UNITARY GROUP AND QUANTUM GROUP INVARIANT q-OSCILLATORS AND FIBONACCIZATION

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Abstract

After a review of different types of q-oscillators and the relationship between their creation/annihilation operators, the derivation of the q generalized version of the Levi-Civita tensor (ϵ) is given. The properties of the quantum group SLq(n) are obtained through the invariance properties of the new q-epsilon tensor. The relations between the inner products of n-particle states of the various q-oscillators are also investigated. Finally, a two parameter generalization of the Coon-Baker-Yu q-oscillator (which we call Fibonaccization) is presented.

Özet

Farklı q-salınımcı çeşitlerinin ve bunların yaratma/yoketme işlemcilerinin arasındaki ilişkilerin gözden geçirilmesinden sonra, Levi-Civita epsilon çoklamının (tensor) q genelleştirilmiş hali bulundu. Yeni q-epsilon çoklamının (tensor) değişmezlik özelliklerinden yararlanılarak SLq(n) quantum grubunun bağıntıları elde edildi. Farklı q-salınımcılarının n parçacıklı durumlarının iç çarpımları arasındaki ilişkiler de araştırıldı. Son olarak Coon-Baker-Yu q-salınımcısının iki parametreli genelleştirilmesi (Fibonaccileştirme adıyla) sunuldu.

List of Symbols

$|\,\rangle$: ground state of an oscillator

$|n\rangle$: n th excited state of an oscillator

$[A, B]$: commutator of A and B

$[n]$: basic number n generalizing integers

a : annihilation operator

$a^\dagger$: creation operator

$\mathcal{H}$: Hamiltonian operator

N: number operator

$N^{i_1 i_2 \cdots i_n}_{k_1 k_2 \cdots k_n}$: a tensor describing the inner products of states

q_1, q_2, q: deformation parameters

SL(n): special linear group

SLq(n): q-deformed special linear quantum group

Δ_{ij}: quantum subdeterminant

Contents

I.INTRODUCTION

Physics is an experimental science which tries to explain the processes in nature with mathematical theories. Generally in a physical theory there is a one to one correspondence between physical concepts (position, velocity, mass) and mathematical symbols $(\vec{x}, \vec{v}, m)$. Then a physical process, can be expressed in terms of these mathematical symbols in an appropriate formalism.

The correspondence between physical concepts of Newtonian mechanics and its mathematical formulation is too obvious and maybe trivial. With the most important conceptual development of the twentieth century, namely Quantum Mechanics, this correspondence became much more tricky. In this physical theory, observables (like position, energy etc.) are represented by hermitian operators, and measured physical quantities are then the eigenvalues of these operators.

As an example of this formulation it is possible to consider the SU(2) Lie algebra and angular momentum operators. In abstract notation the SU(2) algebra consists of 3 generators X_1, X_2, X_3 and their commutation relation

$$[X_i, X_j] = -\epsilon_{ijk} X_k \ .$$

On the other hand i times the angular momentum operator L_i of the quantum mechanics, $L_i \equiv -i\epsilon_{ijk} x_j \partial_k$, has exactly the same commutation relation. Thus one can identify iL_i with the generators of SU(2). As an application of this identification, it is possible to use the matrix representations of the SU(2) generators for the angular momentum operators.

Another example of identification of observables as hermitian operators is the Quantum Harmonic Oscillator (QHO) problem. The algebraic solutions to QHO, (in energy basis) are obtainable through the use of the annihilation operator a and its hemitian conjugate $a^\dagger$, which change the oscillatory energy state of a particle one unit up or down. It is also possible to find a one to one correspondence between "usual" position/momentum operators and $a/a^\dagger$. This correspondence permits us to write the Fock space [1] version of canonical quantization:

$$[q,p] = i\hbar \rightarrow [a, a^\dagger] = 1 \ .$$

If the canonical quantization rules are viewed in terms of position and momentum operators, one can note that the right hand side of the above equation is of the order of $\hbar$. That is, to return to Newtonian mechanics, one has to set $\hbar = 0$. Or more clearly, quantum mechanics is a correction to Newtonian mechanics of the order of $\hbar$.

As a pure speculative theory, it is possible to push quantization one step further and add higher order ($\hbar^2$, $\hbar^3$) terms. The effects of such speculations, in energy basis, would be to change the commutator of a and $a^\dagger$. But in which way? Again in a purely theoretical basis, it is possible to formulate a new commutator, (qumutator!), such that it contains a dimensionless parameter, q. Then for the quantum mechanical limit, one should set $q = 1$. The oscillators containing the generalization parameter q, formulated in this manner, are called "q-oscillators".

Since the first formulation of q-oscillators, there were many applications and different interpretations of this new parameter. For example, in 1975 Arik and in 1992 Arik and Mungan gave an interpretation as a relativistic effect, in 1989 Manko et al. as a nonlinearity in the oscillation frequency, in 1993 Arik and Rador as a discreteness parameter in space-time etc.

A similar step in pure mathematics was made with the discovery of quantum groups. Apart from ordinary groups these are defined through a deformation parameter "q". As the name sounds alike, a relation between q-oscillators and q-groups can be expected. It was found that different q-oscillators' commutation relations were invariant under some quantum groups. This helped to classify the q-oscillators.

In our approach, we consider the tensors created from state vectors, and this yields the full structure of a q-group, GLq(n). As a byproduct , "q" version of the ordinary Levi-Civita tensor is also obtained. With this approach, keeping in mind that tensors created by state vectors are related to q-groups, a search for connections between state vectors of different q-oscillators is performed. The surprising result is that the q-oscillators are only special cases the more general Fibonacci oscillators with two parameters, q_1 and q_2. Then Fibonacci versions of various q-oscillators can be formulated through a well defined procedure. Using a generalization of commutators (qumutators), it is possible to increase the number of deformation parameters to n. The effects of such a parametrization can be thought to have a physical meaning. The related basic number, if associated to the energy levels of a q-oscillator model, permits us to locate the point above which q-quantum effects are to be observed.

These remarks constitute a verbal summary of this present work. Now we can explore the details with the help of mathematical formulae.

II.REVIEW OF q-OSCILLATORS

II.1 QHO and One Dimensional q-Oscillator

The classical one dimensional quantum harmonic oscillator (QHO) problem [2] can be solved with the operator method using the annihilation operator a and its hermitian conjugate $a^\dagger$, the creation operator, both acting on a Hilbert space defined through the inner product of basis bra and ket vectors. The action of these operators can be written as follows

$$a^\dagger|n\rangle = |n+1\rangle \tag{II.1}$$

$$a|n\rangle = |n-1\rangle \tag{II.2}$$

Using the fact that no states exist below the ground state, the action of the annihilation operator can be formulated as

$$a|\,\rangle = 0, \tag{II.3}$$

then a general state is obtained as

$$|n\rangle \equiv (a^\dagger)^n|\,\rangle, \tag{II.4}$$

with the normalization

$$\langle n|m\rangle = n!\delta_{nm} \tag{II.5}$$

The energy of each state is measured with the number operator defined by

$$N = a^\dagger a \tag{II.6}$$

with

$$N|n\rangle = n|n\rangle \tag{II.7}$$

Then the following commutation relations exist between these operators:

$$[a, a^\dagger] = 1 \tag{II.8}$$
$$[N, a^\dagger] = a^\dagger \tag{II.9}$$
$$[N, a] = -a \tag{II.10}$$

where the commutator $[A, B]$ is defined as $[A, B] \equiv AB - BA$.

This notation can be generalized [3, 4] by adding one extra parameter (q) to the first commutation relation, which becomes

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1 \tag{II.11}$$

In this generalization the integer n, representing the energy level of each state is to be replaced by basic number $[n]$ in order to obtain a q deformed number operator. Identifying $a^\dagger a$ by [N] and $aa^\dagger$ by [N+1] equation (II.11) can be put into a difference equation form:

$$[N + 1] = q[N] + 1 \tag{II.12}$$

The solution to the difference equation (II.12), using the fact that there are no states below the ground state, (namely $a|\ \rangle = 0$), is given by Jackson's [5] basic integer formula:

$$[n] = \frac{q^n - 1}{q - 1}. \tag{II.13}$$

The spectrum of this deformed oscillator is given by the sequence $0, 1, 1+q, 1+q+q^2, \cdots$.

There are also other 1-dimensional q-oscillators. For example the q-oscillator used to construct SU$q(2)$ q-deformed Lie algebra is defined by the following commutation relation (first formulated independently by Macfarlane and Biedenharn (1989) and hereafter referred as the MB oscillator [6, 7])

$$aa^\dagger - q^{-1}a^\dagger a = q^N. \tag{II.14}$$

Again $aa^\dagger$ represents $[N+1]$ and $a^\dagger a$ represents $[N]$, and this yields the difference equation of the MB q-oscillator. However there exists a transformation [8] which permits to go from MB q-oscillator to the first one (II.11). If one defines operator c as

$$c \equiv q^{-N/2}\, a \tag{II.15}$$

after some rearrangements one finds the relation

$$cc^\dagger = q^{-2}c^\dagger c + 1 \tag{II.16}$$

which by the replacement $q^{-2} \to q$ becomes the q-oscillator (II.11). During these calculations we used the following properties of a and $a^\dagger$:

$$\begin{aligned} af(N) &= f(N+1)a && \text{(II.17)} \\ a^\dagger f(N) &= f(N-1)a^\dagger && \text{(II.18)} \end{aligned}$$

where $f(N)$ is any analytic function of N. The spectrum of the MB oscillator is given by the symmetric form [9] of the basic integer $[n]$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{II.19}$$

It is important to note that the commutation relations of both 1-dimensional q-oscillators are invariant under U(1). Choosing complex U such that $U\bar{U} = 1$, $\tilde{a}$ can be defined as:

$$\tilde{a} = Ua \quad \text{(II.20)}$$
$$\tilde{a}^{\dagger} = \bar{U}a. \quad \text{(II.21)}$$

Then one can see that the commutation relation is invariant such that:

$$aa^{\dagger} - qa^{\dagger}a = 1 \rightarrow U\bar{U}(aa^{\dagger} - qa^{\dagger}a) = 1. \quad \text{(II.22)}$$

We finally note that since the generalized number operator measures the energy level of a particle/state, the Hamiltonian of a 1-dimensional q-oscillator is given by $[N]$.

II.2 n-Dimensional Commuting q-Oscillator

As the simplest multidimensional version of the above q-oscillator, it is possible to use n commuting copies of the same 1-dimensional q-oscillator (first formulated by Arik and Coon, and hereafter referred as the AC oscillator [3]). Here the lowering and raising operators belonging to the same dimension/particle are related by the formula:

$$a_i a_i{}^{\dagger} = q a_i{}^{\dagger} a_i + 1 \quad \text{(II.23)}$$

However, lowering and/or raising operators belonging to different dimensions/particles commute:

$$[a_i, a_j{}^{\dagger}] = [a_i, a_j] = [a_i{}^{\dagger}, a_j{}^{\dagger}] = 0 \quad i \neq j \quad \text{(II.24)}$$

Thus for example an n particle state is defined as:

$$|ijk\cdots n\rangle \equiv a_i{}^{\dagger}a_j{}^{\dagger}a_k{}^{\dagger}\cdots a_n{}^{\dagger}|\;\rangle\;. \tag{II.25}$$

This definition gives rise to the degeneracy of the q-oscillator states, since now the states corresponding to a different ordering of raising operators are the same due to (II.24). The total energy of such a d-dimensional system is given by the basic number consisting of individual sums of internal degrees of freedom

$$\mathcal{H} = [\sum_{i=1}^{d} N_i] = \frac{q^{N_1+N_2+\cdots+N_d} - 1}{q-1} \tag{II.26}$$

For the same system, the total energy $[N]$ can also be expressed in terms of the energies of each dimension/particle, using the expansion property of basic numbers, as

$$[N] = [n_1 + n_2 + \cdots + n_d] = [n_1] + q^{n_1}[n_2] + \cdots + q^{n_1+n_2+\cdots+n_{d-1}}[n_d]. \tag{II.27}$$

II.3 n-Dimensional CBY q-Oscillator

Another multidimensional version of the 1-dimensional q-oscillator in equation (II.11), which is invariant under U(n) was postulated by Coon Baker and Yu [13] (hereafter referred as CBY). Its defining commutation relation can be written as follows:

$$a_i a_j^{\dagger} = q a_j^{\dagger} a_i + \delta_{ij} \tag{II.28}$$

Here $a^{\dagger}$ and a are the creation and annihilation operators, as usual. The ground state is normalized to 1. Then the multidimensional states are obtained by applying the creation operators on the ground state. Thus,

$$|ijk\cdots n\rangle \equiv a_i^\dagger a_j^\dagger a_k^\dagger \cdots a_n^\dagger|\ \rangle. \tag{II.29}$$

It is important to note that in the CBY q-oscillator , we only have relations between creation and annihilation operators. A commutation type relation which contains only creation operators or only annihilation operators does not exist. Due to this fact, the degeneracy of the oscillator states is lost. For example in two dimensions the states $|21\rangle$ and $|12\rangle$ are linearly independent:

$$\begin{aligned} \langle 12|12\rangle &= \langle 21|21\rangle = 1 && \text{(II.30)} \\ \langle 12|21\rangle &= \langle 21|12\rangle = q && \text{(II.31)} \end{aligned}$$

This implies that $-1 \leq q = Cos(\theta) \leq 1$, where θ can be thought as the angle between the states $|21\rangle$ and $|12\rangle$. Since the states $|21\rangle$ and $|12\rangle$ are shown to be linearly independent, no commutation-like relation can be postulated between a_1 and a_2.

The orthonormalization of these linearly independent states is another problem. For two particle states, this problem is equivalent to the diagonalization of the 2x2 matrix:

$$\begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}.$$

For three particle states, if two of the particles are in the same energy level, (eg. 1,1,2) the diagonalization problem is again solvable. If the states are ordered as $|112\rangle$, $|121\rangle$, $|211\rangle$, the 3x3 matrix to be diagonalized is:

$$\begin{bmatrix} 1+q & q+q^2 & q^2+q^3 \\ q+q^2 & 1+q^3 & q+q^2 \\ q^2+q^3 & q+q^2 & 1+q \end{bmatrix}.$$

The eigenvalues of the above matrix can easily be found using a symbolic computing tool, for example Mathematica. If all the three particles are in different states, the matrix to be diagonalized is 6x6 and Mathematica fails to find the eigenvectors. The solution can be found if proper ordering is choosen for the linearly independent states; and the correct order turns out to be:

$$|123\rangle,\ |213\rangle,\ |132\rangle,\ |321\rangle,\ |312\rangle,\ |231\rangle\ .$$

Then, the inner product matrix is in the form: $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$, where A and B are all 3x3 matrices given by:

$$A = \begin{bmatrix} 1 & q & q \\ q & 1 & q^2 \\ q & q^2 & 1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} q^3 & q^2 & q^2 \\ q^2 & q^3 & q \\ q^2 & q & q^3 \end{bmatrix}.$$

One can write the eigenvectors of the above block matrix as $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. With this step the problem is reduced to finding the eigenvectors of a 3x3 matrix, which was solvable by Mathematica.

The orthonormalization of higher order states is an open problem and to our knowledge a general solution for an n particle state is not yet formulated.

The Hamiltonian for such a non degenerate system, with a degenerate spectrum, expressed in terms of the creation and annihilation operators, turns out to be an infinite series [10].

II.4 n-Dimensional PW q-Oscillator

The multidimensional CBY oscillator is invariant under the non quantum U(n) group. There is also a multidimensional q-oscillator, first formulated by Pusz and Woronowicz [11], (hereafter referred as PW), which is invariant under the Uq(n) quantum group. In fact, the PW oscillator (creation/annihilation operators) can be obtained from the simple multidimensional oscillator discussed in II-2. Recalling the commutation relation of that oscillator, we formulate its q^2 version, i.e.

$$a_i a_i^\dagger = q^2 a_i^\dagger a_i + 1 \tag{II.32}$$

It is possible to define new creation and annihilation operators, c_i and $c_i^\dagger$ such that,

$$\begin{aligned} c_i &\equiv q^{N_1+N_2+\cdots+N_{i-1}}\, a_i && \text{(II.33)} \\ c_i^\dagger &\equiv a_i^\dagger q^{N_1+N_2+\cdots+N_{i-1}} && \text{(II.34)} \end{aligned}$$

where the basic number $[n_i]$ is defined as:

$$[n_i] \equiv \frac{q^{2n_i}-1}{q^2-1} = a_i^\dagger a_i \tag{II.35}$$

It is a straightforward calculation to show that the $c, c^\dagger$ oscillators have the following commutation relations:

$$\begin{aligned} c_i c_j^\dagger &= q c_j^\dagger c_i \qquad i \neq j\ , && \text{(II.36)} \\ c_i c_j &= q^{-1} c_j c_i \qquad i > j\ , && \text{(II.37)} \\ c_i^\dagger c_j^\dagger &= q c_j^\dagger c_i^\dagger \qquad i > j\ , && \text{(II.38)} \\ c_1 c_1^\dagger &= q^2 c_1^\dagger c_1 + 1 \qquad , && \text{(II.39)} \\ c_i c_i^\dagger &= q^2 c_i^\dagger c_i + [c_{i-1}, c_{i-1}^\dagger] \qquad i > 1\ , && \text{(II.40)} \end{aligned}$$

Then the oscillators are Uq(n) invariant PW oscillators. Moreover, there are close connections between the above discussed q-oscillators (CBY, PW, AC), through their inner products. We will discuss this in III-3.

II.5 Two Parameter Generalization of PW q-Oscillator

Up to now, we have seen the basic integer of Jackson (II.13) and the symmetric form of the basic integer (II.19). Noting that in generalizing integers, each integer can be associated with a term in a sequence, we prefer to deal with a simple scheme, namely with the generalized Fibonacci sequence in which the n^{th} term is a weighted linear combination of $(n-1)^{th}$ and $(n-2)^{th}$ terms. For any weights, the generalized Fibonacci sequence is given by a second order difference equation. The solution to such an equation, or more clearly each term in the sequence can be expressed in terms of two distinct parameters, q_1, q_2:

$$[n] = \frac{q_1^n - q_2^n}{q_1 - q_2}\,, \tag{II.41}$$

which is a generalization of the symmetric form of a basic integer. (first considered by Arik et al. [12]). Following the convention of the authors, we will call the above basic number as "Fibonacci basic integer", and we will investigate the "Fibonacci oscillator" which has the spectrum given by Fibonacci basic integers.We start by recalling that the symmetric basic numbers were solutions to MB q-oscillator's difference equation,

$$[n+1] = q^{-1}[n] + q^n \tag{II.42}$$

which follows from (II.14). One can note that this equation is linear, but the solutions to (II.42) are also solutions to a second order linear difference equation:

$$[n+2] = (q+q^{-1})[n+1] - [n]\,. \tag{II.43}$$

It is also possible to write the inhomogenous first order difference equation (II.12), as a homogenous second order difference equation:

$$[n+2] = (q+1)[n+1] - q[n]\,. \tag{II.44}$$

Both of these equations (II.43) and (II.44) are special cases of the equation

$$[n+2] = \alpha[n+1] + \beta[n] \,. \qquad \text{(II.45)}$$

For a specific choice of α and β, it is possible to obtain Fibonacci basic integers. And this choice is:

$$\begin{aligned} \alpha &= q_1 + q_2 \\ \beta &= -q_1 q_2 \end{aligned} \qquad \text{(II.46)}$$

The q-oscillator yielding the Fibonacci basic integers, the Fibonacci q-oscillator, can be expressed in terms of the creation/annihilation operators satisfying

$$aa^\dagger = q_1 a^\dagger a + q_2^N = q_2 a^\dagger a + q_1^N \,. \qquad \text{(II.47)}$$

The construction of multidimensional Fibonacci q-oscillators is also possible. The easiest way to obtain d-dimensional q-oscillators is to take d commuting copies of the same q-oscillator. Then the following equations hold:

$$\begin{aligned} [a_i, a_j] &= [a_i, a_j^\dagger] = 0 \quad i \neq j && \text{(II.48)} \\ a_i^\dagger a_i &= [N_i] && \text{(II.49)} \\ a_i a_i^\dagger &= [N_{i+1}] && \text{(II.50)} \end{aligned}$$

in which the eigenvalue of the number operator $[N_i]$ is defined as in (II.41). The Fibonacci basic number has the decomposition property:

$$[n_1 + n_2 + \cdots + n_d] = [n_1] q_2^{(n_2+\cdots+n_d)} + q_1^{n_1}[n_2] q_2^{(n_3+\cdots+n_d)} + \cdots \,. \qquad \text{(II.51)}$$

If now, new creation and annihilation operators are defined as in (II.33):

$$\begin{aligned} c_1 &\equiv a_1 q_2^{(N_2+\cdots+N_d)/2} \\ c_2 &\equiv q_1^{(N_1)/2} a_2 q_2^{(N_3+\cdots+N_d)/2} \\ &\vdots \\ c_n &\equiv q_1^{(N_1+\cdots+N_{n-1})/2} a_n q_2^{(N_{n+1}+\cdots+N_d)/2} \end{aligned} \tag{II.52}$$

Using equation (II.51), it is possible to sum the individual energies of the c oscillators, and one can write the result in a compact form:

$$\sum_i c_i^\dagger c_i = [n_1 + n_2 + \cdots + n_d] = [N] \tag{II.53}$$

The commutation relations among the c oscillators can be calculated using the relations (II.48) and (II.52) as,

$$c_i c_j{}^\dagger = \sqrt{q_1 q_2} c_j{}^\dagger c_i \qquad i \neq j\ , \tag{II.54}$$

$$c_i c_j = \sqrt{q_1^{-1} q_2} c_j c_i \qquad i > j\ , \tag{II.55}$$

$$c_i{}^\dagger c_j{}^\dagger = \sqrt{q_1 q_2^{-1}} c_j{}^\dagger c_i{}^\dagger \qquad i > j\ , \tag{II.56}$$

$$c_1 c_1{}^\dagger = q_1 c_1{}^\dagger c_1 + q_2^N \qquad , \tag{II.57}$$

$$c_i c_i{}^\dagger = q_1 c_i{}^\dagger c_i + [c_{i-1}, c_{i-1}^\dagger]_{q_2} \qquad i > 1\ , \tag{II.58}$$

One should note that for $q_2 = 1$ and $q_1 = 1$ these oscillators simplify to PW oscillators with parameters $q^{1/2}$ and $q^{-1/2}$ respectively. It is possible to go one step further and define C oscillators as:

$$C_i = q_2^{-N/2} c_i \tag{II.59}$$

Then the commutation relations among C oscillators are exactly the ones of the PW oscillator with $q = \sqrt{q_1/q_2}$. Recalling that the PW oscillators were invariant under Uq(n), one should note that starting from quantum group invariant q-oscillators with one parameter, two parameter q-oscillators are formulated.

III.SLq(n) AND RELATIONS AMONG q-OSCILLATORS

III.1 CBY Oscillator and the q-Epsilon Tensor

In section II-3, we have calculated the inner products of states with the same given number of particles using the defining commutation relation of the CBY q-oscillator. In fact states with different number of particles are orthogonal due to the same relation. Now we would like to generalize the inner product calculation, by starting from:

$$\langle i|j\rangle = \delta^i_j \tag{III.1}$$

$$\langle ij|km\rangle = \delta^i_k\langle j|m\rangle + q\delta^i_m\langle j|m\rangle \tag{III.2}$$

$$\langle ijk|mnp\rangle = \delta^i_m\langle jk|np\rangle + q\delta^i_n\langle jk|mp\rangle + q^2\delta^i_p\langle jk|mn\rangle \tag{III.3}$$

and therefore by induction:

$$\begin{aligned}\langle i_1 i_2 i_3\cdots i_n|k_1k_2k_3\cdots k_n\rangle &= \delta^{i_1}_{k_1}\langle i_2 i_3\cdots i_n|k_2k_3\cdots k_n\rangle \\ &+ q\delta^{i_1}_{k_2}\langle i_2 i_3\cdots i_n|k_1k_3\cdots k_n\rangle \\ &+ q^2\delta^{i_1}_{k_3}\langle i_2 i_3\cdots i_n|k_1k_2k_4\cdots k_n\rangle \\ &+ \cdots \\ &+ q^{n-1}\delta^{i_1}_{k_n}\langle i_2 i_3\cdots i_n|k_1k_3\cdots k_{n-1}\rangle \,.\end{aligned} \tag{III.4}$$

We note that an n particle state inner product, can be cxpressed as a linear combination of (n) inner products of $(n-1)$ particle states. Then we define the tensor N using the inner product

$$\langle i_1 \cdots i_n | k_1 \cdots k_n \rangle \equiv N^{i_1 i_2 \cdots i_n}_{k_1 k_2 \cdots k_n} \equiv N^i_k(n, q). \tag{III.5}$$

The recursive expansion property of the inner product states in equation (III.4) can be written in terms of the N tensor as:

$$\begin{aligned} N^{i_1 i_2 i_3 \cdots i_n}_{k_1 k_2 k_3 \cdots k_n} &= N^{i_1}_{k_1} N^{i_2 i_3 \cdots i_n}_{k_2 k_3 \cdots k_n} \\ &+ q N^{i_1}_{k_2} N^{i_2 i_3 \cdots i_n}_{k_1 k_3 \cdots k_n} \\ &+ q^2 N^{i_1}_{k_3} N^{i_2 i_3 \cdots i_n}_{k_1 k_2 k_4 \cdots k_n} \\ &+ \cdots \\ &+ q^{n-1} N^{i_1}_{k_n} N^{i_2 i_3 \cdots i_n}_{k_1 k_3 \cdots k_{n-1}} . \end{aligned} \tag{III.6}$$

In the definition of the N tensor, we see that it is a function of the real parameter q and the number of particles. Then its simplest nontrivial form, $N(1, q) = N^{i_r}_{k_s}$ becomes the ordinary Kronecker delta, $\delta^{i_r}_{k_s}$. For $q = -1$, the N tensor turns out to be, the generalized Knonecker delta $N(n, -1)$ =det(M), where the elements of the nxn matrix M are given by $M_{rs} = \delta^{i_r}_{k_s}$. Following this we can name N as the q-generalized Kronecker delta.

Now it is possible to define a q-dependent ϵ symbol using the generalized Kronecker delta N.

$$\epsilon_{i_1 i_2 \cdots i_n}(q) = N^{12 \cdots n}_{k_1 k_2 \cdots k_n}(-q) \tag{III.7}$$

In fact, the above q-epsilon symbol is a q-generalized permutation tensor. One can see this by observing that in the $q \rightarrow 1$ limit the q-epsilon tensor reduces to the ordinary epsilon tensor. Writing it down explicitly in two dimensions, one sees that, it can be represented by a 2x2 matrix;

$$\epsilon = \begin{bmatrix} 0 & 1 \\ -q & 0 \end{bmatrix} \tag{III.8}$$

whose square is a constant times the identity matrix:

$$\epsilon^2 = -qI \ . \tag{III.9}$$

Due to its definition, the q-epsilon tensor has a very useful index permutation property: i.e. when two neighboring indices i and j of the q-epsilon tensor are interchanged its value changes by a factor of $-q$ if $j < k$ and $-q^{-1}$ if $j > k$. The following notation expresses this property :

$$\epsilon_{i \cdots jk \cdots n} = -q^{sgn(j-k)} \epsilon_{i \cdots kj \cdots n} \ . \tag{III.10}$$

Another interesting property of the new q-epsilon tensor is the contraction of two epsilon tensors over common indices. Using Einstein convention for summation, one can write this property as follows:

$$\epsilon_{\alpha j_2 j_3 \cdots n} \epsilon_{j_2 j_3 \cdots n \alpha'} = (-q)^{n-1} [n-1]_{q^2}! \delta_{\alpha \alpha'} \ . \tag{III.11}$$

III.2 SLq(n) and the q-Epsilon Tensor

The ordinary (non-quantum) group SL(n) can be defined as the group of matrices with real or complex entries, which leave the ordinary Levi-Civita epsilon tensor invariant. In the same manner, we propose the definition of the quantum group SLq(n) [14] as the group of matrices which leave the q-epsilon tensor invariant. The entries of these matrices, as a generalization of complex fields, probably consist of non-commuting elements. We then propose the invariance of the q-epsilon tensor under the left and the right action of the "quantum" matrix A_{ij}.

$$\epsilon_{i_1 i_2 \cdots i_n} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} = \epsilon_{j_1 j_2 \cdots j_n} \qquad \text{(III.12)}$$

$$A_{j_1 i_1} A_{j_2 i_2} \cdots A_{j_n i_n} \epsilon_{i_1 i_2 \cdots i_n} = \epsilon_{j_1 j_2 \cdots j_n} \qquad \text{(III.13)}$$

where the matrices will probably be the group elements of SLq(n). For $|q| \neq 1$, the above relations cannot be satisfied, with commuting elements of the matrices A. Therefore we assume that the matrices A have non-commuting elements A_{ij}.

III.2.1 The Two Dimensional Case

As the ordinary epsilon tensor, the q-epsilon tensor has the value zero, if two indices are equal. So we have 4 equations for two distinct values of i and j in the two dimensional version of both (III.12) and (III.13):

$$\begin{aligned}
\epsilon_{i_1 i_2} A_{i_1 j_1} A_{i_2 j_2} &= \epsilon_{j_1 j_2} \\
A_{11}A_{21} - qA_{21}A_{11} &= 0 \qquad j_1 = 1, j_2 = 1 &\text{(III.14)}\\
A_{11}A_{22} - qA_{21}A_{12} &= 1 \qquad j_1 = 1, j_2 = 2 &\text{(III.15)}\\
A_{12}A_{21} - qA_{22}A_{11} &= -q \qquad j_1 = 2, j_2 = 1 &\text{(III.16)}\\
A_{12}A_{22} - qA_{22}A_{12} &= 0 \qquad j_1 = 2, j_2 = 2\ . &\text{(III.17)}
\end{aligned}$$

Now, if we represent the 2x2 matrix A as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} , \tag{III.18}$$

these 4 equations can be written as

$$\begin{aligned} ac &= qca & \text{(III.19)} \\ ad &- qcb = 1 & \text{(III.20)} \\ da &- q^{-1}bc = 1 & \text{(III.21)} \\ bd &= qdb\,. & \text{(III.22)} \end{aligned}$$

At this point, we define the q-determinant of a 2×2 matrix as in (III.20) and we see that our matrix A has unit determinant. We note that these are the half of the set of equations which define SL$q(2)$. We can then write down the next 4 equations and obtain the full SL$q(2)$ group. But instead, we prefer to do it with a special trick which only holds for n=2, since for n=2 the epsilon tensor has a matrix representation. The equation (III.12) written in matrix form is

$$A^T \epsilon A = \epsilon\,. \tag{III.23}$$

Using the existence of A^{-1} and the fact that $\epsilon^2 = -qI$, (III.23) can be written as:

$$A \epsilon A^T = \epsilon\,, \tag{III.24}$$

which when solved gives:

$$\begin{aligned} ab &= qba & \text{(III.25)} \\ ad &- qbc = 1 & \text{(III.26)} \\ da &- q^{-1}cb = 1 & \text{(III.27)} \\ cd &= qdc\,. & \text{(III.28)} \end{aligned}$$

These are the relations which complete the defining relations of SL$q(2)$. We note that we have found the complete set of SL$q(2)$ definition relations, using only one equation from our starting propositions.

III.2.2 The Three Dimensional Case

The defining equation is now,

$$\epsilon_{i_1 i_2 i_3} A_{i_1 j_1} A_{i_2 j_2} A_{i_3 j_3} = \epsilon_{j_1 j_2 j_3} \tag{III.29}$$

in which A_{ij} are elements of the matrix A where,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} . \tag{III.30}$$

Proceeding as in the $n \doteq 2$ case, we end up with 27 equations given below:

$$
\begin{array}{lll}
1. & abc - qacb - qbac + q^2bca + q^2cab - q^3cba & = 0 \\
2. & abf - qace - qbaf + q^2bcd + q^2cae - q^3cbd & = 0 \\
3. & abi - qach - qbai + q^2bcg + q^2cah - q^3cbg & = 0 \\
4. & aec - qafb - qbdc + q^2bfa + q^2cdb - q^3cea & = 0 \\
5. & aef - qafe - qbdf + q^2bfd + q^2cde - q^3ced & = 0 \\
6. & aei - qafh - qbdi + q^2bfg + q^2cdh - q^3ceg & = 1 \\
7. & ahc - qaib - qbgc + q^2bia + q^2cgb - q^3cha & = 0 \\
8. & ahf - qaie - qbgf + q^2bid + q^2cge - q^3chd & = -q \\
9. & ahi - qaih - qbgi + q^2big + q^2cgh - q^3chg & = 0 \\
10. & dbc - qdcb - qeac + q^2eca + q^2fab - q^3fba & = 0 \\
11. & dbf - qdce - qeaf + q^2ecd + q^2fae - q^3fbd & = 0 \\
12. & dbi - qdch - qeai + q^2ecg + q^2fah - q^3fbg & = -q \\
13. & dec - qdfb - qedc + q^2efa + q^2fdb - q^3fea & = 0 \\
14. & def - qdfe - qedf + q^2efd + q^2fde - q^3fed & = 0 \\
15. & dei - qdfh - qedi + q^2efg + q^2fdh - q^3feg & = 1 \\
16. & dhc - qdib - qegc + q^2eia + q^2fgb - q^3fha & = q^2 \\
17. & dhf - qdie - qegf + q^2eid + q^2fge - q^3fhd & = 0 \\
18. & dhi - qdih - qegi + q^2eig + q^2fgh - q^3fhg & = 0 \\
19. & gbc - qgcb - qhac + q^2hca + q^2iab - q^3iba & = 0 \\
20. & gbf - qgce - qhaf + q^2hcd + q^2iae - q^3ibd & = q^2 \\
21. & gbi - qgch - qhai + q^2hcg + q^2iah - q^3ibg & = 0 \\
22. & gec - qgfb - qhdc + q^2hfa + q^2idb - q^3iea & = -q^3 \\
23. & gef - qgfe - qhdf + q^2hfd + q^2ide - q^3ied & = 0 \\
24. & gei - qgfh - qhdi + q^2hfg + q^2idh - q^3ieg & = 0 \\
25. & ghc - qgib - qhgc + q^2hia + q^2igb - q^3iha & = 0 \\
26. & ghf - qgie - qhgf + q^2hid + q^2ige - q^3ihd & = 0 \\
27. & ghi - qgih - qhgi + q^2hig + q^2igh - q^3ihg & = 0
\end{array}
$$

Now we have to group these equations in order to extract useful information. The equations $((1,10,19),\ (9,18,27),\ (5,14,23))$ are written as $A\vec{u_i} = 0 \quad (i = 1,2,3)$, where $\vec{u_i}$ denote the column vectors obtained by factoring out the elements of the matrix A from the left hand side. As an example, equations (1,10,19) can be written as :

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{pmatrix} bc - qcb \\ q^2ca - qac \\ q^2ab - q^3ba \end{pmatrix} = 0$$

If A is invertible, then relations of the kind $ab = qba$ are obtained for every 2×2 submatrix of A. For the remaining set of equations, we define the q-subdeterminant of a $n \times n$ matrix, as in the two dimensional case. Thus Δ_{ij} is the q determinant of the remaining submatrix obtained by removing the i^{th} row and the j^{th} column from the original matrix A. For example Δ_{33} is $ae - qbd$. Then the Δ_{ij} can be shown to commute with the elements of the matrix A. Now, we would like to group the set of nine equations as $((2,11,20),\ (3,12,21),\ (6,15,25))$ and write the first triplet as:

$$A\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ q^2 \end{pmatrix}$$

where

$$\vec{v_3} = \begin{pmatrix} \Delta_{31} \\ -q\Delta_{32} \\ q^2\Delta_{33} \end{pmatrix}$$

by factoring out the elements of A from the left hand side. If the existence of A^{-1} is assumed with $A^{-1}A = I$, these sets of equations give A^{-1} as:

$$A^{-1} = \begin{bmatrix} \Delta_{11} & -q^{-1}\Delta_{21} & q^{-2}\Delta_{31} \\ -q\Delta_{12} & \Delta_{22} & -q^{-1}\Delta_{32} \\ q^2\Delta_{13} & -q\Delta_{23} & \Delta_{33} \end{bmatrix} . \qquad \text{(III.31)}$$

For the remaining set of nine equations, grouped as $((4,13,22),\ (8,17,26),\ (7,16,25))$, we would like to make the following definition: Let Δ'_{ij} be the q-subdeterminant of the matrix A, as in equation (III.21), calculated starting from left lower corner, e.g.: $\Delta'_{33} = ea - q^{-1}db$.

Again factoring out the elements of the matrix A from the left hand side, and writing down as vector transformation equations, using the existence of A^{-1}, A^{-1} is obtained in terms of Δ'_{ij} as:

$$A^{-1} = \begin{bmatrix} \Delta'_{11} & -q^{-1}\Delta'_{21} & q^{-2}\Delta'_{31} \\ -q\Delta'_{12} & \Delta'_{22} & -q^{-1}\Delta'_{32} \\ q^2\Delta'_{13} & -q\Delta'_{23} & \Delta'_{33} \end{bmatrix} . \tag{III.32}$$

From the uniqueness of A^{-1} it follows that $\Delta'_{ij} = \Delta_{ij}$ and this holds for every 2x2 submatrix of A. Then we obtain for example $ae - qbd = \Delta_{33} = ea - q^{-1}db$.

We can now postulate the definition of the SL$q(3)$ as the group of 3x3 matrices of determinant one and which have all 2x2 submatrices, elements of GL$q(2)$. Thus we have obtained half of the GL$q(3)$ definition relations. In order to obtain the full set, this time, stating from (III.13) instead of (III.12), we would have another set of 27 equations which can be grouped and treated as in the above case. This second set would give us the rest of the defining relations of GL$q(3)$. Noting that from equation (6), our matrix A has unit determinant, we finally conclude that $A \in$ SL$q(3)$.

III.2.3 Generalization to n-Dimensional Case

Since we obtained SL$q(2)$ from the invariance of $\epsilon(2,q)$ and SL$q(3)$ from the invariance of $\epsilon(3,q)$, we would like to study the invariance of $\epsilon(n,q)$ and show that it gives rise to SL$q(n)$. In these calculations, for the sake of clearity, we prefer not to use the Einstein convention for summation, but instead to write all the summations explicitly. Starting from (III.13), we can write:

$$\sum_{m_1\cdots m_{n-2}} \sum_{r,s} \epsilon_{m_1\cdots m_{n-2}rs} a_{i_1m_1} \cdots a_{i_{n-2}m_{n-2}} a_{jr} a_{ks} = \epsilon_{i_1\cdots i_{n-2}jk} \tag{III.33}$$

and renaming the indices j and k, the same equation reads,

$$\sum_{m_1\cdots m_{n-2}} \sum_{r,s} \epsilon_{m_1\cdots m_{n-2}rs} a_{i_1m_1} \cdots a_{i_{n-2}m_{n-2}} a_{kr} a_{js} = \epsilon_{i_1\cdots i_{n-2}kj} \quad . \tag{III.34}$$

If we now assume $j < k$, we can use the index permutation property (III.10) and, we can join the above two equations as

$$\begin{aligned} \sum_{m_1\cdots m_{n-2}} \sum_{r,s} \epsilon_{m_1\cdots m_{n-2}rs} a_{i_1m_1} \cdots a_{i_{n-2}m_{n-2}} a_{jr} a_{ks} &= \\ -q^{-1} \sum_{m_1\cdots m_{n-2}} \sum_{r,s} \epsilon_{m_1\cdots m_{n-2}rs} a_{i_1m_1} &\cdots a_{i_{n-2}m_{n-2}} a_{kr} a_{js} \quad . \end{aligned} \tag{III.35}$$

We would like to divide the summation in the above equation over r, s into two parts such that

$$\sum_{r,s} = \sum_{r<s} + \sum_{r>s}$$

since $\epsilon_{m_1\cdots m_{n-2}rr}$ is zero. Interchanging the indices r and s in the second summation, one finds:

$$\begin{aligned} &\sum_{m_1\cdots m_{n-2}} \sum_{r<s} \epsilon_{m_1\cdots m_{n-2}rs} a_{i_1m_1} \cdots a_{i_{n-2}m_{n-2}} a_{jr} a_{ks} \quad + \\ &\qquad \sum_{m_1\cdots m_{n-2}} \sum_{s>r} \epsilon_{m_1\cdots m_{n-2}sr} a_{i_1m_1} \cdots a_{i_{n-2}m_{n-2}} a_{js} a_{kr} = \\ -q^{-1}(&\sum_{m_1\cdots m_{n-2}} \sum_{r>s} \epsilon_{m_1\cdots m_{n-2}rs} a_{i_1m_1} \cdots a_{i_{n-2}m_{n-2}} a_{kr} a_{js} \quad + \\ &\qquad \sum_{m_1\cdots m_{n-2}} \sum_{s>r} \epsilon_{m_1\cdots m_{n-2}sr} a_{i_1m_1} \cdots a_{i_{n-2}m_{n-2}} a_{ks} a_{jr}). \end{aligned} \tag{III.36}$$

Since all the summations over s and r are ordered, it is possible to use the index permutation property (III.10) and factor out $\epsilon_{m_1\cdots m_{n-2}rs}$ from the left hand side. Then equation (III.36) reads:

$$\sum_{m_1\cdots m_{n-2}}\sum_{r<s} a_{i_1m_1}\cdots a_{i_{n-2}m_{n-2}}\epsilon_{m_1\cdots m_{n-2}rs}C_{jrks} = 0\ , \tag{III.37}$$

$$C_{jrks} \equiv ((a_{jr}a_{ks} - qa_{js}a_{kr}) + q^{-1}(a_{kr}a_{js} - qa_{ks}a_{jr}))\ .$$

If the invertibility of A is assumed, matrix multiplying from left by A^{-1}, equation (III.37) gives:

$$\sum_{r<s}\epsilon_{m_1\cdots m_{n-2}rs}((a_{jr}a_{ks} - qa_{js}a_{kr}) + q^{-1}(a_{kr}a_{js} - qa_{ks}a_{jr})) = 0\ . \tag{III.38}$$

One should note that the first $n-2$ indices of the epsilon tensor are free, any choice of them such that epsilon is non vanishing uniquely determines both r and s, since $r < s$. If the trivial solution of equation (III.38) is left out, the result can be written as:

$$((a_{jr}a_{ks} - qa_{js}a_{kr}) + q^{-1}(a_{kr}a_{js} - qa_{ks}a_{jr})) = 0 \tag{III.39}$$

with $r < s$ and $j < k$.

If we define an arbitrary 2×2 submatrix of A as follows, we can better understand the meaning of equation (III.39).

$$
A = \begin{array}{c} \\ j \\ \\ k \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cc} r & s \end{array} \\
\begin{bmatrix}
\vdots & \vdots \\
\cdots a \cdots & \cdots b \cdots \\
\vdots & \vdots \\
\cdots c \cdots & \cdots d \cdots \\
\vdots & \vdots
\end{bmatrix}
\end{array}
$$

In this visualization, equation (III.39) simply gives us: $ad - qbc = da - q^{-1}cb$ which is one of the defining equations of GL$q(2)$. Then we can define an arbitrary $n \times n$ matrix A as element of GLq(n), if and only if its any 2×2 submatrix is an element of GL$q(2)$. If we had started from (III.12) instead of (III.13), we could proceed likewise, but this time the counterpart of equation (III.37) would be multiplied by A^{-1} from the right, instead of from the left, and we would obtain

$$((a_{jr}a_{ks} - qa_{kr}a_{js}) = (a_{ks}a_{jr} - q^{-1}a_{js}a_{kr})) \tag{III.40}$$

with $r < s$ and $j < k$ again. The new equation (III.40) when applied to our above visualization of an arbitrary 2x2 submatrix gives: $ad - qcb = da - q^{-1}bc$ .Combining the equations (III.39) and (III.40), we obtain ,

$$a_{js}a_{kr} = a_{kr}a_{js} \tag{III.41}$$

which can be shown by our visualization as $bc = cb$.

Until now we have investigated the relations between diagonal elements of our arbitrary 2x2 submatrix. In order to obtain the relations between row/column elements, we start from (III.33) and postulate that $j = k$, so that the right hand side becomes zero. Then we divide the summation into two parts as we did previously :

$$\sum_{m_1 \cdots m_{n-2}} \sum_{r<s} \epsilon_{m_1 \cdots m_{n-2} rs} a_{i_1 m_1} \cdots a_{i_{n-2} m_{n-2}} a_{jr} a_{js} \quad + \tag{III.42}$$

$$\sum_{m_1 \cdots m_{n-2}} \sum_{r>s} \epsilon_{m_1 \cdots m_{n-2} r s} a_{i_1 m_1} \quad \cdots \quad a_{i_{n-2} m_{n-2}} a_{jr} a_{js} = 0 \ .$$

Interchanging r and s in the second part and using the index permutation property of the epsilon tensor, the above equation gives :

$$\sum_{m_1 \cdots m_{n-2}} \sum_{r<s} a_{i_1 m_1} \cdots a_{i_{n-2} m_{n-2}} \epsilon_{m_1 \cdots m_{n-2} r s} a_{jr} a_{js} - q a_{js} a_{jr} = 0 \ . \qquad \text{(III.43)}$$

Again, if the invertibility of A is assumed, matrix multiplication from left by A^{-1} yields:

$$a_{jr} a_{js} - q a_{js} a_{jr} = 0 \qquad \text{(III.44)}$$

which when applied to our representative 2x2 submatrix gives $ab = qba$. The solution by the same procedure, starting from (III.12) instead of (III.13) would give :

$$a_{jr} a_{kr} - q a_{kr} a_{jr} = 0 \ , \qquad \text{(III.45)}$$

which can be represented as $ac = qca$ for our arbitrary 2x2 submatrix.

Thus we have discovered all the necessary relations to make any arbitrary 2x2 submatrix of A, an element of the quantum group GLq(2). Moreover $\epsilon_{12 \cdots n} = 1$. This property ensures that the matrix A has unit determinant. It follows that, using these results, $A \in$ SLq(n).

III.3 Relations Between Inner Products of States for CBY-PW-AC q-Oscillators

In (II.4), we have seen the procedure for obtaining the creation/annihilation operators of PW oscillators from the creation/annihilation operators of AC oscillators. Moreover, the defining relations of A€ oscillators and CBY oscillators were very similar. In fact there is a very close relation between the CBY oscillator and the PW oscillator through the innner products of states.

If we note by a subindex PW, the states created from vacuum by PW creation operators and by a subindex CBY, the states belonging to CBY oscillator, it is possible to relate these two states by the following formula:

$$|i_1 i_2 \cdots i_n\rangle_{PW} = |i'_1 i'_2 \cdots i'_n\rangle_{CBY} \langle i'_1 i'_2 \cdots i'_n | i_1 i_2 \cdots i_n\rangle_{CBY}\, C_{i_1 \cdots i_n} \tag{III.46}$$

where

$$C_{i_1 i_2 \cdots i_n} \equiv \sqrt{\frac{[r_1]_{q^2}!\,[r_2]_{q^2}!\,\cdots [r_n]_{q^2}!}{([r_1]_q!)^3\,([r_2]_q!)^3\,\cdots([r_n]_q!)^3}}\ . \tag{III.47}$$

Here we note that the states on the right hand side belong only to CBY and the states on the left hand side to PW oscillators. The primed states, $|i'_1 i'_2 \cdots i'_n\rangle$ represent the "normal" ordering of the indices in $|i_1 i_2 \cdots i_n\rangle$ such that $i'_1 \leq i'_2 \leq \cdots \leq i'_n$. In the definition of the totally symmetric symbol C , r_i is the number of times index i occurs among $i_1 i_2 \cdots i_n$. The subindex q or q^2 of basic numbers [r] represent the parameter of each basic number .

As an application of the above formula, we would like to check it for the $|121\rangle_{PW}$ state. We start by calculating its norm from the defining relations of the PW oscillators given previously :

$$\langle 121|121\rangle_{PW} = \langle\ | a_1 a_2 a_1 a_1^\dagger a_2^\dagger a_1^\dagger |\ \rangle = q^2(q^2+1)\ \ , \tag{III.48}$$

recalling that states with all particles different are normalized to unity. The equation (III.46) implies that

$$|121\rangle_{PW} = |112\rangle_{CBY}\langle 112|121\rangle_{CBY} C_{121}\ . \tag{III.49}$$

If we calculate the norm of the right hand side, we will need the following inner products

$$\langle 112|121\rangle_{CBY} = q(1+q)\,, \tag{III.50}$$

$$\langle 112|112\rangle_{CBY} = (1+q) \tag{III.51}$$

and noting that only one of the indices is repeated two times (i.e. $r_1 = 2$, $r_{i\neq 1} = 1$) the number C_{121} can be written as follows:

$$C_{121} = \sqrt{\frac{[2]_{q^2}!}{([2]_q!)^3}} = \sqrt{\frac{1+q^2}{(1+q)^3}}\,. \tag{III.52}$$

Then the norm of the left hand side of (III.49) can be expressed as:

$$q^2(1+q^2) = \frac{1+q^2}{(1+q)^3}q^2(1+q)^3\,. \tag{III.53}$$

Since the left hand side of the above equation is identically equal to its right hand side, one can say that the eqauation (III.46) is verified for this case.

The relationship between the CBY states and the AC states is much more simple. A state of the AC oscillator $|i_1 i_2 \cdots i_n\rangle_{AC}$ can be expressed by CBY states as $|i'_1 i'_2 \cdots i'_n\rangle$ where primes represent the "normal" ordering of the AC indices such that $i'_1 \leq i'_2 \leq \cdots \leq i'_n$.

IV. FIBONACCIZATION AND FURTHER MULTIDIMENSIONAL, MULTIPARAMETER GENERALIZATIONS

IV.1 Two Parameter Generalization of CBY q-Oscillators

In section 5 of chapter 2, we have seen a two parameter generalization of PW multidimensional q-oscillators. The associated basic number and the q-oscillator were named as the Fibonacci basic number and the Fibonacci oscillator respectively. Now we would like to follow a similar procedure and obtain a two parameter version of CBY oscillators. We will call this procedure "Fibonaccization".

IV.1.1 Introduction

For the Fibonaccization of CBY oscillators, we start from equation (II.28) and write it in the following form:

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij} \, . \tag{IV.1}$$

Noting that the right hand side of this equation is a scalar, we multiply it by $a_k^\dagger$ from both sides and obtain

$$a_k^\dagger (a_i a_j^\dagger - q a_j^\dagger a_i) = (a_i a_j^\dagger - q a_j^\dagger a_i) a_k^\dagger \, . \tag{IV.2}$$

Rearranging the terms one can also write the same equation as:

$$a_i a_j^\dagger a_k^\dagger = q a_j^\dagger a_i a_k^\dagger - q a_k^\dagger a_j^\dagger a_i + a_k^\dagger a_i a_j^\dagger \, . \tag{IV.3}$$

Now, we introduce another parameter q_2 by defining a new operator b:

$$
\begin{aligned}
b &= q^{N/2}a \\
b^\dagger &= a^\dagger q^{N/2} .
\end{aligned} \tag{IV.4}
$$

Using the fact that $aa^\dagger = [N+1]$, $a^\dagger a = N$ and also equation (IV.4) one can find:

$$
\begin{aligned}
bf(N) &= f(N+1)b \\
b^\dagger f(N) &= f(N-1)b^\dagger .
\end{aligned} \tag{IV.5}
$$

Writing equation (IV.3) in terms of b and $b^\dagger$, and cancelling $q_2^{-3N/2}$ from both sides one has:

$$
b_i b_j^\dagger b_k^\dagger = qq_2 b_j^\dagger b_i b_k^\dagger - qq_2^2 b_k^\dagger b_j^\dagger b_i + q_2 b_k^\dagger b_i b_j^\dagger . \tag{IV.6}
$$

If we set $q = q_1/q_2$, we obtain the two parameter version of the multidimensional CBY q-oscillator:

$$
b_i b_j^\dagger b_k^\dagger = q_1 b_j^\dagger b_i b_k^\dagger - q_1 q_2 b_k^\dagger b_j^\dagger b_i + q_2 b_k^\dagger b_i b_j^\dagger . \tag{IV.7}
$$

Following the notation in the literature, we call the above defined q-oscillator as the Fibonacci-CBY q-oscillator (hereafter denoted as fCBY). We note that, if in equation (IV.2) a_k was used instead of $a_k^\dagger$, the hermitian conjugate of the above equation could be obtained.

IV.1.2 Relations Between the States of CBY and fCBY Oscillators

Using the equation (IV.7), we now calculate a few inner products of this two parameter oscillator, and name the tensor defined by the inner products as $N_f(q_1, q_2, n)$ where n is the number of particles in a state :

$$\begin{aligned}\langle i|j\rangle &= \delta^i_j = N_f(q_1,q_2,1) \\ \langle ij|km\rangle &= q_2\delta^i_k\langle j|m\rangle + q_1\delta^i_m\langle j|k\rangle = N_f(q_1,q_2,2) \\ \langle ijk|mnp\rangle &= q_2^2\delta^i_m\langle jk|np\rangle + q_1q_2\delta^i_n\langle jk|mp\rangle + q_1^2\delta^i_p\langle jk|mn\rangle = N_f(q_1,q_2,3)\ .\end{aligned} \tag{IV.8}$$

Using the above calculations, we see that, the new Fibonacci $N_f(q_1,q_2,n)$ tensor can be expanded in terms of $N_f(q1,q2,n-1)$:

$$\begin{aligned}{N_f}^{i_1i_2i_3\cdots i_n}_{k_1k_2k_3\cdots k_n} &= q_2^{n-1}{N_f}^{i_1}_{k_1}{N_f}^{i_2i_3\cdots i_n}_{k_2k_3\cdots k_n} \\ &+ q_1q_2^{n-2}{N_f}^{i_1}_{k_2}{N_f}^{i_2i_3\cdots i_n}_{k_1k_3\cdots k_n} \\ &+ q_1^2q_2^{n-3}{N_f}^{i_1}_{k_3}{N_f}^{i_2i_3\cdots i_n}_{k_1k_2k_4\cdots k_n} \\ &+ \cdots \\ &+ q_1^{n-1}{N_f}^{i_1}_{k_n}{N_f}^{i_2i_3\cdots i_n}_{k_1k_3\cdots k_{n-1}}\ .\end{aligned} \tag{IV.9}$$

It is also possible to find a relation between the inner products of the CBY oscillator and the fCBY oscillator. For the two dimensional case

$$N = q_2^{-1}N_f\ |_{q=q_1/q_2}$$

and for the three dimensional case

$$N = q_2^{-3}N_f\ |_{q=q_1/q_2}\quad .$$

We would like to extend this relation to n dimensions and obtain a general formula to relate $N(q,n)$ to $N_f(q_1,q_2,n)$. To accomplish this, we start by noting that N_f is symmetric in $q_1 \leftrightarrow q_2$ and each time, we multiply N_f by the highest power of q_2 in the associated inner product. Then we can write a d-dimensional inner product as:

$$\langle i_1 i_2 \cdots i_d | j_1 j_2 \cdots j_d \rangle = \langle \, | a_{i_d} \cdots a_{i_2} a_{i_1} a^\dagger_{j_1} a^\dagger_{j_2} \cdots a^\dagger_{j_d} | \, \rangle \, . \tag{IV.10}$$

In the above expression each a_{i_k} should be coupled with the rightmost possible $a^\dagger_{j_m}$, giving rise to a factor of q_1 each time. Thus a_{i_1} will couple with $a^\dagger_{j_d}$, picking up (d-1) times q_1. The next one, a_{i_1}, will then couple to $a^\dagger_{j_{d-1}}$ and pick up (d-2) times q_1 etc. We denote the total power of q_1 by $m(d)$. It can be found by the summation

$$m(d) = \sum_{i=1}^{d} (d-i) = \frac{d(d-1)}{2}$$

From the symmetry of powers of q_1 and q_2, we can write the most general relation as

$$\langle i_1 \cdots i_d | j_1 \cdots j_d \rangle_{fCBY} = q_2^{m(d)} \langle i_1 \cdots i_d | j_1 \cdots j_d \rangle_{CBY|q=q_1/q_2} \; . \tag{IV.11}$$

Since the Fibonacci CBY states are related to the CBY states by the above formula, the tensors N and ϵ obtained from both oscillators are also related. Then the group discovered through the invariance properties of the Fibonacci epsilon tensor is again SLq(n), where $q = q_1/q_2$.

IV.1.3 fCBY Oscillator and Basic Numbers

The defining equation of the fCBY oscillator was multidimensional (IV.7). We now want to consider its one dimensional version in order to be able to find the associated basic number. Setting $i = j = k$ one finds:

$$bb^{\dagger 2} = (q_1 + q_2)b^\dagger b b^\dagger - q_1 q_2 b^{\dagger 2} b. \tag{IV.12}$$

The above equation can also be written as a difference equation if one identifies $[N] = b^{\dagger 2}b$, $[N+1] = b^\dagger b b^\dagger$ and $[N+2] = bb^{\dagger 2}$. Then the corresponding equation is of the form as (II.45); and the basic number obtained is (II.41)

$$[n] = \frac{q_1^n - q_2^n}{q_1 - q_2} \quad . \tag{IV.13}$$

IV.2 n Parameter Generalization of CBY Oscillators

In the last section starting from the CBY oscillator, we have obtained its two parameter version and we have named it as the Fibonacci CBY oscillator. In this section we will generalize the commutation relation (IV.7), to contain "n" parameters that we call q_i. We start by noting that the defining equation (IV.7) can be written using the commutator notation (II.12) as [28]:

$$[[b_i, b_j^\dagger]_{q_1}, b_k^\dagger]_{q_2} = 0 \quad . \tag{IV.14}$$

Since the righthand side of the above equation is zero, it is possible to introduce a new parameter, q_3, in the form of a multi-commutator:

$$[[[b_i, b_j^\dagger]_{q_1}, b_k^\dagger]_{q_2}, b_l^\dagger]_{q_3} = 0 \quad . \tag{IV.15}$$

This process can be continued for any number of parameters to obtain:

$$[\cdots[[b_i, b_j^\dagger]_{q_1}, b_k^\dagger]_{q_2}, \cdots b_z^\dagger]_{q_n} = 0 \quad . \tag{IV.16}$$

The resulting equation is the defining relation of a multidimensional n parameter q-oscillator.

One can consider its one dimensional version in order to find the associated basic number. For the n parameter version, the number of required initial conditions is also n. If these are choosen to be the integers up to n, one can obtain a spectrum in which the number of parameters imply the point at which the q effects will be observed.

V. CONCLUSION

After all these calculations, now a few words on them:

Chapter II was a review, an introduction to the subject. Chapters III and IV were original work. In chapter III we derived the q-epsilon tensor and its properties. Using this tensor's invariance properties, the quantum group GLq(n) was constructed. This chapter was mainly based on our February 94 paper [19]. The relations between state vectors of U(n) and Uq(n) invariant oscillators were also investigated.

In chapter IV, a multidiparameter version of one parameter CBY oscillators was constructed. If the number of parameters is set to two, which naturally comes out, the process is called Fibonaccization. For higher number of parameters, by changing the initial conditions and by adjusting the number of parameters it is possible to relate these calculations to fitting the q-oscilator spectrum to a real physical problem (in which q_i will represent the internal, unknown parameters). This may be the next step to this work.

Due to the exponential increase of knowledge, yesterday's calculations made only for the mathematical beauty, are vital for today's science and technology. And hopefully today's mathematical speculations will develop into an experimentally confirmed physical theory in the future.

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