FROM FIVE DIMENSIONAL FLAT SPACETIME TO OUR FOUR DIMENSIONAL BRANEWORLD VIA KALUZA-KLEIN

by

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ABSTRACT

FROM FIVE DIMENSIONAL FLAT SPACETIME TO OUR FOUR DIMENSIONAL BRANEWORLD VIA KALUZA-KLEIN

In five dimensional cosmological models, the convention is to include the fifth dimension in a way similar to the other space dimensions. In this work we attempt to introduce the fifth dimension in a way that a time dimension would be introduced. In our metric ansatz we take the scale factor of three dimensional space, the x, y, z coordinates, to depend on both time and internal space. We allow time and internal space, the extra dimension, to share the same metric coefficient that depends on both dimensions. As such time and internal space play similar roles. From such a metric, we obtain a five dimensional flat spacetime into which all relevant four dimensional cosmologies can be locally embedded. Different cases, such as radiation, matter or dark energy dominated cosmologies, correspond to different choices of the free parameters. Each choice is a different frame. We argue on which frame might correspond to the cosmological frame. From our choice of the cosmological frame we obtain a braneworld scenario by restraining internal space from stretching along the negative direction. In this model all the matter fields are confined to the brane and the bulk is empty. We also see that it is possible for the three dimensional space to shrink to zero away from the brane. Thus our four dimensional world is confined to this four dimensional brane.

ÖZET

KALUZA-KLEIN ARACILIĞI İLE BEŞ BOYUTLU DÜZ UZAY-ZAMANDAN BİZİM DÖRT BOYUTLU UZAY-ZAMANIMIZA

Beş boyutlu kozmoloji modellerinde beşinci boyut genellikle diğer uzay boyutları gibi ele alınır. Bu çalışmada beşinci boyut bir zaman boyutunun metriğe eklenileceği şekilde ele alınılmaktadır. Başlangıç metriğimizde x, y, z koordinatlarından oluşan üç boyutlu uzayın ölçek faktörünü hem zamana hem fazla boyuta, bağlı alıyoruz. Zaman ve fazla boyutun metrik katsayılarını da ortak ve iki boyuta birden bağlı kabul ediyoruz. Bu hali ile fazla boyutun üstlendiği görev zaman ile aynı. Böyle bir metriğe tüm gerekli dört boyutlu kozmolojilerin lokal olarak gömülebileceği, beş boyutlu düz bir uzay-zaman elde ediyoruz. Burada, ışımanın, maddenin veya karanlık enerjinin ağırlıklı olduğu gibi farklı durumlar, serbest parametrelerin belirli değerlerine, her seçenek de farklı bir koordinat sistemine karşılık geliyor. Bu olası koordinat sistemlerinden hangisinin kozmolojik koordinatlar olabileceği sorusunu ele alıyoruz. Kozmik koordinatlar seçimimizde fazla boyutun negatif yönde uzanmamasını şart koşarak bir zar-evren modeli elde ediyoruz. Bu modelde içerisi boş, tüm madde alanları zara hapis olmuş durumda. Üç boyutlu uzay zamanın zardan uzakta sıfıra kadar küçülmesi mümkün. Kısaca bizim dört boyutlu dünyamız bu dört boyutlu zardan oluşmakta.

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LIST OF SYMBOLS

c	speed of light
$C_{\mu\nu\lambda\kappa}$	Weyl tensor
$\mathbb{E}^n$	Euclidean space
E_{ab}	curvature tensor
$g_{\mu\nu}$	metric tensor
G_N	Newton's gravitational constant
$G_{\mu\nu}$	Einstein's tensor
$\hbar$	Planck's constant
K_{ab}	extrinsic curvature tensor
$\mathcal{L}$	Lagrangian
$R^{\mu}{}_{\nu\lambda\kappa}$	Riemann curvature tensor
$R_{\mu\nu}$	Ricci tensor
R	Ricci scalar
$\mathfrak{S}$	Action
$T_{\mu\nu}$	Energy-Momentum tensor
∇_{μ}	covariant derivative
κ	curvature of spacelike sections
Λ	cosmological constant
$d\Sigma^2$	metric of three dimensional space
$w^{\mu}{}_{\nu\lambda}$	connection coefficients
$w^{\mu}{}_{\nu}$	connection one forms
$\Omega^{\mu}{}_{\nu}$	curvature two forms

LIST OF ACRONYMS/ABBREVIATIONS

BW	Braneworld model
CFS	conformally flat spacetime
FRW	Friedmann-Robertson-Walker metric
IM	Induced Matter Theory
STM	Space-Time-Matter Theory

1. INTRODUCTION

We considered ourselves to be living in a three dimensional space until Einstein changed our notion of time from a parameter to a dimension to explain electrodynamics of moving bodies and led us to think in terms of a four dimensional spacetime. The number of dimensions has been increasing ever since. With Kaluza [1] and Klein [2] the four dimensions were augmented to five in an attempt to unite electromagnetism and gravity and explain the quantization of electric charge. While we are plainly aware of our four dimensional surroundings, nobody has been able to observe a fifth dimension yet. Obviously extra dimensions are going to be helpful, but one needs to explain their observational absence.

In time, the spirit of Kaluza-Klein theory grew into questioning of embedding general relativity's solutions into higher dimensions. Unless there was something special to settle the number of dimensions, it would be most natural to consider an N dimensional theory to be linked with a higher (N+1) dimensional one. Thus the idea of embedding brought on the quest to embed four dimensional solutions of Einstein's equations into five dimensional flat solutions. In the original Kaluza-Klein theory, the metric coefficients are independent of the fifth dimension, this is also known as the cylinder condition. Also the internal space is compactified in a natural attempt to explain its lack of observation. By relaxing Kaluza's cylinder condition and allowing components of the metric tensor to depend on the extra dimension, higher dimensional theories became more fruitful. It turns out that standard four dimensional cosmological models are special in that they can be embedded into five dimensional flat spacetimes. As for all of the solutions in general, they can be embedded in the general canonical metric [3].

The usefulness of a fifth dimension grew when Randall and Sundrum [4, 5] used it to explain the hierarchy problem, which brought forth the concept of brane worlds. Although important steps were made with all these works and many others, it seems that there is still much to be done in order to completely understand internal extra dimensions. Today the number of dimensions have gone up to eleven or one can also say that they came down from twenty six via superstrings with string or M-theory's quest to understand quantum effects of gravity and unite all fundamental forces [6].

Something as mysterious as extra dimensions is the dark energy. It was Hubble, who first observed galaxies to be receding from each other. Today we are certain that our universe is accelerating while expanding [7,8]. We have come up with the term dark energy as the source of this accelerated expansion, yet we are not certain what it really is, hence the name "dark". Perhaps the two mysterious concepts, dark energy and extra dimensions, are connected with each other [9]. A recent attitude towards dark energy is to explain it by a modification to the geometric side of Einstein's equations. One successful attempt which includes extra dimensions, is brane-world gravity, where at high energies massive modes of graviton dominate, gravity leaks off the brane where its weakening initiates acceleration [10].

In this work we want to approach this jungle of dimensions with purely cosmological concerns. We want to see what happens when we introduce an extra spacelike dimension into the cosmological metric, in the same way that a timelike dimension would be introduced. This way we will be putting forth symmetries between time and the internal space, which brings up the question whether internal space can be as fundamental as time. Our main motivation is curiosity while our second motivation is to see if we can obtain the effects of dark energy from this five dimensional metric without having to introduce a cosmological constant. In the end we will achieve all relevant four dimensional cosmologies as a four dimensional slice of a flat five dimensional cosmology. Thus we will have pointed out that our internal space is just as fundamental as time and we will have obtained the expansion usually credited to the dark energy, from an extra dimension under certain values for free parameters. We will discuss how fixing the free parameters amounts to choosing different frames. Our choice of the cosmological frame will be the simplest frame that is also sensible in terms of dimensional arguments. This choice will amount to a linearly expanding universe. We will conclude by considering a braneworld version of our model where our four dimensional universe is confined to a brane and the five dimensional bulk is empty.

1.1. A brief review of mathematical and physical concepts that led to Einstein's equations

The first steps in the study of surfaces began with the flat Euclidean space. The Euclidean space is an $n-$dimensional vector space over real numbers $\mathbb{R}$. A vector space $\mathbb{V}$, consists of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ and linear operations which are addition of vectors and multiplication by scalars, $a, b \in \mathbb{R}$ or $\mathbb{C}$. These operations are commutative

(i) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

(ii) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$,

and associative

(iii) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(iv) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(v) $(ab)\mathbf{u} = a(b\mathbf{u})$.

The vector space includes a zero vector $\mathbf{0}$, for which

(vi) $\mathbf{u} + \mathbf{0} = \mathbf{u}$,

an inverse $-\mathbf{u}$, for each of its elements such that

(vii) $(\mathbf{u}) + (-\mathbf{u}) = \mathbf{0}$

and an identity element $\mathbf{1}$ where,

(viii) $\mathbf{1u} = \mathbf{u}$.

Defined on an Euclidean space $\mathbb{E}^n$ are geometric objects such as points, lines, planes and a positive definite inner product. The properties of the geometric objects are given as axioms. The existence of a positive definite inner product allows one to introduce length and orthogonality. Since length is a positive quantity, the positive definiteness of the inner product is crucial in arriving at such a concept. Length in turn, allows one to introduce coordinates.

A topological space (χ, τ), is a collection $\tau = \{U_i | i \in I\}$ of open sets U_i which are also the subsets of a set χ. τ is required to include the empty set and χ itself, the sub collection of another interval J such that for $\{U_j | j \in J\}$ where $\bigcup U_j \in \tau$ and for K a finite sub collection of I, include the intersections of the family $\{U_k | k \in K\}$ with

$\bigcap U_k \in \tau$. The metric $d(x, y)$, to be said more on later, is a notion of length. Once a topological space is furnished by a metric its topology can be given in terms of open balls or cubes where the open ball $B_r(x) = \{y \in \mathbb{R}^n | d(x, y) < r\}$ can be thought as the inside of a ball of radius r, centered at y. Such a topological space is called the metric space [11]. The metric space $\mathbb{R}^n$, which is the n fold Cartesian product of real numbers, can be described as an $n-$dimensional Euclidean space $\mathbb{E}^n$, equipped with a coordinate system. The objects that live in $\mathbb{R}^n$ are ordered n-tuples of real numbers, $(x^1, x^2, ..., x^n)$.

$\mathbb{E}^n$ and $\mathbb{R}^n$ are flat spaces, like the surface of a pond when there is no wind. In general one comes across curved surfaces and spaces with complicated topologies, like a saddle. To study more general spaces we make use of manifolds. An $n-$dimensional manifold M is Hausdorff, which means its points can be separated from each other in the sense that the open sets to which the points belong do not intersect. M has a countable basis of open sets and is locally an $n-$dimensional Euclidean space. That is, although the manifold on the whole is curved and has a more complicated topology, locally the working mechanisms of functions and coordinates on it are the same as on $\mathbb{R}^n$. This is to our advantage because we are more familiar with flat spaces then curved ones and this allows us to express a manifold as patches of $\mathbb{R}^n$ sewn together.

A map $\phi : M \to N$ takes an element of a set M to an element of a set N. If the p^{th} derivative of ϕ exists, and is continuous, ϕ is $p-$times differentiable and it is called a C^p map. When a map is infinitely times differentiable it becomes a C^∞ map. A map that takes each element of M to only one element of N is a one-to-one map. In a one-to-one map N may have elements that do not correspond to any element of M. When all elements of N correspond to some element of M the map is onto. In an onto map two different elements of M can correspond to the same element of N. A map that is both one-to-one and onto is invertible. Since in a one-to-one and onto map each element of M goes to a unique element of N, one can trace back an element of N to which element of M it corresponds to under ϕ. A coordinate chart (U, ϕ) consists of an open subset U of a set M and a one-to-one map ϕ that takes the elements of U to an open region in $\mathbb{R}^n$.

In the definition of manifolds in terms of patches, the manifold is considered as a collection of coordinate charts (U_α, ϕ_α) such that $\bigcup_\alpha U_\alpha = M$. The U_α is said to cover M. Moreover the intersection of charts is nonempty, $U_\alpha \cap U_\beta = \varnothing$. The second condition is what allows the charts to be sewn together smoothly, leaving nothing out. Such a collection of every possible coordinate chart which consists of C^∞ maps is called a differentiable , or a C^∞ manifold.

For a vector space $\mathbb{V}$ on $\mathbb{R}$, a map $\phi : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ that is linear in each variable separately is said to be a bilinear form on $\mathbb{V}$. The bilinearity means for $a, b \in \mathbb{R}$, $\mathbf{v}, \mathbf{v_1}, \mathbf{v_2}, \mathbf{w}, \mathbf{w_1}, \mathbf{w_2} \in \mathbb{V}$

(i) $\phi(a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w}) = a\phi(\mathbf{v}_1, \mathbf{w}) + b\phi(\mathbf{v}_2, \mathbf{w})$

(ii) $\phi(\mathbf{v}, a\mathbf{w}_1 + b\mathbf{w}_2) = a\phi(\mathbf{v}, \mathbf{w}_1) + b\phi(\mathbf{v}, \mathbf{w}_2)$

On a basis $\{e_i\}$ of $\mathbb{V}$ if $\mathbf{v} = \lambda^i e_i$, $\mathbf{w} = \eta^j e_j$ and if $g_{ij} = \phi(e_i, e_j)$ where the repeated indices are summed over

$$\phi(\mathbf{v}, \mathbf{w}) = \phi(\lambda^i e_i, \eta^j e_j)$$

$$= \lambda^1 \phi(e_1, \eta^j e_j) + \lambda^2 \phi(e_2, \eta^j e_j) + ... + \lambda^n \phi(e_n, \eta^j e_j)$$

$$= \lambda^1 \eta^1 \phi(e_1, e_1) + ... + \lambda^1 \eta^n \phi(e_1, e_n) + \lambda^2 \eta^1 \phi(e_2, e_1) + ... + \lambda^n \eta^n \phi(e_n, e_n)$$

$$= \lambda^1 \eta^1 g_{11} + \lambda^1 \eta^2 g_{12} + ... + \lambda^2 \eta^1 g_{21} + ... + \lambda^n \eta^n g_{nn}$$

$$\phi(\mathbf{v}, \mathbf{w}) = g_{ij} \lambda^i \eta^j \tag{1.1}$$

The last expression is similar to one that would arise in the multiplication of a matrix g and two vectors $\mathbf{v}$ and w expressed in basis $\{e_i\}$. This points out a one-to-one correspondence between $n \times n$ matrices and bilinear forms once a basis is specified.

A manifold M is said to be Riemannian if it has a field of symmetric, where $\phi(\mathbf{v}, \mathbf{w}) = \phi(\mathbf{w}, \mathbf{v})$, positive definite bilinear forms defined on it. In that case the bilinear from is called the Riemannian metric. A positive definite symmetric bilinear from $\phi(\mathbf{v}, \mathbf{v}) \geq 0$ equals zero only when $\mathbf{v} = 0$. Such a bilinear form is the inner product. The Riemannian metric is a symmetric positive definite bilinear form, hence it has an inner product defined on it. The length of a C^1 curve $p(t)$ on a Riemannian manifold stretching between $t = [a, b]$ can be defined via the inner product of its infinitesimal segments as

$$L = \int_a^b \left[\phi(\frac{dp}{dt}, \frac{dp}{dt})\right]^{\frac{1}{2}} dt.$$

Since the length is independent of the choice of parametrization one can use the arc length parametrization in a single coordinate chart (U, ϕ) with basis $\{e_{1p}, ...e_{np}\}$ such that $\phi(e_{ip}, e_{jp}) = g_{ij}(x)$ and $\phi(p(t)) = x(t) = (x^1(t), ..., x^n(t)) \in \mathbb{R}^n$ where $p \in U$ and $p = x^i e_i$. This way the length is [12]

$$s = L(t) = \int_a^t \left(g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt}\right)^{\frac{1}{2}} dt \tag{1.2}$$

This is usually interpreted as $(\frac{ds}{dt})^{\frac{1}{2}} = g_{ij}(x(t))\frac{dx^i}{dt}\frac{dx^j}{dt}$ and results in the following abbreviation

$$ds^2 = g_{ij}(x)dx^i dx^j \tag{1.3}$$

which gives the metric the notion of an interval of length.

So far we have been talking about Riemannian spaces. The metric signature is the number of positive and negative eigenvalues of the metric. For Euclidean and

Riemannian metrics the metric signs are all positive. When used in the sense of a Riemannian space Euclidean space does not mean that it is necessarily flat. But we will save the term for flat spaces. If the metric signature includes a single negative sign it is called a Lorentzian or pseudo-Riemannian metric and indefinite if it includes a number of negative and positive signs. The metric coefficients $g_{\mu\nu}$ will change depending on the choice of coordinates. The form in which $g_{\mu\nu} = diag(-1, .., -1, 1, .., 1, 0, .., 0)$ is called the canonical form of the metric. In its canonical form the first derivatives of the metric with respect to the coordinates vanish, the space is locally flat to first order. Yet the second derivatives of the metric with respect to the coordinates remain nonzero. It is these second derivatives that carry the information about the curvature of the manifold. If they vanish also, then the space is globally flat. The coordinate system that gives the canonical form is known as the locally inertial coordinates [13].

The discussion up until now has been about spaces of any dimension. We have not yet said anything about time. In Newtonian physics there is the three dimensional space and the parameter time. It was with Einstein that we began to perceive time not as a parameter but as a dimension on its own. Therefore in general relativity we have a four dimensional spacetime. Of course time is not the same kind of a dimension as space, it governs causality and we still measure changes in space with respect to time. Thus time should be introduced differently then space. As such the four dimensional manifolds of general relativity are Lorentzian manifolds and time is the dimension with a negative metric signature.

We have also been talking about manifolds being curved. We mentioned that the metric carries the information weather a certain manifold is curved or flat. But we have not yet said anything as to what may cause this curvature. The idea that lies at the heart of general relativity is that spacetime is curved because of its matter content. The matter content may be composed of pressure, p, and energy momentum density, ρ, which is expressed in terms of a symmetric $(2, 0)$ tensor, $T^{\mu\nu}$, the stress energy momentum tensor. If we assume the matter content to be free of stress and shear, like a perfect fluid, the off diagonal elements of $T^{\mu\nu}$ all vanish. Its diagonal elements are composed of p and ρ. The conservation of energy and momentum is expressed by the vanishing of its covariant derivative. As for curvature we have to go back to considering the metric.

Let us write down a four dimensional Lorentzian metric

$$ds^2 = g_{ab}dx^a dx^b = -dt^2 + a^2(t, x^k)dx^i dx^j \tag{1.4}$$

with $a, b = 1, 2, 3, 4,, \ i, j, k = 1, 2, 3$ and repeated indices are summed over unless otherwise noted. We can consider the coordinate system in which this metric is written to be

$$e^i = dx^i \tag{1.5a}$$

$$e^4 = dt \tag{1.5b}$$

then the nonzero metric coefficients are

$$g_{44} = -1 \tag{1.6a}$$

$$g_{ij} = a^2(t, x^k). \tag{1.6b}$$

Here we have to use the metric to raise and lower indices, for example $T^4{}_2 = T^{4a}g_{a2}$. On the other hand we can choose the coordinate frame to be,

$$e^i = a(t, x^k)dx^i \tag{1.7a}$$

$$e^4 = idt. \tag{1.7b}$$

In this case $g_{ab} = \delta_{ab}$ and it is easier to raise and lower indices. We will work in the second frame where the coordinates are called orthonormal basis one forms. In general differential forms are completely antisymmetric, $(0, p)$ tensors. A one form corresponds to a dual vector which transforms as $\tilde{\theta}_j = \Lambda^i{}_j \theta_i$ where vectors are $(1, 0)$ tensors that transform as $\tilde{V}^j = \Lambda^i{}_j V^i$, under some transformation $\Lambda^i{}_j$.

Curvature is a measure of how much a space, or in our case spacetime, deviates from a flat manifold. It is expressed by the Riemann curvature tensor, and is invariant of the choice of coordinates. If the components of Riemann curvature tensor $R_{\mu\nu\lambda\kappa}$, vanish in one frame, they vanish in all frames, and the manifold is flat. Otherwise the manifold is curved irrespective of the frame. There is a theorem [12] which states that for a given C^∞ family of coframes $e^1, ...e^n$ defined on a neighborhood U that cover a Riemannian manifold, there exists a uniquely determined set of C^∞ connection forms ${w^i}_j$ that satisfy

(i) $de^\mu + {w^\mu}_\lambda \wedge e^\lambda = 0$

(ii) ${w^\mu}_\nu + {w^\nu}_\mu = 0$

where the wedge product is an antisymmetric tensor product. Given a p form A and a q form B

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]} \tag{1.8}$$

with the square brackets denoting anti symmetrization. The curvature two forms ${\Omega^i}_j$ are defined as

$${\Omega^\mu}_\nu = d{w^\mu}_\nu + {w^\mu}_\kappa \wedge {w^\kappa}_\nu. \tag{1.9}$$

The components of the Riemann curvature tensor can be deduced from the curvature two forms by

$${\Omega^\mu}_\nu = \frac{1}{2} {R^\mu}_{\nu\lambda\kappa} e^\lambda \wedge e^\kappa. \tag{1.10}$$

Since the wedge product is antisymmetric in the indices λ and κ, ${R^\mu}_{\nu\lambda\kappa}$ is antisymmetric under the exchange of its third and fourth indices. ${\Omega^\mu}_\nu$ is antisymmetric due to the antisymmetry of ${w^\mu}_\nu$, therefore ${R^\mu}_{\nu\lambda\kappa}$ is also antisymmetric in its first and second indices. This method is known as the Cartan's formalism.

We have discussed how to express the curvature of a manifold. Now the question is how to relate the matter content with curvature. Einstein wanted to write down a covariant equation, whose form would not change from one frame to another. Tensors allow such a notation. The right hand side of the equation, governing the matter content, was obviously going to be $T_{\mu\nu}$, it was the left hand side, that expresses the curvature, which took more thought in the making. $T_{\mu\nu}$ is a tensor of type $(0,2)$, in order to keep covariance it should be equal to another tensor of the same type. The components of the Riemann tensor $R^{\lambda}{}_{\mu\nu\kappa}$, which is of type $(1,3)$, form the components of the Ricci tensor, $R_{\mu\nu}$ by contracting the first and fourth indices,

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\nu\lambda}. \tag{1.11}$$

The Ricci tensor is of type $(0,2)$, and Einstein first wrote down his equations as

$$R_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.12}$$

The conservation of energy momentum is expressed by the vanishing of the covariant derivative of $T_{\mu\nu}$,

$$\nabla_{\mu} T^{\mu}{}_{\nu} = 0. \tag{1.13}$$

However the covariant derivative of $R_{\mu\nu}$ is nonzero [13]. The left hand side of the equation should have a tensor whose covariant derivative also vanishes. Although $R_{\mu\nu}$ measures curvature, for physical reasons we need another tensor for formulation. This is the Einstein's tensor, $G_{\mu\nu}$, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{1.14}$$

where the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$, is the trace of the Ricci tensor.

Thus the geometry of a spacetime is engraved inside Einstein's tensor, while $T_{\mu\nu}$ expresses the pressure and energy momentum density within that spacetime. As such the phrase that spacetime is curved due to presence of matter, is formulated as

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.15}$$

These are the Einstein's field equations which will be put into use throughout the rest of this work. Later on Einstein, realizing that the equations imply a dynamical spacetime, added a constant to achieve static solutions,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} - g_{\mu\nu}\Lambda. \tag{1.16}$$

With Hubble's discovery of the linear expansion of the universe, it was shown that the universe is indeed dynamic, and Einstein removed this constant Λ. Today the cosmological constant Λ is viewed as a possible explanation of the dark energy.

1.2. Embedding Theorems

According to Campbell's theorem any analytic, n dimensional, Riemannian space can be locally embedded in an $(n+1)$ dimensional, Ricci flat, Riemannian space by adding either an extra space dimension or an extra time dimension. Therefore $n-$dimensional solutions of Einstein's equations to arbitrary energy-momentum tensors can at least locally be embedded to $(n+1)$ dimensional vacuum solutions of Einstein's equations. This, turning the argument other way around, implies that once we have an $(n+1)$ dimensional metric

$${}^{(n+1)}dS^2 = g_{ab}(x^c, L)dx^a dx^b + \epsilon h^2(x^c, L)dL^2 \tag{1.17}$$

that satisfies field equations in vacuum, we can obtain non empty spacetimes on hypersurfaces where the $(n+1)$th dimension is constant, $L = constant$. Here L refers to the extra dimension which can be spacelike, $\epsilon = +1$, or timelike $\epsilon = -1$. In our convention small case Latin indices run from 1 to 4 referring to four dimensional spacetime, and Greek indices run from 1 to 5 referring to five dimensional spacetime. The first three indices (123), also denoted by i or j, refer to xyz, 4 refers to time and 5 refers to internal space or the extra dimension. The locality of the theorem comes from the fact that the n dimensional metric and field equations we obtain by setting $L = constant$, restricts the situation around that hypersurface. We will not consider the global properties of the embedding. As an application, Lidsey et al. [14] state the set of conditions placed on the functional form of the higher dimensional metric coefficients and apply Campbell's theorem to embed four dimensional gravitational and electromagnetic plane waves to five dimensional Ricci flat spacetimes.

The n dimensional field equations are the $(n + 1)$ dimensional field equations on $L = constant$ hypersurfaces. As a less restricted version, the Campbell-Magaard theorem states that any n dimensional manifold can be locally embedded in an $(n+1)$ dimensional Einstein space. A modern and less rigorous version of the proof presents the field equations in terms of three symmetric, n dimensional tensors namely, the induced metric, extrinsic curvature and a tensor that resembles the components of the $(n + 1)$ dimensional curvature tensor outside of this n dimensional hypersurface. These field equations do not contain any change of these three tensors with respect to the internal dimension, meaning the same equations are to be satisfied on each $L = constant$ hypersurface. So in a sense the n dimensional field equations are actually constraint equations. Moreover the number of independent dynamical quantities, which are the elements of these tensors, are more than the number of field equations when there are at least two dimensions, which means there are more free variables then constrained ones. As such the line element on the $L = constant$ hypersurface can be chosen to correspond to any n dimensional Lorentzian manifold while still satisfying the constraint equations. Thus it is possible to embed any n dimensional manifold in an $(n + 1)$ dimensional Einstein space [15].

The Campbell's theorem and other embeddings related with it, discuss the embedding of curved spacetimes to Ricci flat spacetimes. Being Ricci flat means that there is no pressure or energy density present. However, in order to be completely flat the spacetime must also have a zero Weyl tensor in addition to a zero Ricci tensor. Just as a vanishing Ricci tensor indicates that the spacetime does not contain any matter in the usual sense, a vanishing Weyl tensor indicates the absence of gravitational fields. Among the four dimensional solutions of general relativity, FRW metrics are special in that they can be embedded to five dimensional flat spacetimes, flat with both a vanishing Ricci and a vanishing Weyl tensor.Most curved four dimensional solutions cannot be embedded to flat five dimensions [16]. However any solution of the field equations in four dimensions with no ordinary matter, where dark energy is allowed, can be expressed as a five dimensional metric with pure canonical form [3]. In the canonical metric the four dimensional metric ${}^4ds$, is multiplied by the square of the extra dimension and its metric coefficients are allowed to depend on the extra dimension

$$ {}^5dS^2 = \frac{L^2}{L_0^2}(g_{ab}(x^c, L)dx^a dx^b) \pm dL^2 \tag{1.18} $$

where L_0 is just a constant with dimension of length. On the other hand, in the pure canonical form the four dimensional metric tensor is independent of the extra dimension

$$ {}^5dS^2 = \frac{L^2}{L_0^2}(g_{ab}(x^c)dx^a dx^b) \pm dL^2. \tag{1.19} $$

In the case of FRW metrics it is stated, and applied for a few cases that, five dimensional Minkowski metric with zero spatial curvature, M_5, gives the complete FRW metric. It is cautioned that the geodesics for the hypersurface in M_5 appear as parabolas, contary to common intuition for flat spacetimes. This is because of the metric signature and the correct measure of the curvature clearly shows the flatness [17].

1.3. Space-Time-Matter and Braneworld Theories

The ability to embed a curved spacetime into a higher dimensional flat one brings on the possibility to interpret matter as a geometrical effect coming from a higher dimensional theory. According to general relativity spacetime is curved only in the presence of energy and momentum or gravitational fields. A Ricci flat universe means a vacuum universe in terms of energy and momentum, which amount to matter such as dust and radiation. Since we can embed a four dimensional curved spacetime, which is nonempty, in a Ricci flat five dimensional universe we can hold the extra dimension to be responsible of the material effects in four dimensions. This is the idea that lies at the heart of Induced Matter (IM) or Space-Time-Matter (STM) theory. The metric ansatz of Induced Matter theory is

$$^5ds^2 = e^{\nu(t,L)}dt^2 - e^{w(t,L)}(dr^2 + r^2 d\Omega^2) - e^{\mu(t,L)}dL^2 \tag{1.20}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. We write the above metric in the $(+,-,-,-,-)$ convention as it appears in Wesson [18]. The five dimensional Einstein equations for vacuum are

$$R_{\mu\nu} = 0, G_{\mu\nu} = 0$$

where as four dimensional field equations with matter are

$$G_{ab} = 8\pi T_{ab}.$$

Of course the five dimensional $R_{\mu\nu}$ includes terms that depend on the scale factor of the extra dimension, ν, and partial derivatives of the scale factors with respect to the extra dimension L. The four dimensional G_{ab} with a nonzero T_{ab}, corresponding to this five dimensional $G_{\mu\nu}$ in vacuum, is evaluated by collecting out these terms that arise because of the extra dimension. That is to say, these terms in $^{(5)}G_{44}$ are collected out as $-^{(4)}T_{44}$ and the rest are $^{(4)}G_{44}$ and those in $^{(5)}G_{ii}$ are collected out as $-^{(4)}T_{ii}$ and the rest are $^{(4)}G_{ii}$. In this point of view the five dimensional field equations are written as

$$^{(5)}R_{ii} =^{(4)} G_{ii} +^{(4)} T_{ii} = 0 \tag{1.21a}$$

$$^{(5)}R_{44} =^{(4)} G_{44} +^{(4)} T_{44} = 0 \tag{1.21b}$$

$$^{(5)}R_{55} = 0 \tag{1.21c}$$

To make it more clear, the field equations in five dimensions for metric (1.4) in Wesson's convention are

$$G^4{}_4 = e^{-\nu}(-\frac{3\dot{w}^2}{4} - \frac{3\dot{w}\dot{\mu}}{4}) + e^{-\mu}(\frac{3w''}{2} + \frac{3w'^2}{2} - \frac{3\mu' w'}{4}) \tag{1.22a}$$

$$G^4{}_5 = e^{-\nu}(\frac{3\dot{w}'}{2} + \frac{3\dot{w}w'}{4} - \frac{3\dot{w}\nu'}{4} - \frac{3w'\dot{\mu}}{4}) \tag{1.22b}$$

$$G^i{}_i = G^1{}_1 = -e^{-\nu}(\ddot{w} + \frac{3\dot{w}^2}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{w}\dot{\mu}}{2} - \frac{\dot{\nu}\dot{w}}{2} - \frac{\dot{\nu}\dot{\mu}}{4}) \tag{1.22c}$$

$$+e^{-\mu}(w'' + \frac{3w'^2}{4} + \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{w'\nu'}{2} - \frac{\mu' w'}{2} - \frac{\nu'\mu'}{4}) \tag{1.22d}$$

$$G^5{}_5 = -e^{-\nu}(\frac{3\ddot{w}}{2} + \frac{3\dot{w}^2}{2} - \frac{3\dot{\nu}\dot{w}}{4}) + e^{-\mu}(\frac{3w'^2}{4} + \frac{3w'\nu'}{4}). \tag{1.22e}$$

Here the dot denotes differentiation with respect to time and prime denotes differentiation with respect to internal space L. One obtains the apparent matter content of the four dimensional spacetime by collecting out the terms that appear because of the presence of the fifth dimension. Setting the elements of four dimensional energy-momentum tensor as ${}^{(4)}T^4{}_4 = \rho$ and ${}^{(4)}T^1{}_1 = -p$ gives

$$8\pi\rho \equiv -\frac{3}{4}e^{-\nu}\dot{w}\dot{\mu} + \frac{3}{2}e^{-\nu}(w'' + w'^2 - \frac{\mu' w'}{2}) \tag{1.23a}$$

$$8\pi p \equiv e^{-\nu}(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{w}\dot{\mu}}{2} - \frac{\dot{\nu}\dot{\mu}}{4}) \tag{1.23b}$$

$$-e^{-\mu}(w'' + \frac{3w'^2}{4} + \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{w'\nu'}{2} - \frac{\mu' w'}{2} - \frac{\nu'\mu'}{4}) \tag{1.23c}$$

This approach gives us two different ways to look at the same universe. It implies that we can view the universe to be either four dimensional and curved or as five dimensional and Ricci flat without matter.

STM starts out with the geometry of the bulk, which is a solution of five dimensional field equations in vacuum. Our four dimensional world is obtained from the five dimensional bulk by introducing the matter content as a geometric effect. The brane is evaluated from the bulk. In braneworld theories on the other hand, one starts out with a four dimensional brane containing matter, with a presupposed T_{ab} and brane tension which is related to the vacuum energy density, and arrives at the geometry of the bulk by imposing boundary conditions. The bulk effects the brane by a nonlocal Weyl radiation field. This corresponds to the $E_{\mu\nu}$ dependent extra terms that appear in the effective four dimensional G_{ab} due to the presence of internal space. These terms resemble radiation because $E_{\mu\nu}$ is a traceless tensor. STM wants to address matter as the effect of a higher dimensional geometry while BW scenarios attempt to solve the hierarchy problem. Although STM and BW theories arise from different motivations, they share important common features in terms of their working principles. First of all both allow for a nontrivial dependence on the internal space in the metric coefficients and do not force any compactification on the extra dimension. The relaxation of such constraints is what leads to a geometric interpretation of matter, as pointed out by Wesson and Ponce de Leon in related works. In both Scenarios the four dimensional metric, identified with our physical spacetime, is evaluated as a hypersurface in five dimensions by setting $L = constant$ in the solutions of the five dimensional field equations. The matter fields are confined to this hypersurface, or brane, as well as the observers who cannot enter the bulk.

At first sight the two theories are complementary ways of embedding a four dimensional world in a five dimensional one. Ponce de Leon [19] shows that both actually carry the same properties when examined throughly. The STM equations can be considered as the equations of gravity in a braneworld scenario of a Z_2 symmetric brane, where $-L$ is identified with L, with a certain matter content. Both theories arrive at the same effective matter in four dimensions. In this respect STM forms the generating space for braneworld scenarios and can be shown to include the local and nonlocal corrections to four dimensional gravity same as BW models. Therefore the two theories are equal and this equivalence can be turned into an advantage to overcome their shortcomings. In STM there are not enough physical restrictions to determine all of the arbitrary functions that arise in field equations. In BW theories the brane lacks of enough information for the reconstruction of the bulk. It is proposed to use the physics on the brane, coming from the BW point of view, to restrict the abundant freedom in STM.

Ponce de Leon introduces a normal vector orthogonal to spacetime for a metric of the form ${}^{5}dS^2 = g_{ab}(x^c, L)dx^a dx^b + \epsilon\Phi^2(x^c, L)dL^2$, with $\epsilon = \pm 1$ whether the internal space is timelike or spacelike. And writes down the four dimensional Einstein Tensor and conservation equation with the motivation of STM as

$$ {}^{(4)}G_{ab} = \epsilon[K^c{}_c K_{ab} - K_{ac}K^c{}_b + \frac{1}{2}g_{ab}(K_{de}K^{de} - (K^c{}_c)^2) - E_{ab}], \tag{1.24} $$

$$ \nabla_b P^b{}_a = 0. \tag{1.25} $$

Here $K_{ab} = \frac{1}{2\Phi}g'_{ab}$ is the previously mentioned extrinsic curvature, and $E_{ab} = \frac{{}^{(5)}R_{a5b5}}{\Phi^2}$ is the other curvature related symmetric tensor. Written in this form the dependence of the four dimensional Einstein tensor on the curvature of the five dimensional spacetime evaluated at a certain hypersurface is quite clear. Moreover the quantity P_{ab} is also expressible in terms of extrinsic curvature as, $P_{ab} = K_{ab} - g_{ab}K^d{}_d$.

2. THE METRIC AND THE EINSTEIN TENSOR

The Friedmann-Robertson-Walker metric has the following form

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2 \tag{2.1}$$

where $d\Sigma^2$ is the metric of three spacelike dimensions all of which have uniform curvature. We use natural units with $c = \hbar = 1$. The spacelike sections, being scaled by $a(t)$, may expand or contract in time. Therefore the scale factor $a(t)$ is what gives us the dynamics of this four dimensional spacetime. Because all three spatial dimensions have the same scale factor they all change by the same amount, hence this universe expands or contracts isotropically only with time. Here time is the proper time, which is what an observer who sees the universe expand around him measures as time. Since it doesn't have a factor dependent on any of the spacelike dimensions in front of it, it has the same value at every point. In other words the cosmological time is the proper time at every point in this spacetime. The role of time is fundamental here.

We will consider a metric of the form

$$ds^2 = f^2(t)g^2(w)[-dt^2 + dw^2] + a^2(t)b^2(w)\frac{dx^2 + dy^2 + dz^2}{(1 + \frac{\kappa(x^2+y^2+z^2)}{4})^2} \tag{2.2}$$

where κ is the curvature of spacelike sections with the values -1 for negatively curved, 0 for flat, $+1$ for positively curved, we can always make a coordinate transformation so that

$$dT = f(t)dt \tag{2.3a}$$

$$dW = g(w)dw \tag{2.3b}$$

$$ds^2 = -G^2(W)dT^2 + F^2(T)dW^2 + A^2(T)B^2(W)\frac{dx^2 + dy^2 + dz^2}{(1 + \frac{\kappa(x^2+y^2+z^2)}{4})^2}. \tag{2.4}$$

Here T may be called the cosmological time because it is the only coordinate that an observer will measure as time. But the value measured will change for different observers at different points in W, because we cannot get rid of the factor of W in front of time. We cannot get rid of the factor of time in front of W the internal space either. As such, the role of internal space in this five dimensional universe is as fundamental as the role time plays here. We will carry on our calculations in the coordinates where the metric is as it is in Equation 2.2.

The observable three spacelike dimensions share the same scale factor and are again isotropic. Here they do not evolve only in time but in w as well. Although our internal space, w, is a spacelike dimension, it works as a timelike extra dimension would.

Our basis one forms are

$$e^4 = if(t)g(w)dt, \quad i = \sqrt{-1} \tag{2.5a}$$

$$e^5 = f(t)g(w)dw \tag{2.5b}$$

$$e^i = a(t)b(w)\frac{dx^i}{1+\frac{\kappa r^2}{4}} \tag{2.5c}$$

and we use the metric $g_{\mu\nu} = diag(1,1,1,1,1)$ with $i = 1,2,3$. Using Cartan's formalism and leaving the details of the calculation to the appendix, we get the curvature two forms to be

$$\Omega^i{}_j = \left[\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} - \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} + \frac{\kappa}{a^2(t)b^2(w)}\right] e^i \wedge e^j \tag{2.6}$$

$$\Omega^i{}_4 = \left[\frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)} - \frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} + \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)}\right] e^4 \wedge e^i$$

$$+ \left[\frac{\dot{a}(t)b'(w)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)}\right] e^5 \wedge e^i \tag{2.7}$$

$$\Omega^i{}_5 = \left[\frac{b'(w)\dot{a}(t)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)}\right] e^4 \wedge e^i$$

$$+ \left[\frac{b''(w)}{b(w)f^2(t)g^2(w)} - \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)} - \frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)}\right] e^5 \wedge e^i \tag{2.8}$$

$$\Omega^4{}_5 = \left[-\frac{\dot{f}(t)^2}{f^4(t)g^2(w)} + \frac{\ddot{f}(t)}{f^3(t)g^2(w)} + \frac{g'(w)^2}{g^4(w)f^2(t)} - \frac{g''(w)}{g^3(w)f^2(t)}\right] e^4 \wedge e^5 \tag{2.9}$$

where differentiation with respect to w and t are denoted as

$$\dot{h} = \frac{\partial h}{\partial t}$$

$$h' = \frac{\partial h}{\partial w}.$$

We get the Riemann tensor $R_{\mu\nu\lambda x}$ from curvature two forms by

$$\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\lambda x} e^\lambda \wedge e^x$$

and the components of our Einstein tensor by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

where R is the Ricci Scalar $R = g^{\mu\nu}R_{\mu\nu}$. All this gives us the following

$$G_{ii} = -2\frac{b''(w)}{b(w)f^2(t)g^2(w)} + 2\frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} + \frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} - \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)}$$

$$+\frac{\kappa}{a^2(t)b^2(w)} - \frac{\dot{f}^2(t)}{f^4(t)g^2(w)} + \frac{\ddot{f}(t)}{f^3(t)g^2(w)} + \frac{g'^2(w)}{f^2(t)g^4(w)} - \frac{g''(w)}{f^2(t)g^3(w)} \tag{2.10}$$

$$G_{44} = 3\frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)} + 3\frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)} - 3\frac{b''(w)}{b(w)f^2(t)g^2(w)}$$

$$+3\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} - 3\frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} + 3\frac{\kappa}{a^2(t)b^2(w)} \tag{2.11}$$

$$G_{55} = 3\frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} + 3\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} + 3\frac{\kappa}{a^2(t)b^2(w)}$$

$$-3\frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)} - 3\frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} - 3\frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)} \tag{2.12}$$

$$G_{54} = 3[\frac{b'(w)\dot{a}(t)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)}] \tag{2.13}$$

3. VACUUM SOLUTIONS IN FIVE DIMENSIONS

Now let us consider the vacuum solutions for flat spacelike sections, that is solutions to $G_{\mu\nu} = 0$ with $\kappa = o$.

From $G_{ii} = 0$ we get

$$2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} - \frac{\dot{f}^2(t)}{f^2(t)} + \frac{\ddot{f}(t)}{f(t)} = 2\frac{b''(w)}{b(w)} + \frac{b'^2(w)}{b^2(w)} - \frac{g'^2(w)}{g^2(w)} + \frac{g''(w)}{g(w)} \tag{3.1}$$

The right hand side of this equation is purely $w-$dependent, and the left hand side purely $t-$dependent. The only way these two sides are equal to one another is if they are equal to the same constant k. Thus out of G_{ii} we get the following two equations

$$2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} - \frac{\dot{f}^2(t)}{f^2(t)} + \frac{\ddot{f}(t)}{f(t)} = k \tag{3.2}$$

and

$$2\frac{b''(w)}{b(w)} + \frac{b'^2(w)}{b^2(w)} - \frac{g'^2(w)}{g^2(w)} + \frac{g''(w)}{g(w)} = k \tag{3.3}$$

With the same reasoning we get from $G_{44} = 0$

$$\frac{\dot{a}(t)\dot{f}(t)}{a(t)f(t)} + \frac{\dot{a}^2(t)}{a^2(t)} = l \tag{3.4}$$

$$\frac{b''(w)}{b(w)} + \frac{b'^2(w)}{b^2(w)} - \frac{b'(w)g'(w)}{b(w)g(w)} = l \tag{3.5}$$

and from $G_{55} = 0$

$$\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} - \frac{\dot{a}(t)\dot{f}(t)}{a(t)f(t)} = m \tag{3.6}$$

$$\frac{b'^2(w)}{b^2(w)} + \frac{b'(w)g'(w)}{b(w)g(w)} = m \tag{3.7}$$

Thus we have two sets of equations, one set related to t and the other related to w. We will solve these two sets first and check whether the solutions satisfy $G_{54} = 0$, which gives

$$1 - \frac{a(t)\dot{f}(t)}{\dot{a}(t)f(t)} = \frac{g'(w)b(w)}{g(w)b'(w)} = constant. \tag{3.8}$$

Let's first look at the set related to t, whose solution will give us $a(t)$ and $f(t)$

$$2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} - \frac{\dot{f}^2(t)}{f^2(t)} + \frac{\ddot{f}(t)}{f(t)} = k \tag{3.9}$$

$$\frac{\dot{a}(t)\dot{f}(t)}{a(t)f(t)} + \frac{\dot{a}^2(t)}{a^2(t)} = l \tag{3.10}$$

$$\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} - \frac{\dot{a}(t)\dot{f}(t)}{a(t)f(t)} = m \tag{3.11}$$

We can get an equation for $a(t)$ by adding the last two equations,

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} = m + l. \tag{3.12}$$

If we consider a solution of the form $a(t) = a_0 e^{\nu t}$ and plug this in Equation 3.12 we get

$$a(t) = a_0 \exp\left[\sqrt{\frac{(m+l)}{3}}t\right]. \tag{3.13}$$

By imposing this solution on Equation 3.11 we obtain

$$f(t) = f_0 \exp\left[\frac{2l - m}{\sqrt{3(m+l)}} t\right] \tag{3.14}$$

When the solutions in Equation 3.14 and Equation 3.13 are inserted into Equations 3.9, 3.10, 3.11 we find that Equation 3.10 and Equation 3.11 are satisfied identically where as Equation 3.9 imposes the condition

$$m + l = k. \tag{3.15}$$

A similar approach to the w related set of equations,

$$2\frac{b''(w)}{b(w)} + \frac{b'^2(w)}{b^2(w)} - \frac{g'^2(w)}{g^2(w)} + \frac{g''(w)}{g(w)} = k \tag{3.16}$$

$$\frac{b''(w)}{b(w)} + \frac{b'^2(w)}{b^2(w)} - \frac{b'(w)g'(w)}{b(w)g(w)} = l \tag{3.17}$$

$$\frac{b'^2(w)}{b^2(w)} + \frac{b'(w)g'(w)}{b(w)g(w)} = m, \tag{3.18}$$

gives

$$b(w) = b_0 \exp\left[\sqrt{\frac{(m+l)}{3}} w\right] \tag{3.19}$$

and

$$g(w) = g_0 \exp\left[\frac{\sqrt{3}(2m - l)}{\sqrt{m+l}} w\right] \tag{3.20}$$

where Equation 3.16 imposes the same condition $m + l = k$. Moreover our solutions imply that

$$k = m + l \geq 0 \tag{3.21}$$

since they each contain a $\sqrt{(m+l)}$ term. With these solutions $G_{54} = 0$ is satisfied as well.

Thus the vacuum solutions of our five dimensional metric with flat spacelike sections are

$$ds^2 = f_0^2 g_0^2 exp\left[\frac{4l-2m}{\sqrt{3(m+l)}}t + \frac{4m-2l}{\sqrt{3(m+l)}}w\right](-dt^2 + dw^2)$$

$$+a_0^2 b_0^2 exp\left[2\sqrt{\frac{m+l}{3}}(t+w)\right][dx^2 + dy^2 + dz^2] \tag{3.22}$$

By redefining parameters

$$M_1 = \frac{2l-m}{\sqrt{3(m+l)}}, \tag{3.23a}$$

$$M_2 = \frac{2m-l}{\sqrt{3(m+l)}} \tag{3.23b}$$

and rescaling coordinates we can write our metric in its simplest form as

$$ds^2 = e^{2(M_1 t + M_2 w)}[-dt^2 + dw^2] + e^{2(M_1+M_2)(t+w)}[dx^2 + dy^2 + dz^2] \tag{3.24}$$

4. THE EFFECTIVE FOUR DIMENSIONAL SOLUTION

We will now consider the above solution of the vacuum five dimensional spacetime at some $w = w_0$ where w_0 is a constant. Such a way of considering four dimensional hypersurfaces along constant internal space amounts to local embedding of four dimensional spacetimes into five dimensions. At $w = w_0$ spacetime metric becomes

$$ds^2 = f_0^2 g_0^2 exp[\frac{4m-2l}{\sqrt{3(m+l)}} w_0] exp[\frac{4l-2m}{\sqrt{3(m+l)}} t](-dt^2)$$

$$+a_0^2 b_0^2 exp[2\sqrt{\frac{m+l}{3}} w_0] exp[2\sqrt{\frac{m+l}{3}} t][dx^2 + dy^2 + dz^2] \tag{4.1}$$

$f_0 g_0 exp[\frac{2m-l}{\sqrt{3(m+l)}} w_0]$ is just a constant so we can set it equal to another constant F_0. With

$$F_0 = f_0 g_0 exp[\frac{2m-l}{\sqrt{3(m+l)}} w_0],$$

$$A_0 = a_0 b_0 exp[\sqrt{\frac{m+l}{3}} w_0]$$

we can write our solution as

$$ds^2 = -F_0^2 exp[\frac{4l-2m}{\sqrt{3(m+l)}} t] dt^2 + A_0^2 exp[2\sqrt{\frac{m+l}{3}} t][dx^2 + dy^2 + dz^2]. \tag{4.2}$$

To write this in terms of the cosmological proper time consider the following coordinate transformation

$$d\tilde{t} = F_0 exp[\frac{2l-m}{\sqrt{3(m+l)}} t] dt. \tag{4.3}$$

To simplify the notation we will define

$$\beta = \sqrt{\frac{m+l}{3}},$$

and

$$\alpha = \frac{3\beta}{2l-m}.$$

With all this our coordinate transformation gives,

$$\tilde{t} = F_0 \alpha e^{[\frac{t}{\alpha}]} \tag{4.4}$$

and

$$e^{2\beta t} = (\frac{\tilde{t}}{F_0 \alpha})^{2\beta\alpha}. \tag{4.5}$$

This coordinate transformation has turned our solution into

$$ds^2 = -d\tilde{t}^2 + A_0^2 \tilde{t}^{2\alpha\beta}[dx^2 + dy^2 + dz^2]. \tag{4.6}$$

We can always absorb A_0 into $\vec{r}$ by a coordinate transformation. So if we drop the tilde, define $\alpha\beta = n$ our metric in its simplest form becomes

$$ds^2 = -dt^2 + t^{2n}[dx^2 + dy^2 + dz^2]. \tag{4.7}$$

The metric in Equation 4.7 contains all the relevant four dimensional cosmologies with ordinary matter. For $n = \frac{2}{3}$ we have matter dominated universe, for $n = \frac{1}{2}$ we have radiation dominated universe.

Furthermore by setting $m = 2l$ in Equation 4.1 we get

$$ds^2 = f_0^2 g_0^2 e^{2\sqrt{l}w_0}[-dt^2] + a_0^2 b_0 e^{2\sqrt{l}w_0} e^{2\sqrt{l}t}[dx^2 + dy^2 + dz^2]. \tag{4.8}$$

Before explaining what we have obtained let us simplify this metric further first. The factor $e^{2\sqrt{l}w_0}$ is just a constant which can be set to c_0^2. We can also absorb all the constants into dt^2 by the coordinate transformation,

$$d\tau = f_0 g_0 c_0 dt$$

$$\frac{\tau - \tau_0}{f_0 g_0 c_0} = t \tag{4.9}$$

and define $a_0 b_0 c_0 exp[-\frac{\tau_0}{f_0 g_0 c_0}] = A_0^2$ so that we have

$$ds^2 = -d\tau^2 + A_0^2 e^{[\frac{2\sqrt{l}}{f_0 g_0 c_0}\tau]} d\vec{r}^2. \tag{4.10}$$

Let us denote τ by t and set $\alpha = \frac{\sqrt{l}}{f_0 g_0 c_0}$, the constant A_0 can also be absorbed into $d\vec{r}$

$$ds^2 = -dt^2 + e^{2\alpha t} d\vec{r}^2. \tag{4.11}$$

Thus we have obtained an exponential scale factor, a behavior attributed to dark energy with $\alpha = H_0$ where H_0 is approximately today's value of Hubble's parameter.

As such we have shown how it is possible to obtain all relevant four dimensional cosmologies with radiation, matter, inflation and dark energy from our five dimensional metric. Of course each case corresponds to different values of the parameters and we are not yet able to switch from one case to another.

5. THE CURVATURE AND WEYL TENSORS

So far we have arrived at a five dimensional spacetime whose four dimensional hypersurfaces correspond to relevant cosmologies. At this point it is important to consider the flatness of the five dimensional model to gain further insight. Therefore we will now calculate the Ricci tensor, which carries information about the ordinary matter content of the universe, and the Weyl tensor, which informs of the presence of gravitational fields. A zero Ricci tensor corresponds to a Ricci flat metric, and a vanishing Weyl tensor corresponds to a conformally flat metric. A flat metric is the one that is both Ricci flat and conformally flat.

Components of the Weyl tensor in our convention of Ricci tensor $R_{\nu\lambda} = R^{\mu}{}_{\nu\lambda\mu}$, metric sign $(-,+,+,+)$, are calculated as

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \frac{1}{d-2}(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho})$$

$$-\frac{1}{(d-1)(d-2)}(g_{\rho\mu}g_{\nu\sigma} + g_{\rho\nu}g_{\mu\sigma})R \tag{5.1}$$

where d is the number of dimensions.

For our five dimensional solution, in Equation 3.22,

$$R_{ijij} = [f_0^2 g_0^2 exp(\frac{4l-2m}{\sqrt{3(m+l)}}t + \frac{4m-2l}{\sqrt{3(m+l)}}w)]^{-1}(\frac{m+l}{3} - \frac{m+l}{3}) = 0$$

$$R_{i44i} = [f_0^2 g_0^2 exp(\frac{4l-2m}{\sqrt{3(m+l)}}t + \frac{4m-2l}{\sqrt{3(m+l)}}w)]^{-1}(\sqrt{\frac{m+l}{3}}\frac{l+m}{\sqrt{3(m+l)}} - \frac{m+l}{3}) = 0$$

$$R_{i45i} = R_{i54i} = \frac{1}{if^2(t)g^2(w)}[\frac{m+l}{3} - \sqrt{\frac{m+l}{3}}\frac{m+l}{\sqrt{3(m+l)}}] = 0$$

$$R_{i55i} = \frac{1}{f^2(t)g^2(w)}[\frac{m+l}{3} - \sqrt{\frac{m+l}{3}}\frac{m+l}{\sqrt{3(m+l)}}] = 0$$

$$R_{4545} = \frac{1}{f^2(t)g^2(w)}[-\frac{\dot{f}^2}{f^2} + \frac{\ddot{f}}{f} + \frac{g'^2}{g^2} - \frac{g''}{g}] = 0$$

all the components of Riemann curvature tensor are zero. Therefore the Ricci Scalar, all components of $R_{\mu\nu}$, and the Weyl tensor for the Ricci flat five dimensional metric are all zero. Our five dimensional universe is Ricci flat, meaning it contains no energy nor momentum density, and conformally flat, it doesn't contain any gravitational fields either, in short it is flat and empty.

The Ricci flatness of the metric does not guarantee that it will be conformally flat. It is possible to have Ricci flat solutions with nonzero $R_{\rho\sigma\mu\nu}$. Our universe turned out to be conformally flat because all of its $R_{\rho\sigma\mu\nu}$ vanish.

It is a well established fact that the Friedmann-Robertson-Walker (FRW) metric can be put in a conformally flat form [20,21]. It has been further pointed out that [22,23] calculations on the age of the universe and its matter density carried out in conformally flat spacetime (CFS) coordinates agree better with the observations then those carried out in FRW coordinates. With such emphasis on the conformal flatness of our universe, it is an achievement to be able to embed standard four dimensional conformally flat cosmology in a five dimensional flat spacetime in this work on higher dimensional cosmologies.

6. PHYSICAL IMPLICATIONS

6.1. Transformations involving internal space and time

So far we have obtained the following metric for a flat five dimensional universe

$$ds^2 = e^{2(M_1 t + M_2 w)}[-dt^2 + dw^2] + e^{2(M_1+M_2)(t+w)}[dx^2 + dy^2 + dz^2]. \qquad (6.1)$$

We have seen that we can derive all relevant four dimensional cosmological solutions from this metric at some $w = constant$ slice, by adjusting the free parameters M_1 and M_2.

We wish to consider SO(1,1) transformations of the t and w coordinates which leave $-dt^2 + dw^2$ interval invariant. That is,

$$-dt^2 + dw^2 = -d\tilde{t}^2 + d\tilde{w}^2 \qquad (6.2)$$

which is the usual Lorentz transformation with a parameter α, a boost along w where t and w are transformed as

$$\tilde{t} = (cosh\alpha)t + (sinh\alpha)w \qquad (6.3a)$$

$$\tilde{w} = (sinh\alpha)t + (cosh\alpha)w. \qquad (6.3b)$$

Of course we would like to express the general parameter α in terms of the parameters of our metric. The hyperbolic functions are obliged to satisfy the following identity

$$(cosh\alpha)^2 - (sinh\alpha)^2 = 1. \qquad (6.4)$$

If we define $cosh\alpha$ and $sinh\alpha$ as

$$cosh\alpha = \frac{M_1}{\sqrt{(M_1^2 - M_2^2)}} \tag{6.5a}$$

$$sinh\alpha = \frac{M_2}{\sqrt{(M_1^2 - M_2^2)}} \tag{6.5b}$$

the identity is satisfied. Therefore the rapidity for our spacetime, defined in terms of the parameters that appear in our metric is $\alpha = cosh^{-1}[\frac{M_1}{\sqrt{(M_1^2-M_2^2)}}]$. As such t and w expressed in terms of $\tilde{t}$ and $\tilde{w}$ is

$$t = (cosh\alpha)\tilde{t} - (sinh\alpha)\tilde{w} = \frac{M_1}{\sqrt{(M_1^2 - M_2^2)}}\tilde{t} - \frac{M_2}{\sqrt{(M_1^2 - M_2^2)}}\tilde{w} \tag{6.6a}$$

$$w = -(sinh\alpha)\tilde{t} + (cosh\alpha)\tilde{w} = -\frac{M_2}{\sqrt{(M_1^2 - M_2^2)}}\tilde{t} + \frac{M_1}{\sqrt{(M_1^2 - M_2^2)}}\tilde{w} \tag{6.6b}$$

This transformation effects the scale factor of $[-dt^2 + dw^2]$ as

$$M_1 t + M_2 w = \sqrt{(M_1^2 - M_2^2)}\tilde{t} \tag{6.7}$$

and the scale factor of three space as

$$t + w = \sqrt{(M_1^2 - M_2^2)}(\tilde{t} + \tilde{w}). \tag{6.8}$$

Thus the metric $ds^2 = e^{2(M_1 t + M_2 w)}[-dt^2 + dw^2] + e^{2(M_1+M_2)(t+w)}[dx^2 + dy^2 + dz^2]$ transforms into

$$ds^2 = e^{2\sqrt{(M_1^2-M_2^2)}\tilde{t}}[-d\tilde{t}^2 + d\tilde{w}^2] + e^{2\sqrt{(M_1^2-M_2^2)}(\tilde{t}+\tilde{w})}[dx^2 + dy^2 + dz^2]. \tag{6.9}$$

Apparently we can remove the $w-$dependent part of the scale factor in front of $[-dt^2 + dw^2]$ by a boost along w. The scale factor of three spatial dimensions which depends on $(t + w)$ continues to do so as in the form of $(\tilde{t} + \tilde{w})$ with only a change in the coefficient, hence $(t + w)$ is a lightlike coordinate.

We can redefine $M_1^2 - M_2^2$ as μ^2 and write this metric as

$$ds^2 = e^{2\mu\tilde{t}}[-d\tilde{t}^2 + d\tilde{w}^2] + e^{2\mu(\tilde{t}+\tilde{w})}[dx^2 + dy^2 + dz^2]. \tag{6.10}$$

This is as if we have set M_2=0 via a transformation. At this point we would like to point out that setting the parameters M_1 and M_2 to certain values amounts to choosing different frames. These frames aren't all equivalent because we will pick out one of them to be the cosmological frame, whose time dimension will be the time referred to as cosmological time. The choice is the one in which the scale factor of time is unity. This frame is among those where $M_2 = 0$ because the scale factor of time here, as in metric of Equation 6.10, can be set to one by the following coordinate transformation

$$d\tau = e^{\mu\tilde{t}}d\tilde{t} \tag{6.11}$$

which makes $e^{\mu\tilde{t}} = \mu\tau$. We will drop the tilde on w from now on and write the metric in these coordinates

$$ds^2 = -d\tau^2 + \mu^2\tau^2 dw^2 + \mu^2\tau^2 e^{2\mu w}[dx^2 + dy^2 + dz^2]. \tag{6.12}$$

As such the dimensions of $[\tau]$, $[w]$ and the three space coordinates $[x]$,$[y]$,$[z]$ are all equal to length where as, $[\mu w]$, being the variable of the exponential function, is dimensionless. This form of the metric is appropriate as far as the dimensions are concerned. τ is the cosmological time and we pick this frame as the cosmological frame. In time this universe expands linearly and it does not contain dark energy. So in a sense our choice of the cosmological frame, is the simplest cosmological case. On the other hand we were able to obtain dark energy for the four dimensional slice from metric in Equation6.1 by setting $M_1 = 0$, in chapter 4. As we have argued, metric in Equation 6.1 and in Equation 6.12 are different frames, the time coordinate in one is not the same as the time coordinate of the other. If there is a physical reason for us to choose the time of Equation 6.1 as the cosmological time, it may be the dark energy.

6.2. A test particle moving along w in the cosmological frame

We will continue on considering the frame where the metric is of the form written in Equation 6.12. This time we are concerned about the geodesics of test particles.

$$ds^2 = -d\tau^2 + \mu^2\tau^2 dw^2 + \mu^2\tau^2 e^{2\mu w}[d\vec{x}^2] \tag{6.13}$$

where $d\vec{x}^2 = dx^2 + dy^2 + dz^2$. The action of the particle is expressed as $\mathfrak{S} = -\int ds = \int d\tau \mathcal{L}$. Proceeding in this direction

$$ds^2 = -d\tau^2[1 - M_1^2\tau^2\dot{w}^2 - \mu^2\tau^2 e^{2\mu w}\dot{\vec{x}}^2] \tag{6.14}$$

here dot refers to differentiation with respect to τ. Due to our $(-++++)$ choice of the metric signature $ds = \sqrt{-ds^2}$, which gives,

$$ds = d\tau[1 - \mu^2\tau^2\dot{w}^2 - \mu^2\tau^2 e^{2\mu w}\vec{x}^2]^{\frac{1}{2}}. \tag{6.15}$$

For small values of $\mu^2\tau^2\dot{w}^2$ and $\mu^2\tau^2 e^{2\mu w}\dot{\vec{x}}^2$ we can approximate this as

$$ds \simeq d\tau[1 - \frac{1}{2}\mu^2\tau^2\dot{w}^2 - \frac{1}{2}\mu^2\tau^2 e^{2\mu w}\dot{\vec{x}}^2]. \tag{6.16}$$

Thus

$$\mathfrak{S} = -\int ds = \int[-1 + \frac{1}{2}\mu^2\tau^2\dot{w}^2 + \frac{1}{2}\mu^2\tau^2 e^{2\mu w}\dot{\vec{x}}^2]d\tau = \int \mathcal{L} d\tau \tag{6.17}$$

and

$$\mathcal{L} = -1 + \frac{1}{2}\mu^2\tau^2\dot{w}^2 + \frac{1}{2}\mu^2\tau^2 e^{2\mu w}\dot{\vec{x}}^2. \tag{6.18}$$

Let us consider the equations of motion for $\dot{\vec{x}} = 0$, where the x, y, z coordinates of the particle are held fixed.

$$\frac{d}{d\tau}(\frac{\partial\mathcal{L}}{\partial\dot{w}}) = \frac{\partial\mathcal{L}}{\partial w}$$

$$\frac{d}{d\tau}(M_1^2\tau^2\dot{w}) = M_1^3\tau^2 e^{2M_1w}\dot{\vec{x}}^2$$

using the constraint on $\vec{x}$

$$\frac{d}{d\tau}(\mu^2\tau^2\dot{w}) = 0$$

$$\mu^2\tau^2\dot{w} = constant = \mathfrak{C} \tag{6.19}$$

We can solve this as

$$\frac{dw}{d\tau} = \frac{\mathfrak{C}}{\mu^2\tau^2}$$

$$dw = \frac{\mathfrak{C}}{\mu^2}\frac{d\tau}{\tau^2}$$

$$w = w_0 - \frac{\mathfrak{C}}{\mu^2\tau} \tag{6.20}$$

This is the geodesic for a particle moving along the extra dimension in time. Of course this solution is an approximation for small $\dot{w}$. In the limit $\tau \to \infty$ the test particle moving along internal space approaches a certain slice w_0. In a sense, in time the test particles become confined to w_0, as far as their motion along internal space is concerned, and they can never go further behind.

6.3. A braneworld scenario

Let us go back to consider the Lagrangian with the small $\mu^2\tau^2\dot{w}^2$ and $\mu^2\tau^2e^{2\mu w}\dot{\vec{x}}^2$ approximation,

$$\mathcal{L} = -1 + \frac{1}{2}\mu^2\tau^2\dot{w}^2 + \frac{1}{2}\mu^2\tau^2e^{2M_1w}\dot{\vec{x}}^2. \tag{6.21}$$

The $\mu^2\tau^2$ coefficients of $\dot{w}$ and $\dot{\vec{x}}$ are due to linear expansion of the universe. The -1 term is to be interpreted as some potential and velocity squared terms, $\dot{w}^2$ and $\dot{\vec{x}}^2$, as kinetic energy. In this respect, particles moving along w have unit mass where as those moving along the three spatial dimensions have the mass $e^{2\mu w}$ related to w. It is as if particles along three space gain mass via a contribution coming from the internal dimension. The mass $e^{2\mu w}$ will increase as w becomes more and more positive and decrease as w becomes negative. If we are to relate mass with w, it is more convenient to have the mass increase as the particle travels further along w. This means something should prevent the internal dimension stretching in the negative direction. With this motivation in mind we rewrite our metric as

$$ds^2 = -d\tau^2 + \mu^2\tau^2dw^2 + \mu^2\tau^2e^{2\mu|w|}d\vec{x}^2. \tag{6.22}$$

Braneworld scenarios start with the introduction of Z_2 symmetry, which identifies a dimension streching in the negative direction $(-w)$, with its other half streching in the positive direction (w). Our inclusion of absolute value in the metric has the exact effect. Here we will write down the Einstein tensor for this metric right away, leaving the calculations for the appendix. With the following as the nonzero components of Riemann curvature tensor

$$R_{ii} = 2\frac{\delta(w)}{\mu\tau^2} \tag{6.23a}$$

$$R_{55} = 6\frac{\delta(w)}{\mu\tau^2}, \tag{6.23b}$$

the nonzero components of the Einstein tensor are

$$G_{ii} = -4\frac{\delta(w)}{\mu\tau^2} \tag{6.24a}$$

$$G_{44} = -6\frac{\delta(w)}{\mu\tau^2} \tag{6.24b}$$

where index i refers to (x, y, z), 4 refers to τ and 5 refes to w. The restriction of internal space, w, to be positive has created curvature only along the four dimensional spacetime, and this curvature is related to the fifth dimension. $R_{\mu\nu}$, with only spatial and internal space elements being nonzero, is well adjusted to give zero pressure along the fifth dimension with nonzero energy density and pressure along the four dimensions. Since $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$, we see that

$$G_{ii} = -\kappa p \rightarrow p = 4\frac{\delta(w)}{\kappa\mu\tau^2} \tag{6.25a}$$

$$G_{44} = \kappa\rho \rightarrow \rho = -6\frac{\delta(w)}{\kappa\mu\tau^2} \tag{6.25b}$$

$$G_{55} = -\kappa q \rightarrow q = 0 \tag{6.25c}$$

where κ is the five dimensional gravitational constant which is positive. The energy momentum tensor, $T^\mu{}_\nu$, turned out to be proportional to $\delta(w)$ because of the absolute value involved in the metric, which gives rise to the dirac delta via its second derivative, while physically refraining w from becoming negative. So the pressure and energy density we have along the four dimensional spacetime is confined to the $w = 0$ slice. The equation of state for our spacetime is

$$\frac{p}{\rho} = -\frac{2}{3} \tag{6.26}$$

which is the equation of state for a cosmic wall [24, 25]. Our restriction of w to be positive actually meant the introduction of a brane at $w = 0$. We now see that by including a brane in our five dimensional universe we have restricted ourselves to live on that brane. Moreover the bulk of this universe is empty since both p and ρ are proportional to $\delta(w)$ and q is zero.

Notice that unless the constant μ is negative the energy momentum density of this universe is going to be negative which can be problematic. By setting μ to be negative not only do we avoid such possible problems, but we can also achieve a compactification scheme. Therefore let us redefine $\mu = -M < 0$, to obtain the metric

$$ds^2 = -d\tau^2 + M^2\tau^2[dw^2 + e^{-2M|w|}d\vec{x}^2]. \tag{6.27}$$

Now as one moves along the internal space, away from the $w = 0$ brane, the exponential scale factor tends to zero and the three dimensional space shrinks. In fact it disappears by approaching zero in the limit where $w \to \pm\infty$. In this respect we can say that we are not aware of this extra dimension w, because we are confined to the brane. We are confined to the brane because there is no energy density nor pressure outside the brane, moreover the three dimensional space is too small away from the brane for us to be a part of. So in a sense away from the the brane this spacetime is effectively two dimensional with only τ and w. Usually one of the reasons given in explaining the absence of observation of the extra dimension is to suggest that it is too small. Contrary to that convention here we are suggesting that the three dimensional space is too small away from the brane. To make this shrinking of the three dimensional space as small as possible, the value of M should be chosen to be large. This braneworld universe is neither Ricci flat nor conformally flat. It contains both energy momentum density and gravitational fields.

As a last remark we would also like to consider the geodesics of this brane world with the metric in form

$$ds^2 = -d\tau^2 + \mu^2\tau^2 dw^2 + \mu^2\tau^2 e^{2\mu|w|}d\vec{x}^2. \tag{6.28}$$

From $\mathfrak{S} = -\int ds = \int d\tau \mathcal{L}$,

$$\mathcal{L} = -[1 - \mu^2\tau^2\dot{w}^2 - \mu^2\tau^2 e^{2\mu|w|}\dot{\vec{x}}^2]^{\frac{1}{2}}. \tag{6.29}$$

If we again consider the spatial coordinates x, y, z to be fixed, the equations of motion for w, $\frac{d}{d\tau}(\frac{\partial\mathcal{L}}{\partial\dot{w}}) = \frac{\partial\mathcal{L}}{\partial w}$, give

$$\frac{d}{d\tau}(\frac{\mu^2\tau^2\dot{w}}{\sqrt{1-\mu^2\tau^2\dot{w}^2}}) = 0. \tag{6.30}$$

This means

$$\frac{\mu^2\tau^2\dot{w}}{\sqrt{1-\mu^2\tau^2\dot{w}^2}} = \mathcal{C} = constant$$

$$\dot{w} = \frac{\mathcal{C}}{\mu^2\tau^2(\mu^2\tau^2+\mathcal{C}^2)}$$

$$w - w_0 = -\frac{1}{\mu}ln|\frac{\frac{\mathcal{C}}{\mu}+\sqrt{\tau^2+\frac{\mathcal{C}^2}{\mu^2}}}{\tau}| \tag{6.31}$$

If we are to calculate the acceleration of an object falling along the z direction, with x, y, w held fixed we have to solve the following equation for $\ddot{z}$

$$\frac{d}{d\tau}(\frac{\mu^2\tau^2e^{2\mu|w|}\dot{z}}{\sqrt{1-\mu^2\tau^2e^{2\mu|w|}\dot{z}^2}}) = 0$$

which gives

$$\ddot{z} = -2\frac{\dot{z}}{\tau} + \mu^2\tau e^{2\mu|w|}\dot{z}^3. \tag{6.32}$$

We can always absorb the μ in front of $d\vec{x}^2$ into $\vec{x}^2$ by rescaling x, y, z

$$ds^2 = -d\tau^2 + \mu^2\tau^2dw^2 + \tau^2e^{2\mu|w|}d\vec{x}^2. \tag{6.33}$$

Now consider μ set equal to 0,

$$ds^2 = -d\tau^2 + \tau^2 d\vec{x}^2 \tag{6.34}$$

and consider the acceleration along z,

$$\ddot{z} = -2\frac{\dot{z}}{\tau} \tag{6.35}$$

This means

$$ln\dot{z} = -2ln\tau + \mathfrak{a}_{\mathfrak{o}}$$

$$\dot{z} = \frac{\mathfrak{a}}{\tau^2}$$

$$z = z_0 - \frac{\mathfrak{a}}{\tau}$$

where $\mathfrak{a}_{\mathfrak{o}}$ and $\mathfrak{a}$ are constants. We can always shift the origin of the coordinate system and write this as

$$z = z_0 - \frac{\mathfrak{a}}{\tau_0 + \tau}. \tag{6.36}$$

Let us do a further approximation where

$$z = z_0 - \mathfrak{a}(\tau_0 + \tau)^{-1} \simeq z_0 - \mathfrak{a}\tau_0(1 - \frac{\tau}{\tau_0}).$$

A redefinition of constants to ease notation will give

$$z = \mathfrak{c}\tau + \mathfrak{d}. \tag{6.37}$$

This approximation shows us that in a linearly expanding universe the coordinate z evolves linearly as well. In Equation 6.36 as $\tau \to \infty$, z approaches a constant value, as if it is held fixed. In a sense z is stationary, it just evolves as the spacetime evolves.

7. CONCLUSION

As we pointed out in the beginning, it has been shown that four dimensional curved spacetimes can be embedded in five dimensional flat or Ricci flat spacetimes [17]. The former branch has been well studied in literature. It is stated that a matter and radiation dominated four dimensional universe can be embedded in a five dimensional vacuum universe [26, 27] and the accelerated expansion of the universe can be obtained via extra dimensional models [28]. In this work we have obtained all relevant cosmologies, including dark energy dominated cosmology, as four dimensional slices of a flat, five dimensional metric. We were able to do this by allowing the internal dimension to be fundamental, like time. We name the internal space as fundamental because it affects all the scale factors including that of time. Moreover, although the internal space is a spacelike dimension, the linear combinations of time and the internal space may transform as lightlike coordinates.

We should like to point out that this thesis does not present a detailed cosmological model. In standard four dimensional cosmology, the equation of state that governs the expansion of the universe with time, changes for physical reasons. The early universe goes through different phases, starting with radiation dominated moving on to dust dominated, and so forth by a power law, $a(t) = t^n$, where the value of n changes from one era to another with time. In our model the change of n may be obtained by a pseudo rotation involving internal space and time. Although we can embed all these cases into the same five dimensional metric we are not able to switch from one to another because our parameter n does not depend on time. To turn our model into a physically better suited one it would be necessary to find a different reason for the changes of the time variable to explain this change of n with time which gives the correspondence with radiation dominated, dust dominated, dark energy dominated eras.

We have pointed out that fixing the free parameters amounts to choosing different frames. We have picked out the cosmological frame to be the one in which the scale factor of time is unity and this frame gave us a linearly expanding universe. Thus from dimensional arguments we have shown that, in a universe where the only dimensional constant is the speed of light, the preferred change of scale size in time amounts to linear expansion. This inevitably brings to mind the present relationship between Hubble's parameter H_0, and life time of the universe t_0, being $H_0 t_0 = 1$. If this relationship is valid for all times that would indicate a linearly expanding universe. With this in mind our choice of the cosmological frame might indeed be the suitable choice. On the other hand, among the possible frames, the ones that contain dark energy are more complex and have w dependent scale factors for time. A physical reason to choose one of these as the cosmological frame would be dark energy.

We have also discussed a braneworld version of our cosmological frame by putting a brane at $w = 0$. Our world turns out to be a linearly expanding universe, confined to the brane. It is also the largest universe in size. The other worlds at different $w-$branes are also linearly expanding universes, and the Hubble time is the same for all of them. The only difference between our universe and these others is that their scale size is smaller by a factor of $e^{-\mathcal{M}|w|}$. A more detailed discussion of how cosmological models can be incorporated into brane world scenarios can be found in [29]. Dvali and collaborators [30] present a mechanism by which the correct four dimensional gravitational potential may be obtained for static 3-branes embedded in five dimensional Minkowski space. One of the possible cases to where their mechanism can be applied to consists of matter fields confined to the brane. In our braneworld scenario the 3-brane is dynamic yet the matter fields are still confiend to the brane. Therefore it may be possible to apply the same mechanism here and obtain an expression for the four dimensional gravitational potential with cosmic dynamics.

A Ricci flat spacetime is empty in terms of matter, meaning it contains no pressure or energy density. A vanishing Weyl tensor, which represents the conformal flatness, points the absence of gravitational fields. Therefore a flat universe must be both Ricci flat and Conformally flat, containing neither matter nor gravitational fields. The ability to embed conventional four dimensional cosmologies in Ricci flat five dimensional spacetimes not only presents a simpler frame to consider the situation in but also gives a geometric explanation of observed material effects, this has been the subject of the briefly mentioned literature. Conformal flatness, on the other hand, is one of the key properties of standart cosmology. Our five dimensional spacetime in which all relevant cosmologies can be locally embedded, is both Ricci flat and conformally flat. Therefore we have achieved the embedding of all relevant four dimensional cosmologies in a flat five dimensional spacetime.

The common intuition would be to imagine a four dimensional space expanding along time. Instead what we have introduced here, and in [31], is the three dimensional space expanding along both time and internal space. So in a sense we should visualize this as a four dimensional spacetime evolving along the extra dimension.

APPENDIX A: The calculation of curvature two forms in chapter 2

We will now give the details of the calculations done in evaluating the Einstein's tensor, for the metric

$$ds^2 = f^2(t)g^2(w)[-dt^2 + dw^2] + a^2(t)b^2(w)\frac{d\vec{r}^2}{(1+\frac{\kappa\vec{r}^2}{4})^2} \tag{A.1}$$

where $\vec{r}^2 = x^2 + y^2 + z^2$. The basis one forms are

$$e^4 = if(t)g(w)dt \tag{A.2a}$$

$$e^5 = f(t)g(w)dw \tag{A.2b}$$

$$e^i = a(t)b(w)\tilde{F}^i = a(t)b(w)\frac{dx^i}{(1+\frac{\kappa\vec{r}^2}{4})^2}. \tag{A.2c}$$

In order to obtain the connection coefficients $w^{\mu}{}_{\nu\lambda}$, we must first evaluate the differentials of the basis on forms. We will make use of the following

$$\vec{r}^2 = x^i x_i$$

$$\frac{d\vec{r}^2}{dx^j} = \frac{dx^i}{dx^j}x_i + x^i\frac{dx_i}{dx^j} = \delta^i{}_j x_i + x^i\delta^j{}_i = 2x_j$$

and

$$d\vec{r}^2 = 2x_j dx^j$$

$$d\tilde{F}^i = -\frac{1}{(1+\frac{\kappa\vec{r}^2}{4})^2}\frac{\kappa}{4}d\vec{r}^2 \wedge dx^i$$

$$= -\frac{1}{(1+\frac{\kappa\vec{r}^2}{4})^2}\frac{\kappa}{4}2x_j dx^j \wedge dx^i$$

$$d\tilde{F}^i = -\frac{\kappa}{2}x_j \frac{e^j \wedge e^i}{a^2(t)b^2(w)}. \tag{A.3}$$

A sum over j is indicated, and we get

$$de^i = \dot{a}(t)b(w)dt \wedge \tilde{F}^i + a(t)b'(w)dw \wedge \tilde{F}^i + a(t)b(w)d\tilde{F}^i$$

$$de^i = \frac{\dot{a}(t)}{if(t)g(w)a(t)}e^4 \wedge e^i + \frac{b'(w)}{f(t)g(w)b(w)}e^5 \wedge e^i - \frac{\kappa x_j}{2}\frac{e^j \wedge e^i}{a(t)b(w)} \tag{A.4a}$$

$$de^4 = if(t)g'(w)dw \wedge dt = \frac{g'(w)}{g^2(w)f(t)}e^5 \wedge e^4 \tag{A.4b}$$

$$de^5 = \dot{f}(t)g(w)dt \wedge dw = \frac{\dot{f}(t)}{if^2(t)g(w)}e^4 \wedge e^5 \tag{A.4c}$$

We evaluate the connection coefficients by the following formula

$$de^\mu + w^\mu{}_\nu \wedge e^\nu = 0 \tag{A.5}$$

where the connection forms are expanded as $w^\mu{}_\nu = w^\mu{}_{\nu\lambda}e^\lambda$. From e^5 we obtain;

$$de^5 + w^5{}_1 \wedge e^1 + w^5{}_2 \wedge e^2 + w^5{}_3 \wedge e^3 + w^5{}_4 \wedge e^4 = 0 \tag{A.6}$$

$$\frac{\dot{f}(t)}{if^2(t)g(w)}e^4 \wedge e^5 + w^5{}_{12}e^2 \wedge e^1 + w^5{}_{13}e^3 \wedge e^1 + w^5{}_{14}e^4 \wedge e^1 + w^5{}_{15}e^5 \wedge e^1$$

$$w^5{}_{21}e^1 \wedge e^2 + w^5{}_{23}e^3 \wedge e^2 + w^5{}_{24}e^4 \wedge e^2 + w^5{}_{25}e^5 \wedge e^2 + w^5{}_{31}e^1 \wedge e^3 + w^5{}_{32}e^2 \wedge e^3$$

$$w^5{}_{34}e^4 \wedge e^3 + w^5{}_{35}e^5 \wedge e^3 + w^5{}_{41}e^1 \wedge e^4 + w^5{}_{42}e^2 \wedge e^4 + w^5{}_{43}e^3 \wedge e^4$$

$$w^5{}_{45}e^5 \wedge e^4 = 0 \tag{A.7}$$

Collecting coefficients of the same $e^a \wedge e^b$ gives the following equations

$$w^5{}_{45} = \frac{\dot{f}(t)}{if^2(t)g(w)} \tag{A.8a}$$

$$w^5{}_{15} = w^5{}_{25} = w^5{}_{35} = 0 \tag{A.8b}$$

$$w^5{}_{12} = w^5{}_{21} \tag{A.8c}$$

$$w^5{}_{13} = w^5{}_{31} \tag{A.8d}$$

$$w^5{}_{14} = w^5{}_{41} \tag{A.8e}$$

$$w^5{}_{23} = w^5{}_{32} \tag{A.8f}$$

$$w^5{}_{24} = w^5{}_{42} \tag{A.8g}$$

$$w^5{}_{34} = w^5{}_{43} \tag{A.8h}$$

Similarly we get, from $de^4 + w^4{}_1 \wedge e^1 + w^4{}_2 \wedge e^2 + w^4{}_3 \wedge e^3 + w^4{}_5 \wedge e^5 = 0$;

$$w^4{}_{54} = \frac{g'(w)}{g^2(w)f(t)} \tag{A.9a}$$

$$w^4{}_{14} = w^4{}_{24} = w^4{}_{34} = 0 \tag{A.9b}$$

$$w^4{}_{12} = w^4{}_{21} \tag{A.9c}$$

$$w^4{}_{13} = w^4{}_{31} \tag{A.9d}$$

$$w^4{}_{15} = w^4{}_{51} \tag{A.9e}$$

$$w^4{}_{23} = w^4{}_{32} \tag{A.9f}$$

from $de^1 + w^1{}_2 \wedge e^2 + w^1{}_3 \wedge e^3 + w^1{}_4 \wedge e^4 + w^1{}_5 \wedge e^5 = 0$;

$$w^1{}_{41} = \frac{\dot{a}(t)}{ia(t)f(t)g(w)} \tag{A.10a}$$

$$w^1{}_{51} = \frac{b'(w)}{b(w)f(t)g(w)} \tag{A.10b}$$

$$w^1{}_{21} = -\frac{\kappa x_2}{2a(t)b(w)} \tag{A.10c}$$

$$w^1{}_{31} = -\frac{\kappa x_3}{2a(t)b(w)} \tag{A.10d}$$

$$w^1{}_{23} = w^1{}_{32} \tag{A.10e}$$

$$w^1{}_{24} = w^1{}_{42} \tag{A.10f}$$

$$w^1{}_{25} = w^1{}_{52} \tag{A.10g}$$

$$w^1{}_{34} = w^1{}_{43} \tag{A.10h}$$

$$w^1{}_{35} = w^1{}_{53} \tag{A.10i}$$

$$w^1{}_{45} = w^1{}_{54} \tag{A.10j}$$

from $de^2 + w^2{}_1 \wedge e^1 + w^2{}_3 \wedge e^3 + w^2{}_4 \wedge e^4 + w^2{}_5 \wedge e^5 = 0$;

$$w^2{}_{42} = \frac{\dot{a}(t)}{ia(t)f(t)g(w)} \tag{A.11a}$$

$$w^2{}_{52} = \frac{b'(w)}{b(w)f(t)g(w)} \tag{A.11b}$$

$$w^2{}_{12} = -\frac{\kappa x_1}{2a(t)b(w)} \tag{A.11c}$$

$$w^2{}_{32} = -\frac{\kappa x_3}{2a(t)b(w)} \tag{A.11d}$$

$$w^2{}_{13} = w^2{}_{31} \tag{A.11e}$$

$$w^2{}_{14} = w^2{}_{41} \tag{A.11f}$$

$$w^2{}_{15} = w^2{}_{51} \tag{A.11g}$$

$$w^2{}_{34} = w^2{}_{43} \tag{A.11h}$$

$$w^2{}_{35} = w^2{}_{53} \tag{A.11i}$$

$$w^2{}_{45} = w^2{}_{54} \tag{A.11j}$$

and from $de^3 + w^3{}_1 \wedge e^1 + w^3{}_2 \wedge e^2 + w^3{}_4 \wedge e^4 + w^3{}_5 \wedge e^5 = 0$;

$$w^3{}_{43} = \frac{\dot{a}(t)}{ia(t)f(t)g(w)} \tag{A.12a}$$

$$w^3{}_{53} = \frac{b'(w)}{b(w)f(t)g(w)} \tag{A.12b}$$

$$w^3{}_{13} = -\frac{\kappa x_1}{2a(t)b(w)} \tag{A.12c}$$

$$w^3{}_{23} = -\frac{\kappa x_2}{2a(t)b(w)} \tag{A.12d}$$

$$w^3{}_{12} = w^3{}_{21} \tag{A.12e}$$

$$w^3{}_{14} = w^3{}_{41} \tag{A.12f}$$

$$w^3{}_{15} = w^3{}_{51} \tag{A.12g}$$

$$w^3{}_{24} = w^3{}_{42} \tag{A.12h}$$

$$w^3{}_{25} = w^3{}_{52} \tag{A.12i}$$

$$w^3{}_{45} = w^3{}_{54}. \tag{A.12j}$$

$w^\mu{}_{\nu\lambda}$ is antisymmetric in its first two indices because of the antisymmetry of $w^\mu{}_\nu$. From Equations $A.8c$, $A.11g$, $A.10g$

$$w_{512} = w_{521} = -w_{251} = -w_{215} = w_{125} = w_{152} = -w_{512} = 0 \tag{A.13}$$

Similarly all permutations of w_{513}, w_{514}, w_{523}, w_{524}, w_{534}, w_{412}, w_{413}, w_{423}, w_{123} are zero.

Finally we can obtain all of the connection forms from $w^{\mu}{}_{\nu} = w^{\mu}{}_{\nu\lambda} e^{\lambda}$, to be the following

$$w^{i}{}_{j} = \frac{\kappa}{2a(t)b(w)}(x_i e^j - x_j e^i) \tag{A.14a}$$

$$w^{i}{}_{4} = \frac{\dot{a}(t)}{ia(t)f(t)g(w)} e^i \tag{A.14b}$$

$$w^{i}{}_{5} = \frac{b'(w)}{b(w)f(t)g(w)} e^i \tag{A.14c}$$

$$w^{4}{}_{5} = \frac{g'(w)}{g^2(w)f(t)} e^4 - \frac{\dot{f}(t)}{if^2(t)g(w)} e^5. \tag{A.14d}$$

To obtain the curvature two forms we first need to evaluate the following

$$dw^{i}{}_{4} = \frac{\ddot{a}(t)}{ia(t)f(t)g(w)} dt \wedge e^i - \frac{\dot{a}^2(t)}{ia^2(t)f(t)g(w)} dt \wedge e^i$$

$$-\frac{\dot{a}(t)\dot{f}(t)}{ia(t)f^2(t)g(w)} dt \wedge e^i - \frac{\dot{a}(t)g'(w)}{ia(t)f(t)g^2(w)} dw \wedge e^i + \frac{\dot{a}(t)}{ia(t)f(t)g(w)} de^i$$

$$= [-\frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} + \frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} + \frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)}] e^4 \wedge e^i - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)} e^5 \wedge e^i$$

$$-\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} e^4 \wedge e^i + \frac{\dot{a}(t)b'(w)}{ia(t)f^2(t)g^2(w)b(w)} e^5 \wedge e^i - \frac{\kappa}{2i} \frac{\dot{a}(t)}{a^2(t)f(t)g(w)b(w)} x_j e^j \wedge e^i$$

$$dw^{i}{}_{4} = [\frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)} - \frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)}] e^4 \wedge e^i - \frac{\kappa}{2i} \frac{\dot{a}(t)}{a^2(t)f(t)g(w)b(w)} x_j e^j \wedge e^i$$

$$+[\frac{\dot{a}(t)b'(w)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)}] e^5 \wedge e^i. \tag{A.15}$$

Similarly

$$dw^i{}_5 = [\frac{b''(w)}{b(w)f^2(t)g^2(w)} - \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)}]e^5 \wedge e^i - \frac{\kappa}{2}\frac{b'(w)}{a(t)b^2(w)f(t)g(w)}x_j e^j \wedge e^i$$

$$+[\frac{b'(w)\dot{a}(t)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)}]e^4 \wedge e^i \qquad \text{(A.16)}$$

$$dw^4{}_5 = [-\frac{\dot{f}^2(t)}{f^4(t)g^2(w)} + \frac{\ddot{f}(t)}{f^3(t)g^2(w)} + \frac{g'^2(w)}{g^4(w)f^2(t)} - \frac{g''(w)}{g^3(w)f^2(t)}]e^4 \wedge e^5 \qquad \text{(A.17)}$$

$$dw^i{}_j = \frac{\kappa}{a^2(t)b^2(w)}(1+\kappa\frac{\vec{r}^2}{4})e^i \wedge e^j - \frac{\kappa^2}{4a^2(t)b^2(w)}x_i x_k e^k \wedge e^j$$

$$+\frac{\kappa^2}{4a^2(t)b^2(w)}x_j x_k e^k \wedge e^i. \qquad \text{(A.18)}$$

We will evaluate the curvature two forms with the help of the following equation

$$\Omega^\mu{}_\nu = dw^\mu{}_\nu + w^\mu{}_\lambda \wedge w^\lambda{}_\nu. \qquad \text{(A.19)}$$

The calculations will be given explicitly only for $\Omega^2{}_3$

$$\Omega^2{}_3 = dw^2{}_3 + w^2{}_1 \wedge w^1{}_3 + w^2{}_4 \wedge w^4{}_3 + w^2{}_5 \wedge w^5{}_3$$

$$= \frac{\kappa}{a^2(t)b^2(w)}(1+\kappa\frac{\vec{r}^2}{4})e^2\wedge e^3 + \frac{\kappa}{4a^2(t)b^2(w)}(x_3x_1e^1\wedge e^2 + x_3^2e^3\wedge e^2 - x_2x_1e^1\wedge e^3 - x_2^2e^2\wedge e^3)$$

$$+\frac{\kappa}{4a^2(t)b^2(w)}(x_2e^1 - x_1e^2)\wedge(x_1e^3 - x_3e^1) + \frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)}e^2\wedge e^3 - \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)}e^2\wedge e^3$$

$$\Omega^2{}_3 = [\frac{\kappa}{a^2(t)b^2(w)} + \frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} - \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)}]e^2 \wedge e^3. \tag{A.20}$$

On the whole the curvature two forms are

$$\Omega^i{}_j = [\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} - \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} + \frac{\kappa}{a^2(t)b^2(w)}]e^i \wedge e^j \tag{A.21}$$

$$\Omega^i{}_4 = [\frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)} - \frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} + \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)}]e^4 \wedge e^i$$

$$+[\frac{\dot{a}(t)b'(w)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)}]e^5 \wedge e^i \tag{A.22}$$

$$\Omega^i{}_5 = [\frac{b'(w)\dot{a}(t)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)}]e^4 \wedge e^i$$

$$+[\frac{b''(w)}{b(w)f^2(t)g^2(w)} - \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)} - \frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)}]e^5 \wedge e^i \tag{A.23}$$

$$\Omega^4{}_5 = [-\frac{\dot{f}^2(t)}{f^4(t)g^2(w)} + \frac{\ddot{f}(t)}{f^3(t)g^3(w)} + \frac{g'^2(w)}{g^4(w)f^2(t)} - \frac{g''(w)}{g^3(w)f^2(t)}]e^4 \wedge e^5 \tag{A.24}$$

The components of Riemann tensor can be obtained from the following equation

$$\Omega^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\lambda\kappa}e^\lambda \wedge e^\kappa. \tag{A.25}$$

From $\Omega^i{}_j$ we obtain

$$\Omega^i{}_j = [\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} - \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} + \frac{\kappa}{a^2(t)b^2(w)}]e^i \wedge e^j$$

$$= \frac{1}{2}R_{ijij}e^i \wedge e^j + \frac{1}{2}R_{ijji}e^j \wedge e^i = \frac{1}{2}R_{ijij}e^i \wedge e^j + \frac{1}{2}(-R_{ijij})(-e^i \wedge e^j)$$

$$R_{ijij} = [\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} - \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} + \frac{\kappa}{a^2(t)b^2(w)}]. \tag{A.26}$$

In the same way one gets

$$R_{i44i} = [\frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)} - \frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} + \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)}] \tag{A.27}$$

$$R_{i45i} = [\frac{\dot{a}(t)b'(w)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)}] \tag{A.28}$$

$$R_{i54i} = [\frac{b'(w)\dot{a}(t)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)}] \tag{A.29}$$

$$R_{i55i} = [\frac{b''(w)}{b(w)f^2(t)g^2(w)} - \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)} - \frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)}] \tag{A.30}$$

$$R_{4545} = [-\frac{\dot{f}^2(t)}{f^4(t)g^2(w)} + \frac{\ddot{f}(t)}{f^3(t)g^2(w)} + \frac{g'^2(w)}{g^4(w)f^2(t)} - \frac{g''(w)}{g^3(w)f^2(t)}]. \tag{A.31}$$

And the components of Ricci tensor, calculated from

$$R_{\nu\lambda} = R^{\mu}{}_{\nu\lambda\mu} \tag{A.32}$$

are

$$R_{ii} = \frac{b''(w)}{b(w)f^2(t)g^2(w)} - \frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} - 2\frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)}$$

$$+2\frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} - 2\frac{\kappa}{a^2(t)b^2(w)} \tag{A.33}$$

$$R_{44} = 3[\frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)} - \frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} + \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)}]$$

$$+\frac{\dot{f}^2(t)}{f^4(t)g^2(w)} - \frac{\ddot{f}(t)}{f^3(t)g^2(w)} - \frac{g'^2(w)}{g^4(w)f^2(t)} + \frac{g''(w)}{g^3(w)f^2(t)} \tag{A.34}$$

$$R_{55} = 3[\frac{b''(w)}{b(w)f^2(t)g^2(w)} - \frac{b'(w)g'(w)}{b(w)f^2(t)g^3(w)} - \frac{\dot{a}(t)\dot{f}(t)}{a(t)f^3(t)g^2(w)}]$$

$$+\frac{\dot{f}^2(t)}{f^4(t)g^2(w)} - \frac{\ddot{f}(t)}{f^3(t)g^2(w)} - \frac{g'^2(w)}{g^4(w)f^2(t)} + \frac{g''(w)}{g^3(w)f^2(t)} \tag{A.35}$$

$$R_{54} = R_{45} = 3[\frac{b'(w)\dot{a}(t)}{ia(t)b(w)f^2(t)g^2(w)} - \frac{b'(w)\dot{f}(t)}{ib(w)f^3(t)g^2(w)} - \frac{\dot{a}(t)g'(w)}{ia(t)f^2(t)g^3(w)}]. \tag{A.36}$$

And the Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$R = 6[\frac{b''(w)}{b(w)f^2(t)g^2(w)} - \frac{\ddot{a}(t)}{a(t)f^2(t)g^2(w)} - \frac{\dot{a}^2(t)}{a^2(t)f^2(t)g^2(w)} + \frac{b'^2(w)}{b^2(w)f^2(t)g^2(w)} - \frac{\kappa}{a^2(t)b^2(w)}]$$

$$+2[\frac{\dot{f}^2(t)}{f^4(t)g^2(w)} - \frac{\ddot{f}(t)}{f^3(t)g^2(w)} - \frac{g'^2(w)}{g^4(w)f^4(t)} + \frac{g''(w)}{g^3(w)f^2(t)} \qquad \text{(A.37)}$$

APPENDIX B: The Einstein's tensor for our braneworld scenario

$$ds^2 = -d\tau^2 + \mu^2\tau^2 dw^2 + \mu^2\tau^2 e^{2\mu|w|} d\vec{x}^2 \tag{B.1}$$

The basis one forms for this metric are

$$e^i = \mu\tau e^{\mu|w|} \tag{B.2a}$$

$$e^4 = id\tau \tag{B.2b}$$

$$e^5 = \mu\tau dw. \tag{B.2c}$$

Let us calculate the curvature two forms and Einstein tensor for this metric.We should start with the de^ν for this ,

$$de^i = \mu e^{\mu|w|} d\tau \wedge dx^i + \mu^2\tau \frac{d|w|}{dw} e^{\mu|w|} dw \wedge dx^i = \frac{e^4 \wedge e^i}{i\tau} + \frac{d|w|}{dw}\frac{e^5 \wedge e^i}{\tau} \tag{B.3}$$

$$de^4 = id\tau \wedge d\tau = 0 \tag{B.4}$$

$$de^5 = \mu d\tau \wedge dw = \frac{e^4 \wedge e^5}{i\tau} \tag{B.5}$$

We obtain the connection forms by $de^{\mu} + w^{\mu}{}_{\nu} \wedge e^{\nu} = 0$ and $w^{\mu}{}_{\nu} = w^{\mu}{}_{\nu\lambda} e^{\lambda}$ to be

$$\omega^i{}_j = 0 \tag{B.6a}$$

$$\omega^i{}_4 = \frac{e^i}{i\tau} \tag{B.6b}$$

$$\omega^i{}_5 = \frac{d|w|}{dw}\frac{e^i}{\tau} \tag{B.6c}$$

$$\omega^4{}_5 = -\frac{e^5}{i\tau}. \tag{B.6d}$$

These give us the following

$$d\omega^i{}_j = 0 \tag{B.7}$$

$$d\omega^i{}_4 = -\frac{d\tau \wedge e^i}{i\tau^2} + \frac{de^i}{i\tau} = \frac{d|w|}{dw}\frac{e^5 \wedge e^i}{i\tau^2} \tag{B.8}$$

$$d\omega^4{}_5 = \frac{d\tau \wedge e^5}{i\tau^2} - \frac{de^5}{i\tau} = -\frac{e^4 \wedge e^5}{\tau^2} - \frac{1}{i\tau}\frac{e^4 \wedge e^5}{i\tau} = 0 \tag{B.9}$$

$$d\omega^i{}_5 = -\frac{d|w|}{dw}\frac{d\tau \wedge e^i}{\tau^2} + \frac{d^2|w|}{dw^2}\frac{dw \wedge e^i}{\tau} + \frac{d|w|}{dw}\frac{de^i}{\tau}$$

$$= [2\frac{\delta(w)}{\mu\tau^2} + \frac{1}{\tau^2}(\frac{d|w|}{dw})^2]e^5 \wedge e^i = [2\frac{\delta(w)}{\mu\tau^2} + \frac{1}{\tau^2}]e^5 \wedge e^i \tag{B.10}$$

where in the last equation $\frac{d|w|}{dw} = sgn(w)$ and $\frac{d^2|w|}{dw^2} = 2\delta(w)$. For the curvature two forms $\Omega^{\mu}{}_{\nu} = dw^{\mu}{}_{\nu} + w^{\mu}{}_{\lambda} \wedge w^{\lambda}{}_{\nu}$

$$\Omega^{i}{}_{j} = 0 \tag{B.11a}$$

$$\Omega^{i}{}_{4} = 0 \tag{B.11b}$$

$$\Omega^{i}{}_{5} = 2\frac{\delta(w)}{\mu\tau^2} e^5 \wedge e^i \tag{B.11c}$$

$$\Omega^{4}{}_{5} = 0. \tag{B.11d}$$

On the other hand $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\lambda x} e^{\lambda} \wedge e^{x}$, since the only nonzero curvature two forms are $\Omega^{i}{}_{j}$,

$$\Omega^{1}{}_{5} = 2\frac{\delta(w)}{\mu\tau^2} e^5 \wedge e^i = \frac{1}{2} R_{1551} e^5 \wedge e^1 + \frac{1}{2} R_{1515} e^1 \wedge e^5$$

the only nonzero $R_{\mu\nu\lambda x}$ are

$$R_{1551} = R_{i55i} = 2\frac{\delta(w)}{\mu\tau^2}. \tag{B.12}$$

With $R_{\mu\nu} = R^{\mu}{}_{\nu\lambda\mu}$

$$R_{11} = R^{5}{}_{115} + R^{2}{}_{112} + R^{3}{}_{113} + R^{4}{}_{114} = 2\frac{\delta(w)}{\mu\tau^2} = R_{ii} \tag{B.13a}$$

$$R_{44} = R^{1}{}_{441} + R^{2}{}_{442} + R^{3}{}_{443} + R^{5}{}_{445} = 0 \tag{B.13b}$$

$$R_{55} = R^{1}{}_{551} + R^{2}{}_{552} + R^{3}{}_{553} + R^{4}{}_{554} = 6\frac{\delta(w)}{\mu\tau^2}, \tag{B.13c}$$

$R = g^{\mu\nu} R_{\mu\nu} = 12\frac{\delta(w)}{\mu\tau^2}$ and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

$$G_{ii} = R_{ii} - \frac{1}{2}R = -4\frac{\delta(w)}{\mu\tau^2} \tag{B.14a}$$

$$G_{44} = R_{44} - \frac{1}{2}R = -6\frac{\delta(w)}{\mu\tau^2} \tag{B.14b}$$

$$G_{55} = R_{55} - \frac{1}{2}R = 0. \tag{B.14c}$$

Some of the nonzero elements of the Weyl tensor

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \frac{1}{d-2}(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho})$$

$$-\frac{1}{(d-1)(d-2)}(g_{\rho\mu}g_{\nu\sigma} + g_{\rho\nu}g_{\mu\sigma})R, \tag{B.15}$$

for this metric are

$$C_{1551} = -\frac{5}{3}\frac{\delta(w)}{\mu\tau^2} \tag{B.16a}$$

$$C_{4554} = -3\frac{\delta(w)}{\mu\tau^2} \tag{B.16b}$$

$$C_{5454} = \frac{\delta(w)}{\mu\tau^2} \tag{B.16c}$$

$$C_{1441} = -\frac{5}{3}\frac{\delta(w)}{\mu\tau^2} \tag{B.16d}$$

$$C_{1331} = -\frac{7}{3}\frac{\delta(w)}{\mu\tau^2}. \tag{B.16e}$$

Therefore the metric in Equation A.1 is not conformally flat.

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