ANALYSIS OF COLUMN SUPPORTED SLABS BY THE RELAXATION METHOD

A Thesis

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PREFACE

The moments and deflections in a square slab simply supported at the boundaries and on a column at the middle shall be found by the Relaxation method. The aim of the thesis will be to find the moments and deflections in a square three-span slab without drop panels, simply supported at the boundaries and supported on four columns at the interior equidistant from each other.

GENERAL NOTATION

The following notation is used:

R_1, R_2 = Residuals;

Δx, Δy = Change in the values of x and y;

μ = Poisson's ratio, taken as 0.2 (for concrete);

E = Modulus of elasticity, assumed constant; units of kg/cm^2.

h = Thickness of the plate, assumed constant; unit of cm.

M = Summation of moments at each mesh point;

M_x, M_y = Moments in x- and y- axes at each mesh point, respectively;

q = Uniform load on the slab;

∇ = The Laplace operator;

δq, δc = Deflections due to load and column reaction, respectively;

K = $-\delta q/\delta c$;

w = The variable;

a, b = Number of mesh divisions of the sides of the slab;

M_{yx}, M_{xy} = Torsional moment;

Q_x, Q_y = Shear in x- and y- axes, respectively;

A = Distance in direction of span from center of support to the intersection of the center line of the slab thickness with the extreme 45-deg. diagonal line lying wholly within the concrete section of slab and column or other support, including drop panel, capital and bracket.

D = Flexural rigidity of the plate, defined as; units of kg·cm

$$D = \frac{Eh^3}{12(1-\mu^2)}.$$

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CHAPTER 1

BASIC PRINCIPLES

1.1. The Basic Rule of Relaxation. The basic concept of relaxation and its important and widely used modifications can be explained by employing them to solve a pair of algebraic equations, for example:

$$\begin{aligned} 3x + y &= 50 \\ 2x - y &= 10 \end{aligned} \qquad (1.1)$$

or

$$\begin{aligned} 3x + y - 50 &= 0 \\ 2x - y - 10 &= 0 \end{aligned} \qquad (1.2)$$

The solution of the problem can be easily verified to be $x=12$ and $y=14$. In order to solve the above equations by ~~xx~~ the Relaxation Method all the terms shall be written on one side and with only zero on the other side. New quantities called residuals shall be introduced which will take the place of zero in the above equations. For values of x and y which satisfy the equations (1.2), these residuals are equal to zero. For other values of x and y, these residuals will have a value different from zero.

$$\begin{aligned} 3x + y - 50 &= R_1 \\ 2x - y - 10 &= R_2 \end{aligned} \qquad (1.3)$$

The aim of the relaxation is to make these residuals equal to zero or as small as possible. When this is done x and y will automatically satisfy the above equations.

The process of solution is started by the selection of an initial pair of values for x and y equal to zero. Then for the above equations (1.3), $R_1=-50$ and $R_2=-10$. Now if we change x by 1 unit, R_1 is changed by three units while R_2 is changed by 2 units, if on the other hand we differ y by 1 unit, R_1 is increased by 1 unit, and R_2 is decreased by 1 unit. This can be put in tabular form called the "Operations Table"

THE OPERATIONS TABLE ONE

	$\Delta=R_1$	$\Delta=R_2$
$\Delta x=1$	3	2
$\Delta y=1$	1	-1

The process of relaxation consists of the application of the unit operations repeatedly, gradually, to change residuals from their initial value to zero. The first step in the relaxation will be to change the largest residual, in this case R_1=-50 approximately to zero,i.e.,by a change of + 50. From the operations table x=50/3 or about 15. This operation change the value of $-R_2$ to +20. This can be repeated until we obtain zero or close to zero values for our residuals. The total process written in tabular form is as follows:

OPERATIONS TABLE TWO

x	y	R_1	R_2
0	0	-50	-10
Δx=15		- 5	+20
	Δy=10	+ 5	+10
Δx=-2		-1	+ 6
	Δy= 4	+ 3	+ 2
Δx=-1		o	o
x=12	y=14	0	0

The final line in this table both summarizes and checks the calculation In the first column, the solution to the problem is recorded by summing for each unknown the value initially assigned and all increments added to it subsequently,e.g. x= 0(initially) +15-2-1=12. The last line is simply check, it is derived by substituting the values of x and y in the equations (1.2).

1.2. <u>Block Relaxation</u>. Block relaxation consists in the use of operations other than the basic unit operations. They entail the simultaneous application of increments to more than one of the unknowns at the same t In the example of article (1.1) the block unit operation would be $\Delta x=\Delta y=$ In this case the operations table is:

OPERATIONS TABLE THREE

	ΔR_1	ΔR_2
Δ x=1	3	2
Δ y=1	1	-1
$\Delta x=\Delta$ y=1	4	1

The total residual is -60. The unit block operation affects a change of 4+1=5. So, as a first step we shall take a block operation $\Delta x = \Delta y = -\frac{(-60)}{5} = 12$.

OPERATIONS TABLE FOUR

x	y	ΔR1	ΔR2
0 Δx=12	0 Δy=12 Δy=12	650 - 2 0	-10 2 0
x=12	y=14	0	0

The table shows the usefulness of block relaxation as less steps are involved. There are other devices such as group relaxation, over-relaxat used but we are not directly concerned with them though the reader is referred to books on Relaxation Methods.

The simultaneous equations solved above are examples that show the advantage of the method. It is inefficient to solve two simultaneous equations by this method but if there are too many unknowns involved, the advantage of the method can be clearly seen. The most important applicat of the relaxation method has been to obtain particular solutions of parti differential equations in two dimensions.

In this thesis the method has been used in solving the partial differen tial equations encountered in the theory of thin plates. In any applicati of the relaxation method for the solution of a differential equation, finite difference approximations have to be used. It is by solving these finite difference equations that we shall have a solution of the wanted function at a number of equally spaced points.

The relaxation method is a numerical method of solution that has been developed in recent years mainly by R. V. Southwell.

CHAPTER 2

GENERAL THEORY

2.1. <u>The Differential Equation of a Thin Plate</u>. The differential equation of a thin plate loaded laterally with a uniform load of q is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (2.1)$$

where $D = \frac{E\,h^3}{12(1-\mu^2)}$, the flexural rigidity of a plate.

μ = Poisson's ratio.

E = Modulus of elasticity.

h = The thickness of the plate.

The above fourth order partial differential equation can be put into two second order Poisson type differential equations. Thus reducing considerably the work involved in the solution of Eq. (2.1)., especially when difference equations are used.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{q}{D} \qquad (2.2)$$

but
$$M_x + M_y = -D(1+\mu)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad (2.3)$$

putting
$$M = \frac{M_x + M_y}{1+\mu} = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad (2.4)$$

Equation (2.1) can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(M) = -q$$

or
$$\left(\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2}\right) = -q \qquad (2.5a)$$

and
$$\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -\frac{M}{D} \qquad (2.5b)$$

In the case of a simply supported rectangular slab the boundary conditions for equations (2.5a) and (2.5b) are w=0, and $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$ are equal to zero, which makes M=0.

2.2. <u>Finite Difference Approximations</u>. In any application of the relaxation method to solve a differential equation, certain finite difference approximations of derivatives have to be used, and we shall establish two of these approximations which will be needed in this chapt

Suffixes as in (Fig. 2.1) are used to show a typical point of subdivision when working out a general result.

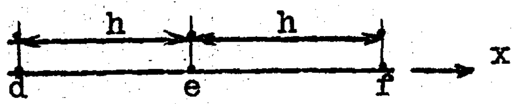


Fig. 2.1.

Corresponding suffixes are used to indicateecvalues of the function w, the transverse deflection, in an expansion of the Taylor series:

$$w=w_e+\left(\frac{dw}{dx}\right)_e(x-x_e)+\frac{1}{2}\left(\frac{d^2w}{dx^2}\right)_e(x-x_e)^2+\frac{1}{3}\left(\frac{d^3w}{dx^3}\right)_e(x-x_e)^3+\ldots\ldots$$

In this relation, if x is put equal, in turn, to (x_e+h) and (x_e-h), we find that,

$$w_f=w_e+h\left(\frac{dw}{dx}\right)_e+\frac{h^2}{2}\left(\frac{d^2w}{dx^2}\right)_e+\frac{h^3}{6}\left(\frac{d^3w}{dx^3}\right)_e+\frac{h^4}{24}\left(\frac{d^4w}{dx^4}\right)_e+\ldots\ldots$$

$$w_d=w_e-h\left(\frac{dw}{dx}\right)_e+\frac{h^2}{2}\left(\frac{d^2w}{dx^2}\right)_e-\frac{h^3}{6}\left(\frac{d^3w}{dx^3}\right)_e+\frac{h^4}{24}\left(\frac{d^4w}{dx^4}\right)_e+\ldots\ldots$$

Adding these two results together, we get

$$w_f+w_d=2w_e+h^2\left(\frac{d^2w}{dx^2}\right)_e+O(h^4).$$

where all the terms containing fourth or higher powers of h are included together as $O(h^4)$. Neglecting this quantity as h is made smaller and smaller, we have the finite-difference approximation to $\left(\frac{d^2w}{dx^2}\right)_e$:

$$h^2\left(\frac{d^2w}{dx^2}\right)_e=w_f+w_d-2w_e \qquad (2.6)$$

In two dimensions as in one, the relaxation solution to a differential equation consists of a finite number of values of the wanted quantity w at a number of points within the region of integration. Whereas in one dimension a range of integration was divided up by points of subdivision into a number of equal intervals, in two dimensions an area of integration is subdivided by a uniform network, and the values of w are calculated at the nodes of the network. Such networks are known as relaxation nets, the commonly used one is the square net, rarely the triangular net is used.

An ordinary differential equation; also any partial differential equation has to be replaced by a set of finite difference equations. For Poisson's equations this can be easily done since it involves only se of the approximation for a second derivative.

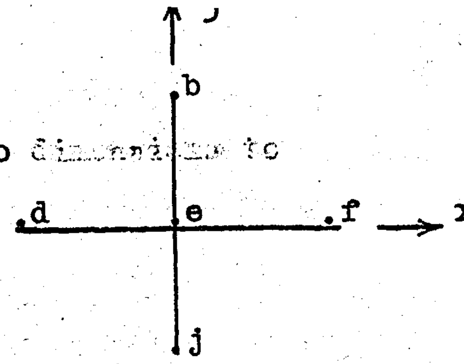


Fig. 2.2.

Fig. 2.2), illustrates the convention used in two imensions to denote a typical node " e "and the urrounding nodes.

$$h^2\left(\frac{\partial^2 w}{\partial x^2}\right)_e = w_f + w_d - 2w_e \qquad (2.7a)$$

as the finite-difference approximation for the econd derivative of w at the point " e ", on the x axis. In the same ay it may be shown that, for the second derivative with respect to y, he approximation is

$$h^2\left(\frac{\partial^2 w}{\partial y^2}\right)_E = w_b + w_j - 2w_e \qquad (2.7b)$$

Adding (2.7a) and (2.7b) together, we obtain the finite-difference approximation for $\nabla^2 w$ at a typical node (node e).

$$h^2(\nabla^2 w)_e = w_f + w_b + w_d + w_j - 4w_e$$

or $$w_f + w_b + w_d + w_j - 4w_e - h^2(\nabla^2 w)_e = 0 \qquad (2.8)$$

There is again an error of $O(h^4)$ in this equation, and the effect of this error again diminishes with the mesh size h. Equation (2.8) must be satisfied by the wanted value of w at every group of five nodes. For any othervallue of w the above equation won't be satisfied and thus a residual (R) will resuat. Putting Eq. (2.8) in the form below,

$$w_j + w_f + w_b + w_d - 4w_e - h^2(\nabla^2 w)_e = R_e \qquad (2.9)$$

we should try to reduce each R_e to a value which will be zero or very close to it by the application of increments w.

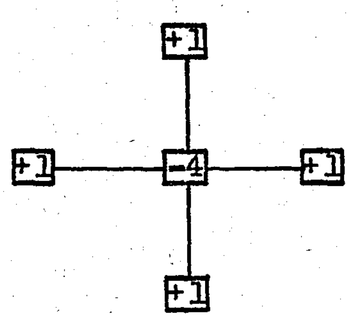


Fig. 2.3.

Examining Eq. (2.9), if w_j is altered by +1, R_e is altered by +1, and similarly for unit alterations to w_f, w_b, w_d. If, wowever, w_e is altered by +1, then R_e is altered by -4. The alterations of the residuals thus follows a definite pattern, and the residuals can be systematically reduced. The amount of work in calculating and changing the residuals can be reduced to some extent by the use of the so called " Relaxation Operator." The relaxation operator is a diagramatical form of the finite-difference equation. The relaxation operator of the present problem is that shown in (Fig. 2.3). Remembering the relaxation operator, the residuals at each point can be computed easily. At the same time, the changing of the residuals is made easier because the operator indicates that a change of the value w at the center point by +1 changes the residual at the center point by -4 and the residuals at each of the four surrounding points by +1.

If we were to put Eq. (2.1) into finite-difference form, the required form would be,

$$20w_e-8(w_b+w_d+w_f+w_j)+2(w_o+w_m+w_r+w_p)+(w_s+w_r+w_t+w_u)=h^4q/D \qquad (2.10)$$

Below is the (Fig. 2.4) that shows the lettered nodes and (Fig. 2.5) is the relaxation operator of Eq. (2.10). It is obvious that Eq. (2.10) is a much more complicated form to deal with than the equivalent finite difference forms of Eqs. (2.5a) and (2.5b).

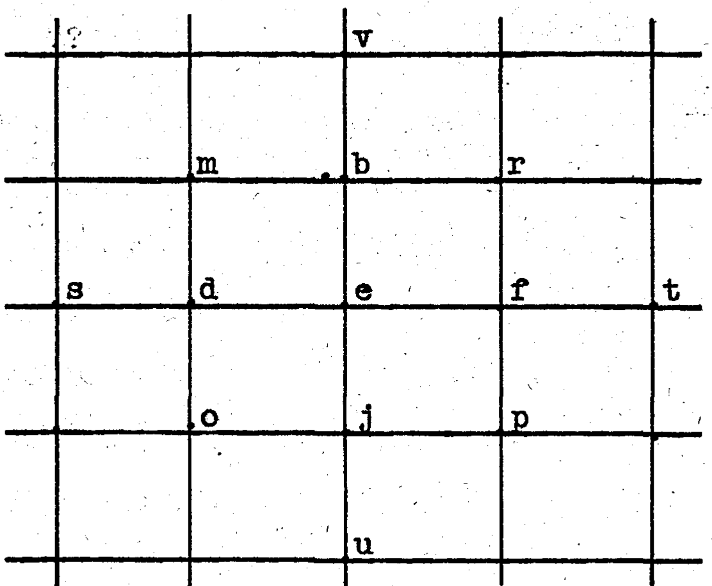


Fig. 2.4.

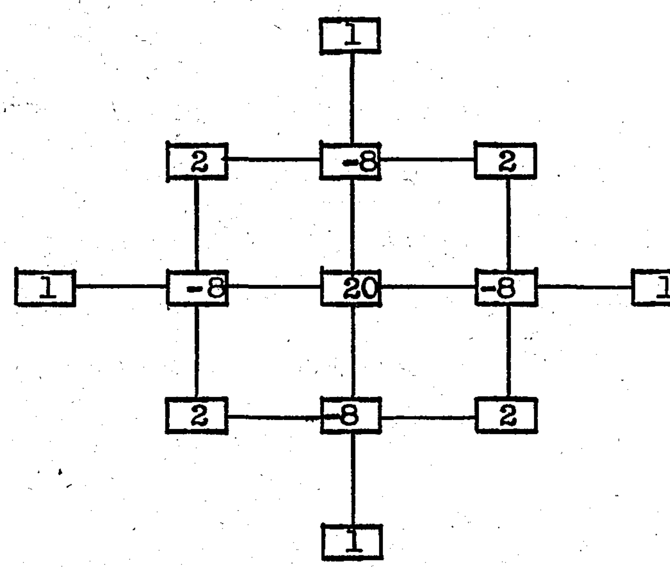


Fig. 2.5.

To get the same result as Eq. (2.10) we have to relax Eq. (2.5a) by puting the q-values at the nodal points and as a result the sum of moments are obtained at the same points. Then Eq. (2.5b) is relaxed by putting the sum of moments as loads at the nodal points until the residuals are minimized. Henceforth the relaxation of Eq. (2.5a) shall be called the <u>M-relaxation</u> and the relaxation of Eq. (2.5b) the <u>w-relaxation</u> which will give the value of the deflection rather w(D) at each nodal point.

CHAPTER 3

THE ANALYSIS OF A SQUARE SLAB WITH A COLUMN AT ITS CENTER

The main purpose of this thesis is to find the moments in a three span slab without drop panels, simply supported at the boundaries and support on four columns at the interior equidistant from each other, and compare the moments with those of the ACI code.

As an introduction a problem of this sort shall be solved. A slab wil be taken with a column supporting the slab in the middle. To find the column reaction and its effects, the slab without a column is analyzed and the deflections in the middle are found due to the loads. This deflection shall be called δq. Then a unit load will be placed at the middle where the column is and the deflections δc will be found. The column reaction will be,

$$K=-\frac{\delta q}{\delta c} \qquad (3.1)$$

The deflection at any node is equal to the deflection of the load+(algeb raically) the deflection due to the reaction at that point. A square slab of 4m. by 4m. will be taken loaded with a uniformly distributed loa of say 1000 kg./m^2. The slab shall be sliced into four equidistant stri with mesh points 1m. from each other. The load per mesh point will be 1000 kg./m^2.(1m.) (1m.)=1000 kg and it shall be placed at each mesh point for relaxation.

In this example the Point Relaxation method is applied, that is to say a unit change in the variable w, will change the residual at each point under consideration by a -4 and the surrounding points by +1 according to the relaxation operator (Fig. 2.3). The result of this relaxation gives the sum of moments at each nodal point according to equation (2.5 By taking these as loads at each nodal point and using point relaxation the deflections of each point are obtained. The moments and shears at each point can be found whan the deflections values at the same points are known.

3.1. Relaxation Method. Now let us start the relaxation process at the center point and reduce the residual there to zero. Thus we have to add +1000/4 or +250 at that point. With this alteration of the variable w, residuals at each of the surrounding four points are increased by an amount of +250. Record the final values of the residuals at the right corner of each nodal point and the change in w, the variable at the left upper corner. This is shown as step 2 in (Fig. 3.2). The nodal points under consideration shall be numbered as shown in (Fig. 3.1).

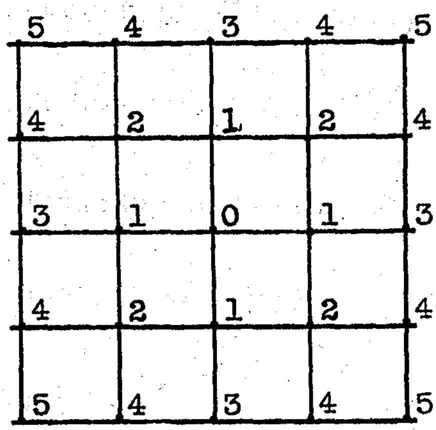


Fig. 3.1.

The method of numbering adopted is used througout on books of relaxation methods.

The residual at point 1 which has become 1250 will be reduced. To reduce it to a value near zero, we shall add a value of +1250/4 or about 300 at this point. Thus the residual at 1 becomes +50, and the residuals at its surrounding points are shown as step 3 in (Fig. 3.2). Point 3 is a boundary point, and the value of the function at this point is determined by the boundary conditions. As the boundary condition is satisfied, the residual there is zero. It is observed that by reducing the residual at 1, the residual at point 0 is again increased. To reduc this additional residual we must increase w_0 again. This always happens whan a point is surrounded by other points with residuals of the same sign. To make the convergence more rapid, instead of reducing R_0 in ste 2 to zero we may increase w_0 in step 2 so that R_0 becomes a negative value. This process is called over-relaxation. The amount of over-rel ation depends on the magnitudes of residuals at the surrounding points. By not oversooting enough or by overshooting too much no harm is done except that some time is lost. Now both points 2 at the right side of the vertical center line have residuals of 1300. To reduce them, let u over-relax the residuals by adding 600 at both points. The changes in

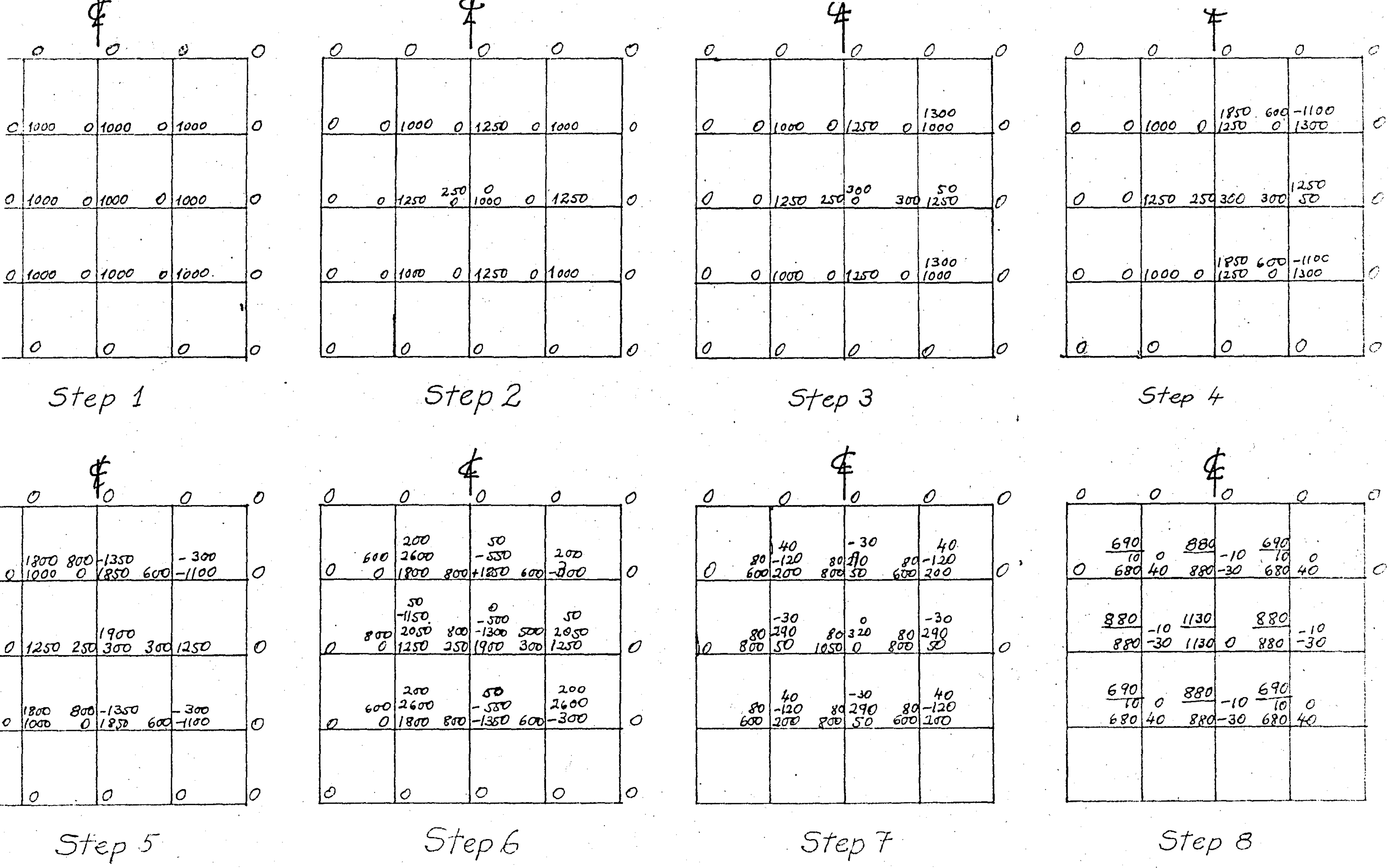


Fig. 3.2. M-relaxation of uniform load on square-slab.

the residuals and the values of w are shown as step 4. Next we add 800 to point above and below 0, respectivelly. The results are shown as step 5. Now, 800 shall be added at point 0 and 1; 600 at points 2 of the left side of the vertical center line and 500 at point 1. The final residuals are shown as step 6. The residuals have now been reduced to a maximum value of 200. If we add 80 at all the inner points, the residuals shall be considerably reduced. The final residuals are shown as step 7. Some further changes as in step 8 reduces the residuals to a minimum.

*3.2. Block Relaxation and Lines of Symmetry. The procedure of relaxation may be altered by the computer for a rapid approach to the final answer of no residuals. We shall now discuss a few short cuts in the relaxation technique which will serve to accelerate the elimination of the residuals.

*C. T. Wang, Applied Elasticity. (New York: Mc. Graw Hill Book Co. Inc., 1953), p. 119.

One of these relaxation techniques is the so-called line and block relaxation. In step 7 of the example considered in Sec. 3.1, we found that the residuals could be reduced by adding 80 to the values of w, at all the net points. Altering simultaneously all the values of the function by the same amount at a group of points in a block of the domain is called block relaxation. Similarly, simultaneously altering the values of the function by the same amount at a group of points along a line in the domain is called line relaxation.

Consider the effect of the simultaneous changing of the function of two adjacent points by the same amount. Obviously this may be carried out by writing down seperately the effects of each displacement and adding them together. In the case of the Laplace operator, by the use of unit operators, we can obtain the two-, three-, and four-point line-relaxation operators, as shown in (Fig. 3.3), and the various block-relaxation operato

*See item No: 5 in bibliography.

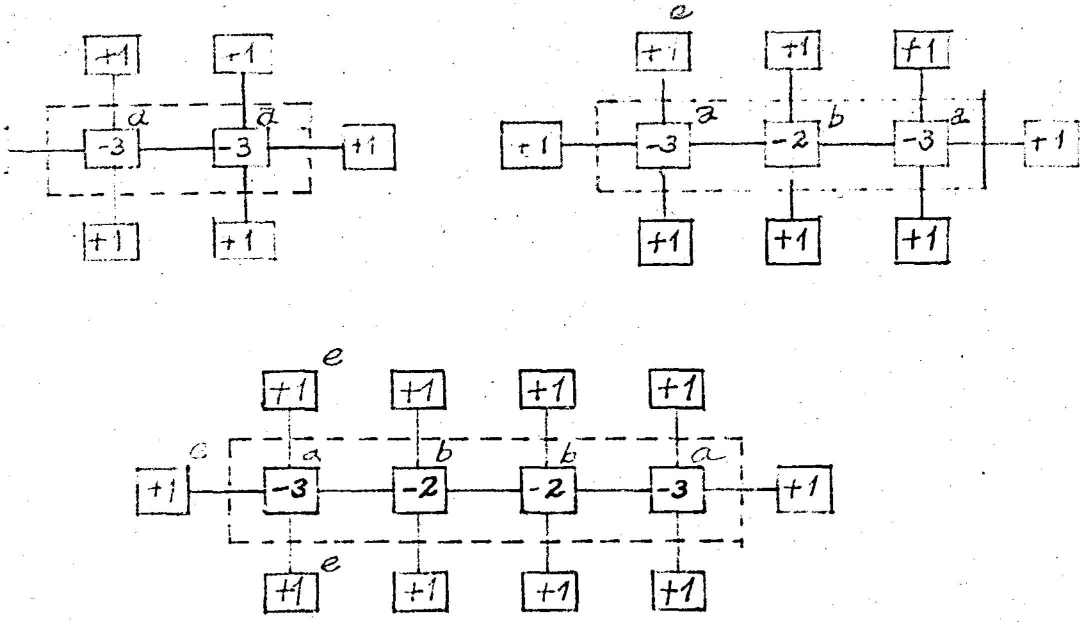


Fig. 3.3. Two-, Three-, and Four-Point Line-Relaxation Operators.

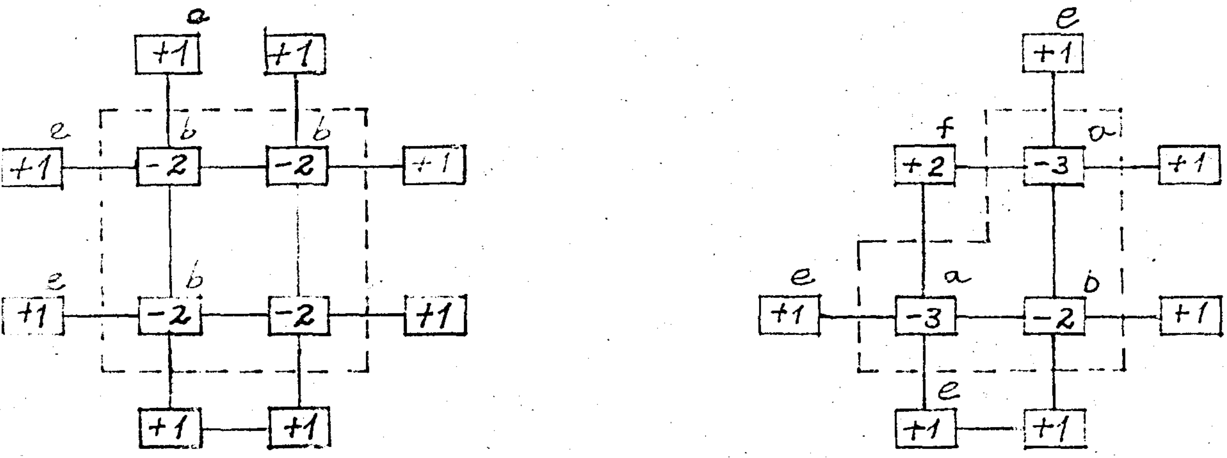


Fig. 3.4. Block Relaxation Operators.

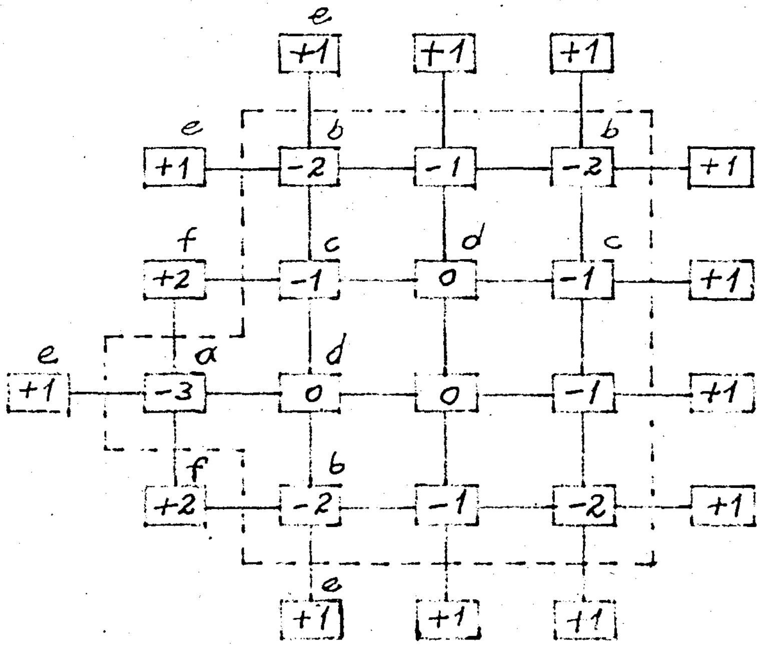


Fig. 3.5. Block Relaxation Operator.

as shown in (Figs. 3.4 and 3.5). Inspecting the operators as shown in (Figs. 3.3, 3.4, and 3.5) carefully, we find that it is possible to obtain a rule by which all such operators can be immediately written down by inspection. The rule for writing the line- and block-relaxation operators for Laplace or Poisson's equations is as follows. By simultaneously altering the values of the function at a group of points in a region along a line or within a block by an amount of +1, the residuals at all points which, like a (Figs. 3.3. and 3.4), are directly connected with three points outside the line- or block-relaxation region are altered by an amount of -3. The residuals at all points which, like b, are connected with two outside points, are altered by an amount of -2. The residual at a point such as c is altered by an amount of -1 when the point is connected with one outside point. There are no changes in the residuals at points such as d which are not directly connected with any outside points. The residuals at all points e, which are outside the line- or block-relaxation region but directly connected to one point within the region, are altered by +1. The residual at an outside point such as f which is directly connected to two outside points is altered by +2. The advantage of the line and block relaxation can easily be seen from (Fig. 3.5). For while the residuals at the points on the boundary of the block are altered, the residuals at points inside the block are not changed. Judicious use of block relaxation can prevent much of the "washing back" of residuals, thus saving much time in obtaining a solution.

Another useful relaxation technique is the observation of the lines of symmetry. In many problems the solution can easily seen to be symmetrical with respect to one or more lines because of the symmetry of the domain and boundary conditions. In solving such problems, it is unnecessary to find the unknown function over the entire area. In the example of Sec. 3.1, there is an eightfold symmetry. Thus it is sufficen to find a solution in one-eight of the domain, as shown in (Fig. 3.6).

There is no markedly new technique involved in solving such problems. It is merely necessary to remember that to preserve symmetry each time a point adjasent to a line of symmetry is altered the point which is symmetrical to this point is altered at the same time. That is, such an operation is accompanied by an automatically equal change on the other side of the line of symmetry. As a result, a point on the line of symmetry will receive a change in its residual from both of the points being altered.

In the problem of Sec. 3.1, in (Fig. 3.2), the computation was deliberately lengthened by ignoring the symmetrical property of the solution . During the relaxation process, the computations should be checked from time to time by evaluating the residuals at all the nodal points. If there are mistakes it is advisable to correct the residuals at the nodal points by using Eq. (2.9). As far as the thesis is concerned,the M- relaxation is accomplished. The w-relaxation can be done by using line relaxation. The numbering of the nodal points will be as in (Fig. 3.1), the center point is 0, the side points as 1, and the corner points as 2. The corresponding change of a variable of +1 put at any one of the three nodal points will be illustrated with the following three diagrams of relaxation operators. The numbering of the points will be written on the lower left corners of each point througout this paper.

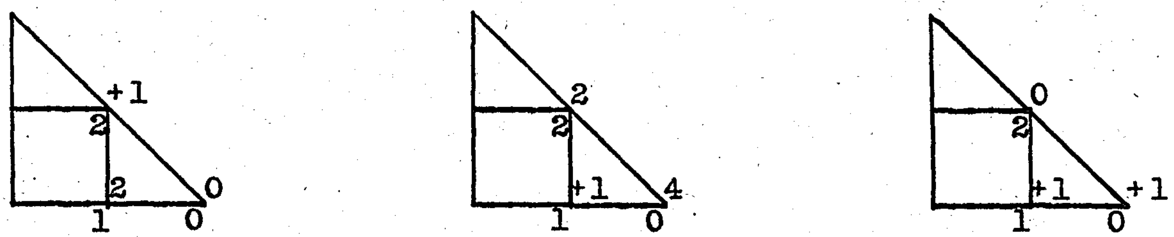


Fig. 3.6.

The steps are not written one by one but the relaxation is carried out on page 18. The 1035 middle deflection can be compared with the result of a similar slab solved by elasticity such that

$$(D)\ w_{max.} = 0.00406qa^4 = 0.00406(1000)(4^4) = 1040$$

The exact and approximate results are in good proximity.

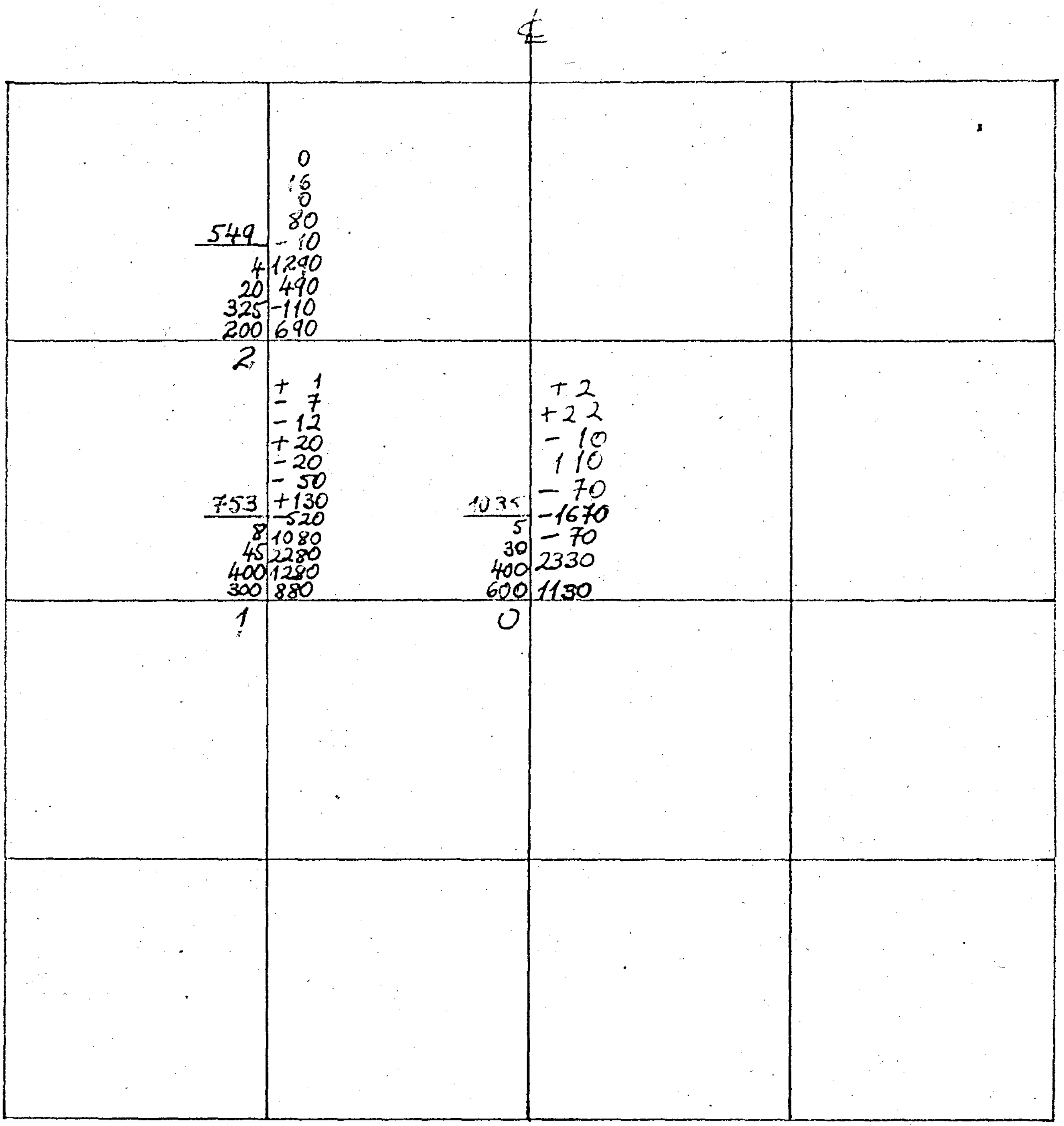


Fig. 3.7. w - relaxation of uniformly distributed load on square-slab simply supported at the boundaries.

The above found values at the nodal points are the deflections of each point due to the uniformly distributed loa[d] To find the column reaction a load of 1000 kg. shall be p[laced] at point 0 for relaxation. The deflections introduced b[y] the column load at 0 will have the opposite sign to [the] deflections of the uniformly distributed load.

Next, the load on the column, supporting the slab from the middle, shall be found.

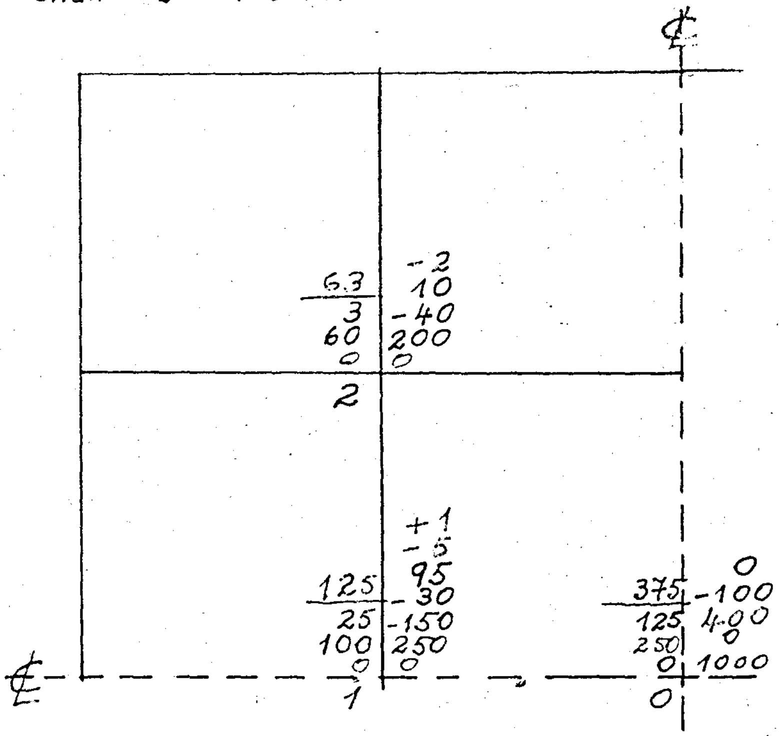


Fig. 3.8. M-relaxation of column load at O.

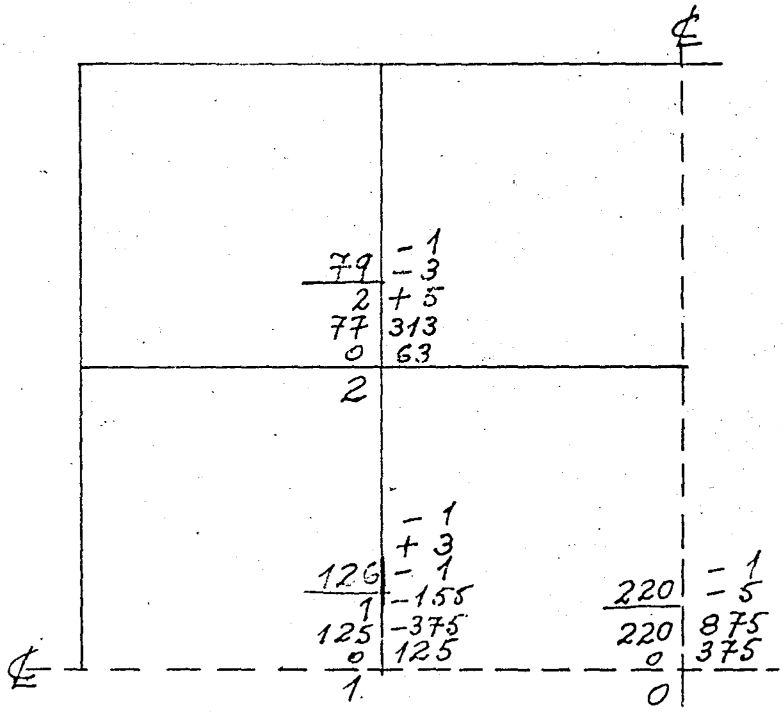


Fig. 3.9. w-relaxation of column load at O.

$$K \cdot 220 = 1035$$
$$K = 4.71$$

Therefore the column reaction is 4710 kg. The final deflection with a column reaction acting are as follows.

$$\omega_2 = 549 - 4.71 \times 79 = 549 - 372 = 177 \text{ cm.}$$

$$\omega_1 = 753 - 4.71 \times 126 = 753 - 594 = 159 \quad ''$$

$$\omega_0 = 1035 - 4.71 \times 220 = 1035 - 1035 = 0 \quad ''$$

The results of the examples solved in Secs. 3.1 and 3.2can be increased in accuracy if the slab is divided into eight meshes. The same process of relaxations shall be done again. The load per mesh point will be $1000 kg./m^2.(0.5m.)(0.5m.)=250kg.$ and it shall be placed at each mesh or nodal point for relaxation. The relaxation operators for the 8 mesh model are shown on page 21 in (Fig. 3.10). The relaxation of the uniform load is performed in two steps. First block relaxation shall be used and the approximate residuals will be carried further by point relaxation. The block relaxation portion of the job is shown in (Fig. 3.11) and the further point relaxation in (Fig. 3.12). Note that block relaxation is carried from the outskirts of the slab to the center in (Fig. 3.11). The 438 kg. of step 1 is found as:

$$\frac{1000\ kg./m^2.(0.5m.)(0.5m.)(\text{No. of mesh points})}{\text{No. of strings}} = \frac{250kg.(49)}{28} = 438kg.$$

or

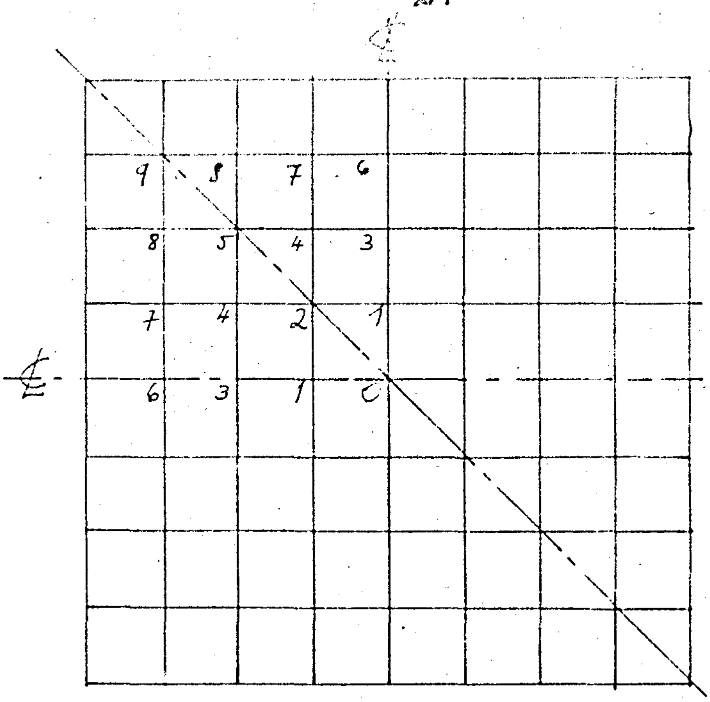
$$\frac{\text{load per node point }(n-1)}{4} \qquad (3.2)$$

where n=no. of division of meshes of a square slab. For any kind of rectangular slab the general formula is:

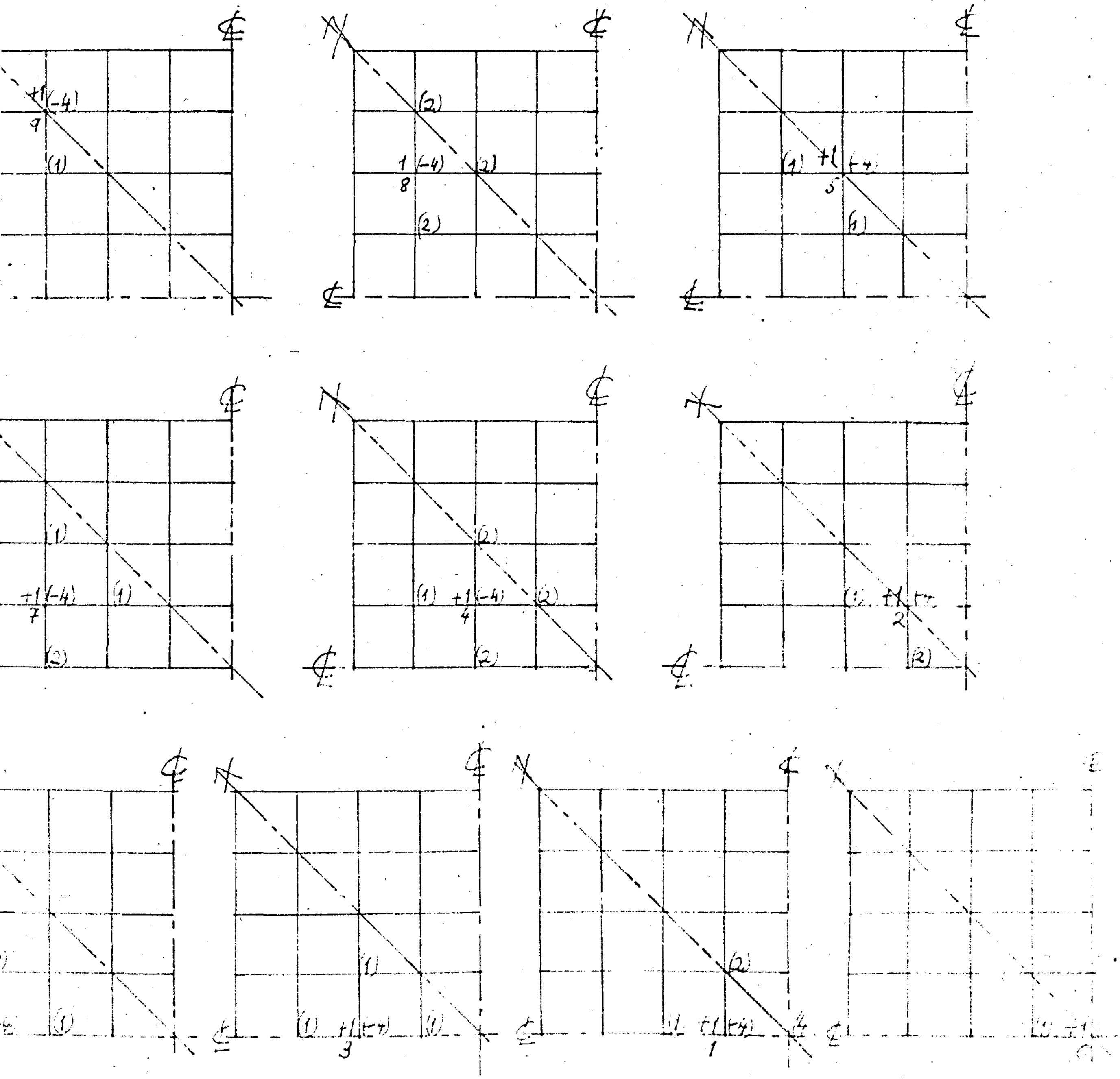
$$\frac{\text{load per node point }(a-1)(b-1)}{2(a+b-2)} \qquad (3.3)$$

where a=no. of mesh divisions of one side of the slab;

b=no. of mesh divisions of the other side of the slab;

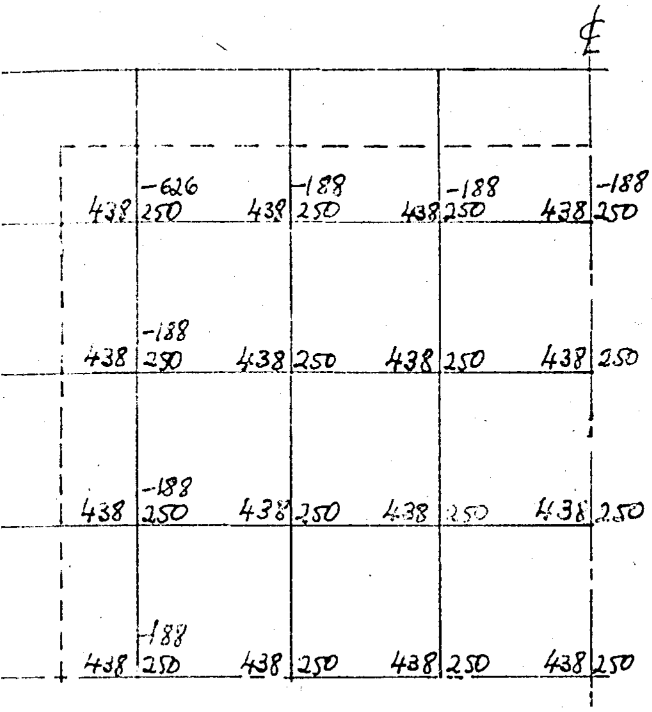


The Relaxation Operators for the 8 mesh model:



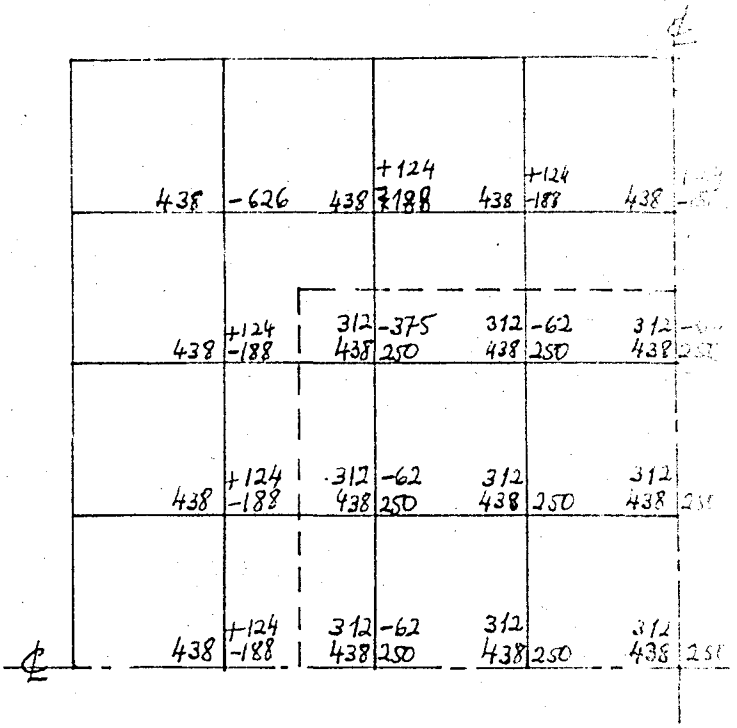
p 1:

$$= \frac{1000 \times \overline{0.5}^2 \times 49}{28} = 438 \text{ kg.}$$



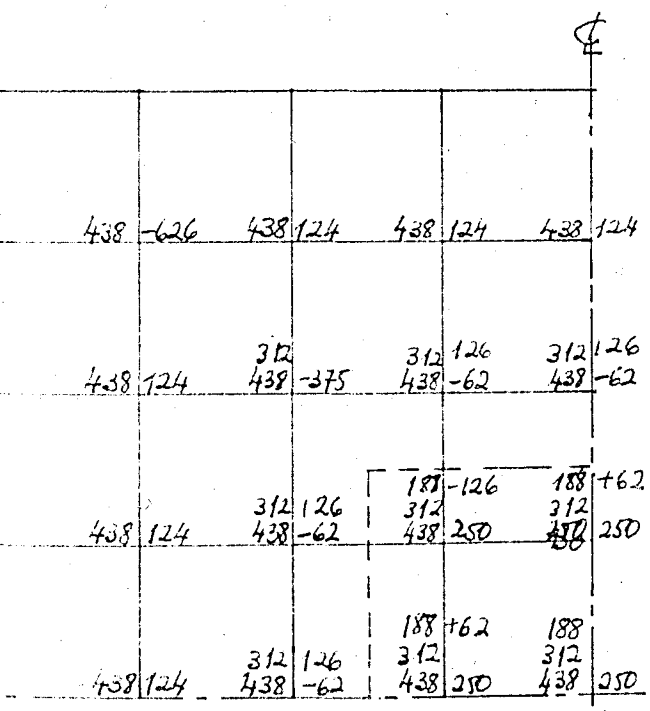
Step 2:

$$= \frac{250 \times 25}{20} = 312 \text{ kg.}$$



ep 3:

$$= \frac{250 \times 9}{12} = 188 \text{ kg.}$$



Step 4:

$$= \frac{250 \times 1}{4} = 62 \text{ kg.}$$

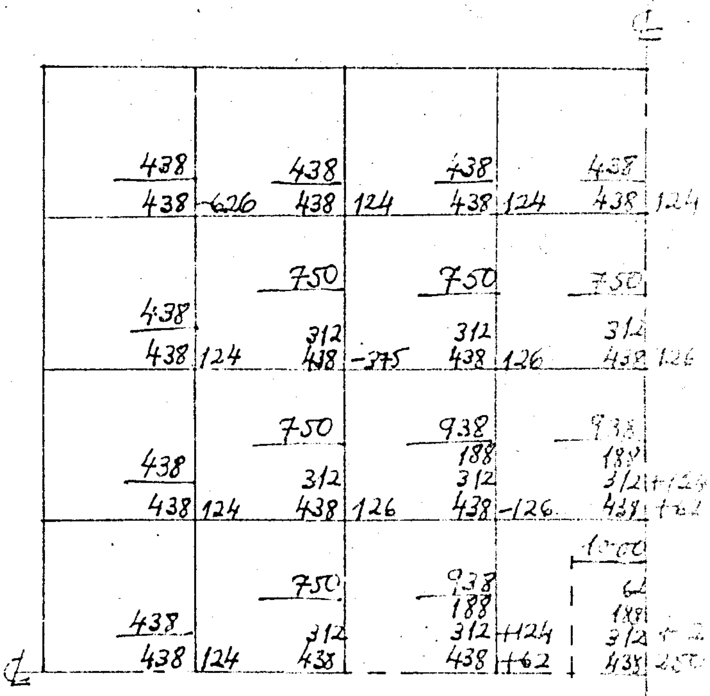


Fig. 3.11. Block relaxation of uniformly distributed load on

8-mesh square-slab simply supported at the

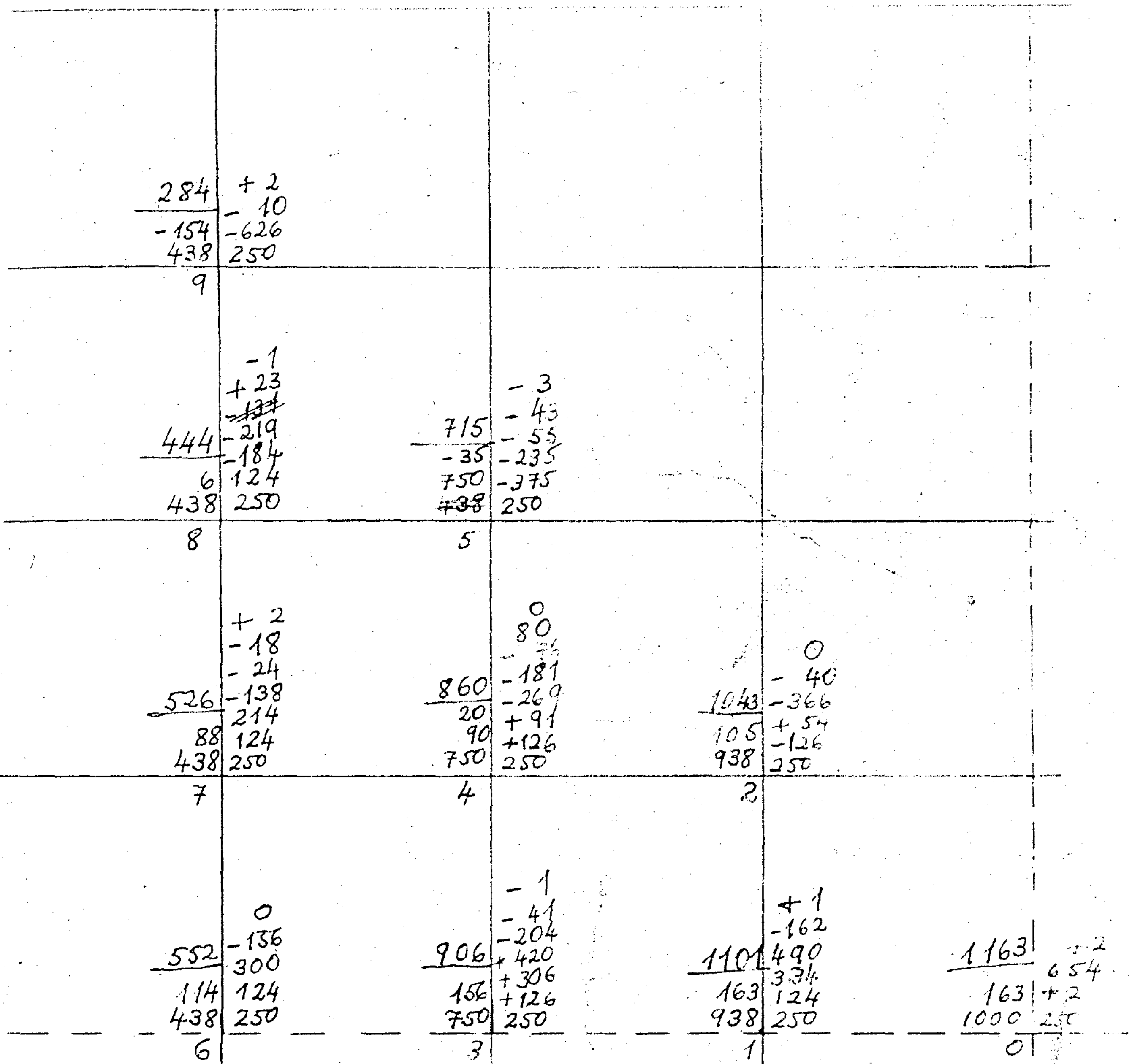


Fig. 3.12. M-relaxation of uniform load on square slab

Above, the residuals and the variables of the Block Relaxation are carried by point relaxation.

The above relaxation results should be multiplied by h^2 so as to find the deflections by the W-relaxation, where h is the space length between mesh lines. It will be cumbersome to do this every time for each slab instead the M_x and M_y moments shall be multiplied by h^2.

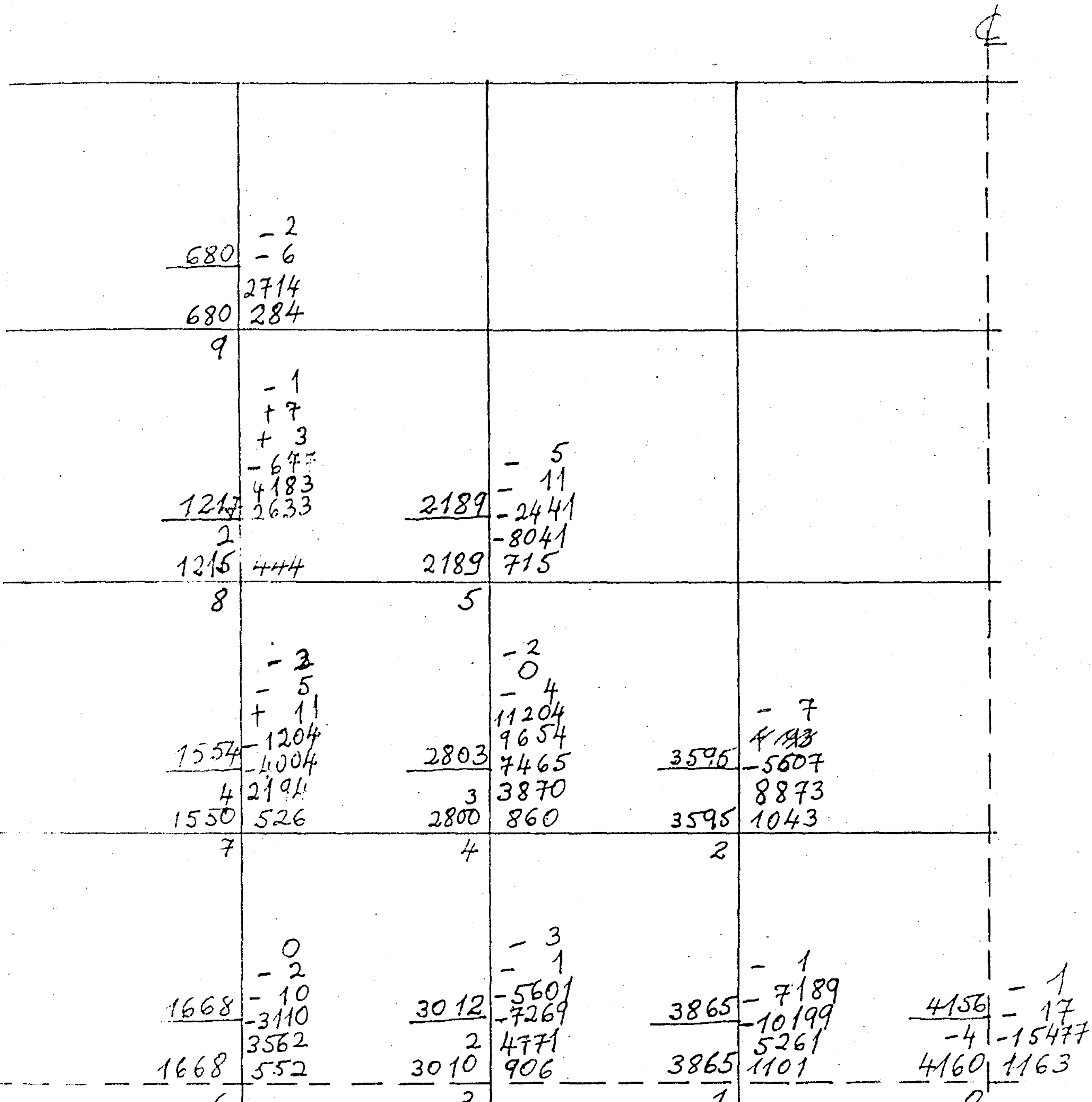


Fig. 3.13. W-relaxation of uniform load on square slab.

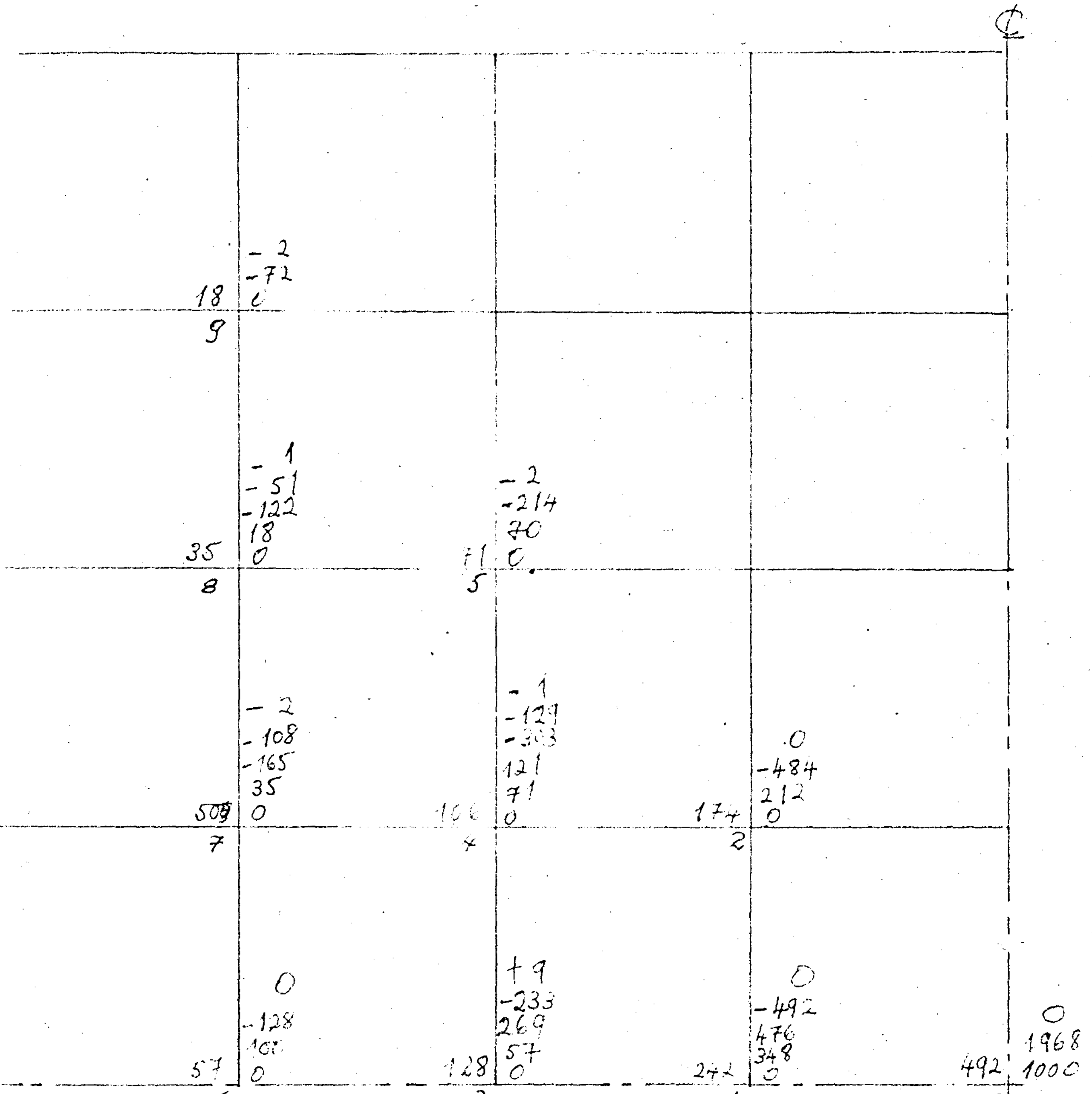


Fig. 3.14. M- relaxation of column load at 0 of square-slab simply supported at the boundaries

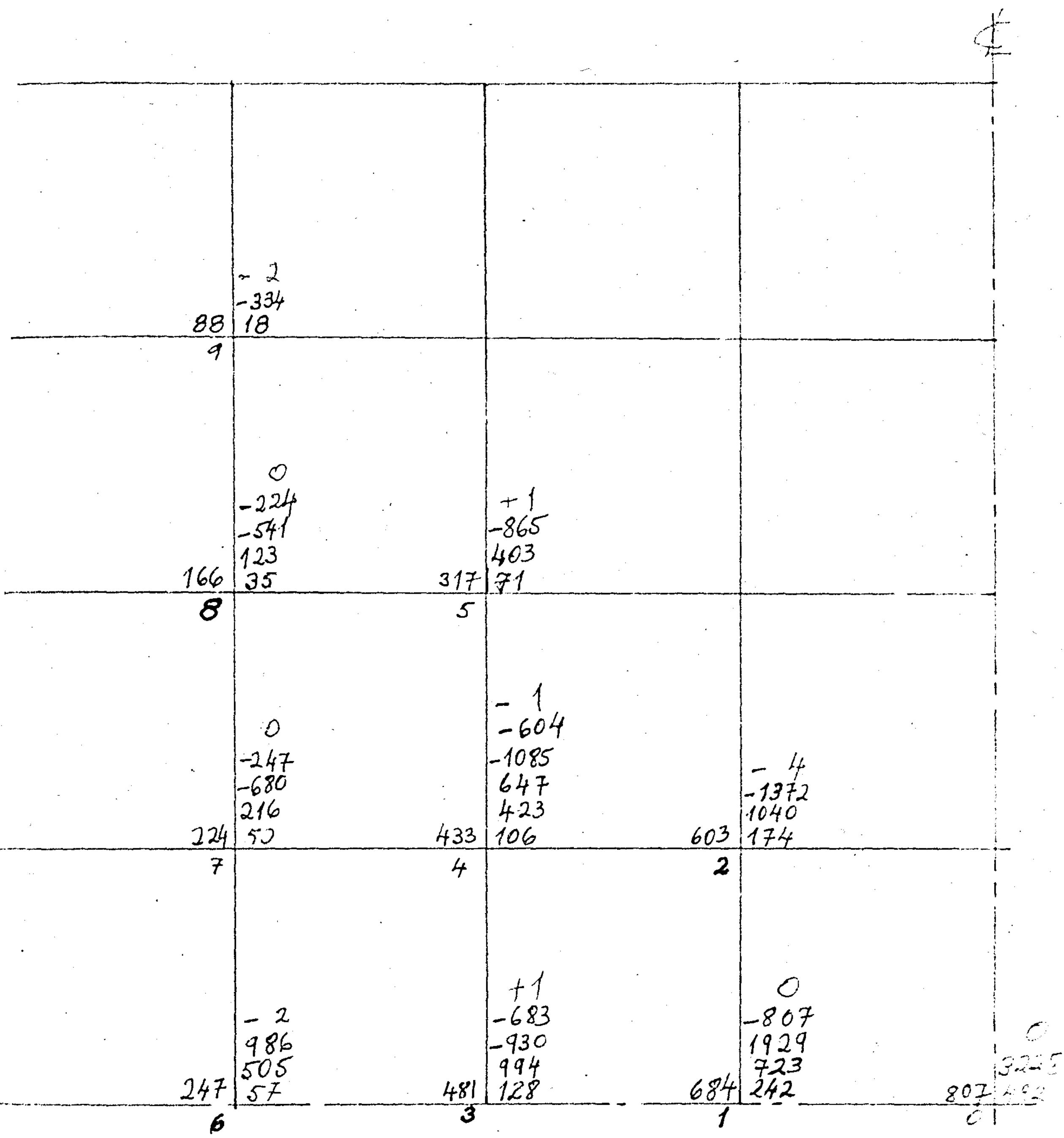


Fig. 3.15. w-relaxation of column load at 0 of square slab simply supported at the boundaries

$$K=\frac{4156}{807}=5.15$$

The column reaction is =5.15(1000)=5150kg.

The deflections of the column load are opposite in sign to the deflections of the uniformly distributed load. The final deflections , i.e. (w)(D)'s with a column reaction acting are assfollows:

w_9= 680-5.15(88)= 680- 453=227 cm.

w_8=1217-5.15(166)=1217- 855=362 cm.

w_7=1554-5.15(224)=1554-1153=401 "

w_6=1668-5.15(247)=1668-1272=396 "

w_5=2189-5.15(317)=2189-1633=556 "

w_4=2803-5.15(433)=2803-2230=573 "

w_3=3012-5.15(481)=3012-2477=535 "

w_2=3595-5.15(603)=3595-3105=490 "

w_1=3865-5.15(684)=3865-3523=342 "

w_0=4156-5.15(807)=4156-4156= 0

The moments M_x and M_y at each nodal point are found by the following formulas:

$$M_x= D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right) \tag{3.4}$$

$$M_y= D\left(\mu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{3.5}$$

The torsional moments are:

$$M_{yx}=M_{xy}=D(1-\mu)\frac{\partial^2 w}{\partial x\partial y} \tag{3.6}$$

where μ= 0.2

The shear Q_x and Q_y are:

$$Q_x=\frac{\partial M_x}{\partial x}-\frac{\partial M_{xy}}{\partial y}=-\frac{\partial}{\partial x}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\right]-\frac{\partial}{\partial y}D(1-\mu)\frac{\partial^2 w}{\partial x\partial y}$$

$$=-D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{3.7}$$

$$Q_y=\frac{\partial M_x}{\partial y}-\frac{\partial M_{xy}}{\partial x}=-\frac{\partial}{\partial y}\left[D\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{x^2}\right)\right]-\frac{\partial}{\partial x}D(1-\mu)\frac{\partial^2 w}{\partial x\partial y}$$

$$=-D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{3.8}$$

The finite difference form of the equations (3.4) and (3.5), where "e" in (Fig. 2.2) will be a typical node with the surrounding nodes b,d,f,j.

$$\frac{M_x}{D}= \frac{(2w_e-w_d-w_f)}{h^2}+\mu\frac{(2w_e-w_b-w_j)}{h^2} \qquad (3.9)$$

$$\frac{M_y}{D}= \mu\frac{(2w_e-w_d-w_f)}{h^2}+ \frac{(2w_e-w_b-w_j)}{h^2} \qquad (3.10)$$

as all the w's are actually w(D), the modified M_x and M_y moments after multiplying by h^2 as noted on p. 23, will be:

$$M_x= (2w_e-w_d-w_f)+\mu(2w_e-w_b-w_j)$$

$$M_y=\mu(2w_e-w_d-w_f)+ (2w_e-w_b-w_j)$$

Now the moments of the 8 mesh model will be found.

$M_{x9}=2(227)-362 \quad +0.2\left[2(227)-362\right]= 110$ kg-m/m.

$M_{x8}=2(362)-556 \quad +0.2\left[2(362)-227-401\right]= 187$ "

$M_{x7}=2(401)-573 \quad +0.2\left[2(401)-396-362\right]= 238$ "

$M_{x6}=2(396)-535 \quad +0.2\left[2(396)-2(401)\right]= 255$ "

$M_{x5}=2(556)-573-362+0.2\left[2(556)-573-362\right]= 212$ "

$M_{x4}=2(573)-401-490+0.2\left[2(573)-535-556\right]=266$ "

$M_{x3}=2(535)-342-396+0.2\left[2(535)-2(573)\right]= 317$ "

$M_{x2}=2(490)-573-342+0.2\left[2(490)-573-342\right]= 78$ "

$M_{x1}=2(342)-535-0 \quad +0.2\left[2(342)-2(490)\right]= 90$ "

$M_{xo}=2(0)-2(342) \quad +0.2\left[2(0)-2(342)\right]=-821$ "

$M_{y9}=$ ----------- $= 110$ kg-m/m.

$M_{y8}= 96+0.2(168)= 130$ "

$M_{y7}= 44+0.2(229)= 90$ "

$M_{y6}= -10+0.2(257)= 41$ "

$M_{y5}=$ ----------- $= 212$ "

$M_{y4}= 55+0.2(255)= 106$ "

$M_{y3}= -76+0.2(332)=- 12$ "

$M_{y2}=$ ----------- $= 72$ "

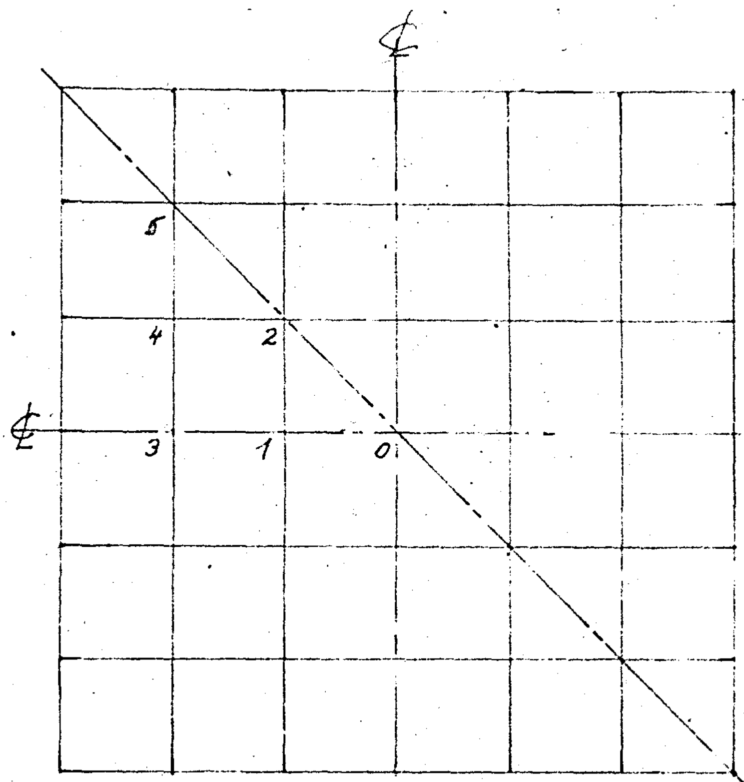
$M_{y1}=-296+0.2(149)=-266$ "

$M_{yo}=$ ----------- $=-821$ "

CHAPTER 4

THE ANALYSIS OF A SIMPLY SUPPORTED SLAB WITH FOUR SYMMETRICALLY PLACED COLUMNS

In the first part of the thesis a slab was solved simply supported at the boundaries and on a column at the middle. For this slab the deflections and moments were found only for the eight mesh model and the deflections for the four mesh one. The aim of the thesis is to collect interest on a slab simply supported at the boundaries and supported internally on four columns equidistant from each other and the boundaries. The slab to be investigated has dimensions of 4m. by 4m.. The relaxation procedure will be similar to the previous ones. The uniform loading will be 1000 kg./m^2,(4/6m.)(4/6m.)=445kg. for the M-relaxation of the uniform load. The relaxation operators for the 6 mesh model are shown in (Fig. 3.16) with the lines of symmetry indicated. The operations tables follow.



The Relaxation Operators for the 6 mesh model:

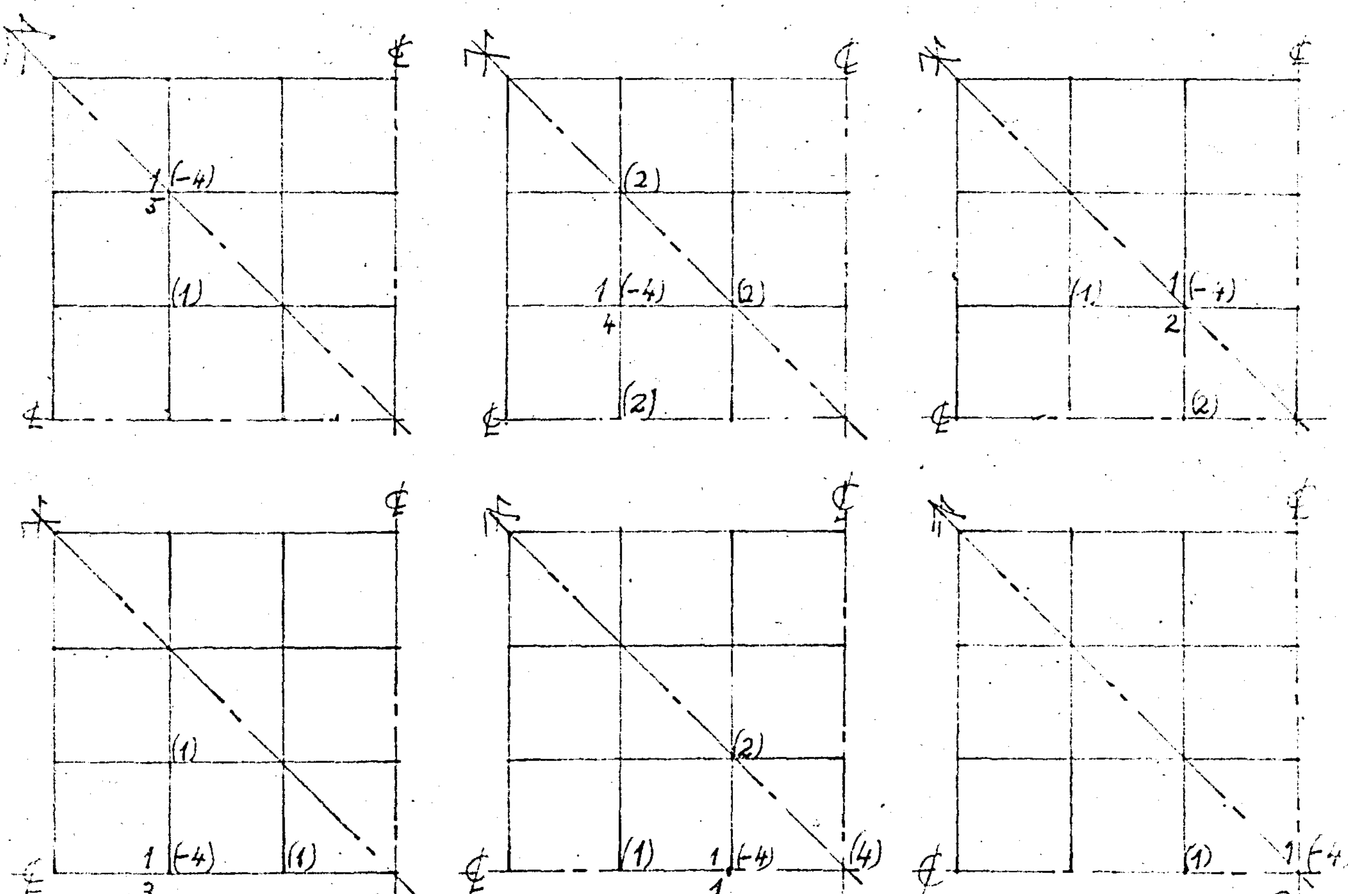


Fig. 4.1.

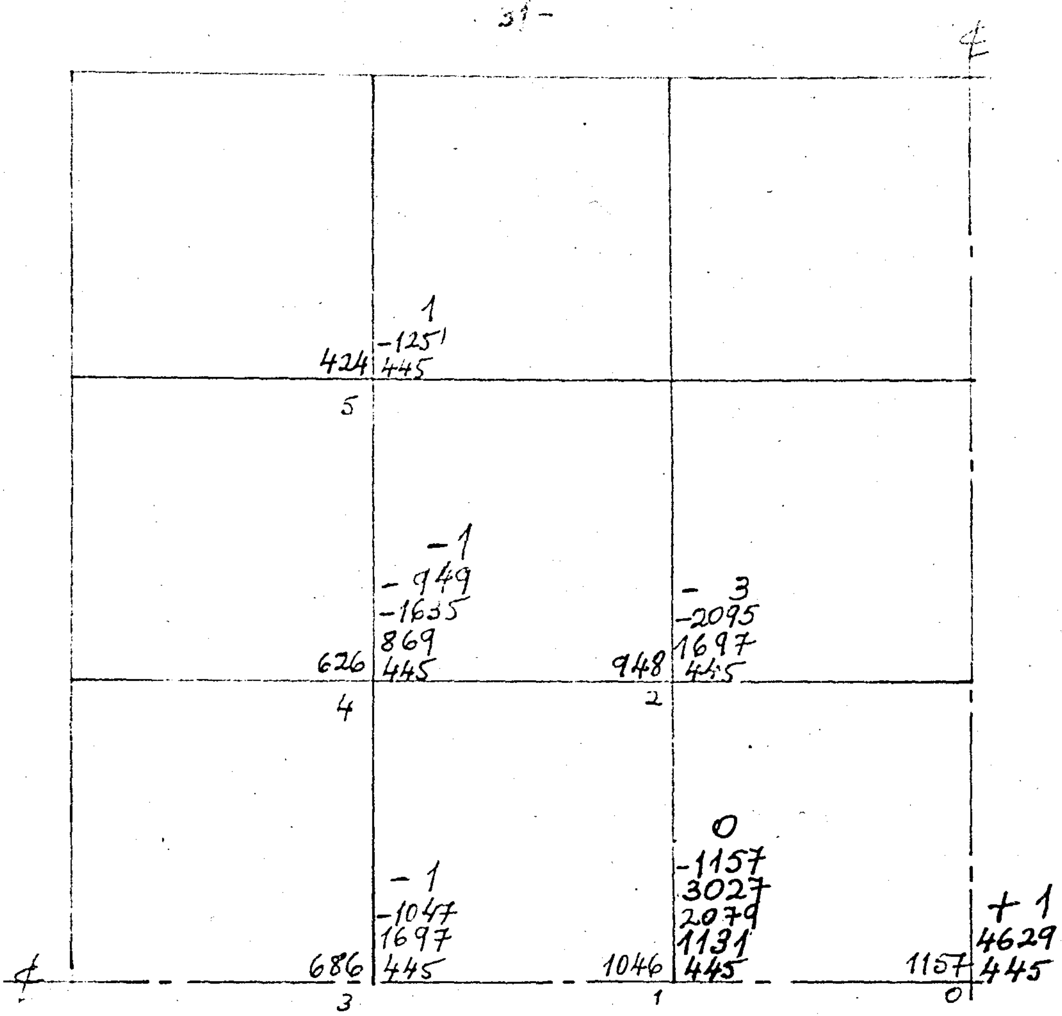


Fig. 4.2. M-relaxation of the uniform load

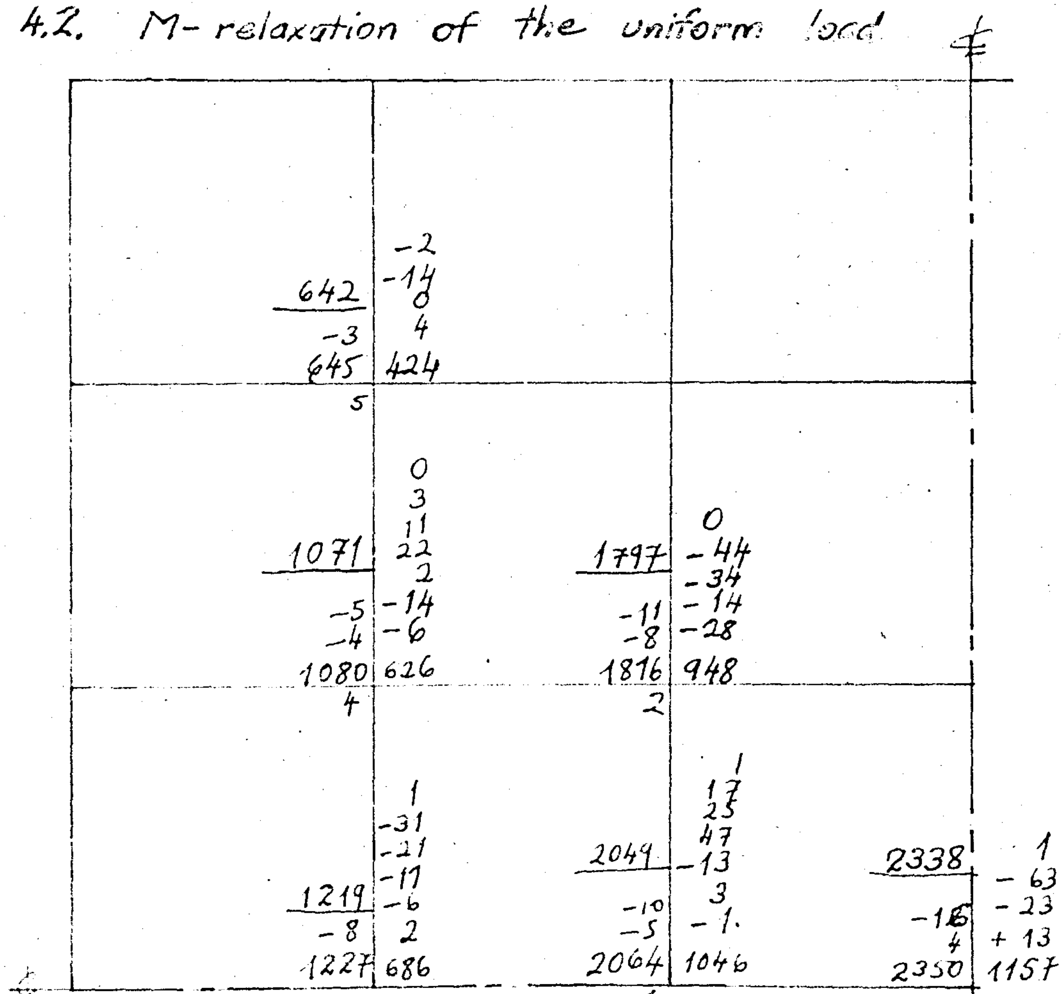


Fig. 4.3. w-relaxation of the uniform load

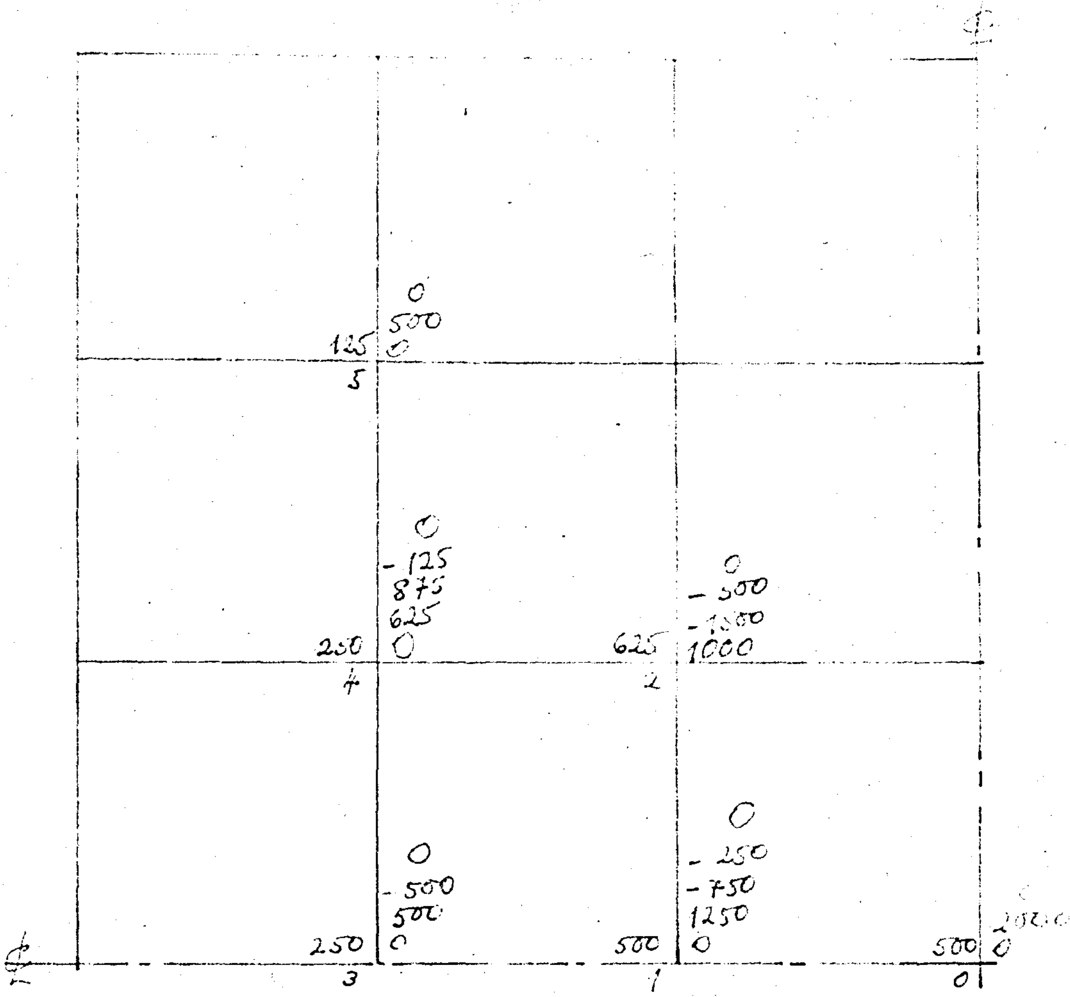


Fig. 4.4. M-relaxation of column load at 2.

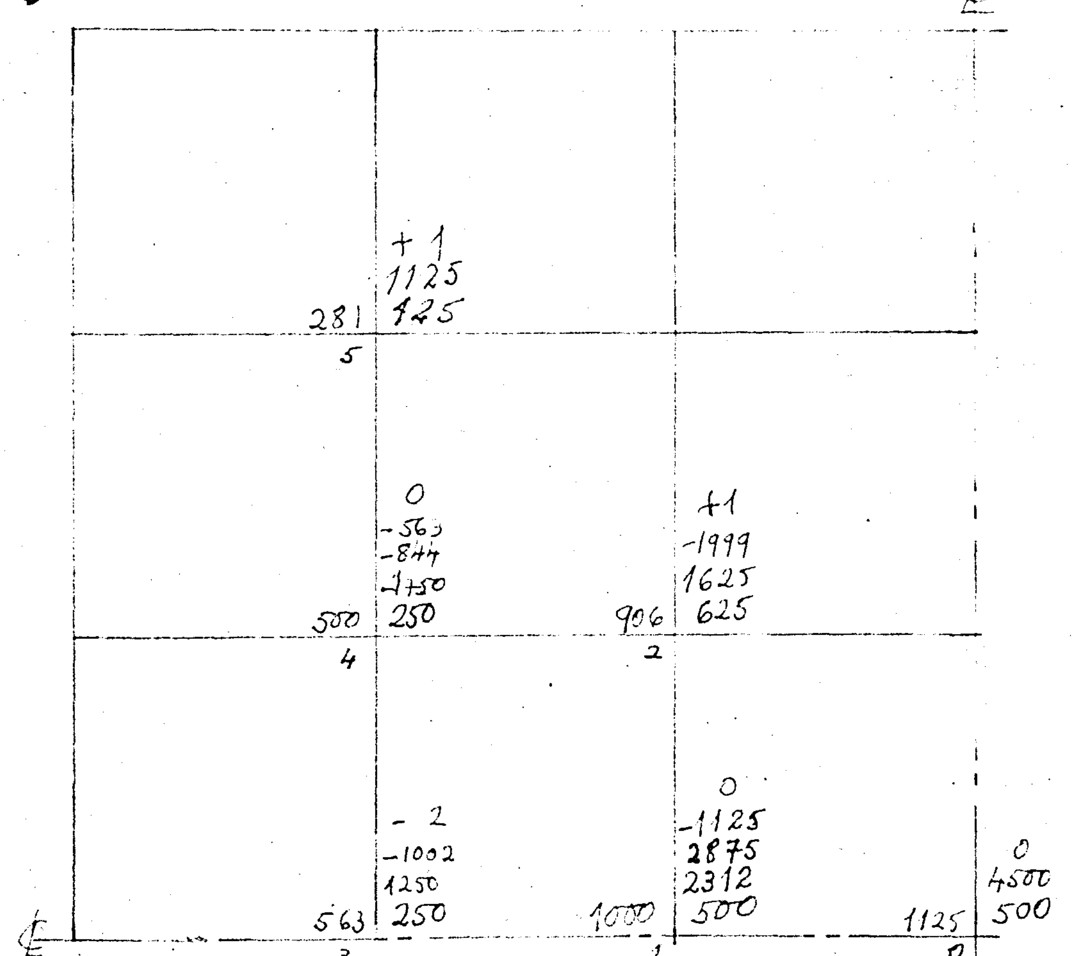


Fig. 4.5. w-relaxation of column load at 3.

It is sufficient to find the load on one column as all four of them are similarly loaded.

$$K=\frac{1797}{906}=1.985$$

The column reaction is = 1.985(1000)=1985 kg.

The final deflections at the mesh points are:

w_5= 6421.9885(281)= 642- 558= 84 cm.

w_4=1071-1.985(500)=1071- 993= 78 ".

w_3=1219-1.985(563)=1219-1118= 101 ".

w_2=1797-1.985(906)=1797-1797= 0

w_1=2049-1.985(1000)=2049-1985= 64 ".

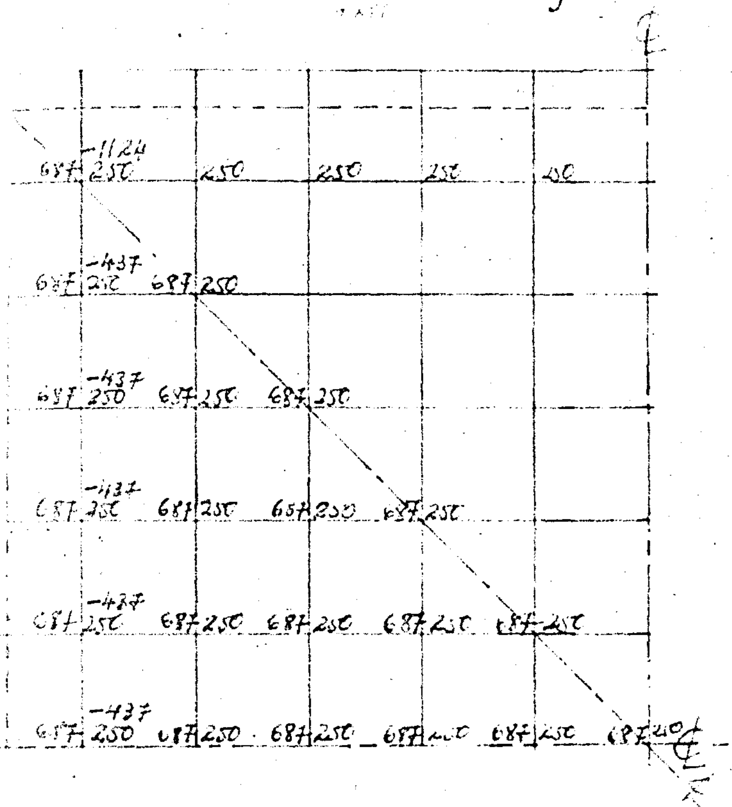
w_0=2338-1.985(1125)=2338-2232= 106 ".

In the previous example, it was not necessary to find the moments since another square slab having dimensions 6m. by 6m. shall be analyzed with the same given conditions but this time the slab shall be divided into 12 meshes so as to minimize the error of the deflections and the moments to be found. To find the moments, strips shall be considered such as the column strip, the middle strip as is done in the analysis of flat slabs in reinforced concrete. This so called strip analysis couldn't be done with the 6 mesh model as there were insufficient nodal points and therefore insufficient deflections. The previous example was done so that it would be a useful guide to the 12 meshed example. In this example the load per nodal point will be 1000 kg./m^2.(0.5m.)(0.5m.)=250 kg. Block relaxation shall be followed by point relaxation. Under each Fig. is written what that Fig. is about so as to facilitate the reading of the thesis.

Step 1:

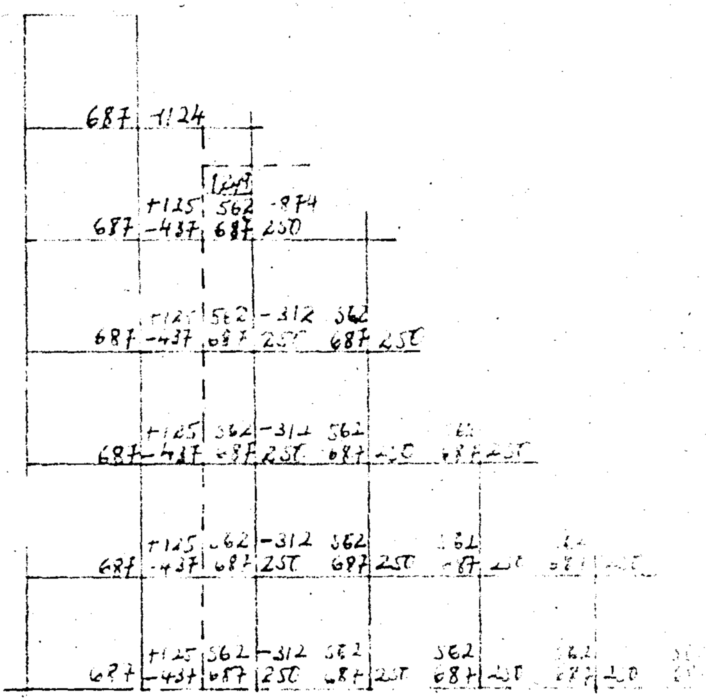
$= \frac{1000 \times 0.5 \ldots}{\ldots}$

$= \frac{250 \times 11 \times 11}{4 \times 11} = 687$ kg.



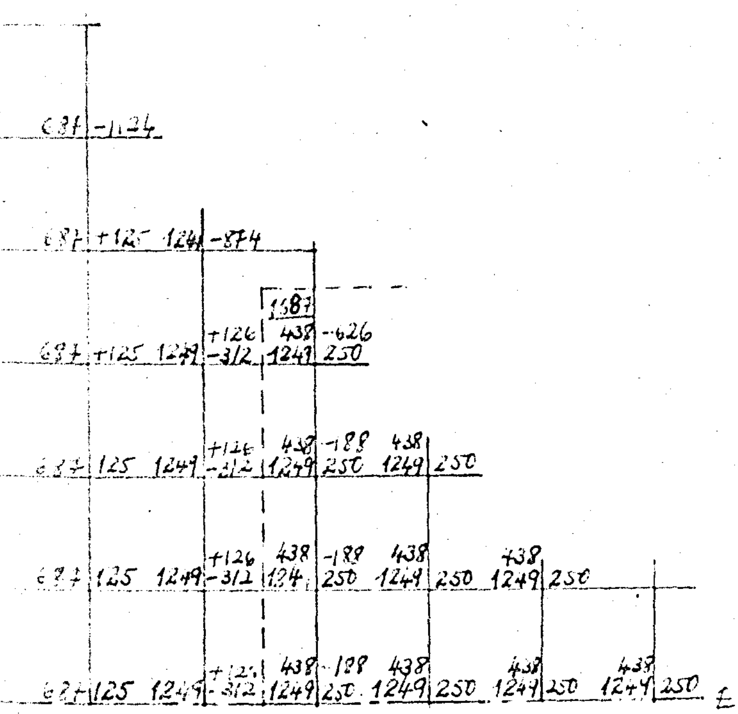
Step 2:

$= \frac{250 \times 9 \times 9}{4 \times 9} = 562$ kg.



Step 3:

$= \frac{250 \times 7 \times 7}{4 \times 7} = 438$ kg.



Step 4:

$= \frac{250 \times 5 \times 5}{4 \times 5} = 312$ kg.

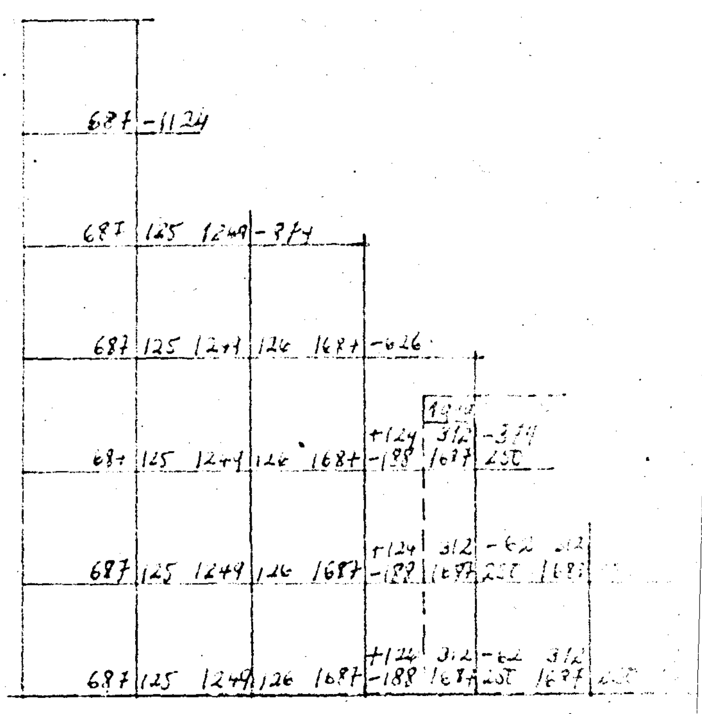


Fig. 4.6. Block relaxation of the uniform load on the square-slab.

Step 5:

$$= \frac{250 \times 3 \times 2}{4 \times 2} = 188 \text{ kg}.$$

Step 6:

$$= \frac{250 \times 1 \times 1}{1 \times 4} = 63 \text{ kg}.$$

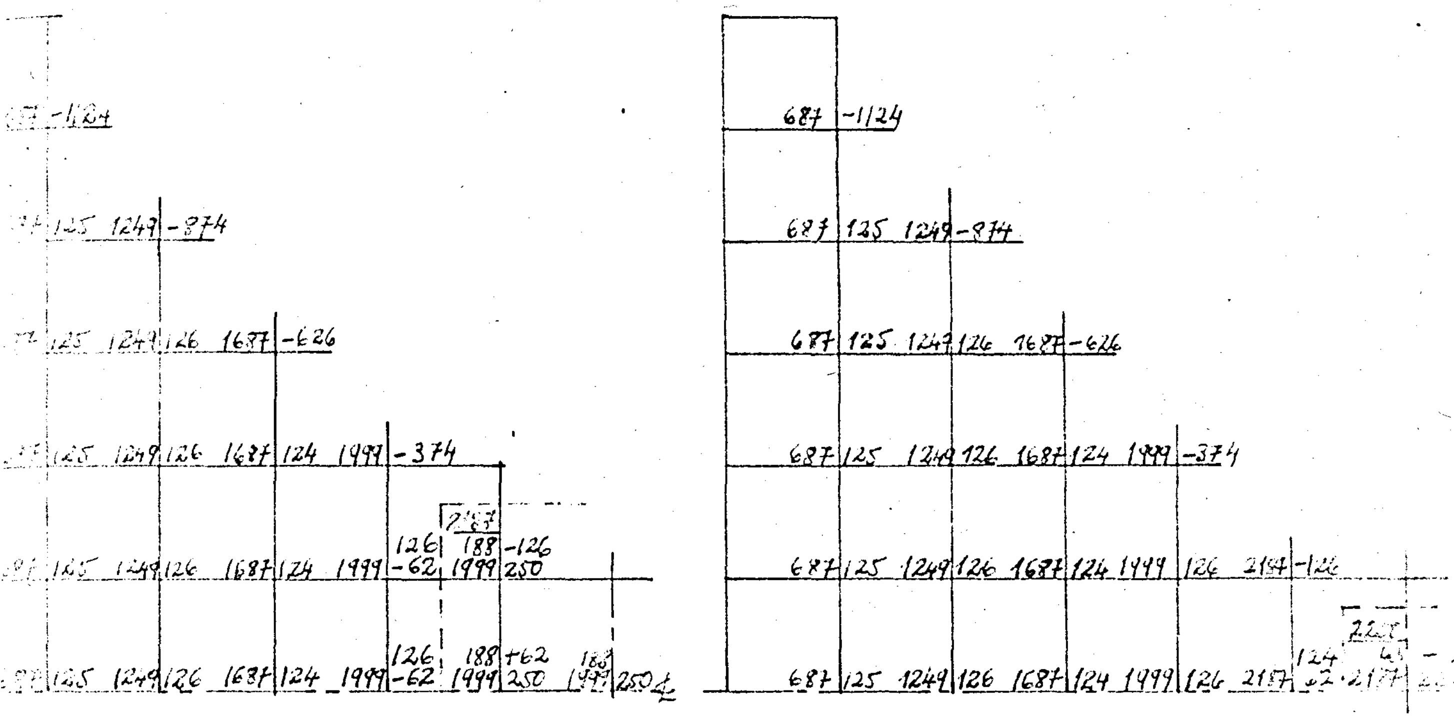
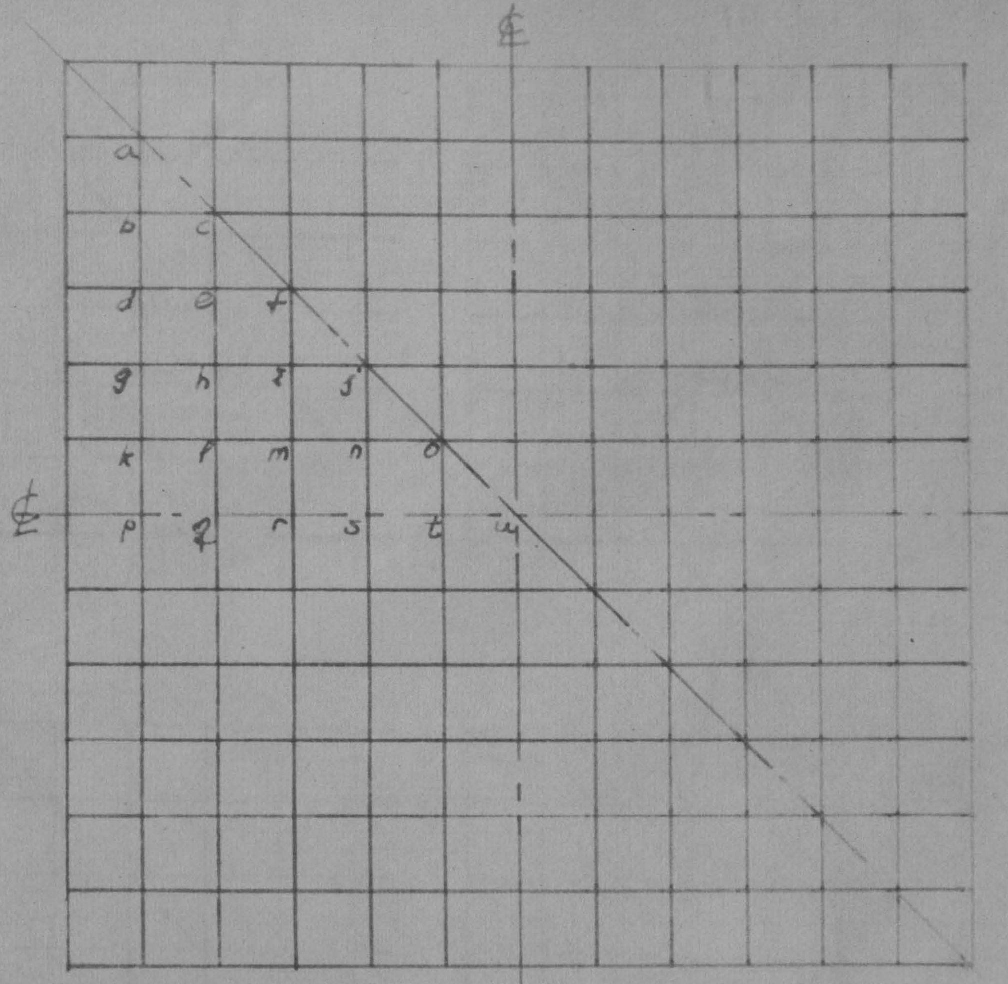


Fig. 4.7. Block relaxation of the uniform load on the square slab.



The Relaxation Operators of the 12 mesh nodes

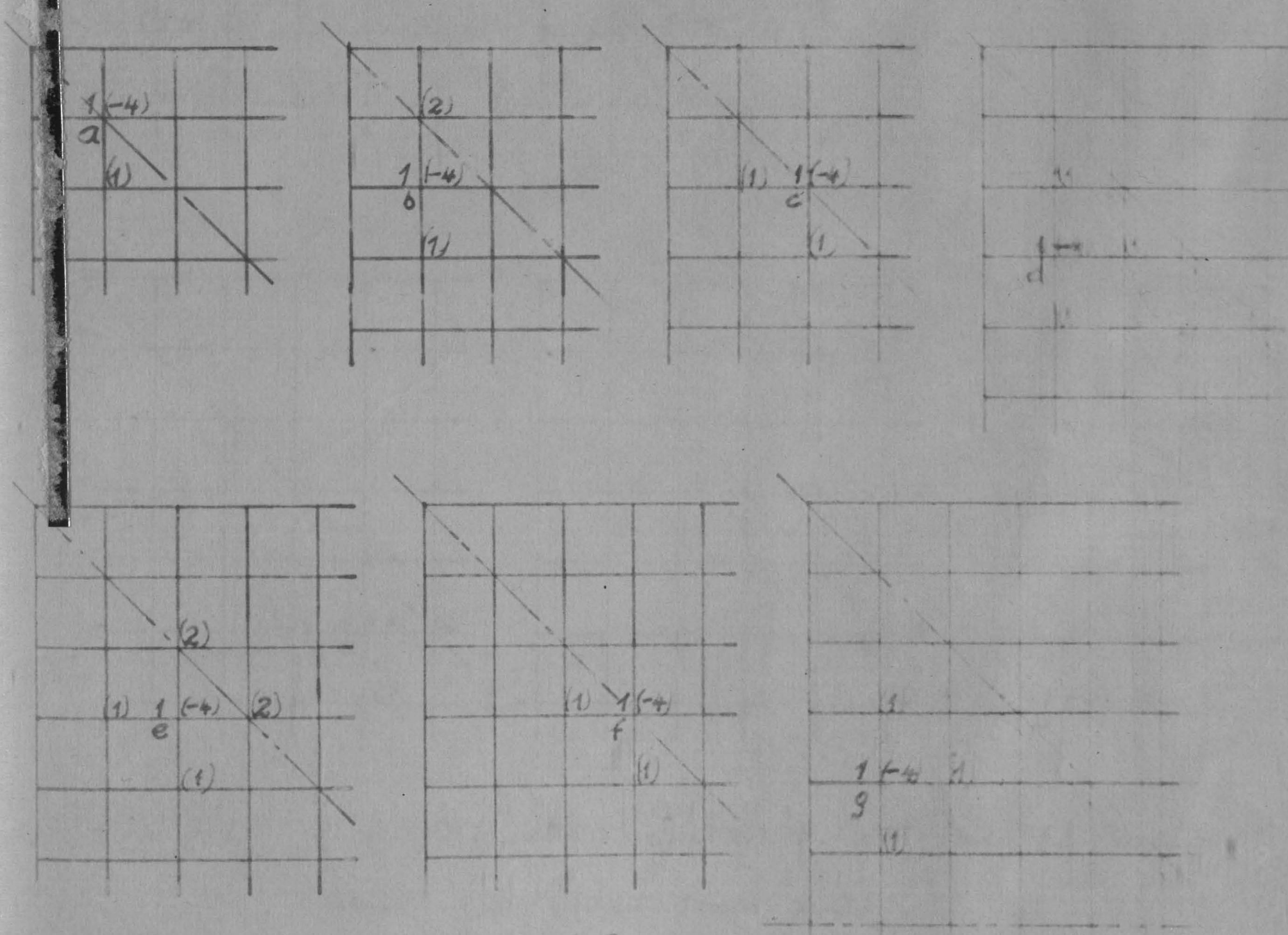


Fig. 4.8

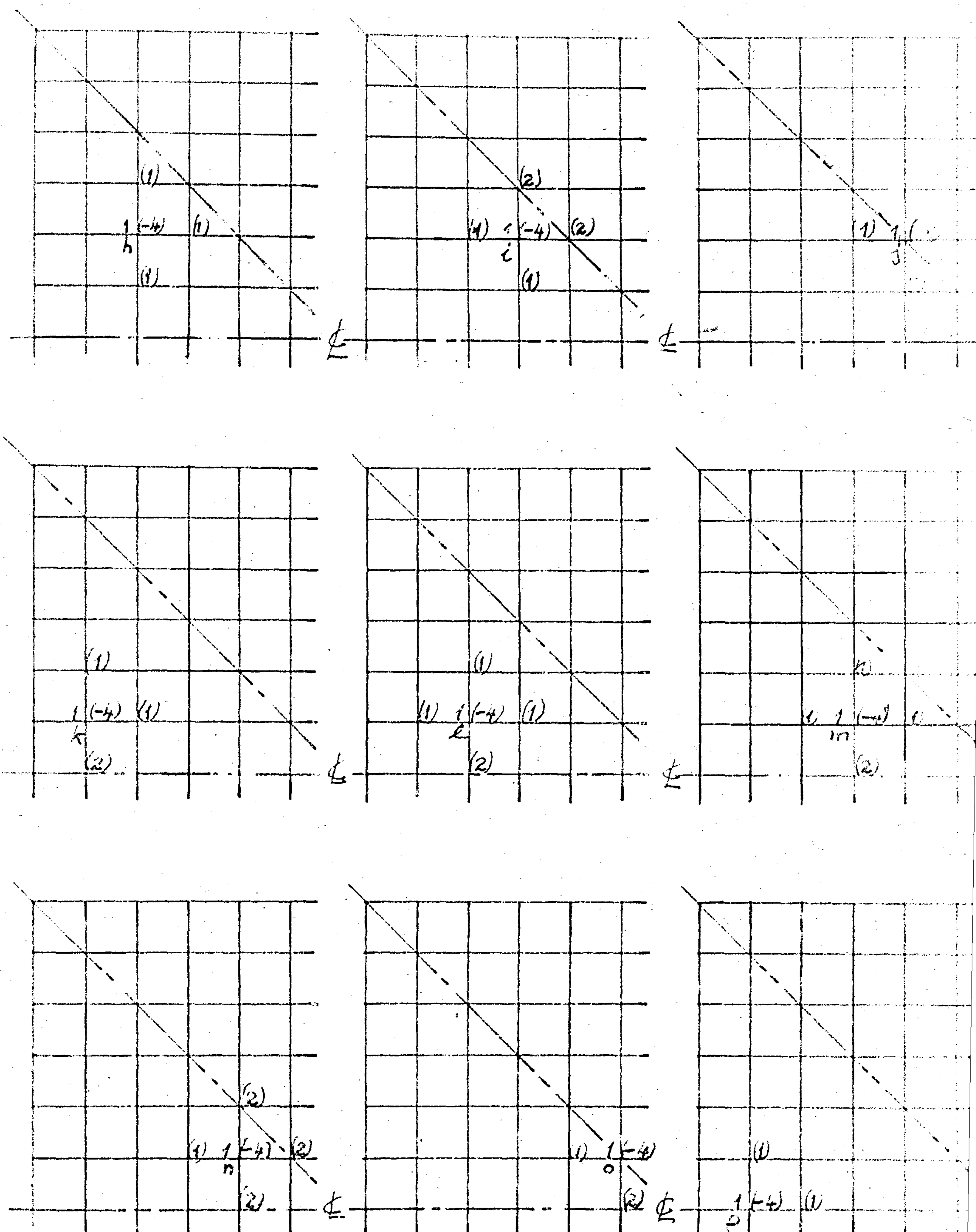


Fig. 4.9. The relaxation operators of the 12 mesh model (continued from p. 36).

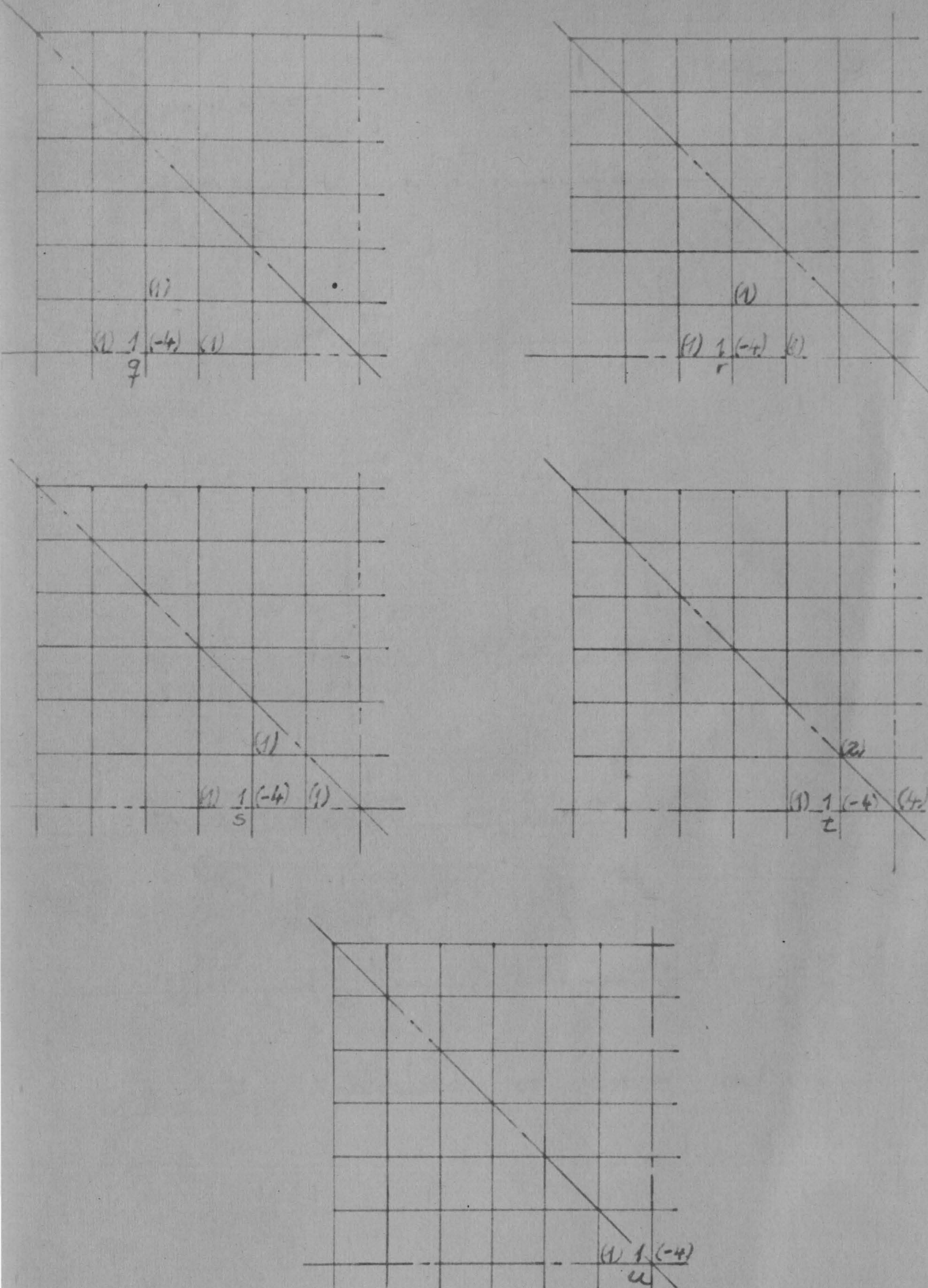


Fig. 4.10. The relaxation operators of the 12 mesh model.

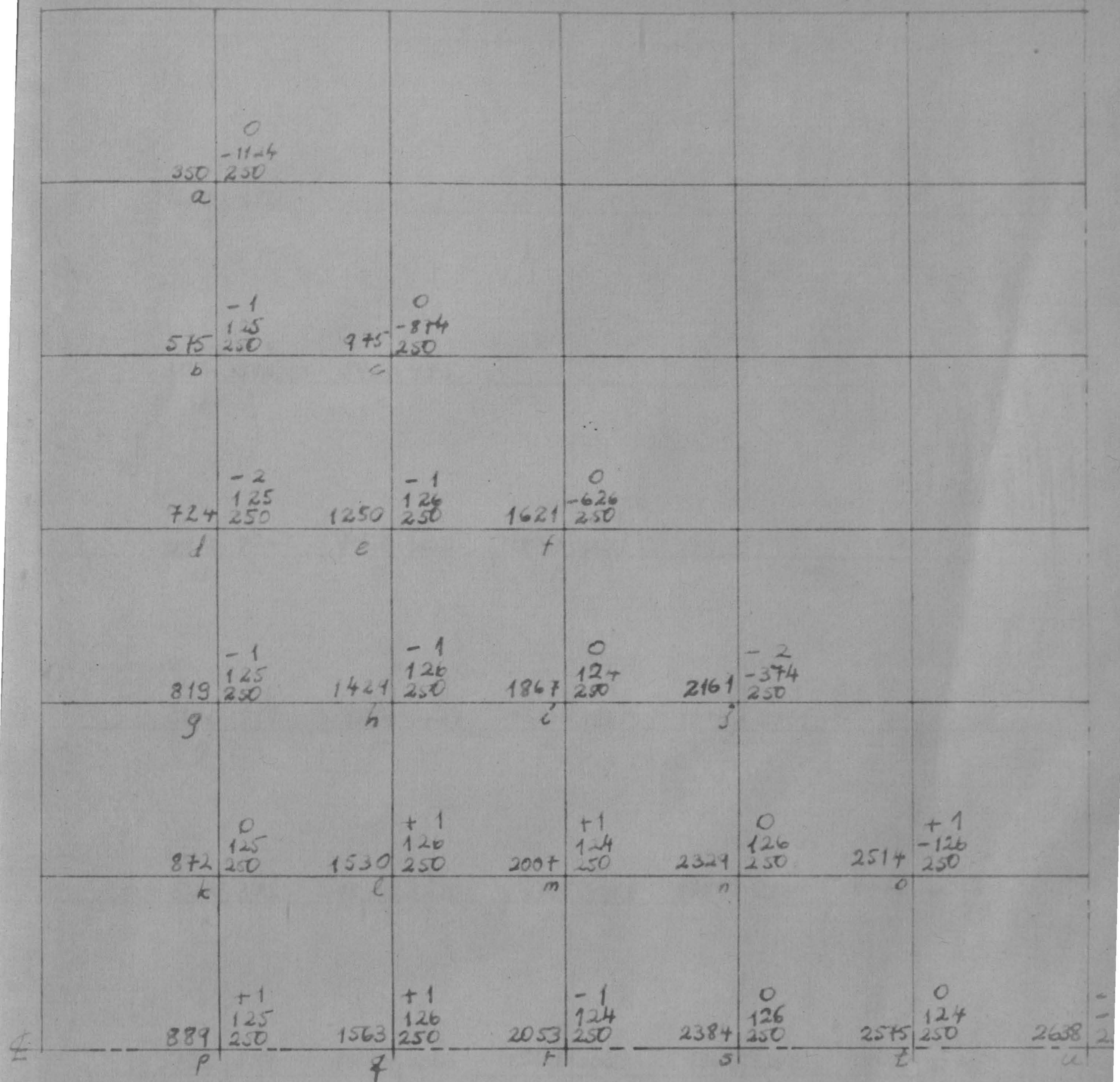


Fig. 4.11 M-relaxation of uniform load on square sla

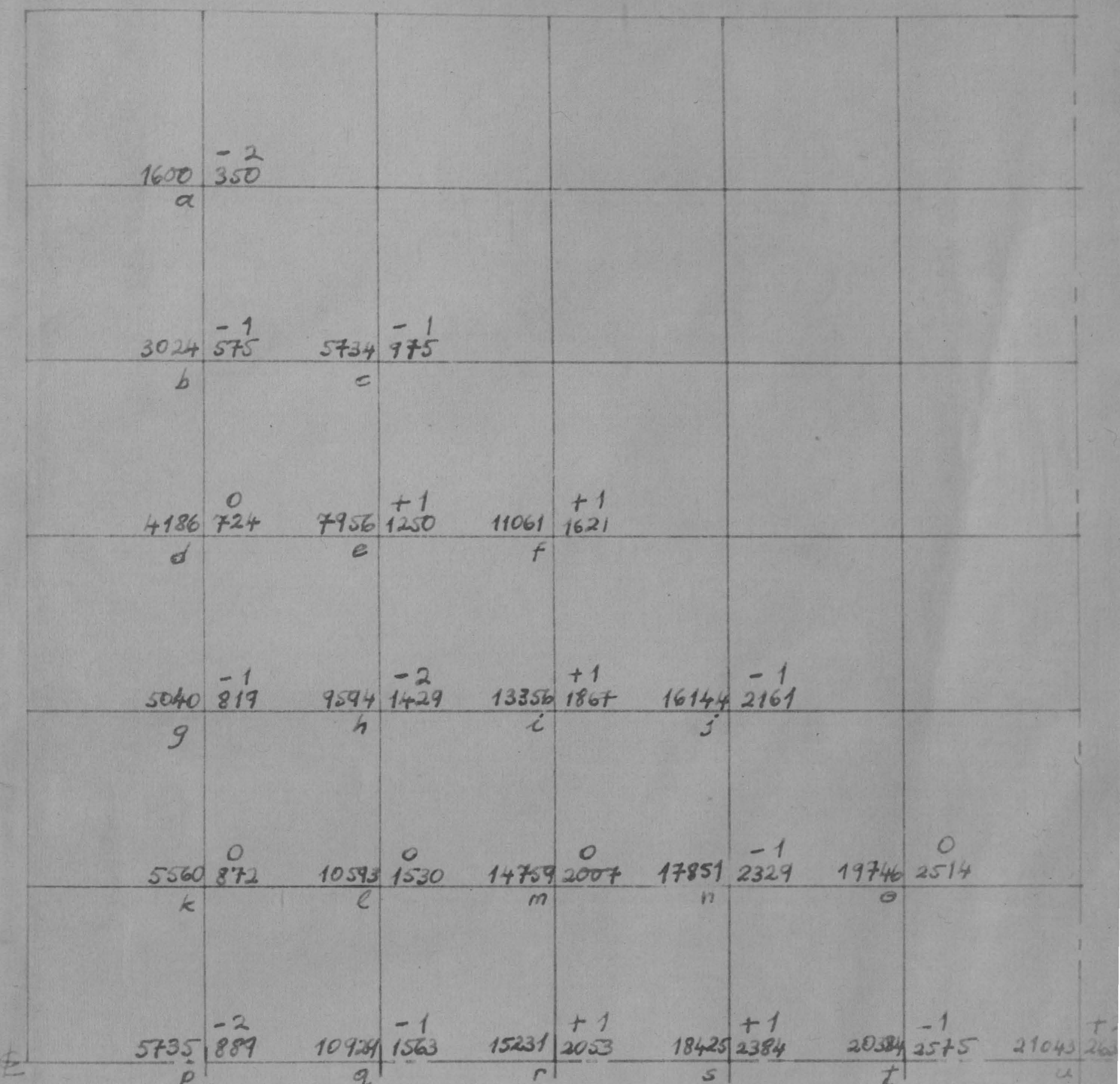


Fig. 4.12 w-relaxation of uniform load on square slab.

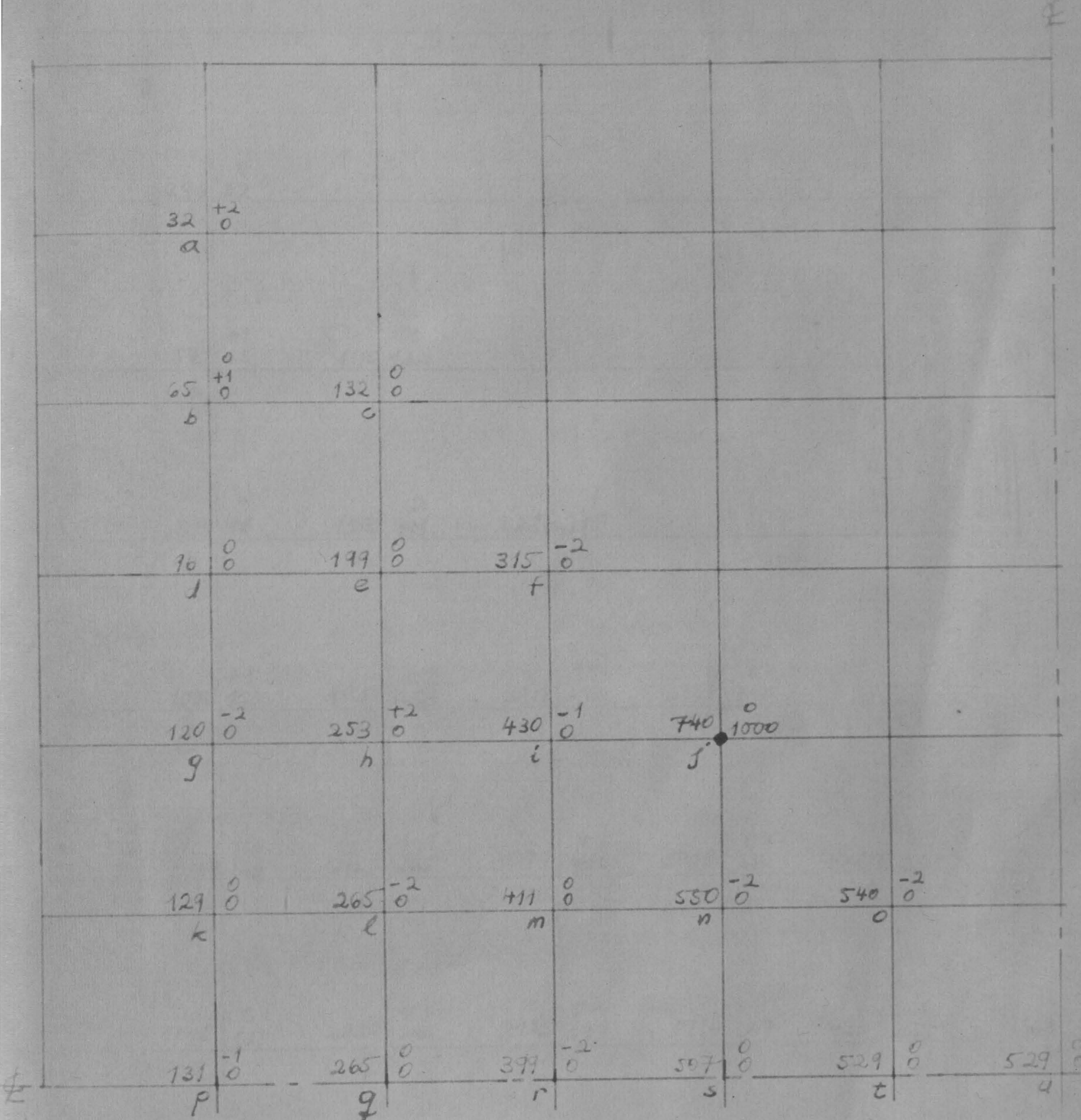


Fig. 4.13 M-relaxation of column load at j.
The column load is 1000 kg.

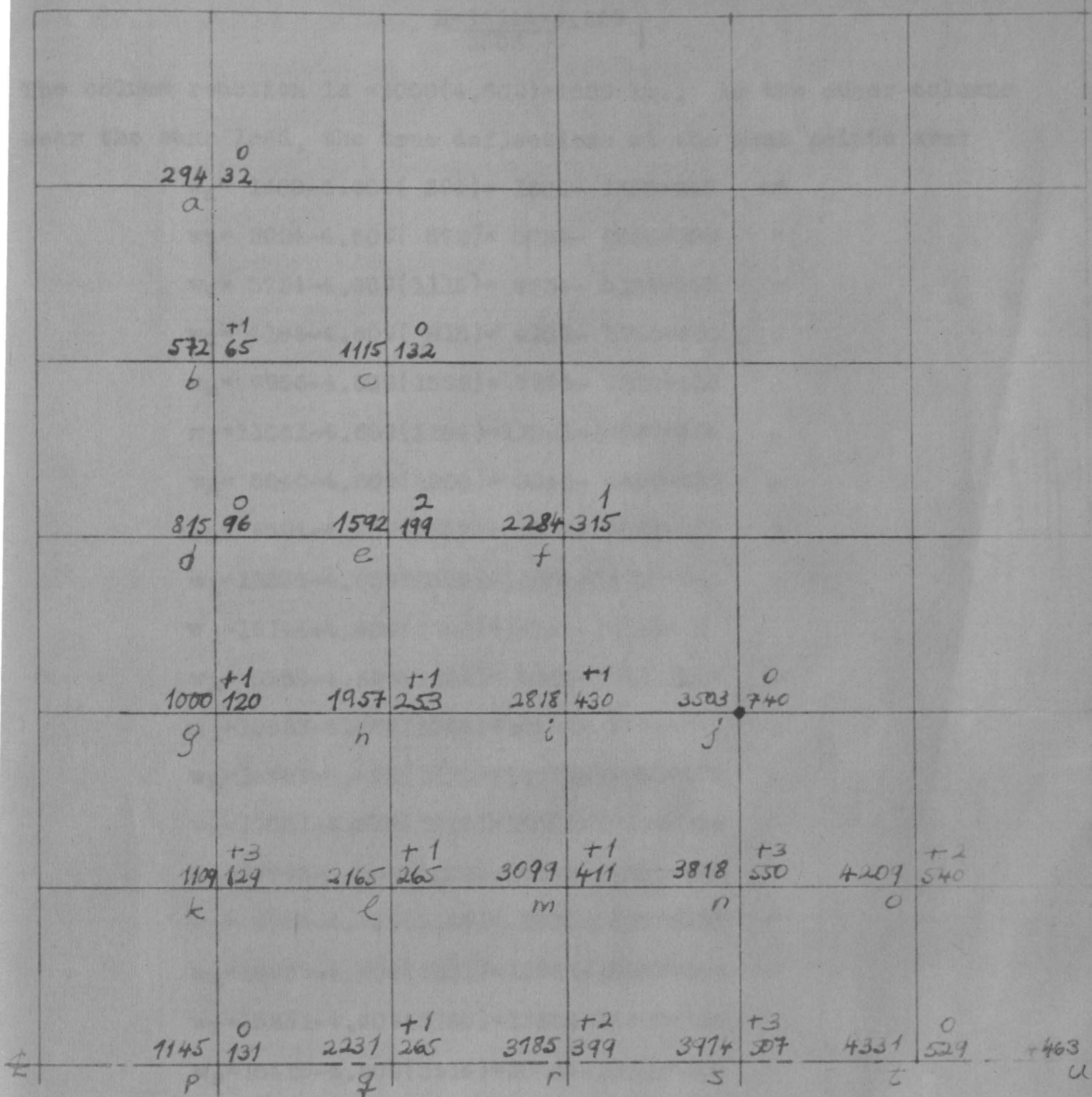


Fig. 4.14 w-relaxation of column load at j

$$K=\frac{16144}{3503}=4.609$$

The column reaction is =1000(4.609)=4609 kg.. As the other columns bear the same load, the true deflections at the mesh points are:

w_a= 1600-4.609(294)= 1600- 1355=245 cm.

w_b= 3024-4.609(572)= 3024- 2636=388 ".

w_c= 5734-4.609(1115)= 5734- 5139=595 ".

w_d= 4186-4.609(815)= 4186- 3756=430 ".

w_e= 7956-4.609(1592)= 7956- 7338=618 ".

w_f=11061-4.609(2284)=11061-10527=534 ".

w_g= 5040-4.609(1000)= 5040- 4609=431 ".

w_h= 9594-4.609(1957)= 9594- 9020=574 ".

w_i=13356-4.609(2818)=13356-12988=368 ".

w_j=16144-4.609(3503)=16144-16144= 0

w_k= 5560-4.609(1109)= 5560- 5111=449 ".

w_l=10593-4.609(2165)=10593- 9979=614 ".

w_m=14759-4.609(3099)=14759-14283=476 ".

w_n=17851-4.609(3818)=17851-17597=254 ".

w_o=19746-4.609(4209)=19746-19399=347 ".

w_p= 5735-4.609(1145)= 5735- 5277=458 ".

w_q=10929-4.609(2231)=10929-10283=646 ".

w_r=15231-4.609(3185)=15231-14680=551 ".

w_s=18425-4.609(3914)=18425-18040=385 ".

w_t=20384-4.609(4331)=20384-19962=422 ".

w_u=21043-4.609(4463)=21043-21570=473 ".

The M_x moments at the mesh points are:

$M_{xa}=2(245)-390 \quad +0.2[2(245)-390] = 120$ kg-m/m.

$M_{xb}=2(388)-595 \quad +0.2[2(388)-245-430] = 201$ "

$M_{xc}=2(595)-388-618+0.2[2(595)-388-618] = 221$ "

$M_{xd}=2(430)-618 \quad +0.2[2(430)-388-431] = 250$ "

$M_{xe}=2(618)-430-534+0.2[2(618)-595-574] = 286$ "

$M_{xf}=2(534)-618-368+0.2[2(534)-618-368] = 98$ "

$M_{xg}=2(431)-574 \quad +0.2[2(431)-430-449] = 285$ "

$M_{xh}=2(574)-431-368+0.2[2(574)-618-614] = 332$ "

$M_{xi}=2(368)-574 \quad +0.2[2(368)-534-476] = 99$ "

$M_{xj}= 0 \quad -368-254+0.2[2(0)-368-254] =-746$ "

$M_{xk}=2(449)-611 \quad +0.2[2(449)-431-458] = 286$ "

$M_{xl}=2(614)-449-476+0.2[2(614)-574-646] = 305$ "

$M_{xm}=2(476)-614-254+0.2[2(476)-368-551] = 91$ "

$M_{xn}=2(254)-476-347+0.2[2(254)-0-385] =-290$ "

$M_{xo}=2(347)-254-422+0.2[2(347)-254-422] = 20$ "

$M_{xp}=2(458)-646 \quad +0.2[2(458)-2(449)] = 274$ "

$M_{xq}=2(646)-458-551+0.2[2(646)-2(614)] = 296$ "

$M_{xr}=2(551)-646-385+0.2[2(551)-2(476)] = 101$ "

$M_{xs}=2(385)-551-422+0.2[2(385)-2(254)] =-151$ "

$M_{xt}=2(422)-385-473+0.2[2(422)-2(347)] = 16$ "

$M_{xu}=2(473)-2(422) +0.2[2(473)-2(422)] = 122$ "

The My moments at the mesh points are:

M_{ya} =	120	kg-m/m
M_{yb} =	137	"
M_{yc} =	221	"
M_{yd} =	89	"
M_{ye} =	121	"
M_{yf} =	98	"
M_{yg} =	41	"
M_{yh} =	14	"
M_{yi} =	-282	"
M_{yj} =	-746	"
M_{yk} =	66	"
M_{yl} =	69	"
M_{ym} =	50	"
M_{yn} =	60	"
M_{yo} =	20	"
M_{yp} =	72	"
M_{yq} =	121	"
M_{yr} =	164	"
M_{ys} =	221	"
M_{yt} =	147	"
M_{yu} =	122	"

CHAPTER 5

DISCUSSION OF RESULTS AND CONCLUSION

5.1. Static Load Check.

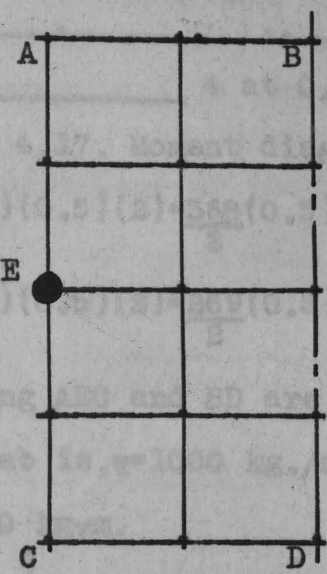


Fig. 4.15. One-half of an interior panel.

Let (Fig. 4.15) represent one-half of an interior panel. If this was a real intermediate panel that is to say an infinite number of panels extended in both directions, then the shear along the lines AB, BD, and CD will be zero. Therefore the sum of moments along BD and AC must be aql^2 where $a=1/8$, no matter what the distribution of moments along this line would be. The sum of moments along AEC will be the area of the moment diagram.

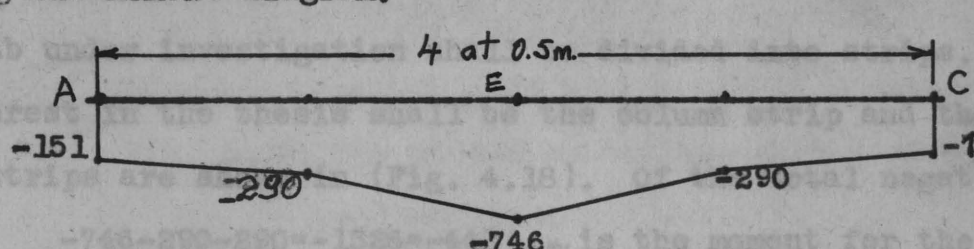


Fig. 4.16. Moment diagram along AEC.

$$\frac{(746+290)}{2}(0.5)(2)=\frac{1036}{2}(0.5)(2)=518 \text{ kg-m.}$$

$$\frac{(290+151)}{2}(0.5)(2)=\frac{441}{2}(0.5)(2)=\underline{220} \quad "$$

$$738 \quad "$$

The sum of moments along BD is.

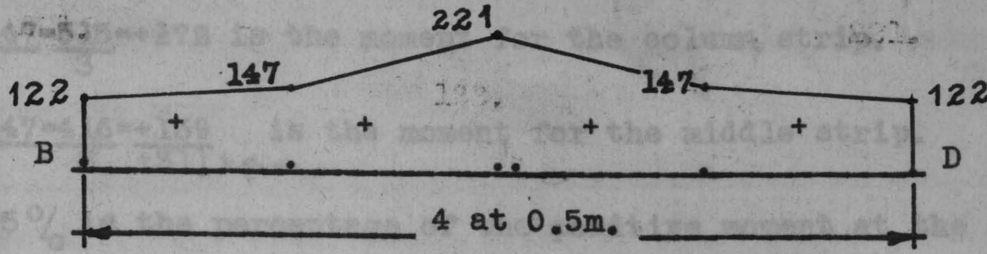


Fig. 4.17. Moment diagram along BD.

$$\left(\frac{221+147}{2}\right)(0.5)(2)=\frac{368}{2}(0.5)(2)=184 \text{ kg-m.}$$

$$\left(\frac{147+122}{2}\right)(0.5)(2)=\frac{269}{2}(0.5)(2)=\underline{135} \text{ "}$$
$$319 \text{ "}$$

The sum of moments along AEC and BD are =738+319=1057 kg-m.

The load on the 2m. slab is, q=1000 kg./m^2.(2m.)=2000 kg./m.

$$M=\frac{1}{8}ql^2=\frac{1}{8}(2000)(2^2)=1000 \text{ kg-m.}$$

The sum of moments of 1057 found by relaxation and the moment of the static load are in good proxmity for statical comparison.

Next we shall find out what percentage of this total moment goes to negative and positive moments. It is observed that of this 1057 total moment,

$$\frac{738}{1057}=0.70=70\,\% \text{ goes to negative moment.}$$

and $$\frac{319}{1057}=0.30=30\,\% \text{ goes to positive moment.}$$

The slab under investigation shall be divided into strips. The strips of interest in the thesis shall be the column strip and the middle strip. These strips are shown in (Fig. 4.18). Of the total negative moments,

$$\frac{-746-290-290}{3}=\frac{-1326}{3}=-441 \text{ kg-m. is the moment for the column strip.}$$

$$\frac{-290-151-290}{3}=\frac{-243}{3}=\underline{-243} \text{ " is the moments for the middle strip.}$$
$$-684 \text{ "}$$

$$\frac{441}{684}=0.645=64.5\,\% \text{ is the percentage of the negative moment at the column strip.}$$

$$\frac{243}{684}=0.355=35.5\,\% \text{ is the percentage of the negative moment at the middle strip.}$$

of the total positive moments,

$\frac{+147+221+147}{3}=\frac{515}{3}=+172$ is the moment for the column strip.

$\frac{+147+122+147}{3}=\frac{416}{3}=\frac{+139}{+311}$ kg-m. is the moment for the middle strip.

$\frac{172}{311}=0.55=55\%$ is the percentage of the positive moment at the column strip.

$\frac{139}{311}=0.45=45\%$ is the percentage of the positive moment at the middle strip.

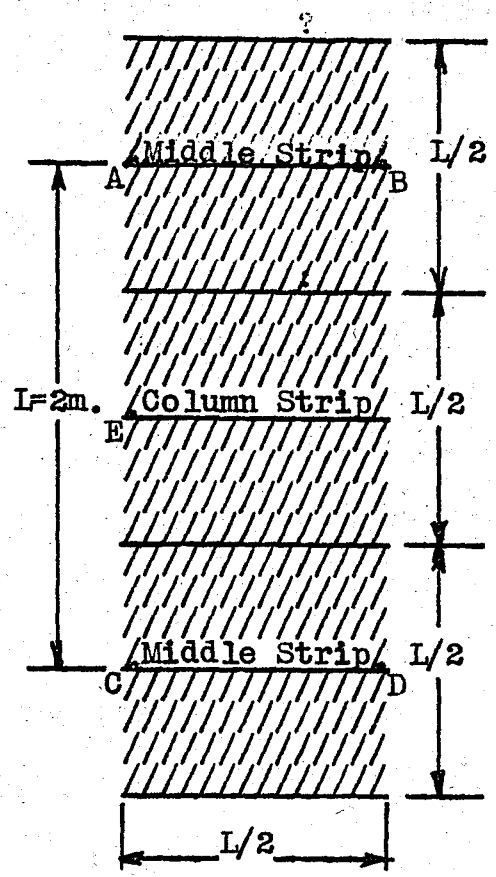


Fig. 4.18. The middle and column strips of a panel O with a column at E.

The results obtained shall be put in tabular form.

TABLE FIVE

PERCENTAGE OF MOMENTS BETWEEN COLUMN STRIPS AND MIDDLE STRIPS IN PERCENT OF TOTAL MOMENTS AT CRITICAL SECTIONS OF A PANEL O.

Strip	Moment Section	
	Negative Moment at Interior Support	Positive Moment
Column Strip	64.5	55
Middle Strip	35.5	45

The slab analyzed above shall be compared by the elastic analysis method of the ACI Code. In design by elastic analysis several assumptions are made.*

*Journal of ACI. Feb. 1962, proc. v.59, No.2, P.229-231, Table 2103(c).

a) Assumptions:

1. The structure may be considered divided into a number of bents, each consisting of a row of columns or supports and strips of supported slabs, each strip bounded laterally by the center line of the panel on either side of the center line of columns or supports. The bents shall be taken longitudinally and transversely of the building.

2. Each such bent may be analyzed in its entirety; or each floor thereof and the roof may be analyzed seperately with its adjacent columns as they occur above and below, the columns beeing assumed fixed at their remote ends. Where slabs are thus analyzed separately, it may be assumed in determining the bending at a given support that the slab is fixed at any support two panels distant therefrom provided the slab continues beyond that point.

3. The joints between columns and slabs may be considered rigid, and this rigidity (infinite moment of inertia) may be assumed to extend in the slabs from the center of the columns to the edge of the capital, and in the column from the top of the slab to the bottom of the capital. The change in length of columns and slabs due to direct stress, and deflections due to shear, may be neglected.

4. Where metal column capitals are used, account may be taken of their contributions to stiffness and resistance to bending and she

5. The moment of inertia of the slab or column at any cross section may be assumed to be that of the cross section of the concrete

*See item No: 6 in bibliography.

Variation in the moments of inertia of the slabs and columns along their axes shall be taken into account.

6. Where the load to be supported is definitely known, the structure shall be analyzed for that load. Where the live load is variable but does not exceed three-quarters of the dead load, or the nature of the live load is such that all panels will be loaded simultaneously, the maximum bending may be assumed to occur at all sections under full live load. For other conditions, maximum positive bending near midspan of a panel may be assumed to occur under three-quarters of the full live load in the panel and in alternate panels; and maximum negative bending in the slab at a support may be assumed to occur under three-quarters of the full live load in the adjacent panels only. In no case, shall the design moments be taken as less than those occuring with full live load on all panels.

b) <u>Critical Sections</u>:

The critical section for negative bending, in both the column strip and middle strip, may be assumed as not more than the distance <u>A</u> from the center of the column or support and the critical negative moment shall be taken to consideration as extending over this distance.

c) <u>Distribution of Panel moments</u>:

Bending at critical sections across the slabs of each bent may be apportined between the column strip and middle strip, as given in Table Six. For design purposes , any of these percentages may be varied by not more than 10 % of its value, but their sum for the full panel width shall not be reduced.

TABLE SIX

PERCENTAGE OF MOMENTS BETWEEN COLUMN STRIPS AND MIDDLE STRIPS IN PERCENT OF TOTAL MOMENTS AT CRITICAL SECTIONS OF A PANEL

Strip	Moment Section	
	Negative Moment at Interior Support	Positive Moment
Column Strip	76	60
Middle Strip	24	40

5.2. Conclusion. It is seen that the values in Tables Five and Table Six are quite close so that the elastic analysis proposed by the ACI Code agrees well with the theoretical analysis. However, as it is seen, the relaxation method is easily applicable specially in cases where the elastic analysis proposed by the ACI Code may give questionable results such as a column in the middle of a slab or slabs with unusual loadings.

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APPENDIX

USEFUL FINITE-DIFFERENCE APPROXIMATIONS

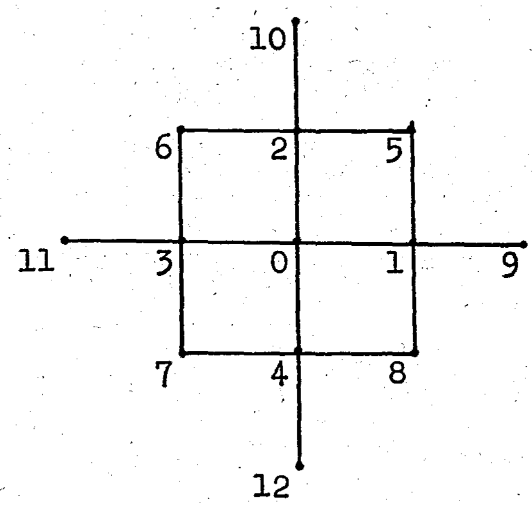

Fig. 4.19. A square net with nodes indicated.

In terms of the w values at the nodes of a square net as indicated in (Fig. 4.19), the following finite-difference approximations to low-order derivatives are frequently needed and have been collected together here for ease of reference:

First Derivatives

$$\left(\frac{\partial w}{\partial x}\right)_0 = \frac{w_1-w_3}{2h},$$

$$\left(\frac{\partial w}{\partial y}\right)_0 = \frac{w_2-w_3}{2h},$$

Second Derivatives

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_0 = \frac{w_1+w_3-2w_0}{h^2},$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_0 = \frac{w_2+w_4-2w_0}{h^2},$$

$$(\nabla^2 w)_0 = \frac{w_1+w_2+w_3+w_4-4w_0}{h^2},$$

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)_0 = \frac{w_5-w_6+w_7-w_8}{4h^2}.$$

Third Derivatives

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_0 = \frac{w_9-2w_1+2w_3-w_{11}}{2h^3},$$

$$\left(\frac{\partial^3 w}{\partial y^3}\right)_0 = \frac{w_{10}-2w_2+2w_4-w_{12}}{2h^3},$$

Fourth Derivatives

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_o = \frac{6w_o - 4w_1 - 4w_3 + w_9 + w_{11}}{h^4},$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_o = \frac{6w_o - 4w_2 - 4w_4 + w_{10} + w_{12}}{h^4},$$

$$\left(\nabla^4 w\right)_o = \frac{20w_o - 8(w_1 + w_2 + w_3 + w_4) + 2(w_5 + w_6 + w_7 + w_8)}{h^4} +$$

$$+ \frac{(w_9 + w_{10} + w_{11} + w_{12})}{h^4}.$$

$$\left(\frac{\partial^4 w}{\partial x^2\, \partial y^2}\right)_o = \frac{4w_o - 2(w_1 + w_2 + w_3 + w_4) + (w_5 + w_6 + w_7 + w_8)}{h^4},$$