi$-axis is mapped into the real t-axis. And simply, for this case $t_A = t_M$ ; so it is understood that Eq.(4.15) is also valid for $\omega = \pi/2$. Hence, note that, the integrals in Eq.(4.15) converge uniformly for all $0 \leq \omega \leq \pi/2$. Changing the order of integration in Eq.(4.15)

$$\bar{I}_0(r,z,s) = \int_{t_A}^{\infty} \{ \frac{2}{\pi} \text{Re} \int_0^{\pi/2} E_1[\xi(t;\omega)] \, \xi(t;\omega)(\frac{d\xi}{dt})_\omega \, d\omega \} e^{-st} \, dt$$
$$\bar{I}_1(r,z,s) = \int_{t_A}^{\infty} \{ \frac{2}{\pi} \text{Im} \int_0^{\pi/2} E_2[\xi(t;\omega)] \, \xi^2(t;\omega)(\frac{d\xi}{dt})_\omega \cos\omega d\omega \, e^{-st} \, dt \qquad (4.17)$$

Note that, if the lower limit of integrations were zero rather than $t_A$, the above equations would become the Laplace transforms of the expressions in curly brackets. Hence, introducing the Heavyside step function and, simply taking the inverse transforms of $I_0$ and $I_1$ by inspection,

$$I_0(r,z,t) = H(t-t_A) \frac{2}{\pi} \text{Re} \int_0^{\pi/2} E_1[\xi(t,\omega)] \, \xi(t,\omega)(\frac{d\xi}{dt})_\omega \, d\omega$$
$$I_1(r,z,t) = H(t-t_A) \frac{2}{\pi} \text{Im} \int_0^{\pi/2} E_2[\xi(t,\omega)] \, \xi^2(t,\omega)(\frac{d\xi}{dt})_\omega \cos\omega d\omega \qquad (4.18)$$

These integrals of $\omega$ can be evaluated numerically.

### 4.1.4. CHANGE OF INTEGRATION VARIABLE $\omega \to \xi$

In calculating the integrals of Eq.(4.18), for each value of t, the values of $\xi$ must to be found for different values of $\omega$, $0 \leq \omega \leq \pi/2$. This is a tedious job to do. Secondly, $(\frac{d\xi}{dt})_\omega$ will have a singularity at some value of $\omega$, $0 \leq \omega \leq \pi/2$, for each value of t. To overcome these complications, another change of variable is needed, (Cagniard [4]), $\omega$ to $\xi$. This Transformation will allow us to transform the finite integral in the

$\omega$-plane into another finite integral in the $\xi$-plane. For this, we will make use of the same transformation given by Eq.(4.6),

$$\cos\omega = \frac{z_p \eta + z_s \zeta - t}{i \xi r} \quad , \quad r \neq 0 \tag{4.19}$$

If $\xi = 0$, then from Eq.(4.6), $t = t_A = z_p + z_s \kappa$ and the above expression becomes indeterminite. Using the L'Hopitals rule, we find that $\omega \to \pi/2$ as $\xi \to 0$. For $\omega \to 0$, Eq.(4.6) yields $\xi = \xi_1(r,z,t)$ where,

$$\begin{aligned} t &= -i\,\xi_1 r + z_p \eta_1 + z_s \zeta_1 \quad , \quad \eta_1 = (\xi_1^2 + 1)^{1/2} \\ \zeta_1 &= (\xi_1^2 + \kappa^2)^{1/2} \end{aligned} \tag{4.20}$$

Hence, with the new limits of integrations, Eq.(4.15) becomes

$$\begin{aligned} I_0(r,z,t) &= H(t-t_A)\,\frac{2}{\pi}\ \mathrm{Re} \int_{\xi_1}^{0} E_1(\xi)\left(\frac{\partial \xi}{\partial t}\right)_\omega \left(\frac{\partial \omega}{\partial \xi}\right)_t \xi\, d\xi \\ I_1(r,z,t) &= H(t-t_A)\,\frac{2}{\pi}\ \mathrm{Im} \int_{\xi_1}^{0} E_2(\xi)\left[\frac{z_p \eta + z_s \zeta - t}{i \xi r}\right]\left(\frac{\partial \xi}{\partial t}\right)\left(\frac{\partial \omega}{\partial \xi}\right)_t \xi^2\, d\xi \end{aligned} \tag{4.21}$$

From Eq.(4.20), it is understood that $\xi_1$ is a complex number on the contour, AML,which is the mapping of the real t-axis back into $\xi$-plane. The integrals of the above equation are along $AM\xi_1$ which is a finite portion of AML, as seen in the below figure.

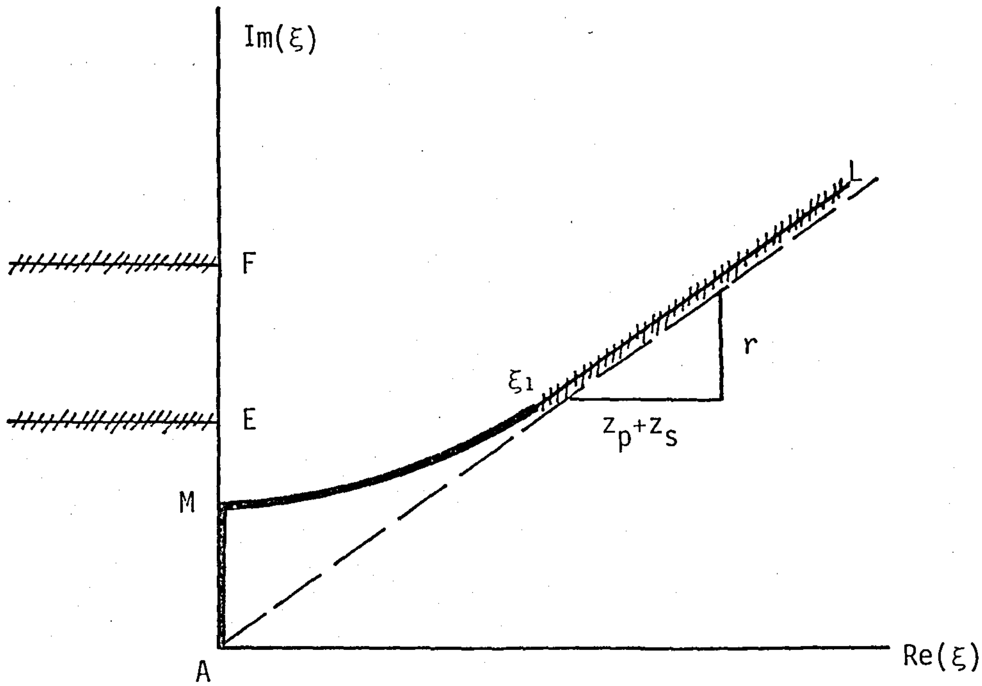



Fig. 4.3. Integration path of direct and reflected rays

From Eq.(4.6),

$$\left(\frac{\partial\omega}{\partial\xi}\right)_t = \frac{-ir\cos\omega + z_p\,\xi/\eta + z_s\,\xi/\zeta}{-ir\,\xi\sin\omega} \tag{4.22}$$

and using Eq.(4.11) we have,

$$\left(\frac{\partial\xi}{\partial t}\right)_\omega \left(\frac{\partial\omega}{\partial\xi}\right)_t = \frac{1}{-ir\xi\,\sin\omega} \tag{4.23}$$

From Eq.(4.19), we obtain the expression for sinω,

$$\sin\omega = \frac{1}{\xi r}\left[\xi^2 r^2 + (z_p\eta + z_s\zeta - t)^2\right]^{1/2} \tag{4.24}$$

Hence Eq.(4.23) becomes,

$$\left(\frac{\partial\xi}{\partial t}\right)_{\omega}\left(\frac{\partial\omega}{\partial\xi}\right)_{t} = -\frac{1}{i\,K(r,z,t,\xi)} \qquad (4.25)$$

where,

$$K(r,z,t,\xi) = \left[\xi r^2 + (z_p\eta + z_s\zeta - t)^2\right]^{1/2} \qquad (4.26)$$

Substituting Eq.(4.25) in Eq.(4.21), we obtain,

$$I_0(r,z,t) = H(t-t_A)\,\frac{2}{\pi}\,\mathrm{Im}\int_0^{\xi_1} E_1(\xi)\,\frac{\xi}{K}\,d\xi \qquad (4.27)$$

$$I_1(r,z,t) = -H(t-t_A)\,\frac{2}{\pi r}\,\mathrm{Im}\int_0^{\xi_1} \xi\,E_2(\xi)\,\frac{z_p\eta + z_s\zeta - t}{K}\,d\xi$$

The above equations are the ones that will be used in numerical calculations. The K function has a branch point at $\xi = \xi_1$, hence, a branch cut must be taken such that, it starts at point $\xi = \xi_1$ and extends along $\xi_1 L$. This branch cut is chosen such that the real part of K is positive when real part of $\xi$ is positive.

## 4.2. ARRIVAL TIMES OF INDIVIDUAL RAYS

The expression for the arrival times of rays, come from the analysis of the stationary point of the transformation given by Eq.(4.6).

From Eq.(4.13),

$$t_M = \ell_M\, r\cos\omega + z_p\,(1 - \ell_M^2)^{1/2} + z_s(\kappa^2 - \ell_M^2)^{1/2} \qquad (4.28)$$

The physical meaning of $t_M$ is made clear by the following analysis of geometry. Let,

$$\ell_M = \sin\alpha = \kappa \sin\beta \qquad , \quad \alpha,\beta > 0 \ , \quad \beta \leq \pi/2 \qquad (4.29)$$

and,

$$z_p \tan\alpha = r_1 \cos\omega \qquad , \quad z_s \tan\beta = r_2 \cos\omega \qquad (4.30)$$

Substituting the above expressions in Eq.(4.11) we get,

$$r = r_1 + r_2 \qquad (4.31)$$

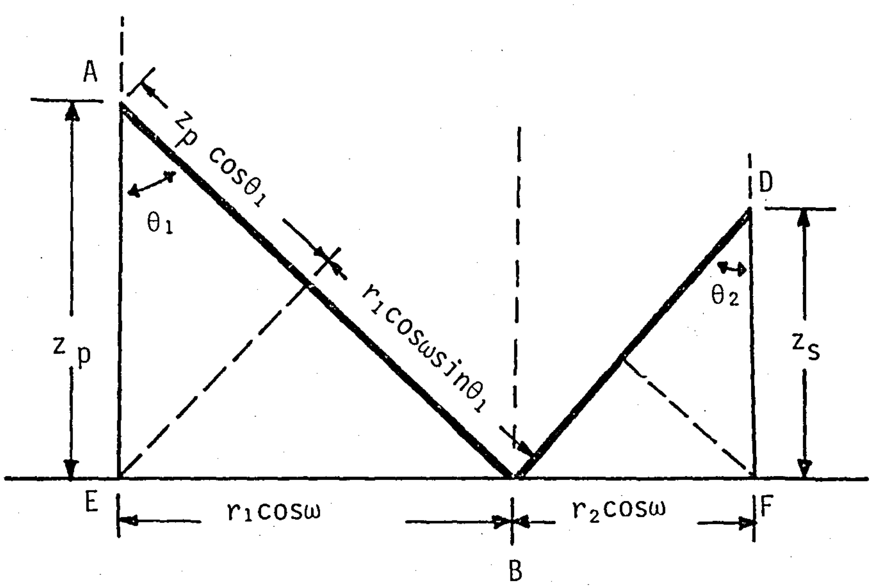



Fig. 4.4. Geometrical interpretation of Eq.(4.35)

From the figure above it is seen that,

$$\tan\theta_1 = \frac{r_1\cos\omega}{z_p} \qquad , \qquad \tan\theta_2 = \frac{r_2\cos\omega}{z_s} \qquad (4.32)$$

Comparing Eq.(4.32) and Eq.(4.30), we get

$$\theta_1 = \alpha \qquad , \qquad \theta_2 = \beta \qquad (4.33)$$

And, Eq.(4.29) becomes,

$$\ell_M = \sin\theta_1 = \kappa \sin\theta_2 \tag{4.34}$$

Now, Eq.(4.28) can be written as,

$$t_M = [r_1 \cos\omega \sin\theta_1 + z_p \cos\theta_1] + \kappa[r_2 \cos\omega \sin\theta_2 + z_s \cos\theta_2] \tag{4.35}$$

Note that, $t_M$ is a function of $\omega$. Then,

$$t_M(\omega) \geq z_p \cos\theta_1 + z_s \cos\theta_2 \tag{4.36}$$

where equality exists for $\omega = \pi/2$. Note that, $t_M$ is a continuous function of $\omega$, for $0 \leq \omega \leq \pi/2$. From Eq.(4.5), we get,

$$\left.\frac{\partial t}{\partial \omega}\right|_{\xi = i\ell_M} = -\ell_M \, r\sin\omega \tag{4.37}$$

thus, $t_M$ is maximum for $\omega = 0$. Now the Fig. 4.5 can be interpreted considering A as the source point, D as the receiver point and EF as a part of the surface of the half-space. And from Eq.(4.34), $\sin\theta_1 = \kappa \sin\theta_2$ is the classical law of reflection of elastic waves. For $\omega = 0$, Eq.(4.32) becomes; $\tan\theta_1 = r_1/z_p$ and $\tan\theta_2 = r_2/z_s$, and the value of $t_M(\omega{=}0)$ is,

$$t_M(0) = [r_1 \sin\theta_1 + z_p \cos\theta_1] + \kappa[r_2 \sin\theta_2 + z_s \cos\theta_2] \tag{4.38}$$

The parameters in the above equation are nondimensional. Restoring the dimensions from Eq.(2.10), we find,

$$t_M(\omega{=}0) = \frac{1}{c}\,(r_1 \sin\theta_1 + z_p \cos\theta_1) + \frac{1}{C}\,(r_2 \sin\theta_2 + z_s \cos\theta_2) \tag{4.39}$$

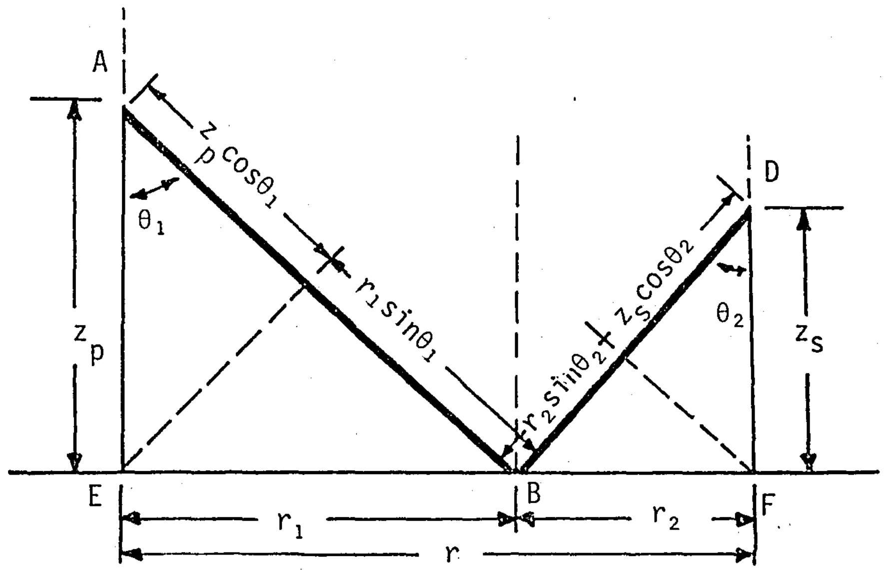



Fig. 4.5. Geometrical interpretation of Eq.(4.38)

This is exactly the arrival time of the wave along the ray path AB as a P-wave and the path BD as a S-wave. Therefore, the value of $t = t_M$ corresponding to the stationary value $\xi = \xi_M$, is the arrival time of the ray whose path has a total vertical projection of $z_p + z_s$, travelling in P and S modes respectively. It is well known that when the angle of incidence of a S-wave is greater than or equal to the critical angle $\alpha_c$, where

$$\alpha_c = \sin^{-1} C/c = \sin^{-1}(1/\kappa) \tag{4.40}$$

there will be a refracted P-wave travelling along the surface of a half-space. It means that the ray travels the path from the source to the surface in S-mode and the rest in P-mode. This wave, for the reason above, reaches to the receiver before the direct S-wave. And, it is called "Head Wave". The arrival time of this wave is obtained by modifying the Eq.(4.38),i.e.,

$$t_{Head}(\omega = 0) = r + z_s\,(\kappa^2 - 1)^{1/2} \tag{4.41}$$

In a bounded elastic medium, a third wave known as the Rayleigh Wave exists which has a velocity of $C_r$, given by

$$C_R = 0.9194\ C \tag{4.42}$$

where C is the shear wave velocity. Since Rayleigh waves decay exponentially with depth, thus are confined to a small region near the surface of the medium. Another feature of these waves is that they decay as $r^{-1/2}$ in the direction of propogation. Unlike the direct waves and head waves, the Rayleigh waves are not associated with a single generalized wave. The arrival time of these waves can be calculated for the case of both the source and the receiver being on the surface.

For a buried source case, the point on the surface where the Rayleigh wave originates due to diffraction is not known. Pekeris and Lifson [17] found that there were no distinct peaks in displacement curves for a buried point force for $r_0/z_0 < 5$.

Now consider the P and S waves travelling from a buried source to a surface receiver. Solving the upper limit of integration from Eq.(4.20), we have,

$$\xi_1 = \frac{z_0}{r^2 + z_0^2}\left[t^2 - \alpha^2(r^2 + z_0^2)\right]^{1/2} + \frac{itr}{r^2 + z_0^2} \tag{4.43}$$

where $\alpha = 1$ , $\alpha = \kappa$ and $z_0 = z_p$ , $z_0 = z_s$ for the P-waves and S-waves respectively.

## 4.3. CONVOLUTION OF RAY INTEGRALS

Note that, displacement equations of Eq.s(4.1) have integrals in the form of $I_0$ and $I_1$, and coefficients of them in the form of $s^n\bar{F}(s)$. In our study, n is either 2 or 1, and the function $\bar{F}(s)$ involves the time dependency of the force input.

Recalling Eq.(2.17), we have,

$$\bar{F}(s) = F_0 \ \bar{f}(s) \ / \ 4\pi^2 \ \kappa^2 \ s^2 \ \mu \ r_0^2 \tag{4.44}$$

then we obtain,

$$s^2\bar{F}(s) = \frac{F_0 \ \bar{f}(s)}{4\pi \ \kappa^2 \ \mu \ r_0^2}$$

$$s \ \bar{F}(s) = \frac{F_0 \ \bar{f}(s)}{4\pi \ \kappa^2 \ \mu \ r_0^2} \left(\frac{1}{s}\right) \tag{4.45}$$

Now, consider the case where the function f(t) is a Delta-Dirac function, then f(s) = 1 and the above expressions reduce to

$$s^2 \ \bar{F}(s) = \frac{F_0}{4\pi \ \kappa^2 \ \mu \ r_0^2}$$

$$s \ \bar{F}(s) = \frac{F_0}{4\pi \ \kappa^2 \ \mu \ r_0^2} \left(\frac{1}{s}\right) \tag{4.46}$$

Hence, the inverse Laplace transform of these quantities are,

$$\mathcal{L}^{-1}(s^2\bar{F}(s)) = \frac{F_0}{4\pi\ \kappa^2\ \mu\ r_0^2}\ \delta(t)$$

$$\mathcal{L}^{-1}(s\bar{F}(s)) = \frac{F}{4\pi\ \kappa^2\ \mu\ r_0^2}\ H(t) \tag{4.47}$$

where $\delta(t)$ is the Delta function and $H(t)$ is Heavyside's step function. Therefore, the integrals with $s^2\bar{F}(s)$ as their coefficients should be convoluted by $\delta(t)$ and those with $s\bar{F}(s)$ should be convoluted by $H(t)$. Recalling the definition of Convolution Theorem, it is well known for any function y,

$$\delta(t) * y(t) = \int_0^t \delta(t-\tau)\ y(\tau)\ d\tau$$

$$= y(t) \tag{4.48}$$

$$H(t) * y(t) = \int_0^t H(t-\tau)\ y(\tau)\ d\tau$$

$$= \int_0^t y(\tau)\ d\tau \tag{4.49}$$

In the view of the above relations, for Delta-Dirac input, only the integrals with $s\bar{F}(s)$ as their coefficients should be integrated from $t=0$ to $t=t$. However, since each integral has a specific arrival time, the lower limit of integration is to be replaced by the corresponding arrival time of the ray, i.e., integration is from $t=t_A$ to $t=t$.

# V. NUMERICAL CALCULATIONS OF FIELD RESPONSES

In this chapter, the numerical procedure used in calculating the ray integrals will be explained and the response of a half-space due to a finite line source with Delta-Dirac function time dependance will be documented. In all of the examples, the Poisson's ratio fo the half-space material is taken to be 0.25 corresponding to $\kappa^2 = 3$.

## 5.1. PROCEDURE IN NUMERICAL CALCULATIONS

First of all, in all of the examples presented in this thesis, we have taken the receiver to be on the surface of the half-space. Therefore, there are two types of rays in our case; one is the incident ray and the other is the refracted ray. The integrals of displacements belonging to these two rays are well documented in the introduction part of Chapter IV.

At this point, we remind the conclusion which we had in section 3.6. To find the response of a vertical line source; the derived integrals, for the bottom point of the line source, are calculated, then the same calculation are done for the top point. The response becomes out as the difference of these calculated results.

To start the numerical calculations, the stationary point of Cagniard's path, $\xi_M$ , and the arrival time of the ray, $t_M$ , are found using Eq.(4.12) and Eq.(4.13) respectively. If there is a head wave effect in the direct S-wave, the arrival time of it can be calculated from Eq.(4.41). The upper limit of integration, $\xi_1$, for each value of time t, is obtained from Eq.(4.43). By combining the modified source functions, receiver functions and reflection coefficients; the integrands of the integrals are formed.

For the P-waves and S-waves with no head wave effect, the stationary point $\xi_M$ is below the branch point $\xi = i$. The upper limit of integration $\xi_1$ moves up the imaginary axis from the origin to the stationary point $\xi_M$. Through this interval, the integrands of the integrals are all real valued and since the imaginary part of the integrals are required as final solution, the response is zero for t less than $t_M$. This is natural because, $t_M$ is the arrival time of the individual ray and no response is expected prior to the arrival of it. At the point $\xi_M$, the integration path leaves the imaginary $\xi$-axis, but stays in the first quadrant of complex $\xi$-plane. However, the integration along this path is very complicated. Therefore, we will use an alternative method, introduced by Pao and Gajevski [16], to solve the integrals. The original path of integration $AM\xi_1$ can be replaced by the path $QM\xi_1$ since the integral along AQ is zero. Consider the closed contour $QM\xi_1P_2P_1Q$ shown in figure below.

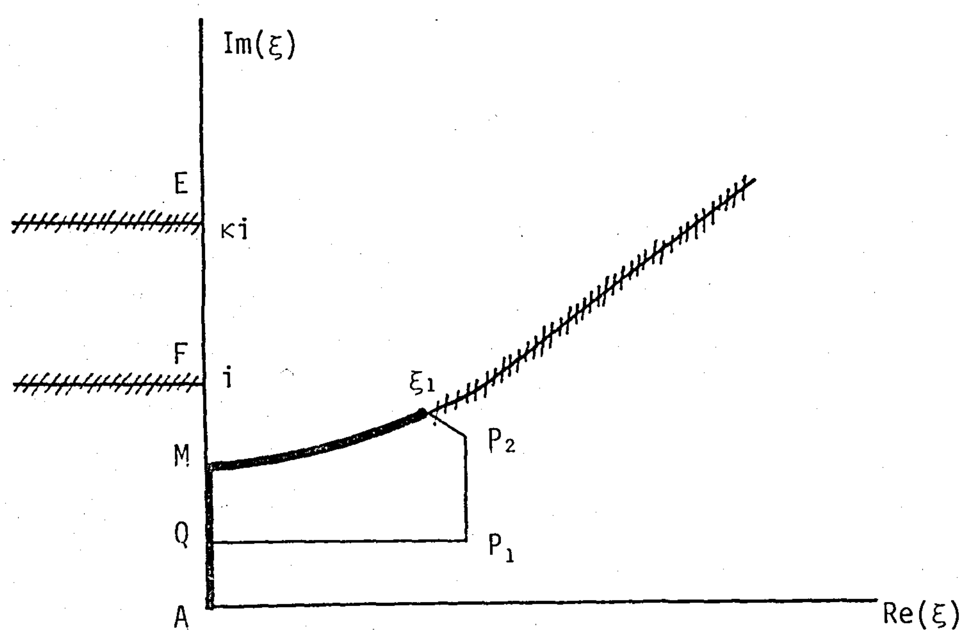



Fig. 5.1. Integration path for the direct P and S-waves

There are no singularities inside this contour. It is known that the integration along QM is zero. Hence, based on Cauchy's theorem, we can replace the integration along $QM\xi_1$ by the sum of the integrals along $QM\xi_1$ by the sum of the integrals along the straight lines $QP_1$, $P_1P_2$, $P_2\xi_1$. This path is much easier to integrate than the original path. The choice of the points Q, $P_1$, $P_2$ were done as follows; let $\xi_1$ be expressed as $\xi_1 = a + ib$, then, if $a < 0.005\ \ell_M$

$$
\begin{aligned}
Q &= i\ 0.8\ \ell_M \\
P_1 &= 0.05\ \ell_M + i\ 0.8\ \ell_M \\
P_2 &= 0.05\ \ell_M + i\ [0.9b + 0.1(0.8\ \ell_M)]
\end{aligned}
\tag{5.1}
$$

and if $a > 0.05\ \ell_M$

$$
\begin{aligned}
Q &= i\ 0.08\ \ell_M \\
P_1 &= (a + 0.03) + i\ 0.8\ \ell_M \\
P_2 &= (a + 0.03) + i\ (b - 0.08)
\end{aligned}
\tag{5.2}
$$

If the S-waves with head wave effect exist in the media, the stationary point M corresponding $\ell_M$ lies between the branch points $\xi = i$ and $\xi = i\kappa$. We know from the previous paragraph that, the integrals are zero for the values of $\xi_1$ below $\xi = i$ which corresponds to the arrival of the head waves. For these rays, the points Q, $P_1$, $P_2$ are chosen as shown in Figure 4.7.

Again representing $\xi_1$ as $(a + ib)$, the points Q, $P_1$, $P_2$ are chosen as,

if $a = 0$, that is $\xi_1$ is on the imaginary axis.

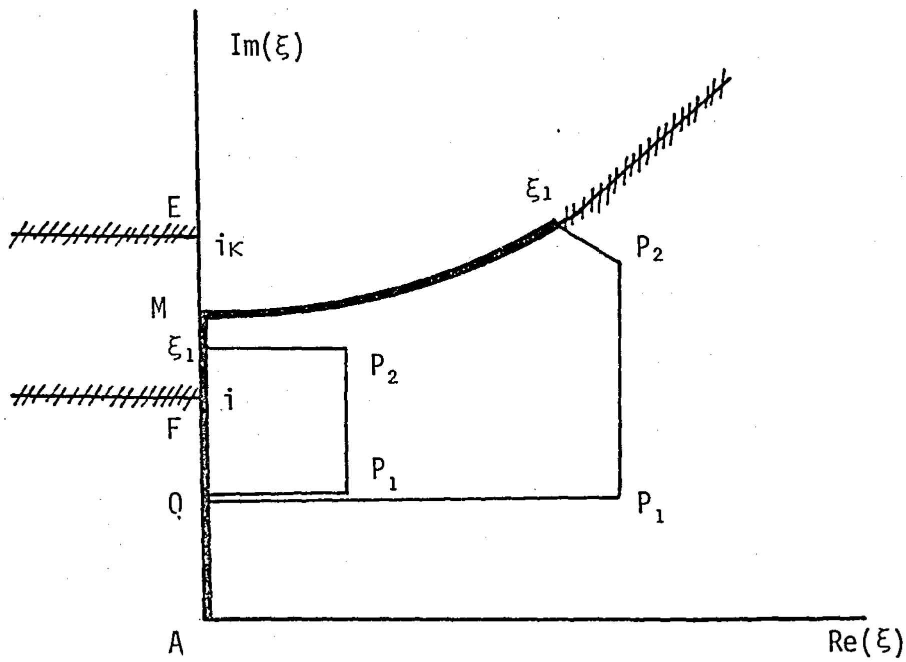



Fig. 5.2. Integration paths for refracted rays

$$Q = i\,0.9$$

$$P_1 = 0.05 + i\,0.9 \tag{5.3}$$

$$P_2 = 0.05 + i\,b$$

if $0 < a < 0.05$

$$Q = i\,0.9$$

$$P_1 = 0.05 + i\,0.9 \tag{5.4}$$

$$P_2 = 0.05 + i\,b$$

if $a > 0.05$

$$
\begin{aligned}
Q &= i\,0.9 \\
P_1 &= (a + 0.03) + i\,0.9 \\
P_2 &= (a + 0.03) + i\,(b - 0.08)
\end{aligned}
\tag{5.5}
$$

These new paths of integration are far from the singularities. However, the singularity at the upper limit of integration, $\xi_1$ , is still on path. To avoid it, a new variable, $\alpha$ , is introduced such that

$$\alpha = (\,\xi^2 - \xi_1^2)^{1/2} \tag{5.6}$$

then,

$$
\begin{aligned}
\alpha\, d\alpha &= \xi\, d\xi \\
\alpha_{p_2} &= (\xi^2{}_{p_2} - \xi_1^2)^{1/2}
\end{aligned}
\tag{5.7}
$$

where $\alpha_{p_2}$ is the value of $\alpha$ at point $P_2$.

This transformation reduces the integrals of Eq.(4.27), along $P_2\xi_1$ to,

$$
\begin{aligned}
I_0 \Big|_{p_2\xi_1} &= H(t-t_A)\,\frac{2}{\pi}\ \mathrm{Im}\int_{\alpha_{p_2}}^{0} E_1[\xi(\alpha)]\,\frac{\alpha}{K[r,z,t;\xi(\alpha)]}\,d\alpha \\
I_1 \Big|_{p_2\xi_1} &= -H(t-t_A)\,\frac{2}{\pi r}\ \mathrm{Im}\int_{\alpha_{p_2}}^{0} E_2[\xi(\alpha)]\,\frac{[z_p\eta(\alpha) + z_s\zeta(\alpha) - t]}{K[r,z,t;\xi(\alpha)]}\,\alpha\, d\alpha
\end{aligned}
\tag{5.8}
$$

Note that, as $\alpha$ approaches to zero, K also goes to zero and the above integrals become indeterminate. To remove this undeterminacy, the function K is expanded into a power series around $\alpha = 0$, and the common factor is cancelled by the $\alpha$ in the numerator, see Appendix B.

Finally, each of the integrals along $QP_1$, $P_1P_2$ and $P_2\xi_1$, in the complex $\xi$-plane can be transformed to an integration with respect to a real variable y in the interval [-1,1], using the following transformations.

Along $QP_1$

$$\xi = \frac{1}{2} (\xi_{P_1} + \xi_Q) + \frac{1}{2} (\xi_{P_1} - \xi_Q)\, y$$

$$\frac{d\xi}{dy} = \frac{1}{2} (\xi_{P_1} - \xi_Q) \tag{5.9}$$

Along $P_1P_2$

$$\xi = \frac{1}{2} (\xi_{P_2} + \xi_{P_1}) + \frac{1}{2} (\xi_{P_2} - \xi_{P_2})\, y$$

$$\frac{d\xi}{dy} = \frac{1}{2} (\xi_{P_2} - \xi_{P_1}) \tag{5.10}$$

Along $P_2\xi_1$

$$\alpha = (1 - y)\, \alpha_{P_2}/2 \qquad , \qquad \frac{d\alpha}{dy} = - \alpha_{P_2}/2$$

$$\xi = (\alpha^2 + \xi_1^2)^{1/2} \qquad , \qquad \frac{d\xi}{d\alpha} = \alpha / \xi \tag{5.11}$$

Hence, Eq.(5.8) yields,

$$I_0 = H(t-t_A) \frac{2}{\pi} \operatorname{Im} \Big[ \int_{-1}^{+1} (E_1 \frac{\xi}{K} \frac{d\xi}{dy})_{QP_1} dy + \int_{-1}^{+1} (E_1 \frac{\alpha}{K} \frac{d\xi}{dy})_{P_1P_2} dy + \int_{-1}^{+1} (E_1 \frac{\alpha}{K} \frac{d\alpha}{dy})_{P_2\xi_1} dy \Big]$$

$$I_1 = H(t-t_A)\,\frac{2}{\pi r}\ \mathrm{Im}\left[\ \int_{-1}^{+1}\left(E_2\,\frac{t - z_p\eta - z_s\zeta}{K}\ \xi\,\frac{d\xi}{dy}\right)_{QP_1} dy + \int_{-1}^{+1}\left(E_2\,\frac{t - z_p\eta - z_s\zeta}{K}\ \xi\,\frac{d\xi}{dy}\right)_{P_1P_2} dy + \int_{-1}^{+1}\left(E_2\,\frac{t - z_p\eta - z_s\zeta}{K}\ \alpha\,\frac{d\xi}{dy}\right)_{P_2\xi_1} dy\ \right] \tag{5.12}$$

These integrals were then calculated using Gaussian quadrature integrations.

## 5.2. NUMERICAL RESULTS AND CONCLUSION

We have considered in this thesis, the case of a buried vertical line force source and the receiver was located on the surface of the media which was taken to be a half-space. The time dependence of source function was chosen to be a Delta-Dirac function.

The response of a media due to arbitrary time dependency of the source can be obtained by applying the principle of superposition. Mathematically, this principle can be written as,

$$h(t) = \int_0^t G(t)\ f(t - \tau)\ dt \tag{5.13}$$

where G(t) is the transfer function of the media due to a $\delta(t)$ input, f(t) is the input time function and h(t) is the output of the system. This equation can be evaluated numerically by breaking the total duration into n intervals of $\Delta t$. In matrix form it can be written as [5],

$$\frac{1}{\Delta t}\ \{h\} = [G]\ \{f\} \tag{5.14}$$

In the calibration of a transducer as a source, the response due to an unknown input force is picked up by a standard transducer. Then finding the transfer function of the media as explained above, the input function can be calculated by deconvoluting Eq.(5.13). In matrix form, using Eq.(5.14), it can be written as,

$$\{f\} = \frac{1}{\Delta t}\,[G]^{-1}\,\{h\} \tag{5.15}$$

In this work, the ray integrals were calculated using the Gaussian quadrature numerical integration method. For each of the line segments $QP_1$, $P_1P_2$, $P_2\xi_1$, eight point Gaussian quadrature integration was used. The convolution of the responses were done simply by employing the numerical integration methods. Throughout the numerical work, two different line sources were considered. One of them lied between $z_1 = 1$ and $z_2 = 1.5$ and the other between $z_1 = 1$ and $z_2 = 2$. The receivers were placed at different $r_0$ values on the surface, namely $r_0 = 1,3,4,6$. And for the different combinations of source and receiver orientations, in the radial and z-direction, i.e., $u_r$, $u_z$, were numerically calculated. As a control mechanism upon all the numerical work done, the source length and receiver orientation were modified to simulate a single point force. For this purpose, the length of the line force was taken as 0.01 unit, i.e., $z_1 = 1$ and $z_2 = 1.01$, and the radial difference between source and receiver was taken as $r_0 = 7$. In this case, the time dependence of source function was introduced as Heaviside's step function. The outcoming time displacement graph was similar to the one obtained by Pekeris [17] for a point force buried in a half-space.

If we look at the time-displacement graphs, we make the following observation. When $r_0 / z_0 > 5$, we can clearly see the great peak of Rayleigh waves. The response dedicated at the points close to the epicentre are strong for all time t. As the receiver point is moved away from the epicentre, the beginning part of the response signal is weaker when compared to the later part.

Thus, the main problem in the study of acoustic emission is the determination of the relations between the recorded signal and the mechanism of the source which emits the signal. It is clear that, if acoustic emission is to be applied to determine plastic deformation and fracture of the materials, and to monitor the safety of a structure, these relations must be known explicetly to calibrate the control equipment.

In this work, we obtained the theoretical solution and numerical solutions for some cases for an isotropic, homogenous and elastic half-space due to application of a vertical line source with a Delta-Dirac input function. Throughout the work, the receiver was placed at the surface. The solutions can be used as a comparison in the calibration of transducers having different sizes.

# REFERENCES

1. Abromovitz, M. and Stegun, I.A., Handbook of Mathematical Functions, U.S. Government Printing Office, Washington, D.C., 1964.

2. Achenbach, J.D., Wave Propogation in Elastic Solids, North-Holland Company, London, 1973.

3. Başakar, A., Response of An Elastic Half-Space to Finite Sized Sources, Boğaziçi University, İstanbul, 1982.

4. Cagniard, L., "Reflection and Refraction of Progressive Seismic Waves", Translated by Flynn, E.A. and Dix, C.H., Mc Graw-Hill Book Co., New York, 1962.

5. Ceranoğlu, A., Acoustic Emission and Propogation of Elastic Pulses in a Plate, Ph.D. Thesis, Cornell University, Ithaca, New York, 1979.

6. Chen, P., "Diffraction of Sound Pulses and Acoustic Emission in a Hollow Elastic Cylinder", Ph.D. Thesis, Cornell University, Ithaca, New York, 1978.

7. Chao, C.C., "Dynamical Response of an Elastic Half-Space to Tangential Surface Loadings, *Journal of Appl. Mechanics*, 27, p.539, 1960.

8. Garwin, W.W., "Exact Transient Solution of the Buried Line Source Problem", *Proc. of the Royal Soc.*, London, Ser.A, p.234, 1956.

9. Lamb, H., "On the Propogation of Tremors Over the Surface of An Elastic Solid", *Phil. Transactions of Royal Soc. of London*, Ser.A, 242, p.63, 1949.

10. Lapwood, E.R., "The Disturbance due to A Line Source in A Semi Infinite Elastic Medium", *Phil. Trans. of Royal Soc. of London,* Ser.A, 242, p.63, 1949.

11. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity, Dover, New York, 1944.

12. Mencher, A.G., "Epicentral Displacement Caused by Elastic Waves in An Infinite Slab", *Journal of Appl. Physics,* 24, p.1240, 1953.

13. Norwood, F.R., "Exact Transient Response of An Elastic Half-Space Loaded over A Rectangular Region of Its Surface", *Journal of Appl. Mechanics,* p.516, 1969.

14. Norwood, F.R., "Response of An Elastic Half-Space to Impulsive Stationary Finite Line Sources", *Journal of Appl. Mechanics,* p.549, June 1971.

15. Pao, Y.H. and Ceranoğlu, A., "Acoustic Emission and Transient Waves in An Elastic Plate", *Journal of Acoustic Soc. Am.,* 1978.

16. Pao, Y.H. and Gajewski, R., "The Generalized Ray Theory and Transient Response of Layered Elastic Solids", *Physical Acoustics,* 13, Ch.6, 1977.

17. Pekeris, C.L. and Lifson, H., "Motion of the Surface of A Uniform Elastic Half-Space Produced by A Buried Pulse", *Journal Acoust. Soc. Am.,* p.1233, 1957.

18. Sherwood, J.W.C., "Transient Sound Propogation in A Layered Liquid Medium", *Journal Acoust. Soc. Am.,* 32, p.1673, 1960.

19. Spencer, T.W., "The Method of Generalized Reflection and Transmission Coefficients", *Geophysics,* 25, p.625, 1960.

20. Sternberg, E., "On the Integration of the Equation of Motion in the Classical Theory of Elasticity", *Archiv. for Rational Mechanics and Analysis,* 6, p.34, 1960.

21. Wu, J.H., Transient Analysis of a Three Dimensional Elastic Plate by the Ray Group Technique, Ph.D. Thesis, University of New Mexico, 1976.

# APPENDIX A

## RESPONSE OF AN UNBOUNDED MEDIUM TO AN ARBITRARILY ORIENTED CONCENTRATED FORCE

In this appendix, a detailed derivation of the displacement potentials used in Chapter II will be discussed. We will use the approach given by Achenbach [2].

The equations of motion for an elastic, isotropic, and homogenous medium are given by,

$$\mu \nabla^2 \underline{u} + (\lambda + \mu) \underline{\nabla}\, \underline{\nabla} \cdot \underline{u} + \rho \underline{F} = \rho \ddot{\underline{u}} \qquad (A.1)$$

where $\underline{u}$ is the displacement vector, $\underline{f}$ is body force per unit mass, $\rho$ is the mass density, and $\lambda$ and $\mu$ are the Lame constants of the medium and a "dot" denotes differentiation with respect to time, t.

Consider a concentrated force of magnitude f(t), direct along the constant unit vector $\underline{a}$, and acting at the point $x_0$ in the cartesian coordinate system. In this case we have

$$\underline{F}(\underline{x},t) = \underline{a}\; f(t) \quad \delta(\underline{x} - \underline{x}_0) \qquad (A.2)$$

where $\delta(\underline{x},x_0)$ is Delta-Dirac function. Now, we wish to decompose both the displacement $\underline{u}$ and the body force vector $\underline{f}$ as,

$$\begin{aligned} \underline{u} &= \underline{\nabla}\, \phi + \underline{\nabla} \times \underline{\Psi}_1 \\ \underline{F} &= \underline{\nabla}\, G + \underline{\nabla} \times \underline{H} \end{aligned} \qquad (A.3)$$

where $\phi$ and F are scalar potentials, and, $\Psi$ and $\underline{G}$ are vector potentials. Above decomposition is known as Helmholtz decomposition. The result of it is,

$$c^2 \nabla^2 \phi + G = \ddot{\phi}$$
$$C^2 \nabla^2 \underline{\Psi}_1 + \underline{H} = \ddot{\underline{\Psi}}_1 \qquad (A.4)$$

If $\underline{F}(\underline{x},t)$ is known; G and $\underline{H}$ can be found. Consider,

$$\nabla^2 \underline{W} = \underline{F} \qquad (A.5)$$

which is known as the vector Poisson's equation. And it has a well known solution,

$$\underline{W} = -\frac{1}{4\pi} \int_V \frac{\underline{F}(\underline{x}')}{[\underline{x}-\underline{x}']} d\underline{x}' \qquad (A.6)$$

where V is volume of the body. Using the identity,

$$\nabla^2 \underline{W} = \underline{\nabla} (\underline{\nabla} \cdot \underline{W}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{W}) \qquad (A.7)$$

in Eq.(A.5), and comparing the resulting expression with the one given by Eq.(A.3), we get,

$$G = \underline{\nabla} \cdot \underline{W}$$
$$\underline{H} = -\underline{\nabla} \times \underline{W} \qquad (A.8)$$

Hence, considering Eq.(A.2) and Eq.(A.8); Eq.(A.4) can be written as,

$$c^2 \nabla^2 \phi - \frac{f(t)}{4\pi} \underline{\nabla} \cdot \left(\frac{\underline{a}}{R}\right) = \ddot{\phi}$$
$$C^2 \nabla^2 \Psi_1 + \frac{f(t)}{4\pi} \underline{\nabla} \times \left(\frac{\underline{a}}{R}\right) = \ddot{\Psi}_1 \tag{A.9}$$

where,

$$R^2 = (x - x_0) + (y - y_0)^2 + (z - z_0)^2 \tag{A.10}$$

Introducing the following non-dimensional quantities

$$\hat{\phi} = \phi/r_0^2 \quad , \quad \hat{R} = R/r_0 \quad , \quad \hat{\underline{\nabla}} = r_0 \underline{\nabla}$$
$$\hat{\underline{\Psi}}_1 = \underline{\Psi}_1/r \quad , \quad \kappa = c/C \quad , \quad \hat{t} = tc/r_0 \tag{A.11}$$

Eq.(A.9) yields,

$$\hat{\nabla}^2\hat{\phi} - f_1(t)\ \hat{\underline{\nabla}} \cdot \left(\frac{\underline{a}}{\hat{R}}\right) = \frac{\partial^2\ddot{\hat{\phi}}}{\partial\hat{t}^2} \tag{A.12}$$
$$\hat{\nabla}^2\hat{\underline{\Psi}}_1 + \kappa^2 f_1(t)\ \hat{\underline{\nabla}} \times \left(\frac{\underline{a}}{\hat{R}}\right) = \kappa^2 \frac{\partial^2\Psi_1}{\partial\hat{t}^2}$$

where,

$$f_1(t) = \frac{f(t)}{4\pi\ c^2 r_0^2} \tag{A.13}$$

again $r_0$ is the radial distance from source to receiver. Dropping the "Hats", and taking Laplace transform after introducing,

$$\phi = \underline{\nabla} \cdot (\underline{a}\ \Phi)$$
$$\underline{\Psi}_1 = -\underline{\nabla} \times (\underline{a}\ \Psi) \tag{A.14}$$

Eq.(A.12) becomes,

$$\nabla^2 \bar{\Phi} - \bar{f}_1(s) \frac{1}{R} = s^2 \bar{\Phi}$$

$$\nabla^2 \bar{\Psi} - \bar{f}_1(s) \frac{1}{R} = s^2 \kappa^2 \bar{\Psi} \qquad \text{(A.15)}$$

where,

$$\Phi(\underline{x},s) = \int_0^\infty \Phi(\underline{x},t)\, e^{-st}\, dt \qquad \text{(A.16)}$$

Since the inhomogenous terms in the above equations show polar symmetry, the solution is conveniently obtained by using the spherical coordinate system. Eq.(A.15) can be written as,

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \bar{\Phi}}{\partial R}\right) - \bar{f}_1(s) \frac{1}{R} = s^2 \bar{\Phi} \qquad \text{(A.17)}$$

Introducing,

$$\bar{\Phi} = \bar{\Phi}_1 / R \qquad \text{(A.18)}$$

we have,

$$\frac{d^2 \bar{\Phi}_1}{dR^2} - s^2 \bar{\Phi}_1 = \bar{f}_1(s) \qquad \text{(A.19)}$$

The complete solution of the above equation is

$$\bar{\Phi}_1 = A\, e^{-sR} - \frac{\bar{f}_1(s)}{s^2} \qquad \text{(A.20)}$$

Since $\bar{\Phi}$ has to be finite at the origin, $R = 0$, $\bar{\Phi}_1$ must vanish at this point. Therefore,

$$\bar{\Phi}(R,s) = \frac{\bar{f}_1(s)}{s^2 R} \left[e^{-sR} - 1\right] \qquad \text{(A.21)}$$

and similarly,

$$\bar{\Psi}_1(R,s) = \frac{\bar{f}_1(s)}{s^2 R} \left[e^{-s\kappa R} - 1\right] \qquad \text{(A.22)}$$

Taking Laplace transform of Eq.(A.3) and Eq.(A.14) we have,

$$\begin{aligned}
\underline{\bar{u}} &= \underline{\nabla}\ \bar{\phi} + \underline{\nabla} \times \underline{\bar{\Psi}}_1 \\
\bar{\phi} &= \underline{\nabla}\cdot(\underline{a}\ \bar{\Phi}) \qquad \text{(A.23)} \\
\underline{\bar{\Psi}}_1 &= -\underline{\nabla} \times (\underline{a}\ \bar{\Psi})
\end{aligned}$$

Thus,

$$\begin{aligned}
\underline{\bar{u}} &= \underline{\nabla}\ (\underline{\nabla}\cdot\underline{a}\ \bar{\Phi}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{a}\ \bar{\Psi}) \\
&= \underline{\nabla}\ (\underline{\nabla}\cdot\underline{a}\ \bar{\Phi}) - \underline{\nabla}(\underline{\nabla}\cdot\underline{a}\ \bar{\Psi}) + \underline{a}\ \nabla^2\bar{\Psi} \\
&= \underline{\nabla}\ \underline{\nabla}\cdot\underline{a}(\bar{\Phi} - \bar{\Psi}) + \underline{a}(\kappa^2 s^2 \bar{\Psi} + \kappa^2\ \bar{f}_1(s)/R) \qquad \text{(A.24)}
\end{aligned}$$

Substituting the expressions for $\bar{\Phi}$ and $\bar{\Psi}$ from Eq.(A.21) and Eq.(A.22), the above equation yields,

$$\underline{\bar{u}} = \frac{\bar{f}_1(s)}{s^2}\left[\underline{\nabla}\,\underline{a}\cdot\underline{\nabla}(g_p - g_s) + \underline{a}\kappa^2 s^2\ g_s\right] \qquad \text{(A.25)}$$

where,

$$g_p = \frac{1}{R}\ e^{-sR} \quad , \quad g_s = \frac{1}{R}\ e^{-s\kappa R} \qquad \text{(A.26)}$$

Note that, R is given by Eq.(A.10). Also note that Eq.(A.25) agrees with Pa and Gajewski [16]. The functions $g_p$ and $g_s$ are called the radial wave functions for longitudinal and shear waves. Using Sommerfield's integral representation [5], $g_p$ and $g_s$ can be written as,

$$g_p = \frac{e^{-sR}}{R} = s \int_0^\infty \frac{\xi}{\eta} e^{-s\eta|z-z_0|} J_0(s\xi r)\, d\xi$$
$$g_s = \frac{e^{-s\kappa R}}{R} = s \int_0^\infty \frac{\xi}{\zeta} e^{-s\zeta|z-z_0|} J_0(s\xi r)\, d\xi \tag{A.27}$$

where,

$$\eta = (\xi^2 + 1)^{1/2} \quad , \quad \zeta = (\xi^2 + \kappa^2)^{1/2} \tag{A.28}$$
$$r^2 = (x - x_0)^2 + (y - y_0)^2$$

From Eq.(A.5), we can obtain the components of the displacement vector, $\underline{\bar{u}}$, as follows,

$$\bar{u}_r(\underline{r},s) = \frac{\bar{f}_1(s)}{s^2} \left[ \left(a_r \frac{\partial^2}{\partial r} + a_z \frac{\partial^2}{\partial r \partial z}\right)(g_p - g_s) + a_r s^2 \kappa^2 g_s \right]$$
$$\bar{u}_\theta(\underline{r},s) = \frac{\bar{f}_1(s)}{s^2} a_\theta \left[ \frac{\partial}{\partial r} (g_p - g_s) + s^2\kappa^2 g_s \right] \tag{A.29}$$
$$\bar{u}_z(\underline{r},s) = \frac{\bar{f}_1(s)}{s^2} \left[ \left(a_r \frac{\partial^2}{\partial r \partial z} + a_z \frac{\partial^2}{\partial z^2}\right)(g_p - g_s) + a_z s^2 \kappa^2 g_s \right]$$

Note that the term $\bar{f}_1(s)/s^2R$ in Eq.(A.21) and Eq.(A.22) does not appear in the expressions for the displacements, so it also won't be seen in the stress expressions. Dropping this term and using Eq.(A.27), we get,

$$\bar{\Phi}(\underline{r},s) = \frac{\bar{f}_1(s)}{s} \int_0^\infty \frac{\xi}{\eta} e^{-s\eta|z-z_0|} J_0(s\xi r)\, d\xi$$
$$\bar{\Psi}(\underline{r},s) = \frac{\bar{f}_1(s)}{s} \int_0^\infty \frac{\xi}{\zeta} e^{-s\zeta|z-z_0|} J_0(s\xi r)\, d\xi \tag{A.30}$$

The relations between the potentials and displacements are,

$$u_r = \frac{\partial\phi}{\partial r} + \frac{\partial^2\Psi}{\partial r\partial z} + \frac{1}{r}\frac{\partial X}{\partial\theta}$$

$$u_z = \frac{\partial\phi}{\partial z} + \frac{\partial^2\Psi}{\partial z^2} - \kappa^2\frac{\partial^2\Psi}{\partial t^2} \qquad (A.31)$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{1}{r}\frac{\partial^2\Psi}{\partial\theta\partial z} - \frac{\partial X}{\partial r}$$

Using the above equations and Eq.(A.29), the expressions for the potentials $\overline{\Psi}$ and $\overline{\chi}$ are obtained as,

$$\overline{\Psi} = -a_z \frac{\overline{f}_1(s)}{s} \int_0^\infty S_v\, e^{-s\zeta|z-z_0|}\, J_0\,(s\,\xi\, r)\, d\xi$$

$$- a_r \frac{\overline{f}_1(s)}{s} \int_0^\infty S'_v\, e^{-s\zeta|z-z_0|}\, J_1\,(s\,\xi\, r)\, d\xi \qquad (A.32)$$

$$\overline{\chi} = -a_\theta\, \overline{f}(s) \int_0^\infty S_H\, e^{-s\zeta|z-z_0|}\, J_1\,(s\,\xi\, r)\, d\xi \qquad (A.33)$$

and using Eq.(A.14) and Eq.(A.30), we get the potential as,

$$\overline{\phi} = \overline{f}_1(s)\, a_z \int_0^\infty S_p\, e^{-s\eta|z-z_0|}\, J_0\,(s\,\xi\, r)\,\xi\, d\xi$$

$$+ \overline{f}_1(s)\, a_r \int_0^\infty S'_p\, e^{-s\eta|z-z_0|}\, J_1\,(s\,\xi\, r)\,\xi\, d\xi \qquad (A.34)$$

where,

$$\overline{f}_1(s) = \overline{f}(s)/4\pi r^2 c^2$$

$$S_p = -\,\varepsilon \quad , \quad S'_p = -\varepsilon/\eta \quad , \quad S_H = \kappa^2/\zeta$$

$$S_v = \xi/\zeta \quad , \quad S'_v = \varepsilon \quad , \quad \varepsilon = \mathrm{sgn}|z-z_0| \qquad (A.35)$$

# APPENDIX B

## THE POWER SERIES EXPANSION OF THE "K" FUNCTION

In this appedix, we expand the "K" function, discussed in section 5.1, in the form of power series in terms of . This is vitally necessary, because, when $\xi$ approaches $\xi_1$, K goes to zero, so the term $\xi/K$ becomes indeterminate.

$$\frac{\xi}{K(r,z,t,\xi)} = \frac{\xi}{[\xi^2 r^2 + (z_p \eta + z_s \zeta - t)^2]^{1/2}} \qquad (B.1)$$

To remove it, a new variable, $\alpha$ , is introduced,

$$\xi^2 = (\alpha^2 + \xi_1^2)^{1/2} \qquad (B.2)$$

Thus, we get,

$$\frac{\xi}{K(r,z,t;\xi)} = \frac{\alpha}{K(r,z,t;\xi(\alpha))} \qquad (B.3)$$

The above equation is also indeterminate when $\alpha$ approaches zero. Therefore, the "K" function will be expanded into power series around $\alpha = 0$. Note that, Eq.(B.1) can be written as,

$$\frac{\xi}{K} = \frac{\xi}{[(i\xi r + z_p \eta + z_s \zeta - t)(-i\xi r + z_p \eta + z_s \zeta - t)]^{1/2}}$$

It is clear that the indeterminacy occurs due to the second expression in the denominator. Thus only bracket will be expanded into a power series. Using Eq.(B.2), we have,

$$-i\xi r + z_p\eta + z_s\zeta - t = -ir(\alpha^2 + \xi_1^2)^{1/2} + z_p(\alpha^2 + \xi_1^2 + 1)^{1/2} + z_s(\alpha^2 + \xi_1^2 + \kappa^2)^{1/2} - t \tag{B.5}$$

Consider the following definition of power series expansion around $\alpha = 0$

$$(\alpha^2 + L^2) = L + \frac{1}{2}\frac{\alpha^2}{L} - \frac{1}{8}\frac{\alpha^4}{L^3} + \frac{1}{16}\frac{\alpha^6}{L^5} - \frac{5}{128}\frac{\alpha^8}{L^7} + \frac{7}{256}\frac{\alpha^{10}}{L^9} \cdots \tag{B.6}$$

Then, applying the above expansion to Eq.(B.5) and Eq.(B.4) we have,

$$\begin{aligned} K(r,z,t,\xi(\alpha)) = \{ & [ir(\alpha^2 + \xi_1^2)^{1/2} + z_p(\alpha^2 + \xi_1^2 + 1)^{1/2} + z_s(\alpha^2 + \xi^2 + \kappa^2)^{1/2} \\ & - t]\alpha^2[-ir(\frac{1}{2\xi_1} - \frac{1}{8}\frac{\alpha^2}{\xi_1^3} + \ldots\ldots) \\ & + z_p(\frac{1}{2\eta_1} - \frac{1}{8}\frac{\alpha^2}{\eta_1^3} + \ldots\ldots) \\ & + z_s(\frac{1}{2\zeta_1} - \frac{1}{8}\frac{\alpha^2}{\zeta_1^3} + \ldots\ldots) - t]\}^{1/2} \end{aligned} \tag{B.7}$$

where,

$$\eta_1 = (\xi_1^2 + 1)^{1/2} \quad , \quad \xi_1 = (\xi_1^2 + \kappa^2)^{1/2} \tag{B.8}$$

Finally, substituting the above equation in Eq.(B.3), $\alpha$'s are cancelled and the uncertainty is removed.

# APPENDIX C

## SOURCE AND RECEIVER FUNCTIONS

### Interior Source Functions

$$S_p = -\,\varepsilon$$

$$S'_p = -\,\xi / \eta$$

$$S_v = \xi / \zeta$$

$$S'_v = \varepsilon$$

$$S_H = \kappa^2 / \zeta$$

### Surface Receiver Functions

$$D^*_{zp} = -2\,\kappa^2\,\eta\,(\xi^2 + \zeta^2)/\,\Delta r$$

$$D^*_{zv} = 4\,\kappa^2\,\eta\,\zeta\,\xi / \Delta r$$

$$D^*_{rp} = D^*_{\theta p} = 4\,\kappa^2\,\eta\,\zeta\,\xi / \Delta r$$

$$D^*_{rv} = D^*_{\theta v} = -2\,\kappa^2\,\zeta(\xi^2 + \zeta^2)/\Delta r$$

$$D^*_{rH} = D^*_{\theta H} = 2$$

Reflection Coefficients

$$R_{pp} = R_{ss} = [4\,\eta\,\zeta\,\xi^2 + (\xi^2 + \zeta^2)^2] / \Delta r$$

$$R_{ps} = -4\,\eta\,\xi\,(\xi^2 + \zeta^2) / \Delta r$$

$$R_{sp} = \left(\frac{\zeta}{\eta}\right) R_{ps}$$

$$R_H = 1$$

Modified Source Function for the Case of; Surface Receiver and Buried Vertical Line Source

$$\underline{S}_p = \left(-\frac{1}{\eta}\right)$$

$$\underline{S}_v = \left(-\frac{\xi}{\zeta^2}\right)$$

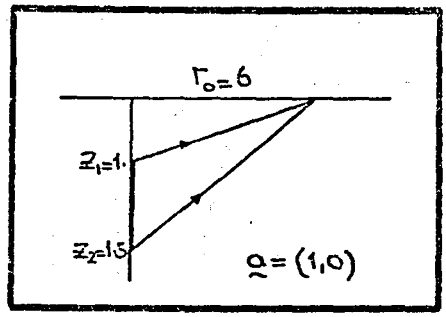
$$\underline{S}'_p = \left(-\frac{\xi}{\eta^2}\right)$$

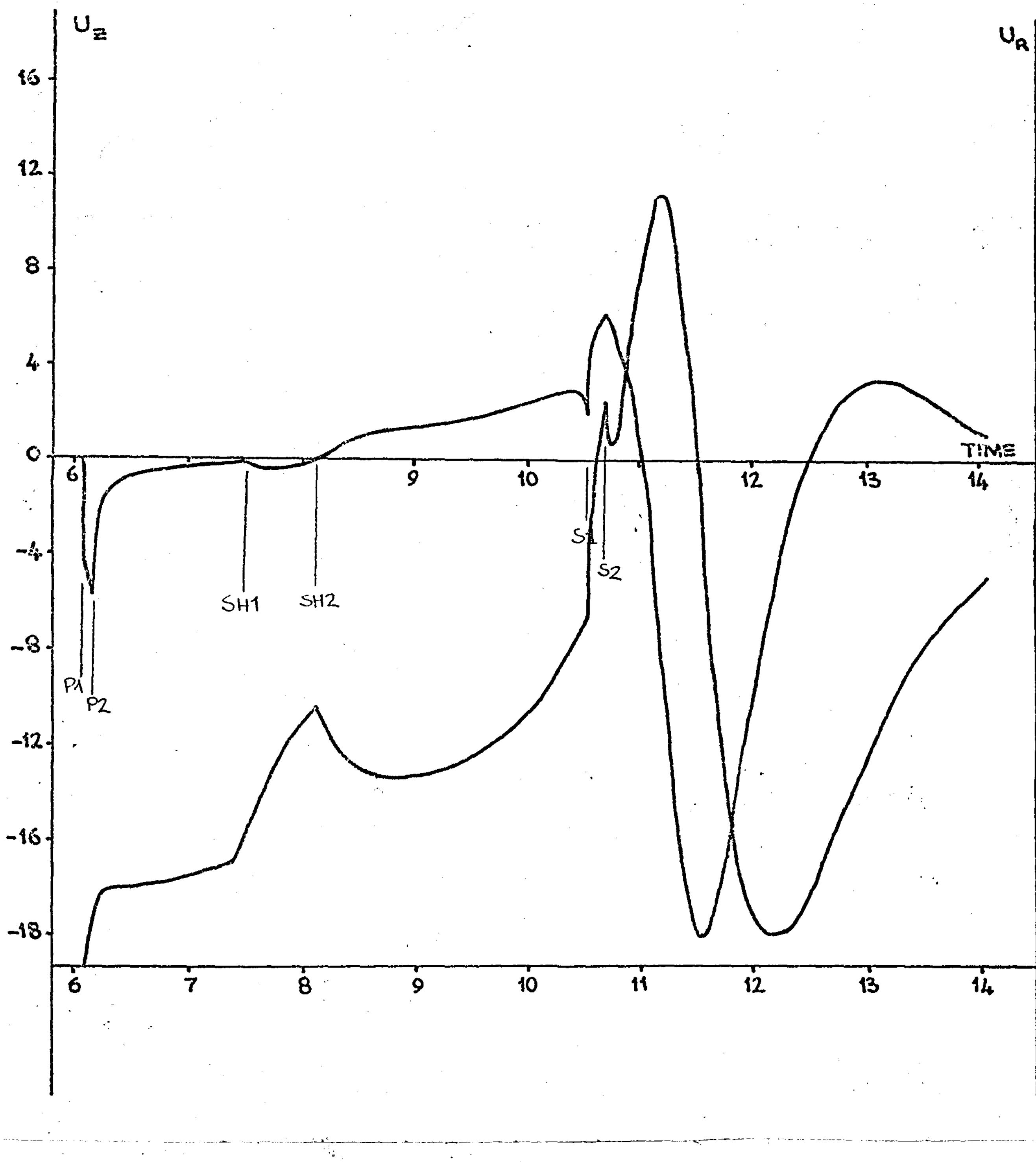
$$\underline{S}'_v = \left(-\frac{1}{\zeta}\right)$$

# APPENDIX D

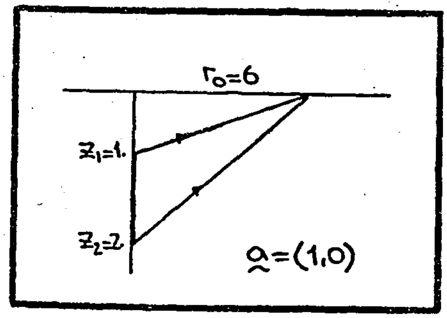
## GRAPHIC EXAMPLES

In all of the following graphs, the displacements and time values are nondimensional quantities nondimensionalized with $\pi \mu r_0^2/F_0$ and $c/r_0$ correspondingly.




igure D.1, Response due to a horizantal line force, $z_1=1$ and $z_2=1.5$, when $r_0=6$.



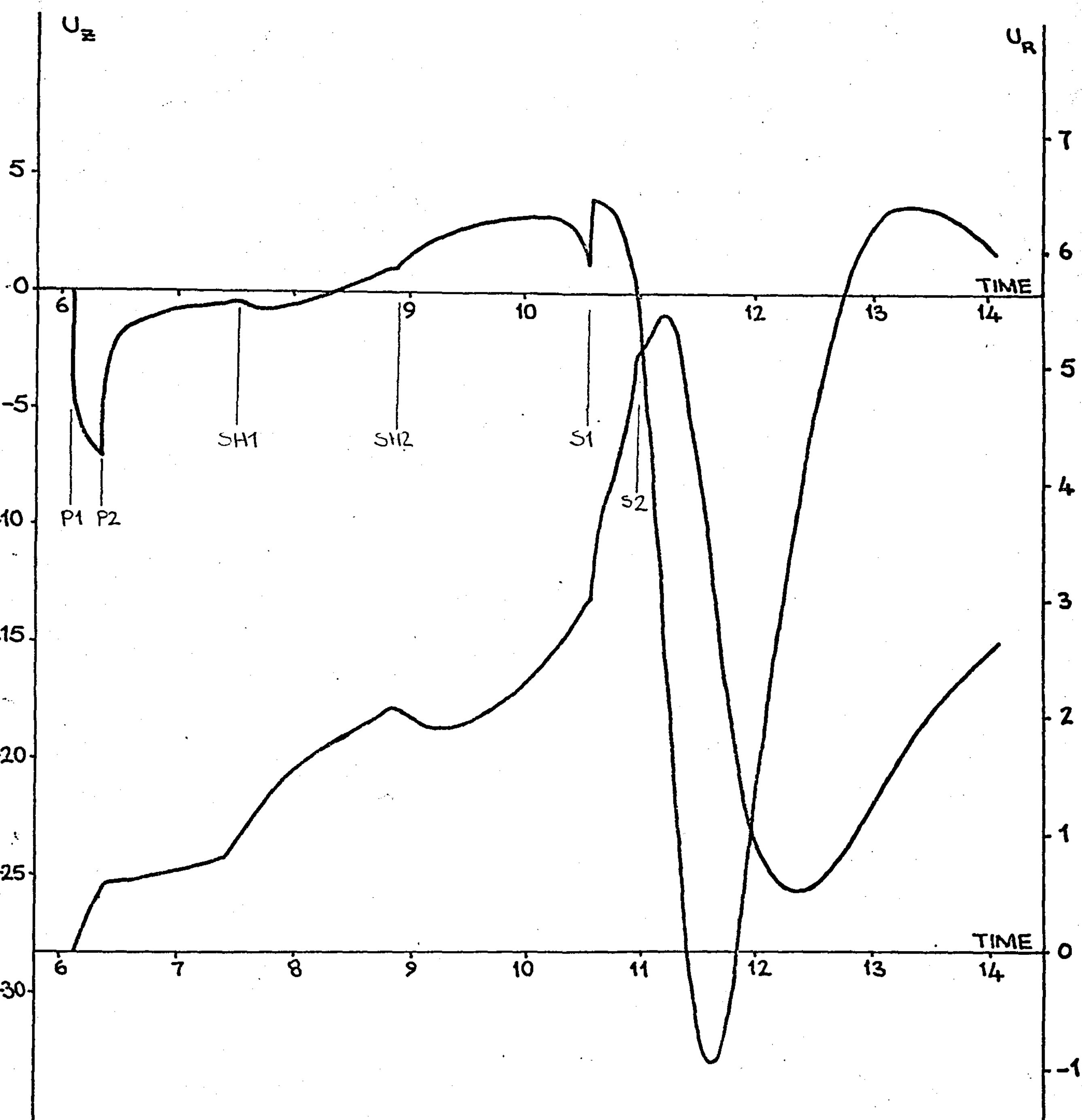
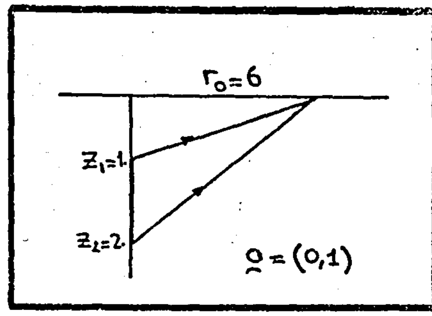
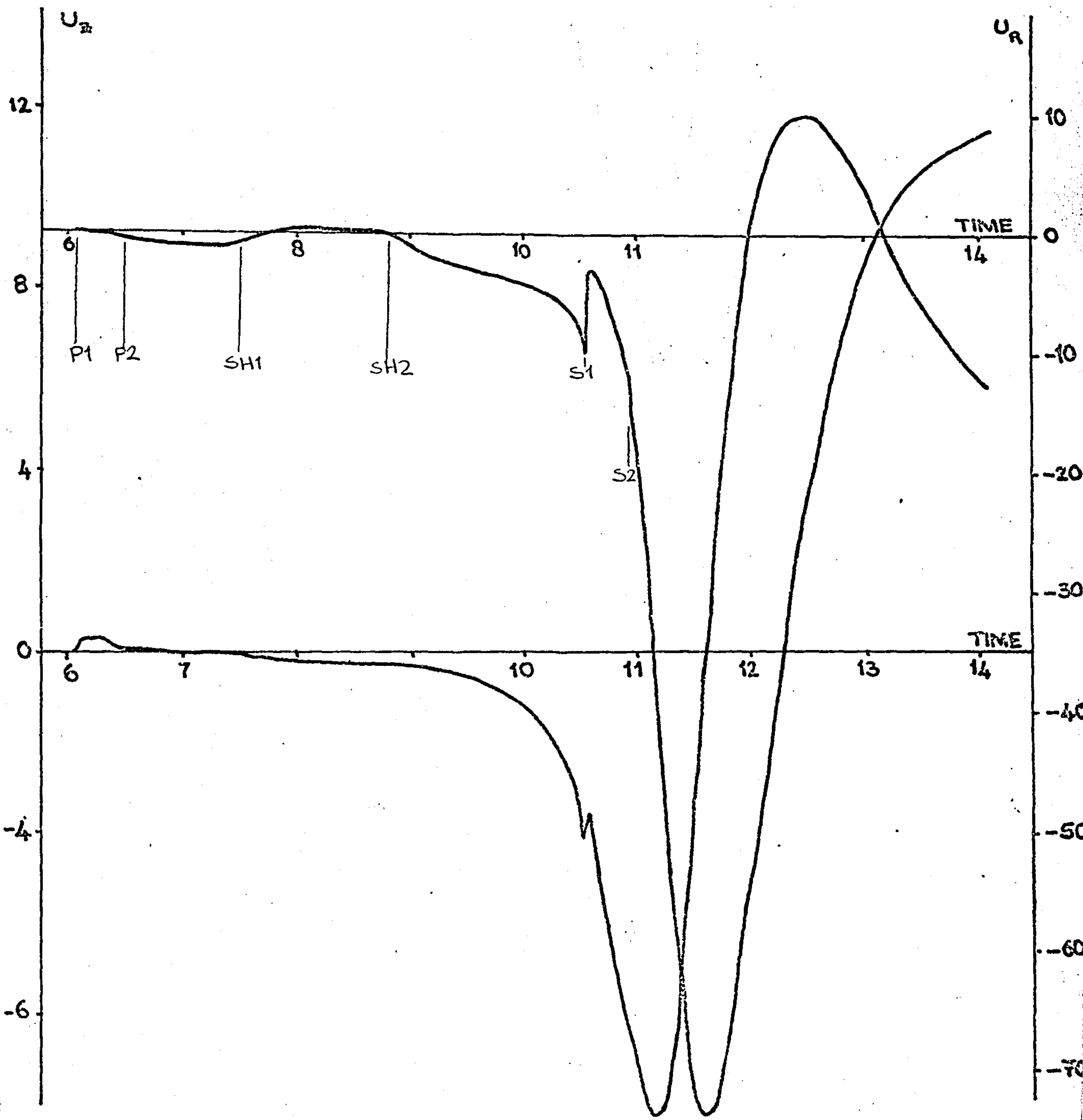




Figure D.3, Response due to a horizantal line force, $z_1=1$ and $z_2=2$, when $r_0=6$.




igure D.4, Response due to a vertical line force, $z_1=1$ and $z_2=2$, when $r_0=6$.

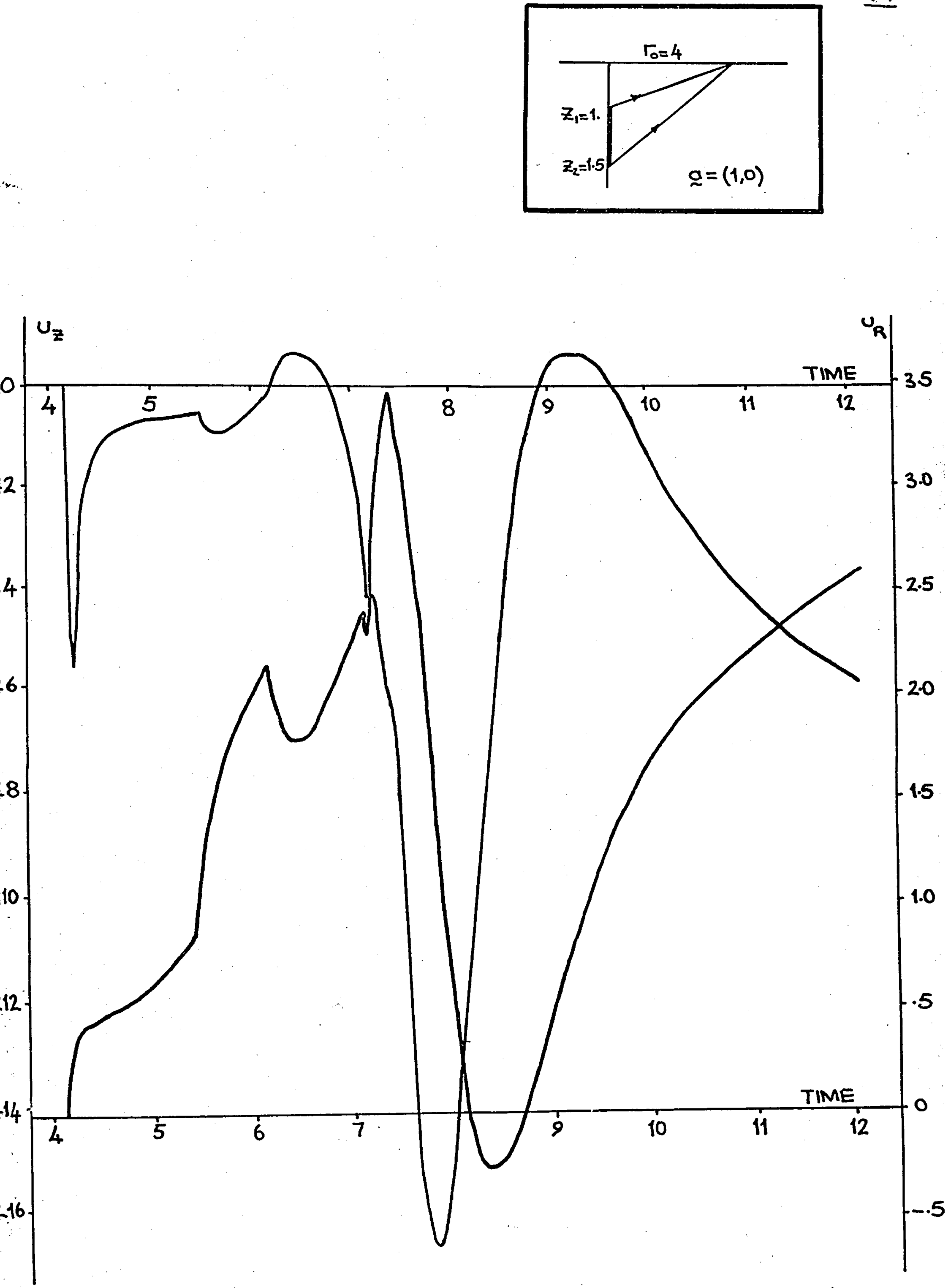



Figure D.5, Response due to a horizantal line force, $z_1=1$ and $z_2=1.5$, when $r_0=4$.

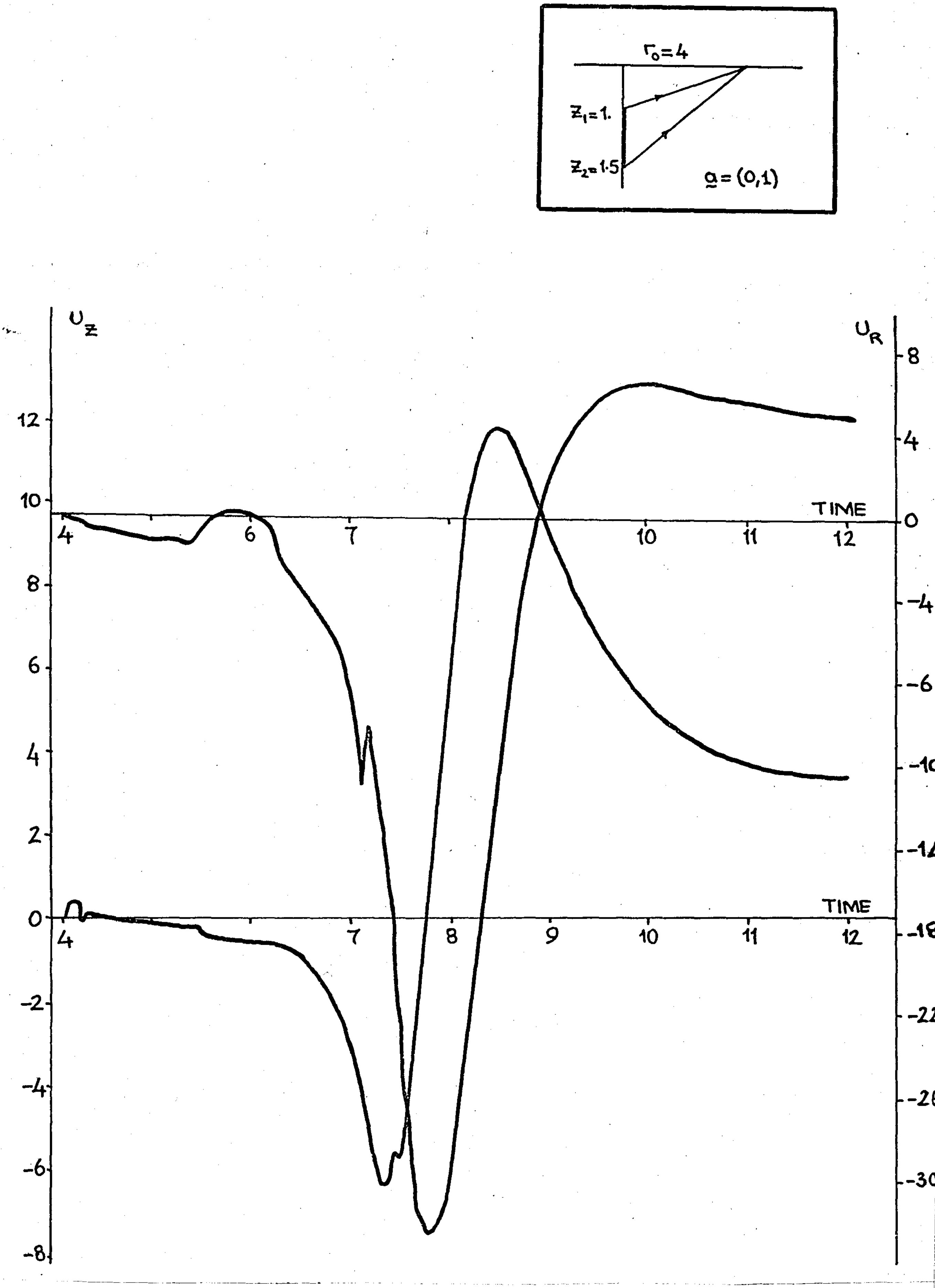
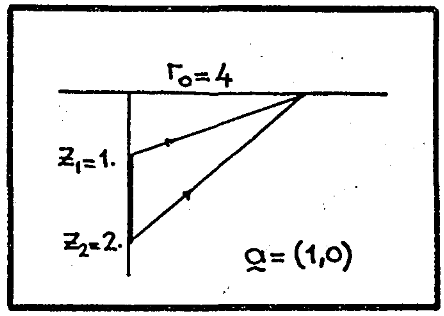



Figure D.6, Response due to a vertical line force, $z_1=1$ and $z_2=1.5$, when $r_0=4$.



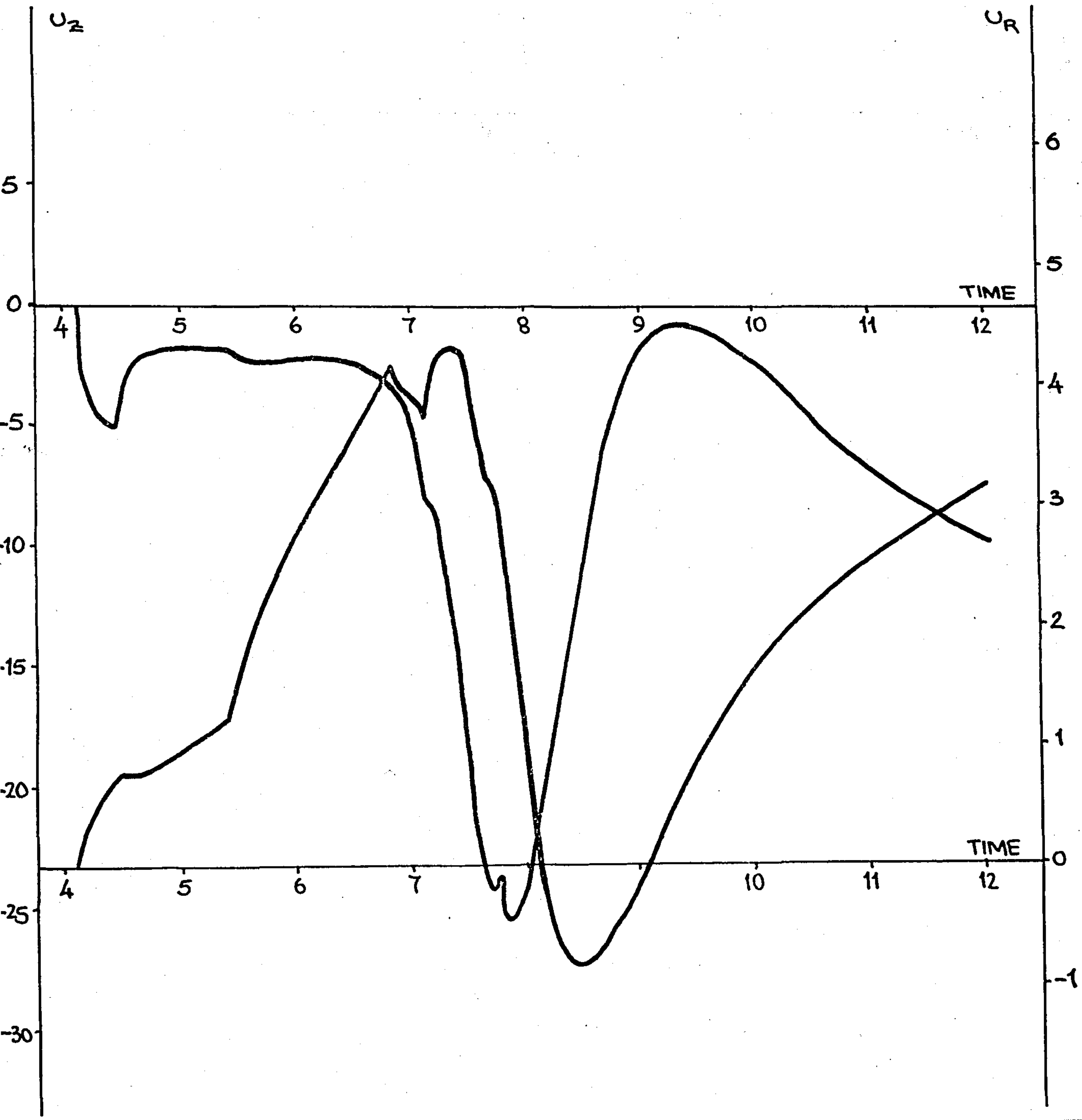
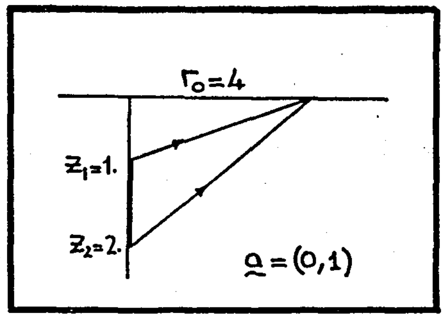




Figure D.8, Response due to a vertical line force, $z_1=1$ and $z_2=2$, when $r_0=4$.



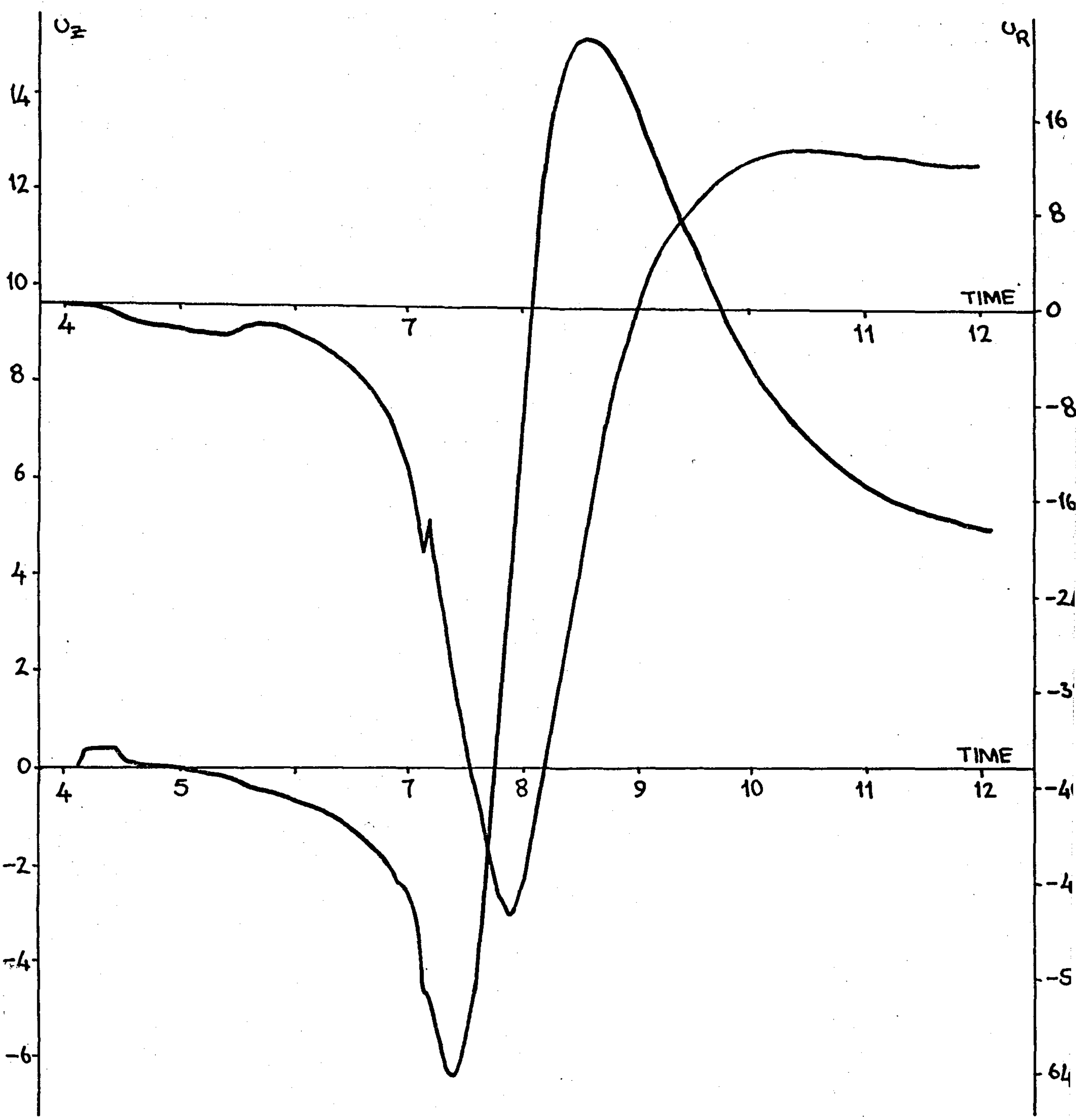
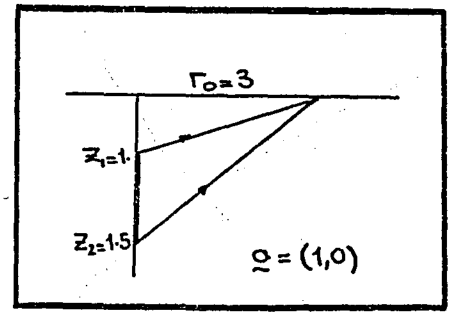




Figure D.9, Response due to a horizantal line force, $z_1=1$ and $z_2=1.5$, when $r_0=3$.



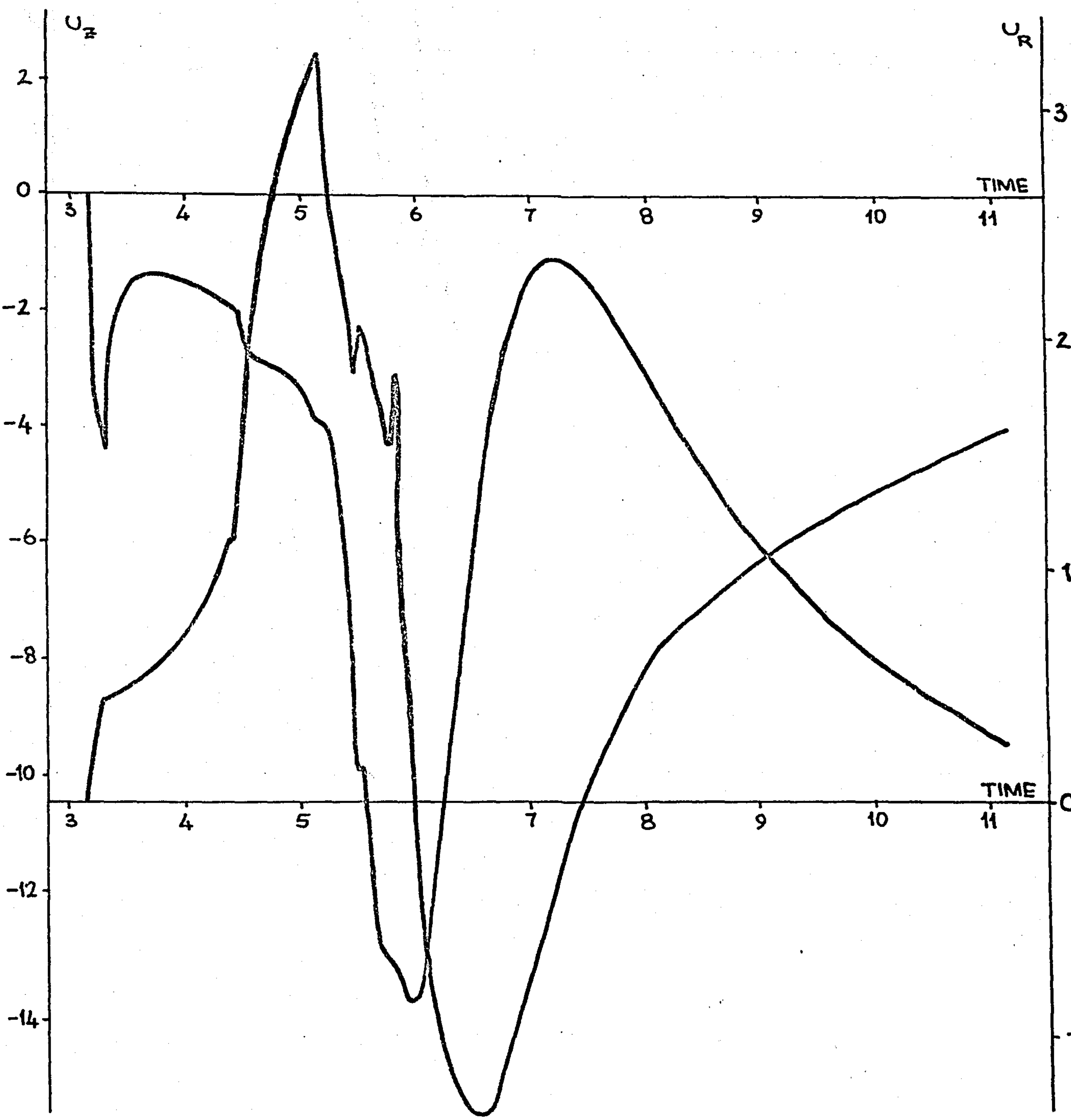




Figure D.10, Response due to a vertical line force, $z_1=1$ and $z_2=1.5$, when $r_0=3$.

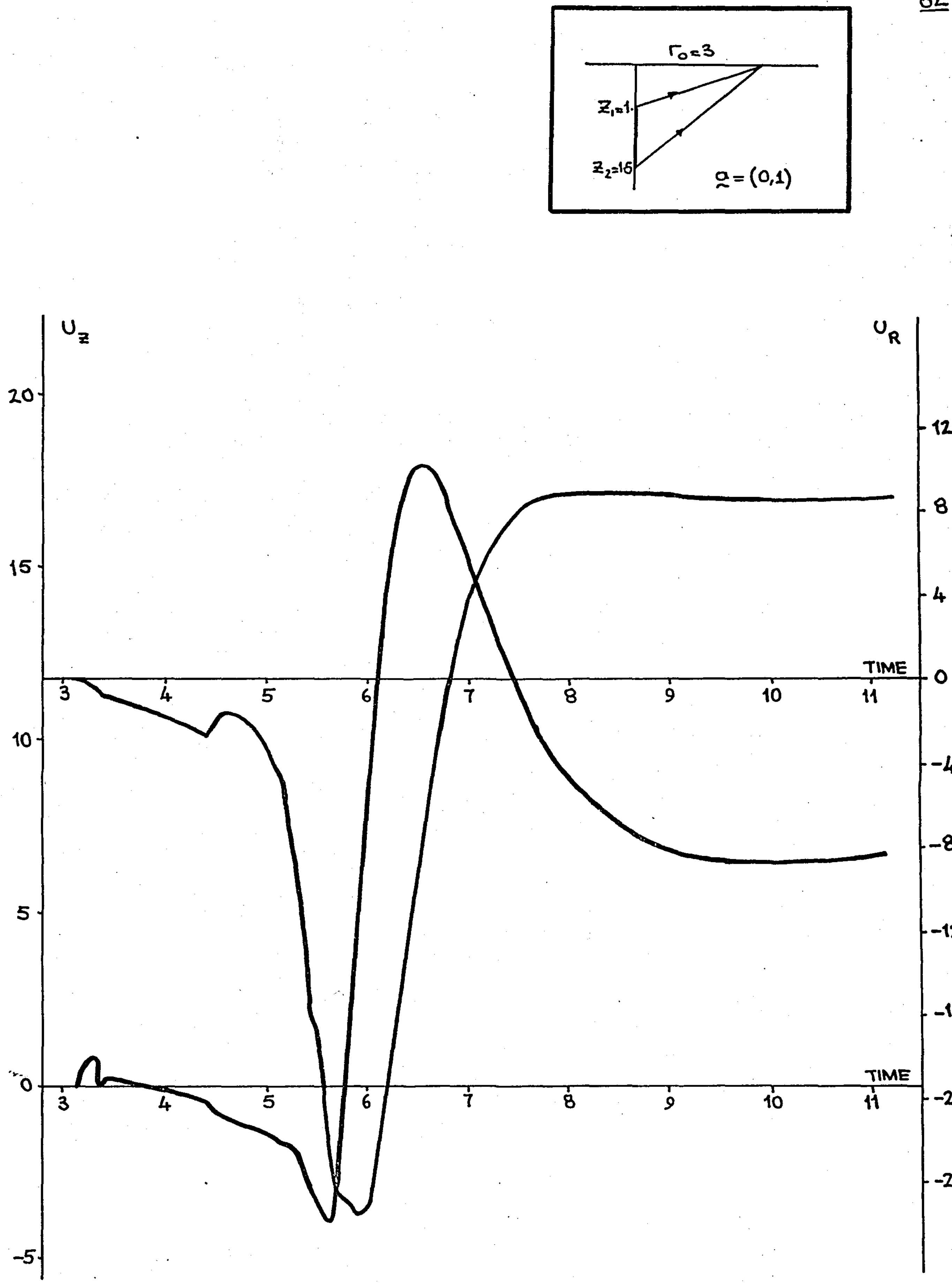
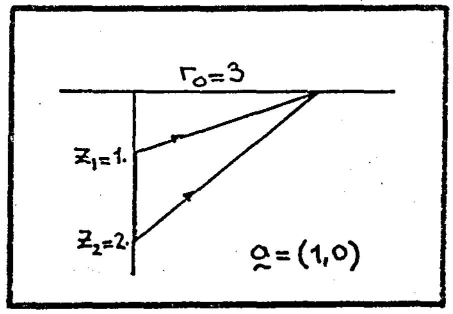



Figure D.11, Response due to a horizantal line force, $z_1=1$ and $z_2=2$, when $r_0=3$.



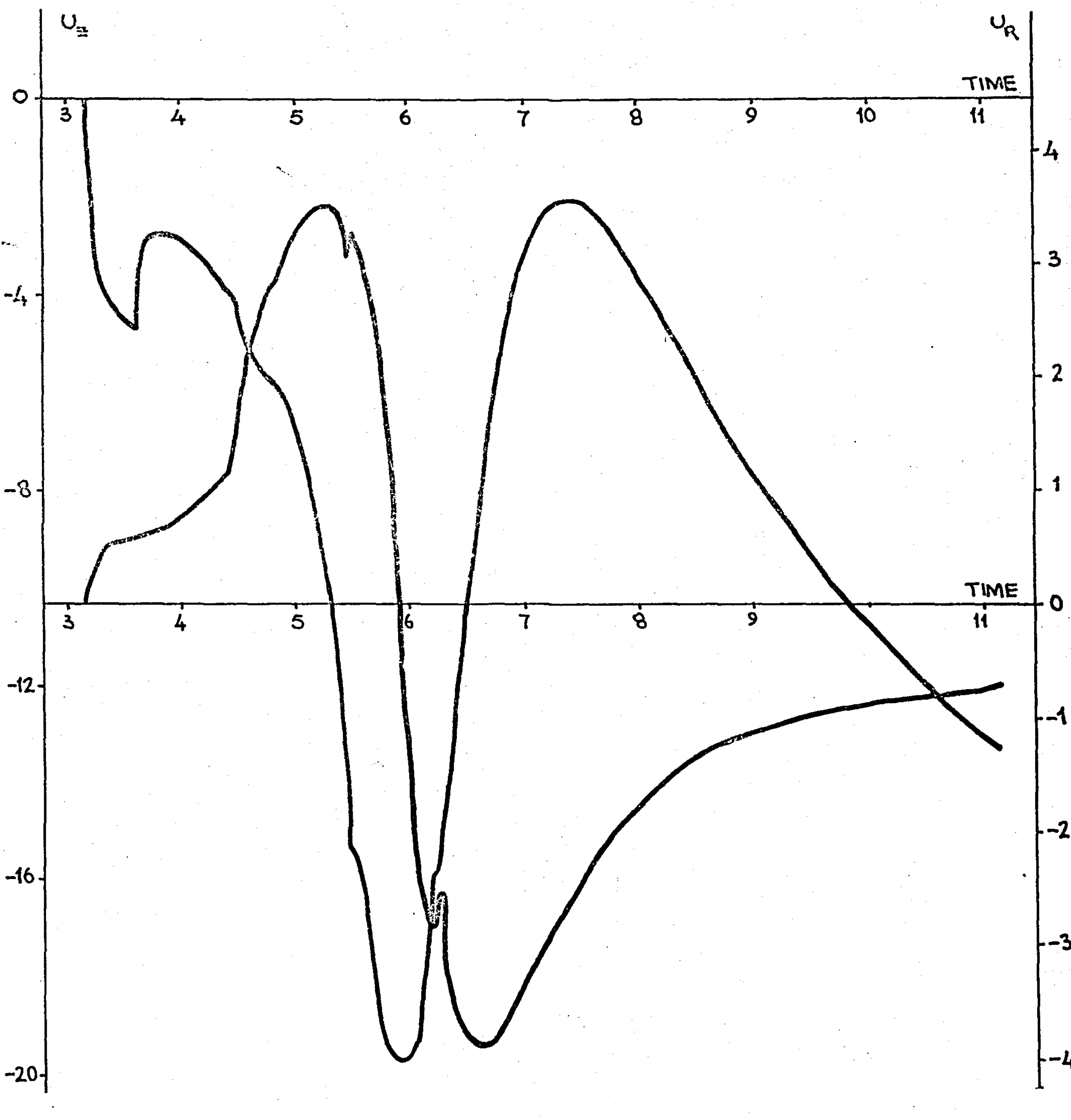
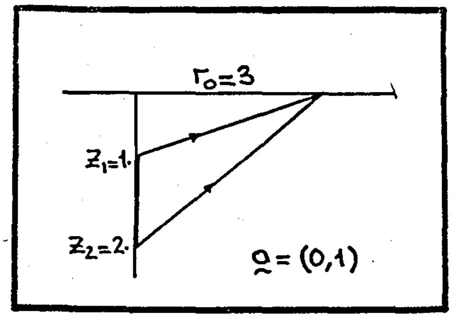




Figure D.12, Response due to a vertical line force, $z_1=1$ and $z_2=2$, when $r_0=3$.



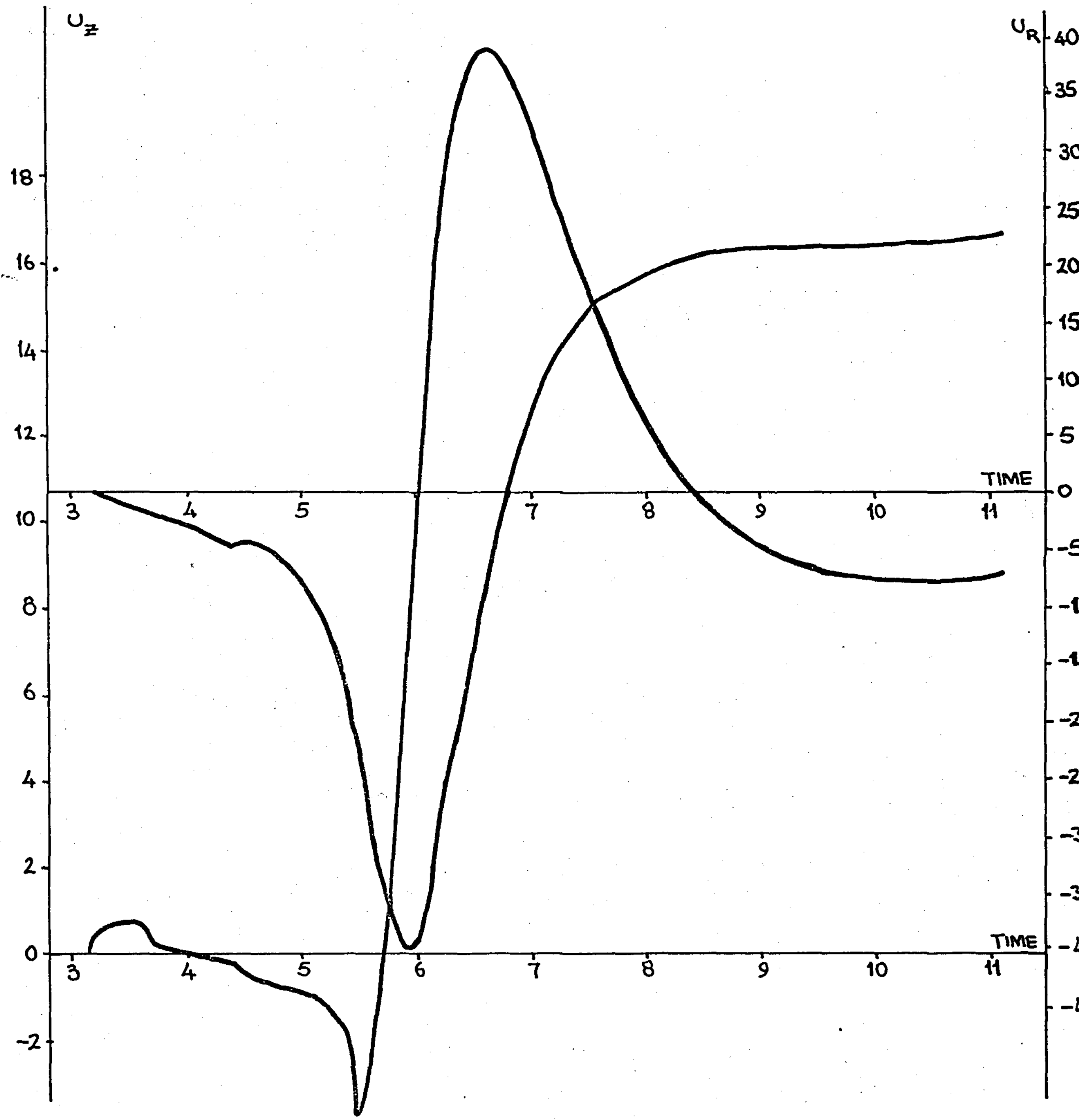
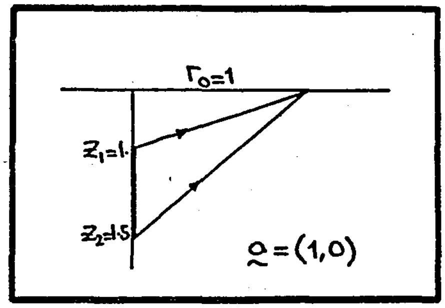




Figure D.13, Response due to a horizantal line force, $z_1=1$ and $z_2=1.5$, when $r_0=7$.



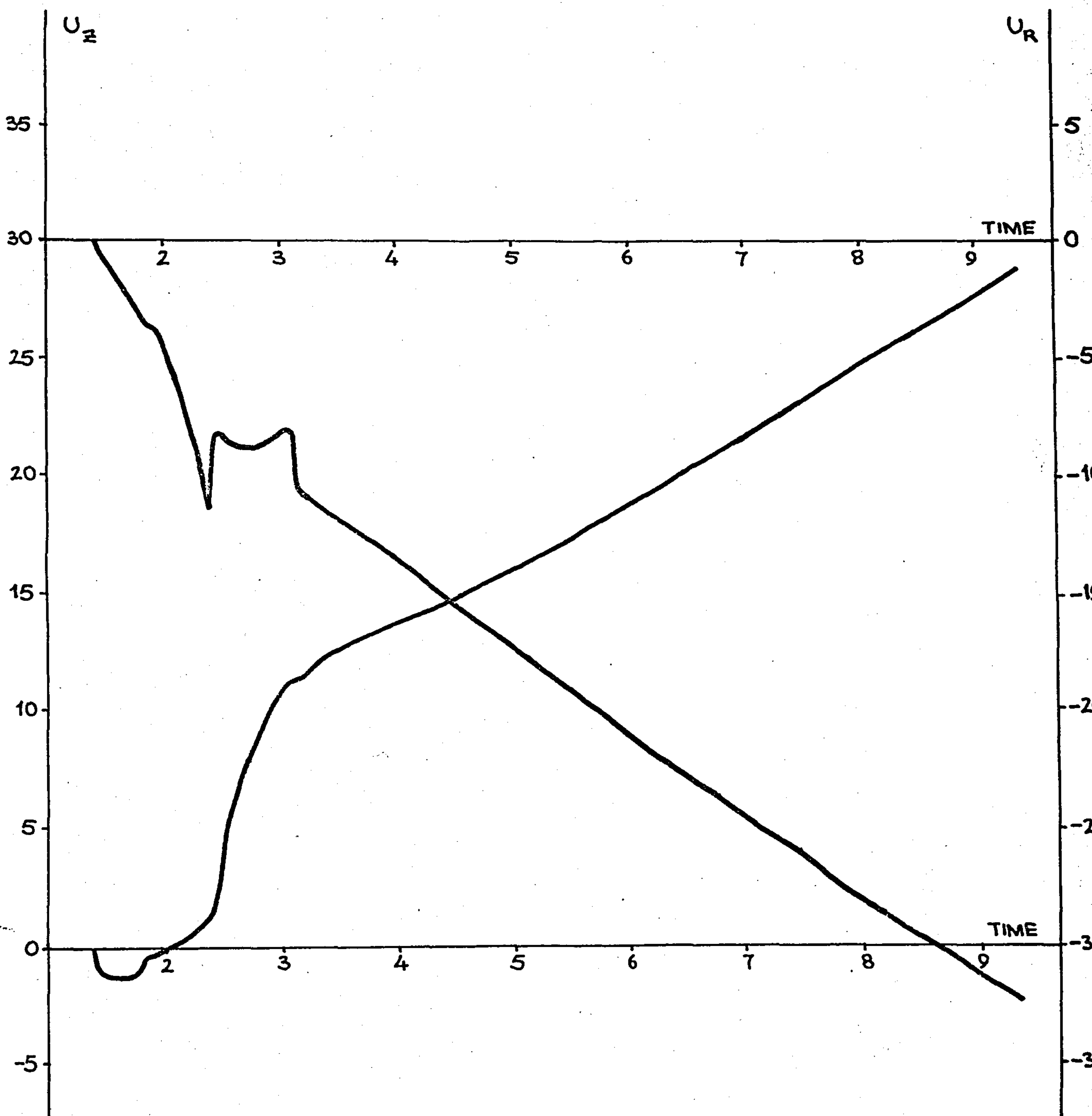
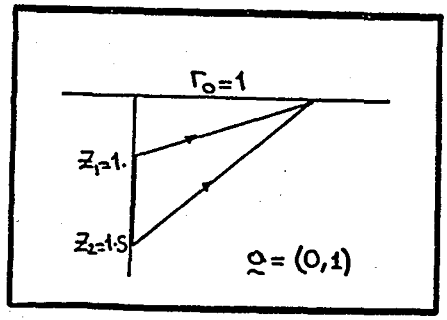




Figure D.13, Response due to a horizantal line force, $z_1=1$ and $z_2=1.5$, when $r_0=1$.



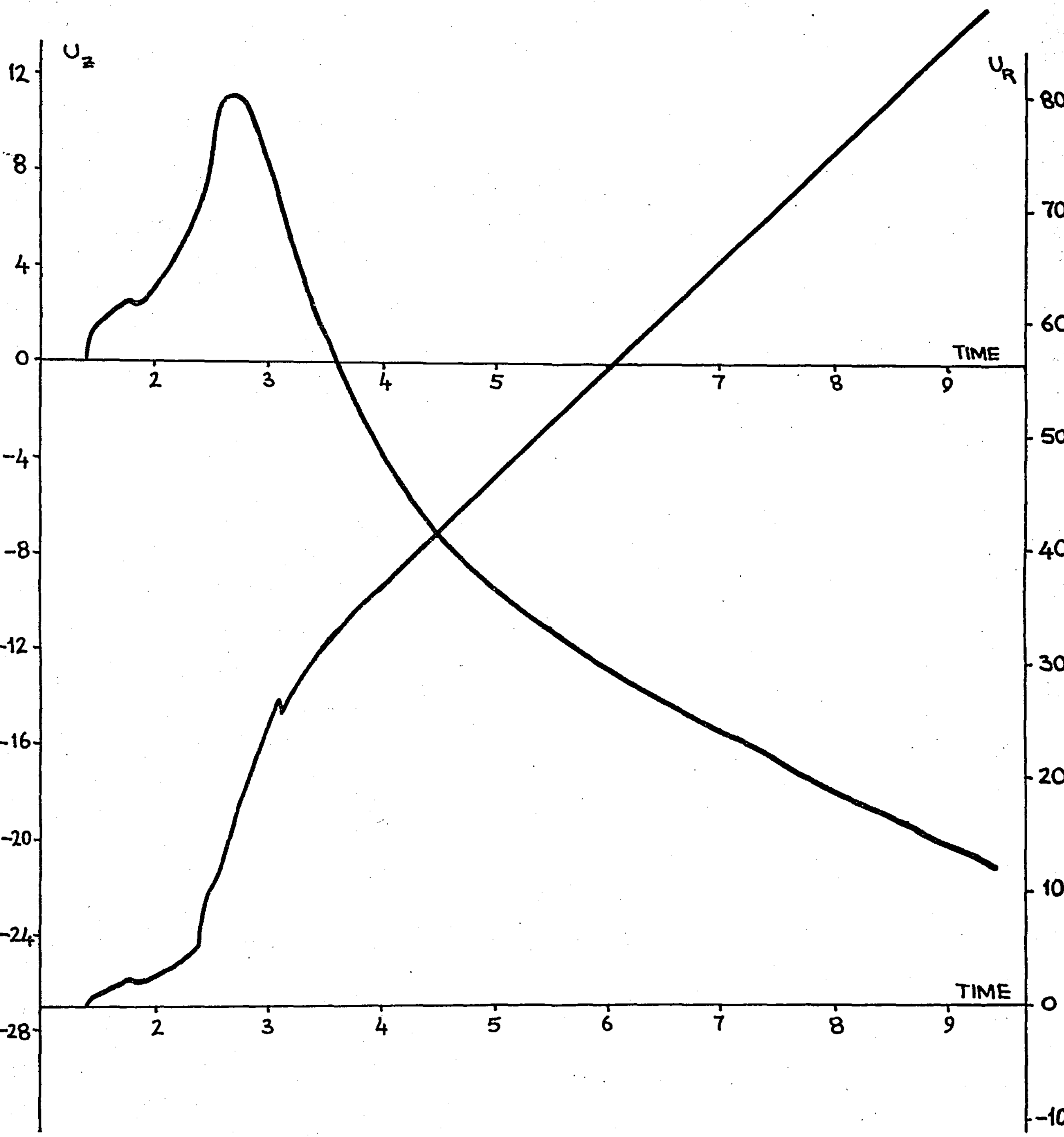
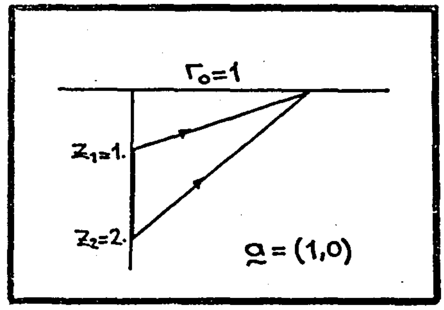




Figure D.14, Response due to a vertical line force, $z_1=1$ and $z_2=1.5$, when $r_0=1$.



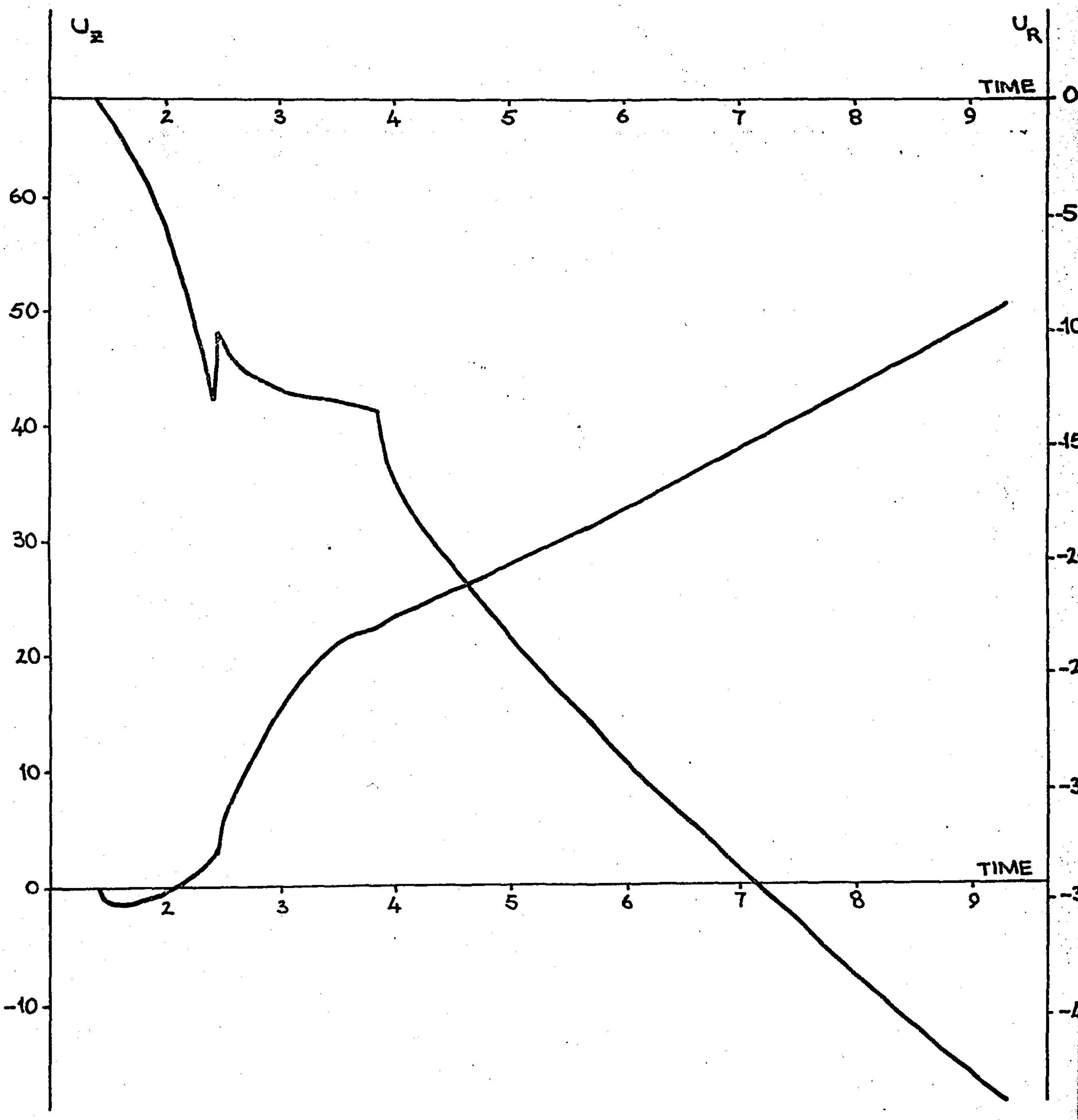
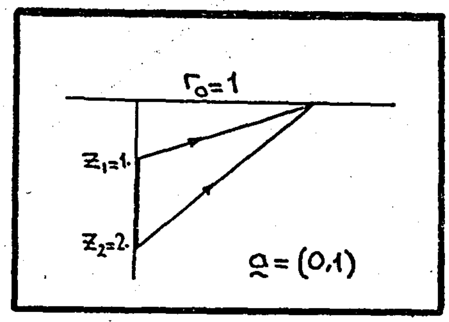




Figure D.15, Response due to a horizantal line force, $z_1=1$ and $z_2=2$, when $r_0=1$.



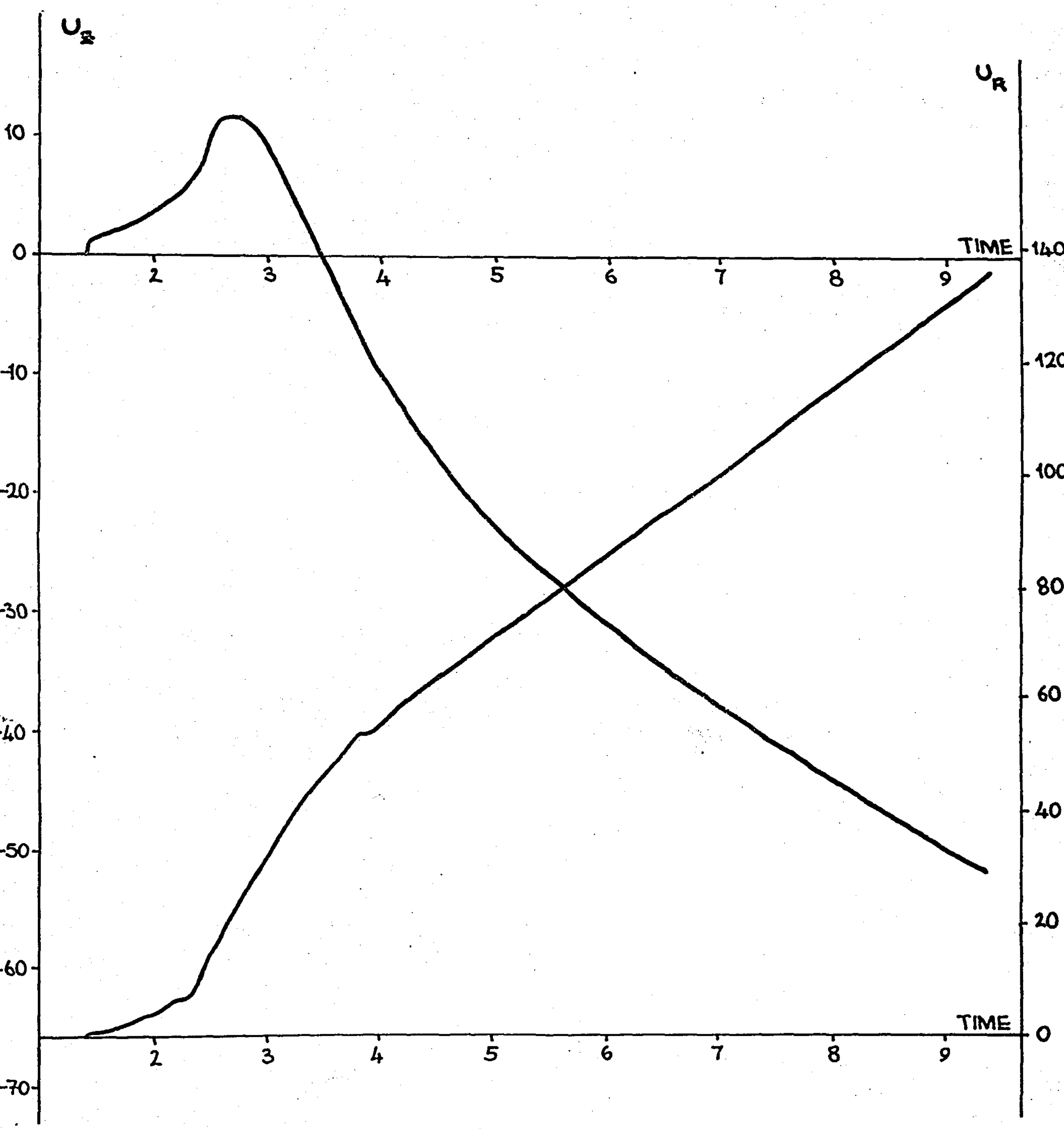




Figure D.16, Response due to a vertical line force, $z_1=1$ and $z_2=2$, when $r_0=1$.