GRAVITATIONAL WAVE SOLUTIONS TO LINEARIZED JORDAN BRANS DICKE THEORY ON A COSMOLOGICAL BACKGROUND

by

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ABSTRACT

GRAVITATIONAL WAVE SOLUTIONS TO LINEARIZED JORDAN BRANS DICKE THEORY ON A COSMOLOGICAL BACKGROUND

In this thesis, approximate vacuum solutions of Jordan-Brans-Dicke theory for perturbed scalar field and perturbed Robertson-Walker metric, were found. First we obtained solutions for the scale factor $a(t)$ and the scalar field $\phi(t)$ in unperturbed JBD theory. The solutions are dependent on JBD constant ω_{JBD} which is the value of how the scalar field is coupled to geometry of space-time. Then we added metric perturbation $h_{\mu\nu}(x)$ to Robertson-Walker metric and perturbation $\delta\phi(x)$ to the scalar field $\phi(t)$ in order to construct linearized JBD equations. After acquiring the metric perturbation and $(\delta\phi/\phi)$ as gravitational wave and scalar gravitational wave respectively, we solved the JBD equations which are first order in $h_{\mu\nu}(x)$ and $\delta\phi(x)$ such that the scale factor and the scalar field solutions are $a \propto t$ and $\phi \propto t^{-2}$ with $\omega_{JBD} = -3/2$. These results are necessary conditions for ordinary and scalar gravitational waves to exist in vacuum case. Despite $\omega_{JBD} > 10^4$ for current solar system environment observations, $\omega_{JBD} = -3/2$ makes JBD theory conformally invariant and fits recent supernovae type Ia data.

ÖZET

LİNEERİZE JORDAN BRANS DİCKE TEORİSİNİN KOZMOLOJİK FONDA GRAVİTASYONEL DALGA ÇÖZÜMLERİ

Bu tez çalışmasında, tedirgenmiş skaler alan ve tedirgenmiş Robertson-Walker metriği için Jordan-Brans-Dicke teorisinin yaklaşık vakum çözümleri bulunmuştur. İlk olarak tedirgenmemiş JBD denklemleri kapsamında ölçek faktörü $a(t)$ ve skaler alan $\phi(t)$ için çözümler elde edilmiştir. Bu çözümler skaler alan ile uzay-zaman geometrisinin nasıl eşleşeceğini belirleyen JBD sabiti ω_{JBD}'nın değerine bağlıdır. Daha sonra lineerize edilmiş JBD denklemlerini oluşturmak için Robertson-Walker metriğine metrik tedirgemesi $h_{\mu\nu}$ ve skaler alana da alan tedirgemesi $\delta\phi$ eklenmiştir. Metrik tedirgemesi ve $\delta\phi/\phi$'ın sırasıyla gravitasyonel ve skaler gravitasyonel dalga olduğu elde edilerek $h_{\mu\nu}$ ve $\delta\phi$ terimleri açısından birinci derece olan JBD denklemleri çözülmüştür. Tüm denklemleri sağlayan çözümler ölçek faktörü için $a \propto t$, skaler alan için $\phi \propto t^{-2}$ ve $\omega = -3/2$ bulunmuştur. Bu sonuçlar vakum durumunda normal gravitasyonel ve skaler gravitasyonel dalgaların var olabilmesi için gerekli şartlardır. Güncel güneş sistemi çevresi gözlemleri JBD sabiti $\omega > 10^4$ olmasını gerektirdiği halde, $\omega = -3/2$ değeri JBD teorisini konformal olarak değişmez yapan değerdir ve son süpernova tip Ia verileriyle uyuşmaktadır.

TABLE OF CONTENTS

LIST OF SYMBOLS

a	Scale factor of the universe
A	Plus polarization wave
B	Cross polarization wave
c	Speed of light in vacuum
$f_{\mu\nu}$	Robertson-Walker metric
g	Determinant of metric $g_{\mu\nu}$
$g_{\mu\nu}$	Metric
G	Newton's gravitational constant
$\hbar$	Planck constant divided by 2π
$h_{\mu\nu}$	Metric perturbation
H_0	Today's Hubble parameter
k	Curvature parameter
k^σ	Wave vector components
$\mathcal{L}$	Lagrangian
$\mathcal{L}_M$	Matter Lagrangian
$\mathcal{L}_\phi$	Scalar field Lagrangian
m	Mass
r	Distance
R	Ricci scalar
$R_{\mu\nu}$	Ricci tensor
$R^\sigma_{\rho\mu\nu}$	Riemann tensor
S_{EH}	Einstein-Hilbert action
S_{JBD}	Jordan-Brans-Dicke action
t	Time
T	Trace of energy momentum tensor
$T_{\mu\nu}$	Energy momentum tensor
$\Gamma^\lambda_{\mu\nu}$	Christoffel symbol

$\delta\phi$	Scalar field perturbation
$\eta_{\mu\nu}$	Minkowski metric
λ	Coupling constant
ρ	Matter density of the universe
ϕ	Scalar field
Φ	Perturbed scalar field
ω	Jordan-Brans-Dicke coupling constant
ω_{JBD}	Jordan-Brans-Dicke coupling constant

LIST OF ACRONYMS/ABBREVIATIONS

EH	Einstein-Hilbert
JBD	Jordan-Brans-Dicke
RW	Robertson-Walker

1. INTRODUCTION

In 1687, gravitational constant G, was first introduced by Isaac Newton in his theory of gravitation which explains the motion of astronomical objects and some phenomena on the earth such as gravitational acceleration and weight. According to the Newton's law, gravitational force between two objects is simply proportional to masses of the objects and inversely proportional to square of the distance between them. The proportionality constant is the Newton's constant which was first measured by British physicist Henry Cavendish in an experiment in 1798 as $G \approx 6.7 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

Another concept which was introduced by Newton is inertial force which is equal to inertial mass times acceleration. Since they are both proportional to mass of the object, inertial force and gravitational force seem parallel to each other. This was the idea for Austrian physicist Ernst Mach to suggest that inertia might be just a phenomenon similar to gravitation and be related to gravitational influence of distant fixed stars or simply general mass distribution of the universe on the object. An example for this is the "Newton's bucket" experiment which states that the surface of water in a bucket can be flat if and only if an observer in the frame of the bucket sees the fixed distant stars as not rotating. In other words, if the bucket-fixed observer sees distant fixed stars as rotating, he must notice that the water surface is not flat because of centrifugal force, which is an inertial force, on the water. According to Mach's principle this should not be regarded as a coincidence.

From this point of wiev, inertial masses of particles do not have to be constant but should be determined by interaction of particles with a cosmic scalar field related to gravity. Scalar field is needed because field should be coordinate independent. Masses of particles can be measured by measuring the gravitational acceleration Gm/r^2 so gravitational constant G has to be related to the average value of scalar field ϕ.

We can make an estimate of the average value of ϕ by computing the central potential of a sphere of dust with density $\rho \approx 10^{-26}$ kg m^{-3}, which is the ordinary matter density of the universe, and radius $R \approx 10^{26}$ m, which is the radius of the observable universe. They give a value

$$\lambda \rho R^2 \backsim \lambda \times 10^{26} \text{ kg m}^{-1} \tag{1.1}$$

where λ is a dimensionless coupling constant. As it is seen, 10^{26} kg m^{-1} is close to the value $c^2/G = 1.35 \times 10^{27}$ kg m^{-1}. Since $\rho R^2 \backsim M/R$, we can also write the following relation

$$\frac{GM}{Rc^2} \backsim 1 \text{ or } \frac{M}{Rc^2} \backsim \frac{1}{G} \tag{1.2}$$

where M is the total mass of the observable universe. M/Rc^2 seems as a potential just like Q/R which is electric potential of charge Q at a distance R. So we can consider M/Rc^2 as cosmic scalar field. The average value of ϕ is

$$\langle \phi \rangle \simeq \frac{1}{G} \tag{1.3}$$

Since we live in an expanding universe, the radius and the mass of the observable universe are time dependent quantities. So the average value of the scalar field should be time dependent as well. The theory which considers reciprocal of G as the scalar field was suggested by Brans and Dicke [1] in 1961 with motivation from Mach's principle. Brans and Dicke made modifications on Einstein's theory to change Newton's constant as time dependent scalar field although Jordan and Thiry had developed their own versions of the theory before Brans and Dicke. So these theories are called as Jordan-Thiry-Brans-Dicke theories or more generally scalar tensor theories. However, in this thesis we prefer common use which is Jordan-Brans-Dicke (JBD) theory.

Another motivation for Brans and Dicke to consider G as a time dependent parameter, comes from "Large Number Hypothesis" which is an observation [2] made by Dirac in 1937. Dirac realised that some combinations of fundamental constants of different branches of physics give values which are comparable to values of some other fundamental quantities. For instance, we can construct a value of mass with G, $\hbar$, c and H_0, which is today's value of Hubble parameter, and it is close to mass of an elementary particle such as pion

$$\left(\frac{\hbar^2 H_0}{Gc}\right)^{\frac{1}{3}} \approx m_\pi \tag{1.4}$$

At first glance, this might seem as a coincidence and it is possible to construct a mass value with some other fundamental quantities from a different combination like

$$\left(\frac{\hbar c}{G}\right)^{\frac{1}{2}} \approx 10^{28}\ \mathrm{eV/c^2} \tag{1.5}$$

However, 10^{28} eV/c^2 is approximately 20 order of magnitude bigger than mass of a typical elementary particle. So it is rational to think that values of not all combinations but some have a real significance. When we consider the equation which gives the mass of pion, we have a problem with being of H_0 not a constant but a time dependent parameter. To handle this kind of problem, Dirac proposed to choose at least one of the other quantities as time dependent. Since making $\hbar$, c or the other ones time dependent requires more serious work like reformulating some areas of physics, the most appropriate one was the Newton's constant.

The reasons we have mentioned above motivated Brans and Dicke to propose that the gravitational constant G in Einstein's field equation should be replaced with the time dependent scalar field ϕ and energy momentum tensor for this field should be added. After that, the field equation becomes

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \frac{8\pi}{\phi}[T_M^{\mu\nu} + T_\phi^{\mu\nu}] \tag{1.6}$$

2. JORDAN BRANS DICKE EQUATIONS

In the theory of general relativity, Einstein field equation can be derived from the action by using variational principle (or variational method), which is simply taking the variance of the action and making it equal to zero. For instance, by using this method, vacuum field equation can be found from Einstein-Hilbert action which is

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \tag{2.1}$$

and, of course, if you want to obtain the equation of more general case which contains energy momentum tensor, you should add the action of this one too.

Equations for Jordan-Brans-Dicke theory can also be obtained from variational principle by simply choosing an appropriate action in which there is a Lagrangian for the scalar field ϕ. Before writing Jordan-Brans-Dicke action, we know that our general Lagrangian can be written as summation of Lagrangians of different sources such as matter and the scalar field. So in JBD theory, the Lagrangian is

$$\mathcal{L} = [\phi R + 16\pi \mathcal{L}_M + \mathcal{L}_\phi] \tag{2.2}$$

and applying variational principle yields

$$\delta \left[\frac{1}{16\pi} \int d^4x \sqrt{-g} [\phi R + 16\pi \mathcal{L}_M + \mathcal{L}_\phi \right] = 0 \tag{2.3}$$

Now, we can start looking for the Lagrangian of the scalar field to be able to apply variational principle. For unit consistency, the Lagrangian of the scalar field ϕ is defined as

$$\mathcal{L}_\phi = -\omega_{JBD} \frac{g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi}{\phi} \tag{2.4}$$

where ω_{JBD} is the Jordan-Brans-Dicke coupling constant. From now on, we will omit JBD subscript and consider it simply as ω. Since we do not theoretically know how the scalar field and geometry are coupled, this constant ω is necessary in the equation. After obtaining the Lagrangian for the scalar field, the Jordan-Brans-Dicke action looks like

$$S_{JBD} = \frac{1}{16\pi} \int d^4x\sqrt{-g} \left(\phi R + 16\pi \mathcal{L}_M - \omega \frac{g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi}{\phi} \right) \tag{2.5}$$

In order to get the JBD equations, we should vary the action with respect to $g^{\mu\nu}$ and ϕ. We do not plan to make these calculations here. However, if you are interested, variation with respect to $g^{\mu\nu}$ is conducted in Appendix. Variation with respect to the scalar field is straightforward and can be operated easily. Variance operations give us the equations as

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} =& \frac{8\pi}{\phi} T_{\mu\nu} + \frac{1}{\phi} \left(\nabla_\mu \partial_\nu \phi - g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \partial_\beta \phi \right) \\ &+ \frac{\omega}{\phi^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) \end{aligned} \tag{2.6}$$

and

$$R + 2\frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \partial_\nu \phi - \frac{\omega}{\phi^2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 0 \tag{2.7}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{2.8}$$

is the energy momentum tensor of matter. If equation (2.6) is contracted with $g^{\mu\nu}$, it becomes

$$-R = \frac{8\pi}{\phi} T - \frac{3}{\phi} \nabla_\mu \partial^\mu \phi - \frac{\omega}{\phi^2} \partial_\mu \phi \partial^\mu \phi \tag{2.9}$$

and substituting equation (2.9) into equation (2.7) gives

$$\frac{(3+2\omega)}{\phi}g^{\mu\nu}\nabla_\mu\partial_\nu\phi = \frac{8\pi}{\phi}T \tag{2.10}$$

Equation (2.6) and equation (2.10) are basic equations we will use in Jordan-Brans-Dicke theory. As can be found in any related textbook, the predictions of Jordan-Brans-Dicke theory are same with the predictions of Einstein field equation when $\omega \to \infty$. Current observational data [3, 4] for solar system environment show that $\omega > 10^4$.

3. VACUUM SOLUTIONS TO JBD THEORY IN RW METRIC

At the end of the chapter 2, we have found the Jordan-Brans-Dicke equations and in this chapter we will focus on the vacuum solutions of the equations on a cosmological background. With word "vacuum", we mean that there is nothing in the environment which we are interested in, no matter, no radiation and no cosmological constant. Besides, of course, the universe we live in, is not steady state universe but it is expanding with the scale factor, so we will use Robertson-Walker metric which is

$$ds^2 = -(dt)^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{3.1}$$

where a is the scale factor of the universe and a function of time.

Before we start calculations, the energy momentum tensor of matter and the trace of it should be set equal to zero in order to attain fundamental two vacuum case equations for JBD theory. We will also consider space as flat which means curvature parameter $k = 0$. For the vacuum case equation (2.6) and equation (2.10) transform into

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} =& \frac{1}{\phi}\left(\nabla_\mu\partial_\nu\phi - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\partial_\beta\phi\right) \\ &+ \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) \end{aligned} \tag{3.2}$$

and

$$g^{\mu\nu}\nabla_\mu\partial_\nu\phi = 0 \tag{3.3}$$

As you can see, equation (3.3) can be substituted into equation (3.2), and the final form of basic JBD equation for vacuum is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{\phi}\nabla_\mu\partial_\nu\phi + \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) \tag{3.4}$$

Now, all we should do is to place the Ricci tensor components and the Ricci scalar of Robertson-Walker metric in the equation in order to construct equations for different μ and ν values. The Ricci tensor and the Ricci scalar of RW metric for $k = 0$ and in cartesian coordinates, which can be easily found in any related textbook, are

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{3.5}$$

$$R_{11} = R_{22} = R_{33} = (a\ddot{a} + 2\dot{a}^2) \tag{3.6}$$

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \tag{3.7}$$

where dots represent time derivatives. We assume that the scalar field ϕ is only function of time, not dependent on spatial coordinates. After substituting equation (3.5), equation (3.6) and equation (3.7) into equation (3.4), we get for $\mu = 0$ and $\nu = 0$

$$3\frac{\dot{a}^2}{a^2} - \frac{\partial_0\partial_0\phi}{\phi} - \frac{\omega}{2}\frac{(\partial_0\phi)^2}{\phi^2} = 0 \tag{3.8}$$

and for $\mu = 1$ and $\nu = 1$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\partial_0\phi}{\phi} - \frac{\omega}{2}\frac{(\partial_0\phi)^2}{\phi^2} = 0 \tag{3.9}$$

Other two equations for cases $\mu = 2$, $\nu = 2$ and $\mu = 3$, $\nu = 3$ are not different from the equation for $\mu = 1$, $\nu = 1$, so they do not give any new information about vacuum case solutions. Subtracting equation (3.9) from equation (3.8) yields

$$4\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} - \frac{\partial_0\partial_0\phi}{\phi} - \frac{\dot{a}}{a}\frac{\partial_0\phi}{\phi} = 0 \tag{3.10}$$

By assuming that the scalar field ϕ and the scale factor a have solutions like power of time t, equation (3.9) and equation (3.10) give us solutions of values of powers which depend on the JBD coupling constant ω. If $\phi \propto t^s$ and $a \propto t^q$, we get

$$-3q^2 + 2q + sq - \frac{\omega}{2}s^2 = 0 \tag{3.11}$$

$$6q^2 - 2q - sq + s - s^2 = 0 \tag{3.12}$$

Solutions to these equations for $\omega \geq -3/2$ and $\omega \neq -4/3$ are

$$q_{\pm} = \frac{1}{3\omega + 4}\left(\omega + 1 \pm \sqrt{\frac{2\omega + 3}{3}}\right) \tag{3.13}$$

$$s_{\pm} = \frac{1 \mp \sqrt{3(2\omega + 3)}}{3\omega + 4} \tag{3.14}$$

They also satisfy the relation

$$3q + s = 1 \tag{3.15}$$

These solutions are same with that of O'Hanlon and Tupper [5]. Once we get the exact value of ω observationally, values of q and s can be determined.

4. VACUUM SOLUTIONS TO LINEARIZED JBD THEORY IN RW METRIC

In this chapter, we will find approximate solutions for the scalar field, the scale factor, ordinary gravitational wave and scalar gravitational wave by regarding perturbed Robertson-Walker metric and perturbed scalar field. So we add first order perturbations to the metric and the scalar field, and neglect all the higher order perturbations in calculations. Since our zeroth order metric is funtion of time, we choose zeroth order scalar field which is ϕ to be function of time for ansatz. In addition, the first order perturbations of the metric and the scalar field are function of time and spatial coordinates. Our perturbed metric and perturbed scalar field are

$$g_{\mu\nu}(x) = f_{\mu\nu}(t) + h_{\mu\nu}(x) \tag{4.1}$$

and

$$\Phi(x) = \phi(t) + \delta\phi(x) \tag{4.2}$$

where

- $f_{\mu\nu}$ is the Robertson-Walker metric
- $h_{\mu\nu}$ is perturbation to the metric and $|h_{\mu\nu}| \ll 1$
- $\delta\phi$ is perturbation to the scalar field and $|\delta\phi| \ll \phi$

In equation (2.6) and equation (2.10), replacing the scalar field ϕ with our new perturbed scalar field Φ leads to

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} =& \frac{8\pi}{\Phi}T_{\mu\nu} + \frac{1}{\Phi}\left(\nabla_\mu\partial_\nu\Phi - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\partial_\beta\Phi\right) \\ &+ \frac{\omega}{\Phi^2}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi\right) \end{aligned} \tag{4.3}$$

and

$$\frac{(3+2\omega)}{\Phi} g^{\mu\nu} \nabla_\mu \partial_\nu \Phi = \frac{8\pi}{\Phi} T \tag{4.4}$$

Since we are dealing with the vacuum solutions of the JBD equations, the energy momentum tensor and the trace of it equal to zero. Thus, these two equations become

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{\Phi} \nabla_\mu \partial_\nu \Phi + \frac{\omega}{\Phi^2} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right) \tag{4.5}$$

and

$$\frac{1}{\Phi} g^{\mu\nu} \nabla_\mu \partial_\nu \Phi = 0 \tag{4.6}$$

If we write them again with explicit form of the field Φ and the metric $g_{\mu\nu}$, they look like

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R (f_{\mu\nu} + h_{\mu\nu}) = & \frac{1}{(\phi + \delta\phi)} [\nabla_\mu \partial_\nu (\phi + \delta\phi)] \\ & + \frac{\omega}{(\phi + \delta\phi)^2} \Big[\partial_\mu (\phi + \delta\phi) \partial_\nu (\phi + \delta\phi) \\ & - \frac{1}{2} (f_{\mu\nu} + h_{\mu\nu}) (f^{\alpha\beta} - h^{\alpha\beta}) \partial_\alpha (\phi + \delta\phi) \partial_\beta (\phi + \delta\phi) \Big] \end{aligned} \tag{4.7}$$

and

$$\frac{1}{(\phi + \delta\phi)} (f^{\mu\nu} - h^{\mu\nu}) \nabla_\mu \partial_\nu (\phi + \delta\phi) = 0 \tag{4.8}$$

We have used inverse of the metric above in equation (4.7) and equation (4.8) as

$$g^{\mu\nu}(x) = f^{\mu\nu}(t) - h^{\mu\nu}(x) \tag{4.9}$$

4.1. Ricci Tensor and Ricci Scalar for Perturbed Robertson Walker Metric

We do not plan to calculate perturbed Ricci tensor explicitly. If you are interested in detailed calculation of it from Christoffel symbols and Riemann tensor, you can check Weinberg's "Gravitation and Cosmology" book [6]. General forms[1] of first order components of the Ricci tensor are,

$$\delta R_{00} = -\frac{1}{2a^2}\left[\partial_0\partial_0 h_{kk} - 2\frac{\dot{a}}{a}\partial_0 h_{kk} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right) h_{kk}\right] \tag{4.10}$$

$$\delta R_{0i} = -\frac{1}{2}\partial_0\left[\frac{1}{a^2}(\partial_i h_{kk} - \partial_k h_{ki})\right] \tag{4.11}$$

$$\begin{aligned} \delta R_{ij} = &- \frac{1}{2a^2}\left[\nabla^2 h_{ij} - \partial_j\partial_k h_{ik} - \partial_i\partial_k h_{jk} + \partial_i\partial_j h_{kk}\right] \\ &- \frac{1}{2}\partial_0\partial_0 h_{ij} + \frac{\dot{a}}{2a}[\partial_0 h_{ij} - \delta_{ij}\partial_0 h_{kk}] + \frac{\dot{a}^2}{a^2}[-2h_{ij} + \delta_{ij}h_{kk}] \end{aligned} \tag{4.12}$$

Now we can consider each component of the Ricci tensor as zeroth order part plus first order part as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu} \tag{4.13}$$

- $R_{\mu\nu}$ is the Ricci tensor for metric $g_{\mu\nu}$
- $\bar{R}_{\mu\nu}$ is the Ricci tensor for the Robertson-Walker metric
- $\delta R_{\mu\nu}$ is the perturbation of Ricci tensor

Before proceeding to compute Ricci tensor components, we can make some simplifications for our sake. As is known, in Minkowski space-time, which is flat, transverse-traceless perturbation $h_{\mu\nu}^{TT}$ represents plane wave solution in cartesian coordinates. The metric perturbation is composed of plus and cross polarization waves. For a plane wave

[1] Since Weinberg used different notation in his book, we multiply first order components of the Ricci tensor with a minus sign.

which is propogating in x^3 direction, it looks like

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{21} & h_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.14}$$

where

$$h_{11}(t-x^3) = h_+ e^{ik_\sigma x^\sigma} \text{ and } h_{22} = -h_{11} \tag{4.15}$$

and

$$h_{12}(t-x^3) = h_\times e^{ik_\sigma x^\sigma} \text{ and } h_{12} = h_{21} \tag{4.16}$$

This solution to Minkowski metric perturbation is for the Einstein equation. If we want to solve the JBD equations for perturbed Minkowski metric, we should choose our scalar field as

$$\Phi(x) = \phi_0 + \delta\phi(x) \tag{4.17}$$

where ϕ_0 is constant and $\delta\phi$ is a function of time and spatial coordinates. As it can be checked in any related textbook [7], the work of Maggiore and Nicolis [8] is recommended, the solution is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{(+)} - \frac{\delta\phi}{\phi_0} & A^{(\times)} & 0 \\ 0 & A^{(\times)} & -A^{(+)} - \frac{\delta\phi}{\phi_0} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.18}$$

where $A^{(+)}(t-x^3) = A_0^{(+)}(\vec{k})\ e^{ik_\sigma x^\sigma}$, $A^{(\times)}(t-x^3) = A_0^{(\times)}(\vec{k})\ e^{ik_\sigma x^\sigma}$ and $\delta\phi/\phi_0$ is scalar gravitational wave. This metric perturbation has trace $\eta^{\mu\nu}h_{\mu\nu} = -2(\delta\phi/\phi_0)$.

Taking the solutions of the JBD equations for perturbed Minkowski metric into consideration, we can assume that perturbation of the Robertson-Walker metric for the JBD theory has trace

$$f^{\mu\nu}h_{\mu\nu} = -2\frac{\delta\phi}{\phi} \tag{4.19}$$

and it is transverse to propagation direction of the wave. For a wave which is propagating in x^3 direction, it looks like

$$h_{\mu\nu} = a^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \left(A - \frac{\delta\phi}{\phi}\right) & B & 0 \\ 0 & B & \left(-A - \frac{\delta\phi}{\phi}\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.20}$$

where A, B and $(\delta\phi/\phi)$ are in wave form. As you can see, we can take $h_{0\nu}$ and $h_{3\nu}$ components of the metric perturbation as zero. This assumption will simplify our calculations and by using equation (4.10)-equation (4.12) components of the Ricci tensor become

$$\begin{aligned} R_{00} = \bar{R}_{00} + \delta R_{00} = &-\frac{3\ddot{a}}{a} - \frac{1}{2a^2}\partial_0\partial_0(h_{11}+h_{22}) + \frac{\dot{a}}{a^3}\partial_0(h_{11}+h_{22}) \\ &+ \left(\frac{\ddot{a}}{a^3} - \frac{\dot{a}^2}{a^4}\right)(h_{11}+h_{22}) \end{aligned} \tag{4.21}$$

$$\begin{aligned} R_{11} = \bar{R}_{11} + \delta R_{11} = &(a\ddot{a} + 2\dot{a}^2) + \frac{1}{2}\partial_0\partial_0 h_{11} - \frac{1}{2a^2}\partial_3\partial_3 h_{11} \\ &+ \frac{\dot{a}}{2a}\partial_0 h_{22} + \frac{\dot{a}^2}{a^2}(h_{11} - h_{22}) \end{aligned} \tag{4.22}$$

$$\begin{aligned} R_{22} = \bar{R}_{22} + \delta R_{22} = &(a\ddot{a} + 2\dot{a}^2) + \frac{1}{2}\partial_0\partial_0 h_{22} - \frac{1}{2a^2}\partial_3\partial_3 h_{22} \\ &+ \frac{\dot{a}}{2a}\partial_0 h_{11} + \frac{\dot{a}^2}{a^2}(h_{22} - h_{11}) \end{aligned} \tag{4.23}$$

$$R_{33} = \bar{R}_{33} + \delta R_{33} = (a\ddot{a} + 2\dot{a}^2) - \frac{1}{2a^2}\partial_3\partial_3(h_{11} + h_{22}) + \frac{\dot{a}}{2a}\partial_0(h_{11} + h_{22}) - \frac{\dot{a}^2}{a^2}(h_{11} + h_{22}) \tag{4.24}$$

$$R_{03} = \bar{R}_{03} + \delta R_{03} = -\frac{1}{2a^2}\partial_3\partial_0(h_{11} + h_{22}) + \frac{\dot{a}}{a^3}\partial_3(h_{11} + h_{22}) \tag{4.25}$$

$$R_{12} = \bar{R}_{12} + \delta R_{12} = \frac{1}{2}\partial_0\partial_0 h_{12} - \frac{1}{2a^2}\partial_3\partial_3 h_{12} - \frac{\dot{a}}{2a}\partial_0 h_{12} + \frac{2\dot{a}^2}{a^2}h_{12} \tag{4.26}$$

$$R_{01} = R_{02} = R_{13} = R_{23} = 0 \tag{4.27}$$

After obtaining components of the Ricci tensor, we can easily compute the Ricci scalar by contracting the Ricci tensor with the inverse of the metric. Again we can regard the Ricci scalar as summation of zeroth and first order parts

$$R = \bar{R} + \delta R \tag{4.28}$$

- R is the Ricci scalar for metric $g_{\mu\nu}$
- $\bar{R}$ is the Ricci scalar for the Robertson-Walker metric
- δR is the perturbation of Ricci scalar

Contracting equation (4.13) with equation (4.9) yields

$$\begin{aligned} R = R_{\mu\nu}g^{\mu\nu} &= (\bar{R}_{\mu\nu} + \delta R_{\mu\nu})(f^{\mu\nu} - h^{\mu\nu}) \\ &= \bar{R}_{\mu\nu}f^{\mu\nu} - \bar{R}_{\mu\nu}h^{\mu\nu} + \delta R_{\mu\nu}f^{\mu\nu} \\ &= \bar{R} - \bar{R}_{\mu\nu}h^{\mu\nu} + \delta R_{\mu\nu}f^{\mu\nu} \end{aligned} \tag{4.29}$$

Since only $\bar{R}_{11}$, $\bar{R}_{22}$ and $\bar{R}_{33}$ have nonzero values, $\bar{R}_{\mu\nu}h^{\mu\nu}$ gives

$$\bar{R}_{\mu\nu}h^{\mu\nu} = (a\ddot{a} + 2\dot{a}^2)(h^{11} + h^{22}) = (\frac{\ddot{a}}{a^3} + \frac{2\dot{a}^2}{a^4})(h_{11} + h_{22}) \tag{4.30}$$

For the third term of equation (4.29), summation can be carried out by using first order parts of equation (4.21)-equation (4.27) and RW metric as

$$\begin{aligned}
\delta R_{\mu\nu}f^{\mu\nu} =& \delta R_{00}f^{00} + \delta R_{11}f^{11} + \delta R_{22}f^{22} + \delta R_{33}f^{33} \\
=& \frac{1}{2a^2}\partial_0\partial_0(h_{11} + h_{22}) - \frac{\dot{a}}{a^3}\partial_0(h_{11} + h_{22}) - \left(\frac{\ddot{a}}{a^3} - \frac{\dot{a}^2}{a^4}\right)(h_{11} + h_{22}) \\
&+ \frac{1}{2a^2}\partial_0\partial_0 h_{11} - \frac{1}{2a^4}\partial_3\partial_3 h_{11} + \frac{\dot{a}}{2a^3}\partial_0 h_{22} + \frac{\dot{a}^2}{a^4}(h_{11} - h_{22}) \\
&+ \frac{1}{2a^2}\partial_0\partial_0 h_{22} - \frac{1}{2a^4}\partial_3\partial_3 h_{22} + \frac{\dot{a}}{2a^3}\partial_0 h_{11} + \frac{\dot{a}^2}{a^4}(h_{22} - h_{11}) \\
&- \frac{1}{2a^4}\partial_3\partial_3(h_{11} + h_{22}) + \frac{\dot{a}}{2a^3}\partial_0(h_{11} + h_{22}) - \frac{\dot{a}^2}{a^4}(h_{11} + h_{22})
\end{aligned} \tag{4.31}$$

Adding equation (3.7), equation (4.31) and subtracting equation (4.30) gives the Ricci scalar for the perturbed RW metric as

$$\begin{aligned}
R =& 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{1}{a^2}\partial_0\partial_0(h_{11} + h_{22}) - \frac{1}{a^4}\partial_3\partial_3(h_{11} + h_{22}) \\
&- 2(\frac{\ddot{a}}{a^3} + \frac{\dot{a}^2}{a^4})(h_{11} + h_{22})
\end{aligned} \tag{4.32}$$

4.2. Derivatives of the Scalar Field in Perturbed Robertson Walker Metric

Up to that point, since we have the Ricci tensor and scalar for metric $g_{\mu\nu}$, we can write the left hand side of the first JBD equation which is equation (4.7). For the other side of the equation, we need to take covariant derivative of partial derivative of perturbed scalar field Φ, which consists of $\phi(t)$ and the perturbation $\delta\phi(x)$. As we have mentioned before, we expect to find a wave solution for $\delta\phi/\phi$, which is a dimensionless quantity. For a scalar gravitational wave with preferred propagation direction x^3, we can consider $\delta\phi$ as it depends on time and x^3 coordinate.

Let us begin with simple formula of covariant derivative of a one-form, which is partial derivative of a scalar.

$$\nabla_\mu \partial_\nu \Phi = \partial_\mu \partial_\nu \Phi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \Phi \tag{4.33}$$

And Christoffel symbol is

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}] \tag{4.34}$$

After writing our metric explicitly, equation (4.34) becomes

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} (f^{\rho\lambda} - h^{\rho\lambda}) [\partial_\mu (f_{\rho\nu} + h_{\rho\nu}) + \partial_\nu (f_{\mu\rho} + h_{\mu\rho}) - \partial_\rho (f_{\mu\nu} + h_{\mu\nu})] \tag{4.35}$$

At this point, we do not have to compute all different combinations of Christoffel symbol. We only need values of superindex λ which are 0 and 3 since perturbed scalar field Φ depends only t, and x^3. So, for $\mu = 0$ and $\nu = 0$, $\nabla_\mu \partial_\nu \Phi$ is

$$\nabla_0 \partial_0 \Phi = \partial_0 \partial_0 \Phi - \Gamma^0_{00} \partial_0 \Phi - \Gamma^3_{00} \partial_3 \Phi \tag{4.36}$$

Since we have

$$\Gamma^0_{00} = \Gamma^3_{00} = 0 \tag{4.37}$$

equation (4.36) is written as

$$\nabla_0 \partial_0 \Phi = \partial_0 \partial_0 \Phi \tag{4.38}$$

For $\mu = 1$ and $\nu = 1$, $\nabla_\mu \partial_\nu \Phi$ is

$$\nabla_1 \partial_1 \Phi = \partial_1 \partial_1 \Phi - \Gamma^0_{11} \partial_0 \Phi - \Gamma^3_{11} \partial_3 \Phi \tag{4.39}$$

Since we have

$$\Gamma^0_{11} = -\frac{1}{2} f^{00} \partial_0 (f_{11} + h_{11}) = \frac{1}{2} \partial_0 (f_{11} + h_{11}) \tag{4.40}$$

and

$$\Gamma^3_{11} = -\frac{1}{2} f^{33} \partial_3 h_{11} = -\frac{1}{2a^2} \partial_3 h_{11} \tag{4.41}$$

equation (4.39) is written as

$$\nabla_1 \partial_1 \Phi = -\frac{1}{2} \partial_0 (f_{11} + h_{11}) \partial_0 \Phi + \frac{1}{2a^2} \partial_3 h_{11} \partial_3 \Phi \tag{4.42}$$

For $\mu = 2$ and $\nu = 2$, $\nabla_\mu \partial_\nu \Phi$ is

$$\nabla_2 \partial_2 \Phi = \partial_2 \partial_2 \Phi - \Gamma^0_{22} \partial_0 \Phi - \Gamma^3_{22} \partial_3 \Phi \tag{4.43}$$

Since we have

$$\Gamma^0_{22} = -\frac{1}{2} f^{00} \partial_0 (f_{22} + h_{22}) = \frac{1}{2} \partial_0 (f_{22} + h_{22}) \tag{4.44}$$

and

$$\Gamma^3_{22} = -\frac{1}{2} f^{33} \partial_3 h_{22} = -\frac{1}{2a^2} \partial_3 h_{22} \tag{4.45}$$

equation (4.43) is written as

$$\nabla_2 \partial_2 \Phi = -\frac{1}{2} \partial_0 (f_{22} + h_{22}) \partial_0 \Phi + \frac{1}{2a^2} \partial_3 h_{22} \partial_3 \Phi \tag{4.46}$$

For $\mu = 3$ and $\nu = 3$, $\nabla_\mu \partial_\nu \Phi$ is

$$\nabla_3 \partial_3 \Phi = \partial_3 \partial_3 \Phi - \Gamma^0_{33} \partial_0 \Phi - \Gamma^3_{33} \partial_3 \Phi \tag{4.47}$$

Since we have

$$\Gamma^0_{33} = -\frac{1}{2} f^{00} \partial_0 f_{33} = \frac{1}{2} \partial_0 f_{33} \tag{4.48}$$

and

$$\Gamma^3_{33} = 0 \tag{4.49}$$

equation (4.47) is written as

$$\nabla_3 \partial_3 \Phi = \partial_3 \partial_3 \Phi - \frac{1}{2} \partial_0 f_{33} \partial_0 \Phi \tag{4.50}$$

For $\mu = 0$ and $\nu = 3$, $\nabla_\mu \partial_\nu \Phi$ is

$$\nabla_0 \partial_3 \Phi = \partial_0 \partial_3 \Phi - \Gamma^0_{03} \partial_0 \Phi - \Gamma^3_{03} \partial_3 \Phi \tag{4.51}$$

Since we have

$$\Gamma^0_{03} = 0 \tag{4.52}$$

and

$$\Gamma^3_{03} = \frac{1}{2} f^{33} \partial_0 f_{33} = \frac{1}{2a^2} \partial_0 f_{33} \tag{4.53}$$

equation (4.51) is written as

$$\nabla_0 \partial_3 \Phi = \partial_0 \partial_3 \Phi - \frac{1}{2a^2} \partial_0 f_{33} \partial_3 \Phi \tag{4.54}$$

For $\mu = 1$ and $\nu = 2$, $\nabla_\mu \partial_\nu \Phi$ is

$$\nabla_1 \partial_2 \Phi = \partial_1 \partial_2 \Phi - \Gamma^0_{12} \partial_0 \Phi - \Gamma^3_{12} \partial_3 \Phi \tag{4.55}$$

Since we have

$$\Gamma^0_{12} = -\frac{1}{2} f^{00} \partial_0 h_{12} = \frac{1}{2} \partial_0 h_{12} \tag{4.56}$$

and

$$\Gamma^3_{12} = -\frac{1}{2} f^{33} \partial_3 h_{12} = -\frac{1}{2a^2} \partial_3 h_{12} \tag{4.57}$$

equation (4.55) is written as

$$\nabla_1 \partial_2 \Phi = -\frac{1}{2} \partial_0 h_{12} \partial_0 \Phi + \frac{1}{2a^2} \partial_3 h_{12} \partial_3 \Phi \tag{4.58}$$

4.3. Solutions to Linearized Jordan Brans Dicke Equations

In this section, we plan to construct and solve the perturbed JBD equations. Since we have found necessary elements in previous sections, we can now place them into the equations, and look for solutions of a, ϕ and perturbations which are consistent with our all equations. We have two basic equations, however the first one which is equation (4.7), will yield more than one due to different components of the Einstein tensor which is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. Let us begin with the second JBD equation which is equation (4.8). Its compact form is equation (4.6) as

$$\frac{1}{\Phi} g^{\mu\nu} \nabla_\mu \partial_\nu \Phi = 0$$

After inserting explicit forms of Φ and $g_{\mu\nu}$, it becomes equation (4.8) which is

$$\frac{1}{(\phi + \delta\phi)} (f^{\mu\nu} - h^{\mu\nu}) \nabla_\mu \partial_\nu (\phi + \delta\phi) = 0$$

There is summation in this relation between metric components and derivatives of Φ. It can be expanded as

$$\begin{aligned}\frac{1}{(\phi+\delta\phi)}[f^{00}\nabla_0\partial_0(\phi+\delta\phi)+(f^{11}-h^{11})\nabla_1\partial_1(\phi+\delta\phi)\\ +(f^{22}-h^{22})\nabla_2\partial_2(\phi+\delta\phi)+f^{33}\nabla_3\partial_3(\phi+\delta\phi)]=0\end{aligned} \tag{4.59}$$

then inserting metric components and covariant derivatives of partial derivatives of Φ, which have been found in the previous section, into equation (4.59) yields

$$\begin{aligned}&-\frac{\partial_0\partial_0\phi}{(\phi+\delta\phi)}-\frac{3\dot{a}}{a}\frac{\partial_0\phi}{(\phi+\delta\phi)}-\frac{1}{2a^2}\frac{\partial_0\phi}{\phi}\partial_0(h_{11}+h_{22})\\ &+\frac{\dot{a}}{a^3}\frac{\partial_0\phi}{\phi}(h_{11}+h_{22})-\frac{\partial_0\partial_0\delta\phi}{\phi}-\frac{3\dot{a}}{a}\frac{\partial_0\delta\phi}{\phi}+\frac{1}{a^2}\frac{\partial_3\partial_3\delta\phi}{\phi}=0\end{aligned} \tag{4.60}$$

In order to separate the zeroth and the first order terms in equation (4.60), we need one more arrangement like

$$(\phi+\delta\phi)^{-1}\simeq\frac{1}{\phi}\left(1-\frac{\delta\phi}{\phi}\right) \tag{4.61}$$

After substituting equation (4.61) into equation (4.60) and separating the zeroth order and the first order terms, the zeroth order equation is

$$-\frac{\partial_0\partial_0\phi}{\phi}-\frac{3\dot{a}}{a}\frac{\partial_0\phi}{\phi}=0 \tag{4.62}$$

Placing $a\propto t^q$ and $\phi\propto t^s$ into this equation yields the relation $3q+s=1$ which is equation (3.15). The first order equation is

$$\begin{aligned}&-\frac{\partial_0\partial_0\delta\phi}{\phi}-\frac{3\dot{a}}{a}\frac{\partial_0\delta\phi}{\phi}+\frac{1}{a^2}\frac{\partial_3\partial_3\delta\phi}{\phi}-\frac{1}{2a^2}\frac{\partial_0\phi}{\phi}\partial_0(h_{11}+h_{22})\\ &+\frac{\dot{a}}{a^3}\frac{\partial_0\phi}{\phi}(h_{11}+h_{22})+\frac{\partial_0\partial_0\phi\delta\phi}{\phi^2}+\frac{3\dot{a}}{a}\frac{\partial_0\phi\delta\phi}{\phi^2}=0\end{aligned} \tag{4.63}$$

Since we know $h_{11}+h_{22}=-2a^2(\delta\phi/\phi)$, by using this relation and integration by parts method, equation (4.63) can be written as

$$-\partial_0\partial_0\left(\frac{\delta\phi}{\phi}\right)-\frac{3\dot{a}}{a}\partial_0\left(\frac{\delta\phi}{\phi}\right)+\frac{1}{a^2}\partial_3\partial_3\left(\frac{\delta\phi}{\phi}\right)-\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right)=0 \tag{4.64}$$

This is the final form of the first order part of equation (4.59) and it has the form of a wave equation for $(\delta\phi/\phi)$. First three terms can be written as $f^{\mu\nu}\nabla_\mu\nabla_\nu(\delta\phi/\phi)$, but the fourth term is unusual.

Now, there is one more equation to solve for the perturbed JBD theory. In total, this one gives six equations because there are six different components of the Einstein tensor $G_{\mu\nu}$ for our perturbed RW metric. Of course, since we have made some assumptions about metric perturbation $h_{\mu\nu}$, the number of nonzero components of the Einstein tensor has been reduced to six. Let us begin with inserting equation (4.21), equation (4.32) and equation (4.38) into equation (4.7) to get equation of $\mu=0$ and $\nu=0$ as

$$\begin{aligned}
&-\frac{3\ddot{a}}{a}-\frac{1}{2a^2}\partial_0\partial_0(h_{11}+h_{22})+\frac{\dot{a}}{a^3}\partial_0(h_{11}+h_{22})+\left(\frac{\ddot{a}}{a^3}-\frac{\dot{a}^2}{a^4}\right)(h_{11}+h_{22})\\
&+3\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)-\left(\frac{\ddot{a}}{a^3}+\frac{\dot{a}^2}{a^4}\right)(h_{11}+h_{22})+\frac{1}{2a^2}\partial_0\partial_0(h_{11}+h_{22})\\
&-\frac{1}{2a^4}\partial_3\partial_3(h_{11}+h_{22})=\frac{1}{(\phi+\delta\phi)}\partial_0\partial_0(\phi+\delta\phi)\\
&+\frac{\omega}{(\phi+\delta\phi)^2}\left[(\partial_0\phi\partial_0\phi+2\partial_0\phi\partial_0\delta\phi)-\frac{1}{2}(\partial_0\phi\partial_0\phi+2\partial_0\phi\partial_0\delta\phi)\right]
\end{aligned} \tag{4.65}$$

Some terms cancel each other, so this equation can be simplified as

$$\begin{aligned}
&3\dot{a}^2-\frac{1}{2a^2}\partial_3\partial_3(h_{11}+h_{22})+\frac{\dot{a}}{a}\partial_0(h_{11}+h_{22})-\frac{2\dot{a}^2}{a^2}(h_{11}+h_{22})=a^2\frac{\partial_0\partial_0\phi}{\phi}\\
&-a^2\frac{(\partial_0\partial_0\phi)\delta\phi}{\phi^2}+a^2\frac{\partial_0\partial_0\delta\phi}{\phi}+\omega a^2\left[\frac{(\partial_0\phi)^2}{2\phi^2}-\frac{(\partial_0\phi)^2\delta\phi}{\phi^3}+\frac{\partial_0\phi\partial_0\delta\phi}{\phi^2}\right]
\end{aligned} \tag{4.66}$$

As we have mentioned before, we can separate this equation in two parts as the zeroth and the first order in $h_{\mu\nu}$ and $\delta\phi$. So the zeroth order part of this equation is

$$3\frac{\dot{a}^2}{a^2} - \frac{\partial_0\partial_0\phi}{\phi} - \omega\frac{(\partial_0\phi)^2}{2\phi^2} = 0$$

This equation is exactly equation (3.8), so it does not give any new information about the solutions of a and ϕ. For the first order part, equation (4.66) can be written as

$$\begin{aligned} &- a^2\frac{\partial_0\partial_0\delta\phi}{\phi} - \frac{1}{2a^2}\partial_3\partial_3(h_{11}+h_{22}) + \frac{\dot{a}}{a}\partial_0(h_{11}+h_{22}) \\ &- \frac{2\dot{a}^2}{a^2}(h_{11}+h_{22}) = -a^2\frac{(\partial_0\partial_0\phi)\delta\phi}{\phi^2} + \omega a^2\left[-\frac{(\partial_0\phi)^2\delta\phi}{\phi^3} + \frac{\partial_0\phi\partial_0\delta\phi}{\phi^2}\right] \end{aligned} \tag{4.67}$$

By using $h_{11}+h_{22} = -2a^2(\delta\phi/\phi)$, equation (4.64) and relations via integration by parts

$$\frac{\partial_0\partial_0\delta\phi}{\phi} = \partial_0\partial_0\left(\frac{\delta\phi}{\phi}\right) + 2\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right) + \frac{\partial_0\partial_0\phi\delta\phi}{\phi^2} \tag{4.68}$$

$$\omega a^2\left[-\frac{(\partial_0\phi)^2\delta\phi}{\phi^3} + \frac{\partial_0\phi\partial_0\delta\phi}{\phi^2}\right] = \omega a^2\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right) \tag{4.69}$$

equation (4.67) can be simplified as

$$a\dot{a}\partial_0\left(\frac{\delta\phi}{\phi}\right) - a^2\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right) = \omega a^2\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right) \tag{4.70}$$

We can write this equation like

$$\frac{\dot{a}}{a} - \frac{\partial_0\phi}{\phi} = \omega\frac{\partial_0\phi}{\phi} \tag{4.71}$$

We have assumed that ϕ and a have power-law solutions like $\phi \propto t^s$ and $a \propto t^q$. So, after we put them into equation (4.71), we get

$$q = s(\omega+1) \tag{4.72}$$

This equation is a new relation of q, s and ω. Now we have three equations to solve for three unknowns. Before finding solutions for q, s and ω, as well as ϕ and a, we should keep going and find consistency of this relation for other equations. Equation (4.7) for $\mu = 1$ and $\nu = 1$ is

$$\begin{aligned}
&(a\ddot{a} + 2\dot{a}^2) + \frac{1}{2}\partial_0\partial_0 h_{11} - \frac{1}{2a^2}\partial_3\partial_3 h_{11} + \frac{\dot{a}}{2a}\partial_0 h_{22} + \frac{\dot{a}^2}{a^2}(h_{11} - h_{22}) \\
&- 3(a\ddot{a} + \dot{a}^2) - \frac{1}{2}\partial_0\partial_0(h_{11} + h_{22}) + \frac{1}{2a^2}\partial_3\partial_3(h_{11} + h_{22}) \\
&+ \left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)(h_{11} + h_{22}) - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) h_{11} = \frac{1}{(\phi + \delta\phi)}\nabla_1\partial_1(\phi + \delta\phi) \\
&+ \frac{\omega}{2(\phi + \delta\phi)^2}\left[a^2(\partial_0\phi\partial_0\phi + 2\partial_0\phi\partial_0\delta\phi) + \partial_0\phi\partial_0\phi h_{11}\right]
\end{aligned} \tag{4.73}$$

Terms of this equation can be arranged. By using equation (4.42) it can be written as

$$\begin{aligned}
&- 2a\ddot{a} - \dot{a}^2 + \frac{\dot{a}}{2a}\partial_0 h_{22} - \frac{1}{2}\partial_0\partial_0 h_{22} + \frac{1}{2a^2}\partial_3\partial_3 h_{22} - \left(\frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a}\right) h_{11} \\
&+ \frac{\ddot{a}}{a} h_{22} = -a\dot{a}\frac{\partial_0\phi}{\phi} + a\dot{a}\frac{\partial_0\phi\delta\phi}{\phi^2} - a\dot{a}\frac{\partial_0\delta\phi}{\phi} - \frac{\partial_0\phi\partial_0 h_{11}}{2\phi} \\
&+ \omega a^2\left[\frac{(\partial_0\phi)^2}{2\phi^2} - \frac{(\partial_0\phi)^2\delta\phi}{\phi^3} + \frac{\partial_0\phi\partial_0\delta\phi}{\phi^2} + \frac{(\partial_0\phi)^2 h_{11}}{2a^2\phi^2}\right]
\end{aligned} \tag{4.74}$$

Before inserting the values of h_{11} and h_{22} to the equation, we separate the zeroth order terms of the equation as

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\partial_0\phi}{\phi} - \frac{\omega}{2}\frac{(\partial_0\phi)^2}{\phi^2} = 0$$

Since this is exactly equation (3.9), it does not have any new information for us. After substituting h_{11} and h_{22} in terms of A and $\delta\phi/\phi$, the first order part of the equation can be arranged as

$$\begin{aligned}
&\frac{1}{2}\partial_0\partial_0 A + \frac{3\dot{a}}{2a}\partial_0 A - \frac{1}{2a^2}\partial_3\partial_3 A + \frac{\partial_0\phi}{2\phi}\partial_0 A + \frac{1}{2}\partial_0\partial_0\left(\frac{\delta\phi}{\phi}\right) \\
&+ \frac{3\dot{a}}{2a}\partial_0\left(\frac{\delta\phi}{\phi}\right) - \frac{1}{2a^2}\partial_3\partial_3\left(\frac{\delta\phi}{\phi}\right) + \frac{\partial_0\phi}{2\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right) = -\frac{\dot{a}}{a}\partial_0\left(\frac{\delta\phi}{\phi}\right) \\
&+ \frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right) + \omega\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right)
\end{aligned} \tag{4.75}$$

As it is seen, the left hand side of the equation is in the form of equation (4.64) for A and $\delta\phi/\phi$ and they are both equal to zero. Again if we use $\phi \propto t^s$ and $a \propto t^q$ relations, the right hand side of the equation gives the equality which we are familiar with,

$$q = s(\omega + 1)$$

As can be checked, equation (4.7) for $\mu = 2$ and $\nu = 2$ is the same for $\mu = 1$ and $\nu = 1$. For $\mu = 2$ and $\nu = 2$, it is

$$\begin{aligned}
&(a\ddot{a} + 2\dot{a}^2) + \frac{1}{2}\partial_0\partial_0 h_{22} - \frac{1}{2a^2}\partial_3\partial_3 h_{22} + \frac{\dot{a}}{2a}\partial_0 h_{11} + \frac{\dot{a}^2}{a^2}(h_{22} - h_{11}) \\
&- 3(a\ddot{a} + \dot{a}^2) - \frac{1}{2}\partial_0\partial_0(h_{11} + h_{22}) + \frac{1}{2a^2}\partial_3\partial_3(h_{11} + h_{22}) \\
&+ \left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)(h_{11} + h_{22}) - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{22} = \frac{1}{(\phi + \delta\phi)}\nabla_2\partial_2(\phi + \delta\phi) \\
&+ \frac{\omega}{2(\phi + \delta\phi)^2}\left[a^2(\partial_0\phi\partial_0\phi + 2\partial_0\phi\partial_0\delta\phi) + \partial_0\phi\partial_0\phi h_{22}\right]
\end{aligned} \tag{4.76}$$

In order to see the similarity between $\mu = 2$, $\nu = 2$ and $\mu = 1$, $\nu = 1$ cases, this equation can also be organised as

$$\begin{aligned}
&- 2a\ddot{a} - \dot{a}^2 + \frac{\dot{a}}{2a}\partial_0 h_{11} - \frac{1}{2}\partial_0\partial_0 h_{11} + \frac{1}{2a^2}\partial_3\partial_3 h_{11} - \left(\frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a}\right)h_{22} \\
&+ \frac{\ddot{a}}{a}h_{11} = -a\dot{a}\frac{\partial_0\phi}{\phi} + a\dot{a}\frac{\partial_0\phi\delta\phi}{\phi^2} - a\dot{a}\frac{\partial_0\delta\phi}{\phi} - \frac{\partial_0\phi\partial_0 h_{22}}{2\phi} \\
&+ \omega a^2\left[\frac{(\partial_0\phi)^2}{2\phi^2} - \frac{(\partial_0\phi)^2\delta\phi}{\phi^3} + \frac{\partial_0\phi\partial_0\delta\phi}{\phi^2} + \frac{(\partial_0\phi)^2 h_{22}}{2a^2\phi^2}\right]
\end{aligned} \tag{4.77}$$

Equation (4.7) for $\mu = 3$ and $\nu = 3$ is

$$\begin{aligned}
&(a\ddot{a} + 2\dot{a}^2) - \frac{1}{2a^2}\partial_3\partial_3(h_{11} + h_{22}) + \frac{\dot{a}}{2a}\partial_0(h_{11} + h_{22}) - \frac{\dot{a}^2}{a^2}(h_{11} + h_{22}) \\
&- 3(a\ddot{a} + \dot{a}^2) - \frac{1}{2}\partial_0\partial_0(h_{11} + h_{22}) + \frac{1}{2a^2}\partial_3\partial_3(h_{11} + h_{22}) \\
&+ \left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)(h_{11} + h_{22}) = \frac{1}{(\phi + \delta\phi)}\nabla_3\partial_3(\phi + \delta\phi) \\
&+ \frac{\omega a^2}{2(\phi + \delta\phi)^2}(\partial_0\phi\partial_0\phi + 2\partial_0\phi\partial_0\delta\phi)
\end{aligned} \tag{4.78}$$

This can be arranged by using equation (4.50) as

$$
\begin{aligned}
&-2a\ddot{a}-\dot{a}^2-\frac{1}{2}\partial_0\partial_0(h_{11}+h_{22})+\frac{\dot{a}}{2a}\partial_0(h_{11}+h_{22})+\frac{\ddot{a}}{a}(h_{11}+h_{22})=\frac{\partial_3\partial_3\delta\phi}{\phi}\\
&-a\dot{a}\frac{\partial_0\phi}{\phi}+a\dot{a}\frac{\partial_0\phi\delta\phi}{\phi^2}-a\dot{a}\frac{\partial_0\delta\phi}{\phi}+\omega a^2\left[\frac{(\partial_0\phi)^2}{2\phi^2}-\frac{(\partial_0\phi)^2\delta\phi}{\phi^3}+\frac{\partial_0\phi\partial_0\delta\phi}{\phi^2}\right]
\end{aligned}
\tag{4.79}
$$

The zeroth order part of this is equation (3.9)

$$
-2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}+\frac{\dot{a}}{a}\frac{\partial_0\phi}{\phi}-\frac{\omega}{2}\frac{(\partial_0\phi)^2}{\phi^2}=0
$$

which we already know, and to find the first order, we can simply write $h_{11}+h_{22}=-2a^2(\delta\phi/\phi)$. This leads to

$$
\begin{aligned}
&\partial_0\partial_0\left(\frac{\delta\phi}{\phi}\right)+\frac{3\dot{a}}{a}\partial_0\left(\frac{\delta\phi}{\phi}\right)-\frac{1}{a^2}\partial_3\partial_3\left(\frac{\delta\phi}{\phi}\right)+\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right)\\
&=-\frac{\dot{a}}{a}\partial_0\left(\frac{\delta\phi}{\phi}\right)+\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right)+\omega\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right)
\end{aligned}
\tag{4.80}
$$

Again the left hand side of the equation is equation (4.64) and it equals to zero. After substituting $\phi \propto t^s$ and $a \propto t^q$, rest of the equation gives equation (4.72) as

$$
q=s(\omega+1)
$$

Let us continue with equations of nondiagonal components of the Einstein tensor. For $\mu=0$ and $\nu=3$, it is

$$
-\frac{1}{2a^2}\partial_3\partial_0(h_{11}+h_{22})+\frac{\dot{a}}{a^3}\partial_3(h_{11}+h_{22})=\frac{\partial_3\partial_0\delta\phi}{\phi}-\frac{\dot{a}}{a}\frac{\partial_3\delta\phi}{\phi}+\omega\frac{\partial_0\phi\partial_3\delta\phi}{\phi^2}
\tag{4.81}
$$

The left hand side of the equation equals to

$$
-\frac{1}{2a^2}\partial_3\partial_0(h_{11}+h_{22})+\frac{\dot{a}}{a^3}\partial_3(h_{11}+h_{22})=\partial_3\partial_0\left(\frac{\delta\phi}{\phi}\right)
\tag{4.82}
$$

and the first term in the right hand side can be written by using integration by parts as

$$\frac{\partial_3\partial_0\delta\phi}{\phi} = \partial_3\partial_0\left(\frac{\delta\phi}{\phi}\right) + \frac{\partial_0\phi}{\phi}\partial_3\left(\frac{\delta\phi}{\phi}\right) \tag{4.83}$$

After inserting last two equations into equation (4.81), the ultimate relation is

$$\frac{\dot{a}}{a}\partial_3\left(\frac{\delta\phi}{\phi}\right) = \frac{\partial_0\phi}{\phi}\partial_3\left(\frac{\delta\phi}{\phi}\right) + \omega\frac{\partial_0\phi}{\phi}\partial_3\left(\frac{\delta\phi}{\phi}\right) \tag{4.84}$$

and using $\phi \propto t^s$ and $a \propto t^q$ gives

$$q = s(\omega + 1)$$

which is equation (4.72). Lastly, for $\mu = 1$ and $\nu = 2$, equation (4.7) is

$$\begin{aligned} &\frac{1}{2}\partial_0\partial_0 h_{12} - \frac{1}{2a^2}\partial_3\partial_3 h_{12} - \frac{\dot{a}}{2a}\partial_0 h_{12} + \frac{2\dot{a}^2}{a^2}h_{12} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{12} \\ = &-\frac{1}{2}\frac{\partial_0\phi\partial_0 h_{12}}{\phi} + \frac{\omega}{2}\frac{\partial_0\phi\partial_0\phi h_{12}}{\phi^2} \end{aligned} \tag{4.85}$$

If we replace h_{12} with $a^2 B$ where B is cross polarization wave and substitute equation (3.9) into equation (4.85), it becomes

$$\frac{1}{2}\partial_0\partial_0 B + \frac{3\dot{a}}{2a}\partial_0 B - \frac{1}{2a^2}\partial_3\partial_3 B + \frac{\partial_0\phi}{2\phi}\partial_0 B = 0 \tag{4.86}$$

which is equation (4.64) for B.

As we have seen, equation (4.64) has a form like a wave equation for $(\delta\phi/\phi)$, A and B. In this section, the relation $q = s(\omega + 1)$ consistently appeared for different μ and ν cases. So now, we can look for solutions for q, s and ω. We have the relations

$$-3q^2 + 2q + sq - \frac{\omega}{2}s^2 = 0$$

$$6q^2 - 2q - sq + s - s^2 = 0$$

$$q = s(\omega + 1)$$

which are equation (3.11), equation (3.12) and equation (4.72) respectively. The solutions that satisfy these relations are $q = 1$, $s = -2$ with $\omega = -3/2$. Thus, we can write

$$a(t) = a_0 \left(\frac{t}{t_0} \right) \tag{4.87}$$

and

$$\phi(t) = \phi_0 \left(\frac{t}{t_0} \right)^{-2} \tag{4.88}$$

Although our finding for ω seems unpleasant because it is a negative coupling constant and solar system observations have shown $\omega > 10^4$, the JBD theory with negative ω value can explain accelerating expansion of the universe [9] without any necessity of cosmological constant [10]. In addition, $\omega = -3/2$ is the value which makes the JBD theory conformally invariant [11]. Also the JBD theory with $\omega = -3/2$ fits recent data of type Ia supernovae [12].

5. CONCLUSION

So far, we have reviewed the motivations that encouraged Jordan, Brans and Dicke to consider Newton's gravitational constant G as a time dependent parameter which is related to cosmic scalar field ϕ. In addition, we have mentioned how Dicke modified Einstein field equation with the idea of that if there is a scalar field, its energy momentum tensor should be included in the field equation. After writing appropriate Lagrangian for the scalar field, which is the function of ϕ and $\partial_\mu \phi$, the action of JBD theory has been easily defined. Then by using variational principle, the equations of JBD theory have been obtained as we have showed detailed calculations in Appendix.

Furthermore, we have solved the JBD equations for unperturbed RW metric and unperturbed scalar field, and as a result we have had two independent equations in terms of a, ϕ and ω. By assuming a and ϕ have power-law solutions like $a \propto t^q$ and $\phi \propto t^s$, values of q and s have been found, depending on the value of ω. The JBD constant ω has been put in the Lagrangian of the scalar field, since we have had no idea how the scalar field is coupled to geometry.

Lastly, we have added perturbations to the metric and the scalar field. Perturbations are dependent on time and spatial coordinates because the metric perturbation and $\delta\phi/\phi$ should have wave solutions. Since scalar wave should be a dimensionless quantity when it is coupled to geometry, not the perturbation of the scalar field, but $\delta\phi/\phi$ should be a scalar wave. In order to simplify our calculations, we have made a gauge choice for the metric perturbation $h_{\mu\nu}$ such that it is transverse to preferred propagation direction of gravitational wave. Then, we have computed the Ricci tensor components, the Ricci scalar and the covariant derivatives of one-forms, which are partial derivatives of perturbed scalar field, for perturbed RW metric.

After we have constructed the JBD equations for perturbed metric and perturbed scalar field, we have separated the equations as the zeroth and the first order equations in terms of $h_{\mu\nu}$ and $\delta\phi$. As we expect, the solutions of the zeroth order equations are not different from the solutions of the unperturbed JBD equations. However, the first order equations have produced two important results. One of them is the form of the wave equation for scalar and ordinary gravitational waves. For $\delta\phi/\phi$, it is equation (4.64) as

$$-\partial_0\partial_0\left(\frac{\delta\phi}{\phi}\right)-\frac{3\dot{a}}{a}\partial_0\left(\frac{\delta\phi}{\phi}\right)+\frac{1}{a^2}\partial_3\partial_3\left(\frac{\delta\phi}{\phi}\right)-\frac{\partial_0\phi}{\phi}\partial_0\left(\frac{\delta\phi}{\phi}\right)=0$$

This equation is similar to ordinary wave equation, since the first three terms can be written as $f^{\mu\nu}\nabla_\mu\nabla_\nu(\delta\phi/\phi)$. However, the last term is unusual. The other result is a new relation of q, s and ω values. We have had three independent equations for three unknowns. Finally, we have found values of q, s, and ω as 1, -2 and $-3/2$ respectively. Thus, we could write the solutions to the scale factor and the scalar field as

$$a(t)=a_0\left(\frac{t}{t_0}\right)$$

and

$$\phi(t)=\phi_0\left(\frac{t}{t_0}\right)^{-2}$$

These values of q, s and ω are the only solution for all the equations to be satisfied. They determine how the scale factor and the scalar field evolve with time for ordinary gravitational and scalar gravitational waves to exist and have the form of the wave equation. Also $\omega=-3/2$ is another necessary condition for being of the JBD theory conformally invariant and compatible with supernovae type Ia data.

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APPENDIX A: JBD EQUATION FROM VARIATIONAL PRINCIPLE

In this Appendix, we will obtain one of the basic Jordan-Brans-Dicke equations from the action by using variational principle. Taking variance of the JBD action with respect to inverse of the metric $g_{\mu\nu}$ and making it equal to zero is the procedure. Variation of the JBD action with respect to the scalar field is easy to operate so we will only focus on the first one. Let us begin with the JBD action which is equation (2.5)

$$S_{JBD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R + 16\pi \mathcal{L}_M - \omega \frac{g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi}{\phi} \right)$$

Since the Ricci scalar is the contraction of the Ricci tensor with the inverse of the metric, this equation becomes

$$S_{JBD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R_{\mu\nu} g^{\mu\nu} + 16\pi \mathcal{L}_M - \omega \frac{g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi}{\phi} \right) \tag{A.1}$$

By taking variation of the action, we can write

$$\begin{aligned} \delta S_{JBD} = \frac{1}{16\pi} \int dx^4 & \Big[\delta\sqrt{-g} \phi R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} \phi \delta R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} \phi R_{\mu\nu} \delta g^{\mu\nu} \\ & - \delta\sqrt{-g} \omega \frac{g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi}{\phi} - \sqrt{-g} \delta g^{\mu\nu} \omega \frac{\nabla_\mu \phi \nabla_\nu \phi}{\phi} \Big] \\ + \delta \int dx^4 & \sqrt{-g} \mathcal{L}_M \end{aligned} \tag{A.2}$$

Now, there are two variations we have to find. One is $\delta\sqrt{-g}$, and the other one is $\delta R_{\mu\nu}$. Let us begin with the first one and write the following relation

$$\ln(\det M) = \mathrm{Tr}(\ln M) \tag{A.3}$$

where M is a square matrix with nonzero determinant. Taking variance of this yields

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M) \tag{A.4}$$

After replacing matrix M with metric $g_{\mu\nu}$ and the determinant of the matrix with the determinant of the metric, which is g, equation (A.4) can be written as

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) \tag{A.5}$$

Since we are looking for variation with respect to $g^{\mu\nu}$, we should raise subindices of $\delta g_{\mu\nu}$. To be able to do that we will consider Kronecker delta which is a constant number. Taking variance of the relation $g_{\mu\sigma}g^{\sigma\nu} = \delta^{\nu}_{\mu}$ gives

$$\delta g_{\mu\sigma} = -g_{\sigma\nu}g_{\mu\sigma}\delta g^{\sigma\nu} \tag{A.6}$$

By using equation (A.6), equation (A.5) can be arranged as

$$\delta g = -g(g_{\mu\nu}\delta g^{\mu\nu}) \tag{A.7}$$

Now if we turn back to

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{\delta g}{\sqrt{-g}} \tag{A.8}$$

and substitute equation (A.7) into it, we obtain

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu} \tag{A.9}$$

For the variance of the Ricci scalar, we start with the relation

$$\delta R_{\mu\nu} = \delta R^{\sigma}_{\mu\sigma\nu} = \delta(\partial_{\sigma}\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \partial_{\nu}\Gamma^{\sigma}_{\sigma\mu} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}) \tag{A.10}$$

and so

$$\delta R^{\sigma}_{\mu\sigma\nu} = \partial_{\sigma}\delta\Gamma^{\sigma}_{\nu\mu} + \delta\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} + \Gamma^{\sigma}_{\sigma\lambda}\delta\Gamma^{\lambda}_{\nu\mu} - \partial_{\nu}\delta\Gamma^{\sigma}_{\sigma\mu} - \delta\Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu} - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\lambda}_{\sigma\mu} \tag{A.11}$$

Since covariant derivatives of the variations of the Christoffel symbols are

$$\nabla_{\sigma}\delta\Gamma^{\sigma}_{\nu\mu} = \partial_{\sigma}\delta\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\sigma}_{\sigma\lambda}\delta\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\sigma\nu}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\lambda}_{\sigma\mu}\delta\Gamma^{\sigma}_{\nu\lambda} \tag{A.12}$$

$$\nabla_{\nu}\delta\Gamma^{\sigma}_{\sigma\mu} = \partial_{\nu}\delta\Gamma^{\sigma}_{\sigma\mu} + \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\lambda}_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu}\delta\Gamma^{\sigma}_{\sigma\lambda} \tag{A.13}$$

then the variation of the Riemann tensor can be written as

$$\delta R^{\sigma}_{\mu\sigma\nu} = \nabla_{\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \nabla_{\nu}\delta\Gamma^{\sigma}_{\sigma\mu} \tag{A.14}$$

By using this, the second term of equation (A.2) in the parenthesis is written like

$$\begin{aligned}\int dx^4\sqrt{-g}\phi g^{\mu\nu}\delta R_{\mu\nu} &= \int dx^4\sqrt{-g}\phi g^{\mu\nu}(\nabla_{\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \nabla_{\nu}\delta\Gamma^{\sigma}_{\sigma\mu}) \\ &= \int dx^4\sqrt{-g}\phi\nabla_{\lambda}(g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} - g^{\mu\lambda}\delta\Gamma^{\sigma}_{\sigma\mu})\end{aligned} \tag{A.15}$$

Regarding how Christoffel symbol is defined, the following relations can be obtained

$$\delta\Gamma^{\lambda}_{\nu\mu} = \frac{1}{2}g^{\lambda\rho}(\partial_{\nu}\delta g_{\rho\mu} + \partial_{\mu}\delta g_{\nu\rho} - \partial_{\rho}\delta g_{\mu\nu}) \tag{A.16}$$

$$\delta\Gamma^{\lambda}_{\nu\mu} = \frac{1}{2}g^{\lambda\rho}(\nabla_{\nu}\delta g_{\rho\mu} + \nabla_{\mu}\delta g_{\nu\rho} - \nabla_{\rho}\delta g_{\mu\nu}) \tag{A.17}$$

Now we will consider the first term in parenthesis in equation (A.15). For the second one, the same procedure should be followed.

$$g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} = \frac{1}{2}(g^{\mu\nu}g^{\lambda\rho}\nabla_{\nu}\delta g_{\rho\mu} + g^{\mu\nu}g^{\lambda\rho}\nabla_{\mu}\delta g_{\nu\rho} - g^{\mu\nu}g^{\lambda\rho}\nabla_{\rho}\delta g_{\mu\nu}) \tag{A.18}$$

By using equation (A.6), this equation can be arranged as

$$g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} = \frac{1}{2}\big(- g^{\mu\nu}g^{\lambda\rho}g_{\rho\lambda}g_{\mu\nu}\nabla_{\nu}\delta g^{\nu\lambda} - g^{\mu\nu}g^{\lambda\rho}g_{\mu\nu}g_{\rho\lambda}\nabla_{\mu}\delta g^{\mu\lambda} \\ + g^{\mu\nu}g^{\lambda\rho}g_{\mu\nu}g_{\mu\nu}\nabla_{\rho}\delta g^{\mu\nu}\big) \tag{A.19}$$

Thus

$$g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} = \frac{1}{2}(-\nabla_{\nu}\delta g^{\nu\lambda} - \nabla_{\mu}\delta g^{\mu\lambda} + g_{\mu\nu}\nabla^{\lambda}\delta g^{\mu\nu}) \tag{A.20}$$

And for the second term, applying same steps gives

$$g^{\mu\lambda}\delta\Gamma^{\sigma}_{\sigma\mu} = \frac{1}{2}(-\nabla_{\sigma}\delta g^{\sigma\lambda} - g_{\rho\sigma}\nabla^{\lambda}\delta g^{\sigma\rho} + \nabla_{\rho}\delta g^{\rho\lambda}) \tag{A.21}$$

By taking difference of them,

$$g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} - g^{\mu\lambda}\delta\Gamma^{\sigma}_{\sigma\mu} = g_{\mu\nu}\nabla^{\lambda}\delta g^{\mu\nu} - \nabla_{\mu}\delta g^{\mu\lambda} \tag{A.22}$$

we can write equation (A.15) as

$$\begin{aligned} &\int dx^4\sqrt{-g}\phi\nabla_{\lambda}(g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} - g^{\mu\lambda}\delta\Gamma^{\sigma}_{\sigma\mu}) \\ &= \int dx^4\sqrt{-g}\phi\nabla_{\lambda}(g_{\mu\nu}\nabla^{\lambda}\delta g^{\mu\nu} - \nabla_{\mu}\delta g^{\mu\lambda}) \\ &= -\int dx^4\sqrt{-g}\nabla_{\lambda}\phi(g_{\mu\nu}\nabla^{\lambda}\delta g^{\mu\nu} - \nabla_{\mu}\delta g^{\mu\lambda}) \\ &= \int dx^4\sqrt{-g}(g_{\mu\nu}\nabla_{\lambda}\nabla^{\lambda}\phi\delta g^{\mu\nu} - \nabla_{\mu}\nabla_{\lambda}\phi\delta g^{\mu\lambda}) \\ &= \int dx^4\sqrt{-g}(g_{\mu\nu}g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\phi - \nabla_{\mu}\nabla_{\nu}\phi)\delta g^{\mu\nu} \end{aligned} \tag{A.23}$$

In the second line of equation (A.23) we have just used the equality in equation (A.22). In the third line, integration by parts has been applied. Also volume element of a covariant divergence has been set equal to a boundary contribution at infinity, which we can set to zero, thanks to Stokes' theorem. In the fourth line, again integration by parts method and Stokes' theorem have been applied. In the last line, some simple index manipulations have been made for nicer looking of it. Substituting equation (A.9) and equation (A.23) into equation (A.2) leads

$$\begin{aligned}\delta S_{JBD} = \frac{1}{16\pi}\int dx^4\sqrt{-g}\delta g^{\mu\nu}\Big[\phi R_{\mu\nu} + (g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi - \nabla_\mu\nabla_\nu\phi)\\ -\frac{1}{2}\phi R g_{\mu\nu} + \frac{1}{2}\omega g_{\mu\nu}\frac{g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi}{\phi} - \omega\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi}\Big]\\ +\delta\int dx^4\sqrt{-g}\mathcal{L}_M\end{aligned} \tag{A.24}$$

Now, by setting the variation of the JBD action equal to zero, the JBD equation can be obtained as in equation (2.6) which is

$$\begin{aligned}R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} =& \frac{8\pi}{\phi}T_{\mu\nu} + \frac{1}{\phi}\left(\nabla_\mu\partial_\nu\phi - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\partial_\beta\phi\right)\\ &+ \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\end{aligned}$$

where for definition of energy momentum tensor of matter, we have used equation (2.8) as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$$