ASPECTS OF NONRELATIVISTIC STRONG GRAVITY

by

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B.S., Physics, Boğaziçi University, 2018

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Physics
Boğaziçi University
2020

ACKNOWLEDGEMENTS

First and foremost, I would also like to thank my family for always being with me emotionally, supporting me financially and most importantly supporting my decisions.

I would like to thank my supervisor Dieter Van den Bleeken for providing me with such a well thought project and for his patient guidance and help throughout this work. Despite being a arduous task, our combined work ethic, fun conversations and constant snack sharing has made this thesis successful.

Finally, I would like to thank my closest friends who have been with me for many years, sharing the most important details of our lives and talking about anything and everything. Our long conversations has been a crucial part of what I have become.

ABSTRACT

ASPECTS OF NONRELATIVISTIC STRONG GRAVITY

We carry out the leading order (LO) large c expansion to GR in the presence of odd powers with the tools provided by Newton-Cartan (NC) gravity. As a result, a set of diffeomorphism invariant LO equations are found. The odd power term is shown to be U(1) vector field that appears in the equations through its field strength. We finally show that the LO equations covers the stationary sector of GR entirely where the rotational degrees of freedom are provided by the U(1) field.

ÖZET

RELATİVİSTİK OLMAYAN GÜÇLÜ YERÇEKİMİ

Genel relativiteye (GR), Newton-Cartan (NC) yerçekimi kuramının sağladığı araçlar kullanılarak ve tekil terimleri de göz önünde bulundurularak ilk mertebe (LO) yüksek ışık hızı açılımı (c) yapılacaktır. Sonuç olarak, difeomorfizmalar altında değişmeyen ilk mertebe denklemler bulunmuştur. Tekil mertebe teriminin U(1) vektör alanı olduğu ve ilk mertebe denklemlerinde kendi oluşturduğu güç alanı altında ortaya çıkışı gösterilecektir. Son olarak ilk mertebe denklemlerinin genel relativitedeki durgun alan denklemlerini tamamen kapladığı ve bu alanın dönüşünü tanımlayan serbestlik derecesinin bu U(1) vektör alanı tarafından sağlandığı gösterilecektir.

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LIST OF SYMBOLS

$T^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}(x)$	Arbitrary spacetime tensor
$T'^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}(x')$	Arbitrary spacetime tensor in a different coordinate basis
$\overset{(n)}{T}{}^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}$	Arbitrary spacetime tensor of order n
Ξ^μ, ξ^μ	Diffeomorphism generating vector field
L_ξ	Lie derivative along vector field ξ^μ
δ	Infinitesimal transformations under all transformations
δ_ξ, δ_Ξ	Infinitesimal transformations under diffeomorphisms
δ_Λ	Infinitesimal transformations under Lorentz rotations
δ_{ad}	Infinitesimal transformations under adjoint transformations
$\mu, \nu, \rho, \lambda, \sigma \dots$	4d Spacetime tensor indices running as $(0,1,2,3)$
$\dot{\mu}, \dot{\nu}, \dot{\rho}, \dot{\lambda}, \dot{\sigma} \dots$	4d Spatial tensor indices running as $(0,1,2,3)$
$i, j, k, l, m \dots$	3d Space tensor indices running as $(1,2,3)$
$A, B, C, D, E \dots$	4d Lorentz tensor indices running as $(0,1,2,3)$
$a, b, c, d, e \dots$	3d Lorentz tensor indices running as $(1,2,3)$
$\tilde{\nabla}_\mu$	Arbitrary connection
$\tilde{\Gamma}^\lambda_{\mu\nu}$	Arbitrary connection coefficients
∇_μ	Levi-Civita connection
$\Gamma^\lambda_{\mu\nu}$	Levi-Civita connection coefficients
$\boldsymbol{\nabla}_\mu$	Newton-Cartan connection of k-metric
$\mathbb{\Gamma}^\lambda_{\mu\nu}$	Newton-Cartan connection coefficients of l-metric
$\dot{\boldsymbol{\nabla}}_\mu$	Newton-Cartan connection of h-metric
$\dot{\mathbb{\Gamma}}^\lambda_{\mu\nu}$	Newton-Cartan connection coefficients of h-metric
$\hat{\boldsymbol{\nabla}}_\mu$	All-compatible Newton-Cartan connection
$\hat{\mathbb{\Gamma}}^\lambda_{\mu\nu}$	All-compatible Newton-Cartan connection coefficients
$\bar{\boldsymbol{\nabla}}_\mu$	Boost invariant Newton-Cartan connection
$\bar{\mathbb{\Gamma}}^\lambda_{\mu\nu}$	Boost invariant Newton-Cartan connection coefficients
χ_μ	Milne Boost parameter
Λ^{AB}	Spacetime Lorentz rotation parameter
Λ^a	Lorentz boost parameter

Λ^{ab}	Spatial Lorentz rotation parameter
λ^a	Galilean boost parameter
$g_{\mu\nu}$	Metric tensor
$g^{\mu\nu}$	Inverse metric tensor
$\tilde{R}^{\rho}{}_{\sigma\mu\nu}$	Riemann tensor of arbitrary connection
$R^{\rho}{}_{\sigma\mu\nu}$	Riemann tensor of Levi-Civita connection
$\tilde{R}_{\mu\nu}$	Ricci tensor of arbitrary connection
$R_{\mu\nu}$	Ricci tensor of Levi-Civita connection
$T_{\mu\nu}$	Conserved energy-momentum tensor
$\mathcal{T}_{\mu\nu}$	Trace reversed energy-momentum tensor
$\mathcal{C}_\mu$	Einstein equations conservation law
c	Speed of light parameter
$\tau_\mu, h^{\mu\nu}$	Newton-Cartan structure fields
$\tau^\mu, h_{\mu\nu}$	Inverse Newton-Cartan structure fields
$\tau_\mu, k^{\mu\nu}$	Conformal Newton-Cartan structure fields
$\tau^\mu, k_{\mu\nu}$	Inverse Conformal Newton-Cartan structure fields
$a_{\dot{\mu}}$	Newton-Cartan torsion

LIST OF ACRONYMS/ABBREVIATIONS

GR	General Relativity
LC	Levi-Civita
LO	Leading Order
NC	Newton-Cartan
NLO	Next to Leading Order
N^2LO	Next to Next Leading Order
PN	Post-Newtonian

1. INTRODUCTION

The theory of Newton-Cartan gravity was originally constructed by E. Cartan [1,2] – hence the name Newton-Cartan – in an attempt to write Newton's equations of gravity in a covariantly in the language resembling general relativity. It was revisited in a further work by Dautcourt [3] to covariantly formulate an approximation to general relativity. The ultimate aim of the initial works [3, 4] was simply to provide a manifestly covariant version of the post-Newtonian expansion. There it was realised such approximation amounts to an expansion of the relativistic metric in inverse powers of the speed of light c and as a result it is referred as the large c expansion.

The original work in [3] has recently been revisited in [4–7, 9–12]. To give a brief summary that is relevant in this work, in [5] it was pointed out that the large c expansion naturally allows the inclusion of strong gravitational effects which are not present in the post-Newtonian expansion. This is no surprise as the post-Newtonian expansion [13] is not only nonrelativistic, but also a weak field approximation. This feature of effectively describing some strong gravitational effects suggests the large c expansion could have interesting phenomenological applications, although this remains largely unexplored.

Previous work on the large c expansion was based on assuming an expansion of the metric in inverse powers of c^2. Under the weak field assumption, together with consideration of some physical constraints on energy-momentum and an appropriate coordinate choice, one can show that odd terms in the relativistic metric can appear only at next to leading order [3, 13]. This is not case when one does not assume weak field, for which the presence of terms with an odd power of c has to date remained unexplored. Its study is initiated in the work [8].

In addition to the inclusion of odd powers, the main difference between the expansions in [8] and in the previous works [3–6] is in the former we let the leading order temporal metric component – will be denoted as $\overset{(-2)}{A}$ – free. It is further noticed that the previous choices of setting $\overset{(-2)}{A} = -1$ is simply a gauge choice. Moreover, this has the implication that one does not need the torsion to describe the strong gravitational effects of the leading order theory that can be taken care of by $\overset{(-2)}{A}$ as opposed to [5].

The Kerr solution explored in [8] – as well as in this thesis – shows that there are real world gravitational phenomena that find themselves in the strong field nonrelativistic regime that is well approximated by the large c expansion. Still, the expansion is only of real practical use for phenomena that are too complicated to describe analytically in general relativity and at the same time require high orders, or a break down in the post-Newtonian approximation. For such phenomena the large c expansion would be an efficient and an analytic tool to compete with numerical relativity.

The main result of [8] was to identify the large c expansion as an expansion around the stationary sector of GR and as will be pointed out it generalizes the post-Newtonian expansion to include metrics that are not nearly flat. Thus, the possible practical applications of the large c expansion concern phenomena where gravity is strong and that are almost stationary. If one chooses the leading order stationary metric to be Minkowski space the large c expansion reduces to the standard post-Newtonian one. But more generally one can choose any stationary metric as the starting point for the expansion, extending it into the strong gravitational regime. We will illustrate this explicitly using the example of the Kerr metric in Chapter 7.

This thesis is mainly based on the paper [8] and in a sense it mainly serves the purpose of demonstration on how each of the intermediate steps and the tedious calculations work. The structure with brief summaries of each chapter goes as follows:

In Chapter 2, we will briefly introduce the basic tools in differential geometry. In doing so, we will mainly follow the sources [18–21]. The chapter mainly serves the purpose of introducing the notation and the formulae that will be used throughout the thesis. They will quite often be referred back when needed. We begin with the definition a tensor and its transformation rules that allows us introduce diffeomorphisms. Further on, we define an inner product by introducing the metric that will help us to describe the space-time. We then introduce a general connection to describe the curvature and provide the general identities.

In Chapter 3, we make the first contact with gravity by initially imposing the metric compatibility conditions. This will allow us to introduce the Levi-Civita connection and its curvature. Upon introducing the matter in the form of the conserved energy-momentum tensor and using the Bianchi identities of the curvature, we will present the familiar Einstein equations. We use trace-reversed equations that will form the basis of our work.

In Chapter 4, we will provide a brief introduction to Newton-Cartan domain and give the covariant form of the Newton's equations, for which, we mainly follow the sources [3, 14, 17]. While doing so, we will introduce the concepts such as Newton-Cartan structure, Newton-Cartan split, Newton-Cartan connection, twistless torsion condition and boost symmetry that will be applied when performing the leading order expansion. In doing so we introduce a convenient dot notation for spatial tensors that will be used throughout this thesis.

Chapter 5 is the point where we combine general relativity and Newton-Cartan theory. Here, we reformulate Einstein equations by splitting it in the temporal and spatial directions using the Newton-Cartan structure and the Newton-Cartan connection introduced previously. In a sense providing a Newton-Cartan description of general relativity that will ease up the messy intermediate calculations of the leading order large c expansion.

Chapter 6 serves as the highlight of this thesis where the final leading order equations are provided in Section 6.4. It is the point where we perform the infamous – odd power including – large c expansion under the twistless torsion condition on the previously introduced split metric and energy-momentum tensor variables and with it on the Einstein equations. This results in a set of leading order Einstein equations subjected to the conservation laws that are not invariant under diffeomorphisms and rescalings. We redefine the initial fields ($\overset{(-2)}{A}$, $\overset{(-1)}{A}_{\dot{\mu}}$, $\overset{(0)}{A}{}^{\dot{\mu}\dot{\nu}}$) to introduce manifestly symmetry invariant

– in a sense stationary – variables given as $(\Psi, C_{\dot{\mu}}, k^{\mu\nu})$. These new dynamical variables bringing us closer to the stationary solutions of GR as well as having the advantage of cleaning up the leading order equations while allowing us to writing them in a final symmetry invariant form.

Finally, in Chapter 7, we discuss the possible gauge choices on the solutions of LO equations and illustrate how they relate to the original works [3]. It will also be shown that one does not really need the existence torsion in order to describe the strong gravitational effects in such a regime. We then proceed by showing how the stationary relativistic metrics provide exact solutions to the LO equations and conclude the discussion with three different limits of Kerr metric. In this section it is realised that the newly introduced variables form a more natural candidate for further expansion to the higher orders where the formulation seems to be more natural.

The appendices will contain technical results that will be referred to when necessary. Finally in Appendix E, in addition to the leading order equations, we also explore the dominion of relativistic and nonrelativistic gauge theory formalisms. We first begin with a discussion of Lie algebras and general gauge theories. We move on with the Poincare algebra formulations of general relativity. Finally we discuss the Bargmann algebra formulation of Newton-Cartan gravity which motivates to the more general nonrelativistic gauge theory formulations. There, the degeneracy in local translations occurring in nonrelativistic gauging procedures as discussed in Appendix E.5. This served as the motivation for the formalism employed here.

2. DIFFERENTIAL GEOMETRY

In this section, we will briefly present the objects that will often used be in this work. To do so, we will roughly follow the sources [18–20]. As we will not go into most of the derivations, a good knowledge in differential geometry is assumed for the readers.

We will restrict the discussion in this thesis to 1+3 dimensions and work with the spacetime coordinates x^μ, $\mu = 0, 1, 2, 3$.

We first begin by introducing tensors and diffeomorphisms on some arbitrary manifold $\mathcal{M}$. Then move on by introducing the metric and a general connection on a manifold $\mathcal{M}$ that is used to define the curvature.

2.1. Tensors and Diffeomorphisms

A (m, n)-tensor is defined to be the object that transforms under the change of coordinates $x^\mu \to x'^\mu(x)$ as

$$T'^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m}(x'(x)) = \frac{\partial x^{\rho_n}}{\partial x'^{\mu_n}} \cdots \frac{\partial x^{\rho_m}}{\partial x'^{\mu_m}} \frac{\partial x'^{\nu_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial x'^{\nu_n}}{\partial x^{\sigma_n}} T^{\sigma_1 \ldots \sigma_\nu}_{\rho_1 \ldots \rho_m}(x) \tag{2.1}$$

We indicate an object in a different basis using primes. For clarity it should be understood that the indices indicating the coordinate functions will be suppressed i.e. $T(x) := T(x^\nu)$. We also employ the Einstein summation convention here and throughout in which repeated indices are summed.

Considering the infinitesimal change of coordinates such that

$$x'^{\mu}(x) - x^{\mu} = -\xi^{\mu}(x) \tag{2.2}$$

results in

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} - \frac{\partial \xi^{\mu}}{\partial x^{\nu}} \qquad \frac{\partial x^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu} + \frac{\partial \xi^{\mu}}{\partial x^{\nu}} \tag{2.3}$$

Combining Equation (2.1) with Equation (2.3), one finds transformation of a tensor under the infinitesimal change:

$$\begin{aligned} \delta_{\xi} T^{\sigma_1 \dots \sigma_\nu}_{\rho_1 \dots \rho_m} &:= T'^{\sigma_1 \dots \sigma_\nu}_{\rho_1 \dots \rho_m}(x) - T^{\sigma_1 \dots \sigma_\nu}_{\rho_1 \dots \rho_m}(x) \\ &= L_{\xi} T^{\sigma_1 \dots \sigma_\nu}_{\rho_1 \dots \rho_m} \end{aligned} \tag{2.4}$$

The transformations under the vector fields ξ^{μ} and Ξ^{μ} are defined to be the diffeomorphisms. Such transformations define the Lie derivative of a tensor as the rule acting on both upper and lower indices:

$$L_{\xi} T^{\sigma_1 \dots \sigma_\nu}_{\rho_1 \dots \rho_m} = \xi^{\lambda} \partial_{\lambda} T^{\sigma_1 \dots \sigma_\nu}_{\rho_1 \dots \rho_m} + T^{\sigma_1 \dots \sigma_\nu}_{\lambda \rho_2 \dots \rho_m} \partial_{\rho_1} \xi^{\lambda} + \dots - T^{\lambda \dots \sigma_\nu}_{\rho_1 \dots \rho_m} \partial_{\lambda} \xi^{\sigma_1} - \dots \tag{2.5}$$

We say that an object is a tensor if it transforms as a Lie derivative under diffeomorphisms.

2.1.1. Metric Tensor

The metric $g_{\mu\nu}$ is a symmetric (2,0)-tensor that transforms under diffeomorphisms as:

$$\delta g_{\mu\nu} = L_{\xi} g_{\mu\nu} \tag{2.6}$$

that is used to take the inner product of two vectors:

$$\langle V, W \rangle_g = g_{\mu\nu} V^{\mu} W^{\nu} \tag{2.7}$$

Through this, one can lower the index via:

$$g_{\mu\nu} V^{\mu} W^{\nu} = V^{\mu} W_{\mu} \quad \text{such that} \quad W_{\mu} = g_{\mu\nu} W^{\nu} \tag{2.8}$$

The metric determines a unique inverse metric $g^{\mu\nu}$ via the equation

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu} \tag{2.9}$$

that transforms as

$$\delta g^{\mu\nu} = L_{\xi} g^{\mu\nu} \tag{2.10}$$

It can be used to raise indices as follows:

$$\omega^{\mu} = g^{\mu\nu} \omega_{\nu} \tag{2.11}$$

2.1.2. Differential Forms

A differential form or a n-form is a $(n,0)$-tensor that is fully antisymmetric in its indices:

$$\omega_{\mu_1 \dots \mu_n} = \omega_{[\mu_1 \dots \mu_n]} \tag{2.12}$$

One can construct a $(n+m)$-form by taking the wedge product of a n-form ω and a m-form η defined through:

$$(\omega \wedge \eta)_{\mu_1 \dots \mu_{n+m}} = \frac{(n+m)!}{n!m!}\omega_{[\mu_1 \dots \mu_n \dots \mu_{n+m}]} \tag{2.13}$$

A n-form ω can be differentiated to give a $(n+1)$-form via the exterior derivative d defined by the rule:

$$(d\omega)_{\mu_1 \dots \mu_{n+1}} = (n+1)\partial_{[\mu_1}\omega_{\mu_2 \dots \mu_{n+1}]} \tag{2.14}$$

Application of this operation twice results in the closedness identity

$$d(d\omega) = 0 \tag{2.15}$$

that applies to any differential form. This result follows from the symmetry of the partial derivatives. A n-form ω is said to be closed if $d\omega = 0$ and exact if $d\omega = \phi$ for a $(n-1)$-form ϕ. Consequently, it follows from Equation (2.15) that all exact forms are closed. Furthermore, the differential of the wedge product of obeys the modified form of the Leibniz rule as:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^n \omega \wedge d\eta \tag{2.16}$$

Additionally, a vector V can act on the first index of a n-form to give a $(n-1)$-form through an operation called interior product

$$(\mathrm{i}_V\omega)_{\mu_1\ldots\mu_{n-1}} = V^\rho \omega_{\rho\mu_1\ldots\mu_{n-1}} \tag{2.17}$$

obeying the very same Leibniz rule:

$$\mathrm{i}_V(\omega \wedge \eta) = \mathrm{i}_V\omega \wedge \eta + (-1)^n \omega \wedge \mathrm{i}_V\eta \tag{2.18}$$

It then follows that the Lie derivative of any form field can be written in terms of the introduced operations in the following manner:

$$L_\Xi\omega = \mathrm{i}_\Xi d\omega + d(\mathrm{i}_\Xi\omega) \tag{2.19}$$

This is called Cartan's formula which will prove useful when discussing the Poincaré algebra formulation of general relativity.

2.2. Connection and Curvature

One can define a general connection $\tilde{\nabla}$ with coefficients $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ on a manifold $\mathcal{M}$ that will transform non-tensorially under diffeomorphisms as:

$$\tilde{\Gamma}'^{\lambda}_{\mu\nu}(x'(x)) = \frac{\partial x^{\lambda}}{\partial x'^{\rho}}\frac{\partial x'^{\sigma}}{\partial x^{\mu}}\frac{\partial x'^{\alpha}}{\partial x^{\nu}}\tilde{\Gamma}^{\rho}_{\sigma\alpha}(x) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \tag{2.20}$$

Or infinitesimally:

$$\delta_{\xi}\Gamma^{\lambda}_{\mu\nu} = L_{\xi}\Gamma^{\lambda}_{\mu\nu} + \partial_{\mu}\partial_{\nu}\xi^{\lambda} \tag{2.21}$$

following the rule for each space-time index

$$\tilde{\nabla}_{\mu}T^{\sigma_1\dots\sigma_\nu}_{\rho_1\dots\rho_m} = \partial_{\mu}T^{\sigma_1\dots\sigma_\nu}_{\rho_1\dots\rho_m} + \tilde{\Gamma}^{\sigma_1}_{\mu\lambda}T^{\lambda\sigma_2\dots\sigma_\nu}_{\rho_1\dots\rho_m} + \dots - \tilde{\Gamma}^{\lambda}_{\mu\rho_1}T^{\sigma_1\dots\sigma_\nu}_{\lambda\rho_2\dots\rho_m} - \dots \tag{2.22}$$

Note that the symbol $\tilde{\nabla}$ refers to the connection itself and $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ refers to the connection coefficients. In most cases however – as in here – $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ is referred as the connection and $\tilde{\nabla}_{\mu}$ is referred as the covariant derivative. Throughout this thesis, as we will define different types of connections and coefficients and label them with various symbols, we will always denote a new connection as a pair $(\tilde{\nabla}_{\mu}, \tilde{\Gamma}^{\lambda}_{\mu\nu})$. It is clear from the definition that the connection coefficients are not tensors. However, the covariant derivative of a tensor is a tensor:

$$\delta_{\xi}(\tilde{\nabla}_{\mu}T^{\sigma_1\dots\sigma_\nu}_{\rho_1\dots\rho_m}) = L_{\xi}(\tilde{\nabla}_{\mu}T^{\sigma_1\dots\sigma_\nu}_{\rho_1\dots\rho_m}) \tag{2.23}$$

In fact the very definition of connection coefficients in Equation (2.20) follows upon demanding this result.

An important consequence is that the covariant derivatives of vector fields in general do not commute:

$$2\tilde{\nabla}_{[\mu}\tilde{\nabla}_{\nu]}V^{\rho} = \tilde{R}^{\rho}{}_{\sigma\mu\nu}V^{\sigma} - \tilde{T}^{\sigma}_{\mu\nu}\tilde{\nabla}_{\sigma}V^{\rho} \tag{2.24}$$

It allows us to introduce the Riemann curvature tensor:

$$\tilde{R}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\tilde{\Gamma}^{\rho}_{\nu\sigma} - \partial_{\nu}\tilde{\Gamma}^{\rho}_{\mu\sigma} + \tilde{\Gamma}^{\rho}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \tilde{\Gamma}^{\rho}_{\nu\lambda}\tilde{\Gamma}^{\lambda}_{\mu\sigma} \tag{2.25}$$

As well as the torsion tensor:

$$\tilde{T}^{\lambda}_{\mu\nu} = 2\tilde{\Gamma}^{\lambda}_{[\mu\nu]} \tag{2.26}$$

The Riemann tensor obeys the so called Bianchi identities:

$$\tilde{R}^{\rho}{}_{[\sigma\mu\nu]} = \tilde{T}^{\lambda}{}_{[\mu\nu}\tilde{T}^{\rho}{}_{\sigma]\lambda} - \tilde{\nabla}_{[\mu}\tilde{T}^{\rho}{}_{\nu\sigma]} \tag{2.27}$$

$$\tilde{\nabla}_{[\lambda}\tilde{R}^{\rho}{}_{|\sigma|\mu\nu]} = \tilde{T}^{\alpha}{}_{[\lambda\mu}\tilde{R}^{\rho}{}_{|\sigma|\nu]\alpha} \tag{2.28}$$

and enjoys the antisymmetry in its last indices

$$\tilde{R}^{\rho}{}_{\sigma\mu\nu} = -\tilde{R}^{\rho}{}_{\sigma\nu\mu} \tag{2.29}$$

Using the Riemann tensor one then defines the Ricci tensor by contracting the first and the third indices:

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\lambda}{}_{\mu\lambda\nu} \tag{2.30}$$

3. GENERAL RELATIVITY

In this chapter we will briefly formulate general relativity in the standard diffeomorphism invariant metric formulation equipped with the Levi-Civita connection. Doing so allows us to introduce the objects that will be used for the rest of this thesis.

Note that a gauge theory formulation of general relativity is also provided in Appendix E which we did not include here for the sake of continuity.

3.1. Metric Formulation

To formulate general relativity we first begin with a Lorentzian metric $g_{\mu\nu}$ and requiring it to be metric compatible and torsion-free:

$$\nabla_\rho g_{\mu\nu} = 0 \qquad\qquad T^\lambda_{\mu\nu} = 2\Gamma^\lambda_{[\mu\nu]} = 0 \tag{3.1}$$

This is called the Levi-Civita connection which will be denoted as a pair $(\nabla_\mu, \Gamma^\lambda_{\mu\nu})$. The Levi-Civita connection coefficients are famously given as:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{3.2}$$

which will admit the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{3.3}$$

obeying the torsionless Bianchi identities

$$R^\rho{}_{[\sigma\mu\nu]} = 0 \tag{3.4}$$

$$\nabla_{[\lambda} R^\rho{}_{|\sigma|\mu\nu]} = 0 \tag{3.5}$$

while enjoying the symmetry properties:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \qquad R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \qquad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \tag{3.6}$$

The Riemann tensor of the Levi-Civita connection can be used to define the symmetric Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} \tag{3.7}$$

Contracting Equation (3.5) twice results in:

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} R_{\rho\sigma} \right) = 0 \tag{3.8}$$

This result defines the Einstein tensor $G_{\mu\nu}$ as:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} R_{\rho\sigma} \quad \text{such that} \quad \nabla^\mu G_{\mu\nu} = 0 \tag{3.9}$$

The Einstein equations are then given by equating the Einstein tensor of the Levi-Civita connection to the energy-momentum tensor :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} R_{\rho\sigma} = \frac{8\pi G_{\mathrm{N}}}{c^4} T_{\mu\nu} \quad \text{such that} \quad \nabla^\mu T_{\mu\nu} = 0 \tag{3.10}$$

These equations follow by varying the Einstein-Hilbert Lagrangian with respect to $g^{\mu\nu}$ given as

$$L = \sqrt{-g} g^{\mu\nu} R_{\mu\nu} + L_{\mathrm{M}} \tag{3.11}$$

For the rest of the thesis, we will work with the trace reversed Einstein equations, as it allows us to split the curvature in a simpler way. We first define the trace reversed energy-momentum tensor as:

$$\mathcal{T}_{\mu\nu} = c^{-4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\lambda}T_{\rho\lambda}\right) \tag{3.12}$$

where the speed of light parameter c is introduced to match the orders of both sides. This can be used to cast the equation into the following form

$$R_{\mu\nu} = 8\pi G_{\mathrm{N}}\mathcal{T}_{\mu\nu} \tag{3.13}$$

The Bianchi identity satisfied by the Ricci tensor $R_{\mu\nu}$ in Equation (3.5) guarantees the conservation of energy-momentum, which is equivalent to

$$\mathcal{C}_{\mu} = \nabla^{\rho}\left(\mathcal{T}_{\rho\mu} - \frac{1}{2}g_{\rho\mu}g^{\lambda\sigma}\mathcal{T}_{\lambda\sigma}\right) = 0 \tag{3.14}$$

4. TWISTLESS TORSIONAL NEWTON-CARTAN GRAVITY

In this chapter we will introduce the structures that will be used throughout this thesis while demonstrating the formulation of Newton-Cartan gravity. The discussion will mostly follow from the sources [3, 14, 17].

Further in this chapter, we will first introduce the Newton-Cartan structure consists of a 1-form τ_μ and a $(0, 2)$-tensor degenerate metric $h^{\mu\nu}$. Then proceed by introducing the non-unique inverse fields τ^μ and $h_{\mu\nu}$ that can be fixed up to a boost symmetry by an introduction of another 1-form m_μ. Then follow up by introducing the torsional Newton-Cartan connection and introduce covariant form of Newton's equations.

4.1. Newton-Cartan Structure and Split

We introduce the Newton-Cartan structure $(\tau_\mu, h^{\mu\nu})$ as a positive semi-definite, symmetric (0,2)-tensor $h^{\mu\nu}$ together with a zero eigencovector τ_μ that is unique up to rescaling:

$$h^{\mu\nu}\omega_\nu = 0 \quad \Leftrightarrow \quad \omega_\mu \sim \tau_\mu \,. \tag{4.1}$$

The one-form τ_μ is often referred to as the clock form as it sets the direction of time, while $h^{\mu\nu}$ encodes a purely spatial metric [1, 2].

Given a Newton-Cartan structure one can introduce additional fields τ^μ and $h_{\mu\nu}$ such that

$$\tau_\mu \tau^\nu + h_{\mu\rho}h^{\rho\nu} = \delta^\nu_\mu \qquad \tau^\mu\tau^\nu h_{\mu\nu} = 0\,. \tag{4.2}$$

which also implies that

$$\tau^\mu \tau_\mu = 1 \qquad \tau^\mu h_{\mu\nu} = 0 \tag{4.3}$$

Equations (4.2) can also be read as the definition of two complementary projectors:

$$\tau_\mu^\nu = \tau_\mu \tau^\nu \qquad h_\mu^\nu = h_{\mu\rho} h^{\rho\nu} \tag{4.4}$$

Equipped with this projector structure, any tensor index can be decomposed into a temporal and spatial part, something we will refer to as a 'Newton-Cartan split'. For an arbitrary 1-form U_μ for example, such decomposition reads

$$U_\mu = \tau_\mu \tau^\nu U_\nu + h_\mu^\nu U_\nu \tag{4.5}$$

Since we will perform such a Newton-Cartan split on essentially any tensor we will encounter it will be useful to introduce some more compact notation. We'll write the temporal part of U_μ as

$$U = \tau^\mu U_\mu \tag{4.6}$$

and we will indicate the spatial projection by putting a dot on the projected index:

$$U_{\dot\mu} = h_\mu^\nu U_\nu \tag{4.7}$$

In this notation the decomposition Equation (4.5) then simplifies to

$$U_\mu = \tau_\mu U + U_{\dot\mu} \tag{4.8}$$

Note that dotted indices contracted with τ vanish, while the spatial components can be lowered and raised by h

$$\tau_\mu U^{\dot{\mu}} = 0 \qquad \tau^\mu U_{\dot{\mu}} = 0 \qquad h_{\mu\nu} U^{\dot{\nu}} = U_{\dot{\mu}} \qquad h^{\mu\nu} U_{\dot{\nu}} = U^{\dot{\mu}} \tag{4.9}$$

This notation can be safely extended to higher rank tensors as long as they are either fully symmetric or anti-symmetric. For example for a (1,2)-tensor that is symmetric in its upper indices we get

$$V_\mu{}^{\nu\rho} = V\tau_\mu\tau^\nu\tau^\rho + 2V^{(\dot{\nu}}\tau^{\rho)}\tau_\mu + V^{\dot{\nu}\dot{\rho}}\tau_\mu + V_{\dot{\mu}}\tau^\nu\tau^\rho + 2V_{\dot{\mu}}{}^{(\dot{\nu}}\tau^{\rho)} + V_{\dot{\mu}}{}^{\dot{\nu}\dot{\rho}} \tag{4.10}$$

Only when there is no ambiguity in the notation, the dotted indices will be raised and lowered with h as in Equation (4.9). For example, note that in the split of $V_\mu{}^{\nu\rho}$ above we have $V^{\dot{\mu}\dot{\nu}} = \tau^\rho h^\mu_\sigma h^\nu_\lambda V_\rho{}^{\sigma\lambda}$ and $V_{\dot{\mu}}{}^{\dot{\nu}} = \tau_\rho h^\sigma_\mu h^\nu_\lambda V_\sigma{}^{\rho\lambda}$ and thus $V^{\dot{\mu}\dot{\nu}} \neq h^{\mu\rho} V_{\dot{\rho}}{}^{\dot{\nu}}$.

Additional note is that in Section 6.2.4 we will switch to a convention where the indices are raised and lowered with the metric k which is conformally related to h.

4.2. Milne Boosts

The additional fields τ^μ and $h_{\mu\nu}$ are not unique for a given Newton-Cartan structure. An equivalent set of solutions to Equation (4.2) is generated through the infinitesimal transformations

$$\delta_\chi \tau^\mu = -\chi^{\dot{\mu}} \qquad \delta_\chi h_{\mu\nu} = \chi_{\dot{\mu}}\tau_\nu + \chi_{\dot{\nu}}\tau_\mu \tag{4.11}$$

To keep these objects invariant under boost transformations, one can additionally introduce a 1-form field m_μ such that

$$\delta_\chi m_\mu = -\chi_{\dot{\mu}} \tag{4.12}$$

This 1-form defines a field strength 2-form as

$$m_{\mu\nu} = 2\partial_{[\mu} m_{\nu]} \tag{4.13}$$

Furthermore, m_μ allows us to define the boost invariant inverse fields through

$$\hat{\tau}^\mu = \tau^\mu - h^{\mu\nu} m_\mu \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} + \tau_\mu m_\nu + \tau_\nu m_\mu \tag{4.14}$$

and a boost invariant Newtonian potential:

$$\Phi = -\tau^\mu m_\mu + \frac{1}{2} h^{\mu\nu} m_\mu m_\nu \tag{4.15}$$

4.3. Newton-Cartan Connection

Defining the Newton-Cartan connection $(\dot{\nabla}_\mu, \dot{\Gamma}^\lambda_{\mu\nu})$ to be compatible with the Newton-Cartan structure $(\tau_\mu, h^{\mu\nu})$:

$$\dot{\nabla}_\mu \tau_\nu = 0 \qquad \dot{\nabla}_\mu h^{\nu\lambda} = 0 \tag{4.16}$$

It should be pointed out that for the class of Newton-Cartan connections, we reserve the notation with a slash inside the nabla. The simplest connection that solves these conditions is

$$\dot{\Gamma}^\lambda_{\mu\nu} = \tau^\lambda \partial_\mu \tau_\nu + \frac{1}{2} h^{\lambda\rho} \left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} \right) \tag{4.17}$$

for which the shifts

$$\dot{\Gamma}^\lambda_{\mu\nu} \to \dot{\Gamma}^\lambda_{\mu\nu} - h^{\lambda\rho} K_{\rho(\mu} \tau_{\nu)} \tag{4.18}$$

preserve the compatibility conditions in Equation (4.16) for an arbitrary 2-form $K_{\mu\nu}$. Note that $K_{\mu\nu}$ is independent of the Newton-Cartan structure as introduced in Equation (4.2). Its existence is mostly related to a higher order field m_μ that is needed to be introduced in addition to the Newton-Cartan structure to keep the connection invariant under the boost symmetry – see [14, 17] –. There are various such connections, see [15, 16] for a discussion of the possibilities. The simplest one is upon identifying $K_{\mu\nu} = m_{\mu\nu}$ in Equation (4.13)

$$\bar{\Gamma}^\lambda_{\mu\nu} = \dot{\Gamma}^\lambda_{\mu\nu} - h^{\lambda\rho} m_{\rho(\mu}\tau_{\nu)} \tag{4.19}$$

denoted by a bar on top. Combining Equations (4.14) and (4.19) it can be written as

$$\bar{\Gamma}^\lambda_{\mu\nu} = \hat{\tau}^\lambda \partial_\mu \tau_\nu + \frac{1}{2} h^{\lambda\rho} \left(\partial_\mu \bar{h}_{\nu\rho} + \partial_\nu \bar{h}_{\mu\rho} - \partial_\rho \bar{h}_{\mu\nu} \right) \tag{4.20}$$

For the rest of the thesis, we will work in the case where $K_{\mu\nu}$ is zero as a boost invariant connection is not necessary for the sake of our discussion. However, when presenting the leading order Einstein equations, it will be discussed that such changes to the connection will not affect the results.

4.4. Twistless Torsion

From this point and on we will assume that the 1-form τ satisfies $\partial_{[\dot{\mu}}\tau_{\dot{\nu}]} = 0$ rather than working with the more standard Newton-Cartan structure in which $\partial_{[\mu}\tau_{\nu]} = 0$. The reason for this choice follows from the fact that the existence of torsion allows to describe strong gravitational regime in a Newtonian manner. This condition can be expressed in three equivalent ways:

$$\tau_{[\mu}\partial_\nu \tau_{\rho]} = 0 \quad \Leftrightarrow \quad \partial_{[\dot{\mu}}\tau_{\dot{\nu}]} = 0 \quad \Leftrightarrow \quad \partial_{[\mu}\tau_{\nu]} = \tau_{[\mu} a_{\nu]}\,, \quad a_\mu = a_{\dot{\mu}} \tag{4.21}$$

Geometrically these conditions guarantee the existence of a foliation by spatial hypersurfaces. Another interpretation is that this condition restricts the torsion of any connection compatible with the Newton-Cartan structure and for this reason Equation (4.21) also goes under the name of twistless torsion. We make this assumption on the Newton-Cartan structure from the beginning, anticipating compatibility with the expanded Einstein equations [5].

As a consequence of Equation (4.21) it follows that

$$a_{\dot{\mu}} = 2\tau^{\rho}\partial_{[\rho}\tau_{\mu]} = L_{\tau}\tau_{\mu} \tag{4.22}$$

where furthermore

$$\partial_{[\dot{\mu}}a_{\dot{\nu}]} = 0 \tag{4.23}$$

so that locally

$$a_{\dot{\mu}} = \partial_{\dot{\mu}}\psi \tag{4.24}$$

The derivation of Equation (4.21) and the other formulae above is shortly reviewed in Appendix D. Note that this implies via Equations (4.22) and (4.24) that

$$\delta_{\chi}a_{\dot{\mu}} = \tau_{\mu}\chi^{\dot{\rho}}a_{\dot{\rho}} \qquad \delta_{\chi}\psi = 0 \tag{4.25}$$

where $\delta_{\chi}(\partial_{\dot{\mu}}\psi) = \tau_{\mu}\chi^{\dot{\rho}}\partial_{\dot{\rho}}\psi + \partial_{\dot{\mu}}\delta_{\chi}\psi$.

We finally would like to point out that this constraint follows from the Lagrangian:

$$L = eh^{\mu\nu}h^{\rho\sigma}(\partial_{\mu}\tau_{\rho} - \partial_{\rho}\tau_{\mu})(\partial_{\nu}\tau_{\sigma} - \partial_{\sigma}\tau_{\nu}) \tag{4.26}$$

where $e = \det e_{\mu}^{A} = \det(\tau_{\mu}, e_{\mu}^{a})$ with $e_{\mu}^{0} = \tau_{\mu}$. Together with the identifications $h^{\mu\nu} = \delta^{ab}e_{a}^{\mu}e_{b}^{\nu}$ and $h_{\mu\nu} = \delta_{ab}e_{\mu}^{a}e_{\nu}^{b}$, the variation on the determinant can be computed as

$$\delta e = -e(\tau_{\mu}\delta\tau^{\mu} + \frac{1}{2}h_{\mu\nu}\delta h^{\mu\nu}) \tag{4.27}$$

via the famous formula: $\ln(\det M) = \mathrm{Tr}(\ln M)$. The variation on the Lagrangian can then collectively be written as

$$\delta L = E^{\tau}_{\mu}\delta\tau^{\mu} + E^{h}_{(\mu\nu)}\delta h^{\mu\nu} \tag{4.28}$$

Note that the spatial projection on the first equation and the temporal projection on the second equation vanish by themselves. As a result they do not contribute to the final equations of motion. Introducing $Y_{\mu\nu} = \partial_\mu\tau_\nu - \partial_\nu\tau_\mu$ for now, one finds the equations to be

$$\begin{aligned} \tau^{\alpha}E^{\tau}_{\alpha} &= \frac{1}{e}\partial_{\rho}\left(eY^{\dot{\rho}\dot{\mu}}\right)\tau_{\mu} - \frac{1}{4}Y^{\dot{\nu}\dot{\sigma}}Y_{\dot{\nu}\dot{\sigma}} = 0 && (4.29) \\ h^{\alpha\lambda}h^{\beta\gamma}E^{h}_{(\alpha\beta)} &= Y^{\dot{\lambda}\dot{\sigma}}Y^{\dot{\gamma}}{}_{\dot{\sigma}} - \frac{1}{4}h^{\lambda\gamma}Y^{\dot{\nu}\dot{\sigma}}Y_{\dot{\nu}\dot{\sigma}} = 0 && (4.30) \end{aligned}$$

Taking the spatial trace of Equation (4.30) equals to $Y^{\dot{\nu}\dot{\sigma}}Y_{\dot{\nu}\dot{\sigma}} = 0$. Solution to this is provided using the linear algebra result for an antisymmetric matrix M:

$$\mathrm{Tr}(M^T M) = 0 \implies M = 0 \tag{4.31}$$

It then follows that this amounts to the twistless torsion condition

$$Y_{\dot{\mu}\dot{\nu}} = 2\partial_{[\dot{\mu}}\tau_{\dot{\nu}]} = 0 \tag{4.32}$$

4.5. Covariant Newton's Equations

The connection introduced in Equation (4.20) admits a Ricci tensor

$$\bar{R}_{\mu\nu} = \partial_\lambda \bar{\Gamma}^\lambda_{\nu\mu} - \partial_\nu \bar{\Gamma}^\lambda_{\lambda\mu} + \bar{\Gamma}^\lambda_{\lambda\sigma}\bar{\Gamma}^\sigma_{\nu\mu} - \bar{\Gamma}^\lambda_{\nu\sigma}\bar{\Gamma}^\sigma_{\lambda\mu} \tag{4.33}$$

The covariant equation Newton-Cartan gravity then follows from the ansatz:

$$\bar{R}_{\mu\nu} = 4\pi G_{\mathrm{N}} \rho \tau_\mu \tau_\nu \tag{4.34}$$

where ρ is the mass density. To show that this actually reproduces the Newton's equation, one can pick the adapted Newtonian coordinates:

$$\tau_\mu = \delta^0_\mu \qquad h^{ij} = \delta^{ij} \qquad \hat{\tau} = \delta^\mu_0 \tag{4.35}$$

Which will result in the Poisson equation

$$\nabla^2 \Phi = 4\pi G_{\mathrm{N}} \rho \tag{4.36}$$

5. NEWTON-CARTAN SPLIT OF GENERAL RELATIVITY

This chapter serves as the intermediate point where we combine general relativity and Newton-Cartan theory where we establish the first contact between the two. While doing so, we will reformulate Einstein equations by splitting it in the temporal and spatial directions using the Newton-Cartan structure and decomposing curvatures with the Newton-Cartan connection introduced previously. An advantage of such decomposition and split is that it eases up the calculations of the leading order large c expansion by explicitly separating the dynamical variables.

5.1. Reformulating General Relativity

We start with the regular old Levi-Civita connection $(\nabla_\mu, \Gamma^\lambda_{\mu\nu})$ and the all-compatible Newton-Cartan connection $(\hat{\boldsymbol{\nabla}}_\mu, \hat{\mathbb{\Gamma}}^\lambda_{\mu\nu})$ derived in Appendix A. The two can be connected via

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu} &= \hat{\mathbb{\Gamma}}^\lambda_{\mu\nu} + S^\lambda_{\mu\nu} - \frac{1}{2}\hat{T}^\lambda_{\mu\nu} && (5.1)\\
\hat{T}^\lambda_{\mu\nu} &= \hat{\mathbb{\Gamma}}^\lambda_{\mu\nu} - \hat{\mathbb{\Gamma}}^\lambda_{\nu\mu} = 2\tau^\lambda \tau_{[\mu} a_{\nu]} - 2\tau_{[\nu} \dot{\hat{\boldsymbol{\nabla}}}_{\mu]} \tau^\lambda && (5.2)\\
S^\lambda_{\mu\nu} &= g^{\lambda\rho}\left(\hat{\boldsymbol{\nabla}}_{(\mu} g_{\nu)\rho} - \frac{1}{2}\hat{\boldsymbol{\nabla}}_\rho g_{\mu\nu} + g_{\sigma(\mu}\hat{T}^\sigma_{\nu)\rho}\right) && (5.3)
\end{aligned}$$

In fact any arbitrary connection of choice can be related to the Levi-Civita connection through these relations. Plugging in relates the Ricci tensors of Levi-Civita connection to the all-compatible Newton-Cartan connection as follows:

$$\begin{aligned}
R_{\mu\nu} &= \hat{\mathbb{R}}_{(\mu\nu)} - \frac{1}{2}\hat{\boldsymbol{\nabla}}_{(\mu}\hat{T}^\rho_{\nu)\rho} + \hat{\boldsymbol{\nabla}}_\rho S^\rho_{\mu\nu} - \hat{\boldsymbol{\nabla}}_{(\mu} S^\rho_{\nu)\rho} && (5.4)\\
&\quad + \frac{1}{4}\hat{T}^\lambda_{\rho(\mu}\hat{T}^\rho_{\nu)\lambda} - \frac{1}{2}S^\lambda_{\mu\nu}\hat{T}^\rho_{\rho\lambda} + \hat{T}^\lambda_{\rho(\mu}S^\rho_{\nu)\lambda} + S^\lambda_{\mu\nu}S^\rho_{\rho\lambda} - S^\rho_{\lambda(\mu}S^\lambda_{\nu)\rho}
\end{aligned}$$

First decomposing all apearing objects in Equation (5.4):

$$R_{\mu\nu} = R\,\tau_\mu\tau_\nu + 2\tau_{(\mu}R_{\dot{\nu})} + R_{\dot{\mu}\dot{\nu}} \tag{5.5}$$

$$\hat{\mathbb{R}}_{\mu\nu} = \hat{\mathbb{R}}\,\tau_\mu\tau_\nu + 2\tau_{(\mu}\hat{\mathbb{R}}_{\dot{\nu})} + \hat{\mathbb{R}}_{\dot{\mu}\dot{\nu}} \tag{5.6}$$

$$S^\lambda_{\mu\nu} = S\,\tau^\lambda\tau_\mu\tau_\nu + 2\tau^\lambda\tau_{(\mu}S_{\dot{\nu})} + \hat{\tau}^\lambda S_{\dot{\mu}\dot{\nu}} + S^{\dot{\lambda}}\tau_\mu\tau_\nu + 2\tau_{(\mu}S^{\dot{\lambda}}_{\dot{\nu})} + S^{\dot{\lambda}}_{\dot{\mu}\dot{\nu}} \tag{5.7}$$

$$\hat{T}^\lambda_{\mu\nu} = 2\tau^\lambda\tau_{[\mu}\hat{T}_{\dot{\nu}]} + 2\tau_{[\mu}\hat{T}^{\dot{\lambda}}_{\dot{\nu}]} \qquad \hat{T}_{\dot{\mu}} = a_{\dot{\mu}} \qquad \hat{T}^{\dot{\nu}}_{\dot{\mu}} = \dot{\hat{\nabla}}_{\dot{\mu}}\tau^\nu \tag{5.8}$$

$$\hat{\nabla}_\mu = \tau_\mu\hat{\nabla} + \hat{\nabla}_{\dot{\mu}} \tag{5.9}$$

Note that identification of $\hat{T}_{\dot{\mu}} = a_{\dot{\mu}}$ and the choice of split in Equation (5.8) amounts to working in a twistless torsional geometry as discussed in Section 4.4. One can also check that $\hat{T}^{\dot{\nu}}_{\dot{\mu}} = \dot{\hat{\nabla}}_{\dot{\mu}}\tau^\nu$ is in fact spatial on the upper index, a result that follows from $\tau_\nu\dot{\hat{\nabla}}_{\dot{\mu}}\tau^\nu = 0$. We further apply a Newton-Cartan split to the rest of the relativistic ingredients introduced in Section 3:

$$\mathcal{T}_{\mu\nu} = \mathcal{T}\,\tau_\mu\tau_\nu + 2\tau_{(\mu}\mathcal{T}_{\dot{\nu})} + \mathcal{T}_{\dot{\mu}\dot{\nu}} \tag{5.10}$$

$$\mathcal{C}_\mu = \mathcal{C}\,\tau_\mu + \mathcal{C}_{\dot{\mu}} \tag{5.11}$$

To proceed with the split of the metric and its inverse, we reserve a different notation:

$$g_{\mu\nu} = A\tau_\mu\tau_\nu + A_{\dot{\mu}}\tau_\nu + A_{\dot{\nu}}\tau_\mu + B_{\dot{\mu}\dot{\nu}} \tag{5.12}$$

$$g^{\mu\nu} = B\tau^\mu\tau^\nu + B^{\dot{\mu}}\tau^\nu + B^{\dot{\nu}}\tau^\mu + A^{\dot{\mu}\dot{\nu}}$$

This A vs B notation is introduced to indicate that we will treat the A variables as independent fields, while the B fields are interpreted as fully determined in terms of the A's through the condition that $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu}$:

$$\begin{aligned} B &= A^{-1}(1 + A^{-1}A_{\dot{\mu}}A_{\dot{\nu}}A^{\dot{\mu}\dot{\nu}}) \\ B^{\dot{\mu}} &= -A^{-1}A_{\dot{\nu}}A^{\dot{\nu}\dot{\mu}} \\ B_{\dot{\mu}\dot{\nu}} &= A_{\dot{\mu}\dot{\nu}} + A^{-1}A_{\dot{\mu}}A_{\dot{\nu}} \end{aligned} \tag{5.13}$$

with $A_{\dot{\mu}\dot{\nu}}$ is the unique. Contrary to Equation (4.2), which has no unique solution for τ^{μ} and $h_{\mu\nu}$ given τ_{μ} and $h^{\mu\nu}$, Equation (5.14) has a unique solution for $A_{\dot{\mu}\dot{\nu}}$ given τ_{μ} , τ^{μ} and $A^{\dot{\mu}\dot{\nu}}$. solution to

$$\tau_{\mu}\tau^{\nu} + A_{\dot{\mu}\dot{\rho}}A^{\dot{\rho}\dot{\nu}} = \delta^{\nu}_{\mu} \tag{5.14}$$

This relation clarifies our initial choice of metric split by relating the A coefficients to the Newton-Cartan structure.

The upshot of this Newton-Cartan split of general relativity is that, the dynamical variables are:

$$A\,, \quad A_{\dot{\mu}}\,, \quad A^{\dot{\mu}\dot{\nu}} \qquad \text{and} \qquad \mathcal{T}\,, \quad \mathcal{T}_{\dot{\mu}}\,, \quad \mathcal{T}_{\dot{\mu}\dot{\nu}}\,.$$

The split Einstein equations are:

$$R = 8\pi G_{\mathrm{N}}\mathcal{T} \qquad R_{\dot{\mu}} = 8\pi G_{\mathrm{N}}\mathcal{T}_{\dot{\mu}} \qquad R_{\dot{\mu}\dot{\nu}} = 8\pi G_{\mathrm{N}}\mathcal{T}_{\dot{\mu}\dot{\nu}} \tag{5.15}$$

$$\mathcal{C} = 0 \qquad \mathcal{C}_{\dot{\mu}} = 0 \tag{5.16}$$

The curvatures and the conservation equations are functions of variables introduced by switching the conneciton. They are explicitly provided in Equations (B.7)-(B.11). One can see the benefit of such a reformulation of Einstein equations is that when expanding both sides of Equations (B.7)-(B.11), the only c dependent coefficients come from the hierarchically constructed split metric and energy-momentum tensor coefficients. This simplifies the computations by making everything more explicit.

6. LEADING ORDER LARGE c EXPANSION OF EINSTEIN EQUATIONS

A manifestly diffeomorphism invariant approximation to GR can be constructed by expanding the relativistic metric and its inverse in inverse powers of the speed of light [3–6]:

$$g_{\mu\nu}(c) = \sum_{k=-2}^{\infty} \overset{(k)}{g}_{\mu\nu} c^{-k} \qquad g^{\mu\nu}(c) = \sum_{k=0}^{\infty} \overset{(k)}{g}{}^{\mu\nu} c^{-k} \tag{6.1}$$

Although it is consistent to assume all coefficients of odd powers to vanish – as was done in [3–6] – we will explore in this work the consequences of relaxing this assumption. The presence of non-vanishing coefficients for odd powers of c is motivated by the fact that these coefficients can be sourced in the equations of motion by certain types of energy-momentum [6].

We will first discuss some generalities and then focus on the leading order, working out explicitly the dynamical equations to this order. Symmetries will play an important role in an appropriate organization of the result.

6.1. Setup and General Observations

We will take a slightly different – but equivalent – approach than the one that was previously taken in [3–6]. We will perform the expansion not directly in terms of the relativistic metric as in Equation (6.3), but rather expand the components as obtained after a Newton-Cartan split as in Equation (5.12). It should be stressed that the Newton-Cartan structure $\tau_\mu\,, h^{\mu\nu}$ is taken to be independent of c.

Our expansion ansatz for the dynamical fields is then

$$A(c) = \sum_{k=-2}^{\infty} \overset{(k)}{A} c^{-k} \qquad A_{\dot\mu}(c) = \sum_{k=-1}^{\infty} \overset{(k)}{A}_{\dot\mu} c^{-k} \qquad A^{\dot\mu\dot\nu}(c) = \sum_{k=0}^{\infty} \overset{(k)}{A}{}^{\dot\mu\dot\nu} c^{-k} \tag{6.2}$$

$$\mathcal{T}(c) = \sum_{k=-2}^{\infty} \overset{(k)}{\mathcal{T}} c^{-k} \qquad \mathcal{T}_{\dot\mu}(c) = \sum_{k=-1}^{\infty} \overset{(k)}{\mathcal{T}}_{\dot\mu} c^{-k} \qquad \mathcal{T}_{\dot\mu\dot\nu}(c) = \sum_{k=0}^{\infty} \overset{(k)}{\mathcal{T}}_{\dot\mu\dot\nu} c^{-k} \tag{6.3}$$

This ansatz is based on a number of starting assumptions, equivalent to those of [3–6] with the exception that odd powers are allowed to be non-vanishing. In those previous works [3–6] the choice $\overset{(-2)}{A} = -1$ was made, but as we will explain below that is simply a choice of gauge for a local scaling symmetry in our formulation. Leaving $\overset{(-2)}{A}$ free has some advantages. In particular it makes manifest the fact that at each order in the expansion there appear two new triplets of fields, $(\overset{(k)}{A}, \overset{(k+1)}{A}_{\dot\mu}, \overset{(k+2)}{A}{}^{\dot\mu\dot\nu})$ and $(\overset{(k)}{\mathcal{T}}, \overset{(k+1)}{\mathcal{T}}_{\dot\mu}, \overset{(k+2)}{\mathcal{T}}_{\dot\mu\dot\nu})$, all of which are a priori free to vary. In the formulation using the Newton-Cartan split it also becomes clear that the proper organization of the orders is not just by counting inverse powers of c, but rather that at a given order the powers of c depend on the number of spatial indices the field carries. The triplet $(\overset{(k)}{A}, \overset{(k+1)}{A}_{\dot\mu}, \overset{(k+2)}{A}{}^{\dot\mu\dot\nu})$ for example consists of coefficients of the powers $(c^{-k}, c^{-k-1}, c^{-k-2})$. This is something which was not properly appreciated in [3–5], but suggested by the results of [6, 7]. That this is indeed the more appropriate point of view is supported by the fact that the equations of motion – which we will further work out below – organize themselves in the following form:

$$\begin{aligned} \overset{(k)}{R}[\overset{(\le k)}{A}, \overset{(\le k+1)}{A}_{\dot\mu}, \overset{(\le k+2)}{A}{}^{\dot\mu\dot\nu}] &= 8\pi G_{\mathrm{N}} \overset{(k)}{\mathcal{T}} \\ \overset{(k+1)}{R}_{\dot\mu}[\overset{(\le k)}{A}, \overset{(\le k+1)}{A}_{\dot\mu}, \overset{(\le k+2)}{A}{}^{\dot\mu\dot\nu}] &= 8\pi G_{\mathrm{N}} \overset{(k+1)}{\mathcal{T}}_{\dot\mu} \\ \overset{(k+2)}{R}_{\dot\mu\dot\nu}[\overset{(\le k)}{A}, \overset{(\le k+1)}{A}_{\dot\mu}, \overset{(\le k+2)}{A}{}^{\dot\mu\dot\nu}] &= 8\pi G_{\mathrm{N}} \overset{(k+2)}{\mathcal{T}}_{\dot\mu\dot\nu} \end{aligned} \tag{6.4}$$

One sees that if the expansion is truncated at order n, i.e. keeping only triplets of the form $(\overset{(k)}{A}, \overset{(k+1)}{A}_{\dot\mu}, \overset{(k+2)}{A}{}^{\dot\mu\dot\nu})$ and $(\overset{(k)}{\mathcal{T}}, \overset{(k+1)}{\mathcal{T}}_{\dot\mu}, \overset{(k+2)}{\mathcal{T}}_{\dot\mu\dot\nu})$ with $k \le n$, then the above set of equations for $k \le n$ is consistent. Furthermore the set of equations has a hierarchic structure, in that one can solve them recursively in the order. The leading order corresponds to $n = -2$ and will be worked out fully below. Note that the structure of

Equation (6.4) follows upon assuming that $\overset{(-4)}{R}_{\mu\nu} = 0$ which amounts to the twistless torsion condition in Equation (4.21), see [5]. But since this is equivalent to the assumption that $\overset{(-4)}{\mathcal{T}}_{\mu\nu} = \overset{(-3)}{\mathcal{T}}_{\mu\nu} = 0$, one sees that this condition is already encoded in Equation (6.3).

Finally we should point out that since $\tau_\mu \overset{(0)}{A}{}^{\dot\mu\dot\nu} = 0$ we can choose to identify the spatial part of the Newton-Cartan structure introduced in Section 4.1 with it:

$$\overset{(0)}{A}{}^{\dot\mu\dot\nu} = h^{\mu\nu} \tag{6.5}$$

Translated back to the more familiar relativistic metric variables this amounts to

$$g_{\mu\nu} = c^2 \overset{(-2)}{A} \tau_\mu \tau_\nu + \mathcal{O}(c) \qquad g^{\mu\nu} = h^{\mu\nu} + \mathcal{O}(c^{-1}) \tag{6.6}$$

So once we start considering the large c expansion, we identify the Newton-Cartan structure introduced in Section 4.1 with the leading coefficients of the relativistic metric and its inverse.

6.1.1. Metric Expansion

The relativistic metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are determined in terms of A and B triplet fields as in Equations (5.12). If one wants to carry out the large c expansion in practice then one will need the expression of the coefficients of the B's in terms of the coefficients of the A's. Expanding Equations (5.13) results in the following recurrence relations:

$$\begin{aligned}
\overset{(k+2)}{B} &= \delta^k_0 \overset{(-2)}{A}{}^{-1} - \overset{(-2)}{A}{}^{-1} \sum_{i=-1}^{k} \left(\overset{(i)}{A} \overset{(k-i)}{B} + \overset{(i)}{A}_{\dot\mu} \overset{(k-i)}{B}{}^{\dot\mu} \right) \\
\overset{(k+2)}{B}{}^{\dot\mu} &= -\overset{(-2)}{A}{}^{-1} \sum_{i=-1}^{k} \left(\overset{(i)}{A} \overset{(k-i)}{B}{}^{\dot\mu} + \overset{(i)}{A}_{\dot\nu} \overset{(k-i)}{A}{}^{\dot\nu\dot\mu} \right) \\
\overset{(k)}{B}_{\dot\mu\dot\nu} &= \delta^k_0 h_{\mu\nu} - \sum_{i=1}^{k+1} \left(\overset{(k-i)}{A}_{\dot\mu} \overset{(i)}{B}{}^{\dot\rho} + \overset{(k-i)}{B}_{\dot\mu\dot\sigma} \overset{(i)}{A}{}^{\dot\sigma\dot\rho} \right) h_{\rho\nu}
\end{aligned} \tag{6.7}$$

To illustrate the use of these recursion relations we work out the first orders of the B fields:

- LO ($k=-2$):

$$\overset{(0)}{B}=0 \qquad \overset{(0)}{B}{}^{\mu}=0 \qquad \overset{(-2)}{B}{}_{\dot{\mu}\dot{\nu}}=0 \tag{6.8}$$

Which is equivalent to

$$\overset{(-2)}{g}{}_{\mu\nu}=\overset{(-2)}{A}\tau_\mu\tau_\nu \qquad \overset{(0)}{g}{}^{\mu\nu}=h^{\mu\nu} \tag{6.9}$$

- $\mathrm{N}^{1/2}$LO ($k=-1$):

$$\overset{(1)}{B}=0 \qquad \overset{(1)}{B}{}^{\dot{\mu}}=\overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}{}^{\dot{\mu}} \qquad \overset{(-1)}{B}{}_{\dot{\mu}\dot{\nu}}=0 \tag{6.10}$$

Which is equivalent to

$$\overset{(-1)}{g}{}_{\mu\nu} = \overset{(-1)}{A}\tau_\mu\tau_\nu+\overset{(-1)}{A}{}_{\dot{\mu}}\tau_\nu+\overset{(-1)}{A}{}_{\dot{\nu}}\tau_\mu \tag{6.11}$$

$$\overset{(1)}{g}{}^{\mu\nu} = -\overset{(-2)}{A}{}^{-1}(\overset{(-1)}{A}{}^{\dot{\mu}}\tau^\nu+\overset{(-1)}{A}{}^{\dot{\nu}}\tau^\mu)+\overset{(1)}{A}{}^{\dot{\mu}\dot{\nu}} \tag{6.12}$$

- NLO ($k=0$):

$$\overset{(2)}{B} = \overset{(-2)}{A}{}^{-2}\overset{(-1)}{A}{}_{\dot{\rho}}\overset{(-1)}{A}{}^{\dot{\rho}}+\overset{(-2)}{A}{}^{-1} \tag{6.13}$$

$$\overset{(2)}{B}{}^{\dot{\mu}} = \overset{(-2)}{A}{}^{-2}\overset{(-1)}{A}\,\overset{(-1)}{A}{}^{\dot{\mu}}-\overset{(-2)}{A}{}^{-1}(\overset{(1)}{A}{}^{\dot{\mu}\dot{\nu}}\overset{(-1)}{A}{}_{\dot{\nu}}+h^{\mu\nu}\overset{(0)}{A}{}_{\dot{\nu}}) \tag{6.14}$$

$$\overset{(0)}{B}{}_{\dot{\mu}\dot{\nu}} = h_{\mu\nu}+\overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}{}_{\dot{\mu}}\overset{(-1)}{A}{}_{\dot{\nu}} \tag{6.15}$$

Which is equivalent to

$$\overset{(0)}{g}{}_{\mu\nu} = \overset{(0)}{A}\tau_\mu\tau_\nu+\overset{(0)}{A}{}_\mu\tau_\nu+\overset{(0)}{A}{}_\nu\tau_\mu+h_{\mu\nu}+\overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}{}_{\dot{\mu}}\overset{(-1)}{A}{}_{\dot{\nu}} \tag{6.16}$$

$$\begin{aligned}\overset{(2)}{g}{}^{\mu\nu} &= (\overset{(-2)}{A}{}^{-2}\overset{(-1)}{A}{}_{\dot{\rho}}\overset{(-1)}{A}{}^{\dot{\rho}}+\overset{(-2)}{A}{}^{-1})\tau^\mu\tau^\nu+\overset{(-2)}{A}{}^{-2}\tau^\nu\overset{(-1)}{A}{}_{\dot{\rho}}(\overset{(-1)}{A}h^{\rho\mu}-\overset{(-2)}{A}\overset{(1)}{A}{}^{\dot{\rho}\dot{\mu}})\\ &\quad+\overset{(-2)}{A}{}^{-2}\tau^\mu\overset{(-1)}{A}{}_{\dot{\rho}}(\overset{(-1)}{A}h^{\rho\nu}-\overset{(-2)}{A}\overset{(1)}{A}{}^{\dot{\rho}\dot{\nu}})-\overset{(-2)}{A}{}^{-1}(h^{\mu\nu}+2\overset{(0)}{A}{}^{(\dot{\mu}}\tau^{\nu)})\end{aligned}$$

The relation of the triplet $(\overset{(k)}{A}, \overset{(k+1)}{A}{}_{\dot{\mu}}, \overset{(k+2)}{A}{}^{\dot{\mu}\dot{\nu}})$ appearing here to the variables introduced in the previous works in the field are listed below:

$$
\begin{array}{llll}
\text{LO} & \overset{(-2)}{A} = -1 & \overset{(-1)}{A}{}_{\dot{\mu}} = 0 & \overset{(0)}{A}{}^{\dot{\mu}\dot{\nu}} = h^{\mu\nu} \\
\text{N}^{1/2}\text{LO} & \overset{(-1)}{A} = 0 & \overset{(0)}{A}{}_{\dot{\mu}} = m_{\dot{\mu}} & \overset{(1)}{A}{}^{\dot{\mu}\dot{\nu}} = 0 \\
\text{NLO (Newtonian)} & \overset{(0)}{A} = -2\Phi & \overset{(1)}{A}{}_{\dot{\mu}} = 0 & \overset{(2)}{A}{}^{\dot{\mu}\dot{\nu}} = \Phi^{\dot{\mu}\dot{\nu}} \\
\text{N}^{3/2}\text{LO} & \overset{(1)}{A} = 0 & \overset{(2)}{A}{}_{\dot{\mu}} = \gamma_{\dot{\mu}} & \overset{(3)}{A}{}^{\dot{\mu}\dot{\nu}} = 0 \\
\text{N}^{2}\text{LO} & \overset{(2)}{A} = -\gamma & \overset{(3)}{A}{}_{\dot{\mu}} = 0 & \overset{(4)}{A}{}^{\dot{\mu}\dot{\nu}} = 0
\end{array}
\tag{6.17}
$$

In these equations all gravitational potentials appearing up to next to next to leading order (N^2LO) in the large c expansion are listed. In the previous literature on this expansion [3–6] most of these potentials have been assumed to vanish, as indicated. The NLO level can be identified as the Newtonian level, since Φ is Newton's potential. Up to this order we have used the nomenclature of [6] for the fields. For the two highest orders we used the nomenclature of [4]. Non-trivial values for the fields of order two have only been considered more recently [5,6]. If the potentials of order zero are assumed to vanish then the potentials in red can be identified with those appearing at first Post-Newtonian order in the standard PN expansion [4]. In this thesis we will focus on the LO only but will allow non-trivial values for the full triplet, see Equation (7.3).

6.1.2. Leading Order Fields

Starting with the expansion ansatz in Equation (6.3), the leading order metric variables can be listed as:

$$
\overset{(-2)}{A} \qquad \overset{(-1)}{A}{}_{\dot{\mu}} \qquad \overset{(0)}{A}{}^{\dot{\mu}\dot{\nu}} \qquad \overset{(0)}{B}{}_{\dot{\mu}\dot{\nu}} \qquad \overset{(1)}{B}{}_{\dot{\mu}} \qquad \overset{(2)}{B}
\tag{6.18}
$$

These fields will lead to the leading order S coefficients appearing in the connection decomposition:

$$\overset{(-2)}{S}{}^{\dot{\mu}} \qquad \overset{(-1)}{S}{}^{\dot{\mu}}_{\ \dot{\nu}} \qquad \overset{(-1)}{S} \qquad \overset{(0)}{S}_{\dot{\mu}} \qquad \overset{(1)}{S}_{\dot{\mu}\dot{\nu}} \qquad \overset{(0)}{S}{}^{\dot{\lambda}}_{\dot{\mu}\dot{\nu}} \tag{6.19}$$

The torsion fields and the curvature $\hat{\mathbb{R}}$ do not carry order of c. The LO expansion enjoy the introduction of the following fields:

$$\begin{aligned}
H_{\dot{\mu}\dot{\nu}} &= h^{\rho}_{\mu}h^{\sigma}_{\nu}(\partial_{\rho}\overset{(-1)}{A}_{\dot{\sigma}} - \partial_{\sigma}\overset{(-1)}{A}_{\dot{\rho}} + \omega_{\dot{\rho}}\overset{(-1)}{A}_{\dot{\sigma}} - \omega_{\dot{\sigma}}\overset{(-1)}{A}_{\dot{\rho}} + \Omega_{\dot{\rho}}\overset{(-1)}{A}_{\dot{\sigma}} - \Omega_{\dot{\sigma}}\overset{(-1)}{A}_{\dot{\rho}}) \\
\omega_{\dot{\mu}} &= -\frac{1}{2}\partial_{\dot{\mu}}\log(-\overset{(-2)}{A}) \\
\Omega_{\dot{\mu}} &= a_{\dot{\mu}} + \omega_{\dot{\mu}}
\end{aligned}$$

Due to Equation (4.24), we can also write

$$\Omega_{\dot{\mu}} = \frac{1}{2}\partial_{\dot{\mu}}\Psi \qquad \Psi = 2\psi - \log(-\overset{(-2)}{A}) \tag{6.20}$$

Together with this, the raw form of the leading equations are:

$$\overset{(-2)}{R} = 8\pi G_{\mathrm{N}}\overset{(-2)}{\mathcal{T}} \qquad \overset{(-1)}{R}_{\dot{\mu}} = 8\pi G_{\mathrm{N}}\overset{(-1)}{\mathcal{T}}_{\dot{\mu}} \qquad \overset{(0)}{R}_{\dot{\mu}\dot{\nu}} = 8\pi G_{\mathrm{N}}\overset{(0)}{\mathcal{T}}_{\dot{\mu}\dot{\nu}} \tag{6.21}$$

$$\overset{(-1)}{\mathcal{C}} = 0 \qquad \overset{(0)}{\mathcal{C}} = 0 \tag{6.22}$$

This result is in accord with our expansion ansatz in Equation (6.3) and the analysis in Equation (6.4). The fields appearing in Equations (6.21) and (6.22) are provided explicitly as

$$\begin{aligned}
\overset{(-2)}{R} &= -\overset{(-2)}{A}(-\dot{\nabla}_{\dot{\lambda}}\Omega^{\dot{\lambda}} + \Omega_{\dot{\lambda}}\Omega^{\dot{\lambda}}) - \frac{1}{4}H^{\dot{\lambda}\dot{\sigma}}H_{\dot{\sigma}\dot{\lambda}} && (6.23)\\
\overset{(-1)}{R}_{\dot{\mu}} &= -\frac{1}{2}\dot{\nabla}_{\dot{\lambda}}H^{\dot{\lambda}}{}_{\dot{\mu}} + \frac{1}{2}\omega_{\dot{\lambda}}H^{\dot{\lambda}}{}_{\dot{\mu}} + \Omega_{\dot{\lambda}}H^{\dot{\lambda}}{}_{\dot{\mu}} + \overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}_{\dot{\mu}}\overset{(-2)}{R} && (6.24)\\
\overset{(0)}{R}_{\dot{\mu}\dot{\nu}} &= \dot{\mathbb{R}}_{(\dot{\mu}\dot{\nu})} + \dot{\nabla}_{(\dot{\mu}}\Omega_{\dot{\nu})} - \Omega_{\dot{\mu}}\Omega_{\dot{\nu}} - \frac{1}{2}\overset{(-2)}{A}{}^{-1}H^{\dot{\lambda}}{}_{\dot{\mu}}H_{\dot{\lambda}\dot{\nu}} + 2\overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}_{(\dot{\mu}}\overset{(-1)}{R}_{\dot{\nu})} && (6.25)\\
&\quad -\overset{(-2)}{A}{}^{-2}\overset{(-1)}{A}_{\mu}\overset{(-1)}{A}_{\nu}\overset{(-2)}{R}
\end{aligned}$$

$$\begin{aligned}
\overset{(-1)}{\mathcal{C}} &= \dot{\nabla}_{\dot{\mu}}(-\overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}{}^{\dot{\mu}} + \overset{(-1)}{\mathcal{T}}{}^{\dot{\mu}}) + \omega_{\dot{\mu}}(-\overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}{}^{\dot{\mu}} + \overset{(-1)}{\mathcal{T}}{}^{\dot{\mu}}) && (6.26)\\
&\quad -2\Omega_{\dot{\mu}}(-\overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}{}^{\dot{\mu}} + \overset{(-1)}{\mathcal{T}}{}^{\dot{\mu}})\\
\overset{(0)}{\mathcal{C}}_{\dot{\mu}} &= -\frac{1}{2}\overset{(-2)}{A}{}^{-1}(1 + \overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}_{\dot{\rho}}\overset{(-1)}{A}{}^{\dot{\rho}})(\dot{\nabla}_{\dot{\mu}}\overset{(-2)}{\mathcal{T}} + 2\omega_{\dot{\mu}}\overset{(-2)}{\mathcal{T}} - 2\Omega_{\dot{\mu}}\overset{(-2)}{\mathcal{T}}) && (6.27)\\
&\quad -2\overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}{}^{\dot{\rho}}(\dot{\nabla}_{[\dot{\rho}}\overset{(-1)}{\mathcal{T}}_{\dot{\mu}]} + \omega_{[\dot{\rho}}\overset{(-1)}{\mathcal{T}}_{\dot{\mu}]} - \Omega_{[\dot{\rho}}\overset{(-1)}{\mathcal{T}}_{\dot{\mu}]})\\
&\quad -\overset{(-2)}{A}{}^{-1}\overset{(-1)}{\mathcal{T}}_{\dot{\mu}}(\dot{\nabla}^{\dot{\rho}}\overset{(-1)}{A}_{\dot{\rho}} + \omega^{\dot{\rho}}\overset{(-1)}{A}_{\dot{\rho}}) + \dot{\nabla}^{\dot{\rho}}\overset{(0)}{\mathcal{T}}_{\dot{\rho}\dot{\mu}} - \frac{1}{2}\dot{\nabla}_{\dot{\mu}}(h^{\rho\sigma}\overset{(0)}{\mathcal{T}}_{\dot{\rho}\dot{\sigma}}) - \Omega^{\dot{\rho}}\overset{(0)}{\mathcal{T}}_{\dot{\rho}\dot{\mu}}
\end{aligned}$$

In these fields all upper indices are raised with $h^{\mu\nu}$ and the Newton-Cartan connection $(\dot{\nabla}_{\mu}, \dot{\Gamma}^{\rho}_{\mu\nu})$ that preserves the Newton-Cartan structure is used. We stress that, as a result of Equations (A.10) and (A.12) for the rest of the thesis we switched back to the Newton-Cartan connection introduced in Equation (4.17) as it is the more natural choice. Note that, Equations (6.23)-(6.27) are not in their final form as they are not in a manifestly symmetry invariant. We will discuss and bring to such form in the next section.

6.2. Symmetries

One can put Equations (6.23)-(6.27) in a manifestly symmetry invariant form by first observing the abundance of $\omega_{\dot{\mu}}$ and $\Omega_{\dot{\mu}}$ fields appearing together with the derivatives of the fields with seemingly unrelated factors. These vector fields were introduced in anticipation of certain symmetries. The field $\omega_{\dot{\mu}}$ can be viewed as an artifact of rescaling invariance of the 1-form τ and the field $\Omega_{\dot{\mu}}$ can be absorbed through the conformal rescaling of the fields. In the case of $\overset{(-1)}{A}_{\dot{\mu}}$ however, the appearance of $\Omega_{\dot{\mu}}$ is related to resemblance of its transformation to U(1) under subleading diffeomorphisms rather than conformal scaling. Another observation is the appearance of lower order terms in higher order equations which is also related to the transformation of equations under subleading diffeomorphisms as well. Additionally, it will help us write the equations in a more efficient manner while bringing the form closer to the 'stationary' equations, allowing us to define new and more natural variables to work with. In depth analysis of the symmetries of the equations and bringing them to a manifestly symmetry invariant form in terms of new variables will be explored in the rest of the section where we will demonstrate how each of these procedures work.

6.2.1. Milne Boost Invariance

Since the relativistic metric $g_{\mu\nu}$ is independent of our choice of τ^{μ} and $h_{\mu\nu}$ used to split it, it follows that the components as defined in Equation (5.12) must transform as

$$\delta_\chi A = -2\chi^{\dot{\mu}} A_{\dot{\mu}} \qquad \delta_\chi A_{\dot{\mu}} = (\tau_\mu A_{\dot{\rho}} - B_{\dot{\mu}\dot{\rho}})\chi^{\dot{\rho}} \qquad \delta_\chi A^{\dot{\mu}\dot{\nu}} = 2B^{(\dot{\mu}}\chi^{\dot{\nu})} \tag{6.28}$$

Since we choose $\tau_\mu\,,\tau^\mu\,,h^{\mu\nu}$ and $h_{\mu\nu}$ to be c independent it follows that the parameter $\chi_{\dot{\mu}}$ will be c independent and under boosts the expanded fields will transform in the same manner:

$$\delta_\chi \overset{(k)}{A} = -2\chi^{\dot{\mu}} \overset{(k)}{A}_{\dot{\mu}} \qquad \delta_\chi \overset{(k)}{A}_{\dot{\mu}} = (\tau_\mu \overset{(k)}{A}_{\dot{\rho}} - \overset{(k)}{B}_{\dot{\mu}\dot{\rho}})\chi^{\dot{\rho}} \qquad \delta_\chi \overset{(k)}{A}{}^{\dot{\mu}\dot{\nu}} = 2\overset{(k)}{B}{}^{(\dot{\mu}}\chi^{\dot{\nu})} \tag{6.29}$$

Still, these are rather complicated transformations due to the rather lengthy expressions for the B's in terms of the A's once the order increases as discussed in Section 6.1.1. On the leading triplet the action is very simple however:

$$\delta_\chi \overset{(-2)}{A} = 0 \qquad \delta_\chi \overset{(-1)}{A}_{\dot\mu} = \tau_\mu \overset{(-1)}{A}_{\dot\rho}\chi^{\dot\rho} \qquad \delta_\chi \overset{(0)}{A}{}^{\dot\mu\dot\nu} = \delta_\chi h^{\mu\nu} = 0 \tag{6.30}$$

The boost transformations of the leading order energy-momentum and curvature triplet are found to be

$$\delta_\chi \overset{(-2)}{\mathcal{T}} = 0 \qquad \delta_\chi \overset{(-1)}{\mathcal{T}}_{\dot\mu} = \tau_\mu \overset{(-1)}{\mathcal{T}}_{\dot\rho}\chi^{\dot\rho} \qquad \delta_\chi \overset{(0)}{\mathcal{T}}_{\dot\mu\dot\nu} = (\tau_\mu \overset{(0)}{\mathcal{T}}_{\dot\nu\dot\rho} + \tau_\nu \overset{(0)}{\mathcal{T}}_{\dot\mu\dot\rho})\chi^{\dot\rho} \tag{6.31}$$

$$\delta_\chi \overset{(-2)}{R} = 0 \qquad \delta_\chi \overset{(-1)}{R}_{\dot\mu} = \tau_\mu \overset{(-1)}{R}_{\dot\rho}\chi^{\dot\rho} \qquad \delta_\chi \overset{(0)}{R}_{\dot\mu\dot\nu} = (\tau_\mu \overset{(0)}{R}_{\dot\nu\dot\rho} + \tau_\nu \overset{(0)}{R}_{\dot\mu\dot\rho})\chi^{\dot\rho} \tag{6.32}$$

Because some of the variables do transform non-trivially under boosts it might appear as if boost invariance is not manifest. However, only objects with lower indices have a non-zero transformation under the Milne boosts and this will furthermore be proportional to τ_μ. It follows that all objects that either have all indices raised or contracted with some other raised indices will automatically be boost invariant. Let us illustrate this argument with an object introduced in Equation (6.20). By Equations (4.25) and (6.30) it follows that Ψ is boost invariant so that $\delta_\chi(\partial_{\dot\mu}\Psi) = \tau_\mu\chi^{\dot\rho}\partial_{\dot\rho}\Psi$. But then observe that

$$\delta_\chi(\partial^{\dot\mu}\Psi) = \delta_\chi(h^{\mu\rho}\partial_{\dot\rho}\Psi) = h^{\mu\rho}\delta_\chi(\partial_{\dot\rho}\Psi) = 0 \tag{6.33}$$

Similarly $\delta_\chi(\partial^{\dot\mu}\Psi\partial_{\dot\mu}\Psi) = \delta_\chi(\dot{\boldsymbol{\nabla}}^{\dot\mu}\partial_{\dot\mu}\Psi) = 0$.

6.2.2. Scale Invariance

A rescaling of the of the 1-form τ_μ can be absorbed in the definitions of the A fields, while leaving $g_{\mu\nu}$ invariant. Infinitesimally these scaling transformations that leave Equations (4.2) and (5.12) invariant are

$$\delta_\gamma \tau_\mu = \gamma \tau_\mu\,, \quad \delta_\gamma \tau^\mu = -\gamma \tau^\mu \tag{6.34}$$

$$\delta_\gamma A = -2\gamma A\,, \quad \delta_\gamma A_{\dot\mu} = -\gamma A_{\dot\mu}\,, \quad \delta_\gamma A^{\dot\mu\dot\nu} = 0 \tag{6.35}$$

Let us point out that this implies that

$$\delta_\gamma a_{\dot\mu} = -\partial_{\dot\mu}\gamma \tag{6.36}$$

As a result of Equation (4.24), this reveals that the twistless torsion degree of freedom in τ_μ is pure gauge, something which we'll discuss further below.

Due to the c independence of γ it follows that the scaling acts straightforwardly on the expansion coefficients:

$$\delta_\gamma \overset{(k)}{A} = -2\gamma \overset{(k)}{A}\,, \quad \delta_\gamma \overset{(k)}{A}_{\dot\mu} = -\gamma \overset{(k)}{A}_{\dot\mu}\,, \quad \delta_\gamma \overset{(k)}{A}{}^{\dot\mu\dot\nu} = 0\,. \tag{6.37}$$

Similarly

$$\delta_\gamma \overset{(k)}{\mathcal{T}} = -2\gamma \overset{(k)}{\mathcal{T}}\,, \quad \delta_\gamma \overset{(k)}{\mathcal{T}}_{\dot\mu} = -\gamma \overset{(k)}{\mathcal{T}}_{\dot\mu}\,, \quad \delta_\gamma \overset{(k)}{\mathcal{T}}_{\dot\mu\dot\nu} = 0\,. \tag{6.38}$$

One can then construct the following first set of scale invariant objects:

$$\begin{array}{ccc} (-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{A}_{\dot\mu} & \overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}} & (-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{\mathcal{T}}_{\dot\mu} \\ \overset{(-2)}{A}{}^{-1}\overset{(-2)}{R} & (-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{R}_{\dot\mu} & (-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{\mathcal{C}}_{\dot\mu} \end{array}$$

The second set of these will be defined in combination with the diffeomorphism invariant objects in the following section. An important note about these is that, they absorb all of the $\omega_{\dot\mu}$ terms appearing in Equations (6.23)-(6.27) as follows:

$$\dot{\boldsymbol{\nabla}}_{\dot\mu}((-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{A}_{\dot\nu}) = -(-\overset{(-2)}{A})^{-1/2}(\dot{\boldsymbol{\nabla}}_{\dot\mu}\overset{(-1)}{A}_{\dot\nu} + \omega_{\dot\mu}\overset{(-1)}{A}_{\dot\nu}) \tag{6.39}$$

Since after such a reformulation all equations and all variables except $\overset{(-2)}{A}$ are scale invariant, it follows that $\overset{(-2)}{A}$, which scales non-trivially has to disappear. However, the derivatives of $\overset{(-2)}{A}$ will still transform non-homogeneously, with the transformation including an extra $\partial_\mu \gamma$. But remembering that the 1-form $a_{\dot{\mu}}$ transforms in the same way – see Equation (6.36) – the previously defined invariant 1-form in Equation (6.20) is scale invariant:

$$\delta_\gamma \Psi = 0 \qquad \delta_\gamma (\partial_{\dot{\mu}} \Psi) = 0 \tag{6.40}$$

As a result, justifying our initial choice of working with it. The upshot of this observation is that the torsion one-form $a_{\dot{\mu}}$ and the field $\overset{(-2)}{A}$ can only appear in the scale invariant equations through the scale invariant field Ψ, or its derivatives.

6.2.3. Diffeomorphism Invariance

One of the key features of GR is its invariance under diffeomorphisms. A priori they can depend arbitrarily on the speed of light c. Compatibility with the expansion ansatz in Equation (6.3) requires however that the generating vector field satisfies [4,5]

$$\xi^\mu(c) = \sum_{k=0}^{\infty} \overset{(k)}{\xi}{}^\mu c^{-k} \tag{6.41}$$

The zeroth order coefficients $\overset{(0)}{\xi}{}^\mu$ generate the diffeomorphisms of the nonrelativistic theory obtained by the expansion, while the higher order coefficients generate additional gauge transformations. The action on the coefficients of an arbitrary c dependent relativistic tensor $U^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_n}(c)$ is

$$\delta_{\overset{(k)}{\xi}} \overset{(l)}{U}{}^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_n} = L_{\overset{(k)}{\xi}} \overset{(l-k)}{U}{}^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_n} \tag{6.42}$$

We define the Newton-Cartan structure τ_μ, $h^{\mu\nu}$ and τ^μ, $h_{\mu\nu}$ to transform as tensors under $\overset{(0)}{\xi}{}^\mu$ but to be invariant under all subleading diffeomorphisms $\overset{(k)}{\xi}{}^\mu$ for $k > 0$. We stress that although such definition is consistent, it implies that the components A, $A_{\dot\mu}$ and $A^{\dot\mu\dot\nu}$ do not transform as tensors under c dependent diffeomorphisms. Rather their transformations are defined as the respective components of the transformed relativistic metric:

$$\delta_{\overset{(k)}{\xi}}\overset{(l)}{g}_{\mu\nu} = \tau_\mu\tau_\nu\delta_{\overset{(k)}{\xi}}\overset{(l)}{A} + 2\tau_{(\mu}\delta_{\overset{(k)}{\xi}}\overset{(l)}{A}_{\dot\mu)} + \delta_{\overset{(k)}{\xi}}\overset{(l)}{B}_{\dot\mu\dot\nu} \qquad (k>0) \tag{6.43}$$

This is equivalent to

$$\begin{aligned}
\delta_{\overset{(k)}{\xi}}\overset{(l)}{A} &= \tau^\rho\tau^\sigma\delta_{\overset{(k)}{\xi}}\overset{(l)}{g}_{\rho\sigma} = \tau^\rho\tau^\sigma L_{\overset{(k)}{\xi}}\overset{(l-k)}{g}_{\rho\sigma} \\
\delta_{\overset{(k)}{\xi}}\overset{(l)}{A}_{\dot\mu} &= \tau^\rho h^\sigma_\mu\delta_{\overset{(k)}{\xi}}\overset{(l)}{g}_{\rho\sigma} = \tau^\rho h^\sigma_\mu L_{\overset{(k)}{\xi}}\overset{(l-k)}{g}_{\rho\sigma} \\
\delta_{\overset{(k)}{\xi}}\overset{(l)}{A}^{\dot\mu\dot\nu} &= h^\mu_\rho h^\nu_\sigma\delta_{\overset{(k)}{\xi}}\overset{(l)}{g}^{\rho\sigma} = h^\mu_\rho h^\nu_\sigma L_{\overset{(k)}{\xi}}\overset{(l-k)}{g}^{\rho\sigma}
\end{aligned} \tag{6.44}$$

which in turn can be rewritten via Equation (5.12) as

$$\begin{aligned}
\delta_{\overset{(k)}{\xi}}\overset{(l)}{A} &= L_{\overset{(k)}{\xi}}\overset{(l-k)}{A} + 2\overset{(l-k)}{A}(\partial\overset{(k)}{\xi} - a_{\dot\rho}\overset{(k)}{\xi}{}^{\dot\rho}) - 2\overset{(l-k)}{A}{}_{\dot\rho}L_{\overset{(k)}{\xi}}\tau^\rho \qquad\qquad (k>0) \\
\delta_{\overset{(k)}{\xi}}\overset{(l)}{A}_{\dot\mu} &= \overset{(l-k)}{A}(\partial_{\dot\mu}\overset{(k)}{\xi} + a_{\dot\mu}\overset{(k)}{\xi}) + h^\sigma_{\dot\mu}L_{\overset{(k)}{\xi}}\overset{(l-k)}{A}{}_{\dot\sigma} + \overset{(l-k)}{A}{}_{\dot\mu}(\partial\overset{(k)}{\xi} - a_{\dot\rho}\overset{(k)}{\xi}{}^{\dot\rho}) - \overset{(l-k)}{B}{}_{\dot\mu\dot\rho}L_{\overset{(k)}{\xi}}\tau^\rho \\
\delta_{\overset{(k)}{\xi}}\overset{(l)}{A}^{\dot\mu\dot\nu} &= 2\overset{(l-k)}{B}{}^{(\dot\mu}h^{\dot\nu)}_\rho L_{\overset{(k)}{\xi}}\tau^\rho + h^\mu_\rho h^\nu_\sigma L_{\overset{(k)}{\xi}}\overset{(l-k)}{A}{}^{\dot\rho\dot\sigma}
\end{aligned}$$

Because $\overset{(k)}{A} = \overset{(k+1)}{A}{}_{\dot\nu} = \overset{(k+2)}{A}{}^{\dot\mu\dot\nu} = 0$ when $k<-2$ it follows that the subleading diffeomorphism $\overset{(k)}{\xi}$ acts non-trivially only on the triplets $(\overset{(l)}{A}, \overset{(l+1)}{A}{}_{\dot\nu}, \overset{(l+2)}{A}{}^{\dot\mu\dot\nu})$ for which $l \geq k-2$. In particular, at leading order only $\overset{(1)}{\xi}$ acts non-trivially, in the simple fashion

$$\delta_{\overset{(1)}{\xi}}\overset{(-2)}{A} = 0 \qquad \delta_{\overset{(1)}{\xi}}\overset{(-1)}{A}{}_{\dot\mu} = \overset{(-2)}{A}(\partial_{\dot\mu}\overset{(1)}{\xi} + a_{\dot\mu}\overset{(1)}{\xi}) \qquad \delta_{\overset{(1)}{\xi}}\overset{(0)}{A}^{\dot\mu\dot\nu} = 0 \tag{6.45}$$

Note that the transformation of $\overset{(-1)}{A}{}_{\dot\mu}$ resembles a U(1) transformation, and indeed we will recast it as such below. Furthermore, the spatial part $\overset{(1)}{\xi}{}^{\dot\mu}$ acts trivially at this order. Let us point out that this simple structure repeats itself at all orders, when one considers only the action of the highest order diffeomorphisms on the highest order triplet:

$$\delta_{\overset{(l+3)}{\xi}}\overset{(l)}{A} = 0 \qquad \delta_{\overset{(l+3)}{\xi}}\overset{(l+1)}{A}_{\dot\mu} = \overset{(-2)}{A}(\partial_{\dot\mu}\overset{(l+3)}{\xi} + a_{\dot\mu}\overset{(l+3)}{\xi}) \qquad \delta_{\overset{(l+3)}{\xi}}\overset{(l+2)}{A}{}^{\dot\mu\dot\nu} = 0 \tag{6.46}$$

A similar analysis reveals the transformations of the leading order energy-momentum triplet to be

$$\delta_{\overset{(1)}{\xi}}\overset{(-2)}{\mathcal{T}} = 0 \qquad \delta_{\overset{(1)}{\xi}}\overset{(-1)}{\mathcal{T}}_{\dot\mu} = \overset{(-2)}{\mathcal{T}}(\partial_{\dot\mu}\overset{(1)}{\xi} + a_{\dot\mu}\overset{(1)}{\xi}) \qquad \delta_{\overset{(1)}{\xi}}\overset{(0)}{\mathcal{T}}_{\dot\mu\dot\nu} = 2\overset{(-1)}{\mathcal{T}}_{(\dot\mu}\partial_{\dot\nu)}\overset{(1)}{\xi} + 2\overset{(-1)}{\mathcal{T}}_{(\dot\mu}a_{\dot\nu)}\overset{(1)}{\xi} \tag{6.47}$$

Let us illustrate how one can indeed make the variables invariant under $\overset{(1)}{\xi}{}^{\mu}$ transformations with an example. First step is to observe that a particular combination of $\overset{(-2)}{A}$, $\overset{(-1)}{A}_{\dot\mu}$, $\overset{(-2)}{\mathcal{T}}$ and $\overset{(-1)}{\mathcal{T}}_{\dot\mu}$ appears in the conservation equation $\overset{(-1)}{\mathcal{C}}$ in Equations 6.23-temporalconservation. By considering the transformation Equation (6.47) of $\overset{(1)}{\mathcal{T}}_{\dot\mu}$ and the transformation in Equation (6.45) of $\overset{(-1)}{A}_{\dot\mu}$ one can show such combination is diffeomorphism invariant:

$$\delta_{\overset{(1)}{\xi}}(\overset{(-1)}{\mathcal{T}}_{\dot\mu} - \overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}_{\dot\mu}) = 0 \tag{6.48}$$

Going a step further, one can make the scale-diffeomorphism invariant combination:

$$\delta_{\overset{(1)}{\xi}}((-\overset{(-2)}{A})^{-1/2}(\overset{(-1)}{\mathcal{T}}_{\dot\mu} - \overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}_{\dot\mu})) = 0 \tag{6.49}$$

Other such combinations can also be made with other variables. One can find all of them by letting Ψ vanish in Section 6.3 .

Further simplification can be obtained by considering invariance of the scale invariant version of $\overset{(-1)}{A}_{\dot\mu}$ – i.e. $(-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{A}_{\dot\mu}$ – under the subleading diffeomorphisms $\overset{(1)}{\xi}{}^{\mu}$. The field transforms under subleading diffeomorphisms in Equation (6.45) as

$$\delta_{\overset{(1)}{\xi}}((-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{A}_{\dot\mu}) = -(\partial_{\dot\mu} + \Omega_{\dot\mu})(-\overset{(-2)}{A})^{1/2}\overset{(1)}{\xi} \tag{6.50}$$

The definition in Equation (6.20) then suggests to further define

$$C_{\dot{\mu}} = -e^{\Psi/2}(-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{A}_{\dot{\mu}} \tag{6.51}$$

so that the rescaled version of $\overset{(-1)}{A}_{\dot{\mu}}$ transforms as a U(1) gauge field under subleading diffeomorphisms:

$$\delta_{\overset{(1)}{\xi}} C_{\dot{\mu}} = \partial_{\dot{\mu}}\zeta \qquad \zeta = e^{-\Psi/2}(-\overset{(-2)}{A})^{\frac{1}{2}}\overset{(1)}{\xi} \tag{6.52}$$

If one can make the other variables and equations manifestly invariant under these subleading diffeomorphisms then it will follow that $C_{\dot{\mu}}$ can only appear through its gauge invariant curvature, that is the field strength of the gauge potential $C_{\dot{\mu}}$.

$$F_{\dot{\mu}\dot{\nu}} = h^{\rho}_{\mu}h^{\sigma}_{\nu}(\partial_{\rho}C_{\sigma} - \partial_{\sigma}C_{\rho}) \quad \text{such that} \quad \delta_{\overset{(1)}{\xi}} F_{\dot{\mu}\dot{\nu}} = 0 \tag{6.53}$$

Schematically this amounts to the replacement $H_{\dot{\mu}\dot{\nu}} \to F_{\dot{\mu}\dot{\nu}}$ in the raw form of leading order equations. As $C_{\mu} = C\tau_{\mu} + C_{\dot{\mu}}$ one can further check that $F_{\dot{\mu}\dot{\nu}} = h^{\rho}_{\mu}h^{\sigma}_{\nu}(\partial_{\rho}C_{\dot{\sigma}} - \partial_{\sigma}C_{\dot{\rho}})$ is equal to Equation (6.53) and the scalar C does not appear. This expression is slightly different from $\partial_{\dot{\mu}}C_{\dot{\nu}} - \partial_{\dot{\nu}}C_{\dot{\mu}} = h^{\rho}_{\mu}\partial_{\rho}C_{\dot{\nu}} - h^{\rho}_{\nu}\partial_{\rho}C_{\dot{\mu}} = F_{\dot{\mu}\dot{\nu}} + 2\tau_{[\mu}h^{\rho}_{\nu]}C_{\dot{\sigma}}\partial_{\rho}\tau^{\sigma}$, which will never appear in our equations.

6.2.4. Conformal Rescaling

In addition to scale and subleading diffeomorphism invariant combinations, the field redefinitions in Section 6.3 contain also the conformal rescalings by powers of e^{Ψ}. Except for the definition of $C_{\dot{\mu}}$, where this is related to making the U(1) gauge invariance manifest, these powers in the definitions of the other fields are chosen to simplify the equations. In particular the choice to redefine the spatial metric with such a factor is related to putting the equations in a form that can naturally be obtained from a variational principle. In addition to this, it also has the benefit of absorbing most of the $\Omega_{\dot{\mu}}V^{\dot{\mu}}$ terms appearing together with the covariant derivative.

The first of these redefinitions is on the spatial metric:

$$k^{\mu\nu} = e^{\Psi} h^{\mu\nu} \qquad k_{\mu\nu} = e^{-\Psi} h_{\mu\nu} \tag{6.54}$$

which preserves the Newton-Cartan structure

$$\tau_\mu \tau^\nu + k_{\mu\rho} k^{\rho\nu} = \delta^\nu_\mu \tag{6.55}$$

It follows that this defines a new connection $(\mathbb{\nabla}_\mu, \mathbb{\Gamma}^\rho_{\mu\nu})$ that preserves the Newton-Cartan structure $(\tau_\mu, k^{\mu\nu})$ and is conformally related on the Levi-Civita-like part of the Newton-Cartan connection:

$$\begin{aligned} \mathbb{\Gamma}^\lambda_{\mu\nu} &= \tau^\lambda \partial_\mu \tau_\nu + \frac{1}{2} k^{\lambda\rho} \left(\partial_\mu k_{\rho\nu} + \partial_\nu k_{\rho\mu} - \partial_\rho k_{\mu\nu} \right) & (6.56) \\ &= \dot{\mathbb{\Gamma}}^\lambda_{\mu\nu} - \Omega_{\dot{\mu}} h^\lambda_\nu - \Omega_{\dot{\nu}} h^\lambda_\mu + \Omega^{\dot{\lambda}} h_{\mu\nu} & (6.57) \end{aligned}$$

That comes with the same torsion tensor

$$\mathbb{T}^\lambda_{\mu\nu} = 2\mathbb{\Gamma}^\lambda_{[\mu\nu]} = 2\tau^\lambda \tau_{[\mu} a_{\nu]} = 2\dot{\mathbb{\Gamma}}^\lambda_{[\mu\nu]} \tag{6.58}$$

Such a conformal transformation relates the spatial Ricci tensors as

$$\dot{\mathbb{R}}_{\dot{\mu}\dot{\nu}} = \mathbb{R}_{\dot{\mu}\dot{\nu}} - h_{\mu\nu} h^{\rho\sigma} (\mathbb{\nabla}_{\dot{\rho}} \Omega_{\dot{\sigma}} + \Omega_{\dot{\rho}} \Omega_{\dot{\sigma}}) - \mathbb{\nabla}_{(\dot{\mu}} \Omega_{\dot{\nu})} + \Omega_{\dot{\mu}} \Omega_{\dot{\nu}} \tag{6.59}$$

In particular such a change will absorb $\dot{\nabla}_{\dot{\mu}} \Omega_{\dot{\nu}}$ term in Equation (6.25). As such terms do not have a candidate Lagrangian, this will prove important later on in having one.

Further simplification with the conformal rescaling in Equations (6.23)-(6.27) can be applied to the (scale invariant) equations for $\overset{(-2)}{R}$ and $\overset{(-1)}{\mathcal{C}}$. Notice that, for the divergence of spatial tensors one can absorb $\Omega_{\dot{\mu}}$ terms appearing together by conformally rescaling in the following manner

$$\begin{aligned}
\dot{\nabla}_{\dot{\mu}} V^{\dot{\mu}} &= h^{\mu\rho} \dot{\nabla}_{\mu} V_{\dot{\rho}} \\
&= h^{\mu\rho} (\nabla_{\dot{\mu}} V_{\dot{\rho}} - \Omega_{\dot{\mu}} V_{\dot{\rho}} - \Omega_{\dot{\rho}} V_{\dot{\mu}} + \Omega_{\dot{\nu}} h^{\lambda\nu} h_{\mu\rho}) \\
&= e^{-\Psi} k^{\mu\rho} (\nabla_{\dot{\mu}} V_{\dot{\rho}} - \Omega_{\dot{\mu}} V_{\dot{\rho}} - \Omega_{\dot{\rho}} V_{\dot{\mu}} + \Omega_{\dot{\nu}} k^{\lambda\nu} k_{\mu\rho})
\end{aligned} \tag{6.60}$$

where we used the result explored in Appendix A, Equation (A.12). It should be stressed that from this point on we will be switching conventions to which the spatial indices are raised with the metric $k^{\mu\nu}$ instead of the previous $h^{\mu\nu}$. It should be understood that all raised objects in Equations (6.23)-(6.27) will become:

$$V^{\dot{\mu}} = h^{\mu\rho} V_{\dot{\rho}} \quad \rightarrow \quad e^{-\Psi} k^{\mu\rho} V_{\dot{\rho}} = e^{-\Psi} V^{\dot{\mu}} \tag{6.61}$$

Then Equation (6.60) becomes

$$h^{\dot{\mu}\dot{\rho}} \dot{\nabla}_{\dot{\mu}} V_{\dot{\rho}} = e^{-\Psi} (\nabla_{\dot{\mu}} V^{\dot{\mu}} + \Omega_{\dot{\mu}} V^{\dot{\mu}}) \tag{6.62}$$

Where the right hand side is raised with k. Now in order to simplify the equation of $\overset{(-2)}{R}$, notice that letting $V_{\dot{\mu}} = \Omega_{\dot{\mu}}$ removes the extra $h^{\mu\rho} \Omega_{\dot{\mu}} \Omega_{\dot{\rho}} = e^{-\Psi} \Omega_{\dot{\mu}} \Omega^{\dot{\rho}}$ term. For the equation of $\overset{(-1)}{\mathcal{C}}$, observe that the following object

$$\mathfrak{T}_{\dot{\mu}} = e^{-\Psi/2} (-\overset{(-2)}{A})^{-1/2} (\overset{(-1)}{\mathcal{T}}_{\dot{\mu}} - \overset{(-2)}{A}{}^{-1} \overset{(-2)}{\mathcal{T}} \overset{(-1)}{A}_{\dot{\mu}}) \tag{6.63}$$

will result in the equation $\nabla_{\dot{\mu}} \mathfrak{T}^{\dot{\mu}} = 0$. This simplifies the leading order scalar conservation law equation massively while remaining scale and diffeomorphism invariant. All of such redefinitions together with other symmetries are collected in Section 6.3.

Together with the fields Ψ and $C_{\dot{\mu}}$ introduced previously, in Chapter 7 it will be shown that the new triplet of dynamical variables $(\Psi, C_{\dot{\mu}}, k^{\mu\nu})$ in fact form a more natural candidate than A fields for the expansion.

6.3. Field Redefinitions

The detailed redefinitions behind the above discussion are collected below:

$$
\begin{aligned}
\Psi &= 2\psi - \log(-\overset{(-2)}{A}) \\
C_{\dot{\mu}} &= e^{\Psi/2}(-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{A}_{\dot{\mu}} \\
k^{\mu\nu} &= e^{\Psi}h^{\mu\nu} \qquad k_{\mu\nu} = e^{-\Psi}h_{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{T} &= -e^{\Psi}\overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}} \\
\mathfrak{T}_{\dot{\mu}} &= e^{-\Psi/2}(-\overset{(-2)}{A})^{-1/2}\left(\overset{(-1)}{\mathcal{T}}_{\dot{\mu}} - \overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}_{\dot{\mu}}\right) \\
\mathfrak{T}_{\dot{\mu}\dot{\nu}} &= \overset{(0)}{\mathcal{T}}_{\dot{\mu}\dot{\nu}} - \overset{(-2)}{A}{}^{-1}(\overset{(-1)}{A}_{\dot{\mu}}\overset{(-1)}{\mathcal{T}}_{\dot{\nu}} + \overset{(-1)}{A}_{\dot{\nu}}\overset{(-1)}{\mathcal{T}}_{\dot{\mu}}) + \overset{(-2)}{A}{}^{-2}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}_{\dot{\mu}}\overset{(-1)}{A}_{\dot{\nu}} \\
&\quad -\frac{1}{2}h_{\mu\nu}h^{\rho\sigma}\left(\overset{(0)}{\mathcal{T}}_{\dot{\rho}\dot{\sigma}} - 2\overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}_{\dot{\rho}}\overset{(-1)}{\mathcal{T}}_{\dot{\sigma}} + \overset{(-2)}{A}{}^{-2}\overset{(-2)}{\mathcal{T}}\overset{(-1)}{A}_{\dot{\rho}}\overset{(-1)}{A}_{\dot{\sigma}} + \frac{1}{3}h_{\rho\sigma}\overset{(-2)}{A}{}^{-1}\overset{(-2)}{\mathcal{T}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R} &= \overset{(-2)}{A}{}^{-1}\overset{(-2)}{R} \\
\mathfrak{R}_{\dot{\mu}} &= (-\overset{(-2)}{A})^{-1/2}\left(\overset{(-1)}{R}_{\dot{\mu}} - \overset{(-2)}{A}{}^{-1}\overset{(-2)}{R}\overset{(-1)}{A}_{\dot{\mu}}\right) \\
\mathfrak{R}_{\dot{\mu}\dot{\nu}} &= \overset{(0)}{R}_{\dot{\mu}\dot{\nu}} - \overset{(-2)}{A}{}^{-1}(\overset{(-1)}{A}_{\dot{\mu}}\overset{(-1)}{R}_{\dot{\nu}} + \overset{(-1)}{A}_{\dot{\nu}}\overset{(-1)}{R}_{\dot{\mu}}) + \overset{(-2)}{A}{}^{-2}\overset{(-2)}{R}\overset{(-1)}{A}_{\dot{\mu}}\overset{(-1)}{A}_{\dot{\nu}} \\
&\quad -\frac{1}{2}h_{\mu\nu}h^{\rho\sigma}\left(\overset{(0)}{R}_{\dot{\rho}\dot{\sigma}} - 2\overset{(-2)}{A}{}^{-1}\overset{(-1)}{A}_{\dot{\rho}}\overset{(-1)}{R}_{\dot{\sigma}} + \overset{(-2)}{A}{}^{-2}\overset{(-2)}{R}\overset{(-1)}{A}_{\dot{\rho}}\overset{(-1)}{A}_{\dot{\sigma}} + \frac{1}{3}h_{\rho\sigma}\overset{(-2)}{A}{}^{-1}\overset{(-2)}{R}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C} &= (-\overset{(-2)}{A})^{-1/2}\overset{(-1)}{\mathcal{C}} \\
\mathfrak{C}_{\dot{\mu}} &= \overset{(0)}{\mathcal{C}}_{\dot{\mu}} - \overset{(-2)}{A}{}^{-1}\overset{(-1)}{\mathcal{C}}\overset{(-1)}{A}_{\dot{\mu}}
\end{aligned}
$$

An additional note about such redefinitions is that in the second lines of the defining equations of $\mathfrak{R}_{\dot{\mu}\dot{\nu}}$ and $\mathfrak{T}_{\dot{\mu}\dot{\nu}}$ are there to revert back our initial choice of trace-reversed Einstein equations. This choice will result in the appearance of a conserved Einstein tensor of the Newton-Cartan connection.

6.4. Leading Order Equations

Finally we are ready to present the leading order Einstein and conservation equations. After the redefinitions discussed in Section 6.3 one finds that each of

$$\mathfrak{R} = 8\pi G_{\mathrm{N}}\mathfrak{T} \qquad \mathfrak{R}_{\dot\mu} = 8\pi G_{\mathrm{N}}\mathfrak{T}_{\dot\mu} \qquad \mathfrak{R}_{\dot\mu\dot\nu} = 8\pi G_{\mathrm{N}}\mathfrak{T}_{\dot\mu\dot\nu} \tag{6.64}$$

$$\mathfrak{C} = 0 \qquad \mathfrak{C}_{\dot\mu} = 0 \tag{6.65}$$

is equivalent to the corresponding final LO equations:

$$\nabla^{\dot\rho}\partial_{\dot\rho}\Psi = \frac{e^{-2\Psi}}{2}F^{\dot\rho\dot\sigma}F_{\dot\rho\dot\sigma} - 16\pi G_{\mathrm{N}}\mathfrak{T} \tag{6.66}$$

$$\nabla_{\dot\rho}(e^{-2\Psi}F^{\dot\rho\dot\mu}) = -16\pi G_{\mathrm{N}}\mathfrak{T}^{\dot\mu} \tag{6.67}$$

$$\begin{aligned}\mathbb{G}^{\dot\mu\dot\nu} = {} & \frac{e^{-2\Psi}}{8}\left(k^{\mu\nu}F^{\dot\rho\dot\sigma}F_{\dot\rho\dot\sigma} - 4F^{\dot\mu\dot\rho}F^{\dot\nu}{}_{\dot\rho}\right) - \frac{1}{4}k^{\mu\nu}\partial_{\dot\rho}\Psi\partial^{\dot\rho}\Psi \\ & + \frac{1}{2}\partial^{\dot\mu}\Psi\partial^{\dot\nu}\Psi + 8\pi G_{\mathrm{N}}\mathfrak{T}^{\dot\mu\dot\nu}\end{aligned} \tag{6.68}$$

$$\nabla_{\dot\rho}\mathfrak{T}^{\dot\rho} = 0 \tag{6.69}$$

$$\nabla_{\dot\rho}\mathfrak{T}^{\dot\rho\dot\mu} = \mathfrak{T}\partial^{\dot\mu}\Psi + \mathfrak{T}_{\dot\rho}F^{\dot\rho\dot\mu} \tag{6.70}$$

with the dynamical variables:

$$\Psi\,, \quad C_{\dot\mu}\,, \quad k^{\mu\nu} \qquad \text{and} \qquad \mathfrak{T}\,, \quad \mathfrak{T}_{\dot\mu}\,, \quad \mathfrak{T}_{\dot\mu\dot\nu}\,. \tag{6.71}$$

For the rest of this work, we will refer to Equations (6.66)-(6.70) as the LO equations. As discussed in Section 6.2.4, in those equations all upper indices are indices raised with $k^{\mu\nu}$ and the connection $(\nabla_\mu, \Gamma^\rho_{\mu\nu})$ is one preserving the Newton-Cartan structure $(\tau_\mu, k^{\mu\nu})$. In the final LO equations also the Einstein tensor of this connection appears as

$$\mathbb{G}_{\dot\mu\dot\nu} = \mathbb{R}_{\dot\mu\dot\nu} - \frac{1}{2}k_{\mu\nu}k^{\rho\sigma}\mathbb{R}_{\dot\rho\dot\sigma}\,, \tag{6.72}$$

Note that actually any connection of the form explored in Appendix C, Equation (C.5), would leave Equations (6.66)-(6.70) invariant. In that sense Equation (6.56) is the minimal choice.

An important consistency check on LO equations is that the conservation equations follow from the first three equations through the Bianchi identity:

$$\boldsymbol{\nabla}_{\dot{\mu}}\mathbb{G}^{\dot{\mu}\dot{\nu}} = 0 \tag{6.73}$$

This result follows upon schematically identifying $R \to \mathbb{R}$ and $T \to \mathbb{T}$ in Equation (2.28), contracting by $\rho = \mu$ and raising twice by k while making use of a curvature identity of Newton-Cartan connections introduced in [6].

7. DISCUSSION

In this chapter we provide a number of remarks and observations on Equations (6.66)-(6.70).

7.1. Invariance

As explored in the Section 6.2, due to the introduction of scale invariant variables – see Section 6.3 – invariance under the scale symmetry has become trivial. The tensorial nature of the equations guarantees invariance under c independent diffeomorphisms $\overset{(0)}{\xi}$. Furthermore, in these new variables the subleading diffeomorphisms only act on $C_{\dot{\mu}}$ as a U(1) transformation. It then follows that the equations are invariant since $C_{\dot{\mu}}$ only appears through its gauge invariant curvature $F_{\dot{\mu}\dot{\nu}}$. The boost invariance is also trivial as follows from the discussion in Section 6.2.1. Note that invariance under boosts as above is possible due to absence of temporal derivatives. At higher order such derivatives will be present. In that case boost invariant variables can be introduced as was demonstrated in Section 4.2 – see [5,6] for further details – . These fields however, do not exist at the leading order.

7.2. Gauge Choices

As we mentioned in Section 6.1, our approach differs from some of the earlier literature [3–6] in that we leave the potential $\overset{(-2)}{A}$ free, rather than choosing it to be -1. This does *not* amount to the introduction of a new degree of freedom, since upon freeing $\overset{(-2)}{A}$ there appears a local scaling symmetry that in turn removes one scalar degree of freedom. The gauge invariant scalar degree of freedom Ψ – defined in Equation (6.20) – is a combination of $\overset{(-2)}{A}$ and the torsion potential ψ defined in Equation (4.24).

7.2.1. Dautcourt Gauge

By a scale transformation on Equation (6.37) one can always make $\overset{(-2)}{A} = -1$, fixing the scaling symmetry. In this choice of gauge our setup reduces to that originally introduced by Dautcourt [3] and followed in [4–6]. In this case the variable Ψ can be identified with the torsion potential ψ, or in other words:

$$\text{Dautcourt gauge:} \qquad \overset{(-2)}{A} = -1 \qquad a_{\dot{\mu}} = \frac{1}{2}\partial_{\dot{\mu}}\Psi \tag{7.1}$$

In this gauge the physical degree of freedom Ψ – describing a nonrelativistic but strong gravitational time dilation – finds itself thus in the (twistless) torsion of the Newton-Cartan structure, as was first emphasized in [5].

7.2.2. Torsion Free Gauge

Alternatively, via Equation (6.36), one can also use a scale transformation to put $\psi = 0$, again fixing this gauge symmetry. In this gauge the vector $a_{\dot{\mu}}$ vanishes and hence the Newton-Cartan structure is torsionless:

$$\text{Torsion free gauge:} \qquad \overset{(-2)}{A} = -e^{-\Psi} \qquad a_{\dot{\mu}} = 0 \tag{7.2}$$

This gauge has the advantage that the nonrelativistic geometry used to express the large c expansion is simpler and that the potentials $\overset{(k)}{A}$ are treated equally at all orders. The field redefinitions for the leading order triplet, see Section 6.3, reduce in this gauge to those

$$\text{Leading order potentials:} \quad \overset{(-2)}{A} = -e^{-\Psi} \qquad \overset{(-1)}{A}_{\dot{\mu}} = e^{-\Psi}C_{\dot{\mu}} \qquad \overset{(0)}{A}{}^{\dot{\mu}\dot{\nu}} = e^{\Psi}k^{\mu\nu} \tag{7.3}$$

7.2.3. Galilean gauge

The main motivation behind the Newton-Cartan formalism is to express nonrelativistic gravity in manifestly 4-diffeomorphism invariant form. Still, it turns out that the dynamics restrict τ_μ to be twistless [5] which means that there is an inherent direction of time in the theory that all observers can agree upon. Furthermore it is clear from Equations (6.66)-(6.70), the equations obtained from the large c expansion take their simplest form in terms of a time vs space split. This all suggests that it might be quite natural to gauge-fix some of the diffeomorphism invariance by working with an adapted time coordinate. As reviewed in Appendix D, the twistless torsion condition guarantees that there exists a function $t(x^\mu)$ such that locally $\tau_\mu = e^{-\psi}\partial_\mu t$. One can thus choose coordinates $x^\mu = (t, x^i)$ such that $\tau_\mu = e^{-\psi}\delta^0_\mu$. Together with a choice of $\tau^\mu = e^\psi \delta^\mu_0$ these conditions are left invariant by a particular combination of Milne boosts and time-dependent spatial diffeomorphisms [10, 17]. This gauge condition can be summarized as

$$\text{Galilean gauge:} \qquad \tau_\mu = e^{-\psi}\delta^0_\mu\,, \quad \tau^\mu = e^\psi\delta^\mu_0\,, \quad k^{\mu 0} = k_{\mu 0} = 0 \qquad k^{il}k_{lj} = \delta^i_j \tag{7.4}$$

The LO equations remain essentially form invariant under this gauge fixing: one simply replaces $\dot{\mu} \to i$ and finds that the covariant derivatives and Einstein tensor become those with respect to the Levi-Civita connection of the 3-metric k_{ij}. Note that in this gauge the relativistic metric becomes

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= -e^{-\Psi}(c\,dt + C_i dx^i)^2 + e^\Psi k_{ij}dx^i dx^j \\ &\quad + \mathcal{O}(c)\,dt^2 + \mathcal{O}_i(c^0)dt dx^i + \mathcal{O}_{ij}(c^{-1})dx^i dx^j \end{aligned} \tag{7.5}$$

where $\mathcal{O}(c^k)$ terms can be determined via Equations (6.11) and (6.16). If one additionally observes that there are no time derivatives in the LO equations, it can be seen that these equations as those for an arbitrary stationary relativistic 4-metric. The key crucial difference is that in the large c expansion we performed we did not assume time independence of the fields. But we can conclude that any stationary metric will solve the LO equations exactly. Conversely it follows that at leading order in the large c expansion any solution to the relativistic Einstein equations takes the form of a solution to the stationary Einstein equations but with time-dependent integration constants. We can conclude that the large c expansion is an expansion around the stationary sector of GR. If one makes the coefficients of odd powers vanish – i.e. take $C_i = 0$ above – the leading order reduces to the static sector, as was already observed in [10].

Finally we point out that the (vacuum) LO equations, just like those for stationary metrics, can be obtained from a Lagrangian, which follows from a time-like Kaluza-Klein reduction of the Einstein-Hilbert Lagrangian:

$$L = \sqrt{k}\left(R - \frac{1}{2}\partial_i\Psi\partial^i\Psi + \frac{e^{-2\Psi}}{4}F_{ij}F^{ij}\right) \tag{7.6}$$

7.3. Kerr Metric

In this section we illustrate the expansion procedure and how stationary relativistic metrics provide exact solutions to the LO equations. Starting with the relativistic Kerr metric as in [18] and working in the torsion free gauge with the choice $\tau_\mu dx^\mu = dt$, one finds for the potentials defined through the Newton-Cartan split Equation (5.12)

$$\begin{aligned} A &= -c^2\left(1 - \frac{2G_{\mathrm{N}}mr}{c^2\Sigma}\right) \\ A_{\dot\mu}dx^\mu &= -\frac{1}{c}\frac{2aG_{\mathrm{N}}mr\sin^2\theta}{\Sigma}d\phi \\ A^{\dot\mu\dot\nu}\partial_\mu\otimes\partial_\nu &= \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\theta^2 + \frac{\Delta\csc^2\theta - a^2}{\Delta\Sigma}\partial_\phi^2 \end{aligned} \tag{7.7}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 + a^2 - \frac{2G_{\mathrm{N}}mr}{c^2}\,, \qquad a = \frac{J}{mc}\,. \tag{7.8}$$

Depending on how one assumes the mass m and angular momentum J to scale with the speed of light c one gets different expansions.

7.3.1. Weakly Massive, Weakly Rotating Kerr Metric

First let us consider the standard Newtonian regime where $\frac{Gm}{r} \ll c^2$ and $\frac{J}{mr} \ll c$, expanding Equation (7.7) and expressing the fields in terms of the variables of Section 6.3 gives

$$\Psi = 0\,, \qquad C_{\mu} dx^{\mu} = 0\,, \qquad k^{\mu\nu}\partial_{\mu} \otimes \partial_{\nu} = \delta^{ij}\partial_i \otimes \partial_j \tag{7.9}$$

In this case at leading order the fields simply provide a nonrelativistic description of Minkowski space, the starting point of a weak gravity approximation to GR. The first correction comes in the form of the Newtonian potential $\overset{(0)}{A} = 2\Phi = \frac{2GM}{r}$, and then follow further subleading post-Newtonian corrections.

7.3.2. Strongly Massive, Weakly Rotating Kerr Metric

Another regime is where $\frac{Gm}{r} \approx c^2$ but $\frac{J}{mr} \ll c$. We can formally implement this regime by defining $m = Mc^2$ and keeping M rather than m fixed as $c \to \infty$. In this way of expanding the Kerr metric the leading order fields become

$$\Psi = -\log\left(1 - \frac{2G_{\mathrm{N}}M}{r}\right)\,, \qquad C_{\mu}dx^{\mu} = 0\,,$$
$$k^{\mu\nu}\partial_{\mu} \otimes \partial_{\nu} = \partial_r^2 + \frac{1}{r^2 - 2G_{\mathrm{N}}Mr}\left(\partial_{\theta}^2 + \csc^2\theta\,\partial_{\phi}^2\right)$$

In this regime we see that the LO fields contain a spatial metric that is not flat and that translates to a relativistic metric which is not approximately Minkowski. This is an example where the large c expansion extends beyond the regime of weak gravity captured by the post-Minkowski/Newtonian expansion. Note that up to the leading order written here, the Kerr solution coincides with that of Schwarzschild [5, 6]. Again there is an infinite series of further subleading corrections. Interestingly the Newtonian potential $\overset{(0)}{A}$ vanishes.

7.3.3. Strongly Massive, Strongly Rotating Kerr Metric

The previous expansions of the Kerr solution are free of odd powers of c and as such fall inside the treatment of [5,6]. If we however consider a regime where $\frac{Gm}{r} \approx c^2$ and $\frac{Jm}{r} \approx c$ we will see the odd powers appear already at leading order. To set up an expansion around this regime we keep $M = m/c^2$ and $a = J/mc$ fixed as $c \to \infty$. In this case one finds at leading order

$$\Psi = -\log\left(1 - \frac{2G_{\mathrm{N}}Mr}{\Sigma}\right), \quad C_{\hat{\mu}}dx^{\mu} = -\frac{2aG_{\mathrm{N}}Mr\sin^2\theta}{\Sigma - 2G_{\mathrm{N}}Mr}d\phi\,,$$
$$k^{\mu\nu}\partial_{\mu}\otimes\partial_{\nu} = \frac{1}{\Sigma - 2G_{\mathrm{N}}Mr}\left(\Delta\partial_r^2 + \partial_{\theta}^2 + (\csc^2\theta - \frac{a^2}{\Delta})\partial_{\phi}^2\right)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 + a^2 - 2G_{\mathrm{N}}Mr\,. \tag{7.10}$$

We see here an explicit example of the situation discussed in this paper, namely one where the whole leading order triplet of fields is non-trivial.

Finally, it should be pointed out that an analogous weakly massive, strongly rotating regime violates the extremality bound and as a result it is unphysical.

8. CONCLUSIONS

We carried out the leading order large c expansion to GR in the presence of odd powers equipped with the tools provided by Newton-Cartan gravity. Together with it, a set of diffeomorphism invariant leading order equations are found and a new set of variables are introduced via its symmetries. The leading odd power term is shown to be U(1) vector field that appears in the equations through its field strength. We have finally shown that the LO equations covers the stationary sector of GR entirely and hence describing a strong gravitational regime.

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APPENDIX A: ALL-COMPATIBLE NEWTON-CARTAN CONNECTION

For computational purposes of the Newton-Cartan split of GR, we worked with a connection that preserves the inverse fields $(\tau^\mu, h_{\mu\nu})$ in addition to the Newton-Cartan structure

One can introduce such a connection by imposing

$$\hat{\nabla}_\mu \tau_\nu = 0 \qquad \hat{\nabla}_\mu h^{\nu\lambda} = 0 \qquad \hat{\nabla}_\mu \tau^\nu = 0 \qquad \hat{\nabla}_\mu h_{\nu\lambda} = 0 \tag{A.1}$$

which we denote as $(\hat{\nabla}_\mu, \hat{\Gamma}^\lambda_{\mu\nu})$, indicating with a hat on top. We can solve this problem by writing

$$\hat{\Gamma}^\lambda_{\mu\nu} = \dot{\Gamma}^\lambda_{\mu\nu} + Q^\lambda_{\mu\nu} \tag{A.2}$$

it follows from $\hat{\nabla}_\mu \tau_\nu = 0$ that

$$Q^\lambda_{\mu\nu} \tau_\lambda = 0 \tag{A.3}$$

and thus

$$Q^\lambda_{\mu\nu} = h^{\lambda\rho} Q_{\mu\nu\rho} \qquad Q_{\mu\nu\rho}\tau^\rho = 0 \tag{A.4}$$

We can then further decompose

$$Q_{\mu\nu\lambda} = Q_{\mu\lambda}\tau_\nu + P_{\mu\nu\lambda} \qquad P_{\mu\nu\lambda}\tau^\lambda = P_{\mu\lambda\nu}\tau^\lambda = 0 \qquad Q_{\mu\lambda}\tau^\lambda = 0 \tag{A.5}$$

It then follows from $\hat{\nabla}_\mu h^{\nu\lambda} = 0$ that

$$P_{\mu\nu\lambda} = -P_{\mu\lambda\nu} \tag{A.6}$$

Imposing that $\hat{\nabla}_\mu \tau^\nu = 0$ gives

$$Q_{\mu\lambda} = -h_{\lambda\rho}\dot{\nabla}_\mu \tau^\rho \tag{A.7}$$

It then follows automatically that $\hat{\nabla}_\mu h_{\nu\lambda} = 0$. Noting that $\tau_\rho \hat{\nabla}_\mu \tau^\rho = 0$, in summary we find

$$\hat{\Gamma}^\lambda_{\mu\nu} = \dot{\Gamma}^\lambda_{\mu\nu} - \tau_\nu \dot{\nabla}_\mu \tau^\lambda + h^{\lambda\rho} P_{\mu\nu\rho} \qquad P_{\mu\nu\rho}\tau^\rho = 0 \qquad P_{\mu\nu\rho} = -P_{\mu\rho\nu} \tag{A.8}$$

The torsion of this connection is

$$\hat{T}^\lambda_{\mu\nu} = 2\tau^\lambda \tau_{[\mu} a_{\nu]} + 2\tau_{[\mu}\dot{\nabla}_{\nu]}\tau^\lambda + 2h^{\lambda\rho} P_{[\mu\nu]\rho} \tag{A.9}$$

Because we are working with this connection as a mere computational tool, for simplicity for this thesis $P_{\mu\nu\rho} = 0$ will be assumed.

Additionally, we would like to write our final equations in terms of the standard Newton-Cartan connection instead of the all-compatible one as it is the more natural choice of the two. For this reason we would like to demonstrate the consequences of such a change in anticipation of the final results. First it will shown in Appendix C that the spatial parts of the Ricci tensors of both connections are the same:

$$\hat{\Gamma}^\lambda_{\mu\nu} = \dot{\Gamma}^\lambda_{\mu\nu} - \tau_\nu \dot{\nabla}_\mu \tau^\lambda \quad \Longrightarrow \quad \hat{\mathbb{R}}_{\dot{\mu}\dot{\nu}} = \dot{\mathbb{R}}_{\dot{\mu}\dot{\nu}} \tag{A.10}$$

This result will prove highly useful when simplifying the leading order equations through symmetries. Another useful result for such a change in connection is

$$\hat{\Gamma}^\lambda_{\mu\dot{\nu}} = \dot{\Gamma}^\lambda_{\mu\dot{\nu}} \tag{A.11}$$

It then follows that the Newton-Cartan covariant derivative and the all-compatible NC covariant derivative of spatial tensors are the same

$$\begin{aligned}\hat{\nabla}_\mu V_{\dot{\nu}} &= \partial_\mu V_{\dot{\nu}} - \hat{\mathbb{\Gamma}}^{\dot{\lambda}}_{\mu\dot{\nu}} V_{\dot{\lambda}} \\ &= \partial_\mu V_{\dot{\nu}} - \dot{\mathbb{\Gamma}}^{\dot{\lambda}}_{\mu\dot{\nu}} V_{\dot{\lambda}} \\ &= \dot{\nabla}_\mu V_{\dot{\nu}} \end{aligned} \tag{A.12}$$

Finally it is a good point to mention that one does not really need to introduce such a all-compatible connection to Newton-Cartan split the Einstein equations. The only reason worked with it initially is that is just for computational simplicity. We chose to include it here as one would have to recompute all of the objects appearing in Appendix B in order to write them in terms of $\dot{\nabla}$ connection. As one notices from this section, both approaches would keep the LO equations the same when performing the large c expansion.

APPENDIX B: OBJECTS IN SPLIT OF EINSTEIN EQUATIONS

In this appendix we provide the objects that follows from the Newton-Cartan split of Einstein equations. The connection used in this is the one that is compatible with the Newton-Cartan structure $(\tau_\mu, h^{\mu\nu})$ as well as the inverses $(\tau^\mu, h_{\mu\nu})$ introduced in Appendix A as $(\hat{\nabla}_\mu, \hat{\Gamma}^\lambda_{\mu\nu})$.

The S coefficients are:

$$\begin{aligned}
S &= \frac{1}{2}B\hat{\nabla}A + B^{\dot{\rho}}\left(\hat{\nabla}A_{\dot{\rho}} - \frac{1}{2}\hat{\nabla}_{\dot{\rho}}A\right) \\
&\quad + B^{\dot{\rho}}\left(A\hat{T}_{\dot{\rho}} + A_{\dot{\sigma}}\hat{T}^{\dot{\sigma}}_{\dot{\rho}}\right)
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
S_{\dot{\mu}} &= \frac{1}{2}B\hat{\nabla}_{\dot{\mu}}A + \frac{1}{2}B^{\dot{\rho}}\left(\hat{\nabla}_{\dot{\mu}}A_{\dot{\rho}} - \hat{\nabla}_{\dot{\rho}}A_{\dot{\mu}} + \hat{\nabla}B_{\dot{\mu}\dot{\rho}}\right) \\
&\quad + \frac{1}{2}B^{\dot{\rho}}\left(\hat{T}_{\dot{\rho}}A_{\dot{\mu}} + B_{\dot{\mu}\dot{\sigma}}\hat{T}^{\dot{\sigma}}_{\dot{\rho}}\right) - \frac{1}{2}B\left(A_{\dot{\sigma}}\hat{T}^{\dot{\sigma}}_{\dot{\mu}} + A\hat{T}_{\dot{\mu}}\right)
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
S_{\dot{\mu}\dot{\nu}} &= B\left(\hat{\nabla}_{(\dot{\mu}}A_{\dot{\nu})} - \frac{1}{2}\hat{\nabla}B_{\dot{\mu}\dot{\nu}}\right) + B^{\dot{\rho}}\left(\hat{\nabla}_{(\dot{\mu}}B_{\dot{\nu})\dot{\rho}} - \frac{1}{2}\hat{\nabla}_{\dot{\rho}}B_{\dot{\mu}\dot{\nu}}\right) \\
&\quad - BA_{(\dot{\mu}}\hat{T}_{\dot{\nu})} - B\hat{T}^{\dot{\sigma}}_{(\dot{\nu}}B_{\dot{\mu})\dot{\sigma}}
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
S^{\dot{\lambda}} &= \frac{1}{2}B^{\dot{\lambda}}\hat{\nabla}A + A^{\dot{\lambda}\dot{\rho}}\left(\hat{\nabla}A_{\dot{\rho}} - \frac{1}{2}\hat{\nabla}_{\dot{\rho}}A\right) \\
&\quad A^{\dot{\lambda}\dot{\rho}}\left(A\hat{T}_{\dot{\rho}} + A_{\dot{\sigma}}\hat{T}^{\dot{\sigma}}_{\dot{\rho}}\right)
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
S^{\dot{\lambda}}_{\dot{\mu}} &= \frac{1}{2}B^{\dot{\lambda}}\hat{\nabla}_{\dot{\mu}}A + \frac{1}{2}A^{\dot{\lambda}\dot{\rho}}\left(\hat{\nabla}_{\dot{\mu}}A_{\dot{\rho}} - \hat{\nabla}_{\dot{\rho}}A_{\dot{\mu}} + \nabla B_{\dot{\mu}\dot{\rho}}\right) \\
&\quad + \frac{1}{2}A^{\dot{\lambda}\dot{\rho}}\left(A_{\dot{\mu}}\hat{T}_{\dot{\rho}} + B_{\dot{\mu}\dot{\sigma}}\hat{T}^{\dot{\sigma}}_{\dot{\rho}}\right) - \frac{1}{2}B^{\dot{\lambda}}\left(A_{\dot{\sigma}}\hat{T}^{\dot{\sigma}}_{\dot{\mu}} + A\hat{T}_{\dot{\mu}}\right)
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
S^{\dot{\lambda}}_{\dot{\mu}\dot{\nu}} &= B^{\dot{\lambda}}\left(\hat{\nabla}_{(\dot{\mu}}A_{\dot{\nu})} - \frac{1}{2}\hat{\nabla}B_{\dot{\mu}\dot{\nu}}\right) + A^{\dot{\lambda}\dot{\rho}}\left(\hat{\nabla}_{(\dot{\mu}}B_{\dot{\nu})\dot{\rho}} - \frac{1}{2}\hat{\nabla}_{\dot{\rho}}B_{\dot{\mu}\dot{\nu}}\right) \\
&\quad - B^{\dot{\lambda}}A_{(\dot{\mu}}\hat{T}_{\dot{\nu})} - B^{\dot{\lambda}}\hat{T}^{\dot{\sigma}}_{(\dot{\nu}}B_{\dot{\mu})\dot{\sigma}}
\end{aligned} \tag{B.6}$$

Where B coefficients can be written in terms of A through the relations in Equation (5.13).

Additionally, plugging in the decomposed metric and connection fields into decomposed curvatures and conservation laws results in:

$$\begin{aligned} R \;=\; & \hat{\mathbb{R}} - \frac{1}{2}\hat{\boldsymbol{\nabla}}\hat{T}^{\dot\rho}_{\dot\rho} - \frac{1}{4}\hat{T}^{\dot\lambda}_{\dot\rho}\hat{T}^{\dot\rho}_{\dot\lambda} \\ & +\hat{\boldsymbol{\nabla}}_{\dot\rho}S^{\dot\rho} - \hat{\boldsymbol{\nabla}}S^{\dot\rho}_{\dot\rho} + \frac{1}{2}S\hat{T}^{\dot\rho}_{\dot\rho} - \frac{3}{2}\hat{T}_{\dot\rho}S^{\dot\rho} - \hat{T}^{\dot\lambda}_{\dot\rho}S^{\dot\rho}_{\dot\lambda} \\ & +S^{\dot\rho}_{\dot\rho}S - S_{\dot\rho}S^{\dot\rho} - S^{\dot\lambda}_{\dot\rho}S^{\dot\rho}_{\dot\lambda} + S^{\dot\rho}_{\dot\rho\dot\lambda}S^{\dot\lambda} \end{aligned} \tag{B.7}$$

$$\begin{aligned} R_{\dot\mu} \;=\; & \hat{\mathbb{R}}_{\dot\mu} - \frac{1}{4}\hat{\boldsymbol{\nabla}}_{\dot\mu}\hat{T}^{\dot\rho}_{\dot\rho} + \frac{1}{4}\hat{\boldsymbol{\nabla}}\hat{T}_{\dot\mu} + \frac{1}{4}\hat{T}^{\dot\rho}_{\dot\mu}\hat{T}_{\dot\rho} \\ & +\frac{1}{2}\hat{\boldsymbol{\nabla}}S_{\dot\mu} + \hat{\boldsymbol{\nabla}}_{\dot\rho}S^{\dot\rho}_{\dot\mu} - \frac{1}{2}\hat{\boldsymbol{\nabla}}_{\dot\mu}(S + S^{\dot\rho}_{\dot\rho}) - \frac{1}{2}\hat{\boldsymbol{\nabla}}S^{\dot\rho}_{\dot\rho\dot\mu} \\ & +\frac{1}{2}\hat{T}_{\dot\mu}S + \frac{1}{2}\hat{T}^{\dot\rho}_{\dot\mu}S_{\dot\rho} - S^{\dot\rho}_{\dot\mu}\hat{T}_{\dot\rho} - \frac{1}{2}S^{\dot\rho}_{\dot\mu\dot\lambda}\hat{T}^{\dot\lambda}_{\dot\rho} + \frac{1}{2}S_{\dot\mu}\hat{T}^{\dot\rho}_{\dot\rho} \\ & +S_{\dot\mu}S^{\dot\rho}_{\dot\rho} + S^{\dot\lambda}_{\dot\mu}S^{\dot\rho}_{\dot\rho\dot\lambda} - S_{\dot\mu\dot\rho}S^{\dot\rho} - S^{\dot\lambda}_{\dot\mu\dot\rho}S^{\dot\rho}_{\dot\lambda} \end{aligned} \tag{B.8}$$

$$\begin{aligned} R_{\dot\mu\dot\nu} \;=\; & \hat{\mathbb{R}}_{(\dot\mu\dot\nu)} + \frac{1}{2}\hat{\boldsymbol{\nabla}}_{(\dot\mu}\hat{T}_{\dot\nu)} - \frac{1}{4}\hat{T}_{\dot\mu}\hat{T}_{\dot\nu} \\ & +\hat{\boldsymbol{\nabla}}S_{\dot\mu\dot\nu} + \hat{\boldsymbol{\nabla}}_{\dot\rho}S^{\dot\rho}_{\dot\mu\dot\nu} - \hat{\boldsymbol{\nabla}}_{(\dot\mu}S_{\dot\nu)} - \hat{\boldsymbol{\nabla}}_{(\dot\mu}S^{\dot\rho}_{\dot\nu)\dot\rho} \\ & +\hat{T}_{(\dot\mu}S_{\dot\nu)} + \hat{T}^{\dot\lambda}_{(\dot\mu}S_{\dot\nu)\dot\lambda} - \frac{1}{2}S^{\dot\rho}_{\dot\mu\dot\nu}\hat{T}_{\dot\rho} + \frac{1}{2}S_{\dot\mu\dot\nu}\hat{T}^{\dot\rho}_{\dot\rho} \\ & +(S + S^{\dot\rho}_{\dot\rho})S_{\dot\mu\dot\nu} + S^{\dot\lambda}_{\dot\mu\dot\nu}(S_{\dot\lambda} + S^{\dot\rho}_{\dot\rho\dot\lambda}) - S_{\dot\mu}S_{\dot\nu} - 2S^{\dot\rho}_{(\dot\mu}S_{\dot\nu)\dot\rho} - S^{\dot\lambda}_{\dot\rho\dot\mu}S^{\dot\rho}_{\dot\lambda\dot\nu} \end{aligned} \tag{B.9}$$

$$\begin{aligned} \mathcal{C} \;=\; & B\hat{\boldsymbol{\nabla}}\mathcal{T} + B^{\dot\sigma}\hat{\boldsymbol{\nabla}}_{\dot\sigma}\mathcal{T} + B^{\dot\sigma}\hat{\boldsymbol{\nabla}}\mathcal{T}_{\dot\sigma} + A^{\dot\rho\dot\sigma}\hat{\boldsymbol{\nabla}}_{\dot\rho}\mathcal{T}_{\dot\sigma} \\ & -\frac{1}{2}(\mathcal{T}\hat{\boldsymbol{\nabla}}B + B\hat{\boldsymbol{\nabla}}\mathcal{T} + 2\mathcal{T}_{\dot\rho}\hat{\boldsymbol{\nabla}}B^{\dot\rho} + 2B^{\dot\rho}\hat{\boldsymbol{\nabla}}\mathcal{T}_{\dot\rho} + \mathcal{T}_{\dot\rho\dot\lambda}\hat{\boldsymbol{\nabla}}A^{\dot\rho\dot\lambda} + A^{\dot\rho\dot\lambda}\hat{\boldsymbol{\nabla}}\mathcal{T}_{\dot\rho\dot\lambda}) \\ & -\Big((BS + 2B^{\dot\rho}S_{\dot\rho} + A^{\dot\rho\dot\sigma}S_{\dot\rho\dot\sigma})\mathcal{T} + (BS^{\dot\lambda} + 2B^{\dot\rho}S^{\dot\lambda}_{\dot\rho} + A^{\dot\rho\dot\sigma}S^{\dot\lambda}_{\dot\rho\dot\sigma})\mathcal{T}_{\dot\lambda}\Big) \\ & -((BS + B^{\dot\rho}S_{\dot\rho})\mathcal{T} + (BS^{\dot\lambda} + B^{\dot\rho}S^{\dot\lambda}_{\dot\rho} + SB^{\dot\lambda} + A^{\dot\rho\dot\lambda}S_{\dot\rho})\mathcal{T}_{\dot\lambda}) \\ & -(B^{\dot\sigma}S^{\dot\lambda} + A^{\dot\rho\dot\sigma}S^{\dot\lambda}_{\dot\rho})\mathcal{T}_{\dot\sigma\dot\lambda} - \frac{1}{2}(B^{\dot\rho}T_{\dot\rho}\mathcal{T} + (B^{\dot\rho}T^{\dot\lambda}_{\rho} + A^{\dot\rho\dot\lambda}T_{\dot\rho})\mathcal{T}_{\dot\lambda} + A^{\dot\rho\dot\sigma}T^{\dot\lambda}_{\dot\rho}\mathcal{T}_{\dot\sigma\dot\lambda}) \end{aligned} \tag{B.10}$$

$$\begin{aligned} \mathcal{C}_{\dot\mu} \;=\; & B\hat{\boldsymbol{\nabla}}\mathcal{T}_{\dot\mu} + B^{\dot\sigma}\hat{\boldsymbol{\nabla}}_{\dot\sigma}\mathcal{T}_{\dot\mu} + B^{\dot\sigma}\hat{\boldsymbol{\nabla}}\mathcal{T}_{\dot\mu\dot\sigma} + A^{\dot\rho\dot\sigma}\hat{\boldsymbol{\nabla}}_{\dot\rho}\mathcal{T}_{\dot\sigma\dot\mu} \\ & -\frac{1}{2}(\mathcal{T}\hat{\boldsymbol{\nabla}}_{\dot\mu}B + B\hat{\boldsymbol{\nabla}}_{\dot\mu}\mathcal{T} + 2\mathcal{T}_{\dot\rho}\hat{\boldsymbol{\nabla}}_{\dot\mu}B^{\dot\rho} + 2B^{\dot\rho}\hat{\boldsymbol{\nabla}}_{\dot\mu}\mathcal{T}_{\dot\rho} + \mathcal{T}_{\dot\rho\dot\lambda}\hat{\boldsymbol{\nabla}}_{\dot\mu}A^{\dot\rho\dot\lambda} + A^{\dot\rho\dot\lambda}\hat{\boldsymbol{\nabla}}_{\dot\mu}\mathcal{T}_{\dot\rho\dot\lambda}) \\ & -((BS + 2B^{\dot\rho}S_{\dot\rho} + A^{\dot\rho\dot\sigma}S_{\dot\rho\dot\sigma})\mathcal{T}_{\dot\mu} + (BS^{\dot\lambda} + 2B^{\dot\rho}S^{\dot\lambda}_{\dot\rho} + A^{\dot\rho\dot\sigma}S^{\dot\lambda}_{\dot\rho\dot\sigma})\mathcal{T}_{\dot\mu\dot\lambda}) \\ & -((BS_{\dot\mu} + B^{\dot\rho}S_{\dot\mu\dot\rho})\mathcal{T} + (BS^{\dot\lambda}_{\dot\mu} + B^{\dot\lambda}S_{\dot\mu} + A^{\dot\lambda\dot\rho}S_{\dot\mu\dot\rho} + B^{\dot\rho}S^{\dot\lambda}_{\dot\rho\dot\mu})\mathcal{T}_{\dot\lambda}) \\ & -(B^{\dot\sigma}S^{\dot\lambda}_{\dot\mu} + A^{\dot\rho\dot\sigma}S^{\dot\lambda}_{\dot\rho\dot\mu})\mathcal{T}_{\dot\sigma\dot\lambda} + \frac{1}{2}(\hat{T}_{\dot\mu}B\mathcal{T} + \hat{T}_{\dot\mu}B^{\dot\lambda}\mathcal{T}_{\dot\lambda} + \hat{T}^{\dot\lambda}_{\dot\mu}B\mathcal{T}_{\dot\lambda} + \hat{T}^{\dot\lambda}_{\dot\mu}B^{\dot\sigma}\mathcal{T}_{\dot\sigma\dot\lambda}) \end{aligned} \tag{B.11}$$

APPENDIX C: SPATIAL RICCI TENSOR INVARIANCE

The main point of this appendix is to show that one can work with both an all-compatible or a class of boost invariant connections, do not change the final results.

Let us consider the Ricci tensors of Newton-Cartan connection and its arbitrary extension:

$$\dot{\mathbb{R}}_{\mu\nu} = \partial_\lambda \dot{\Gamma}^\lambda_{\nu\mu} - \partial_\nu \dot{\Gamma}^\lambda_{\lambda\mu} + \dot{\Gamma}^\lambda_{\lambda\sigma}\dot{\Gamma}^\sigma_{\nu\mu} - \dot{\Gamma}^\lambda_{\nu\sigma}\dot{\Gamma}^\sigma_{\lambda\mu} \tag{C.1}$$

$$\tilde{\mathbb{R}}_{\mu\nu} = \partial_\lambda \tilde{\Gamma}^\lambda_{\nu\mu} - \partial_\nu \tilde{\Gamma}^\lambda_{\lambda\mu} + \tilde{\Gamma}^\lambda_{\lambda\sigma}\tilde{\Gamma}^\sigma_{\nu\mu} - \tilde{\Gamma}^\lambda_{\nu\sigma}\tilde{\Gamma}^\sigma_{\lambda\mu} \tag{C.2}$$

such that $\tilde{\Gamma}^\lambda_{\mu\nu} = \dot{\Gamma}^\lambda_{\mu\nu} + L^\lambda{}_{\mu\nu}$. Inserting in shows that the two Ricci tensors can be related as

$$\tilde{\mathbb{R}}_{\mu\nu} = \dot{\mathbb{R}}_{\mu\nu} + \dot{\nabla}_\lambda L^\lambda{}_{\nu\mu} - \dot{\nabla}_\nu L^\lambda{}_{\lambda\mu} + L^\lambda{}_{\lambda\sigma} L^\sigma{}_{\nu\mu} - \dot{T}^\sigma_{\nu\lambda} L^\lambda{}_{\sigma\mu} - L^\lambda{}_{\nu\sigma} L^\sigma{}_{\lambda\mu} \tag{C.3}$$

After spatial projections on both free indices, one finds that

$$h^{\nu\lambda} h^{\mu\rho} L^\lambda{}_{\mu\nu} = 0 \qquad \text{and} \qquad \tau_\lambda L^\lambda{}_{\mu\nu} = 0 \tag{C.4}$$

are the conditions on the extensions that leaves the spatial Ricci tensor invariant. This also implies that the form of the extension is restricted as

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \dot{\Gamma}^\lambda_{\mu\nu} + L^{\dot{\lambda}} \tau_\mu \tau_\nu + \tau_\mu L^{\dot{\lambda}}{}_{\dot{\nu}} + \tau_\nu L^{\dot{\lambda}}{}_{\dot{\mu}} \tag{C.5}$$

The boost invariant connection in [16]

$$L^\lambda{}_{\mu\nu} = h^{\lambda\rho} \tau_{(\mu} m_{\nu)\rho} + h^{\lambda\rho} \left(m_\mu \partial_{[\nu} \tau_{\rho]} + m_\nu \partial_{[\mu} \tau_{\rho]} - m_\rho \partial_{[\mu} \tau_{\nu]} \right) \tag{C.6}$$

satsfies both conditions, leaving the Ricci tensor invariant. Additionally, letting

$$L^{\lambda}{}_{\mu\nu} = -\tau_{\nu}\dot{\nabla}_{\mu}\tau^{\lambda} \tag{C.7}$$

identifies $\tilde{\Gamma}^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu}$ for which both conditions are satisfied immediately.

APPENDIX D: TWISTLESS TORSION TECHNICALITIES

A first technical result is that for any 2-form $Y_{\mu\nu}$ one has the following equivalences

$$Y_{\dot{\mu}\dot{\nu}} = 0 \qquad \Leftrightarrow \qquad \tau_{[\rho}Y_{\mu\nu]} = 0 \qquad \Leftrightarrow \qquad Y_{\mu\nu} = 2\tau_{[\mu}Y_{\dot{\nu}]} \tag{D.1}$$

The equivalence of the very left and very right follow directly from a decomposition as in Section 4.1:

$$Y_{\mu\nu} = 2\tau_{[\mu}Y_{\dot{\nu}]} + Y_{\dot{\mu}\dot{\nu}}\,, \qquad Y_{\dot{\nu}} = \tau^{\rho}h^{\sigma}_{\nu}Y_{\rho\sigma} \quad Y_{\dot{\mu}\dot{\nu}} = h^{\rho}_{\mu}h^{\sigma}_{\nu}Y_{\rho\sigma} \tag{D.2}$$

Note that the middle equality in Equation (D.1) is a direct consequence of the equality on the far right. Furthermore observe that via the decomposition above

$$\tau_{[\rho}Y_{\mu\nu]} = 0 \quad \Rightarrow \quad \tau_{[\rho}Y_{\dot{\mu}\dot{\nu}]} = 0 \quad \Rightarrow \quad \tau^{\rho}\tau_{[\rho}Y_{\dot{\mu}\dot{\nu}]} = 0 \quad \Rightarrow \quad Y_{\dot{\mu}\dot{\nu}} = 0 \tag{D.3}$$

This then establishes Equation (D.1), which by letting $Y_{\mu\nu} = \partial_{[\dot{\mu}}\tau_{\dot{\nu}]}$ becomes Equation (4.21).

Furthermore, provided that

$$\partial_{[\mu}\tau_{\nu]} = \tau_{[\mu}a_{\nu]}\,, \quad a_{\mu} = a_{\dot{\mu}} \tag{D.4}$$

one can then additionally observe that

$$0 = \partial_{[\rho}\partial_{\mu}\tau_{\nu]} = -\tau_{[\rho}\partial_{\mu}a_{\nu]} \tag{D.5}$$

Combining this with Equation (D.1) by now letting $Y_{\mu\nu} = \partial_{[\mu}a_{\nu]}$ then leads to Equation (4.23), which we reproduce here:

$$\partial_{[\dot{\mu}}a_{\dot{\nu}]} = 0 \tag{D.6}$$

It follows that this is satisfied if $a_\mu = \partial_{\dot{\mu}}\psi$:

$$\partial_{[\dot{\mu}}a_{\dot{\nu}]} = h^{\rho}_{[\mu}h^{\sigma}_{\nu]}\partial_\rho\partial_{\dot{\sigma}}\psi = \partial_{[\dot{\mu}}\tau_{\dot{\nu}]}\tau^{\lambda}\partial_\lambda\psi = 0 \tag{D.7}$$

APPENDIX E: GAUGE THEORY FORMULATIONS

The content of this Appendix is somewhat independent of the main text, it served however as an important motivation for the choice of formalism used there.

In the rest of the chapter we will follow the objects introduced in the sections first to formulate general relativity as a Poincaré gauge theory and then introduce the space and time split form – slightly different than the previous one –. Using an expansion ansatz on the split form, further in this chapter we point out that the link between diffeomorphims and local translations in the nonrelativistic case is more degenerate than in the relativistic case. In the relativistic case this degeneracy can be lifted by expressing the Einstein equations in terms of curvatures only, this is not the case in the nonrelativistic setting.

E.1. Lie Algebras and Gauge Theories

Here we will provide the formulation of gauge theories, to do so we will mainly follow [21].

To introduce the concept of gauge theory and a gauge field, we begin with a Lie algebra that is defined to be the set of generators $t_{\hat{A}}$ that obey the following:

$$[t_{\hat{A}}, t_{\hat{B}}] = f_{\hat{A}\hat{B}}{}^{\hat{C}} t_{\hat{C}} \tag{E.1}$$

$$[[t_{\hat{A}}, t_{\hat{B}}], t_{\hat{C}}] + [[t_{\hat{C}}, t_{\hat{A}}], t_{\hat{B}}] + [[t_{\hat{B}}, t_{\hat{C}}], t_{\hat{A}}] = 0 \tag{E.2}$$

The capital Latin indices with a hat on top indicate that the index can take an arbitrary value. As an example, it will be shown in the case of Poincaré algebra that the hatted index will be split into two separate indices $\hat{A} = \{A, AB\}$ where A indicating indices relating to local translations and AB indicating indices associated to the local Lorentz transformations.

Furthermore, we would like to point out that one can introduce a Lie algebra valued n-form that can be expressed as an element in a product basis of Lie algebra generators and space-time tangent space. Examples of these are borrowed from the gauge theory that is to be discussed further below:

$$\Lambda = \Lambda^{\hat{A}} t_{\hat{A}} \qquad \mathcal{A}_\mu = \mathcal{A}_\mu{}^{\hat{A}} t_{\hat{A}} \qquad \mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}{}^{\hat{A}} t_{\hat{A}} \tag{E.3}$$

The appearing objects are Lie algebra valued 0-1-2-forms respectively. Quite often when working with these objects we will suppress the space-time indices and identify them with their Lie algebra indices. This should not cause any confusion for the reader however as which n-form we are working with will always be indicated within the context.

To demonstrate briefly how gauge formalisms work, we first define a gauge field $\mathcal{A}$ as a Lie algebra valued 1-form that infinitesimally transforms as

$$\delta_{\text{ad}}\mathcal{A} = d\Lambda + [\mathcal{A}, \Lambda] \tag{E.4}$$

under the adjoint action of the Lie algebra with a 0-form local parameter Λ. The name adjoint action refers to the transformations of the Lie algebra fields under the algebra itself. One can then define gauge curvature 2-form of this gauge field as

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \tag{E.5}$$

which under the adjoint action will transform covariantly as

$$\delta_{\text{ad}}\mathcal{F} = [\mathcal{F}, \Lambda] \tag{E.6}$$

An important result can be obtained applying Equation (2.19) to a gauge field:

$$L_\Xi \mathcal{A} = \mathrm{i}_\Xi d\mathcal{A} + d(\mathrm{i}_\Xi \mathcal{A}) \tag{E.7}$$

It follows that

$$L_{\Xi}\mathcal{A} = \delta_{\mathrm{ad}}\mathcal{A} + \mathrm{i}_{\Xi}\mathcal{F} \quad \text{for } \Lambda = \mathrm{i}_{\Xi}\mathcal{A} \tag{E.8}$$

In a sense one can replace the diffeomorphisms with the adjoint transformations of the algebra together with an extra curvature contribution.

E.2. Poincaré Algebra Formulation

To begin with, one can write metric in an orthonormal basis and introduce the so-called vielbein E^A_μ, as

$$g_{\mu\nu} = E^A_\mu E^B_\nu \eta_{AB} \tag{E.9}$$

with respect to a Minkowski metric η_{AB}. We will work in conventions where the capital Latin letters indicate the so-called Lorentz indices running as $A = (0, 1, 2, 3)$. The naming of the indices comes from the fact that the components Minkowski metric is by definition invariant under the local Lorentz transformations as:

$$\eta_{AB} = L^C{}_A L^D{}_B \eta_{CD} \tag{E.10}$$

Infinitesimally – i.e. $L^A{}_B = \delta^A_B - \Lambda^A{}_B$ – the relation becomes

$$\delta_\Lambda \eta_{AB} = \eta_{AC}\Lambda^C{}_B + \eta_{CB}\Lambda^C{}_A = 0 \tag{E.11}$$

We will use the Minkowski metric to raise and lower the Lorentz indices. As a result, it can be shown that the infinitesimal Lorentz parameter is antisymmetric

$$\Lambda_{AB} = -\Lambda_{BA} \tag{E.12}$$

It can then shown that the vielbein transforms under Lorentz transformations as:

$$\delta_\Lambda E^A_\mu = \Lambda^A{}_B E^B_\mu \tag{E.13}$$

One can introduce a spin connection $\Omega_\mu{}^{AB}$ by imposing the so-called vielbein postulate

$$D_\mu E^A_\nu = \partial_\mu E^A_\nu - \tilde{\Gamma}^\lambda_{\mu\nu} E^A_\lambda + \Omega_\mu{}^A{}_B E^B_\nu = 0 \tag{E.14}$$

It follows that:

$$\begin{aligned} \tilde{\Gamma}^\lambda_{\mu\nu} &= E^\lambda_A \partial_\mu E^A_\nu + \Omega_\mu{}^A{}_B E^B_\nu E^\lambda_A & \text{(E.15)} \\ \Omega_\mu{}^{AB} &= \eta^{CB} E^\nu_C \tilde{\Gamma}^\lambda_{\mu\nu} E^A_\lambda - \eta^{CB} E^\nu_C \partial_\mu E^A_\nu & \text{(E.16)} \end{aligned}$$

Note that this relation makes use of the unique inverse vielbeins on both indices, defined as:

$$E^\nu_A E^A_\mu = \delta^\nu_\mu \qquad E^B_\mu E^\mu_A = \delta^A_B \tag{E.17}$$

Also the connection in Equation (E.15) admits a torsion tensor

$$2\tilde{\Gamma}^\lambda_{[\mu\nu]} = 2E^\lambda_A \partial_{[\mu} E^A_{\nu]} + 2\Omega_{[\mu}{}^{AB} E^C_{\nu]} E^\lambda_A \eta_{BC} \tag{E.18}$$

It can be found that since the space-time connection is invariant under infinitesimal Lorentz transformations by definition, the spin connection transforms as

$$\delta_\Lambda \Omega_\mu{}^{AB} = d\Lambda^{AB} + \Omega_\mu{}^{CB} \Lambda_C{}^A + \Omega_\mu{}^{AC} \Lambda_C{}^B \tag{E.19}$$

Thus, in this formulation the two 1-forms E^A and Ω^{AB} fields transform as tensors under diffeomorphisms Ξ^μ and local Lorentz transformations Λ^{AB} given collectively as:

$$\delta E^A = L_\Xi E^A + E^B \Lambda_B{}^A \qquad \delta\Omega^{AB} = L_\Xi \Omega^{AB} + d\Lambda^{AB} + \Omega^{CB}\Lambda_C{}^A + \Omega^{AC}\Lambda_C{}^B \tag{E.20}$$

The advantage of using such variables is apparent when one considers a Poincaré gauge theory formalism. In simple terms, the vielbein equipped with a single Lorentz index can be identified with the gauge field of translational symmetries, the spin connection equipped with the pair of antisymmetric Lorentz indices can be identified with the gauge field of rotational symmetries and the space-time indices are identified with differential forms that transform as diffeomorphisms. To that end one can re-express the transformations in Equation (E.20) in terms of local Poincaré transformations via

$$\delta E^A = \delta_{\mathrm{ad}} E^A + \mathrm{i}_\Xi \mathcal{F}^A \qquad \delta \Omega^{AB} = \delta_{\mathrm{ad}} \Omega^{AB} + \mathrm{i}_\Xi \mathcal{F}^{AB} \tag{E.21}$$

where Cartan's formula for the Lie derivative of a differential form Equation (2.19) is used. Since the Lie derivative generates an infinitesimal diffeomorphism the above equality can be used to translate an adjoint transformation into a diffeomorphism at the cost of an extra curvature contribution. Here the adjoint transformation is computed through Equation (E.4) where both $\mathcal{A}$ and Λ are Poincaré valued:

$$\mathcal{A} = E^A P_A + \frac{1}{2}\Omega^{AB} J_{AB} \qquad \Lambda = Z^A P_A + \frac{1}{2}\bar{\Lambda}^{AB} J_{AB} \tag{E.22}$$

while noting that

$$Z^A = \mathrm{i}_\Xi E^A \qquad \bar{\Lambda}^{AB} = \mathrm{i}_\Xi \Omega^{AB} + \Lambda^{AB} \tag{E.23}$$

In these definitions the objects P_A and J_{AB} are introduced as the generators of local translations and Lorentz transformations respectively. They are subjected to the defining Poincaré commutation relations:

$$[J_{AB}, J_{CD}] = \eta_{AD} J_{BC} - \eta_{BD} J_{AC} - \eta_{AC} J_{BD} + \eta_{BC} J_{AD} \tag{E.24}$$

$$[J_{AB}, P_C] = P_A \eta_{BC} - P_B \eta_{AC} \tag{E.25}$$

Armed with this, the curvature 2-form can be computed via Equation (E.5) and it can be decomposed as Poincaré valued

$$\mathcal{F} = \mathcal{F}^A P_A + \frac{1}{2}\mathcal{F}^{AB} J_{AB} \tag{E.26}$$

From the definitions above the results are listed as

$$\delta_{\text{ad}} E^A = dZ^A + \Omega^{AB} Z_B + E^B \bar{\Lambda}_B{}^A \tag{E.27}$$

$$\delta_{\text{ad}} \Omega^{AB} = d\bar{\Lambda}^{AB} + \Omega^{CB} \bar{\Lambda}_C{}^A + \Omega^{AC} \bar{\Lambda}_C{}^B \tag{E.28}$$

and

$$\mathcal{F}^A = dE^A + \Omega^A{}_B \wedge E^B \tag{E.29}$$

$$\mathcal{F}^{AB} = d\Omega^{AB} + \Omega^{AC} \wedge \Omega_C{}^B \tag{E.30}$$

Observe that the translational curvature of Equation (E.29) is exactly equal to the torsion in Equation (E.18):

$$E^\lambda_A \mathcal{F}_{\mu\nu}{}^A = 2\Gamma^\lambda_{[\mu\nu]} \tag{E.31}$$

Einstein equations then can be reproduced by first one imposing the torsion constraint:

$$\mathcal{F}^A = 0 \tag{E.32}$$

One can solve this by retrieving back the spacetime indices and through the following combination

$$\eta_{AB} E^B_\rho \mathcal{F}_{\mu\nu}{}^A + \eta_{AB} E^B_\nu \mathcal{F}_{\rho\mu}{}^A - \eta_{AB} E^B_\mu \mathcal{F}_{\nu\rho}{}^A = 0 \tag{E.33}$$

gives the unique solution to the spin connection in terms of the vielbein and its inverses:

$$\Omega^{AB} = -\eta^{AC} \mathrm{i}_{E_C} dE^B + \eta^{BC} \mathrm{i}_{E_C} dE^A - \eta^{AC} \eta^{BD} \eta_{KL} E^K \mathrm{i}_{E_C} \mathrm{i}_{E_D} dE^L \tag{E.34}$$

Inserting this relation back in Equation (E.15), one can first identify the Levi-Civita connection $\Gamma^{\lambda}_{\mu\nu}$ with the arbitrary connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ and then connect the second curvature 2-form to the Riemann tensor:

$$R^{\lambda}{}_{\rho\mu\nu} = E^{\lambda}_A E^B_{\rho} \mathcal{F}_{\mu\nu}{}^A{}_B \tag{E.35}$$

and then Ricci tensor:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = E^{\lambda}_A E^B_{\mu} \mathcal{F}_{\lambda\nu}{}^A{}_B \tag{E.36}$$

E.3. Poincaré Algebra Formulation in Space and Time Split Form

To separate space and time, we begin by splitting up the Lorentz index as $A = (0, a)$ with $a = (1, 2, 3)$. Then the metric becomes

$$g_{\mu\nu} = -E^0_{\mu} E^0_{\nu} + E^a_{\mu} E^b_{\nu} \delta_{ab} \tag{E.37}$$

and the invertibility relations become

$$E^{\nu}_0 E^0_{\mu} + E^{\nu}_a E^a_{\mu} = \delta^{\nu}_{\mu} \qquad E^0_{\mu} E^{\mu}_0 = 1 \qquad E^b_{\mu} E^{\mu}_a = \delta^b_a \tag{E.38}$$

By introducing the speed of light parameter $\epsilon = c^{-1}$ with the following redefinitions

$$T_{\mu} = \epsilon E^0 \qquad \Lambda^a = \epsilon^{-1} \Lambda^{a0} \qquad \Omega^a = \epsilon^{-1} \Omega^{a0} \tag{E.39}$$

$$H = \epsilon^{-1} P_0 \qquad G_a = \epsilon J_{a0} \tag{E.40}$$

one can identify Λ^a as the Lorentz boost parameter and Λ^{ab} as the rotation parameter. The invertibility relations and the metric are:

$$T^\nu T_\mu + E^\nu_a E^a_\mu = \delta^\nu_\mu \qquad T_\mu T^\mu = 1 \qquad E^b_\mu E^\mu_a = \delta^b_a \qquad T^\mu E^a_\mu = 0 \qquad T_\mu E^\mu_a = 0 \quad \text{(E.41)}$$

$$g_{\mu\nu} = -\epsilon^{-2} T_\mu T_\nu + E^a_\mu E^b_\nu \delta_{ab} \quad \text{(E.42)}$$

The vielbein postulates become:

$$\begin{aligned} D_\mu T_\nu &= \partial_\mu T_\nu - \tilde{\Gamma}^\lambda_{\mu\nu} T_\lambda - \epsilon^2 \Omega_{\mu a} E^a_\nu = 0 && \text{(E.43)} \\ D_\mu E^a_\nu &= \partial_\mu E^a_\nu - \tilde{\Gamma}^\lambda_{\mu\nu} E^a_\lambda + \Omega^a_{\mu b} E^b_\nu - \Omega_\mu{}^a T_\nu = 0 && \text{(E.44)} \end{aligned}$$

and relation to the space-time connection is then

$$\tilde{\Gamma}^\lambda_{\mu\nu} = T^\lambda \partial_\mu T_\nu + E^\lambda_a \partial_\mu E^a_\nu + \Omega_\mu{}^a{}_b E^b_\nu E^\lambda_a - \Omega_\mu{}^a T_\nu E^\lambda_a - \epsilon^2 \Omega_{\mu a} E^a_\nu T^\lambda \quad \text{(E.45)}$$

and the field transformations of Equation (E.20) become:

$$\begin{aligned} \delta T &= L_\Xi T + \epsilon^2 E^a \Lambda_a && \text{(E.46)} \\ \delta E^a &= L_\Xi E^a + E^b \Lambda_b{}^a + T \Lambda^a && \text{(E.47)} \\ \delta \Omega^a &= L_\Xi \Omega^a + d\Lambda^a + \Omega^b \Lambda_b{}^a + \Omega^{ab} \Lambda_b && \text{(E.48)} \\ \delta \Omega^{ab} &= L_\Xi \Omega^{ab} + d\Lambda^{ab} + \Omega^{cb} \Lambda_c{}^a + \Omega^{ac} \Lambda_c{}^b + \epsilon^2 (\Omega^a \Lambda^b - \Omega^b \Lambda^a) && \text{(E.49)} \end{aligned}$$

These transformations can again be re-expressed in terms of redefined local Poincaré transformations via

$$\delta T = \delta_{\mathrm{ad}} T + \mathrm{i}_\Xi \mathcal{S} \qquad \delta E^a = \delta_{\mathrm{ad}} E^a + \mathrm{i}_\Xi \mathcal{S}^a \quad \text{(E.50)}$$

$$\delta \Omega^a = \delta_{\mathrm{ad}} \Omega^a + \mathrm{i}_\Xi \mathcal{R}^a \qquad \delta \Omega^{ab} = \delta_{\mathrm{ad}} \Omega^{ab} + \mathrm{i}_\Xi \mathcal{R}^{ab} \quad \text{(E.51)}$$

Here the adjoint transformation is again given in Equation (E.4) where both $\mathcal{A}$ and Λ are Poincaré valued 1-form and 0-forms respectively:

$$\mathcal{A} = TH + E^a P_a + \Omega^a G_a + \frac{1}{2}\Omega^{ab} J_{ab} \qquad \Lambda = ZH + Z^a P_a + \bar{\Lambda}^a G_a + \frac{1}{2}\bar{\Lambda}^{ab} J_{ab} \quad \text{(E.52)}$$

Note that

$$Z = \mathrm{i}_\Xi T \quad Z^a = \mathrm{i}_\Xi E^a \qquad \bar{\Lambda}^a = \mathrm{i}_\Xi \Omega^a + \Lambda^a \quad \bar{\Lambda}^{ab} = \mathrm{i}_\Xi \Omega^{ab} + \Lambda^{ab} \quad \text{(E.53)}$$

The curvature is computed through Equation (E.5). It can be decomposed as

$$\mathcal{F} = \mathcal{S}H + \mathcal{S}^a P_a + \mathcal{R}^a G_a + \frac{1}{2}\mathcal{R}^{ab} J_{ab} \quad \text{(E.54)}$$

The redefinitions amount to the following Poincaré commutation relations:

$$[H, G_a] = P_a \qquad [P_a, G_b] = \epsilon^2 \delta_{ab} H \qquad [G_a, G_b] = \epsilon^2 J_{ab} \quad \text{(E.55)}$$

$$[J_{ab}, G_c] = G_a \delta_{bc} - G_b \delta_{ac} \qquad [J_{ab}, P_c] = P_a \delta_{bc} - P_b \delta_{ac} \quad \text{(E.56)}$$

$$[J_{ab}, J_{cd}] = \delta_{ad} J_{bc} - \delta_{bd} J_{ac} - \delta_{ac} J_{bd} + \delta_{bc} J_{ad} \quad \text{(E.57)}$$

The results are then listed as

$$\begin{aligned}
\delta_{\mathrm{ad}} T &= dZ + \epsilon^2 (E^a \bar{\Lambda}_a - \Omega^a Z_a) && \text{(E.58)}\\
\delta_{\mathrm{ad}} E^a &= dZ^a - \Omega^a Z + \Omega^a{}_b Z^b + T\bar{\Lambda}^a + E^b \bar{\Lambda}_b{}^a && \text{(E.59)}\\
\delta_{\mathrm{ad}} \Omega^a &= d\bar{\Lambda}^a - \Omega^b \bar{\Lambda}_b{}^a + \Omega^{ab} \bar{\Lambda}_b && \text{(E.60)}\\
\delta_{\mathrm{ad}} \Omega^{ab} &= d\bar{\Lambda}^{ab} + \Omega^{cb} \bar{\Lambda}_c{}^a + \Omega^{ac} \bar{\Lambda}_c{}^b + \epsilon^2 (\Omega^a \bar{\Lambda}^b - \Omega^b \bar{\Lambda}^a) && \text{(E.61)}
\end{aligned}$$

and

$$\begin{aligned}\mathcal{S} &= dT - \epsilon^2 \Omega_b \wedge E^b && \text{(E.62)}\\ \mathcal{S}^a &= dE^a + \Omega^a{}_b \wedge E^b - \Omega^a \wedge T && \text{(E.63)}\\ \mathcal{R}^a &= d\Omega^a + \Omega^{ab} \wedge \Omega_b && \text{(E.64)}\\ \mathcal{R}^{ab} &= d\Omega^{ab} + \Omega^{ac} \wedge \Omega_c{}^b + \epsilon^2 \Omega^a \wedge \Omega^b && \text{(E.65)}\end{aligned}$$

Einstein equations can be reproduced by first one imposing the constraints:

$$\mathcal{S} = 0 \qquad\qquad \mathcal{S}^a = 0 \tag{E.66}$$

These can be solved as:

$$\begin{aligned}\Omega^a &= -\epsilon^{-2}(\delta^{ac}\mathrm{i}_{E_c} dT + \delta^{ac} T \mathrm{i}_{E_c}\mathrm{i}_T dT) - \mathrm{i}_T dE^a + \delta^{ac}\delta_{bd} E^b \mathrm{i}_{E_c}\mathrm{i}_T dE^d && \text{(E.67)}\\ \Omega^{ab} &= \epsilon^{-2}\eta^{ac}\eta^{bd} T \mathrm{i}_{E_c}\mathrm{i}_{E_d} dT - \eta^{ac}\mathrm{i}_{E_c} dE^b + \eta^{bc}\mathrm{i}_{E_c} dE^a - \eta^{ac}\eta^{bd}\eta_{kl} E^k \mathrm{i}_{E_c}\mathrm{i}_{E_d} dE^l\end{aligned}$$

Inserting these back in Equation (E.45) one can again identify the Levi-Civita connection $\Gamma^\lambda_{\mu\nu}$ with the arbitrary connection $\tilde{\Gamma}^\lambda_{\mu\nu}$ and connect the two curvature 2-forms to the Riemann tensor:

$$R^\lambda{}_{\rho\mu\nu} = E^\lambda_a E^b_\rho \mathcal{R}_{\mu\nu}{}^a{}_b - E^\lambda_a T_\rho \mathcal{R}_{\mu\nu}{}^a - \epsilon^2 T^\lambda E^a_\rho \mathcal{R}_{\mu\nu a} \tag{E.68}$$

and finally the Ricci tensor:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = E^\lambda_a E^b_\mu \mathcal{R}_{\lambda\nu}{}^a{}_b - E^\lambda_a T_\mu \mathcal{R}_{\lambda\nu}{}^a - \epsilon^2 T^\lambda E^a_\mu \mathcal{R}_{\lambda\nu a} \tag{E.69}$$

E.4. Bargmann Algebra and Newton-Cartan Gravity

In the work [14] it was realised that one can view the Newton-Cartan gravity in the language of gauge theory. There, with the following ansatz to the Poincaré formulation:

$$\begin{aligned} T &= \tau + \epsilon^2 m + \mathcal{O}(\epsilon^4) \\ E^a &= e^a + \mathcal{O}(\epsilon^2) \\ \Omega^a &= \omega^a + \mathcal{O}(\epsilon^2) \\ \Omega^{ab} &= \omega^{ab} + \mathcal{O}(\epsilon^2) \end{aligned}$$

one can identify the 1-form m with the one in Equation (4.12). Also expanding the Lorentz parameters and the diffeomorphism generating vector field

$$\begin{aligned} \Lambda^a &= \lambda^a + \mathcal{O}(\epsilon^2) \\ \Lambda^{ab} &= \lambda^{ab} + \mathcal{O}(\epsilon^2) \\ \Xi^\mu &= \xi^\mu + \mathcal{O}(\epsilon^2) \end{aligned}$$

The fields then can be written in an algebra valued gauge field as

$$\mathcal{A} = \tau H + mN + e^a P_a + \omega^a G_a + \frac{1}{2}\omega^{ab} J_{ab} \tag{E.70}$$

subjected the Bargmann commutation relations:

$$\begin{gathered} [H, G_a] = P_a \qquad [P_a, G_b] = \delta_{ab} N \\ [J_{ab}, J_{cd}] = -\delta_{ad} J_{bc} + \delta_{bd} J_{ac} + \delta_{ac} J_{bd} - \delta_{bc} J_{ad} \\ [J_{ab}, X_c] = -X_a \delta_{bc} + X_b \delta_{ac} \qquad X_a \in \{P_a, G_a\} \end{gathered} \tag{E.71}$$

One can furthermore assign a gauge curvature of the form

$$\mathcal{F} = \mathcal{S} H + \mathcal{M} N + \mathcal{S}^a P_a + \mathcal{R}^a G_a + \frac{1}{2}\mathcal{R}^{ab} J_{ab} \tag{E.72}$$

where we make use of the curvature ansatz

$$\begin{aligned} \mathcal{S} &\rightarrow \mathcal{S} + \epsilon^2 \mathcal{M} + \mathcal{O}(\epsilon^4) \\ \mathcal{S}^a &\rightarrow \mathcal{S}^a + \mathcal{O}(\epsilon^2) \end{aligned}$$

Note that in assigning the leading order curvatures, we label the zeroth order curvatures as the same as in the space and time split Poincaré formulation. This notation will be used in the next section as well. Upon imposing the curvature constraints

$$\mathcal{S} = 0 \qquad \mathcal{M} = 0 \qquad \mathcal{S}^a = 0 \tag{E.73}$$

the spin connection fields ω^a and ω^{ab} can be solved exactly in terms of τ, e^a and m fields. As a result, Newton-Cartan equation in Equation (4.34) can be obtained – see [14, 17] for further details –.

E.5. Nonrelativistic Expansion as a Gauge Theory

Motivated by the exploration in the previous section one can attempt to formulate the large c expansion in the language of gauge theories. One can further expand the ansatz to contain the second order for all fields. First consider the following 1-form fields and transformations as in [7]:

$$\begin{aligned}
\delta\tau &= L_\xi \tau && \text{(E.74)} \\
\delta m &= L_\xi m + L_\zeta \tau + \lambda_a e^a && \text{(E.75)} \\
\delta e^a &= L_\xi e^a + \tau\lambda^a + \lambda^a{}_b e^b && \text{(E.76)} \\
\delta\pi^a &= L_\xi \pi^a + L_\zeta e^a + m\lambda^a + \tau\eta^a - \lambda^a{}_b \pi^b + \eta^a{}_b e^b && \text{(E.77)} \\
\delta\omega^a &= L_\xi \omega^a + d\lambda^a + \omega^a{}_b \lambda^b + \lambda^a{}_b \omega^b && \text{(E.78)} \\
\delta\omega^{ab} &= L_\xi \omega^{ab} + d\lambda^{ab} + \omega^{bc}\lambda^a{}_c - \omega^{ac}\lambda^b{}_c && \text{(E.79)} \\
\delta\varpi^a &= L_\xi \varpi^a + L_\zeta \omega^a + d\eta^a + \varpi^a{}_b \lambda^b + \omega^a{}_b \eta^b - \lambda^a{}_b \varpi^b - \eta^a{}_b \omega^b && \text{(E.80)} \\
\delta\varpi^{ab} &= L_\xi \varpi^{ab} + L_\zeta \omega^{ab} + d\eta^{ab} + \varpi^{bc}\lambda^a{}_c - \omega^{bc}\eta^a{}_c + \omega^{ac}\eta^b{}_c - \varpi^{ac}\lambda^b{}_c && \text{(E.81)}
\end{aligned}$$

One can arrive at these transformations by expanding the transformations of 1-form fields in the Poincaré formulation of GR with the following ansatz:

$$
\begin{aligned}
T &= \tau + \epsilon^2 m + \mathcal{O}(\epsilon^4) \\
E^a &= e^a + \epsilon^2 \pi^a + \mathcal{O}(\epsilon^4) \\
\Omega^a &= \omega^a + \epsilon^2 \varpi^a + \mathcal{O}(\epsilon^4) \\
\Omega^{ab} &= \omega^{ab} + \epsilon^2 \varpi^{ab} + \mathcal{O}(\epsilon^4)
\end{aligned}
$$

Also expanding the Lorentz parameters and the diffeomorphism generating vector field

$$
\begin{aligned}
\Lambda^a &= \lambda^a + \epsilon^2 \eta^a + \mathcal{O}(\epsilon^4) \\
\Lambda^{ab} &= \lambda^{ab} + \epsilon^2 \eta^{ab} + \mathcal{O}(\epsilon^4) \\
\Xi^\mu &= \xi^\mu + \epsilon^2 \zeta^\mu + \mathcal{O}(\epsilon^4)
\end{aligned}
$$

and assigning new generators to each of the new gauge fields. The fields then can be written in an algebra valued gauge field as

$$
\mathcal{A} = \tau H + mN + e^a P_a + \pi^a T_a + \omega^a G_a + \varpi^a B_a + \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{2}\varpi^{ab} S_{ab}\,. \tag{E.82}
$$

and the gauge parameter

$$
\begin{aligned}
\Lambda \;=\; & \mathrm{i}_\xi \tau\, H + (\mathrm{i}_\xi m + \mathrm{i}_\zeta \tau) N + \mathrm{i}_\xi e^a\, P_a + (\mathrm{i}_\xi \pi^a + \mathrm{i}_\zeta e^a) T_a \\
& + (\mathrm{i}_\xi \omega^a + \lambda^a) G_a + (\mathrm{i}_\xi \varpi^a + \mathrm{i}_\zeta \omega^a + \eta^a) B_a \\
& + \frac{1}{2}(\mathrm{i}_\xi \omega^{ab} + \lambda^{ab}) J_{ab} + \frac{1}{2}(\mathrm{i}_\xi \varpi^{ab} + \mathrm{i}_\zeta \omega^{ab} + \eta^{ab}) S_{ab}
\end{aligned} \tag{E.83}
$$

The procedure to rewrite diffeomorphisms in terms of the translational adjoint transformations introduced in Equation (E.8) and applied throughout the gauge formalism here is very natural if the curvature vanishes by a combination of constraints and dynamic equations, as for example in a vielbein formulation to general relativity. We show that it is less natural when this is not the case, as for example in the nonrelativistic approximation to general relativity. Nonetheless this approach remains valid and in [6, 7] an algebra was introduced whose translational part reproduces the diffeomorphism symmetries on the gauge field.

This Lie algebra is however not the unique one with this feature since in Equation (E.8) a modification of the adjoint action can be canceled by a modification of the curvature contribution, leading to identical transformations under diffeomorphisms. Demanding that the Lie algebra is consistent and that the boost and rotational part remains the same all such possibilities are:

$$\begin{gathered}
[H, G_a] = P_a \qquad [N, G_a] = T_a \qquad [H, B_a] = T_a \qquad [P_a, G_b] = \delta_{ab} N \\
[G_a, G_b] = -S_{ab} \qquad [S_{ab}, G_c] = -B_a\delta_{bc} + B_b\delta_{ac} \qquad [S_{ab}, P_c] = -T_a\delta_{bc} + T_b\delta_{ac} \\
[J_{ab}, J_{cd}] = -\delta_{ad}J_{bc} + \delta_{bd}J_{ac} + \delta_{ac}J_{bd} - \delta_{bc}J_{ad} \\
[J_{ab}, S_{cd}] = -\delta_{ad}S_{bc} + \delta_{bd}S_{ac} + \delta_{ac}S_{bd} - \delta_{bc}S_{ad} \\
[P_a, P_b] = \alpha S_{ab} \qquad [N, P_a] = \alpha B_a \qquad [H, T_a] = \alpha B_a \\
[H, P_a] = \alpha G_a + \beta T_a + \gamma B_a \\
[J_{ab}, X_c] = -X_a\delta_{bc} + X_b\delta_{ac} \qquad X_a \in \{P_a, G_a, T_a, B_a\}
\end{gathered} \tag{E.84}$$

This is a family of algebra's parameterized by the real numbers α, β and γ, that reproduces the algebra of [6,7] when $\alpha = \beta = \gamma = 0$. Just as that algebra can be obtained by an expansion procedure from the Poincaré algebra [6, 22], the algebras with non-trivial α but $\beta = \gamma = 0$ can be obtained by expansion from the (A)dS algebra. For other values of the parameters there doesn't seem to exist any relativistic algebra that they descent from, making them similar to some of the exotic nonrelativistic algebras found in [23].

For the explicit computation of the adjoint transformations on the fields and the curvature components as determined by the algebra in Equation (E.84), we first decompose the curvature as

$$\mathcal{F} = \mathcal{S}H + \mathcal{M}N + \mathcal{S}^a P_a + \mathcal{M}^a T_a + \mathcal{R}^a G_a + \mathcal{V}^a B_a + \frac{1}{2}\mathcal{R}^{ab} J_{ab} + \frac{1}{2}\mathcal{V}^{ab} S_{ab} \tag{E.85}$$

where we make use of the ansatz

$$\begin{aligned}
\mathcal{S} &\rightarrow \mathcal{S} + \epsilon^2 \mathcal{M} + \mathcal{O}(\epsilon^4) \\
\mathcal{S}^a &\rightarrow \mathcal{S}^a + \epsilon^2 \mathcal{M}^a + \mathcal{O}(\epsilon^4) \\
\mathcal{R}^a &\rightarrow \mathcal{R}^a + \epsilon^2 \mathcal{V}^a + \mathcal{O}(\epsilon^4) \\
\mathcal{R}^{ab} &\rightarrow \mathcal{R}^{ab} + \epsilon^2 \mathcal{V}^{ab} + \mathcal{O}(\epsilon^4)
\end{aligned}$$

Here we again used the same notation where we label the zeroth order field as the same with the split Poincaré curvatures. Then computing each gauge curvature component as

$$\mathcal{S} = d\tau \tag{E.86}$$

$$\mathcal{M} = dm + e^a \wedge \omega_a \tag{E.87}$$

$$\mathcal{S}^a = de^a + \tau \wedge \omega^a + \omega^a{}_b \wedge e^b \tag{E.88}$$

$$\mathcal{M}^a = d\pi^a + m \wedge \omega^a + \tau \wedge \Omega^a + \omega^a{}_b \wedge \pi^b + \Omega^a{}_b \wedge e^b + \beta\tau \wedge e^a \tag{E.89}$$

$$\mathcal{R}^a = d\omega^a - \alpha e^a \wedge \tau + \omega^a{}_b \wedge \omega^b \tag{E.90}$$

$$\mathcal{V}^a = d\varpi^a + \omega^a{}_b \wedge \varpi^b + \varpi^a{}_b \wedge \omega^b - \alpha e^a \wedge m - \alpha\pi^a \wedge \tau + \gamma\tau \wedge e^a \tag{E.91}$$

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega^{bc} \wedge \omega^a{}_c \tag{E.92}$$

$$\mathcal{V}^{ab} = d\varpi^{ab} + \varpi^{bc} \wedge \omega^a{}_c + \omega^{bc} \wedge \varpi^a{}_c - \alpha e^a \wedge e^b \tag{E.93}$$

While the gauge transformations are computed as

$$\begin{aligned}
\delta_{\mathrm{ad}}\tau &= d(\mathrm{i}_\xi\tau) && \text{(E.94)}\\
\delta_{\mathrm{ad}}m &= d(\mathrm{i}_\xi m+\mathrm{i}_\zeta\tau)-\mathrm{i}_\xi(e^a\wedge\omega_a)+e^a\lambda_a && \text{(E.95)}\\
\delta_{\mathrm{ad}}e^a &= d(\mathrm{i}_\xi e^a)-\mathrm{i}_\xi(\tau\wedge\omega^a)+\tau\lambda^a-\mathrm{i}_\xi(\omega^a{}_b\wedge e^b)+\lambda^a{}_b e^b && \text{(E.96)}\\
\delta_{\mathrm{ad}}\pi^a &= d(\mathrm{i}_\xi\pi^a+\mathrm{i}_\zeta e^a)-\mathrm{i}_\xi(m\wedge\omega^a)+m\lambda^a-\mathrm{i}_\zeta(\tau\wedge\omega^a) && \text{(E.97)}\\
&\quad -\mathrm{i}_\xi(\tau\wedge\varpi^a)+\tau\eta^a-\mathrm{i}_\xi(\omega^a{}_b\wedge\pi^b)-\mathrm{i}_\zeta(\omega^a{}_b\wedge e^b)-\eta^a{}_b\pi^b\\
&\quad -\mathrm{i}_\xi(\varpi^a{}_b\wedge e^b)+\eta^a{}_b e^b-\beta\mathrm{i}_\xi(\tau\wedge\epsilon^a)\\
\delta_{\mathrm{ad}}\omega^a &= d(\mathrm{i}_\xi\omega^a+\lambda^a)+\alpha\mathrm{i}_\xi(e^a\wedge\tau) && \text{(E.98)}\\
&\quad +\omega^a{}_b\lambda^b-\mathrm{i}_\xi(\omega^a{}_b\wedge\omega^b)+\lambda^a{}_b\omega^b\\
\delta_{\mathrm{ad}}\varpi^a &= d(\mathrm{i}_\xi\varpi^a+\mathrm{i}_\zeta\omega^a+\eta^a) && \text{(E.99)}\\
&\quad +\alpha\mathrm{i}_\xi(e^a\wedge m)+\alpha(\mathrm{i}_\xi\pi^a\wedge\tau)+\alpha\mathrm{i}_\zeta(e^a\wedge\tau)\\
&\quad +\varpi^a{}_b\lambda^b-\mathrm{i}_\xi(\varpi^a{}_b\wedge\omega^b)-\mathrm{i}_\zeta(\omega^a{}_b\wedge\omega^b)\\
&\quad +\omega^a{}_b\eta^b-\mathrm{i}_\xi(\omega^a{}_b\wedge\Omega^b)-\lambda^a{}_b\varpi^b-\eta^a{}_b\omega^b-\gamma\mathrm{i}_\xi(\tau\wedge e^a)\\
\delta_{\mathrm{ad}}\omega^{ab} &= d(\mathrm{i}_\xi\omega^{ab}+\omega^{bc}\lambda^a{}_c-\mathrm{i}_\xi(\omega^b{}_c\wedge\omega^{ac})-\omega^{ac}\lambda^b{}_c && \text{(E.100)}\\
\delta_{\mathrm{ad}}\varpi^{ab} &= d(\mathrm{i}_\xi\varpi^{ab}+\mathrm{i}_\zeta\omega^{ab}+\eta^{ab})+\alpha\mathrm{i}_\xi(e^a\wedge e^b) && \text{(E.101)}\\
&\quad +\varpi^{bc}\lambda^a{}_c-\mathrm{i}_\xi(\omega^{bc}\wedge\varpi^a{}_c)-\mathrm{i}_\zeta(\omega^{bc}\wedge\omega^a{}_c)-\omega^{bc}\eta^a{}_c\\
&\quad +\omega^{ac}\eta^b{}_c-\mathrm{i}_\xi(\varpi^{bc}\wedge\omega^a{}_c)-\varpi^{ac}\lambda^b{}_c
\end{aligned}$$

It then directly follows from the relations that these transformations can again be re-expressed in terms of gauge transformations and curvatures via

$$\begin{aligned}
\delta\tau &= \delta_{\mathrm{ad}}\tau+\mathrm{i}_\xi\mathcal{S} \qquad & \delta m &= \delta_{\mathrm{ad}}m+\mathrm{i}_\zeta\mathcal{S}+\mathrm{i}_\xi\mathcal{M} && \text{(E.102)}\\
\delta e^a &= \delta_{\mathrm{ad}}e^a+\mathrm{i}_\xi\mathcal{S}^a \qquad & \delta\pi^a &= \delta_{\mathrm{ad}}\pi^a+\mathrm{i}_\zeta\mathcal{S}^a+\mathrm{i}_\xi\mathcal{M}^a && \text{(E.103)}\\
\delta\omega^a &= \delta_{\mathrm{ad}}\omega^a+\mathrm{i}_\xi\mathcal{R}^a \qquad & \delta\varpi^a &= \delta_{\mathrm{ad}}\varpi^a+\mathrm{i}_\zeta\mathcal{R}^a+\mathrm{i}_\xi\mathcal{V}^a && \text{(E.104)}\\
\delta\omega^{ab} &= \delta_{\mathrm{ad}}\omega^{ab}+\mathrm{i}_\xi\mathcal{R}^{ab} \qquad & \delta\varpi^{ab} &= \delta_{\mathrm{ad}}\varpi^{ab}+\mathrm{i}_\zeta\mathcal{R}^{ab}+\mathrm{i}_\xi\mathcal{V}^{ab} && \text{(E.105)}
\end{aligned}$$

that this procedure indeed reproduces Equations (E.74)-(E.81). Note that while the parameters α, β and γ do show up in both δ_{ad} and $\mathcal{F}$ they cancel out in each of

Equations (E.102)-(E.105) leading to the same transformations as Equations (E.74)-(E.81), which coincide with those of [7].