RELATIVISTIC LEE MODEL ON 2+1 DIMENSIONAL RIEMANNIAN MANIFOLDS

by

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”And did you get what you wanted from this life, even so? I did. And what did you want? To call myself beloved, to feel myself beloved on the earth.”
——————————————————- Raymond Carver

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ABSTRACT

RELATIVISTIC LEE MODEL ON 2+1 DIMENSIONAL RIEMANNIAN MANIFOLDS

In this work, we analyze a two-level system coupled to an arbitrary number of bosons with a relativistic dispersion relation in a static background metric while ignoring pair creation processes. One can obtain a non-perturbative formulation of this problem by directly studying the associated resolvent following an idea proposed by Rajeev. The resolvent allows us to estimate the ground state energy from above and below thanks to the fact that it is formulated through an operator, so called principal operator. Whenever the eigenvalues of the principal operator hit a zero, as they flow with the energy parameter, we find a possible pole in the resolvent, which typically corresponds to a bound state in the spectrum. The rigorous study of this principal operator includes showing that this operator is a holomorphic family of type-A in the sense of Kato. This in turn justifies the fact that our resolvent formula defines a self-adjoint quantum Hamiltonian as well as putting our estimates on a firmer ground. The required operator estimates are obtained through recent two sided heat kernel estimates on manifolds.

ÖZET

2+1 BOYUTLU RIEMANN MANİFOLDLARINDA RELATİVİSTİK LEE MODELİ

Bu çalışmada, iki-kademeli bir sistemin, çift oluşumu göz ardı edilerek statik bir arka plan metriğinde relativistik dağılımlı herhangi sayıda bozonla etkileşimi incelenmiştir. Rajeev tarafından sunulan bir fikir yoluyla bu problemin pertürbasyon dışı formülasyonu, ilişikili rezolventin doğrudan çalışılmasıyla elde edilebilir. Rezolventin temel operatör adı verilen bir operatör cinsinden kurulması, temel durum enerjisinin üstten ve alttan kestirilmesini sağlar. Temel operatörün enerji ile akan özdeğerlerinin sıfırı görmesi rezolventin bir kutbuna karşılık gelir. Bu değer tipik olarak spektrumda bağlı durumlara tekabül eder. Temel operatörün dikkatli bir incelemesi bu operatörün Kato manasında A-tipi holomorf bir aile olduğunu göstermeyi gerektirir. Bu, rezolvent formülümüzün kendine-eş bir kuvantum Hamiltonyeni tanımladığını doğruladığı gibi kestirimlerimizi de sağlam bir zemine oturtur. Gerekli operatör kestirimleri yakın zamanda bulunmuş manifoldlar üzerinde çift yönlü ısı çekirdeği kestirimleri kullanılarak elde edilir.

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LIST OF SYMBOLS

$\mathcal{H}$	Hilbert Space
$\mathcal{F}$	Fock Space
H_0	Free Hamiltonian
H_I	Interaction Hamiltonian
$d_g x$	Riemannian volume element with metric structure g
$L^1(V)$	Lebesgue integration space of functions on V
$L^2(M)$	Space of square integrable functions on M
$K_\nu(z)$	Modified Bessel function of the second kind
$R(E)$	Resolvent
$K_t(x,y)$	Heat Kernel
$D(T)$	Domain of T
$\mathbf{H}(U)$	Analytic functions that are holomorphic on U
$V(M)$	Volume of the Riemannian manifold M
n	Number of bosons
λ	Coupling constant
Λ	Cut-off on the eigenvalues of $(-\nabla^2)$
μ_p	Physical mass difference of the two-level system
$\delta_g(x,y)$	Dirac Delta function at position y
$\Phi(E)$	Principle Operator
Δ *or* ∇_g^2	Laplace-Beltrami operator with metric structure g

LIST OF ACRONYMS/ABBREVIATIONS

$dim(M)$	Dimension of Riemannian manifold M
sup	Supremum
$Re(z)$	Real part of the complex number z

1. INTRODUCTION

Lee, in his paper [1] from 1954, investigated the model named after him with the motivation of gaining a deeper insight into the nature of renormalization procedure through some simple problems which are not as rich as quantum electrodynamics but easier to examine since they are both exactly renormalizable and solvable. For an introduction to the theoretical background of the model, we will make use of a more explanatory text by Schweber [2] where it was analysed in detail.

In the original formulation of the Lee model, one has the nucleon that can exist in two different intrinsic states together with an arbitrary number of bosons. The nucleon transforms between the two intrinsic states which are called the N particle and the V particle. The V particle emits a boson which is called the θ particle and transforms to an N particle. The N particle is allowed only to absorb a boson and transform into a V particle. Schematically, the allowed process is:

$$V \rightleftharpoons N \; + \; \theta \tag{1.1}$$

whereas the following is forbidden:

$$N \rightleftharpoons V \; + \; \theta \tag{1.2}$$

It is assumed that the nucleons are spinless and their energies are independent of their momenta and the total momentum is conserved. We write the full Hamiltonian as the sum of the free and interaction Hamiltonian:

$$H = H_0 + H_I \tag{1.3}$$

where,

$$H_0 = m_{V_0} \int \frac{d\mathbf{p}}{(2\pi)^3} V^\dagger(\mathbf{p}) V(\mathbf{p}) + m_{N_0} \int \frac{d\mathbf{p}}{(2\pi)^3} N^\dagger(\mathbf{p}) N(\mathbf{p}) + \int \frac{d\mathbf{k}}{(2\pi)^3} \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) \tag{1.4}$$

$$H_I = \lambda_0 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{f_1(\mathbf{k}^2)}{\sqrt{2\omega_{\mathbf{k}}}} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[V^\dagger(\mathbf{p}) N(\mathbf{p}-\mathbf{k}) a(\mathbf{k}) + N^\dagger(\mathbf{p}-\mathbf{k}) a^\dagger(\mathbf{k}) V(\mathbf{p}) \right] \tag{1.5}$$

One easily recognizes the creation and annihilation operators for each quanta which obey the usual commutation relations. m_{V_0} and m_{N_0} are the *bare* masses of the quanta of V and N fields, $\omega(\mathbf{k}) = \sqrt{m_0^2 + \mathbf{k}^2}$ is the energy of a Bose quanta with mass m_0 and momentum $\mathbf{k}$. λ_0 is the coupling constant and $f_1(\mathbf{k}^2)$ is a cut-off on the range of interaction.

Besides momentum, there are two other conserved charges:

$$\begin{aligned} Q_1 &= \int \frac{d\mathbf{p}}{(2\pi)^3} [V^\dagger(\mathbf{p} V(p) + N^\dagger(\mathbf{p}) N(\mathbf{p})] && (1.6) \\ Q_2 &= \int \frac{d\mathbf{p}}{(2\pi)^3} N^\dagger(\mathbf{p}) N(\mathbf{p}) - \int \frac{d\mathbf{k}}{(2\pi)^3} a^\dagger(\mathbf{k}) a(\mathbf{k}) && (1.7) \end{aligned}$$

The Schrödinger equation can be solved directly despite the divergences in the Hamiltonian. A detailed analysis shows that to remove the divergences one needs to do a mass renormalization as well as a coupling constant renormalization. An interesting interpretation is when one regards N as a neutron, V as a proton and θ as a π^+ meson. Then the allowed transformations are determined by the charge conservation. What is special about Lee model is that the $N-\theta$ sector is decoupled from the many particle channels. However, in a more realistic setup one would have the antiparticle $\bar{\theta}$, which would allow the following transformation:

$$N \rightleftharpoons V + \bar{\theta} \tag{1.8}$$

The coupling constant renormalization of a similar model which has charge conjugation symmetry was analyzed by Wilson [3] through a series expansion. The unrenormalized Hamiltonian of the model is:

$$H = \sum_{j=0}^{\infty} \kappa^j [(a_j^\dagger a_j + b_j^\dagger b_j - 1) + g_0(a_j + b_j^\dagger)\tau^+ + g_0(b_j + a_j^\dagger)\tau^-] \tag{1.9}$$

where g_0 and κ are constants, $a_j^\dagger$ and $b_j^\dagger$ are creation operators for π^+ and π^-, τ^+ and τ^- are the isospin raising and lowering operators for the two level system (the nucleon). This new model is the “cousin” of the Lee model. It is a challenging problem and a landmark in the understanding of renormalization.

Renormalization of the Lee model through dimensional regularization was done by Bender and Nash [4] and the model was also approached by Dittrich [5] using the source techniques presented by Schwinger [6].

In this work, we analyze a simplified version of the Lee model where the two level system is fixed on a Riemannian manifold and it interacts with an arbitrary number of bosons. The non-relativistic Lee model on 2 and 3 dimensional Riemannian manifolds was presented in [7,8] respectively. The main technique employed is a nonperturbative renormalization method proposed by Rajeev [9], where the resolvent is expressed in terms of the "Principal Operator" $\Phi(E)$. Once one has a finite expression for $\Phi(E)$, the spectral information can be obtained from it. The zeros of $\Phi(E)$, for instance, correspond to the eigenvalues of the Hamiltonian. To apply this technique to our model, we make use of an essential mathematical tool: the heat kernel. We will give a brief overview of this tool on Riemannian manifolds here not to interrupt the flow of the main work. Here we follow [10].

The heat operator on a Riemannian manifold M is given as:

$$L = \Delta - \partial/\partial t \tag{1.10}$$

where Δ is the Laplace-Beltrami operator on M. The heat equation is:

$$Lu = 0 \tag{1.11}$$

The keat kernel $K_t(x, y)$, defined on $(0, \infty) \times M \times M$, is a fundamental solution of the heat equation which is C^2 with respect to x and C^1 with respect to t satisfying:

$$L_x K_t(x, y) = 0 \qquad ; \qquad \lim_{t \to 0^+} K_t(\cdot, y) = \delta_{\cdot, y} \tag{1.12}$$

For M compact, there exists a complete orthonormal basis consisting of eigenfunctions ϕ_i of the Laplace-Beltrami operator Δ and the Sturm-Liouville decomposition of the heat kernel reads:

$$K_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y) \tag{1.13}$$

where λ_i's are the corresponding positive eigenvalues. Among main properties are symmetry, positivity and the semi-group property i.e. respectively:

$$\begin{aligned} K_t(x,y) &= K_t(y,x) && (1.14) \\ K_t(x,y) > 0 \quad &; \quad x,y \in M \quad , \quad t \geq 0 && (1.15) \\ \int_M d_g z K_{t_1}(x,z) K_{t_2}(z,y) &= K_{t_1+t_2}(x,y) && (1.16) \end{aligned}$$

The short time asymptotics for the diagonal heat kernel is given by:

$$K_t(x,x) \approx \frac{1}{(4\pi t)^{d/2}} \sum_{i=0}^{\infty} f_i(x,x) t^i \tag{1.17}$$

where d is the dimension of M and the smooth functions $f_i(x,x)$ restricted to the diagonal are given by explicit formulas in terms of local geometric invariants [11]. One recognizes directly the singular nature of the heat kernel near 0. This is an important point to keep in mind throughout the work when giving estimates to some integrals and searching for the sources of possible divergences.

When estimating some quantities in question, the following upper bound for the heat kernel on compact manifolds will be of great importance [12]:

$$K_t(x,x) \leq \frac{1}{V(M)} + Ct^{-d/2} \tag{1.18}$$

for all $t > 0$ and $x \in M$ where $d = dim(M)$, $V(M)$ is the volume of the manifold and C is a positive constant which can be computed explicitly.

The thesis is organized as follows: In the second chapter, we first construct our model and give the Hamiltonian. We work with compact Riemannian manifolds for it not only makes the calculations easier but the results can also be generalized to the non-compact case albeit with some technical complications. We introduce the resolvent in terms of the Principle Operator $\Phi(E)$. We explicitly construct $\Phi(E)$ and observe that the bound state solutions come from the poles of $\Phi(E)^{-1}$. Recognizing the divergence, we first put a cut-off to the allowed eigenvalues of the Laplacian and let the mass difference μ depend on Λ. Imposing the physical mass condition and solving for $\mu(\Lambda)$, we remove the divergence then take the limit $\Lambda \to \infty$.

Having the Principal Operator renormalized, in section 2.2, we start searching for upper and lower bounds to the ground state energy. The variational method is employed to show that the ground state energy is indeed below the trivial guess $nm+\mu_p$ where n is the number of bosons and μ_p is the physical binding energy. E_* below which the Principal Operator is observed to be invertible serves as a lower bound to the ground state energy.

In section 2.3, we conclude the first chapter by showing with the help of a corollary from [13], that $R(E) = \frac{1}{H-E}$ is indeed the resolvent of a densely defined closed operator.

The third chapter, we study the holomorphicity of the Principal Operator. To show that $\Phi(E)$ is a self-adjoint holomorphic family of type-A in the sense of Kato, one first needs to fix a common domain and show that $\Phi(E)$ is closed on it. We show that the domain D_0 of H_0 can be chosen as the common fixed domain D of $\Phi(E)$ and it is indeed closed on D. We then invoke a theorem from [14] stating the conditions for a function in its integral form to be holomorphic. Showing that the matrix elements $\langle f|\Phi(E)|g\rangle$ are holomorphic, we conclude that $\Phi(E)$ is a holomorphic family of type-A.

In section 3.4, we invoke two theorems to show that $\Phi(E)$ is self-adjoint in D. We find a region on which $\Phi(E)$ is self-adjoint by the famous Kato-Rellich theorem [15] then extend it to the whole domain using Wüst's theorem [16]. This concludes that $\Phi(E)$ is a self-adjoint holomorphic family of type-A and justifies the fact that our resolvent formula defines a self-adjoint quantum Hamiltonian as well as putting our estimates on a firmer ground.

We conclude the thesis in Chapter 4 with some comments and a brief description of possible future directions.

2. RELATIVISTIC LEE MODEL ON 2+1 DIMENSIONAL RIEMANNIAN MANIFOLDS

2.1. Construction of the Model

For the relativistic Lee model on a 2+1 dimensional *compact* Riemannian manifold (M, g) , one writes down the formal Hamiltonian as follows:

$$H = H_0 + H_I \tag{2.1}$$

$$H_0 = \sum_{\sigma} \omega_\sigma a_\sigma a_\sigma^\dagger + \mu \frac{1-\sigma_3}{2} \tag{2.2}$$

$$H_I = \lambda \big[\sigma_+ \phi^{(-)}(\bar{x}) + \sigma_- \phi^{(+)}(\bar{x}) \big] \tag{2.3}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, $\omega_\sigma = \sqrt{\sigma + m^2}$, m the mass of the boson, $\bar{x}$ the location of the two level system. Compactness is not an essential restriction for the formalism presented below, but it simplifies the rigorous analysis we attempt in our work. σ's are the eigenvalues of the negative Laplacian:

$$-\nabla_g^2 f_\sigma(x) = \sigma f_\sigma(x) \tag{2.4}$$

Since the manifold we are working on is compact, the Laplacian has a discrete spectrum and there is a family of orthonormal complete eigenfunctions $f_\sigma(x) \in L^2(M)$ which satisfy [17] :

$$\begin{aligned} \int_M d_g x \; f_\sigma^*(x)\, f_{\sigma'}(x) &= \delta_{\sigma\sigma'} \;, \\ \sum_\sigma f_\sigma^*(x) f_\sigma(y) &= \delta_g(x,y) \end{aligned} \tag{2.5}$$

where $d_g x = \sqrt{det[g_{ij}]}dx$ is the volume element. Hence we introduce:

$$\begin{aligned} \phi^{(-)}(x) &= \sum_\sigma \frac{1}{\sqrt{2\omega_\sigma}} f_\sigma^*(x) a_\sigma^\dagger \\ \phi^{(+)}(x) &= \sum_\sigma \frac{1}{\sqrt{2\omega_\sigma}} f_\sigma(x) a_\sigma \end{aligned} \tag{2.6}$$

Since $f_\sigma(x)$'s can be chosen to be real, the complex conjugate will not be important in the following calculations. One can write the Hamiltonian minus energy as:

$$H - E = \begin{bmatrix} H_0 - E & \lambda\phi^{(-)}(\bar{x}) \\ \lambda\phi^{(+)}(\bar{x}) & H_0 - E + \mu \end{bmatrix} = \begin{bmatrix} a & b^\dagger \\ b & d \end{bmatrix} \tag{2.7}$$

Now, let us search for bound state solutions, or simply the poles of the resolvent below the free spectrum. The resolvent is the formal inverse of 2.7 and can be calculated algebraically in an alternative manner as in [9] :

$$R(E) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \tag{2.8}$$

where:

$$\begin{aligned} \alpha &= a^{-1} + a^{-1}b^\dagger\ \Phi^{-1}(E)\ ba^{-1} \\ \beta &= -\Phi^{-1}(E)\ ba^{-1} \\ \gamma &= -a^{-1}b^\dagger\ \Phi^{-1}(E) \\ \delta &= \Phi^{-1}(E) \\ \Phi &= d - ba^{-1}b^\dagger \end{aligned} \tag{2.9}$$

$a^{-1} = \frac{1}{H_0 - E}$ is the resolvent for the free part of the Hamiltonian and its pole structure, in principle, is completely known. Since we are searching for the ground state energy and expect some binding between the two-level system and the bosons, we should be looking at the region below the free spectrum. Hence the poles we are looking for should be coming from Φ^{-1}.

2.1.1. The Principal Operator

The purpose of this chapter is to calculate Φ - the Principle Operator. Substituting a, b, d from 2.7 into 2.9 one gets $\Phi(E)$:

$$\Phi(E) = [H_0 - E + \mu] - \sum_{\sigma,\tau} \frac{\lambda^2}{\sqrt{2\omega_\sigma}} f_\sigma a_\sigma \frac{1}{H_0 - E} \frac{1}{\sqrt{2\omega_\tau}} f_\tau a_\tau^\dagger \tag{2.10}$$

In the approach suggested by Rajeev [9] , the first step is to normal order 2.10. Using the commutation relation:

$$\left[a_\sigma, a_\tau^\dagger\right] = \delta_{\sigma\tau} \tag{2.11}$$

and

$$\frac{1}{H_0 - E} = \int_0^\infty ds e^{-s(H_0 - E)} \tag{2.12}$$

where $H_0 - Re(E)$ is always positive thanks to the $a_\tau^\dagger$ at the right hand side of 2.10, one gets:

$$a_\sigma \frac{1}{H_0 - E} = \frac{1}{H_0 - E + \omega_\sigma} a_\sigma \tag{2.13}$$

Changing the order of a_σ and $a_\tau^\dagger$, then repeating the calculation above for $\frac{1}{H_0-E+\omega_\sigma}a_\sigma^\dagger$, we arrive at:

$$\Phi_\Lambda(E) = \Big[H_0 - E + \mu(\Lambda)\Big] - \sum_{\sigma<\Lambda} \frac{\lambda^2}{2\omega_\sigma}|f_\sigma|^2 \frac{1}{H_0 - E + \omega_\sigma} \tag{2.14}$$
$$- \sum_{\sigma,\tau<\Lambda} \lambda^2 f_\tau \frac{a_\tau^\dagger}{\sqrt{2\omega_\tau}} \frac{1}{H_0 - E + \omega_\sigma + \omega_\tau} \frac{a_\sigma}{\sqrt{2\omega_\sigma}} f_\sigma$$

Note that we introduced a cut-off anticipating a divergence; the second term in 2.14 diverges as $\Lambda \to \infty$. Hence to make sense of all *formal operations*, one should put a cut-off to the allowed eigenvalues of Δ, as we did above.
We will choose $\mu(\Lambda)$ such that we remove the divergence in 2.14. This still leaves out some ambiguity in the finite parts. But if one imposes the condition that μ_p, the physical binding energy, is exactly where $E = \mu_p$, as a result:

$$\Phi_R(E = \mu_p)|0\rangle = 0 \tag{2.15}$$

should be satisfied. This is a renormalization condition typical of any such problem. Solving for $\mu(\Lambda)$ we get :

$$\mu(\Lambda) = \sum_{\sigma<\Lambda} \frac{\lambda^2}{2\omega_\sigma}|f_\sigma|^2 \frac{1}{(\omega_\sigma - \mu_p)} \ + \ \mu_p \tag{2.16}$$

Substituting back into 2.14 , we get our final result:

$$\Phi_\Lambda = (H_0 - E + \mu_p)\Big[1 + \sum_{\sigma<\Lambda} \frac{\lambda^2}{2\omega_\sigma} \frac{|f_\sigma|^2}{(H_0 - E + \omega_\sigma)} \frac{1}{(\omega_\sigma - \mu_p)}\Big]$$
$$- \sum_{\sigma,\tau<\Lambda} \lambda^2 f_\sigma(\bar{x}) \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \frac{1}{H_0 - E + \omega_\sigma + \omega_\tau} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau(\bar{x}) \ . \tag{2.17}$$

Then we could take the limit $\Lambda \to \infty$ since the divergence is removed.

2.2. Bounds for the Ground State Energy

2.2.1. An Upper Bound

Recall that the principal operator becomes after the physical mass condition is imposed,

$$\Phi_R = (H_0 - E + \mu_p)\Big[1 + \sum_\sigma \frac{\lambda^2}{2\omega_\sigma} \frac{1}{(H_0 - E + \omega_\sigma)} \frac{f_\sigma^2(\bar{x})}{(\omega_\sigma - \mu_p)}\Big]$$
$$- \sum_{\sigma,\tau} \lambda^2 f_\sigma(\bar{x}) \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \frac{1}{H_0 - E + \omega_\sigma + \omega_\tau} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau(\bar{x}) \ .$$

We now compute the flow of eigenvalues as we change E *along the real axis* while staying below $nm + \mu$. This can be accomplished by means of Feynmann-Helmann formula [18] (equation 3.18 page 391, assuming Ω_k is also holomorphic and pretending that the operators involved are actually self-adjoint, which is to be proven later):

$$\begin{aligned} \frac{\partial \Omega_k}{\partial E} = \langle \Omega_k | \frac{\partial \Phi(E)}{\partial E} | \Omega_k \rangle \quad &= \quad -1 - \frac{\lambda^2}{2} \sum_\sigma \langle \Omega_k | \frac{f_\sigma^2(\bar{x})}{\omega(\sigma)(H_0 - E + \omega(\sigma))^2} | \Omega_k \rangle \\ & \quad -\frac{\lambda^2}{2} \underbrace{\sum_{\sigma\tau} \langle \Omega_k | \frac{a_\sigma^\dagger}{\sqrt{\omega(\sigma)}} \frac{f_\tau(\bar{x}) f_\sigma(\bar{x})}{(H_0 - E + \omega(\sigma) + \omega(\tau))^2} \frac{a_\tau}{\sqrt{\omega(\tau)}} | \Omega_k \rangle}_{\int_0^\infty s ds \Big|\Big| \sum_\sigma e^{-s(\frac{1}{2} H_0 + \omega(\sigma) - \frac{1}{2} E)} \frac{f_\sigma(\bar{x}) a_\sigma}{\sqrt{\omega(\sigma)}} | \Omega_k \rangle \Big|\Big|^2} \\ \Longrightarrow \frac{\partial \Omega_k}{\partial E} \quad &< \quad 0 \end{aligned} \tag{2.18}$$

<u>2.2.1.1. The Variational Method.</u> We want to show that there is an upper bound to the ground state energy by means of variational principle. We choose a trial function:

$$|\Omega_* \rangle = \frac{1}{\sqrt{n!}} a_0^\dagger ... a_0^\dagger | 0 \rangle \tag{2.19}$$

where one has n creation operators with $\sigma = 0$. This is possible on a compact manifold since $(-\nabla^2)\frac{1}{\sqrt{V(M)}} = 0$ is a constant solution [17], where $\frac{1}{\sqrt{V(M)}}$ is chosen for the sake of normalization $\int |f|^2 dv = 1$ The zero's of the principal operator give us bound state energies since they are the poles of the resolvent. Hence if we can show that :

$$\Omega_0(E_*) \leqslant \langle \Omega_* | \Phi_R(E_*) | \Omega_* \rangle < 0 \tag{2.20}$$

we can deduce, using 2.18, that $E_{gr} < \langle \Omega_* | \Phi_R(E_*) | \Omega_* \rangle$

Making a trivial guess, we set $E_* = nm + \mu_p$, corresponding to the sector $Q = n + 1$.

$$\begin{aligned} \langle \Omega_* | \Phi_R(E_*) | \Omega_* \rangle &= \langle \Omega_* | (H_0 - nm) | \Omega_* \rangle \\ &+ \langle \Omega_* | (H_0 - nm) \sum_\sigma \frac{\lambda^2}{2\omega_\sigma} \frac{|f_\sigma|^2}{(H_0 - nm - \mu_p + \omega_\sigma)} \frac{1}{(\omega_\sigma - \mu_p)} | \Omega_* \rangle \\ &- \langle \Omega_* | \sum_{\sigma,\tau} \lambda^2 \frac{f_\sigma a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \frac{1}{(H_0 - nm - \mu_p + \omega_\sigma + \omega_\tau)} \frac{f_\tau a_\tau}{\sqrt{2\omega_\tau}} | \Omega_* \rangle \end{aligned} \tag{2.21}$$

We investigate each term separately:

$$\begin{aligned} \mathcal{A} &= \langle \Omega_* | (H_0 - nm) | \Omega_* \rangle \\ &= \langle 0 | \ \frac{\omega_0}{n!} (nm - nm) \ a_0 a_0 ... a_0^\dagger a_0^\dagger ... \ | 0 \rangle \\ &= 0 \end{aligned} \tag{2.22}$$

$$\begin{aligned}
\mathcal{B} &= \langle\Omega_*|(H_0 - nm)\sum_\sigma \frac{\lambda^2}{2\omega_\sigma}\frac{|f_\sigma|^2}{(H_0 - nm - \mu_p + \omega_\sigma)}\frac{1}{(\omega_\sigma - \mu_p)}|\Omega_*\rangle \\
&= \langle 0|\ \frac{\omega_0}{n!}(nm - nm)\ a_0 ... \sum_\sigma \frac{\lambda^2}{2\omega_\sigma}\frac{|f_\sigma|^2}{(H_0 - nm - \mu_p + \omega_\sigma)}\frac{1}{(\omega_\sigma - \mu_p)}a_0^\dagger ...\ |0\rangle \\
&= 0 \qquad (2.23)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C} &= -\langle\Omega_*|\sum_{\sigma,\tau}\lambda^2 f_\sigma \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}}\frac{1}{(H_0 - nm - \mu_p + \omega_\sigma + \omega_\tau)}\frac{a_\tau}{\sqrt{2\omega_\tau}}f_\tau|\Omega_*\rangle \\
&= -\frac{\lambda^2}{n!}\langle 0|a_0 ... a_0 \sum_{\sigma,\tau} f_\sigma f_\tau \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}}\frac{1}{H_0 - nm + \mu_p + \omega_\sigma + \omega_\tau}\frac{a_\tau}{\sqrt{2\omega_\tau}}a_0^\dagger ... a_0^\dagger|0\rangle \\
&= -n^2\frac{\lambda^2}{n!}\langle 0|\ \underbrace{a_0 ... a_0}_{n-1}\ \left(\frac{|f_0|^2}{m}\frac{1}{H_0 - nm + \mu_p + m + m}\right)\ \underbrace{a_0^\dagger ... a_0^\dagger}_{n-1}|0\rangle \\
&= -\frac{n\lambda^2}{m(m + \mu_p)}\ |f_0|^2 \qquad (2.24)
\end{aligned}$$

Setting $|f_0| = (\frac{1}{V(M)})^{1/2}$ where $V(M)$ is the volume of the manifold and adding up $\mathcal{A}, \mathcal{B}, \mathcal{C}$ we get:

$$\Omega_0(E_* = nm + \mu_p)\ <\ \langle\Omega_*|\Phi_R(E_*)|\Omega_*\rangle\ =\ -\frac{n\lambda^2}{V(M)}\frac{1}{m(m + \mu_p)} \qquad (2.25)$$

Note that $\Omega_0(nm + \mu_P) < 0$ and we know, $\frac{\partial\Omega}{\partial E} < 0$ (equation 2.18). Hence we need to *reduce* E to get $\Omega_0(E_{gr}) = 0$ which is the sought after solution of the bound state energy. This implies the following inequality for the actual ground state energy:

$$E_{gr} < nm + \mu_p \tag{2.26}$$

2.2.2. A Lower Bound

Recall that the Principal Operator is:

$$\begin{aligned}\Phi(E) &= (H_0 - E + \mu_p)\{1 + \underbrace{\frac{\lambda^2}{2}\sum_\sigma f_\sigma^2(\bar{x})\frac{1}{\omega_\sigma(\omega_\sigma - \mu_p)(H_0 - E + \omega_\sigma)}}_{K(E)}\} \\ &- \frac{\lambda^2}{2}\sum_{\sigma,\tau} f_\sigma(\bar{x})f_\tau(\bar{x})\frac{a_\sigma^\dagger}{\sqrt{\omega_\sigma}}\frac{1}{H_0 - E + \omega_\sigma + \omega_\tau}\frac{a_\tau}{\sqrt{\omega_\tau}}\end{aligned} \tag{2.27}$$

Since we are focusing on the ground state energy, let us work in the sector where $E < nm + \mu_p$. We also know that $\mu_p < m$ since otherwise there would be no binding. Hence one deduces straightforwardly that $K(E)$ is strictly positive, which leads us to the following operator inequality provided that E s real:

$$\begin{aligned}\Phi_R \geq (H_0 - E + \mu_p)^{1/2}\big[1 - \lambda^2\sum_{\sigma,\tau}\frac{f_\sigma f_\tau}{2\sqrt{\omega_\sigma\omega_\tau}}a_\sigma^\dagger\frac{1}{(H_0 - E + \mu_p + \omega_\sigma)^{1/2}} \\ \frac{1}{(H_0 - E + \omega\sigma + \omega_\tau)}\frac{1}{(H_0 - E + \mu_p + \omega_\tau)^{1/2}}a_\tau\big](H_0 - E + \mu_p)^{1/2}\end{aligned} \tag{2.28}$$

Call the second term in the square brackets as $\mathcal{U}$. Then 2.28 is reduced to:

$$\Phi_R \geq (H_0 - E + \mu_p)^{1/2}\left[1 - \mathcal{U}(E)\right](H_0 - E + \mu_p)^{1/2} \tag{2.29}$$

If $||\mathcal{U}(E)|| < 1$, the right hand side is formally invertible and so is the Principal Operator. Hence if we can find E_* below which $||\mathcal{U}(E)|| < 1$, we can deduce directly:

$$E_{gr} \geq E_* \tag{2.30}$$

2.2.2.1. About the Norm of an Operator. Say one has an operator of the following form:

$$\hat{F} = \sum_{\sigma,\tau} F(\sigma,\tau) a^\dagger_\sigma a_\tau \tag{2.31}$$

Take the following normalized state (for we work in the $Q = n + 1$ sector) :

$$|\psi\rangle = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_{n+1}} \frac{1}{\sqrt{(n+1)!}} \psi(\sigma_1, \sigma_2, ..., \sigma_{n+1}) a^\dagger_{\sigma_1} a^\dagger_{\sigma_2} ... a^\dagger_{\sigma_{n+1}} |0\rangle \tag{2.32}$$

where ψ is symmetric in all of its entries. If one lets $\hat{F}$ operate on this state, one finds:

$$\hat{F}|\psi\rangle = \frac{1}{\sqrt{(n+1)!}}\Big(\sum_{\sigma,\tau} F(\sigma,\tau) \sum_{\sigma_2,\ldots,\sigma_{n+1}} \psi(\tau,\sigma_2,\ldots)a^\dagger_\sigma a^\dagger_{\sigma_2}\ldots a^\dagger_{\sigma_{n+1}}|0\rangle$$
$$+ \sum_{\sigma,\tau} F(\sigma,\tau) \sum_{\sigma_1,\sigma_2,\ldots,\sigma_{n+1}} \psi(\sigma_1,\sigma_2,\ldots,\sigma_{n+1})a^\dagger_\sigma a^\dagger_{\sigma_1} a_\tau a^\dagger_{\sigma_2}\ldots|0\rangle\Big) \tag{2.33}$$

If one keeps normal ordering and finally takes the norm:

$$|\hat{F}|\psi\rangle|^2 = \Big[(n+1)\sum_{\sigma_2,\ldots,\sigma_{n+1}} \sum_{\sigma,\tau} F(\sigma,\tau)\psi(\tau,\sigma_2,\ldots)\Big]^2 \tag{2.34}$$

where we made use of the fact that ψ is symmetric in its entries to bring the resulting term at each step to the same form. Applying Cauchy- Schwartz inequality to 2.34, one deduces:

$$sup_\psi \quad |(n+1)\sum_{\sigma_2,\ldots,\sigma_{n+1}} \sum_{\sigma,\tau} F(\sigma,\tau)\psi(\tau,\sigma_2,\ldots)|^2$$
$$\leq \quad (n+1)^2 \sum_{\sigma,\tau} |F(\sigma,\tau)|^2 \underbrace{\sum_{\sigma_1,\sigma_2,\ldots,\sigma_{n+1}} |\psi(\sigma_1,\sigma_2,\ldots,\sigma_{n+1})|^2}_{1} \tag{2.35}$$

Note that if one thinks of $F(\sigma,\tau)$ as the matrix elements of $\hat{F}$, then $\sum_{\sigma,\tau}|F(\sigma,\tau)|^2$ is just the Hilbert-Schmidt norm squared hence one can write the following inequality:

$$||F(\sigma,\tau)a_\sigma^\dagger a_\tau|| < (n+1)||\hat{F}||_{\mathscr{I}_2} \tag{2.36}$$

where the subscript $\mathscr{I}_2$ stands for the Hilbert-Schmidt norm on the Hilbert Space $\mathcal{H}$.

Having completed a short digression, now define $\chi = nm - E$. Noting that $H_0 \geq nm$, one can replace $H_0 - E$'s by χ and bring $\mathcal{U}$ in the form of 2.31 to write down the following inequality, similar to 2.36:

$$||\mathcal{U}(E)|| \leq \frac{(n+1)\lambda^2}{2} \Big[\sum_{\sigma,\tau} \frac{|f_\sigma|^2|f_\tau|^2}{\omega_\sigma\omega_\tau(\chi+\omega_\sigma)(\chi+\omega_\sigma+\omega_\tau)^2(\chi+\omega_\tau)}\Big]^{1/2} \tag{2.37}$$

where we have also omitted μ_p's for convenience. Using the crude inequality:

$$(\chi+\omega_\sigma+\omega_\tau)^2 > (\chi+\omega_\sigma)(\chi+\omega_\tau) \tag{2.38}$$

we decouple σ and τ to get:

$$||\mathcal{U}(E)|| \leq \frac{(n+1)\lambda^2}{2} \sum_{\sigma} \frac{|f_\sigma|^2}{\omega_\sigma(\chi+\omega_\sigma)^2} \tag{2.39}$$

Using Feynman parametrization, exponentiation and subordination identity consecutively one gets:

$$||\mathcal{U}(E)|| \leq \frac{(n+1)\lambda^2}{2}\int_0^1 \xi d\xi \int_0^\infty \frac{s^3 ds}{2\sqrt{\pi}} e^{-s\xi\chi} \int_0^\infty u^{-3/2} e^{-s^2/4u} \underbrace{\sum_\sigma |f_\sigma|^2 e^{-u\omega_\sigma^2}}_{K_u(\bar{x},\bar{x})e^{-um^2}}$$

One has the following estimate for the heat kernel on a compact manifold [12]:

$$K_u(\bar{x},\bar{x}) \leq \frac{1}{V(m)} + \frac{C}{u} \tag{2.40}$$

where $V(M)$ is the volume of the manifold and C is a positive constant. Hence we have:

$$\begin{aligned}
||\mathcal{U}(E)|| \;&\leq\; \frac{(n+1)\lambda^2}{4\sqrt{\pi}} \int_0^1 \xi d\xi \int_0^\infty s^3 ds e^{-s\xi\chi} \int_0^\infty u^{-3/2} e^{-s^2/4u} \Big(\frac{1}{V(M)} + \frac{C}{u}\Big) e^{-mu^2} \\
&\leq\; (n+1)\lambda^2 \Big\{ \int_0^1 \xi d\xi \frac{1}{V(M)} \frac{1}{(\chi\xi+m)^3} \;+\; \int_0^1 \xi d\xi \frac{C(2m+\chi\xi)}{(m+\chi\xi)^2} \Big\} \\
&\leq\; (n+1)\lambda^2 \Big\{ \frac{1}{V(M)2m(m+\chi)^2} + \frac{C}{m+\chi} \Big\} \\
&\leq\; (n+1)\lambda^2 \Big\{ \frac{1}{2mV(M)\chi^2} + \frac{C}{\chi} \Big\}
\end{aligned} \tag{2.41}$$

For $\chi > m$, one can replace one of the χ's by m:

$$||\mathcal{U}(E)|| \leq (n+1)\lambda^2 \Big\{ \frac{1}{2m^2 V(M)} + C \Big\} \frac{1}{\chi} \tag{2.42}$$

If we impose the condition:

$$\frac{(n+1)\lambda^2}{\chi} \Big\{ \frac{1}{2m^2 V(M)} + C \Big\} < 1 \tag{2.43}$$

$||\mathcal{U}(E)|| < 1$ is guaranteed. Substituting $\chi = nm - E$ one has the lower bound for the ground state energy:

$$nm - (n+1)\lambda^2 \Big(\frac{1}{2m^2 V(M)} + C \Big) \ < \ E_{gr} \tag{2.44}$$

which was first presented in [19].

2.3. The Resolvent Issue

Definition 1. *Let Δ be a subset of the complex plane. A family $J(\lambda)$, $\lambda \in \Delta$, of bounded linear operators on X (X being the Banach Space) satisfying:*

$$J(\lambda) - J(\mu) = (\lambda - \mu)J(\lambda)J(\mu) \tag{2.45}$$

is called a pseudo-resolvent on Δ .

For us, $X = \mathcal{F}^{(n+1)} \otimes \chi_\downarrow \oplus \mathcal{F}^{(n)} \otimes \chi_\uparrow$, thus a Hilbert space. (*Remark: We are not working over the full Fock space but over a chosen sector since Q is conserved.*) We will start by showing that $R(E) = \frac{1}{H-E}$ is a pseudo-resolvent and check the conditions under which there exists a densely defined closed linear operator H such that $R(E)$ is the resolvent family of H according to the following corollary [13] :

Corollary. *Let Δ be an unbounded subset of $\mathbb{C}$ and let $R(E)$ be a pseudo-resolvent on Δ. If there is a sequence $E_n \in \Delta$ such that $|E_n| \to \infty$ as $n \to \infty$ and*

$$s - \lim_{n\to\infty} E_n R(E_n)x = -x \quad for\ all\ x \in X \tag{2.46}$$

then R(E) is the resolvent of a unique densely defined closed operator H.

2.3.1. Pseudo-resolvent

To show that $R(E)$ is a pseudo-resolvent, we need to check:

$$R(E_1) - R(E_2) \stackrel{?}{=} (E_1 - E_2)\Big(R(E_1) - R(E_2)\Big) \tag{2.47}$$

which is equivalent, according to 2.8 :

$$\begin{bmatrix} \alpha(E_1) - \alpha(E_2) & \gamma(E_1) - \gamma(E_2) \\ \beta(E_1) - \beta(E_2) & \delta(E_1) - \delta(E_2) \end{bmatrix} =$$
$$\begin{bmatrix} \alpha(E_1)\alpha(E_2) + \gamma(E_1)\beta(E_2) & \alpha(E_1)\gamma(E_2) + \gamma(E_1)\delta(E_2) \\ \beta(E_1)\alpha(E_2) + \delta(E_1)\beta(E_2) & \beta(E_1)\gamma(E_2) + \delta(E_1)\delta(E_2) \end{bmatrix}$$

where one can always refer to 2.9 for the definitions of the terms. We remark that all operators are bounded here hence there are no issues about domains. Noting that the free resolvent $R_0 = \frac{1}{H_0 - E}$ satisfies 2.45, it is straightforward to show that 2.47 reduces to:

$$\begin{aligned} R_0(E_1) b^\dagger \Phi^{-1}(E_2) \, \Big[\Phi(E_1) - \Phi(E_2) + b \Big(R_0(E_1) - R_0(E_2) \Big) b^\dagger \\ + E_1 - E_2 \Big] \, \Phi^{-1}(E_2) b R_0(E_2) \stackrel{?}{=} 0 \end{aligned} \tag{2.48}$$

We can check 2.48 by direct substitution. Calculating the term in square brackets term by term:

$$\begin{aligned} \mathcal{A} = \Phi(E_1) - \Phi(E_2) &= (H_0 - E_1 + \mu_p) - (H_0 - E_2 + \mu_p) \\ &+ \lambda^2 \sum_\sigma \frac{|f_\sigma|^2}{2\omega_\sigma} \frac{E_2 - E_1}{(H_0 - E_1 + \omega_\sigma)(H_0 - E_2 + \omega_\sigma)} \\ &+ \lambda^2 \sum_{\sigma,\tau} f_\sigma(\bar{x}) \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \frac{E_2 - E_1}{(H_0 - E_1 + \omega_\sigma + \omega_\tau)} \\ &\qquad \frac{1}{(H_0 - E_2 + \omega_\sigma + \omega_\tau)} \frac{a_\tau}{\sqrt{2\omega_\tau}} \end{aligned} \tag{2.49}$$

$$\begin{aligned}
\mathcal{B} = b\Big(R_0(E_1) - R_0(E_2)\Big)b^\dagger &= (\lambda \sum_\sigma \frac{f_\sigma(\bar{x})}{\sqrt{2\omega_\sigma}} a_\sigma)\big[\frac{1}{H_0 - E_1} - \frac{1}{H_0 - E_2}\big] \\
&\qquad (\lambda \sum_\tau \frac{f_\tau(\bar{x})}{\sqrt{2\omega_\tau}} a_\tau^\dagger) \\
&= \frac{\lambda^2}{2} \sum_{\sigma,\tau} \frac{f_\sigma f_\tau}{\sqrt{\omega_\sigma \omega_\tau}} (E_1 - E_2) \frac{1}{H_0 - E_1 + \omega_\sigma} \\
&\qquad a_\sigma a_\tau^\dagger \frac{1}{H_0 - E_2 + \omega_\tau} \\
&= \frac{\lambda^2}{2} \sum_{\sigma,\tau} \frac{f_\tau a_\tau^\dagger}{\sqrt{\omega_\tau}} \frac{E_1 - E_2}{(H_0 - E_1 + \omega_\sigma + \omega_\tau)} \\
&\qquad \frac{1}{(H_0 - E_2 + \omega_\sigma + \omega_\tau)} \frac{f_\tau a_\tau}{\sqrt{\omega_\tau}} \\
&+ \frac{\lambda^2}{2} \sum_\sigma \frac{|f_\sigma|^2}{\omega_\sigma} \frac{E_1 - E_2}{(H_0 - E_1 + \omega_\sigma)(H_0 - E_2 + \omega_\sigma)}
\end{aligned}$$

$$\mathcal{C} = E_1 - E_2 \tag{2.50}$$

Trivially,

$$\mathcal{A} + \mathcal{B} + \mathcal{C} = 0 \tag{2.51}$$

Hence $R(E)$ is indeed a pseudo-resolvent.

2.3.2. The Decay Condition

To show that the resolvent R satisfies Corollary 2.3, we choose a series λ_k on the negative real axis such that for every k, $\lambda_k < 0 < E_{gr}$. Since $\lambda_k = -|\lambda_k|$, one can write down the condition 2.46 as:

$$\lim_{k\to\infty} \left|\left|[|\lambda_k|R(-|\lambda_k|) - 1]x\right|\right|_{\mathcal{H}} = 0 \tag{2.52}$$

Substituting the resolvent 2.8 in the previous equation, one gets 2.52 in the following form:

$$\lim_{k\to\infty} \left|\left| \begin{matrix} \big(|\lambda_k|\alpha(-|\lambda_k|) - 1\big)|f^{n+1}\rangle \;+\; |\lambda_k|\;\gamma(-|\lambda_k|)|f^n\rangle \\ |\lambda_k|\beta(-|\lambda_k|)\;|f^{n+1}\rangle \;+\; \Big(|\lambda_k|\delta(-|\lambda_k|) - 1\Big)|f^n\rangle \end{matrix} \right|\right| = 0 \tag{2.53}$$

Applying the triangular inequality twice, one concludes:

$$\begin{aligned} &\left|\left| \Big[|\lambda_k|R(-|\lambda_k|) - 1\Big] \begin{pmatrix} |f^{n+1}\rangle \\ |f^n\rangle \end{pmatrix} \right|\right| \\ &\leq ||\;\big(|\lambda_k|\alpha(-|\lambda_k|) - 1\big)|f^{n+1}\rangle\;|| + ||\;|\lambda_k|\;\gamma(-|\lambda_k|)|f^n\rangle|| \qquad (2.54) \\ &+\;||\;|\lambda_k|\beta(-|\lambda_k|)\;|f^{n+1}\rangle|| + ||\;\Big(|\lambda_k|\delta(-|\lambda_k|) - 1\Big)|f^n\rangle|| \qquad (2.55) \end{aligned}$$

Hence if one can show that as $k \to \infty$, each term in 2.54 and 2.55 goes to zero separately , 2.52 can be directly deduced.

2.3.2.1. Behaviour of the Φ Operator.

$$\Phi(E) = (H_0 - E + \mu_p)\Big\{1 + \underbrace{\frac{\lambda^2}{2}\sum_\sigma f_\sigma^2(\bar{x})\frac{1}{\omega_\sigma(\omega_\sigma - \mu_p)(H_0 - E + \omega_\sigma)}}_{K(E)}$$

$$- \underbrace{\frac{\lambda^2}{2}\sum_{\sigma,\tau} f_\sigma(\bar{x})f_\tau(\bar{x})\frac{1}{H_0 - E + \mu_p}\frac{a_\sigma^\dagger}{\sqrt{\omega_\sigma}}\frac{1}{H_0 - E + \omega_\sigma + \omega_\tau}\frac{a_\tau}{\sqrt{\omega_\tau}}}_{U(E)}\Big\}$$

To understand the behaviour of Φ and make use of it in the upcoming calculations, we estimate K and U here. If one applies Feynman parametrization to K:

$$\begin{aligned} K &= \mathcal{C}\sum_\sigma f_\sigma^2 \int_0^1 du_1du_2du_3 \frac{\delta(1 - \sum_{i=1}^3 u_i)}{[u_1\omega_\sigma + u_2(\omega_\sigma - \mu_p) + u_3(\omega_\sigma + H_0 - E)]^3} \\ &= \mathcal{C}\sum_\sigma f_\sigma^2 \int_0^1 du_1du_2du_3\ \delta(1 - \sum_{i=1}^3 u_i)\int s^2dse^{-sA} \end{aligned} \tag{2.56}$$

where all constants are absorbed into $\mathcal{C}$ and we defined $A = u_1\omega_\sigma + u_2(\omega_\sigma - \mu_p) + u_3(\omega_\sigma + H_0 - E)$. Note that because of Dirac Delta, $u_1 + u_2 + u_3 = 1$ and we have:

$$K = \mathcal{C}\int_0^1 du_1du_2du_3\ \delta(1 - \sum_{i=1}^3 u_i)\int s^2ds[\sum_\sigma f_\sigma^2 e^{-s\omega_\sigma}]e^{s\mu_p u_2}e^{-su_3(H_0-E)} \tag{2.57}$$

Now one can apply the subordination identity to $e^{-s\omega_\sigma}$ and substitute the heat kernel:

$$K = \mathcal{C} \int_0^1 du_1 du_2 du_3 \; \delta(1 - \sum_{i=1}^{3} u_i)$$

$$\int s^3 ds \int d\xi \frac{e^{-\frac{s^2}{4\xi} - m^2\xi}}{\xi^{3/2}} K_\xi(\bar{x}, \bar{x}) e^{s\mu_p u_2} e^{-su_3(H_0 - E)} \quad (2.58)$$

If one substitutes the estimate for the heat kernel 2.40 and takes the norm:

$$||K|| \leq \mathcal{C} \int s^3 ds du_2 du_3 e^{su_2\mu_p} e^{-s(nm - Re(E))u_3}$$

$$\int d\xi \; \frac{m \; e^{-\frac{s^2 m^2}{4\xi}} \; e^{-\xi}}{\xi^{3/2}} \; \left[\frac{C}{\xi/m^2} + \frac{1}{V(M)} \right] \quad (2.59)$$

where we estimated H_0 as (nm) and note that $E \to Re(E)$ when one takes the norm. The two parts of the integral will be examined separately, we call them $||K||_1$ and $||K||_2$. Looking at the first par ($\frac{C}{\xi/m^2}$ term), one directly recognizes the integral representation of the modified Bessel function of the second kind:

$$K_\nu(z) = \frac{1}{2}(\frac{1}{2}z)^\nu \int_0^\infty \frac{1}{\xi^{\nu+1}} e^{-\xi - \frac{z^2}{4\xi}} d\xi \quad (2.60)$$

where in our case $z = ms$ and $\nu = 3/2$. Then the integral can be computed to get:

$$K_{3/2}(ms) = \sqrt{\frac{\pi}{2ms}}e^{-ms}\Big[1 + \frac{1}{ms}\Big] \tag{2.61}$$

Hence we get:

$$||K||_1 \leq \mathcal{C}\int msdsdu_2du_3e^{su_2\mu_p}e^{-s(nm-Re(E))u_3}e^{-ms}\Big[1 + \frac{1}{ms}\Big] \tag{2.62}$$

where we absorb every constant into $\mathcal{C}$. If one finds an upper bound to the most singular part of the integral (i.e. the second term in square brackets), the bound would also be valid for the other part. As we will take the limit $E = \lambda_k \to -\infty$, the bound we are going to find will be enough for our future calculations. Denote the norm coming from the most singular part as $||K||_{1-sing}$. Introducing $u_1 + u_2 + u_3 = 1$:

$$\begin{aligned}
||K||_{1-sing} &\leq \mathcal{C}\int dsdu_1du_2du_3\delta(1-\sum_{i=1}^{3}u_i)e^{su_2\mu_p}e^{-s(nm-Re(E))u_3}e^{-ms(u_1+u_2+u_3)} \\
&\leq \mathcal{C}\int dsdu_1du_2du_3\delta(1-\sum_{i=1}^{3}u_i)e^{-smu_1}e^{-s(m-\mu_p)u_2}e^{-s\big((n+1)m-Re(E)\big)u_3} \\
&\leq \mathcal{C}\int du_1du_2du_3\delta(1-\sum_{i=1}^{3}u_i) \\
&\qquad \frac{1}{mu_1+(m-\mu_p)u_2+[(n+1)m-Re(E)]u_3}
\end{aligned}$$

If one ignores the term $(m-\mu_p)u_3$ and take the u_2 integral:

$$||K||_{1-sing} \leq \mathcal{C} \int\limits_{0 \leq u_1 + u_3 \leq 1} \frac{du_1 du_3}{mu_1 + [(n+1)m - Re(E)]u_3} \tag{2.63}$$

Note that the region $0 \leq u_1 + u_3 \leq 1$ is contained in $u_1^2 + u_3^2 \leq 1$. Hence we can integrate in the latter since the integrand is positive. One can go to polar coordinates where $u_1 = \rho \cos\theta$ and $u_3 = \rho \sin\theta$:

$$||K||_{1-sing} \leq \mathcal{C} \int\limits_0^1 \int\limits_0^{\pi/2} \frac{\rho d\rho d\theta}{m\rho\cos\theta + [(n+1)m - Re(E)]\rho\sin\theta} \tag{2.64}$$

Since in the first quadrant both sine and cosine are positive, one can write the inequalities $\cos\theta \geq \cos^2\theta$ and $\sin\theta \geq \sin^2\theta$. We then replace cosine and sine with the squares and turn the integral to a more familiar form:

$$\begin{aligned} ||K||_{1-sing} &\leq \mathcal{C} \int\limits_0^1 \int\limits_0^{\pi/2} \frac{d\rho d\theta}{m\cos^2\theta + [(n+1)m - Re(E)]\sin^2\theta} \\ &\leq \mathcal{C} \frac{\pi}{\sqrt{m}} \frac{1}{\sqrt{(n+1)m - Re(E)}} \end{aligned} \tag{2.65}$$

Hence in the limit

$$E = \lambda_k \to -\infty \quad , \quad ||K||_1 \to 0 \tag{2.66}$$

For $||K||_2$, we proceed similarly:

$$||K||_2 = \mathcal{C} \int s^3 ds du_1 du_2 du_3 \delta(1 - \sum_{i=1}^{3} u_i) e^{su_2\mu_p} e^{-s(nm-Re(E))u_3}$$
$$\int d\xi \; \frac{m \; e^{-\frac{s^2m^2}{4\xi}} e^{-\xi}}{\xi^{3/2}} \; \Big[\frac{1}{V(M)}\Big]$$

$$\begin{aligned}
||K||_2 &\leq \frac{\mathcal{C}}{V(M)} \int ds du_2 du_3 s^{5/2} e^{su_2\mu_p} e^{-s(nm-Re(E))u_3} \frac{e^{-ms}}{\sqrt{ms}} \\
&\leq \frac{\mathcal{C}}{V(M)} \int ds du_1 du_2 du_3 \delta(1 - \sum_{i=1}^{3} u_i) s^2 e^{-s[mu_1+(m-\mu_p)u_2+((n+1)m-Re(E))u_3]} \\
&\leq \frac{\mathcal{C}}{V(M)} \int_0^1 du_1 du_2 du_3 \delta(1 - \sum_{i=1}^{3} u_i) \\
&\qquad \frac{1}{\big[mu_1 + (m-\mu_p)u_2 + \big((n+1)m - Re(E)\big)u_3\big]^3} \\
&\leq \frac{\mathcal{C}}{V(M)} \int_0^1 du_3 \frac{1}{[(m-\mu_p) + (nm+\mu_p - Re(E))u_3]^3} \\
&\leq \frac{\mathcal{C}}{V(M)} \Big[\frac{1}{2(nm+\mu_p - Re(E))(m-\mu_p)^2}\Big]
\end{aligned} \tag{2.67}$$

It is straightforward to see that:

$$E = \lambda_k \to -\infty \quad , \quad ||K||_2 \to 0 \tag{2.68}$$

Hence we conclude:

$$E = \lambda_k \to -\infty \quad , \quad ||K|| \to 0 \tag{2.69}$$

One also has the bound for $||U||$, see eq 2.42. Substituting back $\chi = nm - Re(E)$, it is straightforward to take the limit and see that:

$$E = \lambda_k \to -\infty \quad , \quad ||U|| \to 0 \tag{2.70}$$

We add one more result to this section for future simplicity. Note that:

$$|\lambda_k| \; ||\Phi^{-1}(-|\lambda_k|)|| = |\lambda_k| \; ||(1 + K(-|\lambda_k|) - U(-|\lambda_k|))^{-1}|| \; || \; (H_0 + \mu_p + |\lambda_k|)^{-1}|| \tag{2.71}$$

When $||A|| < 1$, one can write down the Neumann equality:

$$(1 - A)^{-1} = \sum_{l=0}^{\infty} A^l \tag{2.72}$$

We have the triangle inequality:

$$||U - K|| \;\; \leq \;\; ||U|| + ||K|| \tag{2.73}$$

Hence using 2.69 and 2.70, one can deduce that:

$$\lim_{|\lambda_k|\to\infty} ||U - K|| = 0 \tag{2.74}$$

Hence the Neumann series expansion is applicable:

$$\begin{aligned} ||(1 + K(-|\lambda_k|) - U(-|\lambda_k|))^{-1}|| &= ||1 + \sum_{l=1}^{\infty}(U - K)^l|| \\ &\leq 1 + ||U - K|| + ||U - K||^2 + ... \end{aligned} \tag{2.75}$$

where we have used the triangle inequality and the following inequality for linear operators [20]:

$$||A^n|| \leq ||A||^n \tag{2.76}$$

Hence taking the limit one can deduce:

$$|\lambda_k| \; ||\Phi^{-1}(-|\lambda_k|)|| \leq |\lambda_k| \; |(nm + |\lambda_k|)^{-1}| \tag{2.77}$$

It is obvious that in the limit $|\lambda_k| \to \infty$ the right hand side goes to one. Hence $|\lambda_k| \; ||\Phi^{-1}(-|\lambda_k|)||$ is finite.

<u>2.3.2.2. The β term.</u> One can write the inequality :

$$\Big\| |\lambda_k|\Phi^{-1}(-|\lambda_k|)\phi^{(+)}\frac{1}{H_0+|\lambda_k|}|f^{n+1}\rangle\Big\| \leq |\lambda_k|\ ||\Phi^{-1}||\ ||\phi^{(+)}\frac{1}{H_0+|\lambda_k|}|f^{n+1}\rangle|| \tag{2.78}$$

without any problems since each term on the right hand side is bounded. We know that $|\lambda_k|\Phi^{-1}$ is controllable and its operator norm is finite. We need to work on:

$$||\ \phi^{(+)}\frac{1}{H_0+|\lambda_k|}|f^{n+1}\rangle\ || = ||\ \sum_\sigma \underbrace{\frac{f_\sigma}{\sqrt{2\omega_\sigma}}\frac{1}{H_0+|\lambda_k|+\omega_\sigma}}_{g(\sigma)} a_\sigma|f^{n+1}\rangle\ || \tag{2.79}$$

Expanding $|f^{n+1}\rangle$ as in 2.32 and normal ordering one gets:

$$\begin{aligned} ||\ \sum_\sigma g(\sigma)a_\sigma|f^{n+1}\rangle\ ||^2 &= (n+1)^2||\ \sum_\sigma \sum_{\sigma_2,\ldots,\sigma_{n+1}} g(\sigma)\psi(\sigma,\sigma_2,\ldots\sigma_{n+1})\ ||^2 \\ &\leq (n+1)^2 \sum_\sigma |g(\sigma)|^2 \sum_{\sigma_1,\ldots,\sigma_{n+1}} |\psi(\sigma_1,\ldots,\sigma_{n+1})|^2 \\ &\leq (n+1)^2 \sum_\sigma |g(\sigma)|^2\ ||f^{n+1}||^2 \end{aligned} \tag{2.80}$$

Substituting $g(\sigma)$ and estimating H_0 as nm:

$$\begin{aligned} \Big\|\sum_\sigma \frac{f_\sigma}{H_0+|\lambda_k|+\omega_\sigma}\frac{a_\sigma}{\sqrt{2\omega_\sigma}}|f^{n+1}\rangle\Big\| &\leq (n+1)\Big[\sum_\sigma \frac{|f_\sigma|^2}{(nm+|\lambda_k|+\omega_\sigma)^2\omega_\sigma}\Big]^{1/2}||f^{n+1}|| \\ &\leq (n+1)\Big[\sum_\sigma \frac{|f_\sigma|^2}{(nm+|\lambda_k|)^2+\omega_\sigma^2}\frac{1}{\omega_\sigma}\Big]^{1/2}||f^{n+1}|| \end{aligned}$$

If one applies Feynman parametrization to the term in square brackets:

$$
\sum_\sigma \frac{|f_\sigma|^2}{(nm+|\lambda_k|)^2+\omega_\sigma^2}\frac{1}{(\omega_\sigma^2)^{1/2}} = \frac{1}{2}\int_0^1 \sum_\sigma \frac{d\xi(1-\xi)^{-1/2}|f_\sigma|^2}{[\xi(nm+|\lambda_k|)^2+\omega_\sigma^2]^{3/2}}
$$
$$
= \frac{1}{2}\int_0^1 \frac{d\xi}{\sqrt{1-\xi}}\int_0^\infty ds\sqrt{s}e^{-s\xi(nm+|\lambda_k|)^2}\underbrace{\sum_\sigma |f_\sigma|^2 e^{-s\omega_\sigma^2}}_{K_s(\bar{x},\bar{x})e^{-sm^2}} \tag{2.81}
$$

Substituting the estimate for the heat kernel 2.40 and computing the integrals we arrive at the following inequality:

$$
\begin{aligned}
\sum_\sigma \frac{|f_\sigma|^2}{(nm+|\lambda_k|)^2+\omega_\sigma^2}\frac{1}{(\omega_\sigma^2)^{1/2}} \quad &\leq \quad \frac{1}{2}\underbrace{\int_0^1 \frac{d\xi}{\sqrt{\xi(1-\xi)}}}_{\pi} C \underbrace{\int_0^\infty dse^{-s^2(nm+|\lambda_k|)^2}}_{\frac{1}{nm+|\lambda_k|}} \\
&+ \quad \frac{1}{2}\frac{1}{V(M)}\underbrace{\int_0^1 \frac{d\xi}{\sqrt{\xi(1-\xi)}}}_{\pi}\frac{1}{nm+|\lambda_k|}\tilde{C} \\
&\leq \quad \Big[\frac{C}{2}+\frac{\tilde{C}}{2V(M)}\Big]\frac{\pi}{nm+|\lambda_k|}
\end{aligned} \tag{2.82}
$$

where $\tilde{C}$ is a finite constant introduced for notational simplicity, which can be easily recovered. Now one can substitute everything into 2.78 to get:

$$\left\| |\lambda_k| \Phi^{-1} \phi^{(+)} \frac{1}{H_0 + |\lambda_k|} |f^{n+1}\rangle \right\| \;\; \leq \;\; \underbrace{|\lambda_k| ||\Phi^{-1}||}_{finite} \, C \Big[\frac{1}{nm + |\lambda_k|} \Big]^{1/2} \tag{2.83}$$

where we have collected all constants in C for simplicity and made use of the fact that $|f^{n+1}\rangle$ is normalized. It is straightforward to see:

$$\lim_{|\lambda_k| \to \infty} |\lambda_k| ||\Phi^{-1}|| A \Big[\frac{1}{nm + |\lambda_k|} \Big]^{1/2} \;\; = \;\; 0 \tag{2.84}$$

which implies directly that the left hand side of 2.83 also goes to zero, exactly what we aimed to show:

$$\lim_{|\lambda_k| \to \infty} || \; |\lambda_k| \beta(-|\lambda_k|) \; |f^{n+1}\rangle \; || = 0 \tag{2.85}$$

2.3.2.3. The γ Term. We proceed as we did in the previous section.

$$\left\| |\lambda_k| \frac{1}{H_0 + |\lambda_k|} \sum_\sigma \frac{f_\sigma}{\sqrt{2\omega_\sigma}} a_\sigma^\dagger \Phi^{-1} |f^n\rangle \right\| \leq |\lambda_k| \; ||\Phi^{-1}|| \; || \sum_\sigma \frac{1}{H_0 + |\lambda_k|} \frac{f_\sigma}{\sqrt{2\omega_\sigma}} a_\sigma^\dagger |f^n\rangle ||$$

Focusing on the last term :

$$||\sum_\sigma \frac{1}{H_0+|\lambda_k|}\frac{f_\sigma}{\sqrt{2\omega_\sigma}}a_\sigma^\dagger|f^n\rangle|| \leq \sqrt{n+1}\Big(\sum_\sigma \frac{|f_\sigma|^2}{(nm+|\lambda_k|+\omega_\sigma)^2\omega_\sigma}\Big)^{1/2}||\,|f^n\rangle|| \qquad (2.86)$$

Note that on the right hand side, we have exactly what we had in the previous section up to some constant. Hence we inherit the result in 2.84 directly:

$$\lim_{|\lambda_k|\to\infty}|\lambda_k|||\Phi^{-1}||C\Big(\frac{1}{nm+|\lambda_k|}\Big)^{1/2} = 0 \qquad (2.87)$$

where C is a numerical constant and again, 2.87 implies directly the result that we sought for:

$$\lim_{|\lambda_k|\to\infty}||\;|\lambda_k|\gamma(-|\lambda_k|)\;|f^n\rangle\;|| = 0 \qquad (2.88)$$

2.3.2.4. The α Term. We again start with an inequality:

$$\Big\|\Big[|\lambda_k|\alpha(-|\lambda_k|)-1\Big]|f^{n+1}\rangle\Big\| \leq \Big\|\;\Big[\frac{|\lambda_k|}{H_0+|\lambda_k|}-1\Big]|f^{n+1}\rangle\;\Big\| + |\lambda_k|\;||\Phi^{-1}||\;||\frac{1}{H_0+|\lambda_k|}\phi^{(-)}||\;||\phi^{(+)}\frac{1}{H_0+|\lambda_k|}\;|f^{n+1}\rangle|| \qquad (2.89)$$

Taking the limit, it is straightforward to see that the first term on the right hand side goes to zero. The second term should be worked out in parts:

$$\lim_{|\lambda_k|\to\infty} |\lambda_k|\ ||\Phi^{-1}|| = C \tag{2.90}$$

where C is some finite constant.

$$\begin{aligned}
&||\frac{1}{H_0+|\lambda_k|}\phi^{(-)}|| = sup_\psi||\frac{1}{H_0+|\lambda_k|}\phi^{-1}|\psi\rangle|| = ||\frac{1}{H_0+|\lambda_k|}\phi^{(-)}|f^n\rangle|| \\
&\implies \lim_{|\lambda_k|\to\infty} ||\frac{1}{H_0+|\lambda_k|}\phi^{(-)}|f^n\rangle|| = \lim_{|\lambda_k|\to\infty} C\Big(\frac{1}{nm+|\lambda_k|}\Big)^{1/2} = 0
\end{aligned} \tag{2.91}$$

since we are in $Q = n+1$ sector and $\phi^{(-)}$ can only operate on an n-particle state. Note that this term is exactly the same as we had in 2.86 and hence the result.

$$\lim_{|\lambda_k|\to\infty} ||\phi^{(+)}\frac{1}{H_0+|\lambda_k|}\ |f^{n+1}\rangle|| = \lim_{|\lambda_k|\to\infty} C\Big(\frac{1}{nm+|\lambda_k|}\Big)^{1/2} = 0 \tag{2.92}$$

where C is constant and the result is taken from 2.82. With 2.90, 2.91, 2.92, we are ready to conclude that:

$$\begin{aligned}
&\lim_{|\lambda_k|\to\infty} \Big[\Big|\Big|\ \Big[\frac{|\lambda_k|}{H_0+|\lambda_k|} - 1\Big]|f^{n+1}\rangle\ \Big|\Big| \\
&\qquad + |\lambda_k|\ ||\Phi^{-1}||\ ||\frac{1}{H_0+|\lambda_k|}\phi^{(-)}||\ ||\phi^{(+)}\frac{1}{H_0+|\lambda_k|}\ |f^{n+1}\rangle||\Big] = 0
\end{aligned} \tag{2.93}$$

Hence follows the result we sought for:

$$\lim_{|\lambda_k|\to\infty} \left\| \Big[|\lambda_k| \alpha(-|\lambda_k|) - 1 \Big] |f^{n+1}\rangle \right\| = 0 \tag{2.94}$$

2.3.2.5. The δ Term.

$$\left\| (\ |\lambda_k| \Phi^{-1} - 1 \ ) |f^n\rangle \right\| \ = \ \left\| \ \Big[|\lambda_k| (H_0 + |\lambda_k| + \mu_p)^{-1} (1 + K - U)^{-1} \ - 1 \Big] |f^n\rangle \right\| \tag{2.95}$$

Remember that in the limit $|\lambda_k| \to \infty$, $(1 - (U - K))^{-1}$ can be expanded as Neumann series. Hence one can write:

$$(1 - (U - K))^{-1} + 1 - 1 = 1 + \sum_{l=1}^{\infty} (U - K)^l \tag{2.96}$$

and 2.95 reduces to:

$$\begin{aligned} \left\| (\ |\lambda_k| \Phi^{-1} - 1 \ ) |f^n\rangle \right\| \ &= \ \Big\| \ \Big[|\lambda_k| (H_0 + |\lambda_k| + \mu_p)^{-1} - 1 \Big] |f^n\rangle \\ &\quad \Big[|\lambda_k| (H_0 + |\lambda_k| + \mu_p)^{-1} \sum_{l=1}^{\infty} (U - K)^l \Big] |f^n\rangle \Big\| \end{aligned} \tag{2.97}$$

Using the triangle inequality:

$$\left\|\left(\ |\lambda_k|\Phi^{-1} - 1 \ \right)|f^n\rangle\right\| \ \leq \ \left\| \ \left[|\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} - 1\right]|f^n\rangle\right\| \qquad (2.98)$$

$$+ \underbrace{\left\||\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1}\right\|}_{finite} \ \underbrace{\left\|\sum_{l=1}^{\infty}(U-K)^l\right\|}_{\to 0}$$

where we took the limit $|\lambda_k| \to \infty$ of the second line. Hence we deduce:

$$\lim_{|\lambda_k|\to\infty} \left\| \ |\lambda_k|\delta(-|\lambda_k|) - 1 \ \right\| = \lim_{|\lambda_k|\to\infty} \left\| \ \left[|\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} + 1\right]|f^n\rangle \ \right\| \qquad (2.99)$$

Note that the resulting equation is 2.46 in the Corollary 2.3, $R(E)$ being the free resolvent. Since $\frac{1}{H_0 - E}$ is *indeed* a *resolvent*, it must satisfy 2.46. Hence we deduce:

$$\lim_{|\lambda_k|\to\infty} \left\| \ \left[|\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} + 1\right]|f^n\rangle \ \right\| = 0 \qquad (2.100)$$

2.3.2.6. Conclusion. Having shown that each term on the right hand side of the equations 2.54 and 2.55 goes to zero as $|\lambda_k| \to \infty$, we can conclude that:

$$\lim_{|\lambda_k|\to\infty} \left\| \left[|\lambda_k|R(-|\lambda_k|) - 1\right] \begin{pmatrix} |f^{n+1}\rangle \\ |f^n\rangle \end{pmatrix} \right\| = 0 \qquad (2.101)$$

Hence we have shown that $R(E) = \frac{1}{H-E}$ is indeed a resolvent.

3. HOLOMORPHIC STRUCTURE OF THE PRINCIPLE OPERATOR

It is well-known that to obtain a spectral decomposition of a family of operators in which eigenvalues and the corresponding projections are holomorphic functions of the parameter, we need the notion of a self-adjoint holomorphic family of type-A in the sense of Kato. The rigorous study of this principal operator requires showing that $\Phi(E)$ is a family of this kind. This in turn justifies the fact that our resolvent formula defines a self-adjoint quantum Hamiltonian as well as putting our estimates on a firmer ground.

Definition 2. *A family $T(E) \in \mathcal{C}(X, Y)$ (closed linear operators from X to Y) defined for E in a domain D_0 of the complex plane is said to be holomorphic of type-A if:*

- *$D(T(E)) = D$ is independent of E,*
- *$T(E)u$ is holomorphic for $E \in D_0$ for every $u \in D$.*

We start by showing that the family can be given a common dense domain for $\mathbb{R}(E) < nm + \mu_p$ on which it is closed. To establish self-adjointness of the family $\Phi(E)$, we rely on the Wust's theorem that it is enough to establish this even at a single point. This in turn is true due to Kato-Rellich theorem on self-adjointness when E is sufficiently small on the real axis [21].

3.1. Fixing the Domain

We start by organizing the Principal Operator $\Phi(E)$ in the following way:

$$\begin{aligned}\Phi_R(E) &= \Big[1+\sum_{\sigma}\frac{\lambda^2}{2\omega_\sigma}\frac{|f_\sigma|^2}{(H_0-E+\omega_\sigma)}\frac{1}{(\omega_\sigma-\mu_p)} \\ &- \sum_{\omega,\tau}\lambda^2 f_\sigma\frac{a^\dagger_\sigma}{\sqrt{2\omega_\sigma}}\frac{1}{H_0-E+\omega_\sigma+\omega_\tau}\frac{1}{H_0-E+\omega_\tau+\mu_p}\frac{a_\tau}{\sqrt{2\omega_\tau}}f_\tau\Big] \\ &\qquad (H_0-E+\mu_p)\end{aligned} \tag{3.1}$$

Recall that we are working on a sector of the full Fock space, $\mathcal{H}=\mathcal{F}^{(n+1)}\otimes\chi_\downarrow\oplus\mathcal{F}^{(n)}\otimes\chi_\uparrow$, which is a Hilbert space. Call the domain of H_0 as $D(H_0)$, which is dense in $\mathcal{H}$, i.e. the closure $D(\bar{H}_0)$ is all of $\mathcal{H}$. Moreover, H_0 is closed on this domain being a self-adjoint operator.

Renaming the terms in 3.1, we rewrite $\Phi(E)$ as:

$$\Phi_R(E)=[1+\mathcal{K}(E)+\mathcal{U}(E)](H_0-E+\mu_p) \tag{3.2}$$

To fix $D(H_0)$ to be the common domain of $\Phi(E)$, we want to show that $\mathcal{K}(E)$ and $\mathcal{U}(E)$ are bounded, E being complex. To achieve this, we employ the wave function 2.32 again. Since $\mathcal{K}(E)$ and H_0 commute, the new splitting of $\Phi(E)$ does not effect the bound we found for $K(E)$ previously in 1.3.2, which is:

$$\begin{aligned}||K(E)|| &\leq \mathcal{C}\,\frac{\pi}{\sqrt{m}}\frac{1}{\sqrt{(n+1)m-Re(E)}} \\ &+\frac{\mathcal{C}}{V(M)}\Big[\frac{1}{2(nm+\mu_p-Re(E))(m-\mu_p)^2}\Big]\end{aligned} \tag{3.3}$$

For $\mathcal{U}(E)$, we had previously worked with E chosen on the real axis and the result must be generalized to the complex case. One starts with collecting the terms using Feynman parametrization:

$$\begin{aligned}\mathcal{U}(E) &= \sum_{\sigma,\tau} \lambda^2 f_\sigma \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \int_0^1 \frac{du}{[H_0 - E + (1-u)\mu_p + u\omega_\sigma + \omega_\tau]^2} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau \\ &= \sum_{\sigma,\tau} \lambda^2 f_\sigma \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \int_0^1 du \int_0^\infty s ds e^{-s(H_0-E)-s\mu_p(1-u)} \\ &\qquad e^{-su\omega_\sigma} e^{-s\omega_\tau} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau \end{aligned} \tag{3.4}$$

Taking the norm:

$$\begin{aligned}||\mathcal{U}(E)|| &= \lambda^2 \int_0^\infty s ds \int_0^1 du e^{-s\mu_p(1-u)} \Big|\Big| \sum_\sigma \frac{a_\sigma^\dagger f_\sigma}{\sqrt{2\omega_\sigma}} e^{-su\omega_\sigma} e^{sE} \\ &\qquad e^{-sH_0} \sum_\tau e^{-s\omega_\tau} \frac{a_\tau f_\tau}{\sqrt{2\omega_\tau}} \Big|\Big| \\ &= \lambda^2 \int_0^\infty s ds \int_0^1 du e^{-s\mu_p(1-u)} e^{sRe(E)} \Big|\Big| \phi^{(-)}(f) e^{-sH_0} \phi^{(+)}(g) \Big|\Big| \end{aligned} \tag{3.5}$$

where we have defined:

$$\begin{aligned}\phi^{(-)}(f) &= \int \phi^{(-)}(x) k_{su}(x,\bar{x}) dx = \sum_\sigma \frac{a_\sigma^\dagger f_\sigma(\bar{x})}{\sqrt{2\omega_\sigma}} e^{-su\omega_\sigma} \\ \phi^{(+)}(g) &= \int \phi^{(+)}(x) k_s(x,\bar{x}) dx = \sum_\tau \frac{a_\tau g_\tau(\bar{x})}{\sqrt{2\omega_\tau}} e^{-s\omega_\tau} \end{aligned} \tag{3.6}$$

and as integral kernels we have generalized heat kernels $k_t(x, \bar{x})$ [22]:

$$k_t(x,\bar{x}) = \sum_\sigma f_\sigma(x) f_\sigma(\bar{x}) e^{-t\omega_\sigma} \tag{3.7}$$

Since,

$$\begin{aligned}
\left|\left|\phi^{(-)}(f)e^{-sH_0}\phi^{(+)}(g)\right|\right| &= \Big[\sup_{||\psi||=1} \langle\psi|\phi^{(+)\dagger}(g)e^{-sH_0}\phi^{(-)\dagger}(f)\phi^{(-)}(f) \\
&\qquad\qquad e^{-sH_0}\phi^{(+)}(g)|\psi\rangle\Big]^{1/2} \\
&= \sqrt{n+1}\Big[\sup_{||\tilde{\psi}||=1} \langle\tilde{\psi}|e^{-sH_0}\phi^{(-)\dagger}(f)\phi^{(-)}(f)e^{-sH_0}|\tilde{\psi}\rangle\Big]^{1/2}
\end{aligned}$$

where $|\tilde{\psi}\rangle$ is the normalized state with n bosons. Now we can estimate e^{-sH_0} as e^{-snm}. Since $\phi^{(-)\dagger}(f)\phi^{(-)}(f)$ is positive definite, one ends up with the following inequality:

$$\left|\left|\phi^{(-)}(f)e^{-sH_0}\phi^{(+)}(g)\right|\right| \leq e^{-snm}\Big[\langle\psi|\phi^{(+)\dagger}(f)\phi^{(-)\dagger}(g)\phi^{(-)}(g)\phi^{(+)}(f)|\psi\rangle\Big]^{1/2} \tag{3.8}$$

Substituting this expression into 3.5, we have:

$$||\mathcal{U}(E)|| \;\leq\; \lambda^2 \int_0^\infty sds \int_0^1 due^{s\mu_p u}e^{-s(nm+\mu_p-Re(E))}||\phi^{(-)}(g)\phi^{(+)}(f)|| \tag{3.9}$$

One needs to show that the integral is finite since the generalized heat kernels appearing in $\phi^{(-)}$ and $\phi^{(+)}$ are singular around 0 which could be problematic. First, we estimate $||\phi^{(-)}(g)\phi^{(+)}(g)||$. Employing the wave function 2.32:

$$\begin{aligned}
\phi^{(-)}(g)\phi^{(+)}(f)|\phi\rangle &= \Phi^{(-)}(g)\sum_{\sigma}\frac{f_\sigma(\bar{x})}{\sqrt{2\omega_\sigma}}e^{-s\omega_\sigma}a_\sigma\sum_{\sigma_1,\ldots}\frac{\psi(\sigma_1,\ldots)}{\sqrt{(n+1)!}}\frac{a^\dagger_{\sigma_1}\ldots a^\dagger_{\sigma_{n+1}}}{\sqrt{2\omega_{\sigma_1}\ldots}}|0\rangle \\
&= (n+1)\sum_{\sigma_1,\ldots}\frac{g_{\sigma_1}(\bar{x})}{\sqrt{(n+1)!}}\frac{e^{-s\omega_{\sigma_1}}}{\sqrt{2\omega_{\sigma_1}}} \\
&\qquad \Big[\sum_{\sigma}\frac{\psi(\sigma,\sigma_2,\ldots)}{2\omega_\sigma}f_\sigma(\bar{x})e^{-s\omega_\sigma}\Big]\frac{a^\dagger_{\sigma_1}a^\dagger_{\sigma_2}\ldots a^\dagger_{\sigma_{n+1}}}{\sqrt{2\omega_{\sigma_1}2\omega_{\sigma_2}\ldots}}|0\rangle \quad (3.10)
\end{aligned}$$

Taking the norm and decoupling the sums by involing Cauchy-Schwartz inequality:

$$||\phi^{(-)}(g)\phi^{(+)}\ (f)|| \le (n+1)\ ||f||\ ||g||\ ||\psi|| \quad (3.11)$$

where,

$$||f||^2 = \sum_{\sigma}\frac{|f_\sigma(\bar{x})|^2}{2\omega_\sigma}e^{-2s\omega_\sigma} \quad , \quad ||g||^2 = \sum_{\sigma}\frac{|g_\sigma(\bar{x})|^2}{2\omega_\sigma}e^{-2us\omega_\sigma} \quad , \quad ||\psi|| = 1 \quad (3.12)$$

We first estimate $||f||$ and $||g||$ by employing the subordination identity:

$$||f||^2 = C \int_0^\infty s\ dt\ e^{-s^2m^2/t}\ t^{-3/2} dt \underbrace{\sum_\sigma e^{-s^2\sigma^2/t}\ |f_\sigma(\bar{x})|^2}_{K_{\frac{s^2}{t}}(\bar{x},\bar{x}) \leq \frac{c}{s^2/t} + \frac{1}{V(M)}} \tag{3.13}$$

We work with the most singular part:

$$\begin{aligned} ||f||_{sing} &\leq \frac{C}{s} \int_0^\infty t^{-1/2} e^{-m^2s^2/t^2} dt \\ &\leq \frac{C}{s} e^{-2ms} \end{aligned} \tag{3.14}$$

and similarly,

$$||g||_{sing} \leq \frac{C}{us} e^{-2mus} \tag{3.15}$$

One needs to feed 3.11 into 3.9 to show that $||\mathcal{U}||$ is bounded:

$$\begin{aligned}
||\mathcal{U}(E)||_{sing} &\leq C\lambda^2(n+1)\int_0^\infty s\ ds\int_0^1 du e^{s\mu_p u}e^{-s(nm+\mu_p-Re(E))}\frac{e^{-ms}}{\sqrt{s}}\frac{e^{-mus}}{\sqrt{su}} \\
&\leq C\lambda^2(n+1)\int_0^\infty dse^{-ms}\int_0^1 du\frac{e^{-s(m-\mu_p)u}}{\sqrt{u}}e^{-s(nm+\mu_p-Re(E))} \\
&\leq C\lambda^2(n+1)\int_0^\infty e^{-s[(n+1)m+\mu_p-Re(E)]}ds \\
&\leq C\lambda^2(n+1)\frac{1}{(n+1)m+\mu_p-Re(E)} \qquad (3.16)
\end{aligned}$$

where we have used:

$$\int_0^1 du\frac{e^{-s(m-\mu_p)u}}{\sqrt{u}} \leq \int_0^1 \frac{du}{\sqrt{u}} = 2 \qquad (3.17)$$

Since the most singular part is finite, the rest certainly is. This concludes that $\mathcal{K}(E)$ and $\mathcal{U}(E)$ are bounded hence one can choose the domain of H_0 as the common domain of the family $\Phi(E)$.

3.2. Closedness of $\Phi(E)$

Definition 3. *An operator T is said to be closed if, for any sequence x_n in its domain $D(T)$, $x_n \to x$ and $Tx_n \to y$ implies that $Tx = y$.*

We want to show that $\Phi(E)$ is closed in its domain $D(\Phi(E)) = D = D(H_0)$. Say we have a sequence x_n that converges to x as well as,

$$\Phi(E)\ x_n \ \rightarrow\ y \tag{3.18}$$

When $Re(E) < Re(E_*)$ where $Re(E_*)$ is sufficiently small such that $\Phi(E)$ is invertible below (see section 1.2.2):

$$\begin{aligned} & [1+\mathcal{K}(E)-\mathcal{U}(E)]\ (H_0+\mu_p-E)\ x_n \ \rightarrow\ y \ \ , \ \ x_n \ \rightarrow\ x \\ \Longrightarrow\ \ & (H_0+\mu_p-E)x_n \ \rightarrow\ [1+\mathcal{K}(E)-\mathcal{U}(E)]^{-1}\ y \ \ , \ \ x_n \ \rightarrow\ x \end{aligned} \tag{3.19}$$

Since H_0 is closed on its domain:

$$\begin{aligned} & (H_0+\mu_p-E)x_n \ \rightarrow\ (H_0+\mu_p-E)\ x \ =\ [1+\mathcal{K}(E)-\mathcal{U}(E)]^{-1}\ y \\ \Longrightarrow\ \ & y \ =\ [1+\mathcal{K}(E)-\mathcal{U}(E)]\ (H_0+\mu_p-E)\ x \end{aligned} \tag{3.20}$$

Hence for $Re(E) < Re(E_*)$, $\Phi(E)$ is closed. For $Re(E) > Re(E_*)$, we rearrange according to 2.49 :

$$\begin{aligned}
\Phi(E)-\Phi(E_*) &= T(E,E_*)(E_*-E) \\
&= (E_*-E)\Big[1+\lambda^2\sum_\sigma \frac{|f_\sigma|^2}{2\omega_\sigma}\frac{1}{(H_0-E+\omega_\sigma)(H_0-E_*+\omega_\sigma)} \qquad (3.21)\\
&+ \lambda^2\sum_{\sigma,\tau} f_\sigma(\bar{x})\frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}}\frac{1}{(H_0-E+\omega_\sigma+\omega_\tau)} \\
&\qquad\qquad \frac{1}{(H_0-E_*+\omega_\sigma+\omega_\tau)}\frac{a_\tau}{\sqrt{2\omega_\tau}}f_\tau(\bar{x})\Big]
\end{aligned}$$

One wants to show that $T(E,E_*)$ is bounded for the reasons that will soon become clear. Calling the second term in square brackets A, we proceed as we did before in chapter 1.3.2 and show that it is bounded. Again we apply Feynman parametrization followed by a subordination and take the norm. Estimating $H_0 > nm$ as well as recognizing the heat kernel as before and substituting the bound given in 2.40, we find:

$$\begin{aligned}
||A|| \;\leq\; & \int_0^1 du_1du_2du_3\int_0^\infty s^3 ds e^{-s(nm-Re(E))u_2}e^{-s(nm-Re(E_*))u_3} \\
& \int d\xi\frac{me^{-s^2m^2/4\xi}e^{-\xi}}{\xi^{3/2}}\Big(\frac{C}{\xi/m^2}+\frac{1}{V(M)}\Big)
\end{aligned}$$

We compute the most singular term (the first part) of the integral:

$$
\begin{aligned}
||A||_{sing} &\leq C\int ds du_1 du_2 du_3\ \delta(1-\sum_i u_i)\ e^{-s(nm-Re(E))u_2} \\
&\qquad e^{-s(nm-Re(E_*))u_3}\ e^{-sm(u_1+u_2+u_3)} \\
&\leq C\int du_1 du_2 du_2\ \delta(1-\sum_i u_i) \\
&\quad \frac{1}{mu_1+((n+1)m-Re(E))u_2+((n+1)m-Re(E_*))u_3} \qquad (3.22) \\
&\leq C\int\limits_{0\leq u_2+u_3\leq 1} \frac{1}{[(n+1)m-Re(E)]u_2+[(n+1)m-Re(E_*)]u_3}
\end{aligned}
$$

where in the last line we have ignored a positive term $m(1-u_2-u_3)$ in the denominator. Passing to polar coordinates:

$$
\begin{aligned}
||A||_{sing} &\leq C\int_0^1\int_0^{\pi/2} \frac{\rho d\rho d\theta}{[(n+1)m-Re(E)]\rho cos\theta+[(n+1)m-Re(E_*)]\rho sin\theta} \\
&\leq C\frac{1}{\sqrt{(n+1)-Re(E)}\sqrt{(n+1)-Re(E_*)}} \qquad (3.23)
\end{aligned}
$$

Note that we absorb every constant we encounter into C. If the most singular part is bounded, the other part certainly is. Hence we have shown that A is bounded. We now show the boundedness of the third term in square brackets in 3.21, call it B for simplicity.

$$\begin{aligned} B &= \lambda^2 \sum_{\sigma,\tau} f_\sigma(\bar{x}) \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \frac{1}{(H_0 - E + \omega_\sigma + \omega_\tau)(H_0 - E_* + \omega_\sigma + \omega_\tau)} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau(\bar{x}) \\ &= \lambda^2 \sum_{\sigma,\tau} \int_0^\infty ds \int_0^1 du f_\sigma(\bar{x}) \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} e^{-s\omega_\sigma} e^{-sH_0} e^{s(Eu+E_*(1-u))} e^{-s\omega_\tau} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau(\bar{x}) \end{aligned}$$

Taking the norm and replacing $Re(E_*)$ by $Re(E)$ since $Re(E_*) < Re(E)$:

$$||B|| = \lambda^2 \int_0^\infty ds \int_0^1 du e^{sRe(E)} ||\phi^{(-)}(f) e^{-sH_0} \phi^{(+)}(f)|| \tag{3.24}$$

Using 3.8 and 3.11:

$$\begin{aligned} ||B||_{sing} &\leq (n+1)\lambda^2 \int_0^\infty sds \int_0^1 du\ e^{-s(nm-Re(E))} ||f||^2 \\ &\leq C\ (n+1) \int_0^\infty ds\ e^{-s((n+2)m-Re(E))} \\ &\leq \frac{C\ (n+1)}{(n+2)m - Re(E)} \end{aligned} \tag{3.25}$$

which is finite. As we have shown that $T(E, E_*)$ is indeed bounded and since every bounded operator on a Hilbert space is closable, we conclude that for fixed E_*:

$$\begin{aligned}[\Phi(E)-\Phi(E_*)]\ x_n &\to [\Phi(E)-\Phi(E_*)]x \\ \Phi(E_*)x_n &= [1+\mathcal{K}(E_*)-\mathcal{U}(E_*)]\ H_0x_n \\ &\to [1+\mathcal{K}(E_*)-\mathcal{U}(E_*)]\ H_0x\end{aligned}$$

We can now add them up to see that:

$$y=[(1+\mathcal{K}(E_*)-\mathcal{U}(E_*))H_0+\Phi(E)-\Phi(E_*)]x \implies \Phi(E)\ x\ =\ y \tag{3.26}$$

Hence we conclude that $\Phi(E)$ is closed on its domain $D(\Phi(E))=D=D(H_0)$.

3.3. Holomorphicity of the Matrix Elements

We now want to show that the family $\Phi(E)$ satisfies the second criteria in the Definition 2. For this we invoke a theorem from [23]:

Theorem 3.1. *Let V be a Lebesgue measurable set of positive or infinite measure, U be an open subset of $\mathbb{C}$ and $L^1(V)$ the Lebesgue integration space of complex valued functions on V. Define $\Theta(E):U\to\mathbb{C}$ by:*

$$\Phi(E):=\int_V \phi(t,E)dt \quad , \quad t\in U \tag{3.27}$$

where $\phi(s,E):V\times U\to\mathbb{C}$ satisfies:

- $\phi\ (\cdot,E)\in L^1(V)\quad,\quad E\in U$
- $\phi\ (t,\cdot)\in \mathbf{H}(U)\quad,\quad t\in V$

where $\mathbf{H}(U)$ *denotes all analytic functions that are holomorphic on* U *and* t *stands for all parameters (since there could be more than just one). If the mapping:*

$$E \quad \to \int_V |\phi(t,E)|dt \tag{3.28}$$

is locally bounded on U*, then* $\Theta(E)$ *is said to be holomorphic.*

In our case, $\Theta(E) = \langle\lambda|\Phi(E)|\psi\rangle$ where $|\lambda\rangle, |\psi\rangle \in \mathcal{H} = \mathcal{F}^{(n+1)} \otimes \chi_{\downarrow} \oplus \mathcal{F}^{(n)} \otimes \chi_{\uparrow}$. Recall that the Principal Operator reads as:

$$\begin{aligned} \Phi_R(E) \quad &= \quad \Big[1 + \sum_\sigma \frac{\lambda^2}{2\omega_\sigma} \frac{|f_\sigma|^2}{(H_0 - E + \omega_\sigma)} \frac{1}{(\omega_\sigma - \mu_p)} \\ &- \quad \sum_{\sigma,\tau} \lambda^2 f_\sigma \frac{a_\sigma^\dagger}{\sqrt{2\omega_\sigma}} \frac{1}{H_0 - E + \omega_\sigma + \omega_\tau} \frac{1}{H_0 - E + \omega_\tau + \mu_p} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau \Big] \\ &\qquad\qquad (H_0 - E + \mu_p) \end{aligned} \tag{3.29}$$

$H_0 - E + \mu_p$ is already entire and obviously holomorphic. We will again call the second term in square brackets as $K(E)$ and the third as $U(E)$.

$$
\begin{aligned}
\langle\lambda|K(E)|\psi\rangle &= \langle 0|\sum_{\tau_1,\tau_2,...}\frac{a_{\tau_1}a_{\tau_2}...}{\sqrt{(n+1)!}}\frac{\lambda^*(\tau_1,\tau_2,...)}{\sqrt{2\omega_{\tau_1}2\omega_{\tau_2}...}}K(E) \\
&\qquad \sum_{\alpha_1,\alpha_2,...}\frac{\psi(\alpha_1,\alpha_2,...)}{\sqrt{2\omega_{\alpha_1}2\omega_{\alpha_2}...}}\frac{a^\dagger_{\alpha_1}a^\dagger_{\alpha_2}...}{\sqrt{(n+1)!}}|0\rangle \\
&= \mathcal{C}\int\limits_{0\le u_2+u_3\le 1} du_2du_3\int ds\sum_\sigma f_\sigma^2\ \langle 0|\sum_{\tau_1,\tau_2,...}\frac{a_{\tau_1}a_{\tau_2}...}{\sqrt{(n+1)!}}\frac{\lambda^*(\tau_1,\tau_2,...)}{\sqrt{2\omega_{\tau_1}2\omega_{\tau_2}...}} \\
&\qquad s^2e^{-s\omega_\sigma}e^{s\mu_p u_2}e^{-su_3(H_0-E)}\sum_{\alpha_1,\alpha_2,...}\frac{\psi(\alpha_1,\alpha_2,...)}{\sqrt{2\omega_{\alpha_1}2\omega_{\alpha_2}...}}\frac{a^\dagger_{\alpha_1}a^\dagger_{\alpha_2}...}{\sqrt{(n+1)!}}|0\rangle \\
&= \mathcal{C}\int\limits_{0\le u_2+u_3\le 1} du_2du_3\int s^3ds\int d\xi\frac{e^{-\frac{s^2}{4\xi}-m^2\xi}}{\xi^{3/2}}e^{s\mu_p u_2}K_\xi(\bar{x},\bar{x}) \\
&\qquad \langle\lambda|e^{-su_3(H_0-E)}|\psi\rangle \\
&= \int\limits_{0\le u_2+u_3\le 1} du_2du_3\int ds\int d\xi\ \ \phi(s,u_2,u_3,\xi,E) \qquad (3.30)
\end{aligned}
$$

where we have collected everything into ϕ except the integral measures. To show integrability, we employ the following theorem [14]:

Theorem 3.2. *Let $|\phi|\le g$, ϕ being a measurable function. If g is integrable, so is ϕ.*

Taking the absolute value of $\phi(s,u_2,u_3,\xi,E)$, we get:

$$\begin{aligned}
\int du_2 du_3 \int ds \ \ |\phi| &= \mathcal{C} \int du_2 du_3 \int s^3 ds \int d\xi \\
& \frac{e^{-\frac{s^2}{4\xi}-m^2\xi}}{\xi^{3/2}} e^{s\mu_p u_2} K_\xi(\bar{x},\bar{x}) \ \langle\lambda| e^{-su_3(H_0-|E|)}|\psi\rangle \\
&\leq \mathcal{C} \ \langle\lambda|\psi\rangle \int du_2 du_3 \int s^3 ds \int d\xi \ K_\xi(\bar{x},\bar{x}) \\
& \qquad e^{s\mu_p u_2} \frac{e^{-\frac{s^2}{4\xi}-m^2\xi}}{\xi^{3/2}} e^{s\mu_p u_2} e^{-su_3(nm-|E|)} \\
&= \int du_2 du_3 \int ds \int d\xi \ g(s,u_2,u_3,\xi,E) \qquad (3.31)
\end{aligned}$$

where we have defined $g \geq |\phi|$ by estimating H_0 as nm as usual and $|E|$ is taken to have a fixed value below $nm + \mu_p$. ϕ consists of well-defined continuous functions hence measurable. Note that the integral 3.31 is the same as 2.58 up to some constants and $Re(E)$ replaced by some constant $|E|$. Hence we can estimate it following the same steps and show that it is bounded. For the integral 2.58 is already shown to be finite in 2.3.2, so is 3.31 hence we can apply the Theorem 3.2 and conclude that $\phi(\cdot, E)$ is indeed L^1.

Note that for the parameters u_2, u_3, s, ξ fixed, ϕ is simply an entire function of E where the only factor depending on E is $e^{-s(H_0-E)}$. Hence the holomorphicity of $\phi(t,\cdot)$ is straightforward.

The extra condition stated in Theorem 3.1 does not require more work since we have already shown the integrability through ($\int du_2 du_3 ds d\xi |\phi|$) being bounded above by ($\int du_2 du_3 ds d\xi \ g$), for $|E| \leq nm + \mu_p$. The explicit bound can be found in 3.3.

As the explicit construction was shown above for $\mathcal{K}(E)$, for $\mathcal{U}(E)$ we will make use of some inequalities based on the bounds we found before. An alternative yet equivalent definition for operator norm is [24]:

$$||\Theta|| \quad := \quad \sup_{\lambda,\psi\in\mathcal{H}} |\langle\lambda|\Theta|\psi\rangle| \;/\; ||\lambda||||\psi||$$

$$\Longrightarrow \quad |\langle\lambda|\mathcal{U}(E)|\lambda\rangle| \;\leq\; ||\mathcal{U}(E)|| \;\;;\;\; ||\lambda||, ||\psi|| = 1 \tag{3.32}$$

One can easily see throughout this work that the functions we have in the integral representations of $\mathcal{U}(E)$ and $\mathcal{K}(E)$ and also the limits of integration are all positive as well as the integration variables are all real. Hence showing that $\int dt|\phi(t,E)|$ is bounded is the same as showing that $|\int dt\phi(t,E)|$ is.

As we have already shown that $\mathcal{U}(E)$ is bounded while fixing the domain of $\Phi(E)$, we conclude straightforwardly that $|\langle\lambda|\mathcal{U}(E)|\psi\rangle|$ is finite, the boundedness of $\int dt|\phi(t,E)|$ is directly satisfied . Integrability condition follows from the Theorem 3.2 in the same way as for $\mathcal{K}(E)$. Holomorphicity in E is again straightforward since the only function containing E is an entire one, $e^{-s(H_0-E)}$.

Hence we have shown that $\Phi(E)$ is indeed holomorphic family of type-A.

3.4. Self-Adjointness of $\Phi(E)$

Note that, formally, $\Phi^\dagger(E) = \Phi(\bar{E})$, hence at least, $D(\Phi(E)) \subset D(\Phi^\dagger(E))$. But to conclude self-adjointness, one needs to show that they admit the same domain.
Our strategy will be the following: we will make use of the well-known Kato-Rellich Theorem [15] to show that $\Phi(E)$ is self-adjoint on some region on the real axis for E chosen to be sufficiently small and then employ Wüst's theorem [16] to generalize it to the whole region of concern.

Theorem 3.3. *Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator and $B : D(B) \to \mathcal{H}$ be* *symmetric. For $D(A) \subset D(B)$, if the following is satisfied:*

$$||Bx|| \leq a||Ax|| + b||x|| \quad , \quad \forall x \in \mathcal{H} \tag{3.33}$$

with $a < 1$, $b < \infty$; then $A + B : D(A) \rightarrow \mathcal{H}$ is self-adjoint.

Recall the form of the Principal Operator:

$$\begin{aligned} \Phi(E) &= (1 + K(E) - U(E))(H_0 - E + \mu_p) \\ &= \underbrace{(H_0 - E + \mu_p) + K(E)(H_0 - E + \mu_p)}_{A} \underbrace{-U(E)(H_0 - E + \mu_p)}_{B} \end{aligned} \tag{3.34}$$

If A is invertible, one can write $x = A^{-1}y$ for some $y \in \mathcal{H}$ hence the inequality 3.33 becomes:

$$||BA^{-1}y|| \leq a||y|| + b||A^{-1}y|| \tag{3.35}$$

We will work on the real axis where $E < E_*$, E_* chosen to be sufficiently small such that $K(E)$ becomes strictly positive as stated before in 2.2.2 and is self-adjoint being a function of H_0 without any poles. Note that for $b = 0$, if the following is true in some region:

$$\sup \frac{||BA^{-1}y||}{||y||} = ||BA^{-1}|| < 1 \tag{3.36}$$

then the conditions stated in Theorem 3.3 are satisfied and $A+B$ is self-adjoint in that region. Rearranging:

$$\begin{aligned} BA^{-1} &= -U(E)(H_0 - E + \mu_p)[(1+K(E))(H_0 - E + \mu_p)]^{-1} && (3.37) \\ &= -U(E)[1+K(E)]^{-1} && (3.38) \end{aligned}$$

since work in the sector where $K(E)$ is positive,

$$||U(E)(1+K(E))^{-1}|| \leq ||U(E)|| \tag{3.39}$$

Recall that while searching for a lower bound for the ground state energy in 2.2.2, we showed that $||U(E)|| < 1$ if we impose the condition 2.43. Then $||BA^{-1}y|| \leq a$ where $a < 1$ and hence, $A + B = \Phi(E)$ is self-adjoint at least in some region where $E < E_*$.

Theorem 3.4. *Let U be a domain in the complex plane which is symmetric around the real axis and $\{\Phi(E),\ E \in D_0\}$ be holomorphic family of type-A in $\mathcal{H}$ with dense domain D_0 such that $\Phi(\bar{E}) \subset \Phi^\dagger(E)$. Define M by:*

$$M := \{E \mid E \in U \ , \quad \Phi^\dagger(E) = \Phi(\bar{E})\} \tag{3.40}$$

If M is not empty, it extends to all of U; i.e. $M \neq \emptyset \implies M = U$.

As we have shown previously that at least in some region on the real line below a sufficiently small E_*, $\Phi(E)$ is self-adjoint. Together with Wüst's theorem, the equality (not only formally but in the real sense; meaning that domains are also equal) $\Phi^\dagger(E) = \Phi(\bar{E})$ extends to all $\mathbb{C}$. Hence we conclude that $\Phi(E)$ is self-adjoint holomorphic family of type-A.

4. CONCLUSION

In this work, we start with a simple formally defined Hamiltonian, despite its simplicity the problem requires renormalization due to an infinite term. Using an approach developed by Rajeev, we obtain a finite non-perturbative renormalized form of the many body resolvent. Having found an exact formula for the resolvent, a natural question presents itself: how do we know that the finite formula we obtained for the resolvent actually defines a Hamiltonian?

This is not a redundant question since one could propose a resolvent through formal substractions, yet this may not correspond to the resolvent of any quantum Hamiltonian. To answer this question in the affirmative, we employ operator theory techniques. This is not so straightforward as we need to put together various results from distinct branches of operator theory.

Thanks to the simplicity of our model and various previously established results in the mathematics literature, we could accomplish the task completely. Experience has taught us that understanding simple systems in depth often clarifies the physical meaning of our formal operations as well. The present model has allowed us to accomplish this goal.

Nevertheless, some questions are left unanswered. A more complete understanding demands establishing uniqueness of ground state. Moreover a proper estimate of the large number of bosons is essential to understand the structure of this ground state, hence a mean field estimate is called for. These are some future directions this model offers us to work on.

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