GAUGE THEORIES BASED ON QUANTUM GROUPS, REPRESENTATIONS OF THE BRAID GROUP

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ABSTRACT

Possible formulations of gauge field models where the gauge group is a quantum group are discussed. The exponential map from the generators of the Lie algebra analog of the quantum group $SU_q(2)$ to the quantum group $SU_q(2)$ itself is presented. The q-deformed Yang-Mills theory is introduced via the definition of the q-trace and the q-deformed Yang-Mills lagrangian which is invariant under the quantum group gauge transformations. The gauge field takes values in the quantum universal enveloping algebra of $SU_q(2)$. As a result of this construction a Weinberg type mixing angle which depends on the quantum group deformation parameter q is obtained.

The representations of the n-braid group where generators are given essentially by 2×2 matrices whose elements belong to a noncommutative algebra are presented. The Burau representation arises as a special (commuting) case of this algebra. A closely related algebra to the braid algebra is introduced and it is shown that the generalized oscillator system given by this algebra generates a hydrogen-like spectrum.

ÖZET

Kuantum gruplarına dayalı ayar alan teorisi modelleri tartışıldıldı. $SU_q(2)$ kuantum grubunun Lie cebri analogunun elemanı olan jeneratörler bulundu. Kuantum gurubu ayar dönüşümü altında invaryant kalan yeni bir iz tanımı yapıldı ve bu tanım kullanılarak deforme edilmiş Yang-Mills Lagranjiyeni inşa edildi.

Elemanları komütatif olmayan bir cebre ait olan 2×2 matrisler kullanılarak Artin örgü grubunun temsilleri elde edildi. Burau temsilinin bu cebrin özel bir hali olduğu gösterildi. Örgü cebrine çok yakın olan, "sözde örgü cebri" diye adlandırdığımız cebrin tanımı yapıldı ve bu cebrin verdiği genelleşmiş osilatör sisteminin hidrojen tipi spektrum verdiği gösterildi.

LIST OF SYMBOLS

A_μ, B_μ	Gauge fields
a, b, c, d	Operators in a Hilbert space
a_n, b_n, c_n, d_n	Eigenvalues of the operators
$F_{\mu\nu}$	Curvature
H, X_+, X_-	QUE generators
J_μ	Current density
L	Lie algebra element
L_q	Lagrangian invariant under quantum group gauge transformations
q	Deformation parameter
Tr_q	Quantum trace
∇_μ	Covariant derivative
θ	Weinberg angle

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1. INTRODUCTION

This thesis consists of two seperate studies.The first study involves gauge theories based on quantum groupswhich appeared in the beginning as a mathematical abstraction in completely integrable systems[1] and statistical mechanical models[2] have attracted a lot of attention. Quantum groups found applications in the theory of lattice models[3], string theories[4], conformal field theories[5] and other topics. The opportunity to use quantum groups instead of Lie groups as gauge groups may generalize the symmetry and solve the standard problems of gauge theories, e.g. quark confinement in QCD by the introduction of Higgs scalars and the difficulties of grand unification. Work along these lines was started by Arafeva and Volovich[6] followed by Isaev and Popowicz[7] and Castellani[8]. They took $SU_q(2)$ as the gauge group instead of SU(2) and worked on the possible constructions of gauge theories. In chapter 2 we discuss two of these possibilities. The first possibility is to take the gauge field as an element of the Lie algebra analog of $SU_q(2)$ while in the other approach the gauge field is an element of the quantum universal algebra of su(2) . In section 2.1, we briefly discuss the quantum Lie group. In section 2.2, we will show that quantum groups can be used for the solutions of the equation of motion for chiral fields. The exponential map from the "generators" to the quantum group $SU_q(2)$ is presented in section 2.3. The deformation of the gauge group, i.e. the use of noncommutative matrix elements instead of the commuting ones, requires modification of the definition of trace and covariant derivative as well as the Lagrangian which is invariant under the quantum group action. The q-deformed Yang-Mills theory with the gauge group $\underline{U_q(2)}$ is discussed and the Weinberg type mixing angle depending on the deformation parameter q is introduced in section 2.4.

The second study involves the braid group which is related to quantum groups.The discovery of new algebraic structures related to braids and to knots and links generated by braid closure has attracted a lot of attention in the past few years[9,10]. The developments in this area have brought about relations among the areas of knot invariants, gauge theories[11], statistical mechanical models[12] and quantum groups[13]. In chapter 3, we shall investigate a class of algebras related to the braid group. We will particularly emphasize the representations of this algebra in a Hilbert space. Our motivation for such a representation is to obtain a direct link between mathematics and physics through quantum mechanics where hermitian operators can be identified with physical observables.

In section 3.1 we will construct a Burau-like representation where each generator of the n-braid group will be represented by a $n \times n$ matrix whose nontrivial part is a 2×2 matrix with matrix elements belonging to an associative but noncommutative algebra. This defines a set of commutation-like relations among the four operator elements of the 2×2

matrix. In section 3.1 we search for a representation of the algebra in a Hilbert space. We show that except for trivial representations where one matrix element is identically zero and the other matrix elements are commutative (Burau representation) further relations have to be satisfied. We discuss several representations that can be obtained and show that there are no unitary representations. In section 3.3 we introduce the pseudo-braid algebra by relaxing the conditions found in section 3.2. We show that this algebra has finite dimensional unitary representations. For both types of representations, one obtains a raising operator b^* and a lowering operator b. The hermitian nonnegative operator b^*b has eigenvalues depending on a parameter q. This eigenvalue spectrum, in the limit $q \to 1$, becomes a hydrogen spectrum, and for $q \neq 1$ gives a one parameter generalization. For the hermitian representation, q is a real number whereas for the unitary representations which are finite dimensional q is a root of unity. In section 4 we present a discussion of our results.

2. GAUGE THEORIES BASED ON QUANTUM GROUPS

2.1. Quantum Lie Group

2.1.1. Quantum Group $SL_q(2)$

An element of the SL(2) group in 2 dimensional representation is

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.1.1}$$

where a, b, c and d are complex numbers and the determinant is unity

$$Det\ g = ad - bc = 1. \tag{2.1.2}$$

We have the inverse of g

$$g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{2.1.3}$$

The inverse is also an element of SL(2). We deform SL(2) by taking the entries not as complex numbers but non-commuting objects

$$g = \begin{bmatrix} a & qb \\ c & d \end{bmatrix} \tag{2.1.4}$$

with the relations

$$ab=qba \tag{2.1.5}$$
$$ac=qca \tag{2.1.6}$$
$$bd=qdb \tag{2.1.7}$$
$$bc=cb \tag{2.1.8}$$
$$Det\ g = ad - q^2 bc = da - bc = 1 \tag{2.1.9}$$

and the inverse matrix

$$g^{-1} = \begin{bmatrix} a' & qb' \\ c' & d' \end{bmatrix} = \begin{bmatrix} d & -b \\ -qc & a \end{bmatrix}. \tag{2.1.10}$$

Now the matrix with primed entries satisfies the relations (2.1.5)-(2.1.9), but with $q \to q^{-1}$. In fact, the entries of quantum matrix g^n satisfy the quantum group relations with q^n instead of q.

2. 1. 2. Quantum Group $SU_q(2)$

It is known that an element of SU(2) can be represented in the form

$$g = \begin{bmatrix} a & -c^* \\ c & a^* \end{bmatrix} \tag{2.1.11}$$

where a and c are complex numbers such that the unitarity condition (where (*) means complex conjugate and (†) means hermitian conjugate)

$$gg^\dagger = g^\dagger g = I \tag{2.1.12}$$

gives

$$aa^* + c^* c = 1. \tag{2.1.13}$$

We can take the entries belonging to an associative but noncommutative algebra. If we impose the unitarity condition on $SL_q(2)$

$$g^\dagger = g^{-1} \tag{2.1.14}$$

which reads

$$\begin{bmatrix} a^* & c^* \\ qb^* & d^* \end{bmatrix} = \begin{bmatrix} d & -b \\ -qc & a \end{bmatrix}, \tag{2.1.15}$$
$$d = a^*, \quad b = -c^*.$$

So an element of $SU_q(2)$ is given by

$$g = \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix}. \tag{2.1.16}$$

Equation (2.1.16) is called the canonical form. The relations (2.1.5)-(2.1.9) become

$$ac^* = qc^*a \tag{2.1.17}$$

$$ac = qca \tag{2.1.18}$$

$$c^*a^* = qa^*c^* \tag{2.1.19}$$

$$c^*c = cc^* \tag{2.1.20}$$

$$aa^* + q^2c^*c = a^*a + cc^* = 1. \tag{2.1.21}$$

If we have two quantum matrices g, h $\in$ $SU_q(2)$ with commuting entries, i.e., if $\left[g_{ij}, h_{ke}\right]$ = 0 then gh is also an element of $SU_q(2)$. More explicitly if

$$g = \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix}, \ h = \begin{bmatrix} a' & -qc'^* \\ c' & a'^* \end{bmatrix} \in SU_q(2)$$

then

$$gh = \begin{bmatrix} aa' - qc^*c' & -qac'^* - qc^*a'^* \\ ca' + a^*c' & -qcc'^* + a^*a'^* \end{bmatrix} \tag{2.1.22}$$

satisfies $SU_q(2)$ quantum group relations.

2. 2. Quantum Group Chiral Field

The standard chiral field is a map from R^n to G,

$$g: R^n \to G$$

where G is a Lie group with the equation of motion

$$\partial_\mu (g^\dagger(x) \partial_\mu g(x)) = 0. \tag{2.2.1}$$

By analogy we can define the quantum group chiral field as a map

$$g: R \to G_q$$

satisfying the equation of motion where G_q is a quantum group. If we take $g \in SU_q(2)$

$$g(x)=\begin{bmatrix} a(x) & -qc^*(x) \\ c^*(x) & a^*(x) \end{bmatrix} \tag{2.2.2}$$

then

$$\partial_\mu g(x)=\begin{bmatrix} \partial_\mu a(x) & -q\partial_\mu c^*(x) \\ \partial_\mu c^*(x) & \partial_\mu a^*(x) \end{bmatrix}. \tag{2.2.3}$$

The derivatives, i.e., elements of $\partial_\mu g(x)$ satisfy the relations found by differentiating (2.1.17)-(2.1.20). Let us discuss two examples of quantum group chiral fields.

2. 2. 1. $SU_q(2)$ WZNW (Wess-Zumino-Novikov-Witten) Chiral Field

The equation of motion for $SU_q(2)$ WZNW chiral field model is given by

$$\partial_x (g^\dagger \partial_y g) = 0 \qquad (2.2.4)$$

The general solution g(x, y)=u(x) v(y) where u(x) and v(y) $\in SU_q$ and hence g(x, y) $\in SU_q(2)$ and the matrix elements of u and v commute among themselves, i.e., $\left[u_{ij}, v_{ke}\right]=0$.

2. 2. 2. $SU_q(2)$ Chiral Field

The equation of motion for SU(2) chiral field in the light-cone variables x, y is

$$\partial_x (g^\dagger \partial_y g) + \partial_y (g^\dagger \partial_x g) = 0 \qquad (2.2.5)$$

where g=g(x, y) takes values in SU(2). We can take g=g(x, y) to be an element of $SU_q(2)$ instead of SU(2). After setting,

$$g^{\dagger}\partial_x g = \partial_x X \text{ and } g^{\dagger}\partial_y g = -\partial_y X \tag{2.2.6}$$

the matrix X can be obtained by integration. The currents

$$J_x = (\partial_x + g^{\dagger}\partial_x g)X \text{ and } J_y = (\partial_y + g^{\dagger}\partial_y g)X \tag{2.2.7}$$

satisfy the conservation law

$$\partial_x J_y + \partial_y J_x = 0. \tag{2.2.8}$$

2. 3. Quantum Lie Algebra

Let g(t) be a function of a real variable t and take values in G, where G is a Lie group. Then the tangent vector $L = g^{\dagger}\left.\frac{dg}{dt}\right|_{t=0}$ is an element of the corresponding Lie algebra. Let us follow the same procedure to find the Lie algebra analog of the quantum group $SU_q(2)$.

Let g(t) $\in SU_q(2)$ and g(0)=g, then

$$L = g^{\dagger}\dot{g}(t)\Big|_{t=0}. \tag{2.3.1}$$

In general, L has the form

$$L = \begin{bmatrix} l_1 & l_0 \\ l_2 & l \end{bmatrix}. \tag{2.3.2}$$

If we differentiate both sides of $g^{\dagger}g = \mathrm{I}$ with respect to t we obtain

$$\left.\frac{d}{dt}(g^{\dagger}g)\right|_{t=0} = \left.\frac{d}{dt}g^{\dagger}g\right|_{t=0} + \left.g^{\dagger}\frac{dg}{dt}\right|_{t=0}. \tag{2.3.3}$$

Here, if $g^{\dagger}\frac{dg}{dt}\bigg|_{t=0} = L$ then $\frac{d}{dt}g^{\dagger}g\Big|_{t=0} = L^{\dagger}$

so (2.3.3) gives $L^{\dagger} + L = 0$, which means L is anti-hermitian, i.e.,

$$\begin{bmatrix} l_1^* & l_2^* \\ l_0^* & l^* \end{bmatrix} = \begin{bmatrix} -l_1 & -l_0 \\ -l_2 & -l \end{bmatrix} \tag{2.3.4}$$

from this equality we get

$$l_1^* = -l_1 \quad l_2 = -l_0^* \quad l^* = -l.$$

Using

$$L = g^{\dagger}\frac{dg}{dt}\bigg|_{t=0}$$

we obtain

$$l_1 = a^*\dot{a} + c^*\dot{c} \quad l_0 = c^*\dot{a}^* - qa^*\dot{c}^* \quad l = a\dot{a}^* + q^2 c\dot{c}^* \tag{2.3.5}$$

and

$$gL = g\,g^{\dagger}\frac{dg}{dt}\bigg|_{t=0} = \frac{dg}{dt}\bigg|_{t=0} . \tag{2.3.6}$$

Since $\frac{dg}{dt}\Big|_{t=0}$ is obviously in canonical form, gL is also in canonical form. Setting

$$gL = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

where

$$\begin{array}{ll} x = al_1 + qc^{*}l_0^{*} & y = al_0 - qc^{*}l \\ z = cl_1 - a^{*}l_0^{*} & w = cl_0 + a^{*}l \end{array} \tag{2.3.7}$$

$x = w^{*}$ gives

$$al_1 + qc^{*}l_0^{*} = l_0^{*}c^{*} - la \tag{2.3.8}$$

$y = -qz^{*}$ gives

$$al_0 - qc^{*}l = ql_1c^{*} + ql_0a . \tag{2.3.9}$$

Assuming the linear independence of the generators l_0, l_0^{*}, l_1 in (2.3.8) and (2.3.9) we get

$$qc^* l_0^* = l_0^* c^* \qquad q l_0 c = c l_0 \tag{2.3.10}$$
$$a l_0 = q l_0 a \qquad l_0^* a^* = q a^* l_0^* \tag{2.3.11}$$
$$a l_1 = -la \qquad -l_1 a^* = \kappa a^* l_1 \tag{2.3.12}$$
$$c^* l = -l_1 c^* \qquad -\kappa l_1 c = c l_1 . \tag{2.3.13}$$

It is reasonable to assume

$$l = \kappa l_1 \tag{2.3.14}$$

where κ is a real parameter.To find the relations between l_0 and a^*, l_0 and c^* we assume

$$l_0 a^* = p a^* l_0 \tag{2.3.15}$$
$$l_0 c^* = r c^* l_0 . \tag{2.3.16}$$

Again p and r are real numbers which will be found from the consistency of the algebra. Let us multiply (2.1.21) by l_0 from the left and from the right

$$l_0 (aa^* + q^2 c^* c) = (aa^* + q^2 c^* c) l_0 .$$

After using the above relations we get

$$\left(\frac{p}{q} aa^* + \frac{r}{q} q^2 c^* c \right) l_0 = \left(aa^* + q^2 c^* c \right) l_0 . \tag{2.3.17}$$

The solution for *p* and *r* is *p=r=q*.

The importance of the exponential map from the Lie algebra to the Lie group is well known. By analogy, let us try to find the relations between the elements of g and L which satisfy

$$g(t) = ge^{tL} \qquad g(t) \in SU_q(t) \qquad \text{(t is real).}$$

In series expansion we have

$$g(t) = g\left(1 + tL + \frac{t^2 L^2}{2} + \dots\right)$$

g and gL were discussed above. Let us find the relations for the second order term, gL^2 , to be in the canonical form

$$gL^2 = \begin{bmatrix} f & h \\ k & m \end{bmatrix}$$

where

$$\begin{aligned}
f &= al_1^2 - al_0 l_0^* + qc^* l_0^* l_1 + q\kappa c^* l_1 l_0^* \\
h &= al_1 l_0 + \kappa a l_0 l_1 + qc^* l_0^* l_0 - q\kappa^2 c^* l_1^2 \\
k &= cl_1^2 - cl_0 l_0^* - a^* l_0^* l_1 - \kappa a^* l_1 l_0^* \\
m &= cl_1 l_0 + \kappa c l_0 l_1 - a^* l_0^* l_0 + \kappa^2 a^* l_1^2 .
\end{aligned}$$

From the preservation of the canonical form we have

$$f^* = m$$

$$\left(al_1^2 - al_0l_0^* + qc^*l_0^*l_1 + q\kappa c^*l_1l_0^*\right)^* = \left(cl_1l_0 + \kappa cl_0l_1 - a^*l_0^*l_0 + \kappa^2a^*l_1^2\right)$$

$$l_1^2a^* - l_0l_0^*a^* - ql_1l_0c - q\kappa l_0l_1c = cl_1l_0 + \kappa cl_0l_1 - a^*l_0^*l_0 + \kappa^2a^*l_1^2.$$

Equating the linearly independent terms we get

$$l_1^2a^* = \kappa^2a^*l_1^2 \tag{2.3.18}$$

$$-l_0l_0^*a^* = -a^*l_0^*l_0 \tag{2.3.19}$$

$$-ql_1l_0c - q\kappa l_0l_1c = cl_1l_0 + \kappa cl_0l_1. \tag{2.3.20}$$

Since $h = -qk^*$

$$-q\left(cl_1^2 - cl_0l_0^* - a^*l_0^*l_1 - \kappa a^*l_1l_0^*\right)^* = al_1l_0 + \kappa al_0l_1 + qc^*l_0^*l_0 - q\kappa^2c^*l_1^2$$

$$-ql_1^2c^* + ql_0l_0^*c^* - ql_1l_0a - q\kappa l_0l_1a = al_1l_0 + \kappa al_0l_1 + qc^*l_0^*l_0 - q\kappa^2c^*l_1^2$$

Equating the linearly independent terms one obtains

$$-ql_1^2c^* = -q\kappa^2c^*l_1^2 \tag{2.3.21}$$

$$l_0l_0^*c^* = c^*l_0^*l_0 \tag{2.3.22}$$

$$-ql_1l_0a - q\kappa l_0l_1a = al_1l_0 + \kappa al_0l_1. \tag{2.3.23}$$

Using relations (2.3.10)-(2.3.16) in (2.3.20) and (2.3.23) one can obtain

$$\left(-ql_1l_0 - q\kappa d_0l_1\right)c = \left(-q\kappa d_1l_0 - q\kappa^2 l_0l_1\right)c \tag{2.3.24}$$

$$\left(-ql_1l_0 - q\kappa d_0l_1\right)a = \left(-q\kappa d_0l_1 - q\kappa^2 l_0l_1\right)a. \tag{2.3.25}$$

There are two solutions

i) $\kappa = 1$

ii) $l_1l_0 + \kappa d_0l_1 = 0 \ i.e.\ l_1l_0 = -\kappa d_0l_1.$ (2.3.26)

Let us take the second solution, i.e., (2.3.26) to find the relation between l_1 and l_0^*. By taking the complex conjugate of both sides of (2.3.26) (remember $l_1^* = -l_1$) we obtain

$$l_0^*l_1 = -\kappa d_1l_0^*. \tag{2.3.27}$$

The only remaining relation is between l_0 and l_0^*. To find this one can use (2.3.10) and (2.3.16) with r=q in the equation (2.3.22) and obtain

$$l_0^*l_0 = q^2 l_0l_0^*. \tag{2.3.28}$$

Using these relations and considering the solution (2.3.26) we obtain

$$L^2 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}. \tag{2.3.29}$$

where

$$A = l_1^2 - l_0 l_0^*$$
$$B = \kappa^2 l_1^2 - l_0 l_0^* \qquad (2.3.30)$$

$$e^{tL} = \left(1 + \frac{t^2}{2!}L^2 + \frac{t^4}{4!}L^4 + \ldots\right) + \left(t + \frac{t^3}{3!}L^2 + \ldots\right)L\ .$$

In this expression the first term is given by

$$\begin{bmatrix} 1 + \frac{t^2}{2!}A + \frac{t^4}{4!}A^2 + \ldots & 0 \\ 0 & 1 + \frac{t^2}{2!}B + \frac{t^4}{4!}B^2 + \ldots \end{bmatrix} = \begin{bmatrix} Cosht\sqrt{A} & 0 \\ 0 & Cosht\sqrt{B} \end{bmatrix} \qquad (2.3.31)$$

and the second term given by

$$\begin{bmatrix} t + \frac{t^3}{3!}A + \frac{t^5}{5!}A^2 + \ldots & 0 \\ 0 & t + \frac{t^3}{3!}B + \frac{t^5}{5!}B^2 + \ldots \end{bmatrix} \begin{bmatrix} l_1 & l_0 \\ -l_0^* & \kappa l_1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{Sinh\, t\sqrt{A}}{\sqrt{A}} & 0 \\ 0 & \frac{Sinh\, t\sqrt{B}}{\sqrt{B}} \end{bmatrix} \begin{bmatrix} l_1 & l_0 \\ -l_0^* & \kappa l_1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{Sinh\, t\sqrt{A}}{\sqrt{A}} l_1 & \frac{Sinht\sqrt{A}}{\sqrt{A}} l_1 \\ -\frac{Sinh\, t\sqrt{B}}{\sqrt{B}} l_0^* & \kappa \frac{Sinh\, t\sqrt{B}}{\sqrt{B}} l_1 \end{bmatrix}$$

(2.3.32)

where

$$\frac{Sinh\, t\sqrt{A}}{\sqrt{A}} = t + \frac{t^3}{3}A + \ldots$$

$$Cosh\, t\sqrt{A} = 1 + \frac{t^2}{2}A + \ldots \; .$$

So by using (2.3.31) and (2.3.32) we find

$$ge^{tL} = \begin{bmatrix} \frac{aSinh\, t\sqrt{A}}{\sqrt{A}}l_1 + aCosh\, t\sqrt{A} + qc^* \frac{Sinh\, t\sqrt{B}}{\sqrt{B}}l_0^* & a\frac{Sinh\, t\sqrt{A}}{\sqrt{A}}l_0 - q\kappa c^* \frac{Sinh\, t\sqrt{B}}{\sqrt{B}}l_1 - qc^* Cosh\, t\sqrt{B} \\ c\frac{Sinh\, t\sqrt{A}}{\sqrt{A}}l_1 + cCosh\, t\sqrt{A} - a^* \frac{Sinh\, t\sqrt{B}}{\sqrt{B}}l_0^* & c\frac{Sinh\, t\sqrt{A}}{\sqrt{A}}l_0 + \kappa a^* \frac{Sinh\, t\sqrt{B}}{\sqrt{B}}l_1 + a^* Cosh\, t\sqrt{B} \end{bmatrix}$$

$$ge^{tL} = \begin{bmatrix} p & m \\ r & s \end{bmatrix}. \tag{2.3.33}$$

Since ge^{tL} belongs to $SU_q(2)$ it must be in canonical form. From $s = p^*$ (notice A and B are hermitian) we have

$$-l_1 \frac{Sinh\, t\sqrt{A}}{\sqrt{A}}a^* + Cosh\, t\sqrt{A}\, a^* + ql_0 \frac{Sinh\, t\sqrt{B}}{\sqrt{B}}c = c\frac{Sinh\, t\sqrt{A}}{\sqrt{A}}l_0 + \kappa a^* \frac{Sinh\, t\sqrt{B}}{\sqrt{B}}l_1 + a^* Cosh\, t\sqrt{B} \tag{2.3.34}$$

and from $m = -qr^*$

$$ql_1\frac{Sinh\,t\sqrt{A}}{\sqrt{A}}c^* - qCosh\,t\sqrt{A}\,c^* + ql_0\frac{Sinh\,t\sqrt{B}}{\sqrt{B}}a = a\frac{Sinh\,t\sqrt{A}}{\sqrt{A}}l_0 - q\kappa c^*\frac{Sinh\,t\sqrt{B}}{\sqrt{B}}l_1 - qc^*Cosh\,t\sqrt{B}. \tag{2.3.35}$$

By using (2.3.10)-(2.3.16),(2.3.26)-(2.3.30) we obtain

$$aA = Ba \quad Aa^* = a^*B \quad cA = Bc \quad c^*B = Ac^* \tag{2.3.36}$$

$$l_1A = Al_1 \quad l_0B = Al_0 \tag{2.3.37}$$

and the generalized relation

$$aA^n = B^na \quad B^nc = cA^n \tag{2.3.38}$$

$$l_1A^n = A^nl_1 \quad l_0B^n = A^nl_0 \tag{2.3.39}$$

so that we have

$$\begin{aligned}
&a\frac{Sinh\,t\sqrt{A}}{\sqrt{A}} = \frac{Sinh\,t\sqrt{B}}{\sqrt{B}}a \qquad c\frac{Sinh\,t\sqrt{A}}{\sqrt{A}} = \frac{Sinh\,t\sqrt{B}}{\sqrt{B}}c\\
&\frac{Sinh\,t\sqrt{A}}{\sqrt{A}}l_1 = l_1\frac{Sinh\,t\sqrt{A}}{\sqrt{A}} \qquad \frac{Sinh\,t\sqrt{A}}{\sqrt{A}}l_0 = l_0\frac{Sinh\,t\sqrt{B}}{\sqrt{B}}\\
&aCosh\,t\sqrt{A} = Cosh\,t\sqrt{B}\,a \qquad cCosh\,t\sqrt{A} = Cosh\,t\sqrt{B}\,c\\
&Cosh\,t\sqrt{A}\,l_1 = l_1Cosh\,t\sqrt{A} \qquad Cosh\,t\sqrt{A}\,l_0 = l_0Cosh\,t\sqrt{B}.
\end{aligned} \tag{2.3.40}$$

Applying (2.3.40) to the left hand side of (2.3.34) we obtain

$$ql_0 \frac{Sinh\, t\sqrt{B}}{\sqrt{B}} c = c \frac{Sinh\, t\sqrt{A}}{\sqrt{A}} l_0 .$$

Again by using (2.3.40) and (2.3.10) we get

$$\frac{Sinh\, t\sqrt{A}}{\sqrt{A}} l_0 c = \frac{Sinh\, t\sqrt{B}}{\sqrt{B}} l_0 c . \tag{2.3.41}$$

Following the same procedure for (2.3.35) we obtain

$$ql_0 \frac{Sinh\, t\sqrt{B}}{\sqrt{B}} a = a \frac{Sinh\, t\sqrt{A}}{\sqrt{A}} l_0 .$$

Using (2.3.40) and (2.3.11)

$$\frac{Sinh\, t\sqrt{A}}{\sqrt{A}} l_0 a = \frac{Sinh\, t\sqrt{B}}{\sqrt{B}} l_0 a . \tag{2.3.42}$$

Equations (2.3.41) and (2.3.42) require

$$Al_0 = Bl_0 ,$$

or more explicitly

$$l_1^2 l_0 - l_0 l_0^* l_0 = \kappa^2 l_1^2 l_0 - l_0 l_0^* l_0.$$

There are various solutions for this equation

$$i)\ \ \kappa^2 = 1$$
$$ii)\ \ l_1^2 l_0 = 0$$
$$iii)\ \ l_1^2 l_0 = 0 \qquad\qquad l_0 l_0{}^* l_0 = 0.$$

Let us take the third solution which is the interesting solution. We can define the quantum superplane relations

$$l_1^2 = 0 \quad l_0^2 = 0 \tag{2.3.43}$$

together with the previously found relations

$$l_1 l_0 = -\kappa l_0 l_1$$
$$l_0 l_0^* = q^2 l_0^* l_0.$$

Let us summarize this section. The exponential mapping from the Lie algebra analog L to the quantum group $SU_q(2)$ was constructed. We found that L is of the form

$$L = \begin{bmatrix} l_1 & l_0 \\ -l_0^* & \kappa l_1 \end{bmatrix}$$

where the entries of L satisfy the quantum plane relations

$$l_0^2 = 0 \qquad l_1^2 = 0 \qquad l_1 l_0 = -\kappa l_0 l_1 \qquad l_0^* l_0 = q^2 l_0 l_0^* \tag{2.3.44}$$

together with $l_1^* = -l_1$. If the entries of L satisfy

$$\begin{array}{ll} a l_0 = q l_0 a & l_0^* a^* = q a^* l_0^* \\ c l_0 = q l_0 c & l_0^* c^* = q c^* l_0^* \\ a l_1 = -\kappa l_1 a & l_1 a = -\kappa a^* l_1 \\ c l_1 = -\kappa l_1 c & l_1 c^* = -\kappa c^* l_1 \\ l_0 a^* = q a^* l_0 & a l_0^* = q l_0^* a \\ l_0 c^* = q c^* l_0 & c l_0^* = q l_0^* c \end{array} \tag{2.3.45}$$

then

$$g(t) = g e^{tL} \in SU_q(2) \qquad \text{where} \qquad g = \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} \in SU_q(2).$$

An explicit construction of $g(t) = g e^{tL}$ was done. In fact, the defined quantum superplane relations give rise to finite number of elements in the series expansion. Since L^2 was found to be

$$L^2 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where

$$A = B = -l_0 l_0^*, \tag{2.3.46}$$

$$L^4 = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

because of the quantum plane relations (2.3.44). Hence, only the terms up to L^3 survive in the power series expansion , others vanish. More explicitly

$$\frac{Sinh\, t\sqrt{A}}{\sqrt{A}} = t + \frac{t^3}{3!}A = t - \frac{t^3}{3!}l_0 l_0^*$$

$$Cosh\, t\sqrt{A} = 1 + \frac{t^2}{2}A = 1 - \frac{t^2}{2}l_0 l_0^*.$$

Using (2.3.45) and (2.3.46) in (2.3.33) we find

$$ge^{tL} = \begin{bmatrix} a\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_1 + a\left(1 - \frac{t^2}{2}l_0 l_0^*\right) + qc^*\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_0^* & a\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_0 - q\kappa c^*\left(t - \frac{t^3}{3!}l_0 l_0^*\right) - qc^*\left(1 - \frac{t^2}{2}l_0 l_0^*\right) \\ c\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_1 + c\left(1 - \frac{t^2}{2}l_0 l_0^*\right) - a^*\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_0^* & c\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_0 + \kappa a^*\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_1 + a^*\left(1 - \frac{t^2}{2}l_0 l_0^*\right) \end{bmatrix}$$

Again using the quantum plane relations (2.3.44) we obtain a one parameter group of automorphisms of the quantum group $SU_q(2)$.

$$a \to a\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_1 + a\left(1 - \frac{t^2}{2}l_0 l_0^*\right) + qtc^* l_0^* \qquad (2.3.47)$$

$$c \to c\left(t - \frac{t^3}{3!}l_0 l_0^*\right)l_1 + c\left(1 - \frac{t^2}{2}l_0 l_0^*\right) - ta^* l_0^*. \qquad (2.3.48)$$

2. 4. Quantum Yang-Mills Formulation

2. 4. 1. Possible Formulations

In the quantum group deformation of gauge theory, it seems, at least at the moment, we have only two possible ways to proceed. In the first way, gauge fields take values in the Lie algebra analog of the quantum group $SU_q(2)$. The relations between the generators are well defined. The general form of generators were found to be

$$L = \begin{bmatrix} l_1 & l_0 \\ -l_0^* & \kappa l_1 \end{bmatrix} = l_1 \begin{bmatrix} 1 & 0 \\ 0 & \kappa \end{bmatrix} + l_0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + l_0^* \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \tag{2.4.1}$$

If the gauge field takes values in the universal enveloping algebra then we have

$$A_\mu(x) = A_\mu^1(x) l_1 \begin{bmatrix} 1 & 0 \\ 0 & \kappa \end{bmatrix} + A_\mu^0(x) l_0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + A_\mu^2(x) l_0^* \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + A_\mu^{01}(x) l_0 l_1 \begin{bmatrix} 0 & \kappa \\ 0 & 0 \end{bmatrix} + \dots \tag{2.4.2}$$

In the second approach the gauge field takes values in the quantum deformation of the universal enveloping algebra of *su(2)* i.e. $U_q(su(2))$ generated by X_+, X_- and H satisfying

$$[H, X_\pm] = \pm 2 X_\pm \qquad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} . \tag{2.4.3}$$

For a sensible gauge theory, the usual Lagrangian formulation has to be modified. In the non-deformed case the Lagrangian is invariant under the usual gauge transformations while for the deformed case it is invariant under the quantum gauge group.This means that the Lagrangian is not a complex number but an element of a non-commuting algebra. So a new trace should be introduced to construct a realistic gauge theory. Using this trace we will discuss the q-deformed $SU(2)\times U(1)$ gauge theory.

2. 4. 2. Quantum Trace

If we have two matrices with commuting entries

$$E=\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \qquad T=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

we know that trace remains invariant under the transformation

$$E_{ij}=T_{ik}E_{kl}T_{lj}^{-1}.$$

These matrices are elements of a quantum group, not the ordinary trace but the quantum trace remains invariant

$$Tr_q(E)=q^{-1}E_{11}+qE_{22}=Tr_q\left(TET^{-1}\right). \qquad (2.4.4)$$

Example: Let A be any 2×2 matrix and G be an element of $SU_q(2)$ i.e.

$$A=\begin{bmatrix} x & y \\ z & w \end{bmatrix} \quad G=\begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} \in SU_q(2)$$

$$Tr_q A = q^{-1}x + qw \tag{2.4.5}$$

$$GAG^{-1} = \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} a^* & c^* \\ -qc & a \end{bmatrix}.$$

Using the fact that entries of the matrix A commute with those of G one obtains

$$GAG^{-1} = \begin{bmatrix} xaa^* - qzc^*a^* - qyac + q^2wc^*c & xac^* - qzc^*c^* + yaa - qwc^*a \\ xcc^* + za^*c^* + yca + wa^*c^* & xc^*c + za^*c^* + yca + wa^*a \end{bmatrix}$$

and

$$\begin{aligned} Tr_q GAG^{-1} &= q^{-1}\left(xaa^* - qzc^*a^* - qyac + q^2wc^*c\right) + q\left(xc^*c + za^*c^* + yca + wa^*a\right) \\ &= q^{-1}x\left(aa^* + q^2cc^*\right) + z\left(qa^*c^* - c^*a^*\right) + y\left(qca - ac\right) + qw\left(a^*a + c^*c\right) \end{aligned}$$

Using (2.1.17)-(2.1.21) one obtains

$$Tr_q GAG^{-1} = q^{-1}x + qw \tag{2.4.6}$$

which means that quantum trace remains invariant under quantum group transformations. In fact, for higher dimensional (N dimensional) matrices we have

$$Tr_q(E) = Tr_q(TET^{-1}) = q^{-(N+1)} \sum_{i=1}^{N} q^{2i} E_{ii} \,. \tag{2.4.7}$$

2. 4. 3. The q-Deformed Yang-Mills Theory

Before proceeding let us remember the basic features of the usual Yang-Mills theory. We have a covariant derivative defined by

$$\nabla_\mu = \partial_\mu + A_\mu(x) \tag{2.4.8}$$

where $x_\mu = \left(x_0, x_1, \ldots, x_d\right)$ is the coordinate of the *d+1* dimensional space-time and $A_\mu(x)$ is the potential taking values in the Lie Algebra. We have

$$A_\mu(x) = A_\mu^i(x)\sigma^i \tag{2.4.9}$$

where σ^i are the generators of the gauge group G. For G=U(2)=SU(2) x U(1) we take the identity matrix σ^0 and the Pauli matrices σ^i, i=1,2,3. For this case the covariant derivative is given by

$$\nabla_\mu = \partial_\mu + A_\mu^i\sigma^i + B_\mu\sigma^0 \tag{2.4.10}$$

and the Lagrangian is given by

$$L = Tr\left(F_{\mu\nu}F_{\mu\nu}\right) \tag{2.4.11}$$

where

$$F_{\mu\nu} = \left[\nabla_{\mu}, \nabla_{\nu}\right] = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + \left[A_{\mu}, A_{\nu}\right]. \tag{2.4.12}$$

The Lagrangian is invariant under the gauge transformations

$$\nabla_{\mu} \rightarrow G(x) \nabla_{\mu} G^{-1}(x) \tag{2.4.13}$$

$$F_{\mu\nu} \rightarrow G(x) F_{\mu\nu} G^{-1}(x). \tag{2.4.14}$$

Now let us take the gauge group elements to belong to the quantum group $U_q(2)$. Then the gauge potentials A^i_{μ} are operators i.e. elements of the quantum universal enveloping algebra $U_q(su(2))$. We can find the q-deformed curvature $F_{\mu\nu}$ and the q-deformed Yang-Mills Lagrangian by using the q-trace formula (2.4.4) and using the usual definition of the covariant derivative. The curvature is

$$F_{\mu\nu} = \left[\nabla_{\mu}, \nabla_{\nu}\right] = \left(\partial_{\mu} + A^a_{\mu}\sigma^a + B_{\mu}\sigma^0\right)\left(\partial_{\nu} + A^b_{\nu}\sigma^b + B_{\nu}\sigma^0\right) - \left(\partial_{\nu} + A^b_{\nu}\sigma^b + B_{\nu}\sigma^0\right)\left(\partial_{\mu} + A^a_{\mu}\sigma^a + B_{\mu}\sigma^0\right).$$

After cancellations one obtains

$$F_{\mu\nu} = \left(\partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu}\right)\sigma^a + \left(\partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}\right)\mathrm{I} + \left[A^a_{\mu}, A^a_{\nu}\right]\mathrm{I} + i\varepsilon^{abc}\left(A^a_{\mu} A^b_{\nu} + A^b_{\nu} A^a_{\mu}\right)\sigma^c + \left[B_{\mu}, B_{\nu}\right]\mathrm{I}$$

$$+\left\{\left[A^a_{\mu}, B_{\nu}\right] + \left[B_{\mu}, A^a_{\nu}\right]\right\}\sigma^a = B_{\mu\nu}\mathrm{I} + F^a_{\mu\nu}\sigma^a \tag{2.4.15}$$

where $\sigma's$ are Pauli matrices.

This curvature transforms as (2.4.14), where now G(x) is an element of the quantum group $U_q(2)$ and $\left[A^i_\mu, G_{ij}\right]=0$. To obtain the invariant "abelian" component $F^0_{\mu\nu}$ of $F_{\mu\nu}$ we use the q-trace formula. Note that q-trace of σ^3 is not zero but $q^{-1}-q$. Then we have

$$F^0_{\mu\nu}=Tr_q\left(F_{\mu\nu}\right)\left[2\left(q^{-2}+q^2\right)\right]^{-1/2}=\left\{\partial_\mu B_\nu-\partial_\nu B_\mu+\left[B_\mu,B_\nu\right]+\left[A^a_\mu,A^a_\nu\right]\right\}\frac{q^{-1}+q}{\sqrt{2\left(q^{-2}+q^2\right)}}$$
$$+\left\{\partial_\mu A^3_\nu-\partial_\nu A^3_\mu+i\varepsilon^{ab3}\left(A^a_\mu A^b_\nu-A^a_\nu A^b_\mu\right)+\left[A^3_\mu,B_\nu\right]-\left[A^3_\nu,B_\mu\right]\right\}\frac{q^{-1}-q}{\sqrt{2\left(q^{-2}+q^2\right)}}. \tag{2.4.16}$$

Defining $Tan\,\theta=\dfrac{q^{-1}-q}{q^{-1}+q}$ (2.4.16) becomes

$$F^0_{\mu\nu}=B_{\mu\nu}Cos\,\theta+F^3_{\mu\nu}Sin\,\theta. \tag{2.4.17}$$

Since Tr_q is invariant under quantum group gauge transformation (2.4.14) $F^0_{\mu\nu}$ is also invariant. We see from equation (2.4.16) that in the "quantum" case the U(1) component B_μ mixes with the nonabelian components A^a_μ. Now let us investigate the results of the transformation (2.4.13)

$$\partial_\mu+\vec{A}_\mu\vec{\sigma}+B_\mu\rightarrow G\partial_\mu G^{-1}+G\vec{A}_\mu\vec{\sigma}G^{-1}+GB_\mu G^{-1}. \tag{2.4.18}$$

Let us take the q-trace of the right hand side of (2.4.18)

$$Tr_q\left\{G\partial_\mu G^{-1}+G\vec{A}_\mu\vec{\sigma}G^{-1}+GB_\mu G^{-1}\right\}\left[2\left(q^{-2}+q^2\right)\right]^{-1/2}=\frac{A_\mu^3\left(q^{-1}-q\right)}{\left[2\left(q^{-2}+q^2\right)\right]^{1/2}}+\frac{B_\mu\left(q^{-1}+q\right)}{\left[2\left(q^{-2}+q^2\right)\right]^{1/2}}$$

$$+Tr_q\left(G\partial_\mu G^{-1}\right)\left[2\left(q^2+q^{-2}\right)\right]^{-1/2}. \tag{2.4.19}$$

Using (2.4.12) and defining

$$A_\mu = A_\mu^3 Sin\,\theta + B_\mu Cos\,\theta \tag{2.4.20}$$

we obtain the field A_μ which transforms as the "abelian" field in the quantum group case.We also obtain that the only combination of the operator valued fields A_μ^3 and B_μ defined by (2.4.20) is simply shifted without rotation:

$$A_\mu \to A_\mu + Tr_q\left(G\partial_\mu G^{-1}\right)\left(2q^2+2q^{-2}\right)^{-1/2}. \tag{2.4.21}$$

In addition to (2.4.20) we can define

$$Z_\mu = -B_\mu Sin\,\theta + A_\mu^3 Cos\,\theta. \tag{2.4.22}$$

It is interesting to mention that the formulas (2.4.20), (2.4.22) coincide with the definitions of the photon and Z-boson in the Weinberg-Salam model where θ is the Weinberg angle. Substituting A_μ and Z_μ into (2.4.16) one obtains

$$F^{0}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + Cos\,\theta\left\{\left[A_\mu, A_\nu\right]\left(1+2Sin^2\,\theta\right) + \left[Z_\mu, Z_\nu\right]Cos\,2\theta\right\}$$
$$+Sin\,\theta Cos\,2\theta\left(\left[A_\mu, Z_\nu\right] + \left[Z_\mu, A_\nu\right]\right) + Cos\,\theta\left(\left[A^1_\mu, A^1_\nu\right] + \left[A^2_\mu, A^2_\nu\right]\right) \qquad (2.4.23)$$
$$+iSin\,\theta\left\{A^1_\mu A^2_\nu + A^2_\nu A^1_\mu - A^1_\nu A^2_\mu - A^2_\mu A^1_\nu\right\}.$$

Now let us find the q-analog of the Lagrangian

$$L_q = Tr_q\left(F_{\mu\nu}F_{\mu\nu}\right) = Tr_q\left(B_{\mu\nu}\mathrm{I} + F^a_{\mu\nu}\sigma^a\right)\left(B_{\mu\nu}\mathrm{I} + F^b_{\mu\nu}\sigma^b\right)$$
$$= Tr_q\left\{\left(B_{\mu\nu}\right)^2\mathrm{I} + B_{\mu\nu}F^b_{\mu\nu}\sigma^b + F^a_{\mu\nu}B_{\mu\nu}\sigma^b + F^a_{\mu\nu}F^b_{\mu\nu}\sigma^a\sigma^b\right\}$$
$$= \left[\left(B_{\mu\nu}\right)^2 + \left(F^a_{\mu\nu}\right)^2\right]\left(q^{-1}+q\right) + \left(B_{\mu\nu}F^3_{\mu\nu} + F^3_{\mu\nu}B_{\mu\nu} + i\varepsilon^{ab3}F^a_{\mu\nu}F^b_{\mu\nu}\right)\left(q^{-1}-q\right).$$
$$(2.4.24)$$

Here we used the identity $\sigma^a\sigma^b = i\varepsilon^{abc}\sigma^c$, (2.4.15) , (2.4.24) and the fact that only the identity matrix and σ^3 contribute to the q-trace.

From the construction of L_q it is clear that L_q is invariant under the quantum gauge transformations

$$F_{\mu\nu} \to GF_{\mu\nu}G^{-1} \qquad (2.4.25)$$

where

$$\left[G_{kl}, F^i_{\mu\nu}\right] = 0. \qquad (2.4.26)$$

We can obtain another invariant (q-analog of the abelian theory) by using (2.4.15)

$$L'_q = \left(Tr_q\, F_{\mu\nu}\right)^2 = \left[Tr_q\left(B_{\mu\nu}\mathrm{I} + F^a_{\mu\nu}\sigma^a\right)\right]^2 = \left[B_{\mu\nu}\left(q^{-1}+q\right)^2 + F^3_{\mu\nu}\left(q^{-1}-q\right)\right]^2$$
$$= B^2_{\mu\nu}\left(q^{-1}+q\right)^2 + \left(F^3_{\mu\nu}\right)^2\left(q^{-1}-q\right)^2 + \left(B_{\mu\nu}F^3_{\mu\nu} + F^3_{\mu\nu}B_{\mu\nu}\right)\left(q^{-2}-q^2\right). \quad (2.4.27)$$

A linear combination of the two invariants L_q and L'_q gives us the analog of the q-deformed Yang-Mills Lagrangian,

$$L''_q = \left(L_q - \frac{L'_q}{q+q^{-1}}\right)\frac{1}{q+q^{-1}} = \left[\left(F^a_{\mu\nu}\right)^2 - \left(F^3_{\mu\nu}\right)^2 Tan^2\,\theta\right] + i\left[F^1_{\mu\nu}, F^2_{\mu\nu}\right]Tan\,\theta$$

which is independent of the field $B_{\mu\nu}$ and is invariant under the quantum gauge group transformations.

3. BRAID GROUP RELATED ALGEBRAS, THEIR REPRESENTATIONS AND GENERALIZED HYDROGEN-LIKE SPECTRA

3. 1. The Braid Algebra

The n-strand Artin braid group is defined in terms of n-1 invertible generators σ_i which satisfy the braid group relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \neq 1 \tag{3.1.1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{3.1.2}$$

We can represent each generator by an $n \times n$ matrix whose nontrivial part is 2×2 and is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Where a,b,c,d are "noncommuting" objects.Then we have

$$\sigma_1 = \begin{bmatrix} a & b & 0 & . & . & . \\ c & d & 0 & & & \\ 0 & 0 & 1 & & & \\ . & & & . & & \\ . & & & & . & \\ . & & & & & 1 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . \\ 0 & a & b & 0 & & \\ 0 & c & d & 0 & & \\ 0 & 0 & 0 & 1 & & \\ . & & & & . & \\ . & & & & & 1 \end{bmatrix} \quad \sigma_{n-1} = \begin{bmatrix} 1 & 0 & . & . & . & . \\ 0 & 1 & & & & \\ . & & . & & & \\ . & & & . & & \\ . & & & & a & b \\ . & & & & c & d \end{bmatrix}$$

In such a representation (3.1.1) is automatically satisfied whereas (3.1.2) imposes relations for the elements of the matrix A. To find the relations we put these matrices into equation (3.1.2) and using the fact that in this equality only the 3×3 part gives relations. We obtain

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

$$\begin{bmatrix} a^2 + bac & ab + bad & b^2 \\ ca + dac & cb + dad & db \\ c^2 & cd & d \end{bmatrix} = \begin{bmatrix} a & ba & b^2 \\ ac & ada + bc & adb + bd \\ c^2 & cda + dc & cdb + d^2 \end{bmatrix}.$$

This equality gives us the relations

$$a^2 + bac = a \tag{3.1.3}$$
$$ab + bad = ba \tag{3.1.4}$$
$$ca + dac = ac \tag{3.1.5}$$
$$cb + dad = ada + bc \tag{3.1.6}$$
$$db = adb + bd \tag{3.1.7}$$
$$cd = cda + dc \tag{3.1.8}$$
$$d = cdb + d^2. \tag{3.1.9}$$

We rearrange these equations to obtain

$$bac = a - a^2 \tag{3.1.10}$$
$$cdb = d - d^2 \tag{3.1.11}$$
$$bc - cb = dad - ada \tag{3.1.12}$$
$$ab = ba(1-d) \tag{3.1.13}$$
$$ca = (1-d)ac \tag{3.1.14}$$
$$bd = (1-a)db \tag{3.1.15}$$
$$dc = cd(1-a). \tag{3.1.16}$$

If the elements of A ,i.e., a,b,c and d commute among themselves from (3.1.10)-(3.1.16) we obtain that either a=0 or d=0 ,or b=c=0. If we take a=0 and other elements to be different from zero, then (3.1.11) gives $cb = 1-d$, or

$$d = 1 - bc. \tag{3.1.17}$$

Setting $bc=t$ we obtain

$$A = \begin{bmatrix} 0 & b \\ \frac{t}{b} & 1-t \end{bmatrix}.$$

By the similarity transformation $S^{-1}AS = A'$ we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{b}{t} \end{bmatrix}\begin{bmatrix} 0 & b \\ \frac{t}{b} & 1-t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{t}{b} \end{bmatrix} = \begin{bmatrix} 0 & t \\ 1 & 1-t \end{bmatrix}. \tag{3.1.18}$$

Equation (3.1.18) is just the Burau representation of the Artin braid group. The Burau representation is used to calculate the Alexander polynomial by the use of the Alexander's theorem which states that each link in three-dimensional space is ambient isotopic (i.e. having the same link invariant) to a link in the form of a closed braid[14].

Instead of taking a,b,c,d as numbers, let us take them to be elements of an associative but noncommuting algebra. Using relations (3.1.10)-(3.1.16) together with the existence of the inverse of A one can obtain the expression for A^{-1} as follows. Putting

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \tag{3.1.19}$$

the existence of the right inverse $AA^{-1} = \mathrm{I}$ implies

$$ax + bz = 1 \tag{3.1.20}$$
$$cx + dz = 0 \tag{3.1.21}$$
$$ay + bw = 0 \tag{3.1.22}$$
$$cy + dw = 1. \tag{3.1.23}$$

Multiplying (3.1.20) from left by c, and (3.1.21) by $(1\text{-}d)a$ and using (3.1.14) one obtains

$$\left[(1-d)ad - cb\right]z = -c. \tag{3.1.24}$$

Similarly one can obtain expressions for x, y and w

$$\left[da - (1-d) - cb\right]x = (1-a)d \tag{3.1.25}$$
$$\left[(1-d)ad - cb\right]w = (1-d)a \tag{3.1.26}$$
$$\left[da(1-d) - cb\right]y = -b. \tag{3.1.27}$$

The existence of the left inverse $A^{-1}A = \mathrm{I}$ gives

$$xa + yc = 1 \tag{3.1.28}$$
$$xb + yd = 0 \tag{3.1.29}$$
$$za + wc = 0 \tag{3.1.30}$$
$$zb + wd = 1. \tag{3.1.31}$$

Repeating the calculations as in the right inverse case we obtain

$$z\left[(1-d)ad-cb\right]=-c \tag{3.1.32}$$

$$x\left[(1-d)ad-cb\right]=d(1-a) \tag{3.1.33}$$

$$w\left[da(1-d)-cb\right]=a(1-d) \tag{3.1.34}$$

$$y\left[da(1-d)-cb\right]=-b. \tag{3.1.35}$$

Solving for x,y,w and z in (3.1.24)-(3.1.27) we obtain the right inverse

$$A^{-1}=\begin{bmatrix}\Delta_1^{-1}(1-a)d & -\Delta_1^{-1}b \\ -\Delta_2^{-1}c & \Delta_2^{-1}(1-d)a\end{bmatrix} \tag{3.1.36}$$

where

$$\Delta_1=(1-a)da-bc=da(1-d)-cb \tag{3.1.37}$$

$$\Delta_2=ad(1-a)-bc=(1-d)ad-cb. \tag{3.1.38}$$

Equation (3.1.36) can be rearranged as

$$A^{-1}=\begin{bmatrix}\Delta_1^{-1} & 0 \\ 0 & \Delta_2^{-1}\end{bmatrix}\begin{bmatrix}1-(1-a)(1-d)-a & -b \\ -c & 1-(1-d)(1-a)-d\end{bmatrix}$$

to yield

$$A^{-1}=\Delta_L^{-1}\left(\mathrm{I}-A+\Delta'_L\right) \tag{3.1.39}$$

where

$$\Delta_L = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \quad \Delta'_L = \begin{bmatrix} \Delta'_1 & 0 \\ 0 & \Delta'_2 \end{bmatrix} \tag{3.1.40}$$

with

$$\Delta'_1 = -(1-a)(1-d) \tag{3.1.41}$$

$$\Delta'_2 = -(1-d)(1-a). \tag{3.1.42}$$

Similarly, using (3.1.32)-(3.1.35) we find the left inverse

$$A^{-1} = \begin{bmatrix} d(1-a)\Delta_2^{-1} & -b\Delta_1^{-1} \\ -c\Delta_2^{-1} & a(1-d)\Delta_1^{-1} \end{bmatrix} \tag{3.1.43}$$

and it can be rearranged to give

$$A^{-1} = \left(\mathrm{I} - A + \Delta'_R\right)\Delta_R^{-1} \tag{3.1.44}$$

where

$$\Delta_R = \begin{bmatrix} \Delta_2 & 0 \\ 0 & \Delta_1 \end{bmatrix} \quad \Delta'_R = \begin{bmatrix} \Delta'_2 & 0 \\ 0 & \Delta'_1 \end{bmatrix} \tag{3.1.45}$$

$\Delta_1, \Delta_2, \Delta'_1$ and Δ'_2 can be identifed as "determinants".Now let us find the relations between the "determinants" and a, b, c and d.We have

$$
\begin{aligned}
d\Delta_1 &= d\left[da(1-d) - cb\right] \\
&= d^2 a(1-d) - dcb \\
&= d^2 a(1-d) - cd(1-a)b \\
&= d^2 a(1-d) - cdb + cdab \\
&= d^2 a(1-d) - d - d^2 + cdba(1-d) \\
&= d^2 a(1-d) - d(1-d) + d(1-d)a(1-d) \\
&= \left[d^2 a - d + da - d^2 a\right](1-d) \\
&= \left[d^2 a - d + da - d^2 a\right](1-d) \\
&= -d(1-a)(1-d) \\
&= d\Delta'_1
\end{aligned}
$$

or in a better form

$$d\left(\Delta_1 - \Delta'_1\right) = 0$$

defining

$$\Delta_1 - \Delta'_1 = \Delta_2 - \Delta'_2 \equiv D$$

and repeating the same procedure one finds

$$aD = Da = 0 \tag{3.1.46}$$
$$dD = Dd = 0 \tag{3.1.47}$$
$$[b, \Delta_1] = [b, \Delta'_1] = 0 \tag{3.1.48}$$
$$[c, \Delta_2] = [c, \Delta'_2] = 0 \tag{3.1.49}$$
$$[\Delta_1, \Delta'_1] = [\Delta_2, \Delta'_2] = 0. \tag{3.1.50}$$

The relations (3.1.46)-(3.1.50) strongly suggest that $\Delta_1 = \Delta'_1$ and $\Delta_2 = \Delta'_2$. Now we will show that for a representation of a, b, c, d as linear operators in a Hilbert space this is indeed true. For D to be diagonalizable the following condition has to be satisfied

$$[D^{\dagger}, D]=0. \tag{3.1.51}$$

When we discuss the hermitian and the unitary representations of A we will explicitly show that (3.1.51) is satisfied. From (3.1.46)-(3.1.50) it follows that the operator D commutes with a, b, c and d. We consider an eigenspace of D with eigenvalue $\delta \neq 0$. Since $aD=Da=0$, it follows that in such a subspace $a=d=0$ and $bc=cb$, and the representation is trivial. Thus we need only consider eigenspaces of D with eigenvalue zero. If on this subspace D is diagonalizable then it is identically zero and

$$\Delta_1 = \Delta'_1 \tag{3.1.52}$$
$$\Delta_2 = \Delta'_2 \tag{3.1.53}$$

from (3.1.52)

$$\begin{aligned} da - dad - bc &= (1-a)(1-d) \\ &= -1 + a + d - ad. \end{aligned}$$

So that

$$bc = 1 - a - d + ad + da - dad$$
$$bc = (1-a) - (1-a)d(1-a). \tag{3.1.54}$$

Using (3.1.12) and (3.1.54) one obtains

$$cb = (1-d) - (1-d)a(1-d). \tag{3.1.55}$$

The equality (3.1.53) gives the same relations as in (3.1.54) and (3.1.55). This only leaves the case where D is not diagonalizable.

A related approach is to consider the uniqueness of the determinant of the operator matrix A. We have two candidates Δ_1, Δ_2 for this "determinant". The inverse of A exists only if both Δ_1 and Δ_2 are invertible. If we insist on a unique determinant then

$$\Delta_1 = \Delta_2 = \Delta. \tag{3.1.56}$$

Using (3.1.37) and (3.1.38) we get

$$da(1-d) - cb = (1-d)ad - cb$$

$$[a,d] = 0 \tag{3.1.57}$$

$$[a,\Delta] = [b,\Delta] = [c,\Delta] = [d,\Delta] = 0. \tag{3.1.58}$$

Hence Δ behaves as a "Casimir" operator for the algebra generated by a, b, c and d and the representation of this algebra can be labeled by the eigenvalue of this "Casimir" operator. Calling this eigenvalue $-q$ and recalling that a and d commute we find that for diagonalizable a and d this covers both the hermitian $A^\dagger = A$ and unitary $A^\dagger = A^{-1}$ representations to be discussed below.

Before proceeding let us discuss the consequences of (3.1.56)

$$\Delta_R = \Delta_L = -qI.$$

Multiplying (3.1.39) from left by Δ_L and from right by A one obtains

$$\Delta_L = A - A^2 + \Delta'_L A.$$

Using (3.1.56) we get

$$A^2 = (1-q)A + q. \tag{3.1.59}$$

Again multiply both sides by A

$$\begin{aligned} A^3 &= (1-q)A^2 + qA \\ &= (1-q)\left[(1-q)A + q\right] + qA \\ &= \left(1-q+q^2\right)A + q(1-q) \end{aligned}$$

and proceeding in this manner we obtain

$$A^n = \frac{1-(-q)^n}{1+q}A + q\frac{1-(-q)^{n-1}}{1+q}. \tag{3.1.60}$$

this gives us an interesting result that, when

$$(-q)^n = 1 \qquad\qquad q^n = (-1)^n$$

(3.1.60) becomes

$$A^n = 1$$

so that

$$\sigma^n = 1 \tag{3.1.61}$$

which means that when you apply a braid group generator on the braid n times you obtain a configuration whose representation is the same as the representation of the original configuration of the braid.

3. 2. Representations of the Braid Group

Now let us take the elements of the matrix A which is the nontrivial part of the braid group generators, as operators on a vector space. Consider they have the following effects on a vector $|n\rangle$ in this space

$$b|n\rangle = b_n|n-1\rangle \tag{3.2.1}$$
$$c|n\rangle = c_n|n+1\rangle \tag{3.2.2}$$
$$a|n\rangle = a_n|n\rangle \tag{3.2.3}$$
$$d|n\rangle = d_n|n\rangle. \tag{3.2.4}$$

Notice that b and c are lowering and raising operators respectively and a and d are diagonal operators in this basis. The vectors $|n\rangle$ are the eigenvectors of a number operator N, i.e., $N|n\rangle = n|n\rangle$ where n is an integer. Since $|n\rangle$ is an eigenvector of the operator cb, we can express bc as a function of the number operator and the eigenvalues of bc as a function of n.

$$cb|n\rangle = cb_n|n-1\rangle = b_n c_{n-1}|n\rangle = [n]|n\rangle = [N]|n\rangle \tag{3.2.5}$$
$$bc|n\rangle = bc_n|n+1\rangle = b_{n+1}c_n|n\rangle = [n+1]|n\rangle = [N+1]|n\rangle \tag{3.2.6}$$

Now let us solve a_n, b_n, c_n and d_n using the braid group relations (3.1.10)-(3.1.16).From (3.1.10) we get

$$bac|n\rangle = a - a^2|n\rangle$$

$$b_{n+1}a_{n+1}c_n = a_n - a_n^2$$

$$b_{n+1}c_n = \frac{a_n - a_n^2}{a_{n+1}} \quad \text{if} \quad a_{n+1} \neq 0. \tag{3.2.7}$$

From (3.1.11)

$$cdb|n\rangle = a - a^2|n\rangle$$
$$c_{n-1}a_{n-1}b_n = d_n - d_n^2$$
$$c_{n-1}b_n = \frac{d_n - d_n^2}{d_{n-1}} \quad \text{if} \quad d_{n-1} \neq 0. \tag{3.2.8}$$

From (3.1.12)

$$(bc - cb)|n\rangle = (dad - ada)|n\rangle$$
$$b_{n+1}c_n - b_n c_{n-1} = a_n d_n (d_n - a_n). \tag{3.2.9}$$

From (3.1.13)

$$ab|n\rangle = ba(1-d)|n\rangle$$
$$a_{n-1}b_n = b_n a_n (1 - d_n)$$
$$a_{n-1} = a_n (1 - d_n). \tag{3.2.10}$$

From (3.1.14)

$$ca|n\rangle = (1-d)ac|n\rangle$$
$$c_n a_n = (1 - d_{n+1}) a_{n+1} c_n$$
$$a_n = (1 - d_{n+1}) a_{n+1}. \tag{3.2.11}$$

From (3.1.15)

$$bd|n\rangle = (1-a)db|n\rangle$$
$$b_n d_n = (1-a_{n-1})d_{n-1}b_n$$
$$d_n = (1-a_{n-1})d_{n-1}. \tag{3.2.12}$$

From (3.1.16)

$$dc|n\rangle = cd(1-a)|n\rangle$$
$$d_{n+1}c_n = c_n d_n(1-a_n)$$
$$d_{n+1} = d_n(1-a_n). \tag{3.2.13}$$

The equations (3.2.10) and (3.2.11) and the equations (3.2.12) and (3.2.13) are identical when we replace n by n+1. Substituting n+1 instead of n in (3.2.8) and using (3.2.13) for d_{n+1} and from (3.1.11) i.e. by taking

$$1-d_{n+1} = \frac{a_n}{a_{n+1}}$$

(3.2.8) turns out to be

$$c_n b_{n+1} = \frac{d_{n+1}(1-d_{n+1})}{d_n} = \frac{d_n(1-a_n)}{d_n}\frac{a_n}{a_{n+1}} = \frac{a_n - a_n^2}{a_{n+1}}.$$

This gives us the same equation as in (3.2.7). One can easily show that (3.2.9) can be obtained by using (3.2.11),(3.2.13) and (3.2.7). In the previous section we have obtained that

$$\left(1-a_n\right)\left(1-d_n\right)=\left(1-a_{n+1}\right)\left(1-d_{n+1}\right)=q. \tag{3.2.14}$$

Using this equality it is easy to show that equations (3.2.11) and (3.2.13) are identical. Hence we have only three independent equations (3.2.10),(3.2.7) and (3.2.14). Let us solve a_n by using (3.2.10) and (3.2.14)

$$a_{n-1}=a_n\left(1-d_n\right)$$

$$a_{n-1}=\frac{qa_n}{1-a_n}$$

$$\frac{1}{a_{n-1}}=\frac{1}{q}\left(\frac{1}{a_n}-1\right)$$

Defining

$$\frac{1}{a_n}=U_n$$

$$U_n=1+qU_{n-1} \tag{3.2.15}$$

we find the solution in terms of U_0

$$\begin{aligned}
U_1&=1+qU_0\\
U_2&=1+q+q^2U_0\\
U_3&=1+q+q^2+q^3U_0\\
&\vdots\\
U_n&=1+q+q^2+....+q^{n-1}+q^nU_0
\end{aligned}$$

$$U_n = \frac{1-q^n}{1-q} + q^n U_0 \tag{3.2.16}$$

$$U_n = \frac{1-q^n + q^n U_0 - q^{n+1} U_0}{1-q}$$

$$U_n = \frac{1+q^n\left(-1+U_0-qU_0\right)}{1-q}.$$

Defining
$$C \equiv -1+U_0-qU_0 \tag{3.2.17}$$

we get
$$U_n = \frac{1+Cq^n}{1-q} \tag{3.2.18}$$

$$a_n = \frac{1-q}{1+Cq^n}. \tag{3.2.19}$$

Using (3.2.14) one obtains

$$d_n = \frac{Cq^{n-1}(1-q)}{1+Cq^{n-1}}. \tag{3.2.20}$$

Using (3.2.19) in (3.2.7) we obtain

$$c_n b_{n+1} = \frac{a_n\left(1-a_n\right)}{a_{n+1}}$$

$$c_n b_{n+1} = \frac{q\left(1+Cq^{n-1}\right)\left(1+Cq^{n+1}\right)}{\left(1+Cq^n\right)^2}. \tag{3.2.21}$$

3. 2. 1. Hermitian Representations

Now let us discuss the case where A is hermitian.

$$A^{\dagger}=A \tag{3.2.22}$$

$$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

gives

$$a^* = a \tag{3.2.23}$$
$$b^* = c \tag{3.2.24}$$
$$d^* = d. \tag{3.2.25}$$

For a hermitian representation of the braid group generators, a and d are themselves hermitian operators acting on a Hilbert space and have real eigenvalues. Using (3.2.14) we get

$$(1-a)(1-d)|n\rangle = (1-a_n)(1-d_n)|n\rangle = q|n\rangle. \tag{3.2.26}$$

Since a_n and d_n are real, the parameter q is also real for a hermitian representation. Also we have

$$\overline{\langle n+1|c|n\rangle} = \langle n|c^*|n+1\rangle = \langle n|b|n+1\rangle = b_{n+1}\langle n|n\rangle \tag{3.2.27}$$
$$\overline{\langle n+1|c|n\rangle} = \overline{c_n}\,\overline{\langle n+1|n+1\rangle}. \tag{3.2.28}$$

Since the scalar product is well defined in Hilbert space and $\langle n|n\rangle = \overline{\langle n+1|n+1\rangle} = 1$, from the equality of (3.2.27)-(3.2.28) we can conclude that

$$\overline{c_n} = b_{n+1} \qquad c_n = \overline{b_{n+1}}\,.$$

Using $\left[a,a^*\right]=0$ and $[d,d^*]=0$ together with $\left[a,d\right]=0$ we can take a and d to be simultaneously diagonal. So the eigenvalues of the operators a,b,c,d become

$$a_n = \frac{1-q}{1+Cq^n}$$

$$d_n = \frac{Cq^{n-1}(1-q)}{1+Cq^{n-1}}$$

$$\left|b_{n+1}\right|^2 = \frac{q\left(1+Cq^{n-1}\right)\left(1+Cq^{n+1}\right)}{\left(1+Cq^n\right)^2} \tag{3.2.29}$$

$$c_n = \overline{b_{n+1}} = e^{i\alpha_n}\left[\frac{q\left(1+Cq^{n-1}\right)\left(1+Cq^{n+1}\right)}{\left(1+Cq^n\right)^2}\right]^{1/2}. \tag{3.2.30}$$

Thus we have constructed an infinite dimensional representation of the braid group with hermitian generators where n is an integer. The right hand side of the equation (3.2.29) must be positive definite and this condition is satisfied only when $C\rangle 0$ and $q\rangle 0$.

Now lets investigate if there is a finite dimensional representation with A hermitian. Suppose there is a ground state $\left|0\right\rangle$ which is annihilated by the lowering operator b and a top state $\left|N-1\right\rangle$ (for N dimensions) which is annihilated by the raising operator b^*.We have

$$b\left|0\right\rangle = 0 \tag{3.2.31}$$

$$c\left|N-1\right\rangle = 0 \tag{3.2.32}$$

from which it follows that

$$\left|b_0\right|^2 = \frac{q(1+c)\left(1+Cq^{-2}\right)}{\left(1+Cq^{-1}\right)^2} = 0. \tag{3.2.33}$$

Equation (3.2.33) is satisfied only when $C = -q^2$. With this value of C

$$a_n = \frac{1-q}{1-q^{n+2}} \tag{3.2.34}$$

$$d_n = -\frac{q^{n+1}(1-q)}{1-q^{n+1}} \tag{3.2.35}$$

$$\left|b_n\right|^2 = \frac{q\left(1-q^{n+2}\right)\left(1-q^n\right)}{\left(1-q^{n+1}\right)^2}. \tag{3.2.36}$$

Let us look if the braid group relations are satisfied with these eigenvalues. Equation (3.1.11) gives

$$cdb|0\rangle = \left(d-d^2\right)|0\rangle$$
$$0 = d_0 - d_0^2.$$

This is satisfied only when $d_0 = 0$ or $d_0 = 1$. From (3.2.35) we obtain $d_0 = -q$. The q=0 case makes the representation trivial . Also the q = -1 case is forbidden because it violates the positive definiteness of $\left|b_n\right|^2$. Hence there are no finite dimensional hermitian representations.

In (3.1.51) we considered $[D^\dagger, D]=0$ and promised to show this explicitly for a hermitian representation. We have

$$D = da - dad - cb + 1 - a - d + ad = \Delta_1 - \Delta_1'$$
$$D^\dagger = a^* d^* - d^* a^* d^* - b^* c^* + 1 - a^* - d^* + d^* a^*$$
$$D^\dagger = ad - dad - cb + 1 - a - d + da = \Delta_1 - \Delta_1' = D.$$

Since $D^\dagger = D$,(3.1.51) is trivially satisfied.

3. 2. 2. Unitary Representations

Now let us discuss the case where A is unitary

$$A^\dagger = A^{-1}$$

$$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} \Delta_1^{-1}(1-a)d & -\Delta_1^{-1}b \\ -\Delta_2^{-1}c & \Delta_2^{-1}(1-d)a \end{bmatrix}.$$

Using (3.1.46) and (3.1.47) one obtains

$$\Delta_1^{-1}(1-a)d = \left[-(1-a)(1-d)\right]^{-1}(1-a)d = -(1-d)^{-1}(1-a)^{-1}(1-a)d$$
$$= \frac{d}{d-1}$$

$$\Delta_2^{-1}(1-d)a = \left[-(1-d)(1-a)\right]^{-1}(1-d)a = -(1-a)(1-d)^{-1}(1-d)a$$
$$= \frac{a}{a-1}.$$

Hence

$$a^* = \frac{d}{d-1} \tag{3.2.37}$$

$$d^* = \frac{a}{a-1} \tag{3.2.38}$$

$$c^* = -\Delta_1^{-1} b \tag{3.2.39}$$

$$b^* = -\Delta_2^{-1} c. \tag{3.2.40}$$

Using (3.1.56) and (3.2.14),(3.2.39) becomes

$$c^* = qb \quad \text{or} \quad c = q^* b^* \tag{3.2.41}$$

$$b^* = qc \quad \text{or} \quad b = q^* c^*. \tag{3.2.42}$$

By substituting (3.2.41) into (3.2.42) one obtains

$$b^* = qq^* b^*$$

which gives

$$qq^* = 1 \tag{3.2.43}$$

$$(1-d)(1-a) = q$$

$$d = \frac{1-q-a}{1-a}$$

$$a^* = \frac{\frac{1-q-a}{1-a}}{\frac{q}{a-1}} = \frac{a+q-1}{q}$$

$$a = 1-q+qa^*. \tag{3.2.44}$$

First let us try to find an infinite dimensional unitary representation. Note that (3.2.41),(3.2.43) and (3.2.44) are the unitarity conditions. We follow the same procedure of the hermitian case,but this time with the unitarity conditions.The operators $b^* = q^{-1}c$ and b are the raising and lowering operators respectively. We find that

$$a_n = \frac{1-q}{1+Cq^n} \qquad d_n = \frac{Cq^{n-1}(1-q)}{1+Cq^{n-1}} \tag{3.2.45}$$

$$|b_n|^2 = \frac{\left(1+Cq^{n-2}\right)\left(1+Cq^n\right)}{\left(1+Cq^{n-1}\right)^2}. \tag{3.2.46}$$

The unitarity condition (3.2.44) imposes $CC^* = 1$. But unfortunately the positive definiteness of (3.2.46) is violated. For some values of n i.e. for some states, $|b_n|^2$ becomes negative. So there are no infinite dimensional unitary representations. There may be a possibility to preserve the positive definiteness by cutting the spectrum where it passes from the positive to the negative region. In other words, let us investigate if there is a finite dimensional representation. Let there be a ground state which is annihilated by the lowering operator and a top state annihilated by the raising operator. For N dimensions we have

$$b|0\rangle = 0 \qquad c|N-1\rangle = 0.$$

Using the braid group relation we get

$$cdb|0\rangle = \left(d-d^2\right)|0\rangle$$
$$0 = d_0\left(1-d_0\right).$$

This gives us the solutions $d_0 = 0$ or $d_0 = 1$. The $d_0 = 0$ solution gives $q = 1$ which makes every eigenvalue of a and d zero and there is no value of q satisfying $d_0 = 1$. The same problem arises for the top state. Hence there are no unitary representations satisfying the braid group relations (3.1.10)-(3.1.16).

For unitary representations we have stated that $[D^\dagger, D]=0$ i.e. D is diagonalizable. Let us explicitly show that this is indeed true. From (3.2.39) and (3.2.40)

$$c^* = -\Delta_1^{-1} b \quad b^* = -\Delta_2^{-1} c \quad c = -b^*\left(\Delta_1^*\right)^{-1}. \tag{3.2.47}$$

It follows that

$$c = \Delta_2^{-1} c \left(\Delta_1^*\right)^{-1}.$$

Since c and Δ_2 commute

$$\Delta_1^* = \Delta_2^{-1} \quad \text{or} \quad \Delta_1^{-1} = \Delta_2^* \tag{3.2.48}$$

and for unitary case we have

$$\begin{aligned}\left(\Delta_2'\right)^* &= -\left(1-a^*\right)\left(1-d^*\right) = -\left(1-\frac{d}{d-1}\right)\left(1-\frac{a}{a-1}\right) \\ &= -\left[(1-a)(1-d)\right]^{-1} = \left(\Delta_1'\right)^{-1}\end{aligned} \tag{3.2.49}$$

$$[D^\dagger, D] = \left[\Delta_1 - \Delta_1', \Delta_2^* - \left(\Delta_2'\right)^*\right] \tag{3.2.50}$$

where $D = \Delta_1 - \Delta_1' = \Delta_2 - \Delta_2'$. Using (3.2.48) and (3.2.49) we get

$$D^{\dagger} = \Delta_1^{-1} - \left(\Delta_1^{'}\right)^{-1} = D^{-1}$$

so D is also unitary and $[D, D^{\dagger}] = 0$.

3. 3. The Pseudo Braid Algebra Their Representations And Generalized Hydrogen Spectrum

In the unitary representation of the braid algebra with b the lowering and $b^* = q^{-1}c$ the raising operator, we have defects for the ground state and for the top state. For a finite dimensional representation we should have $\left|b_0^2\right| = 0$.This is satisfied only when $C = -q^2$ in (3.2.46).Using this value of C in (3.2.45) for N dimensional representation we have

$$cdb|0\rangle = \left(d - d^2\right)|0\rangle$$
$$qd_{-1}|b_0|^2|0\rangle = \left(d_0 - d_0^2\right)|0\rangle \ .$$

The left hand side of the equation must be equal to zero (because b annihilates the ground state). But we have $d_{-1} = \infty$ and $|b_0|^2 = 0$ and the product $d_{-1}|b_0|^2 = \infty \cdot 0$ should give a finite value which is just the value of $d_0 - d_0^2$. Hence we have a defect (or inconsistency 0 = finite nonzero value) for the ground state. Similarly for the top state we have

$$bac|N-1\rangle = \left(a - a^2\right)|N-1\rangle$$
$$qa_N|b_N|^2|N-1\rangle = \left(a_{N-1} - a_{N-1}^2\right)|N-1\rangle.$$

The left hand side must be equal to zero (because $b^* = q^{-1}c$ annihilates the top state). But we have $a_N = \infty$ and $|b_N|^2 = 0$ and the product $d_{-1}|b_0|^2 = \infty \cdot 0$ should give a finite value which is just the value of $a_{N-1} - a_{N-1}^2$. We have also a defect for the top state just like the defect for the ground state.

To overcome this difficulty (inconsistency) i.e. to avoid $\infty.0$ = finite value relations we must avoid a_N and d_{-1} in the relations (3.1.10) and (3.1.11).We replace these relations by (3.1.54) and (3.1.55). Thus

$$bac = a - a^2 \text{ is replaced by } bc = (1-a)[1-d(1-a)]$$
$$cdb = d - d^2 \text{ is replaced by } cb = (1-d)[1-a(1-d)]$$

The relations on the left belong to the braid algebra (BA) while the relations on the right belong to what we call the "pseudo braid algebra (PBA)". If a,b,c,d satisfy the braid algebra and a^{-1} (or d^{-1}) exists then a,b,c,d also satisfy the pseudo braid algebra. If a,b,c,d satisfy the PBA and $\left[1-d(1-a)\right]^{-1}$ (and $\left[1-a(1-d)\right]^{-1}$) exist then a,b,c and d also satisfy the BA. Except (3.3.1) and (3.3.2) all of the braid algebra relations remain the same in the pseudo braid algebra, and (3.1.12) is just a consequence of (3.3.1) and (3.3.2).So we have six relations instead of seven in the case of PBA

$$bc = (1-a)\left[1-d(1-a)\right] \tag{3.3.1}$$
$$cb = (1-d)\left[1-a(1-d)\right] \tag{3.3.2}$$
$$ab = ba(1-d) \tag{3.3.3}$$
$$ca = (1-d)ac \tag{3.3.4}$$
$$bd = (1-a)db \tag{3.3.5}$$
$$dc = cd(1-a). \tag{3.3.6}$$

If we repeat the procedure in the previous section we find the eigenvalues of the operators for the finite dimensional unitary representation as

$$a_n = \frac{1-q}{1+Cq^n} \qquad d_n = \frac{Cq^{n-1}(1-q)}{1+Cq^{n-1}}$$

$$|b_n|^2 = \frac{\left(1+Cq^{n-2}\right)\left(1+Cq^n\right)}{\left(1+Cq^{n-1}\right)^2} \tag{3.3.7}$$

where $qq^* = 1$. The ground state is annihilated by b

$$\begin{aligned} cb|0\rangle &= (1-d)\big[1-a(1-d)\big]|0\rangle \\ |b_0|^2 &= (1-d_0)\big[1-a_0(1-d_0)\big] = 0. \end{aligned} \tag{3.3.8}$$

For $|b_0|^2 = 0$

$$1 + Cq^{-2} = 0 \quad \text{i.e.} \quad C = -q^2. \tag{3.3.9}$$

With this value of C we have

$$a_n = \frac{1-q}{1-q^{n+2}} \qquad\qquad d_n = \frac{-q^{n+1}(1-q)}{1-q^{n+1}}$$

$$|b_n|^2 = \frac{\left(1-q^{n+2}\right)\left(1-q^n\right)}{\left(1-q^{n+1}\right)^2} \tag{3.3.10}$$

$$a_0 = \frac{1}{1+q} \qquad\qquad d_0 = -q.$$

Then the right hand side of (3.3.8) becomes

$$(1+q)\left[1-\frac{1}{q+1}(1+q)\right] = 0.$$

Hence (3.3.2) is satisfied for the ground state. Now we will investigate if (3.3.1) is satisfied for the top state $|N-1\rangle$

$$bc|N-1\rangle = (1-a)[1-d(1-a)]|N-1\rangle.$$

Since c annihilates the top state

$$|b_N|^2 = (1-a_{N-1})[1-d_{N-1}(1-a_{N-1})] = 0 \qquad (3.3.11)$$

$$|b_N|^2 = \frac{(1-q^{N+2})(1-q^N)}{(1-q^{N+1})^2} = 0.$$

This is satisfied when

$$q^{N+2} = 1 \qquad q = e^{i\frac{2\pi}{N+2}}. \qquad (3.3.12)$$

Using (3.3.12) we obtain

$$a_{N-1} = -q \qquad d_{N-1} = \frac{1}{q+1}$$

and with these values the right hand side of (3.3.11) becomes

$$(1+q)\left[1-\frac{1}{(1+q)}(1+q)\right] = 0.$$

Hence (3.3.1) is satisfied for the top state and this completes the construction of the finite dimensional unitary representation of the pseudo braid algebra with the spectrum (3.3.10) and $q = e^{i\frac{2\pi}{n+2}}$ for an N dimensional representation. Since $[1-d(1-a)]^{-1}$ does not exist this unitary representation belongs only to the PBA not to the BA. Now let us show that $|b_n|^2$ is positive definite. Expressing q in terms of trigonometric functions and after a few manipulations we obtain

$$|b_n|^2 = 1 - \frac{Sin^2 \frac{\pi}{N+2}}{Sin^2 \frac{n+1}{N+2}\pi} . \qquad (3.3.13)$$

It is obvious that for $n = 0,1,....,N\text{-}1$ (3.3.13) is positive definite.
For the hermitian representation of the PBA we have

$$A^\dagger = A$$

$$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$a^* = a \qquad c = b^* \qquad d = d^* . \qquad (3.3.14)$$

Again identifying b the lowering operator and c the raising operator and repeating the procedure of (3.2.1) we find that

$$a_n = \frac{1-q}{1+Cq^n}$$

$$d_n = \frac{Cq^{n-1}(1-q)}{1+Cq^{n-1}}$$

$$|b_n|^2 = \frac{q\left(1+Cq^n\right)\left(1+Cq^{n-2}\right)}{\left(1+Cq^{n-1}\right)^2}$$

where q is a real parameter. For an infinite dimensional representation where $n = 0,1,2,.....$. the lowering operator b annihilates the ground state

$$cb|0\rangle = (1-d)\left[1-a(1-d)\right]|0\rangle. \tag{3.3.15}$$

The left hand side must be equal to zero

$$|b_0|^2 = \frac{q\left(1+Cq^n\right)\left(1+Cq^{n-2}\right)}{\left(1+Cq^{n-1}\right)^2} = 0.$$

This is satisfied when $C = -q^2$ and by substituting this value we get

$$a_n = \frac{1-q}{1-q^{n+2}} \tag{3.3.16}$$

$$d_n = \frac{-q^{n+1}(1-q)}{1-q^{n+1}} \tag{3.3.17}$$

$$|b_n|^2 = \frac{q\left(1-q^n\right)\left(1-q^{n+2}\right)}{\left(1-q^{n+1}\right)^2}. \tag{3.3.18}$$

The right hand side of (3.3.15) must be equal to zero for consistency, i.e.,

$$(1-d_0)\left[1-a_0(1-d_0)\right] = 0.$$

Using

$$a_0 = \frac{1}{1+q} \quad \text{and} \quad d_0 = -q$$

we find that

$$(1+q)\left[1 - \frac{1}{1+q}(1+q)\right] = 0$$

so (3.3.15) is satisfied. Since there is no value of q satisfying (3.3.1) for the top state i.e.

$$bc|N-1\rangle = (1-a)[1-d(1-a)]|N-1\rangle$$

we have only infinite dimensional hermitian representations. Since $\left[1-a(1-d)\right]$ is not invertible this hermitian representation belongs only to the PBA not to the BA. If we don't have a ground state i.e. if $n = ...,-2,-1,0,1,2,......$ then the hermitian representations belong both to the BA and to the PBA.

Now let us discuss the $q \to 1$ limit. From (3.1.59) we have

$$A^2 = (1-q)A + q.$$

This reduces to

$$A^2 = 1. \tag{3.3.19}$$

Equation (3.3.19) shows that in the $q \to 1$ limit the hermitian and the unitary representations coincide

$$A = A^{-1} = A^{\dagger}.$$

We can also see this from the eigenvalues of the operators . In both representations the eigenvalues reduce to

$$a_n = \frac{1}{n+2} \tag{3.3.20}$$

$$d_n = -\frac{1}{n+1} \tag{3.3.21}$$

$$|b_n|^2 = \frac{(n+2)n}{(n+1)^2} = 1 - \frac{1}{(n+1)^2} \tag{3.3.22}$$

or

$$b_n = \frac{\left[n(n+2)\right]^{1/2}}{n+1} = c_{n-1} . \tag{3.3.23}$$

We can identify (3.3.22) as a hydrogen-like spectrum and interpret the $q \neq 1$ case as a one parameter generalization of the hydrogen spectrum.

3.4. OTHER REPRESENTATIONS

Let us investigate other possible representations of the braid algebra. In section 3.1., we have shown that

$$\left[b,(1-a)(1-d)\right]=0. \tag{3.4.1}$$

Assuming a^{-1} and b^{-1} exist and using the relation (3.1.13) we get

$$\begin{aligned} ab &= ba(1-d) \\ (1-d) &= a^{-1}b^{-1}ab. \end{aligned} \tag{3.4.2}$$

Substituting this value in (3.4.1)

$$\left[b,(1-a)a^{-1}b^{-1}ab\right]=0.$$

This reduces to

$$\left[b,\frac{(1-a)}{a}b^{-1}a\right]=0\ . \tag{3.4.3}$$

In fact this is the only relation to be satisfied. By solving c and d in terms of a and b using (3.1.13) and (3.1.10), that is

$$c=a^{-1}b^{-1}a(1-a) \tag{3.4.4}$$

$$d=1-a^{-1}b^{-1}ab \tag{3.4.5}$$

and substituting into the braid algebra relations, it can be shown that all of the relations (3.1.10)-(3.1.16) are satisfied when (3.4.3) is satisfied. Hence different solutions to (3.4.3) are different representations of the braid algebra. Finite dimensional representations exist for this case but it can be proven that they cannot be made hermitian or unitary.

4. CONCLUSION

The use of the q-deformation of Lie groups -quantum groups- may be an opportunity to solve the standard problems of field theories by generalizing the symmetry. Quantum groups, we have discussed only the $SU_q(2)$ case, can be used as gauge groups and we have two possibilities for the gauge field. One of the possibility is to take the gauge field as an element of the quantum universal enveloping algebra of su(2) while the other possibility is to take the gauge field as an element of the Lie algebra analog of the gauge group $SU_q(2)$. It would be interesting to find the relation between the superplane defined by (2.3.44) and the superplane introduced in the Manin formulation of quantum groups[15].The relation between the exponential mapping (2.3.33) and the exponential representation found in [16] and [17] still remains as an unsolved problem.

To construct the Lagrangian which is invariant under quantum group gauge transformations the usual notion of trace has to be modified. The q-trace which is invariant under quantum group transformations is defined. But the physical meaning of the noncommuting objects in the Lagrangian has to be clarified and this is another problem for future works.

We have defined two closely related associative algebras by considering a 2×2 matrix whose elements satisfy certain commutation-like relations. If the relations of which we have called the braid algebra are satisfied then a representation of the n-braid group can be constructed. Looking for hermitian and unitary representations of the braid algebra in Hilbert space by identifying raising and lowering operators we have found that the braid algebra has such representations without a ground or top state. If the existence of a ground and/or a top state is desired then one has to define a new algebra which we have called "the pseudo-braid algebra".

The pseudo-braid algebra has two physically interesting representations, one of which corresponds to the case where the 2×2 matrix whose elements generate the algebra is hermitian. In this case the structure is that of a generalized oscillator with creation and annihilation operators b^* and b such that the spectrum of b^*b is a one parameter generalization of the hydrogen spectrum. The other interesting representation corresponds to the case where 2×2 matrix with operator elements is unitary. In this case, the representations are finite dimensional. The spectrum of b^*b, although finite, is again a generalization of the hydrogen spectrum since in the limit $q=\exp\left(2\pi i/\left(N+2\right)\right)\to 1$ it becomes hydrogen-like. In this limit N, the dimension of the representation goes to infinity.

For the pseudo-braid algebra one can again use the 2×2 matrix A with operator elements to construct "the pseudo-braid group".In this case the braid group relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ is only approximately satisfied. Both the braid algebra and the pseudo-braid algebra have mathematically and physically interesting properties.

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