OPTIMALLY LOCATING FACILITIES WITH VARIABLE CHARACTERISTICS

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ABSTRACT

OPTIMALLY LOCATING FACILITIES WITH VARIABLE CHARACTERISTICS

Facility location problems aim at optimally locating facilities like plants, warehouses, convenience stores, shopping malls etc. They can have different objectives such as maximizing the profit gained from the customers or minimizing the costs of locating facilities and serving the customers. In this thesis, we mainly focus on competitive facility location problems which constitute a special family. In a competitive facility location problem, a firm or franchise is concerned with installing new facilities to serve customers in a market where existing facilities with known locations and attractiveness levels compete for increasing their market share and profit. We can classify these problems into two groups: those with non-reactive competition and those with reactive competition. Three different types of competitive facility location models are proposed in order to determine the locations and attractiveness levels of the new facilities to maximize the profit in this thesis. The first one belongs to the former class, where the last two models fall into the latter one.

We formulate the first one as a mixed-integer nonlinear programming problem and propose three methods for its solution: a Lagrangean heuristic, a branch-and-bound method with Lagrangean relaxation, and a branch-and-bound method with nonlinear programming relaxation. The computational results obtained on a set of problem instances show that the branch-and-bound method using nonlinear programming relaxation is the most efficient and accurate solution method in order to solve the proposed problem. We consider next an extension of this model by relaxing the assumption that the competitor in the market does not react to the opening of new facilities. In other words, the competitor can react by adjusting the attractiveness levels of its existing facilities with the objective of maximizing its own profit. To this end, a bilevel mixed-integer nonlinear programming model is formulated. We transform this bilevel model into an equivalent one-level mixed-integer nonlinear program and solve it

by a global optimization method. For this problem, we also consider a scenario in which the new entrant firm ignores the reaction of the competitor. The experimental results indicate that anticipating the competitor's reaction by including this into his optimization problem increases the profit of the new entrant firm, whereas the competitor's profit is decreased. The last competitive facility location model relaxes the limitation about the competitor's reaction: now the competitor can also open new facilities, close existing ones and/or adjust their attractiveness. This also formulates a bilevel mixed-integer nonlinear programming problem which we try to solve by combining tabu search with global optimization algorithms. We develop three different tabu search methods and the computational results on a set of problem instances for comparing the performance of the solution methods show that the third tabu search method is the most accurate one, while the second tabu search method is the most efficient solution procedure.

Finally, we consider a different facility location problem which takes the customer preferences into account. The facilities are not necessarily identical and customers visit different types of facilities according to some given probability distribution and the maximum distance which they are willing to travel. We formulate a binary linear programming problem and solve it by three procedures that include a Lagrangean heuristic whose solution is improved further using a local search method. Based on the experimental results carried out on a set of problem instances the third solution method is the most efficient one. However, a statistical analysis on the quality of the solutions states that there is no significant difference between the three solution procedures.

ÖZET

DEĞİŞKEN ÖZELLİKLERİ OLAN TESİSLER İÇİN ENİYİ YER SEÇİMİ

Tesis yer seçimi problemleri fabrika, depo, bakkal, alışveriş merkezi gibi tesisleri eniyi yerlere yerleştirmeyi amaçlar. Bu problemler, bir yandan eniyi yerleri bulmaya çalışırken diğer yandan müşterilerden elde edilecek kazancı enbüyüklemeye ya da tesis açmaktan ve müşterilere hizmet etmekten kaynaklanan giderleri enküçüklemeye çalışır. Bu tezde, bir bölüm dışında tesis yer seçimi problemlerinin özel bir ailesini oluşturan rekabetçi tesis yer seçimi problemlerine odaklanıyoruz. Bir rekabetçi tesis yer seçimi probleminde işletmeler, rakiplerin bulunduğu pazara yeni bir tesis açarak girmeye çalışır. Rakiplere ait tesislerin yerleri ve çekicilikleri önceden bilinir. Tüm tesislerin amacı müşteriler için rekabet ederek pazar payını ve kazancı enbüyüklemektir. Rekabetçi tesis yer seçimi problemlerini tepkisel olanlar ve tepkisel olmayanlar olarak iki kümeye ayırmak olanaklıdır. Bu tezde ele alınan üç rekabetçi tesis yer seçimi probleminden ilki ikinci kümenin, diğer ikisi ise birinci kümenin içindedirler.

İlk problem için rakibin yeni tesisler açılmasına tepki göstermediği varsayımı altında karışık tamsayılı doğrusal olmayan bir programlama gösterimi geliştirmekte ve üç çözüm yöntemi önermekteyiz: bir Lagrange sezgiseli, Lagrange gevşetmesi kullanan bir dal-sınır algoritması ve doğrusal olmayan programlama gevşetmesi kullanan bir dal-sınır algoritması. Örnek problem verisi üzerinde elde edilen bilgisayısal sonuçlar, bu problemi çözmek için en verimli ve kesin yöntemin doğrusal olmayan programlama gevşetmesi kullanan dal-sınır algoritması olduğunu göstermektedir. İkinci problemimizde rakip firma kendi kazancını enbüyüklemek amacıyla varolan tesislerinin çekiciliklerini değiştirerek tepki göstermektedir. Bu amaçla bir çiftdüzeyli karışık tamsayılı doğrusal olmayan programlama gösterimi önermekteyiz. Bu çiftdüzeyli gösterim eşdeğer bir doğrusal olmayan karışık tamsayı programlama gösterimine dönüştürülmekte ve bir genel eniyileme yaklaşımıyla çözülmektedir. Bu problem için pazara yeni giriş yapan firmanın, rakip firmanın tepkisini yoksaydığı bir senaryoyu göz önünde bulundurmaktayız. Elde

edilen bilgisayısal sonuçlar pazara yeni giriş yapan firmanın rakibin tepkisini öngörmesinin bu firmanın kazancını arttırdığına ve rakip firmanın kazancını azaltığına işaret etmektedir. İncelediğimiz son rekabetçi tesis yer seçimi problemi ise rakibin tepkisi üzerindeki önemli bir sınırlamayı kaldırmaktadır. Rakip işletme, pazara yeni giriş yapan işletmenin yeni tesislerinin yerleri ve çekicilikleri belli olduktan sonra varolan tesislerinin çekiciliklerini değiştirerek, yeni tesisler açarak ve/veya varolan tesislerini kapatarak tepki gösterebilmektedir. Geliştirilen çiftdüzeyli karışık tamsayılı doğrusal olmayan programlama gösterimini çözmek için tabu arama ve genel eniyileme yöntemlerini kullanan melez sezgiseller önermekteyiz. Üç değişik melez tabu arama yöntemi geliştirmekteyiz ve bu yöntemlerin başarımlarını karşılaştırmak için yapılan bilgisayısal deneyler üçüncü tabu arama yönteminin en kesin sonuçları verdiğini ve ikinci tabu arama yönteminin ise en verimli yöntem olduğunu göstermektedir.

Son olarak tesislerin değişik türlerde kurulabildiği ve müşterilerin tercihlerinin gözetildiği rekabetçi olmayan bir yer seçimi problemi ele alınmaktadır. Bu problemde müşterilerin tesislere, türlerine ve aralarında olan uzaklıklarına bağlı olarak gittikleri varsayılmaktadır. Geliştirilen doğrusal ikili tamsayı programlama gösteriminin çözümü için sonucu bir yerel arama algoritmasıyla iyileştirilen bir Lagrange sezgiseli içeren üç yöntem önermekteyiz. Örnek problem verisine dayanan bilgisayısal sonuçlar üçüncü yöntemin en verimli olduğunu göstermektedir. Ancak elde edilen sonuçların kalitesi üzerinde yapılan istatistiksel bir analiz üç çözüm yöntemi arasında anlamlı bir fark olmadığına işaret etmektedir.

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A_k	new attractiveness level of the existing facility at site k
$\underline{A}_k$	current attractiveness level of the existing facility at site k
$\overline{A}_k$	maximum attractiveness level of the existing facility at site k
b_k	unit cost of increasing and unit revenue of decreasing the attractiveness of the competitor's existing facility at site k
c_i	unit attractiveness cost at candidate facility site i
$\tilde{c}_k$	unit attractiveness cost or revenue at existing facility site i
d_{ij}	Euclidean distance between site i and point/zone j
$\tilde{d}_{kj}$	Euclidean distance between the existing facility at site k and point j
$\widehat{d_{lj}}$	Euclidean distance between the competitor's candidate site ℓ and point j
e_ℓ	unit attractiveness cost of the competitor's new facility at site ℓ
f_i	new entrant firm's annualized fixed cost of opening and operating a facility at site i or a type-i facility
$\tilde{f}_\ell$	competitor's annualized fixed cost of opening and operating a facility at site ℓ
h_j	annual buying power or income at demand point or customer zone j
M_ℓ	attractiveness level of competitor's new facility at site ℓ
$\overline{M_\ell}$	maximum attractiveness level of competitor's new facility at site ℓ
N_{ik}	the set of customers within the region of influence of type-k facility at candidate site i
o_j	total utility of existing facilities for customers at point j
p_{jk}	probability that customers at zone j visit a type-k facility
Q_i	attractiveness of the facility opened at site i
S_{jk}	maximum distance customers at zone j are willing to visit a facility of type k
t_k	revenue of closing an existing facility at site k

u_i	maximum attractiveness level of a facility to be opened at site i
X_i	binary variable which is equal to one if a facility is opened at site i, and zero otherwise
x_{ik}	binary variable indicating whether a type-k facility is opened at potential site i
y_{ijk}	binary variable indicating whether customers at zone j visit a type-k facility at potential site i
Y_ℓ	binary variable which is equal to one if the competitor opens a new facility at site ℓ, and zero otherwise
w_{ikl}	auxiliary binary variables used to keep track of the relative difference of the average probability pairs for facility types k and l at potential site i
Z_k	binary variable which is equal to one if the competitor's existing facility at site k is kept open, and zero otherwise
α	step size at an iteration of the gradient ascent algorithm
ϵ	a very small number specified by the user
λ	Lagrangean multipliers
μ	user-defined exponential weight in the fuzzy C-means algorithm
ψ_{ik}	average probability for type-k facility at potential site i

LIST OF ACRONYMS/ABBREVIATIONS

BB	Branch-and-Bound
BB-LR	Branch-and-Bound with Lagrangean Relaxation
BB-NLP	Branch-and-Bound with Nonlinear Programming Relaxation
BP	Bilevel Programming
CFL	Competitive Facility Location
KKT	Karush-Kuhn-Tucker
LH	Lagrangean Heuristic
LLP	Lower Level Problem
LR	Lagrangean Relaxation
LS	Local Search
MCI	Multiplicative Competitive Interaction
MILP	Mixed-Integer Linear Program
MINLP	Mixed-Integer Nonlinear Program
NLP	Nonlinear Program
PD	Percent Deviation
PD_{avg}	Percent Deviation of the Average Objective Value
PD_{max}	Percent Deviation of the Best Objective Value
PFL	Probabilistic Facility Location
SSL	Semi-Supervised Learning
TS	Tabu Search
ULP	Upper Level Problem

1. INTRODUCTION

Facility location problems attempt to find the best location for facilities like warehouses, plants, and other industrial facilities by optimizing a certain objective function. They can be categorized first according to the candidate locations as discrete space and continuous space models. In the discrete space models, there are predetermined candidate facility sites which show where new facilities can be launched, whereas in the continuous space models new facilities can be located anywhere in the plane (Love *et al.*, 1988). In this thesis, we only deal with the discrete space models so that discrete space models are given more emphasis in this chapter and also in the literature survey.

The general facility location models in discrete space literature can be grouped as the p-median, set covering, maximal covering, and fixed-charge models (Daskin, 1995). The objective of a p-median model is to minimize the demand weighted total or average distance of customers from their closest facility by opening a fixed number of facilities p. In contrast to a p-median model the aim of a set covering model is to minimize the number of facilities opened by covering all the demand nodes. A maximal covering location problem, on the other hand, tries to capture the most of the demand by locating a predetermined number of facilities. Such a problem assumes that facilities may not cover all the demand nodes and if all the demand nodes cannot be covered, it tries to cover most of the demand. So far, all these models are uncapacitated location models and there is no cost regarding to the opening of new facilities and transportation between the facilities and demand nodes. What makes a fixed-charge location problem different than the other models are that each facility has a capacity to serve the customers, that there is a fixed cost of opening a new facility at a site, and that the transportation costs for serving the demand nodes are calculated as functions of distances between the demand nodes and new facilities.

In such facility location models all facilities considered to be opened identical. There is no difference between the facilities which is not a realistic assumption when today's economic conditions are taken into account. For example, if a company considers opening new warehouses to stockpile its different products, it is likely that not all the warehouses have the same size, capacity or design. Thus, truly real facilities have different characteristics which distinguish them from each other. The aim of the thesis is the optimal location of facilities with different characteristics.

In this thesis, we first concentrate on facility location problems in competitive environment. In a Competitive Facility Location (CFL) problem, a firm or franchise wants to enter into a market where already competing facilities exist. All the facilities compete for the customers and market share. Usually the objective is the maximization of the market share or the profit (Drezner, 1995). In a competitive environment facilities usually have different attractiveness levels which attract different customer classes. These attractiveness levels can be determined using the size of the facility, floor area of the facility, number of servers, or the diversity of the products sold in the facility. Thus, we see that the facilities in a competitive environment are not identical and can be differentiated from each other using the characteristics of the facilities. Each facility may have a different attractiveness level which uniquely identifies that facility. Furthermore, this attractiveness level defines the characteristics of a facility. For example, when the facility considered is a shopping mall, the number of stores, the size of the parking place, food court availability, or the proximity to the public transportation become the characteristics of that shopping mall and determine all together its overall attractiveness level.

We propose three different CFL models in discrete space, where the demand is assumed to be aggregated at certain points in the plane and new facilities can be located at predetermined candidate sites. In each competitive model, the attractiveness levels of new facilities are the decision variables. Our main goal is to find optimal facility attractiveness levels and facility locations that are also important characteristics of facilities beyond their attractiveness levels. So, we reflect the variable characteristics of facilities as its location and attractiveness levels, which is actually a combination of different facility attributes and features.

In the first proposed model, a new market entrant firm wants to locate new facilities in discrete space to compete against already existing facilities that may belong to one or more competitors. The objective of the new entrant firm is to determine the locations of the new facilities and their attractiveness levels so as to maximize the profit, which is calculated as the revenue from customers less the fixed cost of opening facilities and variable cost of setting their attractiveness levels. We formulate a mixed-integer nonlinear programming model for this problem and propose three methods for its solution: a Lagrangean heuristic, a branch-and-bound method with Lagrangean relaxation, and another branch-and-bound method with nonlinear programming relaxation.

Next we consider an extension on the first CFL model. We deal again with a problem in which a firm or franchise enters a market by locating new facilities where there are existing facilities belonging to a competitor. The new entrant firm aims at finding the location and attractiveness of each facility to be opened so as to maximize its profit. The competitor, on the other hand, can react by adjusting the attractiveness of its existing facilities with the objective of maximizing its own profit. To this end, we formulate a bilevel mixed-integer nonlinear programming model where the firm entering the market is the leader and the competitor is the follower of a game. In order to find the optimal solution of this model, we convert it into an equivalent one-level mixed-integer nonlinear program so that it can be solved by global optimization methods.

As one can see we extend the CFL problem proposed at the beginning with the reaction of the competitor. However, the competitor can react only by modifying the attractiveness levels of its existing facilities. Nevertheless, the reaction of a competitor in a market may not be limited only with redesigning the existing facilities. Therefore, with the next proposed model we address one more time such a CFL problem within a game-theoretic frame, where the reaction of the competitor is developed with new assumptions. We assume that the competitor can react by opening new facilities, closing existing ones, and/or adjusting the attractiveness levels of its existing facilities again with the aim of maximizing its own profit. A more complex bilevel mixed-integer nonlinear programming model is formulated where the entering firm is the leader and the competitor is the follower. We propose heuristics that combine tabu search with exact solution methods.

In all the proposed CFL problems the gravity-based rule is employed in modeling the customer behavior where the probability that a customer visits a certain facility is proportional to the facility attractiveness and inversely proportional to the distance between the facility site and demand point. In such models customer preferences are not taken into account. Thus, we address next a new discrete facility location model where the system planner can establish different types of facilities and customers have preferences represented by probabilities for visiting these facilities. In CFL models, this probability increases with the increased facility attractiveness which is an indication of the quality, type, or design of the facility. In real-life, it may be the case that each different type of facility has its own customer segment. So, for a given customer zone the sum of probabilities associated with different types of facilities is equal to one. It is assumed that customers choose the nearest facility among the same type of facilities and they visit this facility if its distance to the customer zone is within a threshold value. This threshold is determined by the customers and it increases as the utility of the facility type to the customers increases. The aim of the system planner is to find the optimal location and types of facilities to be opened so as to maximize the profit which is computed by the annual revenue collected from the visiting customers less the annualized fixed cost of facility establishments. We formulate a binary linear programming model and solve it by a Lagrangean heuristic the solution of which is further improved by a local search procedure implemented using 1-add, 1-drop, and 1-swap moves.

Hence, in the last proposed discrete model customer preferences are taken into consideration by means of the visiting probabilities for different facility types and the threshold distances. These distances show the maximum distances that a customer is willing to travel to a certain facility type. Thus, the visiting probabilities and threshold distances play an important role in determining the types for new facilities that are important variable characteristics of the facilities, since the facility types are decision variables to the proposed optimization model. Most importantly, customer preferences are better reflected in such a scenario considering favorable treatments for its customers and these preferences become apparent based on the customer attributes such as annual income.

In the remainder of the thesis we first review the literature related to the competitive facility location problems and facility location problems with customer preferences in the next chapter. We present a single-level MINLP formulation and its solution procedures. In the fourth chapter a new bilevel MINLP problem is introduced together with its properties and a global optimization method that can be applied when it is converted into an equivalent single-level problem. Then in the fifth chapter another bilevel MINLP problem is presented. Besides the problem, three different tabu search heuristics and an ϵ-optimal solution method are suggested. In the sixth chapter a novel discrete facility location problem considering customer preferences is formulated as a binary linear programming problem and three solution methods are proposed in order to solve the problem where the visiting probabilities of customers are obtained by two suggested artificial learning techniques. In the seventh chapter, the computational results on randomly generated instances for all proposed solution procedures are reported together with the sensitivity analysis. Finally, the eighth chapter concludes the thesis. In Appendices A and B, the pseudo codes for the global optimization methods αBB and GMIN-αBB are given, respectively. The steps of the first two tabu search heuristics given in Chapter 5 are provided in the Appendices C and D.

2. LITERATURE SURVEY

In this chapter we overview the works about the CFL and facility location models with customer preferences. We first survey the CFL studies and divide them into two groups: works considering the non-reactive competition and reactive competition. In CFL models with reactive competition we further differentiate between the simultaneous and sequential games. Then we continue with facility location models with customer preferences which also include the probabilistic location models.

2.1. Competitive Facility Location

In CFL problems, a firm is concerned with installing new facilities to serve customers in a market where existing facilities with known locations and attractiveness levels compete for increasing their market share and profit. In some cases, the firm may be a new entrant with no already existing facilities, while in others the firm may own one or more existing facilities. The choice of the customers as to which facility to visit can be modeled using different approaches. For example, models can be formulated in which customers do not choose a facility solely on the basis of their proximity to its location, but they also take into account some of the characteristics of the facility. The first paper on CFL was by Hotelling (1929) in which he developed a model with two equally attractive ice-cream sellers along a beach strip where customers patronize the closest one. It is further assumed that the buying power or demand is uniformly distributed. This very first model was extended later for unequally attractive facilities, which is a more realistic assumption given the current situation in the market.

Most of the CFL models in the literature assume that the competitors who own the existing facilities do not react after a new entrant firm opens facilities in the market. However, CFL models with competitor reaction become more popular in recent years. Therefore, we can classify the reviewed works into two groups: those with non-reactive competition and those with reactive competition.

2.1.1. Non-reactive Competition

The CFL models with non-reactive competition assume that the competitor(s) in the market does not react to the opening of new facilities by another firm or franchise. It is possible to divide those kind of CFL models into two categories: deterministic utility models and random utility models. In both categories, the attractiveness level of a facility is determined by a function of its attributes, and customers' attraction is modeled by a utility function. The main difference between the two types of models is that in the deterministic utility models the customers patronize only the facility with the highest utility for them, whereas in random utility models customers visit each facility with a certain probability.

2.1.1.1. Deterministic Utility Models. In deterministic utility models the customers choose a facility based on a utility function. This utility function consists of the facility attributes and the distance between the customer and the facility. However, as explained above the all-or-nothing property is still maintained, since all customers at a demand point patronize the same facility with the highest utility for them. The utility function is in the form of $U = F(x_1, x_2, ..., x_m)$, where $x_1, x_2, ..., x_m$ are the facility attributes and the distance between the customer and the facility. In most of the models, F is an additive function such as $U = \sum_{p=1}^{m} w_p f_p(x_p)$ where w_p represent the weight of the attributes (Drezner, 1995). Deterministic models with additive utility function usually employ a *break even distance*. It is defined as the distance at which utilities of the new and existing facilities are equal to each other. Customers at a demand point patronize a new facility if and only if this new facility is located within the break even distance.

The work of Drezner and Drezner (1994) is a perfect example of deterministic utility models where a new facility is opened in continuous space. It is assumed that all customers at a demand point visit only one facility with the highest utility for them which is calculated as an additive utility function. The best location for the new facility is found according to the break-even distance at which the new and existing facilities have the same utility towards the customers. However, the planner can decide to open only a single new facility where its attractiveness is a parameter to the considered optimization problem.

Another study that deserves attention belongs to Plastria and Carrizosa (2004). In this study, they try to open a single new facility where the determination of its best location as well as its best quality is sought. Although more than one consumer group is allowed to reside at a site, each group has its own deterministic behavior for the choice of the facility where each group patronizes the same facility to which it is attracted the most. Such models are called full capture models as well which can be seen as a deficiency when compared to probabilistic behavior models. It is not a sensible assumption for profit making facilities, since today's customers usually patronize more than one facility. These types of models are more suitable for public-oriented facilities such as post offices or primary schools.

2.1.1.2. Random Utility Models. As can be seen in deterministic utility models the probability that a customer patronizes a facility is either 0 or 1. In contrast to deterministic models, random utility models assume that the utility function varies among customers. It is further assumed that each customer draws his/her utility from a random distribution of utility functions. This means that the all-or-nothing property of deterministic utility models is soothed in a sense that each demand point has a probability of patronizing each facility which changes between 0 and 1.

Random utility models are used for discrete choice methods for short-term travel decisions. The term discrete choice is used when there is a finite number of alternatives for decision makers. In our case the alternatives are represented by the facilities and the decision makers by customers who choose which facility to visit. Most of the behavior models of decision makers are based on the random utility theory. In random utility models, the decision rules according to which the decision maker chooses an alternative is discrete. However, the utilities for decision makers are random variables composed of a deterministic and a random part. The unobserved characteristics of decision makers and unobserved attributes of alternatives evoke the probabilistic parts which are frequently represented by multinomial logit models (Hall, 1999).

The most widely used random utility in the CFL literature is the gravity-based model proposed by Reilly (1931) and later used by Huff (1964, 1966). In this model, the probability that a customer patronizes a facility is proportional to the attractiveness of the facility and inversely proportional to a function of the distance between the customer and the facility. A variety of attributes can be used as a proxy for the attractiveness of the facility. The shopping mall example given in Chapter 1 can be mentioned here one more time, because the number of stores, the size of the parking place, food court availability, or the proximity to the public transportation can be the attributes determining the overall attractiveness. Huff (1964, 1966) uses the floor area as the attractiveness of the facility.

The idea of Reilly (1931) and Huff (1964, 1966) is extended later by Nakanishi and Cooper (1974) giving birth to the multiplicative competitive interaction (MCI) model. In this model, different attributes of the facility were used together by taking their product after raising each to a power. Achabal *et al.* (1982) used the MCI model to determine the best location and design of a number of new stores in discrete space under the presence of a certain number of existing stores so that the market share is maximized. The design of a store is to be selected from a set of possible designs, where the parameter estimation is based on consumer surveys and studies.

All the studies cited in the following apply the gravity-based rule unless mentioned otherwise. First, we review the papers in which facility attractivenesses are parameters and the decisions to be made consist of facility locations only. Drezner and Drezner (2006) consider two models in which they opt for optimally locating p facilities. In the first one, the aim is to minimize the distance traveled by the customers as is the case in the well-known p-median problem, whereas in the second one a balance is sought among the facilities so that the variance of the demand served by the facilities is minimized. The authors also combine the objective functions of the two models to obtain a new multi-objective model.

The papers by Drezner and Drezner (2004) and Drezner *et al.* (2002) differ from each other with respect to the number of facilities that are located in the continuous space. The former paper examines the case with a single facility, while the second one addresses a multi-facility case. In both papers the objective is to maximize the market share where some of the existing facilities belong to the franchise which opens new facilities. The latter paper can be considered as an extension of the first one. Drezner and Drezner (2008) consider an extension of the CFL problem where the probability of customers' unwillingness to visit a facility, which ultimately affects the demand seen by the facility, is computed. They use the gravity-based model, but utilities are computed using an exponential decay function of the facility attractiveness represented by a parameter and the distance between the facility and customer sites. The decay parameter increases as the facility attractiveness decreases. The proposed models further assume that if there is a competing facility at zero distance to a demand point, it captures all the buying power at that point. Benati and Hansen (2002) employ a utility function whose deterministic part is a linear function of the facility attractiveness and the distance, while the random part is assumed to follow the Gumble distribution. Aboolian *et al.* (2007a) and Berman and Krass (2002) develop a spatial interaction model with variable expenditures where demand cannibalization and market expansion are taken into account. In standard spatial interaction models the demand is assumed to be constant and known a priori; therefore the demand is said to be inelastic. As the authors point out opening new facilities results in market expansion meaning that customers who are underserviced are attracted now to the new facilities. Thus, the demand increases with the opening of new facilities. The objective is again to optimally locate new facilities in the discrete space.

There are also papers in the literature in which decisions are made not only about the locations of the facilities but also about their attractiveness levels. The first one is the paper by Achabal *et al.* (1982). Aboolian *et al.* (2007b) formulate a similar model to Aboolian *et al.* (2007a), but they also incorporate the design characteristics of the facilities (i.e., the attractiveness levels) into the model. Although they state that the attractiveness of each facility is a continuous decision variable, they employ discrete design scenarios in the solution of the model and one of a finite number of available designs is determined for each open facility. Fernández *et al.* (2004, 2007), and Tóth *et al.* (2009) analyze similar models in which the aim is to optimally locate new facilities in continuous space. They find out the attractiveness levels when some of the existing facilities belong to the firm's own chain. Only one new facility is opened in Fernández *et al.* (2004), whereas two new facilities are located in Tóth *et al.* (2007). The difference of Fernández *et al.* (2007) is the development of a bi-objective model in which maximizing the profit and minimizing the cannibalization are targeted simultaneously. An interesting study is Drezner and Drezner (2002) in which no facility location is considered. Instead, the authors make use of the past data on the preference of the customers including the sales figures and demographical characteristics of the market area and determine the attractiveness of existing facilities using the gravity-base rule. As an indicator of the attractiveness the total sales of facilities and the buying power of customers are used, where the market area is divided into communities and the buying power of a community is determined by the cumulative effective income of all customers residing in that community.

2.1.2. Reactive Competition

In the previous section we review the CFL studies with non-reactive competition where the competitor(s) in the market is assumed not to react to the opening of new facilities. The categorization of such studies is realized with respect to the customer choice rules for patronizing facilities as deterministic utility models and random utility models. Such a categorization is also possible for CFL problems with reactive competition. However, the nature of competition is the primary factor in determining the class of the CFL problem. The two main classes are simultaneous-entry CFL problems and sequential-entry CFL problems. The competing firms simultaneously decide on their facility locations in simultaneous-entry CFL problems, whereas there exists a priority among the competing firms with regard to the timing of the actions in sequential-entry CFL problems. Some of this second class of CFL problems can also be considered as a Stackelberg type of game between two or more competing firms (von Stackelberg, 1934). Thus, all competitors in a market can be considered as players of a game, where players act either simultaneously or sequentially to optimize their individual objective functions. We briefly review the literature for the former class of problems first, and then focus on the papers that study Stackelberg-type of CFL problems.

<u>2.1.2.1. Simultaneous-Entry Competitive Facility Location Problems.</u> Most of the studies in this class consider the competition of the firms for a single homogeneous product as a two-stage game, where in the first stage both firms simultaneously decide where to locate the facilities. As soon as these decisions are made, they become known to both firms and they continue with the second stage of the game by simultaneously deciding either on the production quantities to supply the markets or on their products' prices. Markets are usually located on the vertices of a network and firms aim at maximizing their profits. Based on some conditions, the uniqueness

and existence properties of a Nash equilibrium for quantities or prices are shown. By means of additional conditions, the equilibrium locations are proved to be the vertices of the network. Among the papers including this type of analysis, we can mention Labbé and Hakimi (1991), Lederer and Hurter (1986), Lederer (1986), and Lederer and Thisse (1990). Sarkar *et al.* (1997) extend the work of Labbé and Hakimi (1991) for multiple firms and nonlinear price functions. Wendell and McKelvey (1981) consider only the locational game for two competing firms on a graph, and seek for a locational equilibrium such that a firm can capture at least 50% of the customers regardless where its competitor is located. By making use of the voting theory, necessary and sufficient conditions for locational equilibrium are developed. Rhim *et al.* (2003) extend the spatial competition on a discrete space to an oligopolistic three-stage game, where in the first stage facility locations, in the second stage facility capacities, and in the third stage quantities to be produced are decided. Peréz *et al.* (2004) investigate only the second stage of the game, where the competition between multiple firms takes place only in terms of prices.

<u>2.1.2.2. Sequential-Entry Competitive Facility Location Problems.</u> In sequential-entry CFL problems the competitor having existing facilities react to the new firm subsequent to its market entry. This situation leads to the so-called two-level or bilevel optimization problem in which there are two independent players called leader and follower. These players act in a sequential manner with the aim of optimizing their own objective functions, which are almost always in conflict with each other as pointed out by Moore and Bard (1990). In this setting, the leader first makes a decision (or selects a strategy) to optimize its objective function with the foresight or anticipation that given this decision the follower will optimize its own objective function. A bilevel programming (BP) formulation in optimization corresponds to a Stackelberg game in the context of game theory (Bard, 1998).

When we review the literature by focusing on the game-theoretic formulations within the context of CFL problems, we come across to the following ten studies. Some of them develop BP models, while others solve Stackelberg equilibrium problems. The book by Miller *et al.* (1996) focuses on the equilibrium facility location modeling. The developed mathematical models therein consider a firm which has to simultaneously decide on the production and distribution of a single, homogeneous product such that there exists an equilibrium in the market represented as a network. The entering firm, usually called a Stackelberg or leader firm, anticipates the reaction of the follower firms that have existing facilities in the market. These followers are assumed to be Cournot firms trying to achieve a Nash equilibrium by making changes in their production and distribution levels as a reaction to the leader firm. However, these firms operate under the Cournot assumption that the others do not change their production and distribution levels. The novelty here is that the Cournot-Nash equilibrium of the followers is represented as a variational inequality formulation. In order to compute the Cournot-Nash and Stackelberg-Cournot-Nash equilibria, many heuristic algorithms employing sensitivity analysis are suggested.

Fischer (2002) considers a discrete CFL with two competitors. Each competitor sells the same product to customers which are aggregated at discrete points in space called markets. One of the competitors becomes the leader and the other takes the role of the follower. Both of the decision makers want to determine both the locations of a fixed number of new facilities to be established from a set of potential sites and the price of the product at each market. It is assumed that the product can be sold at different prices at different markets (i.e., discriminatory pricing) where the price at a market depends on the distance from the facility serving the market. Customers prefer to make the purchase from the competitor offering the lowest price. Fischer (2002) formulates two bilevel models: a mixed-integer nonlinear bilevel model in which both players fix their locations and prices once and for all, and a linear bilevel model with binary variables where price adjustment are possible. A heuristic solution procedure is developed to solve the linear bilevel model, but no computational results are provided.

Bhadury *et al.* (2003) suggest a centroid model in the continuous space, where the follower locates extra facilities as a reaction to the leader's action. Drezner (1982) introduces two problems. In the first one, a new facility is located so as to attract most of the buying power from the demand points when there is already an existing facility in the market, whereas the second problem involves again the location of a new facility with the same goal, but this time by taking into account the possibility that the competitor opens a facility in the future. Plastria and Vanhaverbeke (2008) consider the maximal covering model and incorporate into this model the information that a competitor will enter the market with a new facility in the future. The objective of the leader is to locate facilities under a budget constraint in order to maximize the market share after the competitor's entry. The authors formulate three models corresponding to three strategies: worst-case analysis (maximin strategy), the minimum regret strategy, and the Stackelberg strategy which corresponds to taking into account the objective function of the competitor. Serra and ReVelle (1993) develop a model where both the leader and follower locate an equal number of facilities which are visited by customers only if they are the closest ones. In this model, the objective of the leader is to minimize the market share of the follower, and two heuristic algorithms are proposed for its solution.

Pérez and Pelegrín (2003) are concerned with a Stackelberg equilibrium problem and aim at finding all Stackelberg equilibria for a problem where both the leader and follower locate a single facility at any point in the tree, i.e., nodes as well as points on the edges of the tree are candidate facility sites. Customers visit the facility with the smallest total cost that is comprised of the product's price and the transportation cost. In case there is a tie, the closest facility is visited by the customers. If two facilities are equidistant, then the demand is split between two facilities according to a fraction. Pérez and Pelegrín (2003) develop the entire set of Stackelberg solutions to this competition model. Sáiz *et al.* (2009) formulate a nonlinear BP model in continuous space in which both the leader and the follower locate one facility where their market share is maximized. They make use of the gravity-based model, but the attractivenesses of facilities are known in advance. For the solution of their problem they develop a branch-and-bound method that guarantees the global optimum within a specified accuracy. Drezner and Drezner (1998) develop a similar model to that of Sáiz *et al.* (2009) and propose three heuristic methods for its solution.

Serra and ReVelle (1999) suggest a sequential game for a retail firm, which enters into a spatial market with one competitor that has several outlets and produces a single product. The new entrant leader firm tries to maximize its profit by finding the outlet locations and the mill prices and anticipates the reaction of the follower that can change the price of its product. The model considered here is an extension of the famous maximum capture problem where the customers buy from the firm for which the price and the transportation cost is the lowest. The ties are broken in favor of the closest firm. If both firms are the closest, then all of the demand is captured by the existing facility. For the solution of their problem, the authors suggest a tabu search heuristic algorithm.

Steiner (2010) develops a Stackelberg-Nash leader-follower model, where an entrant firm introduces a new brand in a competitive environment with multiple competitors. The leader firm tries to find the optimal product design by anticipating the possible reactions of the competitors. These reactions include price and design changes. However, it is assumed that the followers have Nash or Cournot reaction functions so that they ignore possible reactions of their competitors.

2.2. Facility Location with Customer Preferences

Since we first propose three different CFL problems, we review the literature of the competitive facility location until this section. However, the last proposed model differs from the CFL models in the sense that the customer preferences are taken into account. Therefore, we also review the literature of facility location problems that incorporate customer preferences. The first study belongs to Boots and South (1997). They model the retail trade areas using Voronoi diagrams. These diagrams are geometric procedures which produce theoretic trade areas using the assumptions about customer behavior and store attributes. In their work, Boots and South propose two Voronoi models which use the same assumption where the customer visits from the k ($k = 2, 3, ...$) nearest most attractive facilities. If the customer is indifferent between these k facilities, the trade areas are modeled as order k, multiplicatively weighted Voronoi diagrams. Furthermore, if the customer shows a preference for nearer facilities , then these models are ordered, order k, multiplicatively weighted Voronoi diagrams. In contrast to the most Voronoi diagrams, these diagrams generate overlapping trade areas. In general, multiplicatively weighted Voronoi diagrams consider both locational and non-locational features of the facilities and like in the case of MCI and multinomial logit models the utility of some facility for a customer is found with respect to a functional relationship of the facility attractiveness and the distance between the customer and the facility. In this paper, on the contrary of these general multiplicatively weighted Voronoi diagrams the customer can visit more than one facility according these two proposed models. This paper also shows the customer preferences in the sense that it analyses the cases where the customers prefer nearer facilities or the customers are indifferent between the facilities. However, the goal of the models is neither the location of new facilities nor determining its types, qualities or attractiveness levels. The models try to generate only trade areas. They can only be applied when the type or attractiveness of the facilities are known. These areas are used then to estimate the sales volume of the facilities. In order to produce these areas some data are made use of like the size of the stores or the number of hours during which the stores are open. These are actually the attributes of a store which make up its attractiveness level or quality.

Serra and Colomé (1999) analyze the influence of the distance and the transportation costs on the optimal locations of the facilities for the traditional facility location models. They consider the famous MAXCAP model to define key parameters which show different ways of the distance usage in the customer behavior theory. In all the models considered in the paper, a new entrant firm wishes to locate a certain number of new outlets in a market with existing outlets in the discrete space as to maximize its profit. The considered spatial market is a connected graph whose vertices represent markets and the potential locations for outlets are prespecified. Each key parameter represents the proportion captured by an outlet from a demand node. First, the MAXCAP model is introduced, where the customer visits the outlet of the nearest chain. Then an MCI model is presented. After that, a special case of the MCI model is given where the customer choses the nearer outlet instead of the chain. Finally, a model with partial binary preferences is considered where the proportion of times that a customer chooses an outlet is inversely proportional to a function of the distance. They compare the models by analyzing the deviation in demand and optimal locations by using other models instead of the true models. For the solution of the models, a metaheuristic based on GRASP and a tabu search heuristic are suggested. All the models considered are associated with the competitive facility location. As a result, each facility has an attractiveness level defined by its attributes. However, the only concern in this paper is to analyze the influence of the distance between the customers and facilities so that all facilities are assumed to be similar to each other where their attractiveness levels are taken to be equal to one for simplicity. Thus, the probability that a customer patronizes a facility is dependent on the distance only. However, in our model there is a distance term called "the maximum distance that a customer is willing to go to a facility" which is a realistic assumption for today's market. This distance depends on both the customer and the type of the facility. When a facility is located outside of the region of influence, it is not visited by that customer. Furthermore, the probability that a customer patronizes a facility is associated with the type of the facility. In competitive facility location models, this probability increases when the quality of the facility increases which may not always be the case in reality. These probabilities are actually real probabilities derived from a long time observation of the visiting frequencies of customers.

Berman and Krass (1998) propose a combined CFL model. The authors categorize the way of capturing of the demand by the facilities into two groups: dedicated trips and flow intercepting trips. In CFL literature dedicated trips are considered within the spatial interaction models whereas the flow intercepting trips withing the flow interception models. In this paper, both models are combined where competing facilities can capture the demand from both types of customers. Especially, such a situation is possible for facilities like gas stations and convenience stores. In order to combine these two models, all customers are considered as flow-by customers. In general, the customer can deviate from his/her path when the facility is within a specified distance. For the dedicated trips a dummy path is introduced in order to define such customers as flow-by customers. As a consequence, a nonlinear integer model is presented. A heuristic method and a branch-and-bound method are proposed for the solution of the model. The same criticism about the visiting probabilities concerned in competitive facility location models is valid for this paper, too. In this paper, again each facility has an attractiveness level and using Huff's gravity based-rule, the probability that a customer patronizes a facility increases with the increased attractiveness level which is not the case for our proposed model. Furthermore, our probabilities reflect the customer preferences in a realistic way. Besides, the model is more appropriate for facilities like gas stations, convenience stores etc. as mentioned above. Other than that, our model is more appropriate for facilities like shopping malls, super markets, stores for furniture and accessories etc.

In their paper, Korpela and Lehmusvaara (1999) consider a warehouse network evaluation and design in the context of supply chain management. The aim is the evaluation and selection of the alternative warehouse operators. For this purpose they need a customer driven and holistic approach. To this end, they combine the analytic hierarchy process (AHP) and mixed-integer linear programming (MILP). The AHP based procedure is used to determine customer-specific priorities for each alternative warehouse operator. These priorities are then used in the MILP as parameters. The objective of the model is to maximize the overall system performance under certain constraints and the model is based on both qualitative and quantitative criteria. Like this paper, Korpela *et al.* (2001) also try to find a framework to find the service elements of a supply chain and to include the firm's own strategy by combining the AHP and MIP. When we analyze the model proposed by Korpela and Lehmusvaara (1999), we see that it is different than our model in the sense that it is a capacitated facility location model where the facilities serve the customers by transporting the needed goods to them. In both papers, there are priorities reflecting the customer specific requirements and preferences instead of the probabilities. These priorities are determined using the AHP procedure. This procedure depends on some verbal or numerical views and judgments of persons. Then all alternatives are compared pairwise based on some weights and scores. This procedure has therefore some subjective features. However, the method we apply is not based on subjective features. Instead it relies on the real facts as mentioned above, it makes use of the real visiting frequencies of the customers so that we do not need some interviews or judgments. It is enough to have the attributes of the customer like the annual income, savings and their visiting frequencies. Also, the method we propose for determining the visiting probabilities can make statistical inference without time consuming and expensive data collecting procedure. In the end, the models proposed by Korpela and Lehmusvaara (1999) and Korpela *et al.* (2001) are traditional capacitated facility location problems in which only the priorities are included as weights to customer demands into the objective functions.

By reason of varying probabilities of customers for visiting different facility types, our proposed model falls also into the category of Probabilistic Facility Location (PFL) problems. Therefore, we review the PFL literature as well. All the studies referred in this section are reviews about the PFL problems. The first study belongs to Melo *et al.* (2009) where they review the literature of the facility location problems in the context of the supply chain management, since the facility location decisions play a crucial role. Especially the integration of the location decisions with the other decisions of the supply chain is discussed. Besides, the performance measures and optimization techniques of the supply chain are reviewed and application examples are given. The literature review in this paper contains only the discrete space models. Since the future customer demands and costs are under uncertainty, the stochastic elements of the facility location models are discussed.

Synder (2006) also reviews the stochastic and robust optimization problems. The decision environment is categorized into three groups: certainty, risk, and uncertainty. In models under certainty all parameters are deterministic and known a priory. In risk models there are uncertain parameters like cost, demand, and travel times. However, the probability distribution they follow is known to the decision maker. The parameters of the uncertain models are also uncertain like risk models, but this time the probability distributions are not known to the decision maker. Models under risk are called stochastic programming, whereas the models under uncertainty are called robust optimization. Both models make use of random parameters. These parameters are defined by continuous or discrete scenarios. If the probability distribution is not known, continuous parameters are restricted to be in prespecified intervals. The objective of the stochastic programming models are usually the minimization of the expected cost or the maximization of the expected profit. The models are usually two stage models, where in the first stage locational decisions are taken and the second stage is realized when the uncertainty is resolved. The objective of the robust programming is in general minimax cost or minimax regret.

Daskin *et al.* (2003) emphasize also the critical role of the facility location in the supply chain design. First, the classical fixed charge problem, its extensions and the solution methods are introduced. Then the integrated location/routing models are considered. These models are deterministic in nature and combine the facility location, customer allocation, and vehicle routing. Usually they are MILP problems. Then integrated location/inventory models are presented, which are usually nonlinear models. Finally, models under uncertainty are considered where the cost and the demand can vary in time. Furthermore, stochastic programming and robust optimization are included in the review. Louveaux (1993) reviews the stochastic location on network which includes stochastic elements of Hakimi's theorem and stochastic queuing location models, discrete stochastic location and covering models. The random parameters in the reviewed literature the random parameters are the location of the customers, travel time or travel costs, and queuing effects.

An important characteristic of the last proposed facility location problem is that customers are willing to travel to a facility of a certain type if it is located within a maximum (threshold) distance which is determined by the customer. This characteristic is one of the ways with which the customers show their preferences. The idea of a maximum distance is similar to the concept of "coverage radius" which is employed in set covering and maximal covering facility location problems. In these problems, a demand point is said to be covered if the closest facility to the customer is established within a distance less than or equal to a *coverage radius* r. However, this radius r is fixed once and for all and is determined without the control of the decision-maker.

There are three studies relaxing the constraint on the coverage radius that deserve attention. The first one belongs to Berman *et al.* (2009) where the covering radius is controlled by the decision maker and determined by the size of the facility. Thus, establishing a facility with a greater coverage radius increases the costs regarding the location of new facilities which is composed of a fixed cost and a cost function that is non-decreasing in the coverage radius. Furthermore, the coverage radius is determined by the furthest assigned customer. They formulate a continuous and a discrete version of a facility location problem the aim of which is to find the optimal minimum number, location, and coverage radius of new facilities to cover all the demand points. For the continuous version they develop heuristic approaches. Berman *et al.* (2003) consider a maximal covering problem where two coverage radii l_i and u_i with $l_i < u_i$ are fixed for each demand point i. In contrast to full coverage concept of classical set covering and maximal covering problems where a demand point is fully covered if it is within the coverage radius, they assume gradual covering. Therefore, a demand point i is fully covered if its distance to the closest facility less than l_i and it is partially covered if this distance between l_i and u_i. If the demand point is located further than u_i, it is not covered at all. To this end, two integer programming problems in discrete space are formulated which make use of different coverage decay functions and can find the optimal locations of new facilities for convex coverage decay functions. In order to solve the models, they apply greedy heuristics and branch-and-bound techniques. The study of Karasakal and Karasakal (2004) is similar to the study of Berman *et al.* (2003) where a maximal covering location problem with partial coverage is considered. They solve the problem with Lagrangean relaxation and compare their model with the classic maximal covering location problem.

3. A NON-REACTIVE DISCRETE COMPETITIVE FACILITY LOCATION PROBLEM WITH VARIABLE ATTRACTIVENESS

In this chapter[1] , we address a CFL problem in the discrete space with the objective of maximizing the profit where attractiveness of new facilities is a continuous decision variable as opposed to the case of a discrete design scenario. To the best of our knowledge, the CFL problem with a discrete set of candidate facility sites and continuous attractiveness is not addressed before. To estimate the market share we employ gravity-based rule according to which the probability that a customer patronizes a facility is proportional to the facility attractiveness and inversely proportional to the distance between the facility and customer. We develop a mixed-integer nonlinear (MINLP) programming formulation, and try to solve it using three different methods: a Lagrangean heuristic, a branch-and-bound method based on Lagrangean relaxation, and a branch-and-bound method based on nonlinear programming relaxation. The last two methods are exact methods and they are capable of finding an optimal solution of the formulated model when a sufficient amount of time is allotted.

An important aspect of our model is that the reaction of the competitors is omitted. Namely, the competitors do not open new facilities or close existing ones or change the attractiveness of their facilities as a reaction to the market entering firm. This assumption is realistic in those situations where there exists a static competition between players. For example, consider the competition in a small district of a city which takes place between the existing convenience stores and a supermarket chain aiming at opening new stores. When a big supermarket chain (such as Migros, Real, and Carrefour in Turkey) opens a gigantic store, the existing convenience stores owned by independent entrepreneurs cannot react most of the time even though they know that a new supermarket in the area will reduce their profit considerably. The reason for the lack of the competitive reaction lies in the fact that supermarkets with large sales volumes have the option of offering low prices to customers for a variety of goods compared to convenience stores whose sales volumes are much lower. As a result, customers begin to make their purchases in the supermarkets rather than in the convenience

[1]The paper Küçükaydın *et al.* (2010a) based on this chapter is published in the Journal of the Operational Research Society.

stores, which ultimately leads to the closure of some of them. Certainly, there are cases where they succeed in their efforts to survive.

3.1. Model Formulation

The aim of the proposed optimization problem is to determine the market share and hence the optimal location and attractiveness of the new facilities of a market entrant firm when there are r existing facilities that belong to a competitor or several competitors. The objective of the firm is to maximize the profit which is computed as the revenue from the customers less the fixed costs of opening new facilities plus the variable costs of setting the attractiveness levels of new facilities.

We assume that customers are aggregated at n (demand) points and the number of candidate facility sites is m. First, we define the parameters and decision variables. The points are indexed by $j = 1, 2, ..., n$, the candidate facility sites by $i = 1, 2, ..., m$, and the existing facilities by $k = 1, 2, ..., r$.

Parameters:

h_j : annual buying power at point j,

c_i : unit attractiveness cost at site i,

f_i : annualized fixed cost of opening and operating a facility at site i,

d_{ij} : Euclidean distance between site i and point j,

o_j : total utility of existing facilities depending on their attractiveness and distance from point j

u_i : maximum attractiveness level of a facility to be opened at site i

$\underline{A}_k$: attractiveness of existing facility k

Decision variables:

Q_i : attractiveness of the facility opened at site i,

X_i : binary variable which is equal to one if a facility is opened at site i, and zero otherwise.

When a facility is opened at site i with attractiveness Q_i, the utility of this facility for customers at point j is given by Q_i/d_{ij}^2 using the gravity-based rule. The total utility of existing facilities for customers at point j is given by parameter $o_j = \sum_{k=1}^{r} \underline{A}_k/d_{kj}^2$, where d_{kj} is the Euclidean distance between demand point j and existing facility k. As a result, the probability P_{ij} that customers at point j patronize a new facility at site i is expressed as

$$P_{ij} = \frac{Q_i/d_{ij}^2}{\sum\limits_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + o_j} \,. \tag{3.1}$$

Thus, the market share M_j that all new facilities capture from customers at point j can be calculated as

$$M_j = \frac{\sum\limits_{i=1}^{m} Q_i/d_{ij}^2}{\sum\limits_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + o_j} \,. \tag{3.2}$$

Note that $\sum\limits_{j=1}^{n} P_{ij}$ can be considered as the market share corresponding to facility at site i and its revenue can be computed by the summation $\sum\limits_{j=1}^{n} h_j P_{ij}$. Hence, the total revenue captured by the new facilities is given as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h_j P_{ij} = \sum_{j=1}^{n} h_j \sum_{i=1}^{m} P_{ij} = \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + o_j} \,. \tag{3.3}$$

Now we can formulate our model as the following mixed-integer nonlinear programming problem:

$$\text{P:} \quad \max \quad z = \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m} \frac{Q_i}{d_{ij}^2}}{\sum\limits_{i=1}^{m} \frac{Q_i}{d_{ij}^2} + o_j} - \sum_{i=1}^{m} c_i Q_i - \sum_{i=1}^{m} f_i X_i \tag{3.4}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{3.5}$$

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \tag{3.6}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{3.7}$$

The objective function (3.4) consists of three terms. The revenue collected by the new facilities is represented by the first term, while the cost associated with opening and operating them is expressed by the last two terms. The first cost component is the annualized fixed cost of opening and operating the facilities, and the second one is the annualized variable cost of opening a facility at a certain attractiveness level Q_i. Constraints (3.5) along with constraints (3.7) ensure that if no facility is opened at site i, then the attractiveness Q_i of the facility is zero. On the other hand, when a facility is opened at site i, then its attractiveness cannot exceed an upper bound u_i. Constraints (3.6) and (3.7) are, respectively, the binary and nonnegativity restrictions on the location variables X_i and the attractiveness variables Q_i. We note that the number of facilities to be located is not fixed; its value is to be determined by the solution of the model.

Making the attractiveness a decision variable liberates the decision maker from a great computational burden, especially if there are plenty of potential facility sites. Otherwise, the decision maker has to determine the attractiveness for each potential facility site one by one in order to make the new facilities more desirable against the competing ones. This situation is unavoidable, if we consider retail facilities, grocery stores, or restaurants etc. Another good feature of our model is that Q_i's are continuous decision variable. If they are discrete, then we have to pre-determine possible design scenarios as in the study of Aboolian *et al.* (2007b).

Before proceeding to the next section where solution methods are introduced, we show an important property of the objective function (3.4) of the proposed model. Namely, the objective function (3.4) is concave in the attractiveness Q_i for $Q_i \geq 0$.

Proposition 3.1. $\sum\limits_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+o_j}$ *is concave in* $\mathbf{Q} = (Q_1, Q_2, ..., Q_m)^{\mathrm{T}}$ *for* $\mathbf{Q} \geq \mathbf{0}$.

Proof. Since the sum of concave functions is a concave function, it suffices to show that $h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+o_j}$ is concave for $\mathbf{Q} \geq \mathbf{0}$ for every $j = 1, 2, ..., n$. Let us define $g_j(\mathbf{Q}) = h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+o_j}$. We now show that $g_j(\mathbf{Q})$ is concave for $\mathbf{Q} \geq \mathbf{0}$. The first and second order derivatives of $g_j(\mathbf{Q})$ are given as follows:

$$\frac{\partial g_j(\mathbf{Q})}{\partial Q_k} = h_j \frac{\left(1/d_{kj}^2\right) o_j}{\left[\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + o_j\right]^2}$$

and

$$\frac{\partial g_j(\mathbf{Q})}{\partial Q_l \partial Q_k} = -2h_j \frac{\left(1/d_{kj}^2\right)\left(1/d_{lj}^2\right) o_j}{\left[\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + o_j\right]^3}.$$

Then the Hessian matrix of $g_j(\mathbf{Q})$, denoted by $H_j(\mathbf{Q})$, becomes

$$H_j(\mathbf{Q}) = -r_j \begin{pmatrix} \frac{1}{d_{1j}^4} & \frac{1}{d_{1j}^2 d_{2j}^2} & \cdots & \frac{1}{d_{1j}^2 d_{mj}^2} \\ \frac{1}{d_{1j}^2 d_{2j}^2} & \frac{1}{d_{2j}^4} & \cdots & \frac{1}{d_{2j}^2 d_{mj}^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{d_{1j}^2 d_{mj}^2} & \frac{1}{d_{2j}^2 d_{mj}^2} & \cdots & \frac{1}{d_{mj}^4} \end{pmatrix} \tag{3.8}$$

where $r_j = 2h_j \frac{o_j}{\left[\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+o_j\right]^3}$. We remark that $r_j \geq 0$ for $\mathbf{Q} \geq \mathbf{0}$ since $h_j \geq 0$ and $o_j \geq 0$. Also note that $g_j(\mathbf{Q})$ is concave if and only if $H_j(\mathbf{Q})$ is negative semidefinite for $\mathbf{Q} \geq \mathbf{0}$. To show the latter, we consider $\mathbf{V}^T H_j(\mathbf{Q})\mathbf{V}$ for $\mathbf{V} \in \mathbb{R}^m$ which is expressed as

$$\mathbf{V}^T H_j(\mathbf{Q})\mathbf{V} = -\left(V_1, V_2, \ldots, V_m\right) r_j \begin{pmatrix} \frac{1}{d_{1j}^4} & \frac{1}{d_{1j}^2 d_{2j}^2} & \cdots & \frac{1}{d_{1j}^2 d_{mj}^2} \\ \frac{1}{d_{1j}^2 d_{2j}^2} & \frac{1}{d_{2j}^4} & \cdots & \frac{1}{d_{2j}^2 d_{mj}^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{d_{1j}^2 d_{mj}^2} & \frac{1}{d_{2j}^2 d_{mj}^2} & \cdots & \frac{1}{d_{mj}^4} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix} \tag{3.9}$$

$$= -\left(V_1, V_2, \ldots, V_m\right) r_j \left(\frac{V_1}{d_{1j}^2} + \frac{V_2}{d_{2j}^2} + \ldots + \frac{V_m}{d_{mj}^2}\right) \begin{pmatrix} 1/d_{1j}^2 \\ 1/d_{2j}^2 \\ \vdots \\ 1/d_{mj}^2 \end{pmatrix} \tag{3.10}$$

$$= -r_j \left(\frac{V_1}{d_{1j}^2} + \frac{V_2}{d_{2j}^2} + \ldots + \frac{V_m}{d_{mj}^2}\right)^2 . \tag{3.11}$$

$r_j \geq 0$ and $\left(\frac{V_1}{d_{1j}^2} + \frac{V_2}{d_{2j}^2} + \ldots + \frac{V_m}{d_{mj}^2}\right)^2 \geq 0$ imply that $\mathbf{V}^T H_j(\mathbf{Q})\mathbf{V} \leq 0$ for $\mathbf{Q} \geq \mathbf{0}$. This means that $H_j(\mathbf{Q})$ is negative semidefinite, which proves the concavity of $g_j(\mathbf{Q})$ for every $j = 1, 2, ..., n$. Hence $\sum\limits_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+o_j}$ is concave in $\mathbf{Q}$ for $\mathbf{Q} \geq \mathbf{0}$. □

Proposition 3.2. $\sum\limits_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+o_j} - \sum\limits_{i=1}^{m} f_i X_i - \sum\limits_{i=1}^{m} c_i Q_i$ *is concave in* $\mathbf{Q}$ *for* $\mathbf{Q} \geq \mathbf{0}$.

Proof. The first term is concave in $\mathbf{Q}$ as a consequence of Proposition 1, the second term is a constant, and the last term is a linear function of $\mathbf{Q}$. The result follows since the sum of concave functions is also concave. □

Regardless of the solution method that is employed, an important issue to be taken into consideration in our modeling framework is reverse fitting. This is related to determining for each facility the best values of the underlying attributes that collectively form the attractiveness of the facility. Using the function that gives the relationship between the attribute values and the resulting attractiveness level (e.g., the MCI model of Nakanishi and Cooper (1974) where the product of the attributes raised by a power), one may generate the values of the attributes starting with the best attractiveness level obtained for the facility. The main difficulty in this process is the degrees of freedom associated with the individual attribute values, i.e., different combinations of attribute values may result in the same attractiveness level. Since there is one equation and as many unknown values as the number of attributes, all attribute values but one can be fixed as desired and the remaining one can be determined from the equation. If the number of attributes is relatively low such as two or three, then even a trial-and-error procedure may help in finding suitable values for the attributes to approximately yield the optimal attractiveness. In some situations, however, it might be the case that the recommended attractiveness level for a facility cannot be achieved due to its unreasonably high value. In this case, a possible remedy would be to adjust the maximum attractiveness level for that facility, and resolve the model.

3.2. Solution Procedures

We propose three methods for the solution of the mixed-integer nonlinear model P. The first one is a Lagrangean heuristic; the second and third ones are branch-and-bound (BB) methods. The difference between the two BB methods is that at each node of the BB tree a Lagrangean relaxation (LR) of the original problem P is solved in the former, while a nonlinear programming relaxation is solved in the latter. All the proposed methods exploit the concavity property of the objective function (3.4) that is given in the previous section by Proposition 2.

3.2.1. A Lagrangean Heuristic

The determination of good lower and upper bounds on the optimal objective function value of a mixed-integer programming problem is a crucial step to reach good solutions. An efficient solution procedure to obtain good upper (lower) bounds in maximization (minimization) problems is Lagrangean heuristic (LH) as shown by Guta (2003) and Beasley (1993b). Furthermore, LHs have successfully been applied to various facility location problems (Beasley, 1993a). The LH first relaxes some complicating constraints of an integer or mixed-integer problem into the objective function by introducing Lagrangean multipliers. Then it tries to find lower and upper bounds on the objective function value by solving the resulting relaxed problem. In this thesis, we also devise a LH to solve the first proposed CFL problem.

The proposed CFL model P is not separable because of the complicating constraints $Q_i \leq u_i X_i$ for $i = 1, 2, ..., m$ that include both binary and continuous decision variables. To this end, we relax these constraints and put them into the objective function (3.4) after multiplying with nonnegative Lagrange multipliers $\lambda_i,\ i = 1, \ldots, m$. The Lagrangean subproblem then becomes:

$$\mathrm{P}\left(\boldsymbol{\lambda}\right): \quad \mathrm{z}\left(\boldsymbol{\lambda}\right) = \max \sum_{j=1}^{n} h_j \frac{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2}}{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2} + o_j} - \sum_{i=1}^{m} (c_i + \lambda_i) Q_i - \sum_{i=1}^{m} (f_i - \lambda_i u_i) X_i \tag{3.12}$$

s.t.

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \tag{3.13}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{3.14}$$

$z\left(\boldsymbol{\lambda}\right)$ provides an upper bound on the optimal objective value z^* of the original model P for any value of multiplier vector $\boldsymbol{\lambda} \geq \mathbf{0}$. $\mathrm{P}(\boldsymbol{\lambda})$ can be solved easily since it can be decomposed into two subproblems $\mathrm{P}_1\left(\boldsymbol{\lambda}\right)$ and $\mathrm{P}_2\left(\boldsymbol{\lambda}\right)$ with optimal objective values $z_1\left(\boldsymbol{\lambda}\right)$ and $z_2\left(\boldsymbol{\lambda}\right)$ as follows:

$$\mathrm{P}_1\left(\boldsymbol{\lambda}\right): \quad \mathrm{z}_1\left(\boldsymbol{\lambda}\right) = \max \ \sum_{j=1}^{n} h_j \frac{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2}}{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2} + o_j} - \sum_{i=1}^{m} (c_i + \lambda_i) Q_i \tag{3.15}$$

s.t.

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{3.16}$$

and

$$\mathrm{P}_2\left(\boldsymbol{\lambda}\right): \quad \mathrm{z}_2\left(\boldsymbol{\lambda}\right) = \max \ -\sum_{i=1}^{m} (f_i - \lambda_i u_i) X_i \tag{3.17}$$

s.t.

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \tag{3.18}$$

The sum of $z_1(\boldsymbol{\lambda})$ and $z_2(\boldsymbol{\lambda})$ provides an upper bound on the objective function value z^* of the original problem P. Problems $\mathrm{P}_1(\boldsymbol{\lambda})$ and $\mathrm{P}_2(\boldsymbol{\lambda})$ can be solved, when the Lagrangean multipliers λ_i's are known. Thus, in order to find the best (smallest) upper bound on z^* the so-called Lagrangean dual problem

$$\mathrm{D}: \qquad \min_{\lambda \geq \mathbf{0}} \left[z_1\left(\boldsymbol{\lambda}\right) + z_2\left(\boldsymbol{\lambda}\right) \right], \tag{3.19}$$

is formulated and solved using the iterative "subgradient optimization" procedure (Guta 2003, Beasley 1993b). At each iteration t of this procedure, a step size $\theta^{(t)}$ is computed and the Lagrangean multipliers are updated using the subgradients according to the formula

$$\lambda_i^{(t+1)} = \max \left\{ 0, \lambda_i^{(t)} + \theta^{(t)} (u_i X_i - Q_i) \right\} \tag{3.20}$$

The computation of $\theta^{(t)}$ requires a lower bound on the optimal objective value z^* of P, which is provided by a feasible solution to P. It can be derived from the solution of the

Lagrangean subproblem P($\boldsymbol{\lambda}$) as will be explained subsequently. We use the step size formula that is commonly used in the literature (Held *et al.*, 1974), i.e., $\theta^{(t)} = \pi(UB^{(t)} - LB_{best})/\sum_{i=1}^{m} \|u_i X_i - Q_i\|^2$ where π is the step size parameter, $UB^{(t)}$ is the upper bound at iteration t, and LB_{best} is the best lower bound obtained until iteration t. The initialization of Lagrangean multipliers are not important in subgradient optimization. We can stop the algorithm when we find a feasible solution such that $\sum\limits_{i=1}^{m} \lambda_i \left(u_i X_i - Q_i\right) = 0$. The reason for that is obvious: if the solution is feasible, then $Q_i \leq u_i X_i$ is satisfied for every $j = 1, 2, ..., n$. Then the value of the objective function for the original problem P is calculated as $z = \sum\limits_{j=1}^{n} h_j \frac{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2}}{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2} + o_j} - \sum\limits_{i=1}^{m} c_i Q_i - \sum\limits_{i=1}^{m} f_i X_i$. Furthermore, the objective function value of the Lagrangean dual problem is $z_1\left(\boldsymbol{\lambda}\right) + z_2\left(\boldsymbol{\lambda}\right) = \sum\limits_{j=1}^{n} h_j \frac{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2}}{\sum_{i=1}^{m} \frac{Q_i}{d_{ij}^2} + o_j} - \sum\limits_{i=1}^{m} (c_i + \lambda_i) Q_i - \sum\limits_{i=1}^{m} (f_i - \lambda_i u_i) X_i$. So, if $\sum\limits_{i=1}^{m} \lambda_i \left(u_i X_i - Q_i\right) = 0$, both objective function values are equal to each other and the lower bound is equal to the upper bound. However, there is usually a duality gap so that the lower bound and the upper bound are rarely equal to each other. Therefore, as suggested by Beasley (1993b), the initial value of π is set to two and halved every 20 iterations without an improvement in the best upper bound. When π becomes less than 0.005, the algorithm is terminated, and the best (largest) lower bound generated throughout the iterations constitutes the solution of the LH.

Now we explain the solution procedure of subproblems $\text{P}_1\left(\boldsymbol{\lambda}\right)$ and $\text{P}_2\left(\boldsymbol{\lambda}\right)$. The solution of $\text{P}_2\left(\boldsymbol{\lambda}\right)$ can easily be obtained by inspection. Namely, $X_i = 1$ if $(f_i - \lambda_i u_i) < 0$, $X_i = 0$ otherwise. To solve $\text{P}_1\left(\boldsymbol{\lambda}\right)$ we make use of the concavity of its objective function $\pi(\mathbf{Q}) = \sum\limits_{j=1}^{n} h_j \frac{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + o_j} - \sum\limits_{i=1}^{m} (c_i + \lambda_i) Q_i$ in terms of the attractiveness variables $\mathbf{Q}$, which is a direct consequence of Proposition 2. We add redundant constraints of the form $Q_i \leq u_i, i = 1, 2, ..., m$ to increase the quality of the upper bound. To find the solution of this concave maximization problem, we use the following optimality conditions: $\mathbf{Q}^*$ is a global optimal solution of $\text{P}_1\left(\boldsymbol{\lambda}\right)$ if and only if

i) $\frac{\partial \pi(\mathbf{Q}^*)}{\partial Q_i} \leq 0$ when $Q_i^* = 0$,

ii) $\frac{\partial \pi(\mathbf{Q}^*)}{\partial Q_i} \geq 0$ when $Q_i^* = u_i$,

iii) $\frac{\partial \pi(\mathbf{Q}^*)}{\partial Q_i} = 0$ when $0 < Q_i^* < u_i$.

These optimality conditions are given in Bertsekas (1995) for the optimality of a minimization problem in convex programming. They give rise to a gradient ascent procedure to determine a global maximum of $\pi(\mathbf{Q})$. First, a small positive value is assigned to parameter ϵ that is used for the termination of the procedure. Then, initial values $\mathbf{Q}^{(0)}$ are assigned to variables $\mathbf{Q}$. After setting the iteration counter t to zero, a direction $\mathbf{d}^{(t)}$ and a step size $\alpha^{(t)} = \arg\max_{\alpha} \pi(\mathbf{Q}^{(t)} + \alpha \mathbf{d}^{(t)})$ is determined, and variables $\mathbf{Q}$ are updated as $\mathbf{Q}^{(t+1)} = \mathbf{Q}^{(t)} + \alpha^{(t)} \mathbf{d}^{(t)}$. The iteration counter is increased by one, and the procedure is repeated until the norm of the direction vector $\left\| \mathbf{d}^{(t)} \right\|$ is smaller than ϵ. In finding the optimal step size, we apply the golden section search (Press *et al.*, 1986) with the initial interval $[0, \alpha_{max}]$, where α_{max} is the maximum possible value for the step size α to maintain the feasibility of $\mathbf{Q}$ with respect to its lower and upper bounds. The gradient ascent algorithm we apply can be summarized as follows:

1. choose $\mathbf{Q}^{(0)}$, ϵ, $t \leftarrow 0$
2. determine the direction $\mathbf{d}^{(t)}$ to move
3. determine the step size $\alpha^{(t)} = \arg\max_{\alpha} \pi(\mathbf{Q}^{(t)} + \alpha \mathbf{d}^{(t)})$
4. $\mathbf{Q}^{(t+1)} = \mathbf{Q}^{(t)} + \alpha^{(t)} \mathbf{d}^{(t)}$
5. $t \leftarrow t + 1$
6. until $\left\| \mathbf{d}^{(t)} \right\| < \epsilon$

Figure 3.1. Gradient Ascent Algorithm

Since the decision variables $\mathbf{Q}$ have lower and upper bounds, the direction $\mathbf{e}^{(t)}$ is determined as follows:

i) $e_i^{(t)} = \frac{\partial \Pi(\mathbf{Q}^{(t)})}{\partial Q_i}$ when $0 < Q_i^{(t)} < u_i$ for $i = 1, 2, ..., m$

ii) $e_i^{(t)} = 0$ when $Q_i^{(t)} = 0$ and $\frac{\partial \Pi(\mathbf{Q}^{(t)})}{\partial Q_i} < 0$ for $i = 1, 2, ..., m$

iii) $e_i^{(t)} = \frac{\partial \Pi(\mathbf{Q}^{(t)})}{\partial Q_i}$ when $Q_i^{(t)} = 0$ and $\frac{\partial \Pi(\mathbf{Q}^{(t)})}{\partial Q_i} > 0$ for $i = 1, 2, ..., m$

iv) $e_i^{(t)} = 0$ when $Q_i^{(t)} = u_i$ and $\frac{\partial \Pi(\mathbf{Q}^{(t)})}{\partial Q_i} > 0$ for $i = 1, 2, ..., m$

v) $e_i^{(t)} = \frac{\partial \Pi(\mathbf{Q}^{(t)})}{\partial Q_i}$ when $Q_i^{(t)} = u_i$ and $\frac{\partial \Pi(\mathbf{Q}^{(t)})}{\partial Q_i} < 0$ for $i = 1, 2, ..., m$.

To generate a feasible solution for P we utilize the solution to the Lagrangean subproblem $\mathrm{P}_2(\boldsymbol{\lambda})$. Since this solution gives us the set of facilities to be opened (and closed), i.e., $X_i = 0$ and $X_i = 1$ for $i = 1, 2, ..., m$, we can fix them in the original problem P and solve the remaining problem in terms of $\mathbf{Q}$. Note that if $X_i^* = 0$, then $Q_i^* = 0$, and all the terms with a zero value for Q_i can be dropped from the problem. The latter can be expressed as

$$\max \ z = \sum_{j=1}^{n} h_j \frac{\sum_{i \in S} \left(Q_i / d_{ij}^2\right)}{\sum_{i \in S} \left(Q_i / d_{ij}^2\right) + o_j} - \sum_{i \in S} c_i Q_i \tag{3.21}$$

s.t.

$$Q_i \leq u_i \qquad\qquad i \in S \tag{3.22}$$

where $S = \{i : X_i = 1\}$. Since we already know that the original problem P is concave in $\mathbf{Q}$ when the binary variables $\mathbf{X}$ are fixed, it can be solved optimally by the application of the same gradient ascent procedure that was used for the solution of $\mathrm{P}_1(\boldsymbol{\lambda})$.

The subgradient optimization procedure we apply can be summarized as follows:

1. Let π be a user-defined parameter such that $0 \leq \pi \leq 2$. Initially, we choose $\pi = 2$. Determine the initial Lagrange multipliers $\lambda_i \geq 0$, $i = 1, 2, ...m$.
2. Solve the Lagrangean dual problem with the current set of multipliers to obtain an upper bound z_{UB}. Apply a Lagrangean heuristic to get a feasible solution and a lower bound z_{LB} on the optimal objective function value. If the solution of the Lagrangean dual problem is feasible and satisfies $\sum_{i=1}^{m} \lambda_i (u_i X_i - Q_i) = 0$, then this solution is optimal for the original problem P and STOP. Otherwise go to step 3.
3. Define the subgradient for each relaxed constraint as $u_i X_i - Q_i$, $i = 1, 2, ..., m$.
4. Calculate the step size $\theta^{(t)} = \pi (UB^{(t)} - LB_{best}) / \sum_{i=1}^{m} \|u_i X_i - Q_i\|^2$.
5. Update the Lagrange multipliers using $\lambda_i^{(t+1)} = \max \left\{0, \lambda_i^{(t)} + \theta^{(t)} (u_i X_i - Q_i)\right\}$ for all $i = 1, 2, ...m$.
6. Update π if there is no improvement in z_{UB} for 20 iterations. If $\pi < 0.005$, then STOP. Otherwise, go to step 2.

Figure 3.2. Subgradient Optimization Procedure

3.2.2. A Branch-and-Bound Algorithm using Lagrangean Relaxation

As a second solution procedure we develop a BB method using Lagrangean relaxation in order to find lower and upper bounds on z^*. We refer to this method as BB-LR in the sequel. In other words, at each node of the BB tree, subproblems are solved by LH to obtain upper bounds as well as lower bounds on the optimal objective value z^* of the original problem P.

In a BB algorithm edges impose constraints to the problem as shown by Nemhauser and Wolsey (1998). Thus, we branch each time on a binary variable x_i so that each branch imposes a constraint that fixes a certain binary variable. At any node k of the tree, some of the binary location variables X_i are fixed. Let $F_k^+ = \{i = 1, \ldots, m : X_i = 1\}$ be the set of the sites with a facility and $F_k^- = \{i = 1, \ldots, m : X_i = 0\}$ be the set of the sites without a facility. Also let $G_k = \{1, 2, ..., m\} \setminus \left(F_k^+ \cup F_k^-\right)$ be the set of the sites without a decision with regard to opening a facility. Note that when $X_i = 0$ for site i, then its corresponding Q_i must be equal to zero as well. Therefore, all location and attractiveness variables corresponding to the sites in F_k^- can be discarded from the model at node k. Furthermore, upper bound constraints for Q_i reduce to $Q_i \leq u_i$ for facility sites in F_k^+. As a result, the subproblem to be solved at node k of the BB tree can be formulated as follows:

$$\mathrm{P}_k: \quad \max \sum_{j=1}^{n} h_j \frac{\sum\limits_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2}}{\sum\limits_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2} + o_j} - \sum_{i \in G_k \cup F_k^+} c_i Q_i - \sum_{i \in G_k} f_i X_i - \sum_{i \in F_k^+} f_i \tag{3.23}$$

s.t.

$$Q_i \leq u_i X_i \qquad i \in G_k \tag{3.24}$$

$$Q_i \leq u_i \qquad i \in G_k \cup F_k^+ \tag{3.25}$$

$$Q_i \geq 0 \qquad i \in G_k \cup F_k^+ \tag{3.26}$$

$$X_i \in \{0, 1\} \qquad i \in G_k \tag{3.27}$$

Note that constraint (3.25) for $i \in G_k$ are in fact redundant for the formulation, but they will help to obtain a better upper bound for P_k. Constraint (3.24) for $i \in G_k$ can be dualized with nonnegative Lagrange multipliers λ_i, $i \in G_k$. Then the Lagrangean subproblem at node k becomes

$$\mathrm{P}_k(\boldsymbol{\lambda}): \quad \max \sum_{j=1}^{n} h_j \frac{\sum\limits_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2}}{\sum\limits_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2} + o_j} - \sum_{i \in G_k} (c_i + \lambda_i) Q_i - \sum_{i \in F_k^+} c_i Q_i$$
$$- \sum_{i \in G_k} (f_i - \lambda_i u_i) X_i - \sum_{i \in F_k^+} f_i \tag{3.28}$$

s.t.

$$0 \le Q_i \le u_i \qquad i \in F_k^+ \tag{3.29}$$

$$X_i \in \{0,1\} \qquad i \in G_k \tag{3.30}$$

As was done in the previous subsection, it is possible to decompose $\mathrm{P}_k(\boldsymbol{\lambda})$ into two problems as follows:

$$\mathrm{P}_{k1}(\boldsymbol{\lambda}): \quad \max \sum_{j=1}^{n} h_j \frac{\sum\limits_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2}}{\sum\limits_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2} + o_j} - \sum_{i \in G_k} (c_i + \lambda_i) Q_i - \sum_{i \in F_k^+} c_i Q_i \tag{3.31}$$

s.t.

$$0 \le Q_i \le u_i \qquad i \in G_k \cup F_k^+ \tag{3.32}$$

$$\mathrm{P}_{k2}(\boldsymbol{\lambda}): \quad \max \; - \sum_{i \in G_k} (f_i - \lambda_i u_i) X_i - \sum_{i \in F_k^+} f_i \tag{3.33}$$

s.t.

$$X_i \in \{0,1\} \qquad i \in G_k \tag{3.34}$$

The sum of the optimal objective values to $\mathrm{P}_{k1}(\boldsymbol{\lambda})$ and $\mathrm{P}_{k2}(\boldsymbol{\lambda})$ provides an upper bound on P_k, which is the subproblem to be solved at node k. The Lagrangean multipliers are updated using the subgradient optimization, at each iteration of which $\mathrm{P}_{k2}(\boldsymbol{\lambda})$ is solved by inspection to find the optimal values of the location variables X_i for $i \in G_k$ and $\mathrm{P}_{k1}(\boldsymbol{\lambda})$ is solved by the described gradient ascent procedure to obtain the optimal values of Q_i for $i \in G_k \cup F_k^+$ of this concave maximization problem. The lower bound, which is employed in updating the Lagrange multipliers, is found by generating a feasible solution to P_k. This is accomplished by making use of the solutions to the Lagrangean subproblems $\mathrm{P}_{k1}(\boldsymbol{\lambda})$ and $\mathrm{P}_{k2}(\boldsymbol{\lambda})$. We simply set $Q_i = 0$ corresponding to $X_i = 0$, and keep the values of Q_i corresponding to $X_i = 1$. Then, we evaluate the objective function of P_k given in (3.23). After generating a lower bound and an upper bound at each subgradient optimization iteration, the best upper bound UB_k as well as the best lower bound LB_k obtained throughout the iterations are stored for node k of the BB tree. It must be emphasized that at the leaf nodes of the tree we do not employ the subgradient optimization procedure since all binary variables $\mathbf{X}$ are fixed, i.e., $\left(F_k^+ \cup F_k^-\right) = \{1, 2, ..., m\}$ and $G_k = \varnothing$. At these nodes we simply apply the gradient ascent procedure to find lower bounds corresponding to the feasible solutions of P_k.

Now we want to explain two important issues regarding the implementation of the BB method: branching and pruning. Branching at node k is performed by considering the (feasible) solution providing the best lower bound at that node and selecting variable $X_i, i \in G_k$ for which $\lambda_i (u_i X_i - Q_i)$ is the largest. Two branches emanating from node k are obtained by setting the selected variable equal to one (left branch) and to zero (right branch), which implies that a binary search tree is generated. The rationale behind the above-mentioned selection can be explained by noting that a solution to $\mathrm{P}_k(\boldsymbol{\lambda})$ is optimal for P_k if the relaxed constraint set $Q_i \leq u_i X_i$ and the complementary slackness condition $\lambda_i (u_i X_i - Q_i) = 0$ are satisfied by this solution. Here we apply a heuristic rule and choose to branch on the X_i variable that corresponds to the largest violation in the complementary slackness conditions. In other words, we choose to branch on the X_i variable for which $|\lambda_i (u_i X_i - Q_i)|$ is the largest at the iteration at which the best upper bound is obtained. Pruning of the nodes in the BB tree is accomplished by comparing the upper bound UB_k at a node k with the current best lower bound LB_{best} obtained in the tree (the objective value of the best feasible solution). That is, node k is pruned when $UB_k \leq LB_{best}$.

The BB algorithm using LR relaxation can be given as

1. **Step 1 (Initialization)**: At active node 0, $G_0 = \{1, 2, ..., m\}$, $LB_{Best} = -\infty$. Begin by applying the explained LR procedure with no X_i branched. Let $LB_{Best} = max\{LB_{Best}, LB_0\}$. Go to step 2.
2. **Step 2 (Branching)**: If no active node exists, go to step 5. Otherwise select an active node according to the branching rule. Go to step 3.
3. **Step 3 (Finding bounds)**: Apply the LR procedure with the imposed constraints coming from branching. Obtain LB_k and UB_k. Let $LB_{Best} = max\{LB_{Best}, LB_k\}$. Go to step 4.
4. **Step 4 (Pruning)**: If $UB_k \leq LB_{Best}$, prune node k and backtrack. Go to step 2.
5. **Step 5 (Termination)**: The feasible solution which yielded LB_{Best} is optimal.

Figure 3.3. BB-LR Method

It is also important to emphasize that we use a depth-first search strategy in the binary BB tree. Whenever a node is pruned, we backtrack and consider an unsolved node. Applying Lagrangean relaxation within a BB method is a computationally intensive approach as it involves subgradient optimization at every node. To reduce the computational burden, we apply the approach suggested in Beasley (1993b). Namely, a large number of subgradient iterations are performed at the root node of the tree. This number is reduced to 30 whenever we branch on a new location variable in the tree, and it is doubled when we backtrack.

3.2.3. A Branch-and-Bound Algorithm using Nonlinear Programming Relaxation

The last solution method we propose is also based on the principle of BB; but rather than using LR to solve P_k at node k of the BB tree we relax the binary restrictions of the location variables $X_i, i \in G_k$ in P_k. That is, $X_i \in \{0, 1\}, i \in G_k$ are replaced by $0 \leq X_i \leq 1$ and a continuous nonlinear program P'_k is obtained at node k.

$$
\mathrm{P}'_k: \quad \max \sum_{j=1}^{n} h_j \frac{\sum_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2}}{\sum_{i \in G_k \cup F_k^+} \frac{Q_i}{d_{ij}^2} + o_j} - \sum_{i \in G_k \cup F_k^+} c_i Q_i - \sum_{i \in G_k} f_i X_i - \sum_{i \in F_k^+} f_i \tag{3.35}
$$

s.t.

$$
Q_i \leq u_i X_i \qquad i \in G_k \tag{3.36}
$$

$$
Q_i \leq u_i \qquad i \in G_k \cup F_k^+ \tag{3.37}
$$

$$
Q_i \geq 0 \qquad i \in G_k \cup F_k^+ \tag{3.38}
$$

$$
0 \leq X_i \leq 1 \qquad i \in G_k \tag{3.39}
$$

We call this method BB-NLP. It is clear that solving P'_k provides an upper bound for P_k. When in the relaxed solution to P'_k all X_i variables turn out to be zero or one, then we obtain a feasible solution of the original problem P, which provides a lower bound on the optimal objective value of P. We employ MINOS solver (Murtagh *et al.*, 2004) that is available within GAMS suite (Rosenthal, 2010) to solve the nonlinear programs. Branching at node k is performed by considering the solution at that node and selecting the X_i variable whose value is the closest to 0.5. In other words, the most fractional variable is chosen as the variable to branch on. Pruning of the nodes is based on two conditions: node k is pruned if either a feasible solution to P is obtained with all X_i variables having binary values or the upper bound UB_k at that node is less than or equal to the best lower bound in the tree, i.e., $UB_k \leq LB_{best}$.

A similar algorithm to the BB-LR can be given for BB-NLP as well:

1. **Step 1 (Initialization)**: At active node 0 $G_0 = \{1, 2, ..., n\}$, $LB_{Best} = -\infty$. All X_i are relaxed between 0 and 1. Solve the relaxed model P''_k using NLP relaxation. Go to step 2.
2. **Step 2 (Branching)**: If no active node exists, go to step 5. Otherwise select an active node according to the branching rule. Go to step 3.
3. **Step 3 (Finding bounds)**: Apply the NLP relaxation procedure. Obtain UB_k of the active node from the solution of P''_k. Go to step 4.
4. **Step 4 (Pruning)**: If a feasible solution is obtained, let $LB_k = UB_k$. Update LB_{Best} by $LB_{Best} = max\,\{LB_{Best}, LB_k\}$, prune that node and backtrack. Go to step 2.
5. **Step 5 (Termination)**: The feasible solution which yielded LB_{Best} is optimal.

Figure 3.4. BB-NLP Method

4. A BILEVEL COMPETITIVE FACILITY LOCATION PROBLEM WITH PARTIAL REACTION OF THE COMPETITOR

In this chapter[2] , we address a bilevel CFL problem in the discrete space with the objective of maximizing the profit of the new market entrant. This firm is the leader of a sequential game in which it determines the optimal location and attractivenesses for the new facilities. Given the new facilities of the leader, the competitor firm that becomes the follower in the game, reacts to the action of the leader by adjusting the attractiveness levels of its existing facilities to optimize its own profit. The new CFL model can be regarded as an extension of the first CFL model given in the previous chapter where the competitor can react now to the new entrant firm by redesigning its existing facilities. To the best of our knowledge, the bilevel CFL problem with a discrete set of candidate facility sites and continuous attractiveness of the leader is not addressed before. The gravity-based rule is employed again in order to model customer behavior. The developed model is based on a mixed-integer nonlinear BP formulation, and we try to solve it using the GMIN-αBB algorithm (Adjiman *et al.*, 1997; Adjiman *et al.*, 2000). Although this algorithm can provide an exact solution to our bilevel CFL problem, it requires a considerable amount of computation time even for relatively small problems. Therefore, we adopt a modified version of GMIN-αBB where the nodes of the BB tree are explored until the improvement in two non-successive iterations is less than a user-specified threshold value.

4.1. Model Formulation

The aim of the model is to determine the optimal location and attractiveness of the new facilities to be opened by a firm to maximize its profit when there are r existing facilities that belong to a competitor. As the new entrant tries to open new facilities in the market, it is possible that the competitor reacts to this new situation. Thus the new entrant firm (referred to as the firm) is considered as the leader of the game and the competing firm (referred to

[2]The paper Küçükaydın *et al.* (2011a) based on this chapter is published in the European Journal of Operational Research.

as the competitor) already in the market as the follower of the game. The follower also tries to maximize its own profit which conflicts with the objective of the new entrant. We assume that the possible reaction of the follower is to adjust the attractiveness level of all or some of its existing facilities. The adjustment can be realized in such a way that the attractiveness is either increased provided that it does not exceed an upper limit or decreased to a value between zero and the current level. Decreasing the attractiveness to zero means that the facility is shut down. We further assume that customers are aggregated at n (demand) points, the number of candidate facility sites is m, and the number of existing facilities of the follower is r. First, we define the parameters and decision variables by indexing the points by $j = 1, 2, ..., n$, the candidate facility sites by $i = 1, 2, ..., m$, and the existing facilities by $k = 1, 2, ..., r$.

Parameters:

h_j : annual buying power at point j,
c_i : unit attractiveness cost at site i,
f_i : annualized fixed cost of opening a facility at site i,
u_i : maximum attractiveness level for a facility to be opened at site i,
d_{ij} : Euclidean distance between candidate site i and point j,
$\tilde{d}_{kj}$: Euclidean distance between existing facility site k and point j,
$\underline{A_k}$: current attractiveness level of competitor's facility at site k,
$\overline{A}_k$: maximum attractiveness level of competitor's facility at site k,
$\tilde{c}_k$: unit attractiveness cost or revenue of competitor's facility at site k.

Decision variables:

Q_i : attractiveness of the facility opened at site i,
X_i : binary variable which is equal to one if a facility is opened at site i, and zero otherwise,
A_k : new attractiveness level of competitor's facility at site k.

As given in Section 3.1., when a facility is opened by the firm at site i with attractiveness Q_i, the utility of this facility for customers at point j is given by Q_i/d_{ij}^2 using gravity-based rule.

The total utility of competitor's facilities at point j is given by $\sum_{k=1}^{r} A_k/\tilde{d}_{kj}^2$. Consequently, the proportion P_{ij} of customers at point j who visit a new facility at site i is expressed as

$$P_{ij} = \frac{\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + \sum\limits_{k=1}^{r}(A_k/\tilde{d}_{kj}^2)}. \tag{4.1}$$

The revenue of this facility can be computed by the expression $\sum_{j=1}^{n} h_j P_{ij}$ and the total revenue captured by the new facilities can be given as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} h_j P_{ij} = \sum_{j=1}^{n} h_j \sum_{i=1}^{m} P_{ij} = \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + \sum\limits_{k=1}^{r}(A_k/\tilde{d}_{kj}^2)}. \tag{4.2}$$

In a similar fashion, one can calculate the total revenue captured by the existing facilities of the competitor as

$$\sum_{j=1}^{n} h_j \frac{\sum\limits_{k=1}^{r}(A_k/\tilde{d}_{kj}^2)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + \sum\limits_{k=1}^{r}(A_k/\tilde{d}_{kj}^2)}. \tag{4.3}$$

Now, we can formulate the following bilevel MINLP model BP1:

$$\text{BP1}: \quad \max_{\mathbf{Q},\mathbf{X}} \sum_{j=1}^{n} h_j \frac{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r} (A_k/\tilde{d}_{kj}^2)} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{4.4}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{4.5}$$

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \tag{4.6}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{4.7}$$

$$\max_{\mathbf{A}} \sum_{j=1}^{n} h_j \frac{\sum_{k=1}^{r} (A_k/\tilde{d}_{kj}^2)}{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r} (A_k/\tilde{d}_{kj}^2)} - \sum_{k=1}^{r} \tilde{c}_k \left(A_k - \underline{A_k}\right) \tag{4.8}$$

s.t.

$$A_k \leq \overline{A}_k \qquad k = 1, ..., r \tag{4.9}$$

$$A_k \geq 0 \qquad k = 1, ..., r \tag{4.10}$$

The objective function (4.4) of the firm consists of three summation terms. The first one represents the revenue collected by the new facilities that are opened, while the second and third components represent the fixed cost and attractiveness cost associated with opening the new facilities, respectively. Constraints (4.5) along with the binary restrictions (4.7) on the location variables X_i and nonnegativity restrictions (4.6) on attractiveness variables Q_i ensure that if no facility is opened at site i, then the corresponding attractiveness Q_i of the facility is zero and if a facility is opened at site i, then its attractiveness Q_i cannot exceed the maximum level u_i. We note that the number of facilities to be located is not fixed, its value is to be determined by the solution of the model. The objective function of the competitor (4.8) has two components: the revenue collected by the competitor's facilities and the cost or revenue associated with adjusting the attractiveness levels. Note that when the attractiveness is reduced from its current level $\underline{A_k}$ to a smaller value A_k, which makes $(A_k - \underline{A_k})$ negative, the follower gains a revenue of $\tilde{c}_k(\underline{A_k} - A_k)$. On the other hand, if $\underline{A_k} < A_k \leq \overline{A}_k$, a cost of magnitude $\tilde{c}_k(A_k - \underline{A_k})$ incurs to the follower. Thus, increasing the attractiveness level of an existing facility incurs a cost to the competitor, whereas decreasing the attractiveness level results in a revenue. This situation can be better explained by an example. Increasing the attractiveness level of a facility can be the consequence of expanding the floor area of the facility, increasing the diversity of the products sold in the facility and/or increasing the number of servers, which altogether determine the new attractiveness level of the facility. In such a situation, a natural cost is incurred to the owner(s) of the facility. On the other hand, shrinking the floor area of the facility, reducing the diversity of the products in the facility and/or decreasing the number of servers are indication of reducing the attractiveness level and can result in a saving for the owner(s) of the facility which can be interpreted as a revenue. Constraints (4.9) and (4.10) ensure that the new attractiveness A_k of an existing facility at site k is between zero and an upper limit $\overline{A}_k$. If at an optimal solution of the model BP1, A_k is equal to zero, then it means that the existing facility at site k is closed by the follower.

Before we move on with the proposed solution method for P in the next section, we show a property of the objective function of the lower level problem, i.e., the competitor's profit function.

Proposition 4.1. $\sum_{j=1}^{n} h_j \frac{\sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)}$ *is concave in* $\mathbf{A} = (A_1, A_2, ..., A_r)^{\mathrm{T}}$ *for* $\mathbf{A} \geq \mathbf{0}$.

Proof. Since the sum of concave functions is a concave function, it suffices to show that each of the terms $g_j(\mathbf{A}) = h_j \frac{\sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)}$ is concave for $\mathbf{A} \geq 0$ for $j = 1, 2, ..., n$. The concavity of $g_j(\mathbf{A})$ for $\mathbf{A} \geq \mathbf{0}$ is obtained if its Hessian matrix $H_j(\mathbf{A})$ is negative semidefinite for $\mathbf{A} \geq \mathbf{0}$. The show the latter, we use the fact that $H_j(\mathbf{A})$ for $\mathbf{A} \geq \mathbf{0}$ is negative semidefinite if and only if $\mathbf{V}^{\mathrm{T}} H_j(\mathbf{A})\mathbf{V} \leq 0$ for any $\mathbf{V}$. To this end, we compute the first and second order derivatives of $g_j(\mathbf{A})$ as follows:

$$\frac{\partial g_j(\mathbf{A})}{\partial A_k} = h_j \frac{\left(1/\tilde{d}_{kj}^2\right) \sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\left[\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r} \left(A_k/\tilde{d}_{kj}^2\right)\right]^2}$$

and

$$\frac{\partial g_j(\mathbf{A})}{\partial A_l \partial A_k} = -2h_j \frac{\left(1/\tilde{d}_{kj}^2\right)\left(1/\tilde{d}_{lj}^2\right)\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\left[\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum\limits_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)\right]^3}.$$

The Hessian matrix $H_j(\mathbf{A})$ of $g_j(\mathbf{A})$ is then

$$H_j(\mathbf{A}) = -p_j \begin{pmatrix} \frac{1}{\tilde{d}_{1j}^4} & \frac{1}{\tilde{d}_{1j}^2\tilde{d}_{2j}^2} & \cdots & \frac{1}{\tilde{d}_{1j}^2\tilde{d}_{rj}^2} \\ \frac{1}{\tilde{d}_{1j}^2\tilde{d}_{2j}^2} & \frac{1}{\tilde{d}_{2j}^4} & \cdots & \frac{1}{\tilde{d}_{2j}^2\tilde{d}_{rj}^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\tilde{d}_{1j}^2\tilde{d}_{rj}^2} & \frac{1}{\tilde{d}_{2j}^2\tilde{d}_{rj}^2} & \cdots & \frac{1}{\tilde{d}_{rj}^4} \end{pmatrix} \tag{4.11}$$

where $p_j = 2h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\left[\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum\limits_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)\right]^3}$. We remark that $p_j \geq 0$ holds for $\mathbf{A} \geq \mathbf{0}$ since $h_j \geq 0$ and $\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) \geq 0$. Now, $\mathbf{V}^{\mathrm{T}}H_j(\mathbf{A})\mathbf{V}$ can be expressed as

$$\mathbf{V}^{\mathrm{T}} H_j(\mathbf{A})\mathbf{V} = -\left(V_1, V_2, \ldots, V_r\right) p_j \begin{pmatrix} \frac{1}{\tilde{d}_{1j}^4} & \frac{1}{\tilde{d}_{1j}^2 d_{2j}^2} & \cdots & \frac{1}{\tilde{d}_{1j}^2 \tilde{d}_{rj}^2} \\ \frac{1}{\tilde{d}_{1j}^2 \tilde{d}_{2j}^2} & \frac{1}{\tilde{d}_{2j}^4} & \cdots & \frac{1}{\tilde{d}_{2j}^2 \tilde{d}_{rj}^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\tilde{d}_{1j}^2 \tilde{d}_{rj}^2} & \frac{1}{\tilde{d}_{2j}^2 \tilde{d}_{rj}^2} & \cdots & \frac{1}{\tilde{d}_{rj}^4} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_r \end{pmatrix} \tag{4.12}$$

$$= -\left(V_1, V_2, \ldots, V_r\right) p_j \left(\frac{V_1}{\tilde{d}_{1j}^2} + \frac{V_2}{\tilde{d}_{2j}^2} + \ldots + \frac{V_r}{\tilde{d}_{rj}^2}\right) \begin{pmatrix} 1/\tilde{d}_{1j}^2 \\ 1/\tilde{d}_{2j}^2 \\ \vdots \\ 1/\tilde{d}_{rj}^2 \end{pmatrix} \tag{4.13}$$

$$= -p_j \left(\frac{V_1}{\tilde{d}_{1j}^2} + \frac{V_2}{\tilde{d}_{2j}^2} + \ldots + \frac{V_r}{\tilde{d}_{rj}^2}\right)^2 . \tag{4.14}$$

$p_j \geq 0$ for $\mathbf{A} \geq \mathbf{0}$ and $\left(\frac{V_1}{\tilde{d}_{1j}^2} + \frac{V_2}{\tilde{d}_{2j}^2} + \ldots + \frac{V_r}{\tilde{d}_{rj}^2}\right)^2 \geq 0$ together imply that $\mathbf{V}^{\mathrm{T}} H_j(\mathbf{A})\mathbf{V} \leq 0$ for any $\mathbf{V}$, and hence $H_j(\mathbf{A})$ is negative semidefinite. This means that $g_j(\mathbf{A})$ for $j = 1, 2, \ldots, n$ is concave for $\mathbf{A} \geq \mathbf{0}$, which completes the proof. $\square$

Proposition 4.2. $\sum_{j=1}^{n} h_j \frac{\sum_{k=1}^{r} A_k/\tilde{d}_{kj}^2}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)} - \sum_{k=1}^{r} \tilde{c}_k (A_k - \underline{A_k})$ *is concave for* $\mathbf{A} \geq \mathbf{0}$.

Proof. The first term is concave for $\mathbf{A} \geq \mathbf{0}$ as a consequence of Proposition 1 and the second term is a linear function of $\mathbf{A} \geq \mathbf{0}$. The result follows since the sum of concave functions is also concave. $\square$

When we consider the objective function

$$\sum_{j=1}^{n} h_j \frac{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{4.15}$$

of the leader, we can observe that it is a function of $\mathbf{Q}$ and $\mathbf{X}$ variables. When X_i values are fixed, it can be shown that this function is concave in $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_m)^{\mathrm{T}}$ for $\mathbf{Q} \geq \mathbf{0}$ using

Proposition 3.2. In other words, the leader's objective function without the facility location cost terms is concave.

4.2. Solution Procedure

Different methods are suggested in the literature for the solution of the BP problems. These are mostly heuristics and do not guarantee a global optimal solution. There are also exact solution methods such as branch-and-bound and cutting plane algorithms which can only be applied for certain types of BP problems. One of the earliest methods is due to Wen and Yang (1990) who suggest both an exact and a heuristic BB algorithm for solving linear BP models, whose upper level problem contains binary decision variables and the lower level (i.e., follower's) problem is continuous. The BB algorithm proposed by Edmunds and Bard (1992) can solve BP formulations to global optimality provided that the upper level problem is mixed-integer and nonlinear, and the lower level problem consists of continuous decision variables and the functions involved are convex and quadratic. Jan and Chern (1994) propose two exact algorithms that make use of parametric analysis for solving nonlinear integer BP models, whose functions are required to be separable and monotone in the leader's and follower's decision variables. Gümüş and Floudas (2005) suggest a global optimization algorithm based on a reformulation/linearization technique for general mixed-integer nonlinear BP problems in which the lower-level problem is linear in terms of the continuous decision variables of the follower. As pointed out in the survey papers of Dempe (2003) and Colson *et al.* (2007), existing BB methods require linear or convex quadratic nonlinear functions in the lower-level problem of the BP models. Unfortunately, this is not the case for our bilevel model where the objective function of the follower's problem is nonlinear and not quadratic. As a consequence, we opt for a solution method where the mixed-integer nonlinear BP formulation is first converted into a one-level mixed-integer nonlinear programming (MINLP) problem using the fact that the follower's problem is a continuous concave maximization problem, which makes the Karush-Kuhn-Tucker optimality conditions necessary and sufficient. Although the NLP relaxation of resulting equivalent one-level MINLP problem is not a concave programming problem (i.e., maximization of a concave function over a compact convex set), we transform it by introducing new variables so that it can be solved optimally by applying the GMIN-αBB algorithm.

4.2.1. Transformation of the Bilevel Model Into An Equivalent One-Level Model

As is shown in Proposition 4.2, the competitor's problem, which is the lower level problem in BP1, is a concave maximization problem in terms of the attractiveness variables $A_k \geq 0$. Hence, we can write the *Karush-Kuhn-Tucker* (KKT) *optimality conditions* for this problem and use them for converting the original bilevel model BP1 into an equivalent one-level optimization model. A necessary and sufficient condition for $\mathbf{A}$ to be an optimal solution to the competitor's problem is that there exist Lagrangean multiplier vectors $\boldsymbol{\lambda_1} = (\lambda_{11}, ..., \lambda_{1r})$ and $\boldsymbol{\lambda_2} = (\lambda_{21}, ..., \lambda_{2r})$ which satisfy the following system

$$\begin{aligned}
&\sum_{j=1}^{n} h_j \frac{(1/\tilde{d}_{kj}^2)\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\left[\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)\right]^2} - \tilde{c}_k + \lambda_{1k} - \lambda_{2k} = 0 && k = 1, 2, ..., r \\
&A_k - \overline{A}_k + s_{1k} = 0 && k = 1, 2, ..., r \\
&-A_k + s_{2k} = 0 && k = 1, 2, ..., r \\
&\lambda_{1k} s_{1k} = 0 && k = 1, 2, ..., r \\
&\lambda_{2k} s_{2k} = 0 && k = 1, 2, ..., r \\
&\lambda_{1k}, \lambda_{2k}, s_{1k}, s_{2k} \geq 0 && k = 1, 2, ..., r.
\end{aligned}$$

Here $\mathbf{s}_1 = (s_{11}, s_{12}, ..., s_{1r})$ and $\mathbf{s}_2 = (s_{21}, s_{22}, ..., s_{2r})$ are slack variables corresponding to constraint sets (4.9) and (4.10) of the competitor's problem, respectively. Three of the Karush-Kuhn-Tucker optimality conditions, i.e., $\sum_{j=1}^{n} h_j \frac{(1/\tilde{d}_{kj}^2)\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\left[\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)\right]^2} - \tilde{c}_k + \lambda_{1k} - \lambda_{2k} = 0$, $\lambda_{1k} s_{1k} = 0$ and $\lambda_{2k} s_{2k} = 0$, are nonlinear in nature. However, we can at least remove the nonlinearity caused by the last two conditions, using the *active set strategy* suggested by Grossmann and Floudas (1987) which can be stated as

$$\begin{aligned}
&\lambda_{1k} - MY_{1k} \leq 0 && k = 1, 2, ..., r \\
&s_{1k} - M(1 - Y_{1k}) \leq 0 && k = 1, 2, ..., r \\
&\lambda_{2k} - MY_{2k} \leq 0 && k = 1, 2, ..., r \\
&s_{2k} - M(1 - Y_{2k}) \leq 0 && k = 1, 2, ..., r \\
&Y_{1k}, Y_{2k} \in \{0, 1\} && k = 1, 2, ..., r
\end{aligned}$$

where M is an upper bound on the slack variables s_{1k} and s_{2k}, $k = 1, 2, ..., r$, and Y_{1k} and Y_{2k}, $k = 1, 2, ..., r$ are auxiliary binary variables. If $Y_{1k} = 1$ or $Y_{2k} = 1$ for some k, then the corresponding slack variable $s_{1k} = 0$ or $s_{2k} = 0$, which implies that the kth constraint of the ith constraint set $(i = 1, 2)$ is active. Otherwise, if $Y_{ik} = 0$ for some k, then $\lambda_{ik} = 0$. In both cases the complementary slackness conditions are satisfied.

In order to solve the original bilevel model we transform it into an equivalent one-level model. To this end, we replace the follower's problem with its equivalent Karush-Kuhn-Tucker conditions and make use of the active set strategy. Consequently, the one-level formulation, which is equivalent to the original problem BP1, is expressed as

$$\text{BP1}': \qquad \max \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum\limits_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{4.16}$$

s.t.

$$Q_i \le u_i X_i \qquad i = 1, \ldots, m \tag{4.17}$$

$$X_i \in \{0,1\} \qquad i = 1, \ldots, m \tag{4.18}$$

$$Q_i \ge 0 \qquad i = 1, \ldots, m \tag{4.19}$$

$$\sum_{j=1}^{n} h_j \frac{(1/\tilde{d}_{kj}^2)\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\left[\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum\limits_{k=1}^{r}\left(A_k/\tilde{d}_{kj}^2\right)\right]^2} - \tilde{c}_k + \lambda_{1k} - \lambda_{2k} = 0 \qquad k = 1, \ldots, r \tag{4.20}$$

$$A_k - \overline{A}_k + s_{1k} = 0 \qquad k = 1, \ldots, r \tag{4.21}$$

$$-A_k + s_{2k} = 0 \qquad k = 1, \ldots, r \tag{4.22}$$

$$\lambda_{1k} - MY_{1k} \le 0 \qquad k = 1, \ldots, r \tag{4.23}$$

$$s_{1k} - M(1 - Y_{1k}) \le 0 \qquad k = 1, \ldots, r \tag{4.24}$$

$$\lambda_{2k} - MY_{2k} \le 0 \qquad k = 1, \ldots, r \tag{4.25}$$

$$s_{2k} - M(1 - Y_{2k}) \le 0 \qquad k = 1, \ldots, r \tag{4.26}$$

$$\lambda_{1k}, \lambda_{2k}, s_{1k}, s_{2k} \ge 0, Y_{1k}, Y_{2k} \in \{0,1\} \qquad k = 1, \ldots, r \tag{4.27}$$

The resulting one-level formulation BP1$'$ is a MINLP model. For its solution we employ the GMIN-αBB algorithm that is derived from the αBB algorithm, as will be explained in detail in the next subsection. Both of these algorithms perform a preprocessing step where all the terms existing in the objective function as well as in the constraints are grouped into different classes such as linear, fractional, concave, bilinear, univariate convex, and general nonconcave. Then, a concave overestimator is generated for each term in all classes with the exception of the linear and concave ones. However, the concavification procedure of the terms in the general nonconcave class is different from those in the other classes in the sense that it requires a more difficult and computationally intensive method referred to as the α calculations. In order to avoid the α calculations, we need to get rid of the nonconcave terms in the objective function (4.16) and constraints (4.20). To do so, we define two new variables w_{1j} and w_{2j} for $j = 1, \ldots, n$ as follows.

$$w_{1j} = \frac{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{w_{2j}} \tag{4.28}$$

and

$$w_{2j} = \sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r} \left(A_k/\tilde{d}_{kj}^2\right), \tag{4.29}$$

where w_{1j} gives the proportion of the utility (i.e., the market share) that the leader captures from point j and w_{2j} is the total utility of the facilities belonging to both the leader and the follower to the customers located at point j. Using the new variables, we can write

$$w_{1j} w_{2j} = \sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) \tag{4.30}$$

and

$$\frac{w_{1j}}{w_{2j}} = \frac{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\left[\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r} \left(A_k/\tilde{d}_{kj}^2\right)\right]^2} \tag{4.31}$$

for $j = 1, \ldots, n$. As a result, using the new variables w_{1j} and w_{2j} $BP1'$ can be written in such a form where general nonconcave terms are eliminated at the expense of introducing new bilinear and fractional terms into the formulation. As mentioned above, this helps in avoiding the computationally intensive α calculations used in the concavification procedure of the terms. Hence, the one-level model BP1$'$ can be formulated as

$$\text{BP1}': \quad \max \sum_{j=1}^{n} h_j w_{1j} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{4.32}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{4.33}$$

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \tag{4.34}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{4.35}$$

$$\sum_{j=1}^{n} h_j \frac{1}{\tilde{d}_{kj}^2} \frac{w_{1j}}{w_{2j}} - \tilde{c}_k + \lambda_{1k} - \lambda_{2k} = 0 \qquad k = 1, 2, ..., r \tag{4.36}$$

$$A_k - \overline{A}_k + s_{1k} = 0 \qquad k = 1, \ldots, r \tag{4.37}$$

$$- A_k + s_{2k} = 0 \qquad k = 1, \ldots, r \tag{4.38}$$

$$\lambda_{1k} - MY_{1k} \leq 0 \qquad k = 1, \ldots, r \tag{4.39}$$

$$s_{1k} - M(1 - Y_{1k}) \leq 0 \qquad k = 1, \ldots, r \tag{4.40}$$

$$\lambda_{2k} - MY_{2k} \leq 0 \qquad k = 1, \ldots, r \tag{4.41}$$

$$s_{2k} - M(1 - Y_{2k}) \leq 0 \qquad k = 1, \ldots, r \tag{4.42}$$

$$\sum_{i=1}^{m} \left(Q_i / d_{ij}^2\right) - w_{1j} w_{2j} = 0 \qquad j = 1, \ldots, n \tag{4.43}$$

$$\sum_{i=1}^{m} \left(Q_i / d_{ij}^2\right) + \sum_{k=1}^{r} \left(A_k / \tilde{d}_{kj}^2\right) - w_{2j} = 0 \qquad j = 1, \ldots, n \tag{4.44}$$

$$\lambda_{1k}, \lambda_{2k}, s_{1k}, s_{2k} \geq 0, Y_{1k}, Y_{2k} \in \{0, 1\} \qquad k = 1, \ldots, r \tag{4.45}$$

$$w_{1j}, w_{2j} \geq 0 \qquad j = 1, \ldots, n \tag{4.46}$$

4.2.2. Solution of the One-Level Model

BP1$'$ can be solved to ϵ-optimality by using the GMIN-αBB algorithm, which is a global optimization method proposed by Adjiman *et al.* (1997) and later revised by Adjiman *et al.* (2000) to solve pure-integer or mixed-integer nonlinear optimization problems. In fact, it is derived from the αBB algorithm developed by Androulakis *et al.* (1995) to tackle nonlinear, nonconvex optimization problems with continuous decision variables having lower and upper limits. In the following, we first provide a brief description of this algorithm adapted for the case of a maximization problem, and then give the working mechanism of GMIN-αBB along with the details of how we apply it to solve our problem BP1$'$.

As its name suggests, αBB algorithm is a BB method as well. At each node of the BB tree, a lower bound and an upper bound are generated on the optimal objective function value of the nonconcave nonlinear continuous maximization problem (NLP) at hand. As a preprocessing step, all the terms in the objective and constraint functions of the NLP are identified and grouped in the following classes: linear, concave, bilinear, trilinear, fractional, fractional trilinear, univariate convex, and general nonconcave (Floudas, 2000). Then, a concave overestimator for the terms in each class except the linear and concave classes is generated. This concavification procedure usually requires the addition of new variables and linear inequality constraints to the problem for bilinear, trilinear, fractional, fractional trilinear, and univariate convex terms. As stated earlier, general nonconcave terms are handled by a more difficult method referred to as the α calculations. However, since all such terms are removed from P', we do not resort to α calculations in this work. While the overestimators provide an upper bound on the optimal objective value of the NLP, a lower bound is generated by finding a local optimal solution to the problem using for example a general purpose NLP solver such as MINOS, CONOPT, and KNITRO that can handle a nonlinear programming problem. To guarantee ϵ-convergence from the global maximum, i.e., the difference between the lower and upper bounds is within ϵ of the optimal objective value, the feasible region of the problem is divided at each node such that the rectangles defined by the lower and upper limits on the decision variables are partitioned into smaller ones. This partitioning constitutes the branching mechanism in the BB tree.

At the beginning of the algorithm the best lower bound, z_{LB}, is set to minus infinity while the best upper bound z_{UB} is set to plus infinity. Moreover, the values of the convergence parameter ϵ and feasibility tolerance parameter ϵ_f are selected. The latter determines by how much the constraints can deviate from feasibility. At the root node of the tree, a local optimal solution is found for the original NLP, which gives a lower bound. Then, an upper bound is obtained by solving the concavified problem. If $z_{UB} - z_{LB}$ is less than ϵ, the algorithm is terminated. Otherwise, the two child nodes are created by dividing the initial rectangle into two new subrectangles. The concavified problem is solved in each of these nodes with the corresponding subrectangle to obtain new upper bounds. The maximum of these upper bounds is used to update z_{UB}; a local optimal solution to the original NLP is found at the associated node. If this solution is ϵ_f-feasible, then z_{LB} is updated. Subsequently, this node (subrectangle) is selected for further branching to give rise to two new nodes with new subrectangles. A node is pruned when the upper bound at this node is less than the best lower bound or the concavified problem is infeasible. This procedure generates a nondecreasing sequence of best lower bounds and a nonincreasing sequence of best upper bounds. A sufficient number of iterations guarantees ϵ-convergence to the global optimum solution. The interested reader may refer to Adjiman *et al.* (1998a), Adjiman *et al.* (1998b), and Floudas (2000) for more details of the αBB algorithm. The pseudo-codes for αBB and GMIN-αBB algorithms are given in the Appendices.

In order to employ the αBB algorithm at each node of the BB tree we relax the binary variable vectors $\mathbf{X} = (X_1, X_2, ..., X_m)$, $\mathbf{Y}_1 = (Y_{11}, Y_{12}, ..., Y_{1r})$, and $\mathbf{Y}_2 = (Y_{21}, Y_{22}, ..., Y_{2r})$ of BP1$'$ such that they can take values within the interval $[0, 1]$. To solve the relaxed version of BP1$'$ we need to concavify all the terms in it except the linear and concave ones. The terms for which an overestimator should be constructed are the fractional terms w_{1j}/w_{2j} in constraints (4.36) and the bilinear terms $w_{1j}w_{2j}$ in constraints (4.43). The overestimation involves introducing new variables and constraints. Using the procedure given by Floudas (2000), we let $w_{3j} = w_{1j}w_{2j}$ and $w_{4j} = w_{1j}/w_{2j}$ for $j = 1, 2, ..., n$. We also let w_{ij}^L and w_{ij}^U denote, respectively, the lower and upper bounds on w_{ij} for $i = 1, 2$ and $j = 1, 2, ..., n$. These bounds are easily established as follows: $w_{1j}^L = w_{2j}^L = 0$, $w_{1j}^U = 1$, and $w_{2j}^U = \sum_{i=1}^{m} \left(u_i/d_{ij}^2\right) + \sum_{k=1}^{r} \left(\overline{A}_k/\tilde{d}_{kj}^2\right)$. To overestimate the fractional terms w_{1j}/w_{2j} the following linear constraints are added to BP1$'$:

$$
\begin{aligned}
& w_{1j}^L/w_{2j} + w_{1j}/w_{2j}^U - w_{1j}^L/w_{2j}^U - w_{4j} \leq 0 \quad j = 1, 2, ..., n \\
& w_{1j}^U/w_{2j} + w_{1j}/w_{2j}^L - w_{1j}^U/w_{2j}^L - w_{4j} \leq 0 \quad j = 1, 2, ..., n.
\end{aligned}
$$

The bilinear terms $w_{1j}w_{2j}$ are overestimated using the following linear constraints:

$$
\begin{aligned}
& w_{1j}^L w_{2j} + w_{2j}^L w_{1j} - w_{1j}^L w_{2j}^L - w_{3j} \leq 0 \quad j = 1, 2, ..., n \\
& w_{1j}^U w_{2j} + w_{2j}^U w_{1j} - w_{1j}^U w_{2j}^U - w_{3j} \leq 0 \quad j = 1, 2, ..., n \\
& -w_{1j}^U w_{2j} - w_{2j}^L w_{1j} + w_{1j}^U w_{2j}^L + w_{3j} \leq 0 \quad j = 1, 2, ..., n \\
& -w_{1j}^L w_{2j} - w_{2j}^U w_{1j} + w_{1j}^L w_{2j}^U + w_{3j} \leq 0 \quad j = 1, 2, ..., n.
\end{aligned}
$$

As a consequence of these new constraints, we obtain the concavified problem $\mathrm{BP1_{UB}}$ whose solution yields an upper bound to the original NLP at a given node of the BB tree when binary variables are relaxed. $\mathrm{BP1_{UB}}$ is given as

$$\text{BP1}_{\text{UB}}: \quad \max \sum_{j=1}^{n} h_j w_{1j} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{4.47}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{4.48}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{4.49}$$

$$\sum_{j=1}^{n} h_j \frac{1}{\tilde{d}_{kj}^2} w_{4j} - \tilde{c}_k + \lambda_{1k} - \lambda_{2k} = 0 \qquad k = 1, \ldots, r \tag{4.50}$$

$$A_k - \overline{A}_k + s_{1k} = 0 \qquad k = 1, \ldots, r \tag{4.51}$$

$$-A_k + s_{2k} = 0 \qquad k = 1, \ldots, r \tag{4.52}$$

$$\lambda_{1k} - MY_{1k} \leq 0 \qquad k = 1, \ldots, r \tag{4.53}$$

$$s_{1k} - M(1 - Y_{1k}) \leq 0 \qquad k = 1, \ldots, r \tag{4.54}$$

$$\lambda_{2k} - MY_{2k} \leq 0 \qquad k = 1, \ldots, r \tag{4.55}$$

$$s_{2k} - M(1 - Y_{2k}) \leq 0 \qquad k = 1, \ldots, r \tag{4.56}$$

$$\sum_{i=1}^{m} \left(Q_i / d_{ij}^2\right) - w_{3j} = 0 \qquad j = 1, \ldots, n \tag{4.57}$$

$$\sum_{i=1}^{m} \left(Q_i / d_{ij}^2\right) + \sum_{k=1}^{r} \left(A_k / \tilde{d}_{kj}^2\right) - w_{2j} = 0 \qquad j = 1, \ldots, n \tag{4.58}$$

$$w_{1j}^L w_{2j} + w_{2j}^L w_{1j} - w_{1j}^L w_{2j}^L - w_{3j} \leq 0 \qquad j = 1, \ldots, n \tag{4.59}$$

$$w_{1j}^U w_{2j} + w_{2j}^U w_{1j} - w_{1j}^U w_{2j}^U - w_{3j} \leq 0 \qquad j = 1, \ldots, n \tag{4.60}$$

$$-w_{1j}^U w_{2j} - w_{2j}^L w_{1j} + w_{1j}^U w_{2j}^L + w_{3j} \leq 0 \qquad j = 1, \ldots, n \tag{4.61}$$

$$-w_{1j}^L w_{2j} - w_{2j}^U w_{1j} + w_{1j}^L w_{2j}^U + w_{3j} \leq 0 \qquad j = 1, \ldots, n \tag{4.62}$$

$$w_{1j}^L / w_{2j} + w_{1j} / w_{2j}^U - w_{1j}^L / w_{2j}^U - w_{4j} \leq 0 \qquad j = 1, \ldots, n \tag{4.63}$$

$$w_{1j}^U / w_{2j} + w_{1j} / w_{2j}^L - w_{1j}^U / w_{2j}^L - w_{4j} \leq 0 \qquad j = 1, \ldots, n \tag{4.64}$$

$$\lambda_{1k}, \lambda_{2k}, s_{1k}, s_{2k} \geq 0 \qquad k = 1, \ldots, r \tag{4.65}$$

$$0 \leq Y_{1k}, Y_{2k} \leq 1 \qquad k = 1, \ldots, r \tag{4.66}$$

$$0 \leq w_{1j} \leq 1 \qquad j = 1, \ldots, n \tag{4.67}$$

$$0 \leq w_{2j} \leq \sum_{i=1}^{m} \left(u_i / d_{ij}^2\right) + \sum_{k=1}^{r} \left(\overline{A}_k / \tilde{d}_{kj}^2\right) \qquad j = 1, \ldots, n \tag{4.68}$$

Branching at a node is performed by considering the solution at that node and selecting the relaxed binary variable whose value is the closest to 0.5. Another important remark regarding our solution method is that the αBB algorithm requires lower and upper bounds on decision variables. The lower bound on each decision variable is zero, while the upper bounds are given in Table 4.1 where $M = \overline{A}_k$. The lower and upper bounds on variables w_{3j} and w_{4j} are slightly more involved and requires the solution of the optimization problems with objective functions min w_{ij} and max w_{ij} for $i = 3, 4$ and constraints of problem $\mathrm{P_{UB}}$. It is also to be emphasized that in our algorithm branching occurs only on the variables which participate in nonconcave terms, namely w_{1j} and w_{2j} for $j = 1, 2, ..., n$.

Table 4.1. Upper bounds on the decision variables in the αBB algorithm.

Variable	Q_i	X_i	A_k	s_{1k}	s_{2k}	λ_{1k}	λ_{2k}	Y_{1k}	Y_{2k}
Upper bound	u_i	1	$\overline{A}_k$	M	M	M	M	1	1

5. A BILEVEL COMPETITIVE FACILITY LOCATION PROBLEM WITH FULL REACTION OF THE COMPETITOR

In Chapter 4, we formulate a BP problem for the situation where a firm, with the aim of maximizing its profit, enters a market in which a competitor has existing facilities. The firm entering the market becomes the leader and the competing firm becomes the follower of the game. After the launch of the new facilities the reaction of the competitor is to adjust (i.e., increase or decrease) the attractiveness of its existing facilities as to maximize its own profit. However, it cannot open new facilities and/or close existing ones, which is a rather restrictive assumption. In this chapter[3] , we relax this assumption and extend the mentioned work by letting the reaction of the competitor include opening new facilities, closing existing ones in addition to adjusting the attractiveness levels of the existing facilities. This extension has a major impact on the structure of the BP model developed in Chapter 4 since the lower level problem of the competitor, which is a continuous nonlinear programming problem, becomes a mixed-integer nonlinear programming problem. Given that the upper level problem is also an MINLP, the formulated BP model turns out to be one of the most difficult types of BP models. The main contribution of this chapter is threefold. The first one is to develop a Stackelberg game between two firms with a realistic set of competitor reactions; the second one is to propose hybrid heuristics based on tabu search (TS); the third one is to devise a method that guarantees an ϵ-optimal solution.

5.1. Model Formulation

In our problem setting, the market entrant firm (referred to as the firm) is the leader and the competing firm (referred to as the competitor) already existing in the market is the follower. The objective of the firm is to find out the optimal location and attractiveness of the new facilities in such a way that its profit is maximized when there are r_1 existing facilities and r_2 candidate facility sites belonging to the competitor. It is assumed that the competitor reacts

[3]The paper Küçükaydın *et al.* (2011b) based on this chapter is accepted for publication in the Turkish Journal of Industrial Engineering. The papers of Küçükaydın *et al.* (2010b) and Küçükaydın *et al.* (2012) based on this chapter are published in the proceedings of HM2010 and in the Computers and Operations Research.

to the market entry of the firm by opening new facilities, closing existing ones, and adjusting the attractiveness of its existing facilities with the aim of maximizing its profit. Note that closing an existing facility at a certain location and opening a new facility at another location can be recognized as the relocation of an existing facility. Thus, our model takes into account all possible reactions of the competitor to the market entry of the new firm. Moreover, the adjustment is assumed to be such that the attractiveness of an existing facility is increased or decreased by the competitor provided that it remains positive and does not exceed a certain upper limit.

We further assume that the customers are aggregated at n (demand) points, the number of candidate facility sites of the leader is m, the number of existing facilities of the competitor is r_1, and the number of candidate facility sites of the competitor is r_2. First, we define the parameters and decision variables by indexing the points by $j = 1, 2, ..., n$, the firm's candidate facility sites by $i = 1, 2, ..., m$, the competitor's existing facilities by $k = 1, 2, ..., r_1$, and its candidate facility sites by $\ell = 1, 2, ..., r_2$.

Parameters:

h_j : annual buying power at point j,

c_i : unit attractiveness cost of the firm's new facility at site i,

e_ℓ : unit attractiveness cost of the competitor's new facility at site ℓ,

b_k : unit cost of increasing and unit revenue of decreasing the attractiveness of the competitor's existing facility at site k,

f_i : firm's annualized fixed cost of opening a facility at site i,

$\widetilde{f}_\ell$: competitor's annualized fixed cost of opening a facility at site ℓ,

t_k : revenue of closing an existing facility at site k,

u_i : maximum attractiveness level of the firm's new facility at site i,

$\overline{M}_\ell$: maximum attractiveness level of the competitor's new facility at site ℓ,

$\overline{A}_k$: maximum attractiveness level of the competitor's existing facility at site k,

$\underline{A}_k$: current attractiveness level of the competitor's existing facility at site k,

d_{ij} : Euclidean distance between the firm's candidate site i and point j,

$\widehat{d}_{\ell j}$: Euclidean distance between the competitor's candidate site ℓ and point j,

$\widetilde{d}_{kj}$: Euclidean distance between the competitor's existing facility at site k and point j.

Decision variables:

X_i : binary variable which is equal to one if a facility is opened at site i, and zero otherwise,

Q_i : attractiveness of the firm's facility opened at site i,

Z_k : binary variable which is equal to one if competitor's existing facility at site k is kept open, and zero otherwise,

A_k : new attractiveness level of competitor's existing facility at site k,

Y_ℓ : binary variable which is equal to one if the competitor opens a new facility at site ℓ, and zero otherwise,

M_ℓ : attractiveness level of competitor's new facility at site ℓ.

By the same reasoning given in Chapter 3 and 4, the probability P_{ij} that customers at point j visits a new facility of the firm at site i can be written as

$$P_{ij} = \frac{\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + \sum\limits_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right) + \sum\limits_{l=1}^{r_2}\left(M_l/\widehat{d}_{lj}^2\right)}. \tag{5.1}$$

Hence, the total revenue captured by the new facilities can be given as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} h_j P_{ij} = \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + \sum\limits_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right) + \sum\limits_{l=1}^{r_2}\left(M_l/\widehat{d}_{lj}^2\right)}. \tag{5.2}$$

Similarly, the total revenue captured by the existing and new facilities of the follower can be computed as

$$\sum_{j=1}^{n} h_j \frac{\sum\limits_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum\limits_{l=1}^{r_2}\left(M_l/\widehat{d}_{lj}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum\limits_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum\limits_{l=1}^{r_2}\left(M_l/\widehat{d}_{lj}^2\right)}. \tag{5.3}$$

Now we can formulate the problem as the following bilevel MINLP model:

$$\text{BP2} : \max_{\mathbf{Q},\mathbf{X}} \ \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum\limits_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum\limits_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)} - \sum_{i=1}^{m} c_i Q_i - \sum_{i=1}^{m} f_i X_i \tag{5.4}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{5.5}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{5.6}$$

$$X_i \in \{0,1\} \qquad i = 1, \ldots, m \tag{5.7}$$

$$\begin{aligned}\max_{\mathbf{A},\mathbf{M},\mathbf{Z},\mathbf{Y}} \sum_{j=1}^{n} h_j & \frac{\sum\limits_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum\limits_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}{\sum\limits_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum\limits_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum\limits_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)} + \sum_{k=1}^{r_1} t_k(1-Z_k) \\ & - \sum_{k=1}^{r_1} b_k(A_k - \underline{A}_k Z_k) - \sum_{\ell=1}^{r_2} e_\ell M_\ell - \sum_{\ell=1}^{r_2} \widetilde{f}_\ell Y_\ell \end{aligned} \tag{5.8}$$

s.t.

$$A_k \leq \overline{A}_k Z_k \qquad k = 1, \ldots, r_1 \tag{5.9}$$

$$A_k \geq 0 \qquad k = 1, \ldots, r_1 \tag{5.10}$$

$$M_\ell \leq \overline{M}_\ell Y_\ell \qquad \ell = 1, \ldots, r_2 \tag{5.11}$$

$$M_\ell \geq 0 \qquad \ell = 1, \ldots, r_2 \tag{5.12}$$

$$Z_k \in \{0,1\} \qquad k = 1, \ldots, r_1 \tag{5.13}$$

$$Y_\ell \in \{0,1\} \qquad \ell = 1, \ldots, r_2 \tag{5.14}$$

The objective function (5.4) of the firm has three components. The first one represents the revenue collected by the new facilities that are opened, while the second and third components represent the fixed cost and attractiveness cost associated with opening the new facilities, respectively. Constraints (5.5) along with the binary restrictions (5.7) on the location variables X_i and nonnegativity restrictions (5.6) on attractiveness variables Q_i ensure that if no facility is opened at site i, then the corresponding attractiveness Q_i of the facility is zero and if a facility is opened at site i, then its attractiveness Q_i cannot exceed the maximum level u_i. We note that the number of facilities to be located is not fixed as in the previous CFL models, its value is to be determined by the solution of the model. The objective function of the competitor (3.23) is comprised of five terms: the first one represents the revenue collected by the existing and new facilities; the second one shows the revenue which occurs when existing facilities are closed; the third one represents the costs or revenue associated with adjusting attractiveness levels of existing facilities; finally the last two terms indicate the costs associated with opening new facilities. Constraints (3.24) and (3.25) ensure that the new attractiveness A_k of an existing facility at site k will be between zero and an upper bound $\overline{A}_k$. If $A_k = \underline{A}_k$ in an optimal solution, then it means that the attractiveness level of the facility at site k is not adjusted, but the facility is kept open. In such a case, no attractiveness cost is incurred since $A_k - \underline{A}_k Z_k = 0$. On the other hand, if $\underline{A}_k < A_k \leq \overline{A}_k$, a cost of $b_k \left(A_k - \underline{A}_k\right)$ is obtained, and if $0 \leq A_k < \underline{A}_k$, a revenue of $b_k \left(\underline{A}_k - A_k\right)$ is gained. If the existing facility at site k is closed, the values of variables A_k and Z_k are equal to zero and no attractiveness cost incurs. Constraints (5.11) along with constraints (5.12) ensure that if no facility is opened at site ℓ, then the corresponding attractiveness M_ℓ of the facility is zero and if a facility is opened at site ℓ, then its attractiveness M_ℓ cannot exceed the maximum level $\overline{M}_\ell$. Finally, constraints (5.13) and (5.14) are the binary restrictions on competitor's location variables, and constraints (5.10) and (5.12) are the nonnegativity restrictions on its attractiveness variables A_k and M_ℓ.

The proposed solution procedures exploit a property of the objective function (5.8) of the competitor, namely it is concave in the attractiveness variables A_k and M_ℓ for $A_k \geq 0$ and $M_\ell \geq 0$. Besides the objective function (5.4) of the leader is concave in its attractiveness variables Q_i for $Q_i \geq 0$, which is already proven in Chapter 3. Before proceeding with the details of the solution procedures in the next section we show these properties with the following propositions.

Proposition 5.1. $\sum_{j=1}^{n} h_j \frac{\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}$ *is concave in* $\mathbf{A}$ *and* $\mathbf{M}$ *for* $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$, *when* $\mathbf{Q} = (Q_1, Q_2, ..., Q_m)$, $\mathbf{Z} = (Z_1, Z_2, ..., Z_{r_1})$, *and* $\mathbf{Y} = (Y_1, Y_2, ..., Y_{r_2})$ *are fixed.*

Proof. Since the sum of concave functions is a concave function, it suffices to show that each of the terms $g_j(\mathbf{A},\mathbf{M}) = h_j \frac{\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}$ is concave for $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$ for every $j = 1, 2, ..., n$. $g_j(\mathbf{A},\mathbf{M})$ is concave for $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$, if its Hessian matrix $H_j(\mathbf{A},\mathbf{M})$ is negative semidefinite for $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$. Showing the negative semidefiniteness of $H_j(\mathbf{A},\mathbf{M})$ is equivalent to showing $\mathbf{V}^T H_j(\mathbf{A},\mathbf{M})\mathbf{V} \leq 0$ for any $\mathbf{V}$ when $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$. The first and second order derivatives of $g_j(\mathbf{A},\mathbf{M})$ are given, respectively, as

$$\frac{\partial g_j(\mathbf{A},\mathbf{M})}{\partial A_p} = h_j \frac{\frac{1}{\widetilde{d}_{pj}^2}\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\left(\sum_{i=1}^{m}(Q_i/d_{ij}^2)+\sum_{k=1}^{r_1}(A_k/\widetilde{d}_{kj}^2)+\sum_{\ell=1}^{r_2}(M_\ell/\widehat{d}_{\ell j}^2)\right)^2},$$

$$\frac{\partial g_j(\mathbf{A},\mathbf{M})}{\partial M_q} = h_j \frac{\frac{1}{\widehat{d}_{qj}^2}\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\left(\sum_{i=1}^{m}(Q_i/d_{ij}^2)+\sum_{k=1}^{r_1}(A_k/\widetilde{d}_{kj}^2)+\sum_{\ell=1}^{r_2}(M_\ell/\widehat{d}_{\ell j}^2)\right)^2},$$

$$\frac{\partial g_j(\mathbf{A},\mathbf{M})}{\partial A_p \partial A_q} = -2h_j \frac{\frac{1}{\widetilde{d}_{pj}^2\widetilde{d}_{qj}^2}\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\left(\sum_{i=1}^{m}(Q_i/d_{ij}^2)+\sum_{k=1}^{r_1}(A_k/\widetilde{d}_{kj}^2)+\sum_{\ell=1}^{r_2}(M_\ell/\widehat{d}_{\ell j}^2)\right)^3},$$

$$\frac{\partial g_j\left(\mathbf{A},\mathbf{M}\right)}{\partial M_p \partial M_q} = -2h_j \frac{\frac{1}{\widehat{d}_{pj}^2 \widehat{d}_{qj}^2} \sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\left(\sum_{i=1}^{m}(Q_i/d_{ij}^2) + \sum_{k=1}^{r_1}(A_k/\widetilde{d}_{kj}^2) + \sum_{\ell=1}^{r_2}(M_\ell/\widehat{d}_{\ell j}^2)\right)^3},$$

$$\frac{\partial g_j\left(\mathbf{A},\mathbf{M}\right)}{\partial M_q \partial A_p} = -2h_j \frac{\frac{1}{\widehat{d}_{qj}^2 \widetilde{d}_{pj}^2} \sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\left(\sum_{i=1}^{m}(Q_i/d_{ij}^2) + \sum_{k=1}^{r_1}(A_k/\widetilde{d}_{kj}^2) + \sum_{\ell=1}^{r_2}(M_\ell/\widehat{d}_{\ell j}^2)\right)^3}.$$

Defining $\eta_j = 2h_j \frac{\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\left(\sum_{i=1}^{m}(Q_i/d_{ij}^2) + \sum_{k=1}^{r_1}(A_k/\widetilde{d}_{kj}^2) + \sum_{\ell=1}^{r_2}(M_\ell/\widehat{d}_{\ell j}^2)\right)^3}$, the (p,q)th entry of the Hessian matrix $H_j(\mathbf{A},\mathbf{M})$ of $g_j(\mathbf{A},\mathbf{M})$ becomes

$$[H_j(\mathbf{A},\mathbf{M})]_{p,p} = \begin{cases} -\frac{\eta_j}{\widetilde{d}_{pj}^4}, & \text{if } 1 \le p \le r_1 \\ -\frac{\eta_j}{\widehat{d}_{pj}^4}, & \text{if } r_1 < p \le r_1 + r_2 \end{cases}$$

$$[H_j(\mathbf{A},\mathbf{M})]_{p,q} = \begin{cases} -\frac{\eta_j}{\widetilde{d}_{pj}^2 \widetilde{d}_{qj}^2}, & \text{for } p \ne q \text{ and } 1 \le p,q \le r_1 \\ -\frac{\eta_j}{\widetilde{d}_{pj}^2 \widehat{d}_{qj}^2}, & \text{for } p \ne q,\ 1 \le p \le r_1, \text{ and } r_1+1 \le q \le r_1+r_2 \\ -\frac{\eta_j}{\widehat{d}_{pj}^2 \widetilde{d}_{qj}^2}, & \text{for } p \ne q,\ r_1 < p \le r_1+r_2 \text{ and } 1 \le q \le r_1 \\ -\frac{\eta_j}{\widehat{d}_{pj}^2 \widehat{d}_{qj}^2}, & \text{for } p \ne q \text{ and } r_1 < p,q \le r_1+r_2. \end{cases}$$

We remark that $\eta_j \ge 0$ for $(\mathbf{A},\mathbf{M}) \ge \mathbf{0}$ since $h_j \ge 0$ and $\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right) \ge 0$. $g_j(\mathbf{A},\mathbf{M})$ is concave if and only if $H_j(\mathbf{A},\mathbf{M})$ is negative semidefinite for all values of $(\mathbf{A},\mathbf{M}) \ge \mathbf{0}$. To show the latter, we consider $\mathbf{V}^T H_j(\mathbf{A},\mathbf{M})\mathbf{V}$ which can be expressed as

$$\mathbf{V}^T H_j(\mathbf{A},\mathbf{M})\mathbf{V} = -\eta_j \left(\frac{V_1}{\widetilde{d}_{1j}^2} + \frac{V_2}{\widetilde{d}_{2j}^2} + ... + \frac{V_{r_1}}{\widetilde{d}_{r_1j}^2} + \frac{V_{r_1+1}}{\widehat{d}_{1j}^2} + ... + \frac{V_{r_1+r_2}}{\widehat{d}_{r_2j}^2} \right)^2. \tag{5.15}$$

$\eta_j \geq 0$ for $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$ and $\left(\frac{V_1}{\widetilde{d}_{1j}^2} + ... + \frac{V_{r_1}}{\widetilde{d}_{r_1j}^2} + ... + \frac{V_{r_1+r_2}}{\widehat{d}_{r_2j}^2} \right)^2 \geq 0$ imply together that $\mathbf{V}^T H_j(\mathbf{A},\mathbf{M})\mathbf{V}$ ≤ 0 for any $\mathbf{V}$. This means that $H_j(\mathbf{A},\mathbf{M})$ is negative semidefinite, which proves the concavity of $g_j(\mathbf{A},\mathbf{M})$ for every $j = 1, 2, ..., n$. Hence

$\sum_{j=1}^{n} h_j \frac{\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}$ is concave in $(\mathbf{A},\mathbf{M})$ for $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$. □

Proposition 5.2. $\sum_{j=1}^{n} h_j \frac{\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)} + \sum_{k=1}^{r_1} t_k(1 - Z_k)$ $- \sum_{k=1}^{r_1} b_k(A_k - \underline{A}_k Z_k) - \sum_{\ell=1}^{r_2} e_\ell M_\ell - \sum_{\ell=1}^{r_2} \widetilde{f}_\ell Y_\ell$ *is concave in* $\mathbf{A}$ *and* $\mathbf{M}$ *for* $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$, *when* $\mathbf{Q} = (Q_1, Q_2, ..., Q_m)$, $\mathbf{Z} = (Z_1, Z_2, ..., Z_{r_1})$, *and* $\mathbf{Y} = (Y_1, Y_2, ..., Y_{r_2})$ *are fixed.*

Proof. The first term is concave for $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$ as a consequence of Proposition 5.1. and the rest of the terms are linear functions of $(\mathbf{A},\mathbf{M}) \geq \mathbf{0}$. The result follows since the sum of concave functions is also concave. □

Proposition 5.3. states that the firm's objective function given in (5.4) is also concave.

Proposition 5.3. $\sum_{j=1}^{n} h_j \frac{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)} - \sum_{i=1}^{m} c_i Q_i - \sum_{i=1}^{m} f_i X_i$ *is concave for* $\mathbf{Q} \geq \mathbf{0}$, *when* $\mathbf{A} = (A_1, A_2, ..., A_{r_1})$, $\mathbf{M} = (M_1, M_2, ..., M_{r_2})$, *and* $\mathbf{X} = (X_1, X_2, ..., X_m)$ *are fixed.*

Proof. We can show the (p, q)th entry of the Hessian matrix $H_j(\mathbf{Q})$ of the function $h_j \frac{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)+\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}_{kj}^2\right)+\sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}_{\ell j}^2\right)}$ is

$$[H_j(\mathbf{Q})]_{p,q} = \begin{cases} -\frac{\psi_j}{d_{pj}^4}, & \text{for } p = q \\ -\frac{\psi_j}{d_{pj}^2 d_{qj}^2}, & \text{for } p \neq q \end{cases}$$

for $1 \leq p, q \leq m$ where $\psi_j = 2h_j \frac{\sum_{k=1}^{r_1} (A_k/\widetilde{d}_{kj}^2) + \sum_{\ell=1}^{r_2} (M_\ell/\widehat{d}_{\ell j}^2)}{\left(\sum_{i=1}^{m} (Q_i/d_{ij}^2) + \sum_{k=1}^{r_1} (A_k/\widetilde{d}_{kj}^2) + \sum_{\ell=1}^{r_2} (M_\ell/\widehat{d}_{\ell j}^2)\right)^3}$. The rest of the proof follows the same steps as the proofs of Proposition 5.2. and Proposition 5.3. □

5.2. Solution Procedures

To find good feasible solutions to our bilevel problem BP2, we develop three tabu search (TS) heuristics, which perform a search in the upper level problem (ULP) over the location variables $\mathbf{X}$. TS is a metaheuristic algorithm that guides the local search to prevent it being trapped in premature local optima or in cycling (Glover and Laguna, 2007). In our implementation, this is managed by forbidding the locations of the firm's opened facilities in previous solutions so that they are not selected in later iterations. To evaluate the quality of the solutions obtained by the TS heuristics, we also devise an exact (ϵ-optimal) solution procedure which combines complete enumeration in terms of competitor's location variables $\mathbf{Y}$ and $\mathbf{Z}$ with the global optimization algorithm GMIN-αBB. It is computationally prohibitive and can only solve small problem instances in a reasonable CPU time.

5.2.1. First Tabu Search Heuristic

The first TS heuristic called TS-1 starts with an initial solution, and at each iteration the neighbors of the current solution are generated by executing three types of moves. After all possible neighboring solutions have been created, we check whether they are in the tabu list which consists of the location variables $\mathbf{X}$ and the attractiveness variables $\mathbf{Q}$ of the firm's facilities in the ULP. The best neighboring solution, namely the solution which provides the highest objective function value of the firm is selected as the next current solution if it is not in the tabu list. Both the incumbent and the tabu list are updated subsequently.

To find an initial solution, we calculate the average distance of each candidate facility site of the firm from the set of demand points and find their minimum. The candidate facility site $i^{'}$ with the minimum average distance is chosen as the location of a facility. Thus, the initial solution consists of only a single facility located at site $i^{'}$, i.e., $X_{i^{'}} = 1$ and $X_i = 0$ for all $i \neq i^{'}$. We set the attractiveness $Q_{i^{'}}$ arbitrarily to the maximum attractiveness level $u_{i^{'}}$. By fixing the attractiveness levels of the firm's facilities ($Q_{i^{'}} = u_{i^{'}}$ and $Q_i = 0, i \neq i^{'}$), we can solve the competitor's lower level problem (LLP) to global optimality using a branch-and-bound algorithm with NLP relaxation as a consequence of Proposition 5.2. At each node of the BB tree, we relax the binary restrictions of the location variables Y_k and Z_l, and solve a continuous NLP problem, which provides an upper bound for the LLP. If all Y_k and Z_l variables turn out to be binary in the relaxed solution, then we obtain a feasible solution for the LLP that constitutes a lower bound on the optimal objective value of the competitor. The relaxed LLP at the initial solution is as follows:

$$\max_{\mathbf{A},\mathbf{M},\mathbf{Z},\mathbf{Y}} \sum_{j=1}^{n} h_j \frac{\sum_{k=1}^{r_1}\left(A_k/\widetilde{d}^2_{kj}\right) + \sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}^2_{\ell j}\right)}{\left(u_{i^{'}}/d^2_{i^{'}j}\right) + \sum_{k=1}^{r_1}\left(A_k/\widetilde{d}^2_{kj}\right) + \sum_{\ell=1}^{r_2}\left(M_\ell/\widehat{d}^2_{\ell j}\right)} + \sum_{k=1}^{r_1} t_k(1 - Z_k)$$

$$- \sum_{k=1}^{r_1} b_k(A_k - \underline{A}_k Z_k) - \sum_{\ell=1}^{r_2} e_\ell M_\ell - \sum_{\ell=1}^{r_2} \widetilde{f}_\ell Y_\ell \tag{5.16}$$

s.t.

$$A_k \leq \overline{A}_k Z_k \qquad k = 1, \ldots, r_1 \tag{5.17}$$

$$A_k \geq 0 \qquad k = 1, \ldots, r_1 \tag{5.18}$$

$$M_\ell \leq \overline{M}_\ell Y_\ell \qquad \ell = 1, \ldots, r_2 \tag{5.19}$$

$$M_\ell \geq 0 \qquad \ell = 1, \ldots, r_2 \tag{5.20}$$

$$0 \leq Z_k \leq 1 \qquad k = 1, \ldots, r_1 \tag{5.21}$$

$$0 \leq Y_\ell \leq 1 \qquad \ell = 1, \ldots, r_2 \tag{5.22}$$

We employ KNITRO 6.0 (Waltz and Plantenga, 2009) to solve the NLPs. Branching at a node is carried out by considering the solution at that node and choosing one of the Y_k and Z_l

variables whose value is the closest to 0.5, i.e., the most fractional variable is selected as the branching variable. The pruning of the nodes is based on two rules: a node is pruned if either a feasible solution is obtained or the upper bound at that node is less than or equal to the best lower bound in the tree. The BB algorithm is utilized not only for the initial solution of TS-1, but also for any solution generated throughout the iterations. When the LLP is solved via the BB method, the locations of the competitor's facilities $\mathbf{Y}'$, $\mathbf{Z}'$ as well as the attractiveness levels $\mathbf{M}'$, $\mathbf{A}'$ of these facilities are found. As a result, the firm's objective value corresponding to the initial solution can be computed as $\sum_{j=1}^{n} h_j \frac{Q_{i'}/d^2_{i'j}}{\left(Q_{i'}/d^2_{i'j}\right)+\sum_{k=1}^{r1}\left(A_{k'}/\widetilde{d}^2_{k'j}\right)+\sum_{\ell=1}^{r2}\left(M_{\ell'}/\widehat{d}^2_{\ell'j}\right)} - c_i Q_{i'} - f_{i'}$, where $Q_{i'} = u_{i'}$ and $f_{i'}$ is the corresponding fixed cost. This BB method with NLP relaxation follows the same steps of the BB method which we propose in Subsection 3.2.3.

Neighboring solutions are generated from the current solution by executing 1-*Add*, 1-*Drop*, and 1-*Swap* moves. A 1-Add move opens a new facility at one of the candidate sites where no facility exists. A 1-Drop move closes an open facility and 1-Swap move closes an open facility and opens a new one at another candidate site without a facility. In order to calculate the firm's profit of a neighboring solution with known locations $\mathbf{X}$, we need to find attractiveness values $\mathbf{Q}$ of the firm and then determine the optimal reaction of the competitor using the BB algorithm with NLP relaxation. To this end, we randomly generate $\mathbf{Q}''$ corresponding to the open facilities in the neighboring solution such that $\sum_{i=1}^{m} \left|Q''_i - Q'_i\right| < \varepsilon$, where ε is a parameter. However, we also need to have rules for a facility added or removed in the neighboring solution. It is obvious that the attractiveness of a closed facility will be zero. The attractiveness of a new facility, on the other hand, is assigned a random value in the interval $[0, u_i]$.

To prevent cycling in TS-1 heuristic, we utilize a tabu list containing location variables $\mathbf{X}$ and the randomly generated attractiveness levels $\mathbf{Q}$ of the firm's facilities. In fact, there is no need to store the location variables $\mathbf{X}$ because an attractiveness level at value zero for a facility implies that there is no facility opened at that location and a positive attractiveness value implies that the corresponding location variable is one. If the values $\mathbf{X}$ of a newly generated neighboring solution coincide with those of a solution in the tabu list *and* the attractiveness values $\mathbf{Q}$ of the firm's facilities are within a hypercube of side length ρ whose center is the point defined by the $\mathbf{Q}$ values of the same solution in the tabu list, then this neighboring solution is declared as tabu.

At each iteration the best neighboring solution becomes the current solution for the next iteration. Moreover, if the firm's profit corresponding the best neighboring solution is higher than of the incumbent, the incumbent is updated as well. We use two termination criteria: the maximum number of iterations performed (*max_iter*) and the maximum number of iterations without an improvement in the incumbent (*max_nonimp_iter*).

5.2.2. Second Tabu Search Heuristic

Here, we only explain the differences of the second TS heuristic (TS-2) from TS-1. When a neighboring solution is created from the current solution by executing one of the 1-Add, 1-Drop, and 1-Swap moves, the attractiveness levels $\mathbf{Q}$ of the firm's facilities in the neighboring solution are not generated randomly, but obtained by the gradient ascent algorithm. Note that when the firm's location variables $\mathbf{X}$ in the *neighboring* solution and the attractiveness levels $\mathbf{A}$ and $\mathbf{M}$ of the competitor's facilities in the *current* solution are fixed, the firm's objective function in (5.4) is concave in terms of $\mathbf{Q} \geq \mathbf{0}$ as a result of Proposition 5.3. To find the solution of this concave maximization problem subject to the constraints $0 \leq Q_i \leq u_i$ we use the following necessary and sufficient conditions, which can be immediately deduced from the Karush-Kuhn-Tucker conditions (Bertsekas, 1995). $\mathbf{Q}^*$ is a global optimal solution of profit function Π if and only if

i) $\left.\frac{\partial \Pi(\mathbf{Q})}{\partial Q_i}\right|_{\mathbf{Q}^*} \leq 0$ when $Q_i^* = 0$,

ii) $\left.\frac{\partial \Pi(\mathbf{Q})}{\partial Q_i}\right|_{\mathbf{Q}^*} \geq 0$ when $Q_i^* = u_i$,

iii) $\left.\frac{\partial \Pi(\mathbf{Q})}{\partial Q_i}\right|_{\mathbf{Q}^*} = 0$ when $0 < Q_i^* < u_i$.

These conditions allow us to device a gradient ascent procedure that can find a global maximum of $\Pi(\mathbf{Q})$. Note that this procedure is the same gradient ascent algorithm given in Subsection 3.2.1. Starting from randomly chosen initial values $\mathbf{Q}^{(0)}$, a direction $\mathbf{e}^{(t)}$ and a step size $\mu^{(t)} = \arg\max_{\mu} \Pi(\mathbf{Q}^{(t)} + \mu \mathbf{e}^{(t)})$ are determined at each iteration of the procedure, which are used to update the value of $\mathbf{Q}$ at the next iteration, i.e., $\mathbf{Q}^{(t+1)} = \mathbf{Q}^{(t)} + \mu^{(t)}\mathbf{e}^{(t)}$. The iterations are repeated until the norm of the direction vector $\left\|\mathbf{e}^{(t)}\right\|$ is smaller than a user-specified value. The important issue here is the determination of the direction $\mathbf{e}^{(t)}$ and the step size $\mu^{(t)}$ at each iteration t. Since the decision variables $\mathbf{Q}$ have lower and upper bounds, the direction $\mathbf{e}^{(t)}$ is determined as follows:

i) $e_i^{(t)} = \frac{\partial\Pi(\mathbf{Q}^{(t)})}{\partial Q_i}$ when $0 < Q_i^{(t)} < u_i$ for $i = 1, 2, ..., m$

ii) $e_i^{(t)} = 0$ when $Q_i^{(t)} = 0$ and $\frac{\partial\Pi(\mathbf{Q}^{(t)})}{\partial Q_i} < 0$ for $i = 1, 2, ..., m$

iii) $e_i^{(t)} = \frac{\partial\Pi(\mathbf{Q}^{(t)})}{\partial Q_i}$ when $Q_i^{(t)} = 0$ and $\frac{\partial\Pi(\mathbf{Q}^{(t)})}{\partial Q_i} > 0$ for $i = 1, 2, ..., m$

iv) $e_i^{(t)} = 0$ when $Q_i^{(t)} = u_i$ and $\frac{\partial\Pi(\mathbf{Q}^{(t)})}{\partial Q_i} > 0$ for $i = 1, 2, ..., m$

v) $e_i^{(t)} = \frac{\partial\Pi(\mathbf{Q}^{(t)})}{\partial Q_i}$ when $Q_i^{(t)} = u_i$ and $\frac{\partial\Pi(\mathbf{Q}^{(t)})}{\partial Q_i} < 0$ for $i = 1, 2, ..., m$.

The step size calculation is carried out by applying golden section search (Press *et al.*, 1986) with the initial interval $[0, \mu_{max}]$, where μ_{max} is the maximum possible value for the step size μ to maintain the feasibility of vector $\mathbf{Q}$ with respect to its lower and upper bounds.

As a consequence of setting the values of the attractiveness levels $\mathbf{Q}$ of the firm's facilities, the tabu list in TS-2 should be constructed differently. Hence, we utilize a tabu list containing the locations $\mathbf{X}$ of the firm and the attractiveness levels $\mathbf{A}$ and $\mathbf{M}$ of the competitor encountered at the previous iterations. There is no need to store the location variables Y_k and Z_l of the competitor because an attractiveness level at value zero for a facility implies that there is no facility opened at that location and a positive attractiveness value implies that the corresponding location variable is one. After executing each move and generating the location vector $\mathbf{X}$ of the neighboring solution, the tabu list is checked to see whether $\mathbf{X}$ is already in the list. If it is not, then the neighboring solution is not tabu active. Otherwise, another check should be performed for the attractiveness levels $\mathbf{A}$ and $\mathbf{M}$ of the competitor. The solution is declared to be tabu active if the values of $\mathbf{A}$ and $\mathbf{M}$ are within a hypercube of side length ρ whose center is the point defined by the $\mathbf{A}$ and $\mathbf{M}$ values of the solution in the tabu list. The reason for keeping track of the values for $\mathbf{A}$ and $\mathbf{M}$ is that we obtain similar attractiveness levels $\mathbf{Q}$ of the firm's facilities for similar values of $\mathbf{A}$ and $\mathbf{M}$, which deteriorates the diversification in the search. As is the case in TS-1, TS-2 makes use of the BB method with NLP relaxation to compute the objective value of the LLP, i.e., the competitor's profit, given the location variables **X** and attractiveness variables **Q** of the firm's facilities. The pseudo codes for both TS heuristics (TS-1, TS-2) are given in the Appendices C and D.

5.2.3. Third Tabu Search Heuristic

TS-3 is the third TS heuristic we propose. It is simply TS-1 initiated at the solution generated by TS-2. This means that the solution TS-3 computes is at least as good as that of TS-2 heuristic.

5.2.4. An ϵ-Optimal Solution Method

It is necessary to fix the binary variables **Z** and **Y** in the LLP of the competitor to be able to use the GMIN-αBB algorithm for the solution of BP2. The combinations of the values that binary variables Z_k and Y_ℓ can take indicate all possible reactions of the competitor, i.e., keeping or closing existing facilities and opening new ones. Since the number of Z_k variables is r_1 and that of Y_ℓ variables is r_2, the competitor has a total of $2^{r_1+r_2}$ possible reactions to the leader firm in terms of locating facilities (excluding the decision about the attractiveness levels of these facilities). For example, one can think of a scenario in which the competitor has $r_1 = 2$ existing facilities and $r_2 = 2$ candidate facility sites. In such a case, the combination $Z_1 = 1$, $Z_2 = 0$, $Y_1 = 0$, $Y_2 = 1$ represents the situation in which the competitor keeps the first existing facility open, closes its second existing facility, and opens a new facility at the second candidate site only.

When the values of the binary variables **Z** and **Y** are fixed in the LLP, the attractiveness variables A_k and M_ℓ have zero values if the corresponding Z_k and Y_ℓ are zero. Thus they can be discarded from the LLP. Suppose that in an arbitrary reaction of the competitor, the number of Z_k (Y_ℓ) variables whose value is equal to one is o_1 (o_2) where $0 \leq o_1 \leq r_1$ ($0 \leq o_2 \leq r_2$). As a result, problem BP2 becomes a bilevel MINLP problem called BP2$'$ whose LLP is an NLP which contains only continuous attractiveness variables **A** and **M**. BP2$'$ can be formulated as

$$\text{BP2}' : \max_{\mathbf{Q},\mathbf{X}} \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\sum\limits_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum\limits_{k=1}^{o_1} \left(A_k/\widetilde{d}_{kj}^2\right) + \sum\limits_{\ell=1}^{o_2} \left(M_\ell/\widehat{d}_{\ell j}^2\right)} - \sum_{i=1}^{m} c_i Q_i - \sum_{i=1}^{m} f_i X_i \quad (5.23)$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \qquad (5.24)$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \qquad (5.25)$$

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \qquad (5.26)$$

$$\max_{\mathbf{A},\mathbf{M}} \sum_{j=1}^{n} h_j \frac{\sum\limits_{k=1}^{o_1} \left(A_k/\widetilde{d}_{kj}^2\right) + \sum\limits_{\ell=1}^{o_2} \left(M_\ell/\widehat{d}_{\ell j}^2\right)}{\sum\limits_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum\limits_{k=1}^{o_1} \left(A_k/\widetilde{d}_{kj}^2\right) + \sum\limits_{\ell=1}^{o_2} \left(M_\ell/\widehat{d}_{\ell j}^2\right)} - \sum_{k=1}^{o_1} b_k(A_k - \underline{A}_k)$$
$$- \sum_{\ell=1}^{o_2} e_\ell M_\ell \qquad (5.27)$$

s.t.

$$A_k \leq \overline{A}_k \qquad k = 1, \ldots, o_1 \qquad (5.28)$$

$$A_k \geq 0 \qquad k = 1, \ldots, o_1 \qquad (5.29)$$

$$M_\ell \leq \overline{M}_\ell \qquad \ell = 1, \ldots, o_2 \qquad (5.30)$$

$$M_\ell \geq 0 \qquad \ell = 1, \ldots, o_2 \qquad (5.31)$$

Note that the second summation term $\sum_{k=1}^{r_1} t_k(1-Z_k)$ representing the revenue obtained by closing the existing facilities and the last term $\sum_{\ell=1}^{r_2} \widetilde{f}_\ell Y_\ell$ corresponding to the fixed cost of new facilities in the competitor's objective function (5.8) of BP1 do not exist in (5.27) because they become constants after setting $\mathbf{Z}$ and $\mathbf{Y}$. Then it becomes possible to obtain the optimal solution to BP2 by solving BP2$'$ using the GMIN-αBB method explained below for each of the $2^{r_1+r_2}$ combinations.

It is important to emphasize that the solutions of BP2$'$ will not be bilevel feasible to the original problem BP2 for some of the combinations. This means that when we fix the location variables $\mathbf{X}$ and attractiveness variables $\mathbf{Q}$ obtained by solving BP2$'$ in the ULP of BP2, and solve the competitor's LLP using the BB algorithm with NLP relaxation (as described in Subsection 5.2.1), the optimal values of $\mathbf{Z}$ and $\mathbf{Y}$ may not coincide with the values of these variables that are fixed in solving BP2$'$. Moreover, even if the optimal values of $\mathbf{Z}$ and $\mathbf{Y}$ in the solution of the LLP of BP2 coincide with the values fixed in the combination, the optimal values of the attractiveness variables $\mathbf{A}$ and $\mathbf{M}$ may not be the same. In these cases, the solutions generated for BP2$'$ with the given combination will be bilevel infeasible.

Now, we can give the details of the GMIN-αBB method used to solve BP2$'$. The application of GMIN-αBB requires that the BP2 model of BP2$'$ be transformed into an equivalent one-level model.

<u>5.2.4.1. Transformation of the Bilevel Model Into An Equivalent One-Level Model.</u> As Proposition 5.2. states, the competitor's problem in BP2$'$ is a concave maximization problem in terms of the continuous attractiveness variables $(\mathbf{A}, \mathbf{M}) \geq \mathbf{0}$. Thus, a necessary and sufficient condition for $(\mathbf{A}, \mathbf{M})$ to be an optimal solution is that there exist nonnegative Lagrange multiplier vectors $\boldsymbol{\lambda_1} = (\lambda_{11}, ..., \lambda_{1o_1})$, $\boldsymbol{\lambda_2} = (\lambda_{21}, ..., \lambda_{2o_1})$, $\boldsymbol{\lambda_3} = (\lambda_{31}, ..., \lambda_{3o_2})$, and $\boldsymbol{\lambda_4} = (\lambda_{41}, ..., \lambda_{4o_2})$ which satisfy the following KKT optimality conditions:

$$\sum_{j=1}^{n} h_j \frac{(1/\widetilde{d}_{kj}^2) \sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\left[\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{o_1} \left(A_k/\widetilde{d}_{kj}^2\right) + \sum_{\ell=1}^{o_2} \left(M_\ell/\widehat{d}_{\ell j}^2\right)\right]^2} - b_k + \lambda_{1k} - \lambda_{2k} = 0 \qquad k = 1, \ldots, o_1$$

$$\sum_{j=1}^{n} h_j \frac{(1/\widehat{d}_{\ell j}^2) \sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\left[\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{o_1} \left(A_k/\widetilde{d}_{kj}^2\right) + \sum_{\ell=1}^{o_2} \left(M_\ell/\widehat{d}_{\ell j}^2\right)\right]^2} - e_\ell + \lambda_{3\ell} - \lambda_{4\ell} = 0 \qquad k = 1, \ldots, o_2$$

$$A_k - \overline{A}_k + S_{1k} = 0 \qquad k = 1, ..., o_1$$

$$-A_k + S_{2k} = 0 \qquad k = 1, \ldots, o_1$$

$$M_\ell - \overline{M}_\ell + S_{3\ell} = 0 \qquad \ell = 1, \ldots, o_2$$

$$-M_\ell + S_{4\ell} = 0 \qquad \ell = 1, \ldots, o_2$$

$$\lambda_{1k} S_{1k} = 0 \qquad k = 1, \ldots, o_1$$

$$\lambda_{2k} S_{2k} = 0 \qquad k = 1, \ldots, o_1$$

$$\lambda_{3l} S_{3\ell} = 0 \qquad \ell = 1, \ldots, o_2$$

$$\lambda_{4l} S_{4\ell} = 0 \qquad \ell = 1, \ldots, o_2$$

$$\lambda_{1k}, \lambda_{2k}, \lambda_{3\ell}, \lambda_{4\ell}, S_{1k}, S_{2k}, S_{3\ell}, S_{4\ell} \geq 0 \qquad k = 1, \ldots, o_1,$$

$$\ell = 1, \ldots, o_2.$$

Here, $\mathbf{S}_1 = (S_{11}, S_{12}, \ldots, S_{1o_1})$, $\mathbf{S}_2 = (S_{21}, S_{22}, \ldots, S_{2o_1})$, $\mathbf{S}_3 = (S_{31}, S_{32}, \ldots, S_{3o_2})$, and $\mathbf{S}_4 = (S_{41}, S_{42}, \ldots, S_{4o_2})$ are nonnegative slack variables introduced for the constraint sets (5.28), (5.29), (5.30), and (5.31), respectively. Six of the KKT optimality conditions, namely the first two and the last four, are nonlinear in nature. However, it is possible to linearize the last four conditions using the active set strategy again by introducing auxiliary binary variables $\{V_{1k}, V_{2k} : k = 1, \ldots, o_1\}$, $\{V_{3\ell}, V_{4\ell} : \ell = 1, \ldots, o_2\}$ and an upper bound θ (its determination is explained in the next section) on the slack variables $\mathbf{S}$ as follows:

$$
\begin{aligned}
&\lambda_{1k} - \theta V_{1k} \leq 0 && k = 1, \ldots, o_1 \\
&S_{1k} - \theta(1 - V_{1k}) \leq 0 && k = 1, \ldots, o_1 \\
&\lambda_{2k} - \theta V_{2k} \leq 0 && k = 1, \ldots, o_1 \\
&S_{2k} - \theta(1 - V_{2k}) \leq 0 && k = 1, \ldots, o_1 \\
&\lambda_{3\ell} - \theta V_{3\ell} \leq 0 && \ell = 1, \ldots, o_2 \\
&S_{3\ell} - \theta(1 - V_{3\ell}) \leq 0 && \ell = 1, \ldots, o_2 \\
&\lambda_{4\ell} - \theta V_{4\ell} \leq 0 && \ell = 1, \ldots, o_2 \\
&S_{4\ell} - \theta(1 - V_{4\ell}) \leq 0 && \ell = 1, \ldots, o_2 \\
&V_{1k}, V_{2k}, V_{3\ell}, V_{4\ell} \in \{0, 1\}
\end{aligned}
$$

If any one of the $(V_{1k}, V_{2k}, V_{3\ell}, V_{4\ell})$ variables is equal to one, then the corresponding slack variable S is zero. Otherwise, the corresponding Lagrange multiplier is zero. The resulting formulation of BP2$'$, where the LLP of the competitor is replaced by the KKT optimality conditions augmented with the active set strategy, is a (one-level) MINLP problem with the nonlinearities occurring in the objective function and the first two KKT optimality conditions. We can solve this problem by the GMIN-αBB algorithm that is explained in Subsection 4.2.1. Furthermore, note that this transformation of the bilevel model BP2$'$ into an equivalent one-level model follows the same track that of the bilevel model BP1$'$ given in Subsection 4.2.1.

In order to get rid of general non-concave terms like we did in Subsection 4.2.1 and avoid applying the α calculations, we define variables

$$w_{1j} = \frac{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{w_{2j}}$$

and

$$w_{2j} = \sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{o_1} \left(A_k/\widetilde{d}_{kj}^2\right) + \sum_{\ell=1}^{o_2} \left(M_\ell/\widehat{d}_{\ell j}^2\right)$$

for $j = 1, \ldots, n$. The new variables help to convert the general non-concave terms in the objective function as well as in the first two KKT optimality conditions into the following bilinear and fractional terms.

$$\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) = w_{1j} w_{2j},$$

and

$$\frac{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\left[\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{o_1} \left(A_k/\widetilde{d}_{kj}^2\right) + \sum_{\ell=1}^{o_2} \left(M_\ell/\widehat{d}_{\ell j}^2\right)\right]^2} = \frac{w_{1j}}{w_{2j}}$$

for $j = 1, \ldots, n$. Then we obtain the following formulation BP2$''$.

$$\text{BP2}'' : \max \ \sum_{j=1}^{n} h_j w_{1j} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{5.32}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{5.33}$$

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \tag{5.34}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{5.35}$$

$$\sum_{j=1}^{n} h_j \frac{1}{\widetilde{d}_{kj}^2} \frac{w_{1j}}{w_{2j}} - b_k + \lambda_{1k} - \lambda_{2k} = 0 \qquad k = 1, \ldots, o_1 \tag{5.36}$$

$$\sum_{j=1}^{n} h_j \frac{1}{\widehat{d}_{\ell j}^2} \frac{w_{1j}}{w_{2j}} - e_\ell + \lambda_{3\ell} - \lambda_{4\ell} = 0 \qquad \ell = 1, \ldots, o_2 \tag{5.37}$$

$$A_k - \overline{A}_k + S_{1k} = 0 \qquad k = 1, \ldots, o_1 \tag{5.38}$$

$$-A_k + S_{2k} = 0 \qquad k = 1, \ldots, o_1 \tag{5.39}$$

$$M_\ell - \overline{M}_\ell + S_{3\ell} = 0 \qquad \ell = 1, \ldots, o_2 \tag{5.40}$$

$$-M_\ell + S_{4\ell} = 0 \qquad \ell = 1, \ldots, o_2 \tag{5.41}$$

$$\lambda_{1k} - \theta V_{1k} \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.42}$$

$$S_{1k} - \theta(1 - V_{1k}) \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.43}$$

$$\lambda_{2k} - \theta V_{2k} \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.44}$$

$$S_{2k} - \theta(1 - V_{2k}) \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.45}$$

$$\lambda_{3\ell} - \theta V_{3\ell} \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.46}$$

$$S_{3\ell} - \theta(1 - V_{3\ell}) \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.47}$$

$$\lambda_{4\ell} - \theta V_{4\ell} \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.48}$$

$$S_{3\ell} - \theta(1 - V_{4\ell}) \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.49}$$

$$\sum_{i=1}^{m} (Q_i / d_{ij}^2) - w_{1j} w_{2j} = 0 \qquad j = 1, \ldots, n \tag{5.50}$$

$$\sum_{i=1}^{m} (Q_i / d_{ij}^2) + \sum_{k=1}^{o_1} (A_k / \widetilde{d}_{kj}^2) + \sum_{\ell=1}^{o_2} (M_\ell / \widehat{d}_{\ell j}^2) - w_{2j} = 0 \qquad j = 1, \ldots, n \tag{5.51}$$

$$\lambda_{1k}, \lambda_{2k}, S_{1k}, S_{2k} \geq 0, V_{1k}, V_{2k} \in \{0,1\} \qquad k = 1, \ldots, o_1 \tag{5.52}$$

$$\lambda_{3\ell}, \lambda_{4\ell}, S_{3\ell}, S_{4\ell} \geq 0, V_{3l}, V_{4l} \in \{0,1\} \qquad \ell = 1, \ldots, o_2 \tag{5.53}$$

$$w_{1j}, w_{2j} \geq 0 \qquad j = 1, \ldots, n \tag{5.54}$$

5.2.4.2. Solution of the One-Level Model. Given that BP2$''$ is in a form that is suitable for the GMIN-αBB, we can relax all binary variables $\mathbf{X} = (X_1, \ldots, X_m)$, $\mathbf{V}_1 = (V_{11}, \ldots, V_{1o_1})$, $\mathbf{V}_2 = (V_{21}, \ldots, V_{2o_1})$, $\mathbf{V}_3 = (V_{31}, \ldots, V_{3o_2})$, and $\mathbf{V}_4 = (V_{41}, \ldots, V_{4o_2})$ in the interval $[0,1]$ and the αBB algorithm is then employed for solving the resulting relaxed (continuous) NLP problem BP2$''$. The rules for pruning a node and branching at a node are the same as the ones of the αBB algorithm given in Subsection 4.2.2. Branching occurs again only on the variables which participate in non-concave terms, namely w_{1j} and w_{2j} for $j = 1, 2, ..., n$.

We define $w_{3j} = w_{1j}w_{2j}$ and $w_{4j} = w_{1j}/w_{2j}$ for $j = 1, 2, ..., n$. The overestimation of bilinear terms is accomplished by including the following constraints in BP2$''$

$$\begin{aligned}
w_{1j}^L w_{2j} + w_{2j}^L w_{1j} - w_{1j}^L w_{2j}^L - w_{3j} &\leq 0 && j = 1, 2, ..., n \\
w_{1j}^U w_{2j} + w_{2j}^U w_{1j} - w_{1j}^U w_{2j}^U - w_{3j} &\leq 0 && j = 1, 2, ..., n \\
-w_{1j}^U w_{2j} - w_{2j}^L w_{1j} + w_{1j}^U w_{2j}^L + w_{3j} &\leq 0 && j = 1, 2, ..., n \\
-w_{1j}^L w_{2j} - w_{2j}^U w_{1j} + w_{1j}^L w_{2j}^U + w_{3j} &\leq 0 && j = 1, 2, ..., n,
\end{aligned}$$

while the two constraints below help to overestimate the fractional terms.

$$\begin{aligned}
w_{1j}^L / w_{2j} + w_{1j} / w_{2j}^U - w_{1j}^L / w_{2j}^U - w_{4j} &\leq 0 && j = 1, 2, ..., n \\
w_{1j}^U / w_{2j} + w_{1j} / w_{2j}^L - w_{1j}^U / w_{2j}^L - w_{4j} &\leq 0 && j = 1, 2, ..., n.
\end{aligned}$$

After defining the new variables w_{3j}, w_{4j} and adding the above constraints the concavified problem BP2$'''$ is obtained.

$$\text{BP2}''' : \max \sum_{j=1}^{n} h_j w_{1j} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{5.55}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{5.56}$$

$$Q_i \geq 0, 0 \leq X_i \leq 1 \qquad i = 1, \ldots, m \tag{5.57}$$

$$\sum_{j=1}^{n} h_j \frac{1}{\widetilde{d}_{kj}^2} w_{4j} - b_k + \lambda_{1k} - \lambda_{2k} = 0 \qquad k = 1, \ldots, o_1 \tag{5.58}$$

$$\sum_{j=1}^{n} h_j \frac{1}{\widetilde{d}_{\ell j}^2} w_{4j} - e_\ell + \lambda_{3\ell} - \lambda_{4\ell} = 0 \qquad \ell = 1, \ldots, o_2 \tag{5.59}$$

$$A_k - \overline{A}_k + S_{1k} = 0 \qquad k = 1, \ldots, o_1 \tag{5.60}$$

$$-A_k + S_{2k} = 0 \qquad k = 1, \ldots, o_1 \tag{5.61}$$

$$M_\ell - \overline{M}_\ell + S_{3\ell} = 0 \qquad \ell = 1, \ldots, o_2 \tag{5.62}$$

$$-M_\ell + S_{4\ell} = 0 \qquad \ell = 1, \ldots, o_2 \tag{5.63}$$

$$\lambda_{1k} - \theta V_{1k} \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.64}$$

$$S_{1k} - \theta(1 - V_{1k}) \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.65}$$

$$\lambda_{2k} - \theta V_{2k} \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.66}$$

$$S_{2k} - \theta(1 - V_{2k}) \leq 0 \qquad k = 1, \ldots, o_1 \tag{5.67}$$

$$\lambda_{3\ell} - \theta V_{3\ell} \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.68}$$

$$S_{3\ell} - \theta(1 - V_{3\ell}) \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.69}$$

$$\lambda_{4\ell} - \theta V_{4\ell} \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.70}$$

$$S_{4\ell} - \theta(1 - V_{4\ell}) \leq 0 \qquad \ell = 1, \ldots, o_2 \tag{5.71}$$

$$\sum_{i=1}^{m} (Q_i / d_{ij}^2) - w_{3j} = 0 \qquad j = 1, \ldots, n \tag{5.72}$$

$$\sum_{i=1}^{m} (Q_i / d_{ij}^2) + \sum_{k=1}^{o_1} (A_k / \widetilde{d}_{kj}^2) + \sum_{\ell=1}^{o_2} (M_\ell / \widehat{d}_{\ell j}^2) - w_{2j} = 0 \qquad j = 1, \ldots, n \tag{5.73}$$

$$w_{1j}^L w_{2j} + w_{2j}^L w_{1j} - w_{1j}^L w_{2j}^L - w_{3j} \leq 0 \qquad j = 1, \ldots, n \qquad (5.74)$$

$$w_{1j}^U w_{2j} + w_{2j}^U w_{1j} - w_{1j}^U w_{2j}^U - w_{3j} \leq 0 \qquad j = 1, \ldots, n \qquad (5.75)$$

$$-w_{1j}^U w_{2j} - w_{2j}^L w_{1j} + w_{1j}^U w_{2j}^L + w_{3j} \leq 0 \qquad j = 1, \ldots, n \qquad (5.76)$$

$$-w_{1j}^L w_{2j} - w_{2j}^U w_{1j} + w_{1j}^L w_{2j}^U + w_{3j} \leq 0 \qquad j = 1, \ldots, n \qquad (5.77)$$

$$w_{1j}^L / w_{2j} + w_{1j} / w_{2j}^U - w_{1j}^L / w_{2j}^U - w_{4j} \leq 0 \qquad j = 1, \ldots, n \qquad (5.78)$$

$$w_{1j}^U / w_{2j} + w_{1j} / w_{2j}^L - w_{1j}^U / w_{2j}^L - w_{4j} \leq 0 \qquad j = 1, \ldots, n \qquad (5.79)$$

$$\lambda_{1k}, \lambda_{2k}, s_{1k}, s_{2k} \geq 0 \qquad k = 1, \ldots, o_1 \qquad (5.80)$$

$$\lambda_{3\ell}, \lambda_{4\ell}, s_{3\ell}, s_{4\ell} \geq 0 \qquad \ell = 1, \ldots, o_2 \qquad (5.81)$$

$$0 \leq V_{1k}, V_{2k} \leq 1 \qquad k = 1, \ldots, o_1 \qquad (5.82)$$

$$0 \leq V_{3\ell}, V_{4\ell} \leq 1 \qquad \ell = 1, \ldots, o_2 \qquad (5.83)$$

$$0 \leq w_{1j} \leq 1 \qquad j = 1, \ldots, n \qquad (5.84)$$

$$0 \leq w_{2j} \leq \sum_{i=1}^{m} \left(u_i / d_{ij}^2\right) + \sum_{k=1}^{o_1} (\overline{A}_k / \widetilde{d}_{kj}^2) + \sum_{\ell=1}^{o_2} (\overline{M}_\ell / \widehat{d}_{\ell j}^2) \qquad j = 1, \ldots, n \qquad (5.85)$$

The lower bounds of the decision variables excluding w_{ij} are zero. Their upper bounds (UB) are given in Table 5.1, where θ is an upper bound on the slack variables. Hence, θ is chosen to be the maximum of $\overline{A}_k$ and $\overline{M}_\ell$. The lower and upper bounds of w_{1j} and w_{2j} can be set as $w_{1j}^L = w_{2j}^L = 0$, $w_{1j}^U = 1$, and $w_{2j}^U = \sum_{i=1}^{m} \left(u_i / d_{ij}^2\right) + \sum_{k=1}^{o_1} \left(\overline{A}_k / \widetilde{d}_{kj}^2\right) + \sum_{\ell=1}^{o_2} \left(\overline{M}_\ell / \widehat{d}_{\ell j}^2\right)$. The bounds on w_{3j} and w_{4j} requires the solution of the optimization problems with objective functions $\{\min w_{ij}\}$ and $\{\max w_{ij}\}$ for $i = 3, 4$ and constraints of problem BP2$''$. The concavified problem BP2$'''$ is then solved using the GMIN-αBB algorithm which is given in detail in Subsection 4.2.2 and Appendix B.

Table 5.1. Upper bounds on the decision variables in the αBB algorithm.

Variable	Q_i	X_i	A_k	M_l	S_{1k}	S_{2k}	$S_{3\ell}$	$S_{4\ell}$	λ_{1k}	λ_{2k}	$\lambda_{3\ell}$	$\lambda_{4\ell}$	V_{1k}	V_{2k}	$V_{3\ell}$	$V_{4\ell}$
UB	u_i	1	$\overline{A}_k$	$\overline{M}_\ell$	$\overline{A}_k$	$\overline{A}_k$	$\overline{M}_\ell$	$\overline{M}_\ell$	θ	θ	θ	θ	1	1	1	1

6. A DISCRETE FACILITY LOCATION PROBLEM WITH CUSTOMER PREFERENCES

So far we have only considered CFL problems that employ the gravity-based rule to model customer behavior. In contrast to these CFL models where customer preferences are not taken into account, the model proposed in this chapter incorporates the customer preferences by means of visiting probabilities. While in CFL problems the visiting probability is assumed to increase with the facility attractiveness, in real-life it may be the case that each different type of facility has its own customer segment. As an example, we can consider Migros Ticaret Inc., one of the major supermarket chains in Turkey. There exist different types of Migros supermarket stores available in Turkey: M, MM, MMM, and 5M sorted in an increasing order of attractiveness. While MMM and 5M Migros stores provide the most extensive shopping experience with a wide variety of items and parking place among others, some customers with especially low income may still prefer M or MM Migros stores to MMM or 5M Migros stores due to the fact that some of the items are cheaper in the former stores everyday. As a result, the probability that a customer with low income patronizes an M Migros store can be higher than that of visiting an MM, MMM or 5M Migros store. Hence, it is more realistic to make the visiting probabilities depend on the customers' attributes such as financial income. Another important aspect of the proposed model is that for each type of facility there is a maximum (threshold) distance customers are willing to travel to visit that facility. When the distance to a facility is beyond this maximum distance, customers will not visit that facility. The proposed model is a binary integer linear program. We solve it using a Lagrangean heuristic the solution of which is further improved by a local search procedure.

6.1. Model Formulation

In the proposed model, there are m potential facility sites indexed by $i = 1, \dots, m$, n customer zones indexed by $j = 1, \dots, n$, and r facility types indexed by $k = 1, \dots, r$. We assume that the larger is the index of a facility type, the higher is the product or the service diversity offered by the facility (e.g., an M Migros store is of type 1, whereas an MMM Migros store is of type 3). We assume that the probability p_{jk} that customers at zone j visit a type-k facility is known. Furthermore, there is a maximum distance S_{jk} that customers at zone j are willing to travel to a type-k facility. We assume that S_{jk} is increasing in the facility type for all customer zones, because people are willing to travel more for higher-level facilities in order to find a large diversity of products or services. Using the concept of maximum distance, we can define a "region of influence", N_{ik} for each pair of potential facility site i and facility type k as follows: $N_{ik} = \{j : d_{ij} \leq S_{jk}\}$, where d_{ij} is the distance between customer zone j and potential facility site i. This implies that N_{ik} includes the zones whose customers would like to go to type-k facility at potential site i. This also means that if a facility is opened at a site with a certain type, it can only serve those customers within the region of influence.

There is a fixed cost f_k associated with opening a type-k facility. It is clear that facilities with a higher k value have a larger fixed cost. The total income or buying power of customers at zone j is denoted by h_j. Note that in our problem setting, when the system planner decides to open a new facility at potential site i, the type of the facility should also be determined. This issue is related to the average visiting probabilities for each facility type of the customers who are within region of influence of potential site i. This can be best explained by an example. Suppose that there is one potential facility site ($i = 1$) and two possible facility types ($k = 1, 2$) to be opened at this site. Customers at zone j have a known visiting probability for each facility type, i.e., p_{j1} and p_{j2}. For this potential site there are two regions of influence, N_{11} and N_{12}. Each region includes the customer zones that are willing to go to this site for the first and second facility types. Now, we have to find out the average probability ψ_{1k} with which the customers in the region of influence visit each facility type. These probabilities are given as $\psi_{11} = (\sum_{j \in N_{11}} p_{j1})/|N_{11}|$ for the first facility type and $\psi_{12} = (\sum_{j \in N_{12}} p_{j2})/|N_{12}|$ for the second facility type. If $\psi_{11} > \psi_{12}$ ($\psi_{11} < \psi_{12}$), then a type-1 (type-2) facility should be established at this potential site. In the case of equality, the system planner would be indifferent between the two facility types. Obviously, the total revenue collected from the customers must exceed the fixed cost of opening the facility to be economically feasible. Otherwise, the best decision would be not to open any facility at the corresponding site.

Before we present our binary integer programming model BIP, we list all the parameters and decision variables used in the model for the sake of clarity.

d_{ij} : Euclidean distance between potential site i and customer zone j

S_{jk} : maximum distance that customers at zone j are willing to visit a type-k facility

N_{ik} : the set of customers within the region of influence of type-k facility at potential site i

p_{jk} : the probability that customers at zone j visit a type-k facility

h_j : annual income or buying power of customers at zone j

f_k : fixed cost of opening a type-k facility

ψ_{ik} : average probability for type-k facility at potential site i computed as $(\sum_{j \in N_{ik}} p_{jk})/|N_{ik}|$

M : a large positive number

x_{ik} : binary variable indicating whether a type-k facility is opened at potential site i

y_{ijk} : binary variable indicating whether customers at zone j visit a type-k facility at potential site i

w_{ikl} : auxiliary variable used to keep track of the relative difference of average probability pairs for facility types k and l at potential site i

$$\text{BIP}: \ \max \ z = \sum_{i=1}^{m} \sum_{k=1}^{r} \sum_{j \in N_{ik}} h_j p_{jk} y_{ijk} - \sum_{i=1}^{m} \sum_{k=1}^{r} c_k x_{ik} \tag{6.1}$$

s.t.

$$\psi_{ik} \leq \psi_{il} + M w_{ikl} \qquad i = 1, \ldots, m, k = 1, \ldots, r-1, l = k+1, \ldots, r \tag{6.2}$$

$$\psi_{il} \leq \psi_{ik} + M(1 - w_{ikl}) \qquad i = 1, \ldots, m, k = 1, \ldots, r-1, l = k+1, \ldots, r \tag{6.3}$$

$$x_{ik} \leq w_{ikl} \qquad i = 1, \ldots, m, k = 1, \ldots, r-1, l = k+1, \ldots, r \tag{6.4}$$

$$x_{il} \leq (1 - w_{ikl}) \qquad i = 1, \ldots, m, k = 1, \ldots, r-1, l = k+1, \ldots, r \tag{6.5}$$

$$y_{ijk} \leq x_{ik} \qquad i = 1, \ldots, m, j \in N_{ik}, k = 1, \ldots, r \tag{6.6}$$

$$\sum_{i: j \in N_{ik}} y_{ijk} \leq 1 \qquad j = 1, \ldots, n, k = 1, \ldots, r \tag{6.7}$$

$$\sum_{h: d_{hj} > d_{ij}} y_{hjk} \leq 1 - x_{ik} \qquad i = 1, \ldots, m, j \in N_{ik}, k = 1, \ldots, r \tag{6.8}$$

$$x_{ik}, y_{ijk}, w_{ikl} \in \{0, 1\} \qquad i = 1, \ldots, m, j \in N_{ik}, k = 1, \ldots, r \tag{6.9}$$

The first term in the objective function (6.1) represents the revenue that the facilities collect, while the second term is the annualized fixed cost of opening facilities. In order to compute the revenue of a facility, the income or the buying power of customers who are in the region of influence are taken into account. However, an opened facility of type-k cannot capture all the buying power at a customer zone in its region of influence. Since each customer has a probability of visiting a facility type, the facilities can capture only a portion of this income, which is obtained by multiplying the income with this probability. For example if the buying power of customers at a zone which is in the region of influence of a type-2 facility is 100 dollars and furthermore if those customers visit a type-2 facility with probability 0.7, then that facility can capture only 70 dollars. As mentioned before, ψ_{ik} computes the average of the probabilities p_{jk} at potential site i with the region of influence of a type-k facility. Thus, constraints (6.2) and (6.3) are either-or constraints used to compare the average probability corresponding to each facility type-k to be opened at each potential facility site i. If ψ_{ik} is greater than ψ_{il} with $l > k$, then constraint (6.2) is redundant and $w_{ikl} = 1$. This means that a type-k facility is more appropriate than a type-l facility at site i since customers who are in the region of influence of a type-k facility are more likely to visit such a facility at this site. If ψ_{il} is greater than ψ_{ik}, then constraint (6.3) is redundant and $w_{ikl} = 0$ meaning that type-l is more appropriate than type-k at site i. Constraints (6.4) and (6.5) link the location variables x with the z variables. For example, if $w_{ikl} = 1$ (type-k is more appropriate than type-l), then $x_{il} = 0$ because of constraint (6.5) so that no type-l facility is opened at site i. Moreover, $x_{ik} \leq 1$ due to constraint (6.4), which means that a type-k facility can be opened at site i depending on the trade-off between the revenue it can generate and its fixed cost. Thus, constraints (6.2), (6.3), (6.4), and (6.5) determine together the best facility type at every potential site. As one can see, the best type is determined by the region of influence, whose average probability ψ_{ik} is the largest. The reason for that is to serve the customers within the region of influence in the best way. As an example, assume that for a potential site i there are two possible facility types and $\psi_{i1} = 0.8$ and $\psi_{i2} = 0.2$. If a type-2 facility is opened at site i, customers within the region of influence will visit the facility with probability 0.2 on the average. This probability 0.2 can be the consequence of transportation costs of customers to the facility, or type-2 is a higher-level facility offering more expensive products so that customers do not prefer to go to that facility because of their income. In other words, if a type-2 facility is opened at site i, these customers will be underserviced and cannot meet their needs. Therefore, we determine the best type for each potential facility site in order to prevent customers being underserviced. With constraints (6.6) it is ensured that customers are not serviced by a facility type at a site if no facility of that type is opened at that site. Constraints (6.7) guarantee that customers at a zone visit at most one facility of each type. Constraints (6.8) make sure that if a type-k facility is opened at site i, i.e., $x_{ik} = 1$, then customers at zone j will not visit other type-k facilities that are further away to them than site i. Consequently, constraints (6.7) and (6.8) ensure together that if more than one facility of the same type are opened at different sites, then customers at some zone j visit only the nearest one of these facilities. Finally, constraints (6.9) are the binary restriction on locations, assignment, and auxiliary variables.

6.2. Determining the Visiting Probabilities

An important aspect of the proposed model that makes it novel in the facility location literature are the visiting probabilities of customers for different facility types. As mentioned above, these probabilities do not necessarily increase with the increased attractiveness level of the facilities; they rather depend on the customer attributes. However, the visiting probabilities of customers are parameters to the optimization problem BIP given in the previous section and their values should be determined before solving the model. For this purpose, we use two different procedures one of which is based on the fuzzy C-means clustering (Bezdek, 1981) and the other one is a parametric Bayesian classification.

Before proceeding with the details of the two methods, we first give a brief introduction about the techniques in machine learning. Machine learning can be seen as a tool which can obtain useful information from an available set of data (Alpaydın, 2004). In machine learning, there are two main learning paradigms: “supervised” and “unsupervised” learning. Supervised learning methods, like classification and regression, use a priori information about the labels of data instances, i.e. the discrete class labels of each instance and the number of classes are known. If the method is regression, then we also know the continuous labels of instances.

In contrast to supervised learning, unsupervised learning can be applied when any label information is not available. Thus, one cannot have a priori information about the labels of instances and the number of classes. In such a case, we try to divide the available data instances into meaningful groups or clusters.

In recent years, semi- supervised learning (SSL) techniques started to become more popular in machine learning. The reason for that is obvious: SSL is between the supervised and unsupervised learning where there are a small labeled and a large unlabeled data set and the task is to make statistical inference from the partially labeled data (Chapelle *et al.*, 2006). SSL has some advantages when compared with the two traditional groups of machine learning. In supervised learning, all instances should be labeled in order to make statistical inference. However, labeling all the available data is an expensive and time consuming task, since we usually need a large data set in order to make, for example, predictions by reducing the classification or regression errors. Nevertheless, labeling only a small portion of the available data is much easier and helps us by giving some supervision. In contrast to supervised learning, we do not have to label any data instance for unsupervised learning. Thus, SSL provides some information about the data set to be used as supervision with the help of the few labeled instances so that we can know, for example, the number of different classes or which instances should be in the same or different clusters.

6.2.1. A Fuzzy C-Means Algorithm

In order to infer the visiting probabilities of customers for different facility types, we assume that each customer is an input or a feature vector and each facility type forms a cluster to which an input, i.e. a customer, can belong. Each customer has attributes that influence the pertaining of the customer to a certain cluster. In this thesis, we assume that these attributes of the customers are the annual income and the threshold distances for each facility type, which altogether determine the dimensions of an input vector. Thus, for example, if there are three possible facility types, then each input vector (customer) has four dimensions, three dimensions for three threshold distances and one dimension for customer's annual income.

Since fuzzy C-means algorithm is an unsupervised technique, we do not have to label any of the input vectors. All we have to do is to divide the customers into C meaningful clusters each of which represents a facility type. Hard clustering techniques also divide the data instances into clusters. However, in hard clustering an instance either belongs to a cluster or not. We can alternatively explain this situation such that an instance belongs to a cluster either with probability one or with probability zero. This is similar to the all-or-nothing property of deterministic utility models in the CFL literature. We assume that the probability that a customer belongs to a cluster embodies the probability of that customer for visiting the facility type that is represented by that cluster. Furthermore, in practice the probability that a customer visits a facility type is not always either zero or one, on the contrary it should lie in the closed interval $[0, 1]$ and the sum of the probabilities for all facility types of a customer should sum up to one. As a consequence, we choose to apply a fuzzy clustering technique which allows an instance to belong to two or more clusters. In order to apply fuzzy C-means clustering, we further assume that each customer has an ideal facility type and customers with similar attributes visit the same facility type with a higher probability.

Fuzzy C-means clustering is introduced by Bezdek (1981) and later revised by Bezdek and Pal (1995). In this algorithm, for each input x_i, $i = 1, 2, ..., n$, there is a degree of membership u_{ij} in the cluster j, which represents the visiting probabilities. The membership degree u_{ij} takes values from the interval $[0, 1]$ and for each input vector x_i, $\sum_{j=1}^{C} u_{ij} = 1$. Each cluster j is represented by a d-dimensional cluster center c_j. The similarity between any input vector x_i and cluster center c_j is expressed by a norm function that represents the distance between the input and the cluster. The goal of the algorithm is to represent all the input data by cluster centers. Therefore, it tries to find the best C cluster centers by minimizing the sum of the weighted squared errors between the clusters. The algorithm starts with arbitrarily assigned membership degrees u_{ij}. At each iteration first the cluster centers are calculated, then the memberships are updated. The algorithm stops when the difference between the membership degrees of the current iteration and the previous iteration is less than a user-specified threshold value ϵ. In fact, the algorithm aims at finding optimal cluster centers by applying a local gradient descent algorithm (Bezdek and Pal, 1995). The most ambiguous issue in this algorithm is the choice of an exponential weight μ $(1 < \mu < \infty)$ that is used in the updates of both the cluster centers and the values of membership degrees. The selection of μ will be discussed in Section 7.4. The fuzzy C-means algorithm can be summarized as follows.

1. Initialize the membership degree $U = [u_{ij}]$ randomly.
2. At the tth step, calculate the cluster centers $c_j^{(t)} = \frac{\sum_{i=1}^{n} \left(u_{ij}^{(t)}\right)^{\mu} x_i}{\sum_{i=1}^{n} u_{ij}^{(t)}}$ for $j = 1, ..., C$.
3. Update $u_{ij}^{(t)} = \frac{1}{\sum_{k=1}^{C} \left(\frac{\left\| x_i - c_j^{(t)} \right\|}{\left\| x_i - c_k^{(t)} \right\|} \right)^{\frac{2}{\mu-1}}}$ for $i = 1, ..., n$, $j = 1, ..., C$.
4. Until $\left\| U^{(t+1)} - U^{(t)} \right\| < \epsilon$.

Figure 6.1. Fuzzy C-Means Clustering Algorithm

6.2.2. A Parametric Bayesian Classification Algorithm

We also develop a procedure following the multivariate classification in order to predict the probabilities of customers for visiting the facility types. The multivariate classification is a parametric technique which assumes that the input vectors are drawn from a known probability distribution like Gaussian (Alpaydın, 2004). In order to make the inference from a labeled data set of instances $\{\mathbf{x}_i, \mathbf{a}_i\}$, we assume that the conditional probabilities $P(\mathbf{x}|C_j)$ are normally distributed, where $\mathbf{x}$ is an $(n \times d)$ data matrix whose rows represent the n customers and whose columns represent the d attributes of the customers, C_j is the jth class, and $\mathbf{a}_i$ shows the class label of a given instance $\mathbf{x}_i$. In other words, a_{ij} is equal to one if the customer $\mathbf{x}_i$ belongs to the jth class and zero otherwise, and we assume that $P(\mathbf{x}|C_j) \sim N_d(\mathbf{M}_j, \Sigma_j)$ such that $P(\mathbf{x}|C_j) = \frac{1}{(2\pi)^{d/2}|\Sigma_j|^{1/2}} \exp[-\frac{1}{2}(\mathbf{x} - \mathbf{M}_j)^T \Sigma_j^{-1} (\mathbf{x} - \mathbf{M}_j)]$, where each component $\mathbf{M}_j$ represents the mean of the jth class and Σ is the $d \times d$ covariance matrix with Σ_j representing the covariance matrix of the jth class. The conditional probability $P(\mathbf{x}_i|C_j)$ that is interpreted as the probability of an input belonging to the jth class has the observed values $\mathbf{x}_i$ for $i = 1, ..., d$ is called the "class likelihood". Furthermore, the probabilities $P(\mathbf{C})$ and $P(\mathbf{x})$ are called the "prior probability" and the "evidence", respectively (Alpaydın, 2004).

Since we observe that the attributes of the input vectors (customers) $\mathbf{x}$, we need the conditional probabilities $P(C_j|\mathbf{x}_i)$ which tells us the probability that the customer $\mathbf{x}_i$ belongs to the jth class. For our problem, we assume that these conditional probabilities which are called the "posterior probabilities" represent the probabilities of customers to visit the different facility types, since each facility type can be represented by a class and each class has an ideal customer profile exemplified by the customer attributes. The posterior probability $P(C_j|\mathbf{x})$

can be calculated as $P(C_j|\mathbf{x}) = \frac{P(\mathbf{x}|C_j)P(C_j)}{p(\mathbf{x})} = \frac{P(\mathbf{x}|C_j)P(C_j)}{\sum_{j=1}^{k} P(\mathbf{x}|C_j)P(C_j)}$ using the Bayes' rule when there are $k > 1$ classes. In order to find the posterior probabilities, we need to find the estimates of $P(C_j)$, $\mathbf{M}_j$, and Σ_j for all classes $j = 1, ..., k$. The estimates of prior probabilities, mean vectors, and covariance matrices for each class can be given as follows:

$$\widehat{P}(C_j) = \frac{\sum_{i=1}^{n} a_{ij}}{n} \quad \text{for } j = 1, ..., k, \tag{6.10}$$

$$\widehat{\mathbf{M}}_j = \frac{\sum_{i=1}^{n} a_{ij}\mathbf{x}_i}{\sum_{i=1}^{n} a_{ij}} \quad \text{for } j = 1, ..., k, \tag{6.11}$$

$$\widehat{\Sigma}_j = \frac{\sum_{i=1}^{n} a_{ij}(\mathbf{x}_i - \widehat{\mathbf{M}}_j)(\mathbf{x}_i - \widehat{\mathbf{M}}_j)^T}{\sum_{i=1}^{n} a_{ij}} \quad \text{for } j = 1, ..., k. \tag{6.12}$$

In a Bayes' classifier, the sum of the posterior probabilities of all the classes is equal to one for each input vector: $\sum_{j=1}^{k} P(C_j|\mathbf{x}) = 1$. One has to predict the class labels of all input vectors, i.e. one has to find out which input vector belongs to which class. In the classification method, this is done simply by choosing the class the posterior probability of which is the greatest among all the classes for each input vector $\mathbf{x}$. In other words, we choose the class j for $\mathbf{x}$ if $P(C_j|\mathbf{x}) = \max_{h=1,...,k} P(C_h|\mathbf{x}) = \max_{h=1,...,k} \frac{P(\mathbf{x}|C_h)P(C_h)}{p(\mathbf{x})} = \max_{h=1,...,k} \frac{P(\mathbf{x}|C_h)P(C_h)}{\sum_{j=1}^{k} P(\mathbf{x}|C_j)P(C_j)}$. Since the expression $p(\mathbf{x}) = \sum_{j=1}^{k} P(\mathbf{x}|C_j)P(C_j)$ is common to all classes, we can discard it from the equation. Then, we simply select the class j for $\mathbf{x}$ if $P(C_j|\mathbf{x}) = \max_{h=1,...,k} P(\mathbf{x}|C_h)P(C_h)$. The expression $P(\mathbf{x}|C_j)P(C_j)$ for a class j actually defines a function $f_j(\mathbf{x})$ of $\mathbf{x}$ that is called a "discriminant function" (Alpaydın, 2004). Therefore, we select for $\mathbf{x}$ the class j, if $P(C_j|\mathbf{x}) =$

$\max_{h=1,\ldots,k} f_h(\mathbf{x})$. However, since $P(\mathbf{x}|C_j)$ is assumed to be equal to $\frac{1}{(2\pi)^{d/2}\left|\widehat{\Sigma}_j\right|^{1/2}} \exp[-\frac{1}{2}(\mathbf{x} - \widehat{\mathbf{M}}_j)^T\widehat{\Sigma}_j^{-1}(\mathbf{x} - \widehat{\mathbf{M}}_j)]$ for $j = 1, ..., k$, we take the logarithm of discriminant functions f_j for all $j = 1, ..., k$. Thus, the set of discriminant functions become as follows, when the constant term $-\frac{d}{2}\log(2\pi)$ is removed.

$$f_j = -\frac{1}{2}\log\left|\widehat{\Sigma}_j\right| - \frac{1}{2}(\mathbf{x} - \widehat{\mathbf{M}}_j)^T\widehat{\Sigma}_j^{-1}(\mathbf{x} - \widehat{\mathbf{M}}_j) + \log\widehat{P}(C_j) \tag{6.13}$$

The multivariate classification technique explained above can be applied to our problem when the class labels of all customers are known to the system planner. This also means that it can be implemented when we know the visiting probabilities of all customers. However, determining the probabilities of all customers is a time consuming task. On the other hand, it is possible to identify the visiting probabilities of few customers by measuring their visit frequencies and making some interviews with them. Therefore, we assume that we can obtain a partially labeled data set of customers whose probabilities are known in advance. When we know the visiting probabilities of the few customers, then we can assign each of them to a class representing a facility type by determining the maximum visiting probability among all facility types.

As a consequence, we decide to apply the given classification technique in the framework of a semi-supervised approach. As explained before, in semi-supervised learning there are a small labeled and a large unlabeled data set. First, we apply the Bayes' classifier on the small labeled data set with α customers ($\alpha < n$) and estimate the statistics we need, i.e. the estimates of prior probabilities, mean vectors, and covariance matrices of all classes. Then, employing these estimates we predict the labels of the unlabeled customers by calculating their posterior probabilities for all classes. As a result, we obtain the class labels and posterior probabilities for all customers. We next select β customers from the newly labeled set that the Bayes' classifier labels most confidently and add them to the initial small labeled data set. In other words, we choose β customers from the initial unlabeled set whose posterior probabilities are the largest, where β is a user-specified integer number. The Bayes' classifier is now implemented on a larger labeled data set with $\alpha + \beta$ customers and the estimates are calculated on this new larger set. This procedure is repeated until all $n - \alpha$ customers who were unlabeled at the beginning are added to the initial labeled set. At the end, the posterior probabilities and the class labels of all customers are obtained and the posterior probabilities can be used as the probabilities of customers for visiting the facility types. The idea of stepwise labeling the customers is based on an SSL method called "Co-Training" introduced by Blum and Mitchell (1998) and later enhanced by Balcan *et al.* (2004). The implemented parametric Bayesian classification approach within the framework of the semi-supervised learning can be abstracted as follows.

1. Implement the Bayes' classifier on the labeled data set with $\alpha + \beta$ customers. At the initial step, $\beta = 0$.
2. Obtain the estimates $\widehat{P}(C_j)$, $\widehat{M_j}$, and $\widehat{\Sigma}_j$ for all classes $j = 1, ..., k$
3. Calculate the posterior probabilities $P(C_j|\mathbf{x})$ and predict the class labels of remaining $n - (\alpha + \beta)$ customers.
4. Select β customers whose posterior probabilities showing the chosen class are the largest and add them to the labeled set.
5. Repeat the procedure until all unlabeled instances at the beginning are added to the initial labeled set.

Figure 6.2. Parametric Bayesian Classification Algorithm

6.3. Solution Procedure

We propose a hybrid solution method combining a Lagrangean heuristic (LH) and a local search for the solution of the model BIP. The solution method starts with the LH and continues with a local search (LS), which takes the best feasible solution (lower bound) provided by the LH as the initial solution. The LH we apply is based on the relaxation of two constraint sets, namely constraints (6.6) and (6.8). To this end, Lagrangean multipliers $\lambda_{ijk} \geq 0$ and $\mu_{ijk} \geq 0$ with $i = 1, 2, ..., m$, $k = 1, 2, ..., r$, and $j \in N_{ik}$ are defined. Thus, we add the terms $\sum_{i=1}^{m} \sum_{k=1}^{r} \sum_{j \in N_{ik}} \lambda_{ijk}(x_{ik} - y_{ijk})$ and $\sum_{i=1}^{m} \sum_{k=1}^{r} \sum_{j \in N_{ik}} \mu_{ijk}(1 - x_{ik} - \sum_{h=1,...,m: d_{hj} > d_{ij}} y_{hjk})$ into the objective function. The Lagrangean subproblem becomes then

$$\text{BIP}\,(\boldsymbol{\lambda},\boldsymbol{\mu}):\ \max z\,(\boldsymbol{\lambda},\boldsymbol{\mu}) = \sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}(h_j p_{jk}-\lambda_{ijk})y_{ijk} - \sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}\mu_{ijk}(\sum_{h:d_{hj}>d_{ij}} y_{hjk})$$
$$+\sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}(\lambda_{ijk}-\mu_{ijk}-c_k)x_{ik}+\sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}\mu_{ijk}$$

s.t.

$$\psi_{ik}\le\psi_{il}+Mw_{ikl}\qquad i=1,\ldots,m,k=1,\ldots,r-1,l=k+1,\ldots,r$$
$$\psi_{il}\le\psi_{ik}+M(1-w_{ikl})\qquad i=1,\ldots,m,k=1,\ldots,r-1,l=k+1,\ldots,r$$
$$x_{ik}\le w_{ikl}\qquad i=1,\ldots,m,k=1,\ldots,r-1,l=k+1,\ldots,r$$
$$x_{il}\le(1-w_{ikl})\qquad i=1,\ldots,m,k=1,\ldots,r-1,l=k+1,\ldots,r$$
$$\sum_{i:j\in N_{ik}} y_{ijk}\le 1\qquad j=1,\ldots,n,k=1,\ldots,r$$
$$x_{ik},y_{ijk},w_{ikl}\in\{0,1\}\qquad i=1,\ldots,m,j\in N_{ik},k=1,\ldots,r$$

We can decompose the Lagrangean subprolem BIP$(\boldsymbol{\lambda},\boldsymbol{\mu})$ into two separate subproblems BIP$_1\,(\boldsymbol{\lambda},\boldsymbol{\mu})$ and BIP$_2\,(\boldsymbol{\lambda},\boldsymbol{\mu})$ with optimal objective values $z_1\,(\boldsymbol{\lambda},\boldsymbol{\mu})$ and $z_2\,(\boldsymbol{\lambda},\boldsymbol{\mu})$ as follows:

$$\text{BIP}_1\,(\boldsymbol{\lambda},\boldsymbol{\mu}):\ \max z_1\,(\boldsymbol{\lambda},\boldsymbol{\mu}) = \sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}(h_j p_{jk}-\lambda_{ijk})y_{ijk} - \sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}\mu_{ijk}(\sum_{h:d_{hj}>d_{ij}} y_{hjk})$$

s.t.

$$\sum_{i:j\in N_{ik}} y_{ijk}\le 1\qquad j=1,\ldots,n,k=1,\ldots,r$$
$$y_{ijk}\in\{0,1\}\qquad i=1,\ldots,m,j\in N_{ik},k=1,\ldots,r$$

$$\text{BIP}_2\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right): \ \max z_2\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right) = \sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}(\lambda_{ijk}-\mu_{ijk}-c_k)x_{ik}$$

s.t.

$$\begin{aligned}
&\psi_{ik} \le \psi_{il} + Mw_{ikl} && i=1,\ldots,m, k=1,\ldots,r-1, l=k+1,\ldots,r\\
&\psi_{il} \le \psi_{ik} + M(1-w_{ikl}) && i=1,\ldots,m, k=1,\ldots,r-1, l=k+1,\ldots,r\\
&x_{ik} \le w_{ikl} && i=1,\ldots,m, k=1,\ldots,r-1, l=k+1,\ldots,r\\
&x_{il} \le (1-w_{ikl}) && i=1,\ldots,m, k=1,\ldots,r-1, l=k+1,\ldots,r\\
&x_{ik}, w_{ikl} \in \{0,1\} && i=1,\ldots,m, k=1,\ldots,r
\end{aligned}$$

Note that $\sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}\mu_{ijk}$ is a constant term and discarded from both subproblems. After both subproblems are solved, the value of the term with optimal multipliers $\boldsymbol{\mu}^*$ should be added to the sum of $z_1\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$ and $z_2\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$ to determine $z^*\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$. Notice that $\text{BIP}_1\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$ contains only the decision variables y_{ijk}, whereas $\text{BIP}_2\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$ consists of x_{ik} and w_{ikl}. $z^*\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$ provides an upper bound on z^*, the optimal value of the original problem BIP. In order to find the best (smallest) upper bound, we need to solve the Lagrangean dual problem using the subgradient optimization procedure. The steps of the subgradient optimization procedure is given in Subsection 3.2.1 as well. At each iteration t of the subgradient optimization procedure, a step size T is computed for the Lagrangean multipliers λ_{ijk} and μ_{ijk}. Using the update formulas $\lambda_{ijk}^{(t+1)} = \max\left(0, \lambda_{ijk}^{(t)} - T^{(t)}(x_{ik}-y_{ijk})\right)$ and $\mu_{ijk}^{(t+1)} = \max\left(0, \mu_{ijk}^{(t)} - T^{(t)}(1-x_{ik}-\sum_{h:d_{hj}>d_{ij}} y_{hjk})\right)$, the Lagrangean multipliers are updated by making use of the subgradients. A step size formula

$$T = \frac{\pi\left(UB^{(t)} - LB_{best}\right)}{\left(\sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}\left\|x_{ik}-y_{ijk}\right\|^2 + \sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}\|(1-x_{ik}-\sum_{h:d_{hj}>d_{ij}} y_{hjk})\|^2\right)}$$

is employed, where $UB^{(t)}$ is the upper bound at iteration t, LB_{best} is the best lower bound obtained until iteration t, and π is the step size parameter. As given in Subsection 3.2.1, π is halved every 20 iterations when there is no improvement in the best bound. The algorithm is terminated as soon as π is less than a threshold value (e.g., 0.005) or the best upper bound and best lower bound are equal to each other with zero duality gap. The best (largest) lower bound obtained throughout the iterations becomes the output of the LH.

The most important part of the subgradient optimization procedure is how to solve the subproblems and how to obtain a feasible solution the objective value of which yields a good lower bound on the objective value of the original problem BIP. Both subproblems $\mathrm{BIP}_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $\mathrm{BIP}_2(\boldsymbol{\lambda}, \boldsymbol{\mu})$ can be solved by inspection. When the objective function of $\mathrm{BIP}_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is factored out, it can be expressed in the following form:

$$\sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j\in N_{ik}}\left(h_j p_{jk} - \lambda_{ijk} - (\sum_{h:d_{hj}<d_{ij}, j\in N_{hk}} y_{hjk})\right) y_{ijk}$$

After fixing the indices i and k, we can focus on each region of influence N_{ik} and determine which customer zones are in N_{ik}. Hence, we can set $y_{ijk} = 1$ if

$$\left(h_j p_{jk} - \lambda_{ijk} - (\sum_{h:d_{hj}<d_{ij}, j\in N_{hk}} y_{hjk})\right) > 0,$$

$y_{ijk} = 0$ otherwise. We also have to satisfy the constraints $\sum\limits_{i:j\in N_{ik}} y_{ijk} \leq 1$. Thus, for each j and k if more than one $y_{ijk} = 1$, we let $y_{i^*jk} = 1$ for which the term $\left(h_j p_{jk} - \lambda_{i^*jk} - (\sum\limits_{h:d_{hj}<d_{i^*j}, j\in N_{hk}} y_{hjk})\right)$ is the largest, and $y_{ijk} = 0$ for $i \neq i^*$.

The solution of $\mathrm{BIP}_2(\boldsymbol{\lambda}, \boldsymbol{\mu})$ can also be obtained by inspection. With the help of the probabilities ψ_{ik}, we can determine the best facility type for each potential site. If the best type at site i is k', then all $x_{ik} = 0$ for $k \neq k'$ and $x_{ik'} \leq 1$. Then the problem $\mathrm{BIP}_2(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is solved by inspection as follows. If $(\sum_{j\in N_{ik'}} (\lambda_{ijk'} - \mu_{ijk'})) - c_{k'} > 0$, then $x_{ik'} = 1$; otherwise it is equal to zero.

In order to generate a feasible solution and hence a lower bound on z, we make use of the solution of $\mathrm{BIP}_2(\boldsymbol{\lambda}, \boldsymbol{\mu})$ which provides the set of opened and closed facilities with their types. If $x_{ik} = 1$, we know that a type-k facility is opened at site i. Otherwise, there is no type-k facility at that site. So, we first let $y_{ijk} = 0$ for which $x_{ik} = 0$ and $j \in N_{ik}$. Then the remaining y_{ijk} are set to one for which $x_{ik} = 1$ and $j \in N_{ik}$. Hence, if $x_{ik} = 1$ and $x_{hk} = 1$, where both $j \in N_{ik}$ and $j \in N_{hk}$, which implies that the same facility type is opened at two different sites, then $y_{ijk} = 1$ and $y_{hjk} = 0$ if $d_{ij} < d_{hj}$. Accordingly, $y_{ijk} = 0$ and $y_{hjk} = 1$ if $d_{ij} > d_{hj}$. Ties can be broken arbitrarily.

The best lower bound obtained by the LH constitutes an initial solution for the local search (LS) procedure. At each iteration of the LS procedure, neighbors of the current solution are generated by executing 1-*Add*, 1-*Drop*, and 1-*Swap* moves. A 1-Add move opens a new facility at one of the potential sites where no facility exists. A 1-Drop move closes an open facility and a 1-Swap move closes an open facility and opens a new one at another potential site without a facility. By executing these moves we can obtain only the location of the opened facilities. As explained before, we can determine the best type for each potential facility site using the average probabilities ψ_{ik}. If the best facility type for site i is k', then $x_{ik} = 0$ for $k \neq k'$ and $x_{ik'} = 1$. Thus, right after executing one possible move we can determine the values of the location variables x_{ik}. In order to calculate the profit we determine also the values of the assignment variables y_{ijk} by the same procedure implemented by the LH. The best neighboring solution, namely the solution that provides the highest objective value is selected as the next current solution. The LS procedure is terminated when the current solution does not improve from one iteration to the next.

7. EXPERIMENTAL RESULTS

As there is no benchmark problem in the literature on CFL problems and location problems considering customer preferences, we generate random test instances to obtain computational results. These computational experiments on randomly created data sets are used to assess both the accuracy and efficiency of the proposed solution procedures. Furthermore, for some models a sensitivity analysis is carried out on model parameters to investigate the effect of these parameters on the solution.

All solution procedures have been coded in C# and the computations have been performed on a server with Intel Xeon 3.16 GHz processor with 16 GB of RAM working under the Windows 2003 Server operating system. For the first CFL problem we employ the MINOS solver (Murtagh *et al.*, 2004), that is available within GAMS suite to solve the nonlinear programs of the BB method using NLP relaxation. To find the lower bounds at the nodes of the BB trees in both bilevel CFL problems, we utilize KNITRO 6.0 (Waltz and Plantenga, 2009), which is an efficient commercial solver for NLP solver.

7.1. A Non-Reactive Discrete Competitive Facility Location With Variable Attractiveness

In order to assess the performance of the three methods proposed in the Section 3.2 to solve the CFL model P given in Chapter 3, we use nine data sets where the number of demand points (n) and the number of candidate facility sites (m) are equal to each other, and take on values from the set $\{10, 15, 20, 25, 30, 35, 40, 45, 50\}$. For each data set, the number of existing facilities belonging to competitors, r, is assigned a value between one and five. As a result, we obtain 45 instances. The x and y-coordinates of the demand points, the candidate facility sites, and the existing facility sites are integer numbers generated from a uniform distribution defined in the interval $[0, 100]$. The distance d_{ij} between site i and demand point j is then calculated as the Euclidean distance. We pay attention not to overlap the demand points with either the candidate facility sites or the existing facility sites because of the fact that a facility coinciding with a demand point captures almost all of the available buying power at that point since the distance is zero. The annualized buying power a_j of customers at point j, the unit attractiveness cost c_i at candidate site i and the attractiveness q_k of existing facility at site k are integer-valued parameters generated from uniform distributions as: $h_j \sim U(100, 10000)$, $c_i \sim U(1, 10)$, and $q_k \sim U(100, 1000)$. The fixed costs f_i which effectively determine the optimal number of new facilities to be opened are set to three different values as follows: $f_i = 100c_i$, $f_i = 1000c_i$, and $f_i = 10000c_i$. In other words, they are chosen as 100, 1000, and 10,000 times as large as the attractiveness costs. Upper bound u_i for the attractiveness of a facility at candidate site i is assigned a value of $u_i = 100c_i$.

7.1.1. Comparison of the Solution Procedures

First, we compare the performance of the three methods in terms of accuracy and efficiency on the set of 45 instances. The results are presented in Tables 7.1–7.3. A time limit of 7200 seconds is allotted for each solution method. BB-NLP is the most efficient among the three approaches as it is able to obtain an optimal solution for every instance. The accuracy of the solutions given by LH and BB-LR is therefore measured by computing the percent relative deviation (PD) of the best lower bound LB_{best} obtained by these methods from the optimal objective value z^* provided by BB-NLP. PD is identified using the formula

$$100 \times \frac{z^* - LB_{best}}{z^*}. \tag{7.1}$$

The average percent deviation and average CPU time requirement computed over all the instances are given in the last rows of the tables. For all fixed cost values, LH performs better than BB-LR on the average in terms of accuracy. For example, at the lowest fixed cost level ($f_i = 100c_i$) the average PD is 0.3% for LH and 1.2% for BB-LR, while the CPU times are 2786.9 seconds for LH and 6110.3 seconds for BB-LR. When we consider the results for the medium level fixed costs presented in Table 7.2, a similar observation can be made where LH again performs better than BB-LR in terms of accuracy with an average percent deviation of 0.6% against 6.2%. LH beats BB-LR also in terms of efficiency: the CPU time requirement of

LH is 2641.4 seconds whereas BB-LR spends 6413.3 seconds on the average. It is remarkable that BB-NLP solves the instances in this table in 464.9 second on the average. We observe that LH finds an optimal value for 19 instances, whereas BB-LR obtains the optimal solution for only nine instances. For the highest fixed cost values, the picture is not different.

The numbers in the "NF" columns of the three tables indicate that the number of new facilities to be opened decreases as the fixed costs increases, which is an expected outcome. Results generated by BB-NLP for the high fixed cost level ($f_i = 10000c_i$) presented in Table 7.2 reveal that it is optimal not to open any new facilities in some instances. This is shown by "–" in the NF column with a corresponding z^* value equal to zero. In these instances, the fixed and variable costs of opening facilities do not justify the opening of new facilities.

We note that although BB-LR is an exact technique, it provides the worst solutions among the three methods. The reason lies in the fact that the allowed time limit of 7200 seconds is not sufficient for the BB tree to explore all the nodes. This can be seen in the columns of Tables 7.1–7.3 labeled as "NN". We observe that at each fixed cost level all the nodes of the BB tree can only be explored for problem instances with $n = 10$ customers, and furthermore for instances $(15, 3)$, $(15, 4)$, and $(15, 5)$ at the lowest fixed cost level. In other words, BB-LR provides an optimal solution in these cases. However, it is clear from the results in Table 7.1 that only the root node is solved for problems with $n = 45$ (NN is equal to 1), and even the root node cannot be solved for problem instances with $n = 50$, which is shown by "NA" in the corresponding cells of the table. Similar situations arise at the medium and high fixed cost levels too. BB-LR can only solve the root node for instances with $n = 50$ when $f_i = 1000c_i$ (see Table 7.2), and fails to do so for instances with $n = 50$ when $f_i = 10000c_i$ (see Table 7.3). We also would like to mention that there are several cases in Table 7.3 where BB-LR obtains a feasible solution with zero objective value in which no facilities are opened even though it is optimal to open some facilities. In these cases, the percent deviations are 100%. Finally, a PD value above 100 implies that the best feasible solution provided by BB-LR has a negative profit.

One may consider comparing the number of nodes explored under the two BB methods. When we do this, we can conclude that BB-NLP is much more efficient than BB-LR in pruning

Table 7.1. Comparison of the solution methods: $f_i = 100c_i$.

Instance	BB-NLP				BB-LR				LH			
(n, r)	z^*	CPU	NN	NF	PD(%)	CPU	NN	NF	PD(%)	CPU	Gap(%)	NF
(10,1)	41653.0	1.2	3	4	0	325.8	757	4	0	6.7	0.26	4
(10,2)	33499.4	1.2	3	5	0	112.1	571	5	0	7200	1.76	5
(10,3)	29260.5	1.2	3	4	0	101.4	577	4	0	7200	2.01	4
(10,4)	22542.1	1.2	3	4	0	97.1	585	4	0	4.7	0.77	4
(10,5)	19500.0	1.3	3	4	0	91.1	575	4	0	6.8	0.75	4
(15,1)	55969.8	7.3	19	5	0	7200	9404	5	0	23.3	0.41	5
(15,2)	52346.4	8.8	23	6	0	7200	11623	6	0	58.1	0.68	6
(15,3)	47213.9	2.9	7	7	0	4467.3	4914	7	0	19.6	0.32	7
(15,4)	46364.9	2.8	7	7	0	4754.5	6256	7	0	21.3	0.18	7
(15,5)	44908.9	2.8	7	7	0	4063.9	5190	7	0	32.4	0.02	7
(20,1)	82155.1	8.1	21	5	0.79	7200	1518	7	0	193.3	0.65	5
(20,2)	67557.8	10.7	27	7	0.06	7200	645	7	0	177.7	0.89	7
(20,3)	55354.3	15.2	39	7	0.01	7200	1461	8	0	155.8	0.5	7
(20,4)	49372.2	12.6	31	8	0.15	7200	1509	8	0	172.6	0.81	8
(20,5)	45441.3	9.6	25	8	0.8	7200	1332	8	0	165.9	0.78	8
(25,1)	118876.3	190.3	491	9	1.02	7200	132	13	0.07	1882.7	1.86	8
(25,2)	105115.5	143.1	373	10	0.82	7200	207	12	0	911	2.6	10
(25,3)	89590.0	103.3	263	12	1.32	7200	252	17	0.44	706.3	4.02	14
(25,4)	86162.3	82.0	211	13	0.49	7200	386	15	1	737.7	4.79	12
(25,5)	83629.0	82.5	213	13	0.77	7200	371	15	3.81	579.8	4.88	14
(30,1)	118770.6	25.3	65	11	0.73	7200	32	14	0	1716.6	1.41	11
(30,2)	106362.7	26.2	67	12	0.15	7200	98	13	0.09	1218.9	1.62	12
(30,3)	93515.8	34.8	89	13	0.08	7200	35	14	0	1483.3	1.93	13
(30,4)	86389.9	23.0	59	13	0.53	7200	60	13	0.09	777.7	1.77	14
(30,5)	82325.4	17.7	45	13	0.96	7200	133	16	0	943.5	1.89	13
(35,1)	131527.7	114.5	313	9	0.47	7200	52	12	0.00	1958.5	0.62	10
(35,2)	121441.5	40.5	111	11	0.48	7200	63	14	0	1982.1	0.53	11
(35,3)	111374.6	24.5	67	12	0.24	7200	65	14	0	1915.9	0.26	12
(35,4)	104418.4	9.7	21	13	0.004	7200	63	13	0	1836.9	0.23	13
(35,5)	99002.7	6.5	15	12	0.27	7200	64	14	0	1746.3	0.19	12
(40,1)	177402.1	67.2	181	10	1.39	7200	5	10	0	6955.1	22.67	10
(40,2)	163006.1	99.3	267	11	0.96	7200	7	14	0	2773.4	31.71	11
(40,3)	146687.2	53.3	143	14	0.55	7200	9	14	0.12	2833.6	31.56	14
(40,4)	142895.3	38.0	101	14	1.49	7200	8	14	0.13	1405.0	37.35	13
(40,5)	140349.3	28.2	75	15	0.66	7200	7	14	0.07	3609.2	25.19	14
(45,1)	209102.6	310.0	811	12	3.82	7200	1	8	0	7200	17.41	12
(45,2)	181111.1	266.0	705	15	7.04	7200	1	13	0.59	7200	28.66	17
(45,3)	170406.8	129.4	341	15	6.22	7200	1	16	0.29	7200	32.97	16
(45,4)	164095.8	96.0	251	15	4.74	7200	1	16	0.43	7200	31.35	18
(45,5)	158001.9	92.6	241	15	8.79	7200	1	17	0.1	7200	40.85	17
(50,1)	209621.6	325.7	817	13	NA	NA	NA	NA	1.18	7200	17.62	17
(50,2)	180258.3	1527.8	3787	16	NA	NA	NA	NA	0.66	7200	41.66	15
(50,3)	168533.4	3056.0	7479	17	NA	NA	NA	NA	0.36	7200	33.99	19
(50,4)	157999.1	2354.5	5771	18	NA	NA	NA	NA	1.08	7200	36.19	22
(50,5)	148419.3	1633.5	4007	19	NA	NA	NA	NA	1.01	7200	41.85	21
Average		246.4	613.4	10.7	1.2	6110.3	1224.3	10.9	0.3	2786.9	11.3	11.2

Table 7.2. Comparison of the solution methods: $f_i = 1000c_i$.

Instance	BB-NLP				BB-LR				LH			
(n, r)	z^*	CPU	NN	NF	PD(%)	CPU	NN	NF	PD(%)	CPU	Gap(%)	NF
(10,1)	34142.7	3.8	9	2	0	188.2	1323	2	0	5.9	5.53	2
(10,2)	24360.4	3.0	7	2	0	132.3	917	2	0	6.4	3.2	2
(10,3)	20524.3	3.0	7	2	0	79.0	921	2	0	8.4	3.28	2
(10,4)	13734.3	3.8	9	2	0	97.0	777	2	0	4.7	5.32	2
(10,5)	10791.6	2.9	7	2	0	103.5	1027	2	0	4.8	8.46	2
(15,1)	45167.0	18.1	43	2	0	7200	9968	2	0	40.4	6.94	2
(15,2)	39642.7	23.8	57	3	0.66	7200	10511	4	0	42.3	7.2	3
(15,3)	33036.4	22.3	53	3	0	7200	10247	3	0	28.8	5.98	3
(15,4)	31989.6	13.9	33	3	0	7200	10762	3	0	33.4	5.47	3
(15,5)	30251.1	11.4	27	3	0	7200	774	3	0	32.2	5.02	3
(20,1)	69794.6	51.9	123	4	0.34	7200	1481	5	0	256.9	4.49	4
(20,2)	51856.7	23.3	55	5	5.34	7200	1119	5	0	209.4	3.58	5
(20,3)	36322.0	14.1	33	5	12.97	7200	1550	7	0	178.9	4.27	5
(20,4)	30847.4	12.2	29	5	2.41	7200	1623	5	0	143.0	4.01	5
(20,5)	27188.2	11.3	27	5	8.6	7200	1613	5	0	126.4	3.82	5
(25,1)	104249.1	380.2	889	6	11.19	7200	113	6	1.23	1282.9	5.91	4
(25,2)	83780.4	295.4	695	7	4.15	7200	170	7	0.86	961.0	9.64	5
(25,3)	62782.3	435.0	1027	6	1.64	7200	225	6	0.26	932.8	14.54	7
(25,4)	56760.9	318.1	753	7	9.39	7200	325	7	0.43	648.9	17.81	6
(25,5)	53735.7	279.5	661	7	17.69	7200	311	7	0.6	692.9	18.46	6
(30,1)	104850.5	169.3	397	7	6.08	7200	29	8	3.58	2821.4	6.83	5
(30,2)	88375.3	152.3	357	8	6.45	7200	89	8	0.18	1287.7	4.54	6
(30,3)	71347.1	279.0	655	8	12.67	7200	33	11	1.08	1073.5	8.01	6
(30,4)	61971.6	155.7	363	8	11.81	7200	61	10	5.01	708.9	12.79	6
(30,5)	56824.7	146.8	345	8	15.48	7200	146	10	0	867.1	8.99	8
(35,1)	116892.7	145.6	319	5	1.34	7200	45	5	1.33	2565.4	4.05	4
(35,2)	101088.8	71.9	167	5	0.95	7200	63	6	0.09	1843.5	3.58	5
(35,3)	87153.4	51.2	119	6	1.09	7200	63	7	0.17	2080.1	3.96	6
(35,4)	79060.7	34.7	81	6	1.71	7200	56	6	0	1510.6	3.54	6
(35,5)	72875.3	28.9	67	6	1.61	7200	63	8	0	1705.1	3.31	6
(40,1)	157685.0	266.7	611	6	0.62	7200	7	6	0.05	4438.9	3.06	5
(40,2)	137842.0	263.6	603	6	4.21	7200	6	6	0.06	7200.0	16.11	7
(40,3)	118149.8	186.1	427	7	0.44	7200	7	7	0.49	3869.3	33.03	6
(40,4)	113466.0	237.3	547	7	6.03	7200	7	7	0.1	5188.2	33.25	7
(40,5)	110328.4	205.0	471	7	3.06	7200	7	8	0	4064.9	38.68	7
(45,1)	188660.0	734.2	1689	6	21.06	7200	2	4	0.17	7200.0	4.57	7
(45,2)	152066.4	1016.3	2349	8	9.9	7200	3	6	1.54	7200.0	20.85	8
(45,3)	137612.3	767.8	1771	8	6.68	7200	3	7	1.01	7200.0	23.94	11
(45,4)	129040.3	1222.4	2823	9	7.1	7200	3	7	0.61	7200.0	39.97	9
(45,5)	121796.2	1383.7	3183	8	13.2	7200	3	7	1.66	7200.0	26.09	10
(50,1)	189038.4	821.2	1327	8	8.1	7200	1	5	1.5	7200.0	7	8
(50,2)	146512.9	5570.1	10175	10	16.85	7200	1	7	0.33	7200.0	32.87	10
(50,3)	131183.1	2536.0	5053	10	17.4	7200	1	11	1.69	7200.0	19.15	11
(50,4)	117765.9	1342.3	2839	11	12.17	7200	1	8	0.91	7200.0	47.94	13
(50,5)	107110.2	1205.7	2555	11	20.66	7200	1	10	1.49	7200.0	51.28	11
Average		464.9	974.2	6.0	6.2	6413.3	1254.6	6.0	0.6	2641.4	13.3	5.9

Table 7.3. Comparison of the solution methods: $f_i = 10000c_i$.

Instance	BB-NLP				BB-LR				LH			
(n, r)	z^*	CPU	NN	NF	PD(%)	CPU	NN	NF	PD(%)	CPU	Gap(%)	NF
(10,1)	10930.0	3.9	11	1	0	314.2	1985	1	61.22	11.8	78.05	2
(10,2)	0	3.2	9	–	0	193.0	2047	–	0	3.0	100	–
(10,3)	0	2.5	7	–	0	183.9	2047	–	0	2.4	100	–
(10,4)	0	1.1	3	–	0	170.0	2047	–	0	2.1	100	–
(10,5)	0	1.1	3	–	0	152.8	2047	–	0	0.6	100	–
(15,1)	10217.5	14.6	41	1	NA	7200	8467	–	2.23	61.7	62.84	1
(15,2)	1586.2	9.5	27	1	NA	7200	11808	–	38.06	44.1	94.41	1
(15,3)	0	7.5	21	–	0	7200	10468	–	0	35.9	100	–
(15,4)	0	6.0	17	–	0	7200	10656	–	0	25.4	100	–
(15,5)	0	5.3	15	–	0	7200	10344	–	0	23.0	100	–
(20,1)	29258.0	41.3	119	1	31.61	7200	1184	1	31.61	370.7	52.13	1
(20,2)	4413.6	27.9	71	1	0	7200	938	1	0	181.1	75.2	1
(20,3)	0	7.5	21	–	0	7200	1382	–	0	136.2	100	–
(20,4)	0	5.3	15	–	0	7200	1364	–	0	49.6	100	–
(20,5)	0	5.5	15	–	0	7200	1471	–	0	44.8	100	–
(25,1)	55642.7	109.1	303	2	23.12	7200	81	2	3.93	855.1	28.85	2
(25,2)	25159.5	101.4	291	2	47.36	7200	125	1	5.81	545.0	51.01	2
(25,3)	4567.0	59.1	163	2	40.52	7200	213	1	0	359.4	83.26	2
(25,4)	2586.4	40.1	115	1	2206.52	7200	230	1	0	384.3	87.09	1
(25,5)	1756.0	30.5	87	1	480.92	7200	305	1	0	347.4	89.13	1
(30,1)	52656.7	108.9	309	2	55.79	7200	37	1	0	1630.9	29.59	2
(30,2)	28399.7	49.7	141	2	100	7200	71	–	49.78	1177.4	71.16	1
(30,3)	13389.5	39.0	111	2	100	7200	40	–	0	687.9	60.27	2
(30,4)	5371.7	23.0	65	1	100	7200	71	–	0	770.6	78.03	1
(30,5)	1703.4	19.4	55	1	100	7200	156	–	0	663.2	91.93	1
(35,1)	57028.6	127.3	361	1	0	7200	33	1	0	3668.3	30.08	1
(35,2)	30528.8	86.5	245	1	55.06	7200	48	1	1.65	2612.2	43.62	1
(35,3)	17875.8	38.9	109	1	187.57	7200	42	1	0	2280.1	48.14	1
(35,4)	12014.2	18.8	53	1	170.24	7200	41	2	0	1946.2	53.66	1
(35,5)	8681.5	11.9	33	1	197.2	7200	57	2	0	1412.7	60.09	1
(40,1)	72559.4	189.6	527	3	87.31	7200	3	3	4.38	3164.0	35.68	2
(40,2)	39978.0	226.0	623	3	8.22	7200	5	3	0	2311.9	49.28	3
(40,3)	20182.3	138.7	387	3	28.16	7200	6	1	14.66	2452.4	70.26	2
(40,4)	16228.6	120.6	337	2	33.64	7200	4	1	1.27	1778.9	70.04	2
(40,5)	13519.7	108.5	303	2	41.53	7200	6	1	25.02	1756.0	79.73	2
(45,1)	116801.8	401.5	1097	3	24.73	7200	2	1	16.87	2883.0	35.89	3
(45,2)	57344.7	521.2	1397	4	28.45	7200	3	3	11.43	3522.8	44.73	4
(45,3)	38496.8	409.4	1089	4	67.4	7200	3	2	0	2592.7	46.19	4
(45,4)	32622.3	311.8	831	3	104.9	7200	4	3	7.14	2049.3	51.66	4
(45,5)	28992.0	160.5	443	3	78.07	7200	4	1	7.61	2526.3	52.52	3
(50,1)	114507.8	364.3	889	3	NA	NA	NA	NA	10.37	4919.0	27.2	2
(50,2)	47511.9	383.7	951	3	NA	NA	NA	NA	34.6	4077.1	58.25	3
(50,3)	30740.2	156.4	387	2	NA	NA	NA	NA	51.32	4674.3	72.97	3
(50,4)	16413.4	94.8	259	2	100	7200	2	–	90.4	4825.3	95.94	3
(50,5)	10938.1	65.4	179	1	360.96	7200	2	1	96.26	3082.5	98.78	1
Average		103.5	278.6	1.5	115.0	6325.4	1746.1	0.9	12.6	1487.7	70.2	1.5

the nodes of the tree due to the fact that it can generate good feasible solutions so that many nodes can be pruned early in the algorithm. This can be observed in Tables 7.1–7.3 for problem instances up to $n = 25$. For larger instances, one may wonder why the number of nodes solved under BB-NLP is more than that under BB-LR. It is due to the efficiency of BB-NLP in solving the problems at each node. To summarize, the overall effectiveness of the BB-NLP method can be attributed to both the speed at which the concave nonlinear problems can be solved at each node of the BB tree and the generation of good feasible solutions which helps to prune many nodes early in the method.

We also report the percent relative deviation between the best lower and upper bounds for the Lagrangean heuristic LH, which is computed as

$$100 \times \frac{(UB_{best} - LB_{best})}{UB_{best}}. \tag{7.2}$$

The average deviations are 11.3%, 13.3%, and 70.2 for low, medium, and high fixed cost levels, respectively. The large deviations associated with the high fixed cost level can be attributed to the relatively small number of iterations that can be performed in the subgradient optimization when $f_i = 10000c_i$. It is equal to 260.6, whereas the average number of iterations is 349.2 when $f_i = 1000c_i$, and 351.1 when $f_i = 100c_i$. Therefore, for all instances of the high fixed cost level, the step size parameter π is halved every five iterations instead of the suggested value of 30 (Beasley, 1993b).

7.1.2. Assessing the Accuracy and Performance

To see the effectiveness of the recommended solution approach, i.e., BB-NLP, we employ two commercial solvers for MINLP problems that are available within GAMS Suite. The first solver, DICOPT (Grossmann *et al.*, 2004) is based on the outer approximation method in which a sequence of mixed-integer programs and nonlinear programs are solved. It is expected to perform better on models that have a significant and difficult combinatorial part. The second solver, OQNLP, is a multi-start heuristic algorithm designed to find global optima of constrained nonlinear programs that are smooth. By "multi-start" it is meant that the algorithm calls a nonlinear programming solver from multiple starting points which are determined by a scatter search implementation called OptQuest (Laguna and Martí, 2003). We choose the instances at the medium fixed cost level and solve them using these two solvers. The results are reported in Table 7.4 in terms of the percent deviation from the optimal objective value produced by BB-NLP. It is clear that DICOPT outperforms OQNLP with respect to both accuracy and efficiency. In fact, with the exception of two instances OQNLP provides only trivial solutions in which there is no open facility and the corresponding profit is zero. Therefore, the resulting percent deviations are 100%. DICOPT, on the other hand, yields solutions that are 16.58% worse than BB-NLP on the average.

7.1.3. Sensitivity Analysis

To investigate the effect of the model parameters on the captured market share and the realized profit by the new facilities, we carry out further experiments by selecting instance $(40, 3)$. We use BB-NLP as it is the most effective solution method and also can yield the optimal solution and the optimal objective value needed for the sensitivity analysis within a reasonable amount of computation time.

The parameters we consider are the number of existing facilities (r), the unit attractiveness cost (c_i), the fixed cost of new facilities (f_i), and the upper bound for attractiveness (u_i). First, we vary r in the range 1–10, and analyze the resulting market share; the change in the market share and profit displayed in Figure 7.1. We observe that they both decrease as r increases, as expected.

To see the effect of the fixed costs, we change the multiplier that relates the fixed cost to the unit attractiveness cost. Recall that $f_i = 1000c_i$ in the base case scenario. After assigning the values $\{500, 750, 1000, 1250, 1500, 1750, 2000, 2250, 2500\}$ to the multiplier, we obtain the market share and profit as illustrated in Figure 7.2. As can be observed, the profit is monotonically decreasing function of the fixed cost represented by the multiplier. The market share, on the other hand, either decreases or remains the same with an increase

Table 7.4. Comparative results with DICOPT and OQNLP solvers on the instances for $f_i = 1000c_i$.

Instance	DICOPT		OQNLP	
(n,r)	PD(%)	CPU	PD(%)	CPU
(10,1)	0	0.07	98	0.08
(10,2)	0	0.12	100	0.06
(10,3)	19.68	0.07	100	0.06
(10,4)	0	0.08	100	0.06
(10,5)	100	0.06	100	0.06
(15,1)	0.36	0.11	100	0.13
(15,2)	21.86	0.09	100	0.14
(15,3)	27.2	0.06	100	0.14
(15,4)	13.02	0.04	100	0.11
(15,5)	26.48	0.04	100	0.13
(20,1)	5.84	0.10	100	0.22
(20,2)	25.07	0.03	100	0.20
(20,3)	29.39	0.05	27.82	0.16
(20,4)	17.48	0.04	100	0.17
(20,5)	33.99	0.05	100	0.17
(25,1)	7.61	0.05	100	0.34
(25,2)	12.67	0.06	100	0.36
(25,3)	3.79	0.12	100	0.33
(25,4)	21.16	0.13	100	0.34
(30,5)	18.4	0.12	100	0.39
(30,1)	17.23	0.12	100	0.58
(30,2)	27.31	0.09	100	0.59
(30,3)	52.28	0.09	100	0.52
(30,4)	26.86	0.11	100	0.52
(30,5)	22.95	0.11	100	0.55
(35,1)	2.85	0.13	100	0.88
(35,2)	9.93	0.14	100	0.88
(35,3)	13.73	0.12	100	0.84
(35,4)	5.46	0.11	100	0.84
(35,5)	8.31	0.09	100	0.84
(40,1)	16.65	0.14	100	1.08
(40,2)	2.93	0.12	100	1.14
(40,3)	10.57	0.18	100	1.19
(40,4)	16.02	0.11	100	1.08
(40,5)	15.95	0.15	100	1.14
(45,1)	6.63	0.21	100	1.56
(45,2)	17.31	0.16	100	1.42
(45,3)	9.84	0.12	100	1.45
(45,4)	9.81	0.12	100	1.53
(45,5)	13.82	0.14	100	1.50
(50,1)	8.11	0.29	100	0.88
(50,2)	11.1	0.26	100	0.75
(50,3)	12.5	0.20	100	0.84
(50,4)	11.49	0.20	100	0.77
(50,5)	12.42	0.23	100	0.77
Avg.	16.58	0.12	98.35	0.62

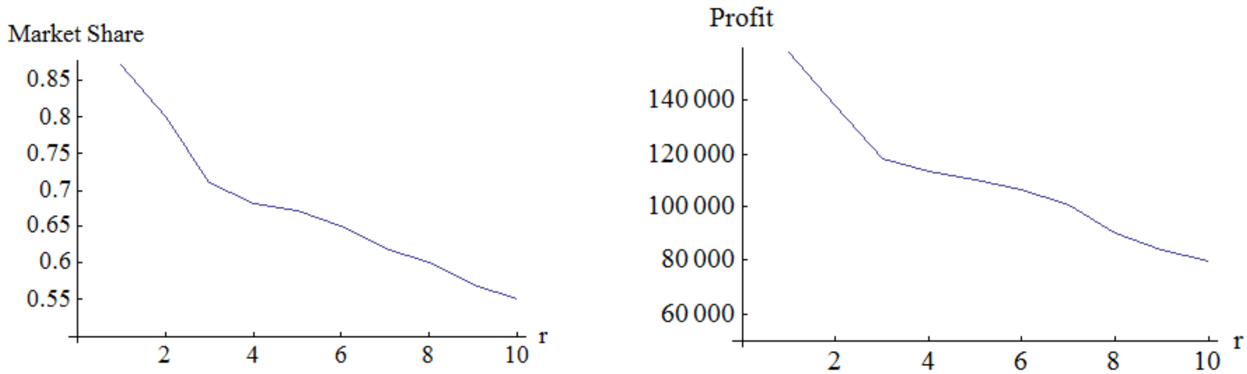

Figure 7.1. Effect of r on the market share and profit

in the fixed cost. We explain this pattern on the basis of an observation we make from the optimal solutions. Namely, as the fixed costs increase, the number of opened facilities is either reduced or remains the same. Furthermore, the opened facilities have always the same optimal attractiveness levels and the same locations. When the number of facilities remains the same with an increase in the fixed costs, the market share is not affected since the attractiveness of neither the competitors nor the new facilities change. However, the resulting profit is reduced since a higher fixed cost is incurred. When the number of open facilities decreases, the market share is also affected in a negative way since the competitors can capture the customers' buying power more. This results in diminishing revenue from the customers. The decrease in the profit can be attributed to the outcome that the reduction in the revenue outweighs the decrease in the fixed costs.

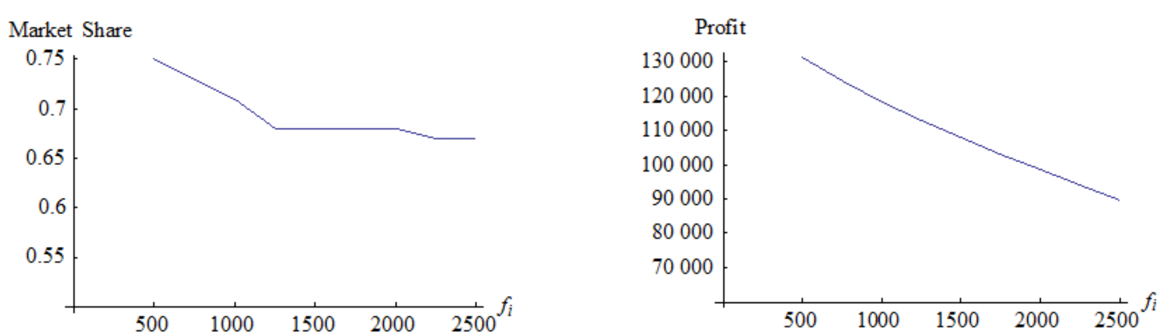

Figure 7.2. Effect of f_i on the market share and profit

In the base scenario, the unit attractiveness cost c_i was randomly generated from the interval $[1, 10]$. The sensitivity of our model to c_i is measured by changing this interval as follows: $[11, 20]$, $[21, 30]$, $[31, 40]$, $[41, 50]$, $[51, 60]$, $[61, 70]$, $[71, 80]$, $[81, 90]$, and $[91, 100]$. The plots in Figure 7.3 where the labels in the x-axis denote the upper limits of the intervals (i.e. 10, 20,..., 100) show that the market share as well as the profit decline with an increase in the unit attractiveness costs. This is an expected result because as c_i increases, either some of the facilities are not opened or the attractiveness levels of some open facilities are reduced resulting a loss in the market share and profit.

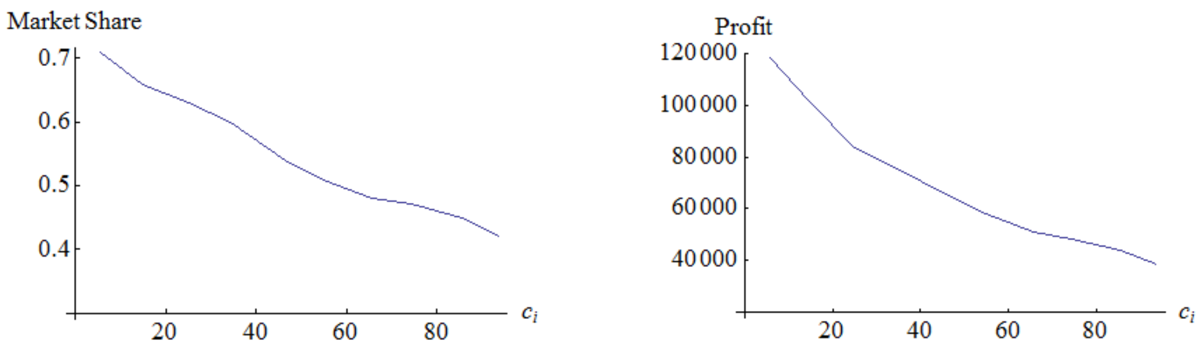

Figure 7.3. Effect of c_i on the market share and profit

Now, we turn our focus on the sensitivity with respect to the parameter u_i. In the base case scenarios, u_i values were set equal to a multiplier times c_i where the multiplier is 100. When we conduct new experiments by varying this multiplier between 100 and 25000, and plot the market share and profit values corresponding to the optimal solutions, we obtain Figure 7.4. The first part of the figure shows that the market share exhibits a slowly increasing trend with some drops and stabilizes after u_i reach very large values. When we examine the solutions, we observe that as the maximum attractiveness levels increase, the optimal attractiveness of some facilities are increased and that of some others are reduced. The facilities with increased attractiveness steal from the market share of the facilities with a reduced attractiveness, which implies that the market share of different facilities may increase or decrease. As a consequence, the overall market share exhibits a fluctuating pattern. It turns out that the resulting profit always shows an increasing trend that asymptotically converges to a limiting value.

To shed further light on the sensitivity analysis with respect to the parameter u_i and find out whether there persist unacceptably low optimal attractiveness levels for some facilities, we carry out the following experiment. Using the same $(40, 3)$ instance with $f_i = 1000c_i$, we only change the maximum attractiveness levels of two facilities (facilities at sites 10 and 37) among seven opened ones. We assign the same values to the multipliers of u_{10} and

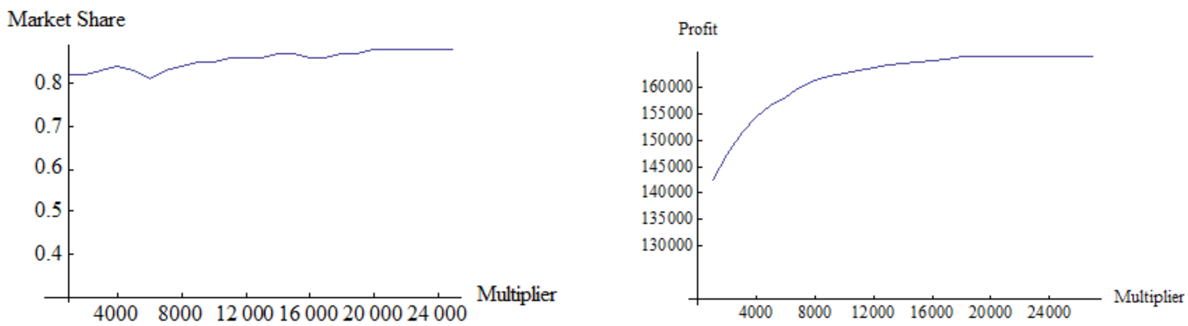

Figure 7.4. Effect of u_i on the market share and profit

u_{37} from the set {1000, 2000, 3000, 4000,..., 30000} and examine the optimal attractiveness levels of the two facilities. As is illustrated in Figure 7.5, the optimal attractiveness Q^*_{10} of the facility at site 10 decreases and the optimal attractiveness Q^*_{37} of the facility at site 37 increases monotonically as the multiplier and hence the maximum attractiveness levels of the two facilities increase. Q^*_{10} is initially equal to 2303.1 when $u_{10} = 1000c_{10}$, and decreases down to 725.5 when $u_{10} = 15000c_{10}$. For larger values of the multiplier, no facility is opened at site 10. This means that unacceptably low optimal levels of the attractiveness are not likely to be seen. The reason is that the cost savings resulting from facility closures is not compensated by the additional revenue that the firm can earn by keeping the facility open at a low attractiveness level. On the other hand, the facility at site 37 has a nondecreasing optimal attractiveness level Q^*_{37} as a result of increasing multiplier values. When the multiplier hits 23000, Q^*_{37} becomes 22940.1 and this value remains constant for further increases in the multiplier value. The maximum attractiveness level is a parameter which is determined by the firm, and in order to prevent situations in which the optimal attractiveness level is regarded as unacceptably high the firm should set reasonable upper bounds that can be achieved. Our experiments with other problem instances have shown that there might persist unacceptably high optimal attractiveness levels, but unacceptably low optimal levels do not occur.

7.2. A Bilevel Competitive Facility Location Problem with Partial Reaction of the Competitor

We use eighteen data sets where the number of demand points $n \in \{10, 20, 30\}$, the number of candidate facility sites $m \in \{2, 3, 5\}$, and the number of existing facilities of the

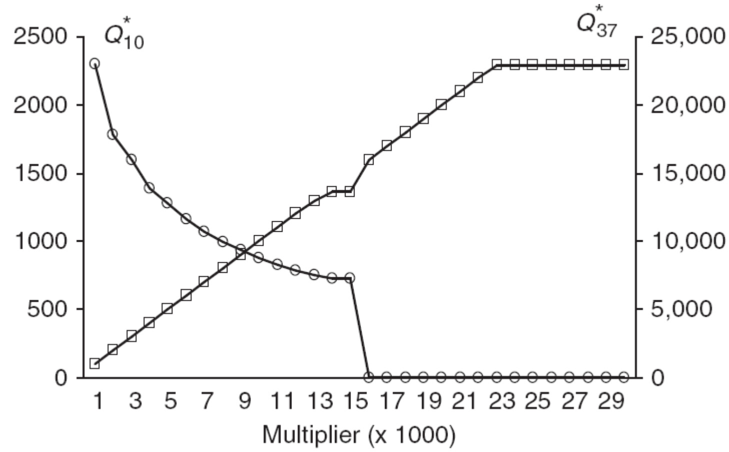

Figure 7.5. A closer look at Q^* values with varying u_i values

competitor $r \in \{1, 2\}$ for the evaluation of the performance of the proposed procedure to solve the CFL problem BP2 given in Section 4.2. For each data set, ten different data instances are created. As a result, we obtain 180 problem instances. The x and y-coordinates of the demand points, the candidate facility sites, and the existing facility sites are integer numbers generated from a uniform distribution defined in the interval $[0, 100]$. The annualized buying power h_j of customers at point j, the unit attractiveness cost c_i at candidate site i for the leader, $\tilde{c}_k$ for the follower, and the current attractiveness $\tilde{A}_k$ of an existing facility at site k are real-valued quantities generated from uniform and normal probability distributions as given in Table 7.5. Note that the means of the parameter values are equal to each other, and that the standard deviation of the normal distribution is equal to one sixth of the range of the uniform distribution.

Table 7.5. The values of parameters h_j, c_i, $\tilde{c}_k$, and $\tilde{A}_k$.

	$U(a,b)$	$N(\mu,\sigma^2)$
h_j	$a = 100, b = 100000$	$\mu = 50050, \sigma^2 = 16650^2$
c_i	$a = 1, b = 10$	$\mu = 5.5, \sigma^2 = 1.5^2$
$\tilde{c}_k$	$a = 1, b = 10$	$\mu = 5.5, \sigma^2 = 1.5^2$
$\tilde{A}_k$	$a = 100, b = 1000$	$\mu = 550, \sigma^2 = 150^2$

The first five data instances of each data set are generated from uniform distributions, whereas the remaining five instances are from normal distributions. The fixed costs f_i are set 1000 times as large as the attractiveness costs, i.e., $f_i = 1000c_i$. Upper bound u_i for the attractiveness of a new facility at candidate site i is taken as $u_i = 10000c_i$ and upper bound $\overline{A}_k$ for the attractiveness of an existing facility is assigned a value of $\overline{A}_k = 10000\tilde{c}_k$.

7.2.1. The Performance of the Solution Methods

As can be seen from the discussion about the solution procedure for the one-level MINLP model in Subsection 4.2.2, GMIN-αBB is a branch-and-bound method at each node of which another branch-and-bound method, αBB algorithm, is employed. Hence, GMIN-αBB is computationally expensive especially for very small values of the convergence parameter ϵ. To improve the running time efficiency of GMIN-αBB, we adopt the approximation strategy suggested by Floudas (2000). Namely, instead of terminating the αBB algorithm when the difference $z_{UB} - z_{LB}$ is less than $\epsilon\%$, we stop the iterations as soon as the improvement in the gap $z_{UB} - z_{LB}$ between two non-successive iterations becomes less than a user-specified threshold value δ. In other words, the BB tree is explored until

$$\frac{\left(z_{UB}^{(t-k)} - z_{LB}^{(t-k)}\right) - \left(z_{UB}^{(t)} - z_{LB}^{(t)}\right)}{z_{UB}^{(t-k)} - z_{LB}^{(t-k)}} \leq \delta \tag{7.3}$$

where $z_{UB}^{(t)}$ and $z_{UB}^{(t-k)}$ are the upper bounds at iterations t and $t-k$, respectively, while $z_{LB}^{(t)}$ and $z_{LB}^{(t-k)}$ are the best lower bounds at iterations t and $t-k$, respectively. k is a user-specified parameter that represents the number of iterations during which the improvement in the gap between z_{UB} and z_{LB} is measured. We choose $k = 3$ and $\delta = 0.1$. We present the results on the test instances in Table 7.6. Here, z_L^{best} indicates the best lower bound of the leader obtained by the GMIN-αBB algorithm run with the above-mentioned approximation strategy, and z_F^{best} stands for the profit realized by the follower as a reaction to the leader.

To check that a solution $(\mathbf{Q}, \mathbf{X}, \mathbf{A})$ is bilevel feasible for problem BP1, i.e., the values of the decision variables $\mathbf{A}$ of the competitor are really those that would be obtained when the competitor's problem were solved given the values of the decision variables $\mathbf{Q}$ and $\mathbf{X}$ of the leader firm, we utilize another procedure which makes use of the result given in Proposition 4.2. Namely, when the values of the leader's variables $\mathbf{Q}$ and $\mathbf{X}$ are fixed, the competitor's (follower's) objective function given in (4.8) is concave in terms of the variables $\mathbf{A}$ for $\mathbf{A} \geq \mathbf{0}$. To find the solution of this concave maximization problem subject to the constraints $0 \leq A_k \leq \overline{A}_k$, we use the same optimality conditions for the CFL problem P (see Subsection 3.2.1) and for the solution of ULP (see Subsection 5.2.2) of the CFL problem BP2 when the leader's and

Table 7.6. Results obtained by GMIN-αBB on 180 instances

Instance (m,n,r)	z_L^{best}	z_F^{best}	CPU (sec.)	Instance (m,n,r)	z_L^{best}	z_F^{best}	CPU (sec.)
	267,010.87	69,255.14	860.9		915,719.32	476,825.06	4692.8
	335,943.92	115,773.54	420.7		753,691.76	526,968.34	6388.5
	186,763.23	60,835.24	336.7		578,965.07	345,570.87	3646.2
	49,935.34	438,952.89	1097.1		58,0488.02	634,724.53	19,161.8
(2,10,1)	203,957.01	125,862.05	561.1	(2,30,2)	697,307.10	341,943.95	43,987.2
	220,743.30	47,021.85	1389.3		493,958.06	407,153.34	2423.4
	247,848.93	87,371.04	860.1		322,204.89	791,142.08	12,318.1
	240,484.48	134,679.55	498.6		639,371.29	464,802.55	34,732.7
	116,794.54	238,660.83	210.5		379,143.38	824,629.68	13,127.6
	176,552.51	98,471.63	354.1		761,507.49	350,377.39	7594.2
	161,544.78	133,136.44	310.5		228,259.02	53,819.60	7042.8
	192,229.36	263,558.59	528.1		165,671.35	265,561.93	574.7
	77,668.09	384,928.51	894.7		257,171.17	155,351.36	1593.8
	7,435.78	510,544.92	983.3		162,328.08	158,149.31	633.8
(2,10,2)	223,314.89	202,936.07	2358.9	(3,10,1)	154,451.37	141,923.08	1463.0
	78,067.33	347,779.22	2362.1		58,807.89	332,796.99	1283.6
	197,665.72	61,130.02	9144.0		155,270.41	96,878.62	2108.0
	258,342.85	70,869.07	751.4		166,037.18	170,484.38	1141.6
	298,722.27	139,039.79	1690.8		125,168.06	127,938.20	364.2
	181,988.23	138,569.61	2983.6		371,037.55	7,288.60	1576.6
	398,665.25	436,325.37	1010.8		131,228.67	66,039.21	8505.1
	452,296.42	56,539.21	2093.5		280,765.55	58,358.53	2735.9
	630,081.90	141,755.83	6437.6		344,335.67	67,861.20	1623.5
	516,210.54	388,477.60	378.6		178,411.07	180,203.29	1363.6
(2,20,1)	402,880.83	287,034.92	3789.5	(3,10,2)	188,883.49	189,441.25	917.5
	247,627.96	325,239.64	2260.6		186,226.52	34,613.38	2084.4
	435,920.82	195,990.83	1510.1		187,386.08	66,437.19	5251.0
	289,196.11	287,155.39	4315.6		120,097.36	117,653.56	1778.0
	494,575.32	79,084.76	3316.5		205,558.36	126,925.35	3704.4
	457,788.39	181,500.86	1162.2		157,548.61	127,425.67	3402.4
	574,507.41	310,152.91	3523.0		629,034.57	110,134.81	6084.6
	302,725.24	540,465.66	5135.8		315,860.37	248,255.75	5320.0
	400,646.47	242,346.84	7706.1		535,827.39	264,489.08	10,430.4
	330,271.66	463,269.57	2556.3		436,059.00	186,186.00	2240.4
(2,20,2)	355,334.63	356,293.55	8854.5	(3,20,1)	407,525.98	361,528.97	6108.6
	283,881.25	257,491.62	5958.5		526,156.63	90,306.13	2193.1
	264,351.96	443,117.13	9031.4		385435.90	242,084.78	9257.2
	330,858.54	475,695.24	3212.7		438,195.24	318,325.22	13,949.5
	288,297.94	229,851.74	4456.9		344,948.95	274,320.99	19,262.9
	487,186.91	193,933.33	4698.2		323,607.40	347,194.48	3595.0
	752,572.57	314632.18	30,184.2		260,735.93	518,138.83	6403.5
	875,376.07	295150.25	10,017.6		404,062.27	182,917.33	18,610.7
	717,126.55	383153.80	6746.2		351,595.92	338,560.73	12,368.7
	442,038.26	543878.04	5383.4		378,316.63	436,454.96	4885.3
(2,30,1)	1,007,622.14	310346.99	22,828.1	(3,20,2)	545,011.58	282,956.24	3,919.3
	940,080.32	389662.24	7,001.3		319,545.82	409,766.56	8920.1
	567,539.34	194918.84	9100.7		548,713.58	203,732.87	10,585.2
	569,393.48	503432.62	3210.4		372,727.00	424,587.75	5288.2
	339,309.67	574505.19	3982.0		266,330.65	334,488.65	2753.7
	638,993.97	580656.08	5336.5		413,352.17	231,286.92	12,962.3

follower's binary variables are fixed: $\mathbf{A}^*$ is a global optimal solution if and only if

Table 7.6. Results obtained by GMIN-αBB on 180 instances (cont)

Instance (m,n,r)	z_L^{best}	z_F^{best}	CPU (sec.)	Instance (m,n,r)	z_L^{best}	z_F^{best}	CPU (sec.)
	835,061.12	488,990.58	15,315.8		672,365.11	3,611.49	27,703.8
	811,046.60	337,442.95	8043.2		473,540.18	208,328.44	8367.6
	517,004.92	593,696.84	35,891.1		359,701.47	252,409.50	12,028.4
	505,948.64	530,607.27	4633.6		548,478.05	73,988.35	19,071.0
(3,30,1)	604,591.14	502,021.78	6885.0	(5,20,1)	682,476.91	32,099.26	33,603.7
	644,524.52	464,323.82	21,568.7		372,465.90	190,345.78	32,108.3
	651,910.16	634,321.65	7459.8		663,044.90	27,409.80	92,467.3
	788,674.95	137,931.81	23,095.6		552,790.06	82,537.72	11,608.2
	413,185.92	667,182.24	14,911.6		390,237.57	277,364.19	20,864.7
	431,963.82	598,780.72	90,056.4		758,732.72	45,151.92	10,103.4
	846,062.44	228,603.39	21,069.7		290,974.18	476,051.38	35,161.6
	635,799.05	623,162.76	11,109.6		621,817.65	190,232.53	7770.7
	521,425.24	464,823.72	9644.3		441,071.33	330,489.54	33,356.7
	543,487.70	420,650.40	13,365.8		404,722.95	276,869.25	41,345.9
(3,30,2)	288,169.71	979,509.89	25,989.1	(5,20,2)	305,967.62	399,617.09	12,377.4
	446,110.07	469,603.84	8549.5		439,461.19	254,353.54	89,688.8
	557,318.55	327,500.94	32,579.1		451,996.77	143,802.41	3888.1
	587,954.85	528,840.45	21,342.6		319,399.81	148,668.99	141,674.7
	405,880.55	427,124.72	54,878.6		469,049.52	239,900.76	6504.9
	480,123.91	497,784.71	8132.5		520,882.15	110,916.44	74,425.8
	197,119.20	102,639.02	2991.8		790,312.43	164,010.99	32,522.0
	462,601.91	77,836.15	5728.9		470,979.20	292,450.46	50,706.3
	326,614.77	86,940.81	5795.9		986,899.79	148,858.56	33,176.7
	281,523.31	50,297.76	1428.6		1,447,961.03	53,358.87	86,994.5
(5,10,1)	115,792.55	172,422.45	1826.5	(5,30,1)	756,829.04	246,190.09	233,923.7
	187,148.00	107,756.67	1091.4		790,851.79	347,410.43	81,344.6
	308,421.16	68,012.42	9810.4		613,196.46	432,247.56	57,985.3
	320,974.56	46,353.79	5427.3		719,137.88	507,555.03	96,536.5
	163,679.11	135,353.72	1177.1		800,173.37	229,847.26	52,636.3
	227,629.14	108,384.69	8941.1		611,546.67	414,744.14	73,755.4
	211,745.70	181,780.36	508.5		743,523.39	401,218.65	318,935.0
	454,426.20	25,073.60	3312.5		892,473.09	41,755.66	1241.1
	191,304.36	66,146.83	5907.8		575,502.75	376,892.55	19,185.1
	313,304.80	22,450.45	2941.4		536,837.14	242,787.66	55,833.4
(5,10,2)	312,192.06	68,398.25	573.0	(5,30,2)	572,308.48	264,832.44	31,938.1
	219,090.94	63,487.06	5397.8		687,330.16	242,709.53	15,2682.5
	233,710.04	113,805.26	5011.5		470,213.53	658,745.14	144,363.4
	207,027.94	86,380.87	12,454.2		736,612.93	248,211.10	243,978.3
	293,448.80	9,481.84	9890.4		676,171.58	347,924.61	333,029.7
	184,962.45	164,563.85	4375.4		642,127.55	106,636.43	85,295.3

i) $\left.\frac{\partial \Pi(\mathbf{A})}{\partial A_k}\right|_{\mathbf{A}^*} \leq 0$ when $A_k^* = 0$,

ii) $\left.\frac{\partial \Pi(\mathbf{A})}{\partial A_k}\right|_{\mathbf{A}^*} \geq 0$ when $A_k^* = \overline{A}_k$,

iii) $\left.\frac{\partial \Pi(\mathbf{A})}{\partial A_k}\right|_{\mathbf{A}^*} = 0$ when $0 < A_k^* < \overline{A}_k$.

These conditions give again rise to the same gradient ascent algorithm we develop in Subsection 3.2.1. The application of this procedure shows that all the solutions found by the GMIN-αBB algorithm are bilevel feasible.

7.2.2. Benefit of Anticipating the Competitor's Reaction

In this subsection our aim is to quantify the benefit (gain) of the leader and the disbenefit (loss) of the competitor when the leader takes into consideration the reaction of the competitor. To this end, we utilize a two-stage method. In the first stage, we solve a version of BP1 where the attractiveness of the existing facilities belonging to the competitor are assumed to be unchanged, i.e., the competitor does not or cannot react to the leader firm. This one-level nonlinear mixed-integer concave maximization model analyzed in Chapter 3 and can be given as follows.

$$\mathrm{P}_1: \qquad \max \sum_{j=1}^{n} h_j \frac{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right)}{\sum_{i=1}^{m} \left(Q_i/d_{ij}^2\right) + \sum_{k=1}^{r} \left(\tilde{A}_k/\tilde{d}_{kj}^2\right)} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{7.4}$$

s.t.

$$Q_i \leq u_i X_i \qquad i = 1, \ldots, m \tag{7.5}$$

$$X_i \in \{0, 1\} \qquad i = 1, \ldots, m \tag{7.6}$$

$$Q_i \geq 0 \qquad i = 1, \ldots, m \tag{7.7}$$

where $\tilde{A}_k$ is the current attractiveness level for the existing facility at site k. Note that in this model the attractiveness levels of competitor's facilities remain at their current values $\tilde{A}_k$ after the market entrant firm opens its new facilities. Since the best performing algorithm for P_1 is a branch-and-bound method with NLP relaxation (BB-NLP) among the three algorithms proposed for the solution of CFL problem P, we employ it in solving all the instances of the previous subsection. The solution of P_1 yields the optimal locations $\mathbf{X}'$ of the new facilities and their attractiveness levels $\mathbf{Q}'$, which completes the first stage. By fixing the values of these variables, in the second stage of our method, we determine the optimal reaction of the competitor by solving the following model.

$$\mathrm{P}_2: \qquad \max \sum_{j=1}^{n} h_j \frac{\sum\limits_{k=1}^{r} \left(A_k/\tilde{d}_{kj}^2\right)}{\sum\limits_{i=1}^{m} \left(Q_i'/d_{ij}^2\right) + \sum\limits_{k=1}^{r} \left(A_k/\tilde{d}_{kj}^2\right)} - \sum_{k=1}^{r} \tilde{c}_k (A_k - \tilde{A}_k) \tag{7.8}$$

s.t.

$$0 \leq A_k \leq \overline{A}_k \qquad k = 1, ..., r \tag{7.9}$$

Note that this model is the same as the one which was used earlier for checking the bilevel feasibility of the solution generated by GMIN-αBB. Hence, it can be solved to global optimality by the gradient ascent procedure explained in previously. As soon as we obtain the optimal reaction $\mathbf{A}'$ of the competitor from P_2, it is possible to calculate the profit z_L' realized by the market entrant firm as well as the profit z_F' by the competitor using the values of $(\mathbf{X}', \mathbf{Q}', \mathbf{A}')$ as follows:

$$z_L' = \sum_{j=1}^{n} h_j \frac{\sum\limits_{i=1}^{m} \left(Q_i'/d_{ij}^2\right)}{\sum\limits_{i=1}^{m} \left(Q_i'/d_{ij}^2\right) + \sum\limits_{k=1}^{r} \left(A_k'/\tilde{d}_{kj}^2\right)} - \sum_{i=1}^{m} f_i X_i' - \sum_{i=1}^{m} c_i Q_i' \tag{7.10}$$

$$z_F' = \sum_{j=1}^{n} h_j \frac{\sum\limits_{k=1}^{r} \left(A_k'/\tilde{d}_{kj}^2\right)}{\sum\limits_{i=1}^{m} \left(Q_i'/d_{ij}^2\right) + \sum\limits_{k=1}^{r} \left(A_k'/\tilde{d}_{kj}^2\right)} - \sum_{k=1}^{r} \tilde{c}_k \left(A_k' - \tilde{A}_k\right) \tag{7.11}$$

Thus, $\left(z_L^{\text{best}} - z_L'\right)$ describes the gain of the market entrant firm associated with foreseeing the reaction of the competitor. To put it differently, it is the reduction in the profit of the leader due to the carelessness of the capability of the competitor for redesigning its facilities. Similarly, $\left(z_F^{\text{best}} - z_F'\right)$ is the loss of the competitor due to the anticipation of its reaction by the leader. The results given in Table 7.7 are obtained for the same 180 instances considered before. For each instance, the gain (loss) of the leader (competitor) is computed as the percentage relative deviation $100 \times \left(z_L^{\text{best}} - z'_L\right)/z'_L$ $(100 \times \left(z_F^{\text{best}} - z'_F\right)/z'_F)$.

The average gain of the leader computed over all the instances is 58.33%, which indicates that there is a substantial gain for this firm when it incorporates the reaction of the competitor into his profit maximization model. Similarly, the average loss of the competitor is 45.31%, which proves that the competitor is better off if its reaction is not taken into account by the leader. It is worth mentioning that there are some instances in Table 7.2.1 (e.g. the 5th instance with $(m, n, r) = (2, 10, 1)$) where the gain of the leader turns out to be zero, or equivalently z_L^{best} and z'_L have the same value. A closer examination of these cases reveals that the leader firm opens the same facility at the maximum attractiveness level with or without anticipating the competitor's reaction, and hence the reaction of the competitor becomes the same in both cases. We think that the occurrence of these cases are due to the small size of the instances where the number of alternative decisions is very limited from the perspective of the leader, i.e., there are only two or three candidate sites for opening facilities, and only one facility is opened. Indeed, the gain is not zero for any of the relatively larger instances in Table 7.7.

An interesting observation we make from the results is that not only the leader but also the competitor sometimes takes advantage of the leader's effort to take competitor's reaction into account. For example, consider the fourth instance when $(m, n, r) = (2, 10, 1)$. The profit of the leader increases from 43,310.35 to 49,935.34 (15.30%) as the result of anticipation of the competitor's reaction. This also helps the competitor in increasing its profit from 379,836.31 to 438,952.89 (15.56%). A positive value in the loss columns in Table 7.7 indicates the existence of this rare but interesting situation.

7.2.3. An Instance Based On Real-World Data

In this subsection, we use the data set of a real-world example provided by Drezner and Drezner (2002) for the solution of BP1. The data include the locations of 28 communities in Orange County, California, United States, and their effective buying income, the locations of

Table 7.7. The gain of the leader and the loss of the competitor

Instance (m,n,r)	z'_L	z'_F	Gain (%)	Loss (%)	Instance (m,n,r)	z'_L	z'_F	Gain (%)	Loss (%)
	202,468.4	177,588.5	31.88	-61.00		893,840.5	588,580.6	2.45	-18.99
	334,653.7	125,754.5	0.39	-7.94		719,718.9	612,667.2	4.72	-13.99
	142,141.2	148,635.5	31.39	-59.07		511,025.3	485,390.2	13.29	-28.81
	43,310.4	379,836.3	15.30	15.56		527,892.4	862,852.4	9.96	-26.44
(2,10,1)	203,957.0	125,862.1	0.00	0.00	(2,30,2)	520,950.8	726,412.5	33.85	-52.93
	155,789.8	198,145.4	41.69	-76.27		320,230.4	790,428.1	54.25	-48.49
	140,107.2	295,768.9	76.90	-70.46		301,628.5	837,833.6	6.82	-5.57
	143,715.9	265,355.8	67.33	-49.25		561,511.2	644,774.7	13.87	-27.91
	787,42.3	273,113.9	48.32	-12.61		365,620.6	873,853.8	3.70	-5.63
	126,106.7	241,478.6	40.00	-59.22		505,003.3	709,197.6	50.79	-50.60
	108,896.8	249,594.0	48.35	-46.66		154,276.6	223,351.3	47.95	-75.90
	175,062.0	271,110.1	9.81	-2.79		165,671.4	265,561.9	0.00	0.00
	70,624.2	430,108.4	9.97	-10.50		213,394.8	262,136.8	20.51	-40.74
	203.0	425,831.5	3563.31	19.89		89,621.8	317,989.4	81.13	-50.27
(2,10,2)	209,889.5	267,169.8	6.40	-24.04	(3,10,1)	64,974.1	324,124.2	137.71	-56.21
	37,120.9	333,331.2	110.31	4.33		41,090.3	356,673.4	43.12	-6.69
	187,395.3	166,102.7	5.48	-63.20		105,681.1	245,017.5	46.92	-60.46
	221,654.6	143,789.3	16.55	-50.71		141,533.5	225,487.9	17.31	-24.39
	218,243.9	209,704.3	36.88	-33.70		71,999.5	244,842.2	73.85	-47.75
	165,951.9	247,001.0	9.66	-43.90		246,246.8	112,799.0	50.68	-93.54
	256,496.7	734,908.5	55.43	-40.63		131,134.3	85,517.2	0.07	-22.78
	234,016.3	497,314.8	93.28	-88.63		237,200.0	178,892.7	18.37	-67.38
	527,681.3	249,830.4	19.41	-43.26		278,015.9	129,659.4	23.85	-47.66
	462,047.0	547,046.5	11.72	-28.99		163,197.1	181,560.3	9.32	-0.75
(2,20,1)	331,338.2	482,863.5	21.59	-40.56	(3,10,2)	149,895.1	232,789.5	26.01	-18.62
	223,754.9	441,818.8	10.67	-26.39		129,571.7	150,235.3	43.72	-76.96
	243,262.9	500,256.9	79.20	-60.82		127,238.8	200,680.7	47.27	-66.89
	159,216.8	579,699.3	81.64	-50.46		99,014.4	198,795.7	21.29	-40.82
	282,392.9	383,848.0	75.14	-79.40		110,969.9	271,048.0	85.24	-53.17
	271,397.3	514,130.6	68.68	-64.70		150,392.8	155,662.4	4.76	-18.14
	462,682.7	548,574.2	24.17	-43.46		452,086.0	438,131.0	39.14	-74.86
	302,725.2	540,465.7	0.00	0.00		256,230.6	392,925.3	23.27	-36.82
	367,728.4	459,997.7	8.95	-47.32		404,327.3	541,022.8	32.52	-51.11
	188,253.8	806,459.8	75.44	-42.56		366,233.6	343,609.1	19.07	-45.81
(2,20,2)	244,588.7	680,229.3	45.28	-47.62	(3,20,1)	269,286.2	616,682.3	51.34	-41.38
	225,992.1	491,382.7	25.62	-47.60		305,942.9	410,199.2	71.98	-77.98
	248,894.7	439,203.0	6.21	0.89		300,216.7	511,344.7	28.39	-52.66
	288,731.8	550,308.6	14.59	-13.56		279,141.0	631,972.6	56.98	-49.63
	248,038.0	405,867.3	16.23	-43.37		236,638.5	526123.4	45.77	-47.86
	352,303.3	435,238.2	38.29	-55.44		290,778.1	516,788.3	11.29	-32.82
	665,117.5	466,695.6	13.15	-32.58		176,952.6	545,066.3	47.35	-4.94
	692,450.9	604,187.2	26.42	-51.15		307,241.4	472,582.8	31.51	-61.29
	583,168.5	677,519.4	22.97	-43.45		292,480.0	572,893.1	20.21	-40.90
	368,764.1	779,630.8	19.87	-30.24		310,838.1	609,895.2	21.71	-28.44
(2,30,1)	852,215.5	482,172.0	18.24	-35.64	(3,20,2)	480,233.0	414,692.4	13.49	-31.77
	631,237.2	851,375.5	48.93	-54.23		212,067.6	599,876.4	50.68	-31.69
	280,596.9	861936.7	102.26	-77.39		328,196.1	544,770.1	67.19	-62.60
	461,875.0	738,125.0	23.28	-31.80		332,306.3	561,927.7	12.16	-24.44
	248,281.7	857,877.1	36.66	-33.03		206,937.2	565,712.9	28.70	-40.87
	468,017.1	934,989.3	36.53	-37.90		285,351.7	477,666.3	44.86	-51.58

Table 7.7 The gain of the leader and the loss of the competitor (cont)

Instance (m,n,r)	z'_L	z'_F	Gain (%)	Loss (%)	Instance (m,n,r)	z'_L	z'_F	Gain (%)	Loss (%)
	708,852.6	704,964.0	17.80	-30.64		343,070.3	379,001.4	95.98	-99.05
	564,020.4	832,798.6	43.80	-59.48		411,193.6	334,127.4	15.16	-37.65
	464,970.8	822,731.1	11.19	-27.84		191,845.4	540,676.5	87.50	-53.32
	395,822.7	792,958.0	27.82	-33.09		314,899.6	392,828.4	74.18	-81.17
(3,30,1)	500,084.4	751,846.7	20.90	-33.23	(5,20,1)	375,830.4	364,714.7	81.59	-91.20
	372,558.4	900,031.5	73.00	-48.41		248,560.3	471,773.1	49.85	-59.65
	618,994.3	756,969.0	5.32	-16.20		393,575.0	389,380.7	68.47	-92.96
	527,612.9	564,615.5	49.48	-75.57		298,452.5	507,313.6	85.22	-83.73
	308,615.7	866,520.7	33.88	-23.00		231,307.0	467,592.5	68.71	-40.68
	398,768.6	687,308.4	8.32	-12.88		581,865.8	230,477.4	30.40	-80.41
	562,512.5	734,778.9	50.41	-68.89		250,947.7	447,996.3	15.95	6.26
	497,548.0	865,110.5	27.79	-27.97		514,161.9	347,493.6	20.94	-45.26
	364,240.4	746,203.1	43.15	-37.71		352,817.9	454,030.1	25.01	-27.21
	481,581.6	662,125.9	12.85	-36.47		243,929.4	567,656.5	65.92	-51.23
(3,30,2)	280,660.6	1,051,812.0	2.68	-6.87	(5,20,2)	247,026.1	510,245.2	23.86	-21.68
	350,359.3	670,350.7	27.33	-29.95		368,766.5	483,874.6	19.17	-47.43
	379,473.4	759,000.5	46.87	-56.85		244,443.9	461,552.1	84.91	-68.84
	406,924.8	757,076.5	44.49	-30.15		213,975.4	370,305.4	49.27	-59.85
	363,254.1	737,528.3	11.73	-42.09		353,327.6	427,663.6	32.75	-43.90
	366,055.3	771,376.3	31.16	-35.47		343,706.7	459,675.9	51.55	-75.87
	153,131.7	194,126.2	28.73	-47.13		539,053.4	443,182.2	46.61	-62.99
	301,308.6	204,990.4	53.53	-62.03		213,279.5	744,092.0	120.83	-60.70
	231,668.4	239,574.2	40.98	-63.71		777,654.1	371,083.1	26.91	-59.89
	171,490.6	161,474.3	64.16	-68.85		1,293,990.8	175,540.7	11.90	-69.60
(5,10,1)	62,965.3	257,666.8	83.90	-33.08	(5,30,1)	520,283.8	650,635.7	45.46	-62.16
	149,693.6	220,154.3	25.02	-51.05		481,680.7	824,040.7	64.19	-57.84
	229,160.6	175,594.7	34.59	-61.27		469,933.4	715,358.7	30.49	-39.58
	227,649.6	177,708.0	41.00	-73.92		474,788.9	981,497.1	51.46	-48.29
	97,264.1	293,585.9	68.28	-53.90		386,712.3	784,124.6	106.92	-70.69
	154,747.7	244,826.2	47.10	-55.73		394,333.8	835,329.5	55.08	-50.35
	98,560.6	401,199.6	114.84	-54.69		671,861.2	538,055.8	10.67	-25.43
	321,126.5	160,438.6	41.51	-84.37		661,892.2	690,837.6	34.84	-93.96
	131,690.3	166,152.2	45.27	-60.19		489,740.5	654,684.7	17.51	-42.43
	244,715.4	109,253.3	28.03	-79.45		375,678.9	538,468.7	42.90	-54.91
(5,10,2)	237,161.1	179,912.3	31.64	-61.98	(5,30,2)	396,781.3	573,022.5	44.24	-53.78
	154,032.5	170,137.8	42.24	-62.68		462,313.8	644,535.6	48.67	-62.34
	169,307.9	221,011.2	38.04	-48.51		338,422.2	814,529.6	38.94	-19.13
	179,555.9	146,705.5	15.30	-41.12		399,593.5	719,141.1	84.34	-65.49
	185,010.6	168,677.9	58.61	-94.38		561,874.5	570,644.3	20.34	-39.03
	179,040.6	229,764.8	3.31	-28.38		414,306.1	641,367.9	54.99	-83.37

10 shopping malls, their inferred attractiveness levels, and the Euclidean distances between the communities and the shopping malls corrected using the areas of the communities. We take the communities as the demand points, and randomly choose the locations of two malls as the existing facility sites of the competitor (South Cost Plaza and Main Place). The locations of the remaining eight malls constitute the candidate facility site for the leader. The attractiveness of the existing facilities are set equal to the inferred attractiveness of the two malls after multiplying by 1000 for scaling purposes. Since the data set does not include every parameter used in BP2, we need to assign the values of the following parameters. The attractiveness cost of the existing facilities as well as the new facilities to be opened by the leader firm is set to five. As in the case of randomly generated instances, the fixed costs of the new facilities are taken as 1000 times the unit attractiveness cost (i.e., 5000) and the maximum attractiveness levels of new facilities are equated to 10,000 the unit attractiveness cost (i.e., 50,000). The problem and the optimal solution are depicted in Figure 7.6 and Figure 7.7. The leader firm enters the market with three facilities opened at the second, fourth, and sixth candidate sites at attractiveness levels 3573, 3035, and 7133, respectively. The competitor reacts by increasing the attractiveness of its facilities from 3629 to 6880, and from 748 to 7340, respectively.

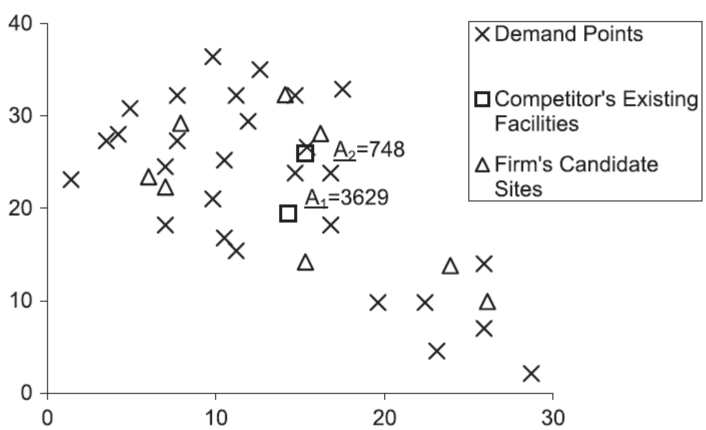

Figure 7.6. A problem with real-world data

7.3. A Bilevel Competitive Facility Location Problem with Full Reaction of the Competitor

In this section, we first evaluate the performance of the proposed TS heuristics for the solution of the bilevel CFL model BP2 examined in Chapter 5 in comparison with the ϵ-

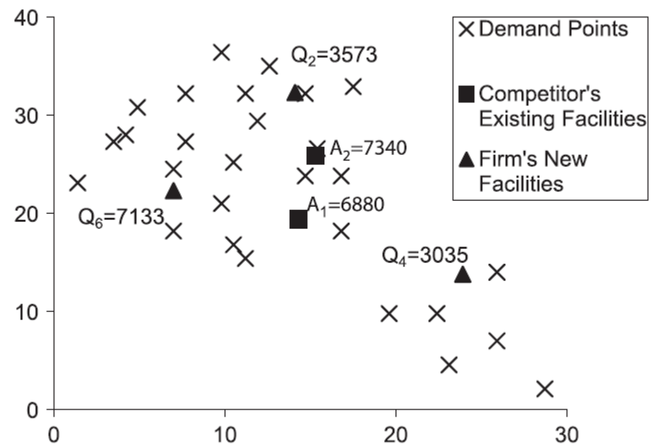

Figure 7.7. Optimal solution of the problem with real-world data

optimal solution method proposed in the same chapter. We use eight data sets where the number of demand points (n) is set to 10 and 20, the number of existing facilities of the competitor (r_1) and the number of the candidate facility sites of the competitor (r_2) to one and two, while the number of candidate facility sites of the firm (m) is equal to five. The first $r_1 + r_2$ candidate sites of the firm coincide with the candidate and existing facility sites of the competitor. This enables us to capture the possibility of the firm to co-locate its facilities with those of the competitor if it is profitable to do so. Three different instances are generated for each dataset, which makes a total of 24 problem instances in total. The reason why we choose relatively small values for m and n is due to the fact that the GMIN-αBB can only handle instances of this size within reasonable computation times. Furthermore, we have to also apply the same modified version of the GMIN-αBB algorithm given in Subsection 7.2.1, where the gap between the lower bound and upper bound on the objective values becomes less than a user-specified threshold value.

The x and y-coordinates of the demand points, the candidate facility sites of both parties, and the existing facility sites of the competitor are integer numbers generated from a discrete uniform distribution defined in the interval $[0, 100]$. The annual buying power h_j of customers at point j, the unit attractiveness cost c_i of the firm's new facility at site i, unit attractiveness cost e_l of competitor's new facility at site l, unit cost of increasing or unit revenue of decreasing the attractiveness of competitor's existing facility at site (b_k), and the current attractiveness $\underline{A}_k$ of competitor's existing facility at site k are integer-valued parameters generated from the uniform distributions as: $h_j \sim U(100, 100000)$, $c_i \sim U(1, 10)$, $e_\ell \sim U(1, 10)$, $b_k \sim U(1, 10)$, and

$\underline{A}_k \sim U(100, 1000)$. The fixed costs f_i and $\widetilde{f}_\ell$ are set 1000 times as large as the attractiveness costs, i.e., $f_i = 1000c_i$ and $\widetilde{f}_\ell = 1000e_\ell$. The upper bounds u_i, $\overline{A}_k$, and $\overline{M}_\ell$ on the attractiveness of a new or existing facility are taken as $u_i = 10000c_i$, $\overline{A}_k = 10000b_k$, and $\overline{M}_l = 10000e_l$, respectively. Finally, the revenue t_k that incurs to the competitor by closing an existing facility at site k is set to 500 times as large as the attractiveness cost b_k, i.e. $t_k = 500b_k$. The maximum number of iterations performed (*max_iter*) and the maximum number of iterations without an improvement in the incumbent (*max_nonimp_iter*) are set to 1000 and 100, respectively.

Having compared the TS heuristics with the ϵ-optimal solution method, we create larger problem instances and compare their accuracy with each other. To this end, we generate five instances for each of the 18 data sets, where $n \in \{50, 60, 70, 80, 90, 100\}$, $r_1 \in \{2, 3, 4\}$, $r_2 = r_1$, and $m = 2(r_1 + r_2)$. This makes 90 problem instances in total.

7.3.1. Comparing the Tabu Search Heuristics and the ϵ-Optimal Solution Method

In this subsection, the computational results of the comparison of the TS heuristics with the ϵ-optimal solution method are given. For each of the 24 created instances we provide the corresponding CPU time and the accuracy of each TS heuristic measured as a percent relative deviation (PD), which is computed as $100 \times (\phi^* - \phi)/\phi^*$, where ϕ indicates the objective value of a feasible solution found by a TS heuristic. Since TS-3 starts with the best feasible solution given by TS-2, the CPU time for each instance is obtained as the sum of the corresponding CPU time spent by TS-2 to find the starting solution and the CPU time of running TS-1. Moreover, as TS-1 and TS-3 generate the attractiveness levels $\mathbf{Q}$ of the firm randomly, we run them five times for each instance, and report the percent relative deviation of the average objective value (PD_{avg}) and that of the best objective value (PD_{max}) over five runs. As can be observed, the average deviation of TS-2 is 1.10%. The averages of PD_{avg} and PD_{max} are 0.69% and 0.19% for TS-1 and 0.29% and 0.09% for TS-3. This indicates that all TS heuristics produce solutions of good quality, and in line with our expectations TS-3 outperforms TS-1 and TS-2. This, however, comes at the expense of additional CPU time. The average CPU times are 64.1 s, 192.5 s, and 213.0 s for TS-2, TS-1, and TS-3, respectively. Nevertheless, the efficiency of all TS heuristics is considerably better than that of the ϵ-optimal solution method which requires 7140.0 s on the average.

Table 7.8. Efficiency of the TS heuristics: comparison with the ϵ-optimal solution method

Dataset	ϵ-optimal		TS-2		TS-1			TS-3		
(m, n, r_1, r_2)	Profit (ϕ^*)	CPU (s)	PD (%)	CPU (s)	PD_{avg} (%)	PD_{max} (%)	CPU (s)	PD_{avg} (%)	PD_{max} (%)	CPU (s)
	205,592.2	638.7	0.00	2.0	0.81	0.24	106.4	0.00	0.00	66.5
(5,10,1,1)	133,326.9	874.8	<0.01	1.2	1.13	<0.01	70.5	<0.01	<0.01	43.3
	256,934.6	1777.3	3.90	230.4	2.21	0.54	65.3	1.64	0.39	303.4
	114,496.0	1718.0	3.74	1.6	0.97	<0.01	164.4	0.27	0.00	126.1
(5,10,1,2)	122,742.9	2157.5	1.34	98.3	0.28	0.04	123.8	0.21	0.02	221.8
	170,216.5	2972.7	0.00	2.5	0.32	0.09	195.5	0.00	0.00	111.6
	270,919.7	1576.5	3.65	57.7	0.08	0.01	98.8	0.11	0.03	158.9
(5,10,2,1)	209,598.4	2206.5	2.60	50.4	0.11	0.04	131.9	0.14	0.01	208.6
	188,913.0	4967.7	1.50	192.5	0.96	0.11	148.1	0.97	0.35	339.7
	72,214.6	10,601.5	0.48	2.0	0.05	<0.01	207.9	0.05	0.00	197.7
(5,10,2,2)	155,149.8	6369.5	0.00	3.8	0.61	0.07	291.9	0.00	0.00	175.2
	155,957.3	9180.3	0.00	163.6	0.68	0.32	400.2	0.00	0.00	377.6
	207,269.6	5391.8	0.00	33.4	0.00	0.00	58.8	0.00	0.00	89.2
(5,20,1,1)	488,628.2	942.4	1.54	1.8	0.71	0.30	88.1	0.75	0.16	102.4
	511,244.0	1595.8	0.00	2.2	0.40	0.14	104.9	0.00	0.00	76.7
	458,578.0	8053.5	0.00	3.0	<0.01	0.00	204.6	0.00	0.00	136.1
(5,20,1,2)	316,072.2	5667.2	0.72	54.6	0.54	0.36	158.7	0.60	0.42	203.3
	276,636.3	3482.4	0.00	1.7	0.00	0.00	137.1	0.00	0.00	139.5
	317,272.3	12,466.9	0.00	75.2	1.85	0.61	286.2	0.00	0.00	211.8
(5,20,2,1)	593,588.8	2071.5	0.00	56.0	1.68	0.79	192.9	0.00	0.00	176.8
	270,621.5	5106.7	0.01	60.0	0.31	0.13	194.6	0.01	0.01	183.3
	492,045.9	32,863.9	2.78	272.2	1.22	0.06	390.3	1.30	0.38	652.3
(5,20,2,2)	503,150.1	24,378.1	0.00	8.7	0.39	0.03	459.7	0.00	0.00	278.9
	313,793.8	24,299.1	3.00	164.9	0.64	0.15	340.2	0.72	0.28	532.0
Average		7140.0	1.10	64.1	0.69	0.19	192.5	0.29	0.09	213.0

7.3.2. Comparing the Tabu Search Heuristics on Larger Instances

Being satisfied with the performance of the three TS heuristics we create the larger problem instances mentioned at the beginning of the section and compare their accuracy with each other. For the ease of comparison, we highlight the best objective value in boldface for each problem instance. We observe that out of 90 instances TS-3 provides the best solution for 70 instances. Among these, TS-2 and TS-3 find the same solution in 38 instances. In other words, TS-3 cannot improve the solution provided by TS-2. TS-1 is the best performing heuristic for 20 instances. The comparison of the TS heuristics in terms of efficiency reveals that TS-2 spends 6177.5 s on the average, and is much faster than TS-1 and TS-3 which require an average of 41383.8 s and 27174.2 s, respectively.

7.4. A Discrete Facility Location Problem with Customer Preferences

We generate ten data sets where the number of customer zones $n \in \{50, 100, 150, 200, 250, 300, 350, 400, 450, 500\}$ and the number of facility types is three for the evaluation of the performance of the proposed solution procedures given in Section 6.3. The potential facility sites coincide with the customer zones so that each new facility is opened at one of the locations where customers are assumed to be aggregated. For each

Table 7.9. Efficiency comparison of the TS heuristics for larger instances

Dataset	TS-2		TS-1			TS-3		
(m,n,r_1,r_2)	ϕ^2	CPU (s)	ϕ^1_{avg}	ϕ^1_{max}	CPU (s)	ϕ^3_{avg}	ϕ^3_{max}	CPU (s)
	708,017.7	29.8	750,366.7	766,382.6	2452.5	760,005.2	**766,907.3**	1574.2
	1,263,561.1	41.7	1,270,250.1	**1,290,220.1**	2983.0	1,277,559.5	1,288,133.8	1567.3
(8,50,2,2)	1,100,169.1	700.8	1,128,690.3	**1,132,967.5**	2237.6	1,128,219.0	1,131,741.1	3203.3
	1,435,969.4	47.1	1,415,429.4	1,425,493.0	2911.8	1,435,969.4	**1,435,969.4**	1034.5
	974,414.7	35.4	992,894.4	**998,766.5**	2210.8	991,264.0	996,140.2	1875.7
	1,295,452.5	775.9	1,297,641.0	1,300,481.3	3266.1	1,299,308.0	**1,301,186.1**	2709.0
	1,298,793.4	1241.6	1,281,583.3	1,289,867.2	3552.9	1,298,793.4	**1,298,793.4**	2211.7
(8,60,2,2)	1,765,233.6	17.3	1,805,048.4	1,819,376.9	2862.5	1,816,865.7	**1,822,650.2**	1723.6
	1,082,383.6	21.4	1,075,318.1	1,078,942.6	2320.4	1,082,383.6	**1,082,383.6**	1020.7
	1,032,622.6	772.3	1,117,480.9	1,122,044.4	4074.6	1,097,013.1	**1,122,392.2**	3088.8
	1,531,871.9	66.7	1,550,223.1	**1,555,918.1**	3364.3	1,547,673.0	1,551,500.9	2978.9
	1,315,932.8	5737.1	1,306,634.1	1,315,198.4	3153.0	1,316,403.0	**1,317,913.0**	7765.6
(8,70,2,2)	**1,346,942.7**	1004.4	1,323,815.2	1,343,446.7	3574.5	1,346,942.7	**1,346,942.7**	2859.6
	1,732,022.6	1379.1	1,736,160.3	**1,739,953.3**	3449.7	1,736,766.2	1,738,901.7	3676.8
	2,051,814.9	54.3	2,024,865.5	2,033,849.9	1489.8	2,051,814.9	**2,051,814.9**	755.2
	1,988,501.0	55.8	2,011,973.6	**2,025,662.7**	2471.0	2,007,983.1	2,019,337.1	1980.6
	1,712,239.1	945.4	1,693,575.8	1,709,185.3	5014.2	1,712,394.2	**1,713,014.4**	2755.3
(8,80,2,2)	**1,438,617.7**	361.0	1,431,195.0	1,435,705.6	2072.7	1,438,617.7	**1,438,617.7**	1458.5
	1,607,937.7	53.1	1,603,809.4	1,606,703.3	2641.5	1,607,937.7	**1,607,937.7**	1194.4
	1,826,378.6	1547.7	1,793,954.2	1,810,550.0	2380.0	1,831,595.1	**1,841,547.9**	3013.4
	1,837,377.8	1030.2	1,788,463.0	1,822,999.3	2704.3	1,838,015.3	**1,840,565.3**	2960.4
	1,542,366.6	142.3	1,546,174.9	**1,547,088.2**	4468.1	1,544,318.8	1,546,254.5	2847.7
(8,90,2,2)	2,093,375.7	606.0	2,099,596.4	2,102,111.4	2991.2	2,096,825.7	**2,102,277.0**	2180.4
	1,979,438.3	70.5	1,907,056.9	1,934,454.6	1916.6	1,979,438.3	**1,979,438.3**	1339.7
	1,972,675.6	1908.7	1,913,771.5	1,929,643.4	1862.8	1,972,675.6	**1,972,675.6**	3018.2
	2,217,403.3	66.4	2,202,827.6	2,233,622.0	4438.3	2,229,775.2	**2,249,225.1**	3642.3
	2,182,111.6	261.8	2,163,313.4	2,175,738.1	4268.7	2,182,111.6	**2,182,111.6**	2414.2
(8,100,2,2)	**2,650,566.0**	923.1	2,598,623.8	2,616,071.2	3446.1	2,650,566.0	**2,650,566.0**	1934.0
	1,740,434.0	2154.4	1,758,666.5	1,771,011.5	7460.8	1,761,096.4	**1,778,636.2**	7237.0
	2,351,607.3	81.7	2,340,927.3	2,346,270.3	5611.3	2,351,607.3	**2,351,607.3**	2445.8
	1,202,746.0	415.7	1,157,165.4	1,163,015.9	15969.0	1,202,746.0	**1,202,746.0**	7148.2
	995,525.1	2928.8	1,009,569.7	1,019,116.3	16,660.3	1,021,895.7	**1,023,174.7**	15955.7
(12,50,3,3)	922,530.2	5377.4	931,495.8	**934,413.7**	8701.4	930,332.2	934,377.5	17500.4
	1,285,707.0	330.2	1,324,135.4	**1,336,619.3**	12,466.0	1,297,047.0	1,324,586.5	7787.9
	950,105.6	1801.1	1,008,943.3	**1,013,564.6**	23,117.2	1,000,318.7	1,012,702.5	14,126.6
	1,411,604.0	323.0	1,417,183.8	**1,421,999.4**	17,828.5	1,414,756.4	1,417,655.4	12214.4
	968,010.8	357.0	948,126.6	959,606.1	22,704.2	968,010.8	**968,010.8**	9788.0
(12,60,3,3)	**1,015,085.7**	2159.4	1,007,736.1	1,010,069.7	14,220.8	1,015,085.7	**1,015,085.7**	9922.3
	1,386,021.3	6298.1	1,387,592.4	1,396,547.8	18,463.0	1,399,161.7	**1,410,252.6**	22,154.6
	1,354,679.3	441.2	1,316,385.2	1,350,391.1	19,350.0	1,354,679.3	**1,354,679.3**	8898.9
	1,348,284.2	4795.9	1,346,114.5	**1,371,741.6**	39,165.0	1,358,581.8	1,369,849.1	23,283.1
	1,226,156.9	7626.5	1,245,489.7	1,249,736.3	18,536.9	1,245,489.7	**1,249,736.3**	26,163.4
(12,70,3,3)	1,284,975.6	491.1	1,316,810.7	**1,320,915.0**	33,125.2	1,315,775.1	1,320,157.2	24,751.9
	1,733,072.7	6305.8	1,711,289.8	1,729,202.4	30,162.0	1,733,229.9	**1,733,857.6**	23,930.0
	1,517,700.3	2663.1	1,505,453.4	1,510,444.0	25,333.3	1,517,966.9	**1,519,033.6**	13,964.0

data set, five different instances are created so that 50 problem instances are obtained. The x and y-coordinates of the customer zones are integer numbers generated from a uniform distribution defined in the interval $[0, 100]$. The distance d_{ij} between two zones i and j is then calculated as the Euclidean distance. The annualized buying power h_j of customers at zone j are integer-valued parameters generated from a uniform distribution as $h_j \sim U(100, 1000)$.

Table 7.9 Efficiency of the TS heuristics for larger instances (cont)

Dataset	TS-2		TS-1			TS-3		
(m, n, r_1, r_2)	ϕ^2	CPU (s)	ϕ^1_{avg}	$\phi^1_{\max}$	CPU (s)	ϕ^3_{avg}	$\phi^3_{\max}$	CPU (s)
(12,80,3,3)	1,807,982.8	451.2	1,799,054.2	1,811,408.3	15,347.8	1,815,419.5	**1,819,257.9**	9543.3
	1,966,959.0	742.3	1,969,586.6	1,971,853.4	22,315.6	1,970,126.8	**1,971,893.7**	13,289.9
	1,210,034.7	2370.5	1,242,274.0	1,261,668.3	28,247.4	1,263,116.4	**1,269,319.2**	25,062.2
	1,861,643.7	902.2	1,938,686.6	**1,951,310.1**	50,540.4	1,880,390.9	1,948,330.5	55,491.7
	1,691,014.3	1072.8	1,698,982.8	1,725,638.1	41,742.7	1,736,483.6	**1,761,950.9**	25,091.1
(12,90,3,3)	**1,631,391.7**	487.2	1,591,817.4	1,618,763.9	2,2,985.3	1,631,391.7	**1,631,391.7**	10,958.4
	1,407,237.5	389.4	1,398,962.7	1,410,811.0	30,074.3	1,408,636.0	**1,411,299.2**	16,548.1
	1,602,355.3	851.7	1,591,084.3	1,606,459.4	28,230.3	1,605,425.4	**1,613,503.9**	34,748.1
	1,692,652.6	4777.7	1,677,305.6	1,687,095.9	39,745.2	1,692,699.4	**1,692,886.4**	15,868.4
	1,973,023.8	5964.6	1,943,298.0	1,955,377.3	33,777.5	1,973,023.8	**1,973,023.8**	22,000.7
(12,100,3,3)	2,113,551.3	9656.6	2,191,193.0	2,196,573.7	37,459.7	2,190,583.0	**2,206,240.1**	27,571.0
	2,416,126.0	2475.7	2,346,621.2	2,388,047.8	17,250.7	2,418,711.8	**2,424,838.4**	11,498.6
	2,190,723.0	19,139.8	2,181,531.3	2,194,188.4	63,565.4	2,193,710.6	**2,196,029.3**	55,764.8
	1,907,006.4	637.2	1,872,539.8	1,906,978.0	23,487.6	1,907,006.4	**1,907,006.4**	11,461.3
	1,738,030.6	12,127.1	1,726,331.0	**1,739,827.6**	37,725.1	1,738,030.6	1,738,030.6	24,725.1
(16,50,4,4)	**831,941.3**	10,846.0	818,224.2	828,829.4	30,932.5	831,941.3	**831,941.3**	24,456.4
	1,075,374.0	2216.1	1,050,324.6	1,059,006.0	49,974.4	1,075,374.0	**1,075,374.0**	28,254.6
	1,228,906.2	1148.7	1,169,779.0	1,219,764.2	72,241.1	1,228,906.2	**1,228,906.2**	21,030.9
	889,945.0	25,614.5	877,563.3	912,234.5	169,656.3	910,301.2	**917,429.0**	149,206.8
	1,388,153.4	3360.4	1,384,622.7	1,387,409.4	36,136.0	1,388,153.4	**1,388,153.4**	18,348.6
(16,60,4,4)	1,188,730.1	1518.3	1,205,472.7	1,236,350.9	78,457.6	1,228,316.7	**1,236,710.6**	51,553.5
	1,350,667.2	1965.4	1,334,471.7	1,344,647.3	59,011.3	1,350,667.2	**1,350,667.2**	23,065.6
	1,304,864.6	1291.2	1,298,186.4	1,300,657.6	64,362.4	1,304,864.6	**1,304,864.6**	25926.8
	1,125,113.0	3492.0	1,111,024.7	1,119,612.8	26,732.1	1,125,113.0	**1,125,113.0**	17,314.9
	1,484,921.4	17,283.4	1,463,697.7	1,477,639.9	82,369.5	1,484,921.4	**1,484,921.4**	54,799.2
(16,70,4,4)	**1,046,319.9**	2306.5	1,038,168.2	1,043,405.8	68,579.3	1,046,319.9	**1,046,319.9**	35,846.5
	1,340,241.8	749.7	1,313,879.2	1,331,300.4	25,549.8	1,340,241.8	**1,340,241.8**	12,276.1
	1,572,186.9	2531.7	1,529,865.9	1,556,600.2	50,645.1	1,572,186.9	**1,572,186.9**	26,353.6
	1,777,202.2	5898.4	1,826,590.9	**1,857,900.2**	134,719.0	1,840,527.0	1,857,800.9	99,440.9
	1,403,871.5	37,158.8	1,399,256.9	1,417,502.4	61,952.2	1,409,779.1	**1,422,276.1**	79,339.9
(16,80,4,4)	**1,523,885.7**	50,852.5	1,496,961.4	1,511,858.5	171,382.6	1,523,885.7	**1,523,885.7**	119,179.8
	1,852,580.4	2218.8	1,823,922.4	1,843,171.1	133,996.1	1,853,135.6	**1,855,356.5**	62,840.1
	2,361,338.3	18,685.7	2,299,384.2	2,326,639.6	35,256.4	2,374,530.9	**2,380,121.5**	34,925.7
	1,692,032.3	42,632.0	1,647,057.3	1,672,870.4	62,770.0	1,692,032.3	**1,692,032.3**	81,357.8
	1,533,233.6	15,262.9	1,527,088.1	1,530,445.0	82,762.6	1,533,233.6	**1,533,233.6**	59,722.3
(16,90,4,4)	1,689,441.3	5919.2	1,707,920.0	**1,747,068.2**	153,454.2	1,719,113.1	1,737,895.8	110,117.1
	1,674,399.7	26,362.3	1,704,154.0	**1,717,501.1**	64,010.5	1,704,060.6	1,715,728.0	80,106.9
	1,715,332.0	28,304.9	1,688,454.9	1,704,803.6	118,251.7	1,715,403.6	**1,715,690.1**	61,501.3
	1,740,497.3	3037.4	1,722,340.3	1,731,245.3	100,225.0	1,740,497.3	**1,740,497.3**	54,314.2
	1,829,994.1	2033.8	1,784,381.1	1,821,404.4	73,333.2	1,829,994.1	**1,829,994.1**	27,416.9
(16,100,4,4)	**2,279,358.9**	41,045.9	2,212,650.5	2,269,695.9	158,760.9	2,279,358.9	**2,279,358.9**	98,687.6
	1,927,504.9	28,553.8	1,895,525.7	1,902,469.4	147,667.9	1,938,080.9	**1,941,143.9**	87,935.1
	2,283,729.3	20,213.3	2,299,793.7	**2,314,636.8**	117,829.6	2,306,047.8	2,312,266.5	97,196.2
	2,562,060.3	21,592.5	2,538,983.1	**2,590,208.1**	157,623.7	2,562,317.1	2,563,344.5	60,443.9
	2,195,552.5	5386.0	2,153,934.2	2,169,545.5	229,946.1	2,195,552.5	**2,195,552.5**	67,037.8

The fixed costs f_k which determine the optimal number of new facilities to be opened are set to three different values for each facility type in the increasing order: 1500, 3000, 6000. The threshold distances S_{jk} are integer-valued quantities generated from uniform distributions based on the annual buying power h_j of customers as given in Table 7.10.

Table 7.10. The values of parameters S_{jk}

	$U(a,b)$ for $k=1$	$U(a,b)$ for $k=2$	$U(a,b)$ for $k=3$
$100 \leq h_j \leq 300$	$a=10$,$b=25$	$a=S_{j1}$, $b=45$	$a=S_{j2}$, $b=60$
$300 < h_j \leq 700$	$a=5$,$b=20$	$a=S_{j1}$, $b=65$	$a=S_{j2}$, $b=70$
$700 < h_j \leq 1000$	$a=0, b=15$	$a=S_{j1}$, $b=50$	$a=S_{j2}$, $b=80$

As pointed out in Section 6.3, the proposed solution method begins with the LH. On the basis of the observations we made, the LH is terminated after 30 iterations since the best feasible solution is usually found by the LH within this number of iterations. The best feasible solution provided by the LH is given as the initial solution to the LS procedure which generates neighbors from the current solution by 1-Add, 1-Drop, and 1-Swap moves. However, the number of all neighbors is very large when the number of potential facility sites for our test instances is taken into account. When s facilities are opened at the current solution, then there are $m-s$ possible 1-Add moves and $s \times (m-s)$ possible 1-Swap moves. To keep the computation time within reasonable limits, only half of the 1-Add moves is selected at each LS iteration. This is done deterministically by computing the average of the annual buying powers of customer zones that are in the region of influence N_{ik} of the facility opened by a 1-Add move. The first half of the moves which have the highest average buying power are selected to be used in our implementation. Furthermore, only one-third of the possible 1-Swap moves are selected randomly to generate neighboring solutions and also we develop another procedure which selects half of the 1-Add moves by following the same selection criterion and only one-third of the possible 1-Swap moves randomly. So, totally we develop three solution procedures starting with the LH and then taking half of the 1-Add moves, one-third of the possible 1-Swap moves, and half of the 1-Add moves plus one-third of the possible 1-Swap moves.

We assess the performance of the proposed solution methods in comparison with the commercial solver CPLEX 11.0 (ILOG, 2007). The results are presented in Table 7.11 and Table 7.14. The accuracy of the two-stage heuristic (LH+LS) is measured by the percent relative deviation (PD) of the best lower bound z_H obtained for each test instance from the objective value z_{Cplex} provided by CPLEX, which is computed by the formula

$$100 \times \frac{(z_{Cplex} - z_H)}{z_{Cplex}}. \tag{7.12}$$

7.4.1. Computational Results Using the Fuzzy C-Means Algorithm

As explained in Section 6.2, we develop two procedures to determine the visiting probabilities of customers, namely a fuzzy C-means algorithm and a semi-supervised procedure. In this subsection, we give the computational results using the probabilities obtained by the fuzzy C-means algorithm. As pointed out in Subsection 6.2.1, the most problematic issue in applying the fuzzy C-means algorithm is the selection of an exponential weight μ which should be greater than one. Based on our observations on the randomly generated instances, the values of membership degree u_{ij} depend dramatically on the choice of μ. As the value of μ is increased, the values of u_{ij} become similar to each other for all clusters given the ith customer. Thus, as we increase the value of μ, the visiting probabilities p_{jk} of the jth customer becomes closer to $1/3$ for all 3 facility types $k = 1, 2, 3$. This situation can be interpreted as if the value of μ increases, customers become indifferent between the facility types and the model becomes a traditional maximal covering problem. In such a problem only facilities of the first type will be opened at the solution of the problem, since there is no distinction between the facilities by capturing the revenue from the customers and the first type facilities have the lowest fixed costs. However, by formulating such a problem we expect that customers differentiate between the facility types and customers with similar properties visit the same type facility with higher probabilities which naturally varies from customer to customer. As a result, we set the value of μ to 2, 3, and 5 in order to perform the computations and apply all solution procedures for these three values. The results are given in Table 7.11.

Since we fix the exponential weight μ to three different values, we run all the solution procedures three times for each of the 50 instances, which makes totally 150 different runs for every solution procedure. For the ease of comparison of three heuristic methods with Cplex, we highlight the least percent relative deviation (PD) which indicates the best objective value in boldface for each problem instance. We observe that out of 150 runs, the second heuristic finds the best solution for 67 runs, whereas the first heuristic provides the best solution for 43 runs and the third heuristic for 62 runs. An important distinction can be realized especially for the first heuristic. For smaller instances reported in Table 7.14, the first heuristic performs better than the other two heuristics in terms of accuracy. It can provide the optimal solution for 29 runs out of 75 instances, while the second heuristic finds it for 17 runs and the third heuristic for only 11 runs. However, when the PDs of the three solution procedures are compared for all runs, then it turns out that the second heuristic has the least PD with 1.72% on the average. The average PDs of the first and third heuristics are 2.48% and 2.07%, respectively. When we compare the three heuristics in terms of efficiency, we see that the third heuristic spends 6226.66 seconds on average and is a little faster than the first and second heuristics where average CPU times are 6800.46 seconds and 6781.33 seconds, respectively.

Since the average PDs of the three heuristic methods are close to each other, we decide to make a statistical analysis on the PDs provided by these methods in order to understand the quality of the solutions. Therefore, a paired T-test and a two-sample pooled T-test with unknown and unequal variances are realized on the three heuristics with 0.01 level of significance. For the paired T-test and the two-sample pooled T-test, we constitute three groups of hypothesis testing which are given in Table 7.12 and Table 7.13, respectively, where PD_i shows the average PD of the ith heuristic with $i = 1, 2, 3$ on 50 instances. We arbitrarily choose the results provided by the three heuristics when the exponential weight μ is chosen to be equal to 5.

The computed test statistic is given by

$$T_i = \frac{\bar{d}_i}{s_{di}/\sqrt{n}} \tag{7.13}$$

for the paired T-test, where $\bar{d}_i$ shows the average of the differences between PDs, s_{di} the standard deviation of the differences between PDs, n is the number of instances we have (i.e. n is equal to 50), and the index i indicates the order of the comparisons. When i is

Table 7.11. Efficiency of the heuristics using the fuzzy C-means algorithm

Instance	CPLEX		Heuristic 1		Heuristic 2		Heuristic 3	
(n, μ)	z^*	CPU (s)	PD (%)	CPU (s)	PD (%)	CPU (s)	PD (%)	CPU (s)
(50,2)	4004.03	0.56	**0**	0.64	13.06	0.53	13.06	0.28
(50,3)	2425.78	0.48	**0**	0.42	14.21	0.39	15.12	0.25
(50,5)	1675.65	0.48	**0**	0.41	18.69	0.39	**0**	0.25
(50,2)	2562.48	0.69	**0**	0.61	**0**	0.52	**0**	0.42
(50,3)	1160.11	0.55	**0**	0.30	**0**	0.27	**0**	0.25
(50,5)	575.49	0.53	**0**	0.42	0.73	0.27	0.73	0.23
(50,2)	773.29	0.63	**0**	0.41	**0**	0.30	12.17	0.20
(50,3)	780.47	0.56	**0**	0.44	**0**	0.30	**0**	0.30
(50,5)	706.43	0.52	**0**	0.47	17.29	0.38	78.44	0.23
(50,2)	8552.14	0.56	**0**	0.77	**0**	0.92	2.39	0.55
(50,3)	6173.69	0.63	**0**	0.61	12.38	0.30	**0**	0.45
(50,5)	4147.28	0.81	**0**	0.48	13.18	0.31	**0**	0.36
(50,2)	8283.66	0.80	**0**	0.77	**0**	0.48	**0**	0.30
(50,3)	6571.93	0.59	**0**	0.64	**0**	0.45	**0**	0.44
(50,5)	4593.44	0.56	**0**	0.52	**0**	0.45	**0**	0.28
(100,2)	14685.48	2.61	**0**	12.91	**0**	12.70	6.29	7.20
(100,3)	12671.12	1.61	**0**	12.06	2.02	12.36	1.23	8.78
(100,5)	11036.33	1.56	**0**	12.86	**0**	10.08	5.03	7.06
(100,2)	25487.20	1.81	**0.80**	12.89	1.88	15.55	2.64	10.39
(100,3)	22759.78	1.95	**1.61**	14.89	2.66	10.53	4.99	8.17
(100,5)	19320.40	1.98	**0**	13.98	3.01	15.31	3.96	9.63
(100,2)	22479.69	2.06	1.06	9.77	**0**	8.70	1.26	7.00
(100,3)	11416.22	1.70	**0**	9.17	0.68	10.41	**0**	8.88
(100,5)	7649.87	1.61	**0**	8.86	6.41	13.59	4.63	5.97
(100,2)	11909.46	1.59	**0**	9.03	4.74	10.59	0.06	5.91
(100,3)	10851.20	1.72	**0**	12.73	**0**	13.47	2.29	9.50
(100,5)	9238.28	1.69	**0**	10.48	5.57	9.17	5.96	6.47
(100,2)	19840.36	1.78	1.35	19.38	**0**	25.19	4.87	5.75
(100,3)	21221.24	2.03	5.62	16.86	**0**	12.06	7.00	7.31
(100,5)	17625.24	1.75	**0.58**	16.75	13.91	9.97	9.76	6.64
(150,2)	26438.46	7.70	**0.61**	98.77	2.28	91.25	1.43	78.70
(150,3)	25180.96	7.84	1.51	125.91	**1.50**	96.49	2.96	52.70
(150,5)	23732.95	7.56	0.61	109.80	1.10	126.55	**0.45**	52.20
(150,2)	31262.84	10.27	2.15	79.53	**0.06**	84.22	0.13	52.78
(150,3)	27515.39	9.89	1.26	67.89	**0.82**	68.28	**0.82**	64.02
(150,5)	23307.80	9.78	**0**	94.75	1.12	112.97	0.95	64.14
(150,2)	20693.94	8.98	2.24	90.28	**1.84**	69.60	2.91	36.61
(150,3)	18821.38	8.69	2.71	138.60	1.58	76.22	**0.52**	62.08
(150,5)	16983.25	7.63	**0**	98.88	2.33	99.11	0.78	80.77
(150,2)	26722.02	9.34	0.15	53.97	**0**	67.88	0.41	52.50
(150,3)	17654.97	9.06	6.88	60.83	2.94	89.70	**2.14**	48.84
(150,5)	16358.16	12.97	7.45	56.00	**0**	141.94	0.32	83.88
(150,2)	21113.05	8.67	4.72	72.77	**0.01**	118.99	1.20	36.39
(150,3)	18594.95	8.69	**1.08**	65.63	2.58	52.74	1.09	51.88
(150,5)	16678.88	8.50	**0.50**	95.88	9.26	74.45	**0.50**	61.58
(200,2)	42597.35	18.81	1.27	476.39	**0.46**	457.34	3.12	188.80
(200,3)	38795.12	17.92	**0.02**	444.34	0.18	374.70	1.06	400.09
(200,5)	34393.67	17.58	0.85	343.96	**0.57**	364.04	2.17	210.40
(200,2)	61166.13	16.84	2.52	528.00	0.52	300.71	**0.09**	263.05
(200,3)	53321.02	20.89	1.23	477.79	**0.65**	707.44	0.82	316.74
(200,5)	40573.17	18.03	1.45	387.18	0.76	446.90	**0.21**	461.68
(200,2)	51342.77	20.06	1.64	330.81	**0.62**	298.59	1.12	222.27
(200,3)	35695.15	17.17	2.87	300.74	**0**	430.34	**0**	290.40
(200,5)	25404.17	17.13	2.18	388.62	**1.32**	343.60	3.52	217.66
(200,2)	26473.66	19.89	4.45	367.90	0.60	352.62	**0.04**	394.70
(200,3)	31355.72	17.17	**1.50**	307.45	2.29	335.57	2.10	212.90
(200,5)	26138.87	17.58	2.43	290.16	**1.51**	246.91	2.20	394.09
(200,2)	50264.72	18.25	1.42	431.96	0.52	275.48	**0**	239.82
(200,3)	43638.35	16.64	**0.92**	495.48	1.02	301.04	1.07	264.83
(200,5)	31181.70	17.22	**0.47**	474.86	2.31	331.96	1.36	173.18
(250,2)	34754.02	37.88	6.65	1136.99	3.62	1083.68	**0.60**	888.67
(250,3)	34497.72	48.44	7.93	1005.63	5.31	1021.16	**2.59**	590.46
(250,5)	33728.72	39.61	3.09	1035.18	4.01	715.29	**2.59**	995.35
(250,2)	84522.53	39.13	5.74	953.07	**0.39**	712.29	2.37	706.56
(250,3)	75979.72	51.92	0.88	1328.76	**0.18**	985.27	0.47	908.62
(250,5)	64559.99	40.41	2.48	1115.50	**0.51**	1369.58	1.90	754.77
(250,2)	41205.07	45.99	8.99	351.85	0.42	929.83	1.31	526.92
(250,3)	38601.88	44.31	8.64	732.53	0.50	1060.65	1.04	833.50
(250,5)	35398.81	66.45	7.56	735.75	1.04	1108.91	**0.84**	740.92
(250,2)	84825.18	59.67	3.04	1087.47	**0.34**	1276.09	0.37	654.89
(250,3)	74090.87	46.09	2.37	888.36	0.47	918.21	**0.30**	695.00
(250,5)	52563.54	59.42	2.01	1227.87	**0.04**	1178.98	1.34	608.50
(250,2)	52659.02	54.60	1.21	1226.70	0.58	892.52	**0.35**	654.66
(250,3)	48600.66	60.49	5.70	950.35	0.66	1238.38	**0.32**	991.85
(250,5)	34488.93	54.08	8.48	1014.63	1.47	1050.55	**0.59**	1080.88

equal to 1, then the first heuristic and second heuristic are paired; when it is equal to 2, the first and the second heuristics are paired. Finally, the second and third heuristics are paired, when the index i is equal to 3. Critical regions are constructed using the t-distribution with $n-1$ (=49) degrees of freedom. Thus, we reject a hypothesis when $T_i < -2.68$ or $T_i > 2.68$. The computed statistics T_i turns out to be -0.212, -0.430, and -0.401 for the three testings, respectively. Hence, we do not reject any of the hypotheses. This finding indicate that there is no significant difference between the three heuristic methods with respect to the quality of the solutions they provide.

The test statistics for the two-sample pooled T-test are given by

$$T_1 = \frac{PD_1 - PD_2}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}} \tag{7.14}$$

$$T_2 = \frac{PD_1 - PD_3}{\sqrt{\frac{S_1^2}{n} + \frac{S_3^2}{n}}} \tag{7.15}$$

$$T_3 = \frac{PD_2 - PD_3}{\sqrt{\frac{S_2^2}{n} + \frac{S_3^2}{n}}} \tag{7.16}$$

where S_i is the standard deviation of the samples obtained by the ith heuristic with $i = 1, 2, 3$. These statistics have an approximate degrees of freedom $\frac{((s_1^2/50)+(s_2^2/50))^2}{[((s_1^2/50)^2/49)+((s_2^2/50)^2/49)]}$, which is approximately equal to 98 for the three comparisons. Hence, the tests are not rejected when $-t_{0.005,98} < T_i < t_{0.005,98}$ for all $i = 1, 2, 3$, where the critical value $t_{0.005,98}$ is equal to 2.627. The statistics T_i are computed as -0.239, -0.443, and -0.322 resulting in not rejecting any of the hypotheses. Hence, the two-sample pooled T-test points out the same conclusion with the paired T-test so that there is no significant difference between the heuristics with respect to the quality of the solutions.

Another important observation based on the optimal objective function values z^* provided by Cplex is that the profit of the firm usually decreases as the value of the exponential weight μ increases. As explained before, as the value of μ increases the membership degree u_{ij}, which represents the visiting probabilities, decreases for a customer who certainly belongs to a specific cluster. This situation can be better explained with an example. A customer whose annual buying power is 100 is more likely to visit a first type facility such that its visiting probability for that facility type turns out be close to 1 when μ is chosen to be 2. However, if μ is set to 5, then that probability can decrease down to for example 0.65 which ultimately results in a decrease in the profit when a first type facility is constructed whose region of influence covers that customer. On the other hand, in rare occasions this could result in constructing another facility type such as the second or third since the visiting probabilities increase for them as the probability for the first type is decreased. As a consequence, the profit can increase because it is possible that a higher type facility captures more revenue from the customers.

7.4.2. Computational Results Using the Parametric Bayesian Classification Algorithm

The parametric classification algorithm we develop can be applied only when we have a partially labeled data set. Therefore, we label one-fifth of customers for each problem instance with the exception of instances with 50 customer zones. As given in Subsection 6.2.2, the algorithm begins with α labeled customers and applies the initial weak Bayes' classifier on this small data set. When α is a small number, it is possible that the number of labeled instances for each class is tiny so that the covariance matrices are singular and the inverses do not exist. This situation can occur when the labeled data set is small as given by Alpaydın (2004). In order to prevent this situation, we label 20 customers of the first five problem instances with 50 customer zones so that the initial weak classifier can be applied for these problem instances.

The computational results obtained by using the parametric Bayesian classification algorithm are summarized in Table 7.14. As in the case of the fuzzy C-means algorithm, we compare the three heuristic methods with Cplex and highlight the least PD in boldface for all problem instances. We notice that the second heuristic provides the best solution for 25 instances, while the first heuristic finds the best solution for 16 instances and the last heuristic for 18 instances. When we only compare the number of best solutions obtained by each heuristic, the second heuristic turns out to be the most accurate one. However, when the percent deviations (PDs) from z_{Cplex} provided by all the heuristics for all problem instances are examined, the third heuristic method provides the least PD with 2.11% on average. The average PDs for the first and second heuristic methods are 3.67% and 2.69%, respectively. A similar analysis on the computational results using the fuzzy C-means algorithm indicates that the first heuristic outperforms the other two methods for smaller instances, where for example it can find the optimal solution for all instances with 100 customer zones. Another important observation can be made with respect to the average CPU times, where the three heuristics spend 3017.40 seconds, 3232.55 seconds, and 2375.29 seconds on average, respectively. These comparisons based on the average CPU time and average PD indicate that the third heuristic is the best solution procedure in terms of the efficiency.

The same statistical analysis is carried out on the PDs provided by the three heuristic methods as we do Subsection 7.4.1. Again the paired T-test and the two-sample pooled T-test are realized on 50 instances with the same level of significance and the critical regions. For the paired T-test, the statistics are computed as 0.821, 1.693, and 1.081 for the three heuristic methods, respectively. Besides, the statistics are found as 0.828, 1.806, and 0.491 when the two-sample pooled T-test is applied. Both hypothesis testing procedures indicate that there is no significant difference between the three heuristic methods with respect to the quality of the solutions, which is a consistent result with the solutions when the fuzzy C-means algorithm is used in order to determine the visiting probabilities.

A comparison can also be carried out based on the visiting probabilities determined by the fuzzy C-means and the parametric Bayesian classification algorithms. The fuzzy C-means algorithm provides more realistic probabilities than the parametric classification algorithm using the Bayesian classifier. This situation relies on the fact that the former algorithm is fuzzy in the sense that it obtains smoother probabilities. In contrast to this algorithm, the

Bayes' classifier groups the customers into classes and the posterior probability is usually one or close to one for the selected class, while the rest of the posterior probabilities for the remaining not chosen classes are zero or close to zero. However, this situation affects both the best and optimal objective function values and the CPU times for each randomly generated problem instance. An important distinction between the algorithms is that the average CPU times for problem instances with probabilities determined by the fuzzy C-means algorithm are considerably larger than the average CPU times for the same instances with probabilities obtained by the parametric Bayesian classification algorithm. Another difference between two algorithms can be detected in the objective function values. The computational results for the same problem instances can indicate different objective function values, i.e. profits. However, it is not possible to conclude which algorithm results in higher objective function values. For one problem instance the first algorithm causes a higher objective function value, whereas the second algorithm causes for another problem instance a higher profit. This situation can be explained with an example as follows. A customer with annual buying power 900 is more likely to visit a third type facility. The visiting probability of that customer for the third type facility turns out to be 1 when the parametric Bayesian classification algorithm is applied and is 0.85 when the fuzzy C-means algorithm with $\mu = 3$ is applied. If a facility of the third type is constructed the region of influence of which contains the customer, the revenue captured by the parametric Bayesian classification algorithm is higher than the revenue captured by the fuzzy C-means algorithm. On the other hand, the same customer is also in the region of influence of another facility of the second type which is also to be opened. Then the revenue captured by the parametric Bayesian classification algorithm is zero, although the customer is also likely to visit that facility of the second type. However, the fuzzy C-means algorithm can result in a revenue which is equal to for example 0.12 times 900. Therefore, there is a large trade-off between the two algorithms so that the profits differ from each other and it is difficult to conclude which algorithm results in higher profits.

Table 7.11. Efficiency of the heuristics using the fuzzy C-means algorithm (cont)

Instance	CPLEX		Heuristic 1		Heuristic 2		Heuristic 3	
$(n,\ \mu)$	z^*	CPU (s)	PD (%)	CPU (s)	PD (%)	CPU (s)	PD (%)	CPU (s)
(300,2)	44577.86	67.22	1.48	1940.59	**0.78**	2226.37	2.41	2021.17
(300,3)	78003.97	69.50	**0.22**	2524.46	1.24	1885.35	2.65	1169.50
(300,5)	42754.40	64.88	2.66	2297.93	**2.06**	3761.14	3.27	1833.83
(300,2)	50403.93	78.83	7.80	1156.79	**0.35**	1633.27	0.79	1429.95
(300,3)	47889.41	82.74	6.11	1660.36	**0.29**	1991.25	0.87	1067.09
(300,5)	45350.27	80.55	12.57	1468.62	**0.32**	3145.78	1.59	1498.43
(300,2)	76556.08	60.77	1.66	1961.05	1.28	2120.43	**0.22**	3301.22
(300,3)	71574.17	60.63	1.08	2882.70	**0.37**	2406.34	2.29	3016.85
(300,5)	66416.67	61.61	1.20	2990.95	0.48	2902.73	**0.15**	2470.50
(300,2)	48065.73	78.36	9.06	1176.63	2.68	944.50	**0.27**	2093.45
(300,3)	45379.93	77.36	4.43	1316.92	1.36	43371.73	**1.23**	44763.28
(300,5)	42766.62	77.99	4.14	1393.67	**2.51**	1609.19	4.39	910.80
(300,2)	45998.54	89.50	6.13	1772.72	1.23	1273.03	**0.97**	1381.71
(300,3)	45131.03	104.11	5.80	2332.64	0.61	2314.45	**0.29**	1753.06
(300,5)	43511.01	97.64	4.90	1930.99	3.17	1754.50	**0.58**	2093.99
(350,2)	101289.14	144.66	0.74	7661.13	0.89	4155.94	**0.68**	4006.98
(350,3)	100084.88	214.58	**0.49**	9542.48	0.70	6441.56	0.89	4576.37
(350,5)	93576.22	122.94	1.02	6341.07	**0.22**	6280.90	0.72	4209.48
(350,2)	68125.05	99.92	2.23	6427.03	1.26	5490.78	**0.11**	6180.32
(350,3)	92448.33	156.36	**0.20**	8054.53	0.23	5564.31	3.03	2150.46
(350,5)	88256.88	161.46	1.73	6173.29	**0**	9792.38	0.39	6530.03
(350,2)	62904.67	221.14	2.84	4713.55	**0.02**	6249.66	0.15	4578.95
(350,3)	59032.36	187.00	5.17	3736.06	1.55	3891.95	**0**	3030.12
(350,5)	55723.45	150.91	7.69	2898.12	**0.48**	4664.14	1.71	3547.38
(350,2)	65538.87	119.71	5.13	3954.90	**0.57**	3476.52	1.39	3708.26
(350,3)	64673.43	124.81	5.71	5747.97	0.80	4849.06	**0.14**	4351.80
(350,5)	61814.33	124.66	3.91	3816.20	0.96	7322.36	**0.69**	5041.24
(350,2)	101249.41	102.17	0.49	6188.34	**0.33**	6409.64	4.69	3353.89
(350,3)	99116.99	106.66	**0.02**	8083.73	0.68	5655.84	0.25	5229.85
(350,5)	94468.91	159.02	0.46	6586.24	0.43	7705.25	**0.20**	6353.40
(400,2)	130355.98	161.92	0.27	11061.35	**0.23**	7995.56	0.25	9685.51
(400,3)	122977.16	213.36	1.48	8070.76	**0.20**	8247.41	0.43	6590.50
(400,5)	66838.90	248.05	2.87	10045.97	**0.56**	9342.37	2.20	9140.03
(400,2)	63708.88	234.60	4.12	4348.57	**1.12**	8378.15	1.54	5683.37
(400,3)	64277.01	307.38	4.63	6844.68	**0.78**	10793.05	0.93	6461.53
(400,5)	64044.41	209.97	3.81	9248.04	1.19	7425.53	**0.40**	8645.01
(400,2)	71701.16	164.13	5.39	9609.81	**0.24**	7729.55	1.02	7183.97
(400,3)	70109.74	196.60	1.60	10417.18	1.52	8996.26	**1.25**	10517.26
(400,5)	68147.22	175.86	2.37	9527.17	0.49	9225.08	**0.40**	8661.76
(400,2)	67669.69	155.77	2.14	6934.98	0.54	14243.01	**0.24**	8554.96
(400,3)	67406.80	174.33	0.77	11937.32	**0.74**	9933.33	1.54	8638.48
(400,5)	66402.61	211.10	1.69	11815.32	1.65	8454.38	**1.55**	9334.73
(400,2)	71721.02	161.28	1.77	19384.23	**1.02**	16503.50	1.10	13115.16
(400,3)	67637.10	208.68	1.32	11552.10	2.78	6784.79	**0.09**	10354.75
(400,5)	68053.42	229.52	1.58	10934.68	0.81	14085.58	**0.61**	11387.12
(450,2)	139885.13	400.62	4.83	11270.90	0.61	13839.94	**0.48**	18670.19
(450,3)	80608.73	398.38	3.81	18395.06	1.72	13121.48	**0.31**	13422.62
(450,5)	78870.67	406.95	6.28	12846.28	1.33	13191.07	**1.26**	11812.23
(450,2)	172457.78	371.26	2.31	25771.30	**0.60**	14916.70	0.83	10177.78
(450,3)	154480.51	319.91	1.22	22882.08	0.19	20980.61	**0.14**	18910.24
(450,5)	110885.05	398.07	2.14	22701.72	0.18	26703.80	**0**	16056.21
(450,2)	80090.60	322.12	4.37	14557.42	1.03	12877.43	**0.67**	13079.03
(450,3)	79777.85	273.94	4.52	15724.83	3.48	16330.28	**1.39**	13839.69
(450,5)	80486.14	260.18	2.61	14335.51	1.56	14362.62	**0.53**	21321.58
(450,2)	72691.61	250.38	2.48	32084.79	**1.01**	19993.05	1.68	21242.18
(450,3)	77257.46	284.90	0.67	20762.08	**0.59**	18703.13	0.95	22740.92
(450,5)	82004.19	357.69	**0.81**	25967.13	1.30	13093.90	1.73	16323.73
(450,2)	80812.38	321.15	1.35	15081.85	0.73	13708.23	**0.36**	18221.10
(450,3)	109019.45	374.35	**0.08**	25694.41	0.27	13628.40	0.35	14292.26
(450,5)	82111.08	363.05	0.95	26169.54	**0.84**	15232.51	1.55	14045.32
(500,2)	90407.56	431.06	4.68	30535.95	**0.27**	34110.13	0.66	32167.25
(500,3)	90756.95	444.66	1.53	42918.62	**0.29**	32595.97	0.84	18747.28
(500,5)	91616.39	422.76	3.58	23100.01	0.90	26000.74	**0.88**	35999.00
(500,2)	193531.97	549.15	**0.53**	37395.37	0.67	29177.73	0.70	17514.71
(500,3)	183937.18	445.04	0.33	32603.75	**0.23**	31697.43	0.27	25241.82
(500,5)	134850.40	366.24	0.40	48400.03	0.61	31751.99	**0.20**	34178.64
(500,2)	91049.87	630.04	5.11	22037.58	0.55	26693.90	**0.03**	22355.17
(500,3)	88625.57	504.48	2.54	43028.49	1.64	33899.92	**0**	33043.64
(500,5)	87282.56	480.27	1.34	33811.95	**0.77**	44636.58	1.64	32067.92
(500,2)	89060.21	587.11	3.73	14709.61	**0.33**	16715.05	1.34	20839.29
(500,3)	90159.27	624.45	1.94	17026.59	**0.39**	25308.31	0.89	25010.53
(500,5)	93124.01	427.73	3.64	34353.51	2.01	42427.34	**0.15**	43083.03
(500,2)	88653.86	445.76	6.06	13203.20	**0.19**	25081.27	0.25	18044.15
(500,3)	89064.38	556.09	3.82	14846.35	**0.88**	20141.12	1.40	28275.67
(500,5)	87444.64	478.49	8.83	15042.65	**1.09**	24176.04	4.01	14787.72

Table 7.12. Hypothesis Testing for the paired T-test

	$\mathbf{H_1}$ & $\mathbf{H_2}$	$\mathbf{H_1}$ & $\mathbf{H_3}$	$\mathbf{H_2}$ & $\mathbf{H_3}$
$\mathbf{H_0}$:	$PD_{d1} = PD_1 - PD_2 = 0$	$PD_{d2} = PD_1 - PD_3 = 0$	$PD_{d3} = PD_2 - PD_3 = 0$
$\mathbf{H_1}$:	$PD_{d1} = PD_1 - PD_2 \neq 0$	$PD_{d2} = PD_1 - PD_3 \neq 0$	$PD_{d3} = PD_2 - PD_3 \neq 0$

Table 7.13. Hypothesis Testing for the two-sample pooled T-test

	$\mathbf{H_1}$ & $\mathbf{H_2}$	$\mathbf{H_1}$ & $\mathbf{H_3}$	$\mathbf{H_2}$ & $\mathbf{H_3}$
$\mathbf{H_0}$:	$PD_1 - PD_2 = 0$	$PD_1 - PD_3 = 0$	$PD_2 - PD_3 = 0$
$\mathbf{H_1}$:	$PD_1 - PD_2 \neq 0$	$PD_1 - PD_3 \neq 0$	$PD_2 - PD_3 \neq 0$

Table 7.14. Efficiency of the heuristics using the parametric Bayesian classification algorithm

Instance	CPLEX		Heuristic 1		Heuristic 2		Heuristic 3	
n	z^*	CPU (s)	PD (%)	CPU (s)	PD (%)	CPU (s)	PD (%)	CPU (s)
50	6557.35	0.50	**0**	0.59	49.06	0.30	29.16	0.53
50	3547.85	0.69	20.49	0.28	**0**	0.44	**0**	0.36
50	5554.55	0.80	**0**	0.27	**0**	0.23	**0**	0.22
50	7099.03	0.61	14.52	0.47	14.52	0.41	**0**	0.50
50	9404.28	0.59	**0**	0.53	1.28	0.36	**0**	0.39
100	11717.56	3.61	**0**	11.86	**0**	11.63	1.78	5.97
100	35384.09	3.11	**0**	14.98	4.29	8.50	2.26	7.27
100	20658.26	2.78	**0**	8.41	0.58	7.06	0.58	5.88
100	20656.75	4.03	**0**	9.06	5.99	10.20	5.70	6.36
100	23406.97	2.73	**0**	12.45	**0**	9.50	2.88	10.11
150	22445.37	10.73	12.98	71.86	**0.40**	133.58	0.42	63.52
150	44172.32	11.16	5.12	40.80	**3.14**	70.72	3.42	43.49
150	30820.36	12.08	2.35	48.92	**1.55**	79.41	3.78	68.02
150	17646.51	10.11	2.39	78.42	5.92	65.92	**1.11**	63.74
150	20802.89	11.45	9.69	40.74	5.53	77.74	**2.23**	51.03
200	60455.18	26.20	1.32	499.46	1.59	318.07	**1.21**	200.88
200	67076.88	31.14	**0.99**	243.24	1.63	140.30	4.18	46.91
200	64208.23	24.19	1.49	323.83	**0.46**	235.49	2.96	289.76
200	25816.63	25.45	**0.46**	352.76	3.53	146.82	3.12	124.83
200	25763.24	25.23	9.45	266.79	0.54	287.66	**0.07**	175.44
250	39888.35	44.53	12.88	645.12	**0.44**	577.01	2.13	500.85
250	63363.95	47.74	**0.35**	474.74	**0.35**	438.56	2.82	360.35
250	54617.04	49.05	**0.04**	626.67	**0.04**	469.48	0.10	459.15
250	86141.89	45.88	5.61	1380.90	**1.14**	982.60	**1.14**	898.69
250	41121.77	45.36	4.08	833.71	**0.04**	858.80	1.09	540.46
300	49077.97	87.92	8.38	1100.30	**2.50**	1292.82	3.94	1154.43
300	48265.59	97.08	3.08	1381.60	**0.13**	1257.51	0.89	1192.49
300	46623.17	105.52	**0.94**	2533.92	2.86	1511.34	2.49	823.38
300	46313.72	102.52	5.31	1643.80	**2.43**	974.75	2.57	810.73
300	51939.58	92.05	**0.56**	1203.21	0.94	1169.61	0.75	887.35
350	55174.81	173.14	2.66	1499.37	**0**	2223.04	0.31	1255.66
350	99046.59	160.52	**0.80**	1520.54	0.83	1129.15	**0.80**	1821.19
350	57125.45	160.52	2.59	2360.58	**1.22**	2048.62	2.05	1734.49
350	65824.44	155.03	4.05	3655.37	2.94	2286.20	**0.35**	2128.63
350	60871.15	161.74	4.59	1895.96	1.16	1420.92	**0.46**	2008.74
400	68095.28	376.15	2.67	3609.98	1.23	4306.39	**0.06**	6039.79
400	64921.70	247.90	1.48	4966.95	1.88	2272.45	**0.82**	2628.05
400	68142.48	198.97	2.02	3937.34	1.32	4421.54	**0.31**	4207.60
400	67750.53	228.07	6.86	4337.99	0.92	5705.11	**0.36**	5098.18
400	68295.69	225.74	0.16	5417.12	0.37	6025.68	**0**	2552.92
450	70666.77	458.10	5.04	4587.78	**0.13**	8316.35	0.63	5965.90
450	87379.04	508.84	8.49	7275.44	**6.52**	31180.37	7.73	4591.18
450	82939.90	381.05	2.94	10279.49	**0.07**	8312.28	0.25	7289.91
450	76058.82	465.09	5.02	6064.55	0.53	4568.31	**0.29**	9091.74
450	77420.64	268.93	2.26	7462.46	**1.09**	6605.36	1.90	4514.09
500	93872.28	676.67	2.57	10005.11	**0.89**	10261.71	3.06	11846.93
500	91767.09	639.65	1.83	10135.07	**0.10**	8465.82	0.93	6334.59
500	90339.91	931.05	**0**	21917.00	1.96	8404.55	0.79	6906.96
500	85217.42	578.68	3.91	12095.41	**0**	15312.16	0.19	13174.54
500	89719.85	511.59	0.95	13996.55	**0.21**	17224.77	1.39	10780.18

8. CONCLUSIONS

In this thesis we first focused on three different types of CFL problems that build a subgroup of facility location problems. Initially, we considered a non-reactive variant of the discrete CFL problem in which both the locations as well as the attractiveness levels of new facilities have to be determined simultaneously to maximize the profit of the new entrant firm. To solve the problem, we formulate a mixed-integer nonlinear programming model and propose three solution methods. One of them is a heuristic based on the Lagrangean relaxation of the model (LH), while the others are exact procedures based on the branch-and-bound (BB) technique. The difference between BB-based methods is that one relaxes the integrality restrictions on the binary variables and solves the nonlinear programming relaxation at each node of the BB tree (BB-NLP), whereas the other solves the Lagrangean relaxation (BB-LR). All of the three solution procedures make use of the concavity of the objective function in terms of the attractiveness variables when the binary location variables are fixed or relaxed.

The computational results obtained on a set of problem instances indicate that BB-NLP is the most efficient method and provides the optimal solution for all instances within the allowed time limit of 7200 seconds. LH is also quite accurate in the sense that the average percent deviation of the solutions generated by LH is 0.3% and 0.6% away from the optimal objective values when the fixed cost levels are low and medium, respectively. We also make sensitivity analysis by changing the four main parameters of the model. An interesting finding of our experiments is that unacceptly low optimal attractiveness levels do not occur; the firm is better off when it does not open a facility instead of opening a facility with low attractiveness. Unacceptably high attractiveness levels, however, can occur especially when the maximum attractiveness levels are set to high values. But, since it is a parameter that is determined by the firm, it would never result in a situation where the firm has to open a facility with an unrealistic attractiveness level.

Next we consider a discrete bilevel CFL problem where again the new entrant firm determines both the locations and attractiveness levels of its new facilities so as to maximize its profit. It is assumed that there is a competitor in the market and it can react to the opening of the new facilities by adjusting the attractiveness levels of its existing facilities with the objective of maximizing its own profit. We formulated a bilevel mixed-integer nonlinear programming model (MINLP) and solve it using a global optimization method called GMIN-αBB after converting the bilevel model into an equivalent one level MINLP model. This conversion is possible since competitor's subproblem, which is the follower's subproblem of the corresponding leader-follower game, is a concave programming problem in terms of the attractiveness variables when the locations and attractiveness levels of the leader are fixed.

The solution method is implemented on a set of problem instances of varying sizes. We also investigated a scenario in which the new entrant firm ignores the competitor's reaction while solving its profit maximizing optimization problem. The results indicate that anticipating the competitor's reaction and incorporating this into his decision making frame enables the leader firm to increase its profit by 58.33% on the average. This affects the profit of the competitor in a negative way, an average loss of 45.31% is incurred to the competitor.

We then address another discrete CFL problem by taking into account the sequential game between the market entrant firm and its competitor. As usual, the market entrant firm wants to determine the location and attractiveness levels of its new facilities considering the reaction of its competitor in the market in order to maximize its profit. For once, we assumed that the competitor can react to the opening of the new facilities by opening new facilities, adjusting the attractiveness levels of its existing facilities and/or closing them completely. We formulate a bilevel MINLP, where both the upper level as well as the lower level problems contain continuous and binary decision variables. In order to find feasible solutions we developed three tabu search (TS) heuristics.

First, we assessed the performance of these heuristics by comparing them with an ϵ-optimal solution method on a small set of problem instances. Since their performance are quite satisfactory, we applied them on larger problem instances. The results indicate that TS-3 performs better than TS-1 and TS-2 in terms of accuracy. TS-2 is also completely satisfying in the sense that for more than 50% of the instances, for which TS-3 gives the best solution, TS-2 can also find it.

Finally, we propose a discrete facility location problem, where customer preferences are explicitly taken into consideration. Customers at demand points called zones determine the probability for visiting different facility types. In contrast to the traditional assumption in which the visiting probability increases as the facility attractiveness increases, we assume that these probabilities can differ with respect to customer zones based on the customer attributes such as financial income. Furthermore, customers set a maximum distance to travel to a certain facility type. A binary integer linear programming model is formulated and solved using a solution methods that combines a Lagrangean heuristic and a local search.

In order to determine the visiting probabilities of customers, which constitute important parameters to our model and make the proposed model different from the facility location problems in the literature we developed an unsupervised fuzzy C-means algorithm and a parametric classification algorithm making use of a Bayes' classifier for multivariate input vectors. In order to apply these two algorithms, we set the customer attributes that influence the visiting probabilities as the annualized buying power and the maximum distances for different facility types of customers. The computational results obtained on the randomly generated instances using the visiting probabilities provided by the two developed algorithms show that the third solution method is the most efficient one. In order to understand the quality of the solutions provided by the three solution methods, we carry out a statistical analysis based on the percent deviations from the optimum. A paired T-test and a two-sample pooled T-test are applied on the percent deviations which indicate that there is no significant difference between the solution methods in terms of the accuracy. Besides the accuracy and efficiency of the solution methods, we also had the opportunity to compare the probabilities determined by the two algorithms. It turns out that the visiting probabilities obtained by the fuzzy C-means algorithm are more realistic than the probabilities determined by the parametric Bayesian classification algorithm.

As a future research direction one can consider employing different distance functions and different utility models for CFL models. All the facility location problems considered in this thesis are discrete models. The formulations can be easily adapted for the continuous space, but of course one has to develop different solution methods for the continuous version of the problems. Another important extension to CFL models is that the customers perceive the facilities different so that the probabilities for patronizing facilities vary from customer to customer. For the last proposed discrete facility location model it is also possible to consider other distributions than Gaussian for class likelihoods when determining the visiting probabilities with the parametric Bayesian classification. In addition to these proposed extensions, a substantial progress is needed for developing an efficient exact solution method for the CFL model with full reaction of the competitor and a more efficient solution procedure in order to obtain good results for the last considered facility location problem.

An important future research direction is to investigate the use of hybrid metaheuristics for the solution of a bilevel mixed-integer linear or nonlinear programming model, whose lower-level includes integer decision variables and is a concave maximization problem when the integer variables are fixed. Developing such solution procedures will be an important contribution to the literature of mathematical programming.

APPENDIX A: αBB ALGORITHM

i. Initialization: Set $z_{LB} = -\infty$, $z_{UB} = \infty$. Select a convergence tolerance ϵ and a feasibility tolerance ϵ_f for constraints.

ii. Solve the single-level problem P_{LB} using a nonlinear solver to obtain a local solution. If the solution at hand is ϵ_f-feasible, update z_{LB}.

iii. Solve the concavified problem P_{UB}. If it is infeasible, then STOP. Otherwise update z_{UB}, select the branching variable and obtain two new subrectangles.

While $z_{UB} - z_{LB} > \epsilon$ repeat the following steps:

iv. Find the subrectangle which gives the maximum of upper bounds and update z_{UB}. This subrectangle is used as the current rectangle and the solution which gives the maximum of upper bounds is the current point.

v. Solve the problem P_{LB} to get a local solution using the current point as the starting point. If the local solution is ϵ_f-feasible, z_{LB} is updated.

vi. The current rectangle is partitioned into two new subrectangles by selecting a branching variable. The concavified problem P_{UB} is solved in each subrectangle. If the solution in a subrectangle turns out to be infeasible or the objective function value of P_{UB} is less than z_{LB}, then that subrectangle is removed from further consideration. Report the solution with the objective value z_{LB} as the ϵ-optimal solution.

APPENDIX B: GMIN-αBB ALGORITHM

i. Set $z_{LB} = -\infty$. Obtain the relaxed problem P' by relaxing all integer variables.

While there are active nodes in the tree, repeat the following steps:

ii. Select an active node according to some branching rule.

iii. Apply the αBB algorithm and obtain an upper bound UB at the active node from the solution of P'.

iv. If a feasible solution is obtained, let $UB = LB$, where LB indicates a lower bound to the problem. Update z_{LB} by $z_{LB} = \max\{z_{LB}, LB\}$, prune that node and backtrack. Otherwise, if an infeasible solution is obtained or $UB \leq z_{LB}$, prune that node and backtrack. Report the solution with the objective value z_{LB} as the optimal solution.

APPENDIX C: STEPS OF THE FIRST TABU SEARCH HEURISTIC

Notation:

num_iter: number of iterations performed

max_iter: maximum number of iterations

num_nonimp_iter: number of iterations through which the incumbent does not improve

max_nonimp_iter: maximum number of iterations through which the incumbent does not improve

num_neigh: number of neighbors generated in the current iteration

$size\_neigh_i$: number of neighbors generated in the current iteration using the ith move

Obj: objective value of a newly generated neighboring solution for the firm

Obj_Best_Neigh: objective value of the best neighboring solution for the firm

Obj^* : objective value of the incumbent for the firm

$\mathbf{Q}$, $\mathbf{X}$: value of the decision variables for the firm for the newly generated neighboring solution

$\mathbf{Q}_{best}$, $\mathbf{X}_{best}$: value of the decision variables for the firm for the best neighboring solution

$\mathbf{Q}^*$, $\mathbf{X}^*$: value of the decision variables for the firm for the incumbent

ϵ: radius of the ϵ-ball

Initialization:

Find an initial solution. Set it as the incumbent as well as the current solution.

This way, Obj^*, $\mathbf{Q}^*$, and $\mathbf{X}^*$ are initialized.

Set *num_iter*:=0 and *num_nonimp_iter*:=0.

Search:

While (*num_iter*< *max_iter*) AND (*num_nonimp_iter*< *max_nonimp_iter*)

Set *Obj_Best_Neigh*:=0.

For each move type i = 1-Swap, 1-Add, 1-Drop **do**

Calculate $size\_neigh_i$ and set *num_neigh*:=0

While (*num_neigh* < $size\_neigh_i$) **do**

Generate a new neighboring $\mathbf{X}$ by executing move i on the current solution.

Fix **X** in the upper level problem and generate **Q** randomly in the ϵ ball centered at the **Q** values of the current solution.

If the new solution is not tabu active

Fix the firm's variables in the competitor's problem and solve it using the BB algorithm with NLP relaxation.

Get the values for **A** and **M**.

Return to the firm's objective function, fix **Q**, **X**, **A**, and **M** and calculate the objective function value *Obj* of the firm.

If *Obj*> *Obj_Best_Neigh* then **do**

Set *Obj_Best_Neigh*:= *Obj* and $\mathbf{Q}_{best}$:=**Q**, $\mathbf{X}_{best}$:=**X**.

If *Obj*>Obj^* then **do**

Set this neighbor as the incumbent, and update Obj^*:=*Obj* and $\mathbf{Q}^*$:=**Q**, $\mathbf{X}^*$:=**X**.

Set *num_nonimp_iter*:=0.

End **If**

End **If**

Put the newly generated solution into the tabu list, but only the location values **X** and the attractiveness levels **Q** of the firm.

End **If**

num_neigh:=*num_neigh*+1.

End **While**

End **For**

Set the best neighboring solution as the current solution and use the attractiveness levels **Q** of the firm in the next iteration for all moves.

num_nonimp_iter:=*num_nonimp_iter*+1.

num_iter:=*num_iter*+1.

End **While**

APPENDIX D: STEPS OF THE SECOND TABU SEARCH HEURISTIC

Initialization:

Find an initial solution. Set it as the incumbent as well as the current solution. This way, Obj^*, $\mathbf{Q}^*$, and $\mathbf{X}^*$ are initialized.

Record $\mathbf{A}$, $\mathbf{M}$ values and use them in the next iteration for the current solution.

Set *num_iter*:=0 and *num_nonimp_iter*:=0.

Search:

While (*num_iter*< *max_iter*) AND (*num_nonimp_iter*< *max_nonimp_iter*)

Set *Obj_Best_Neigh*:=0.

For each move type i = 1-Swap, 1-Add, 1-Drop **do**

Calculate $size\_neigh_i$ and set *num_neigh*:=0

While ($num\_neigh < size\_neigh_i$) **do**

Generate a new neighboring $\mathbf{X}$ by executing move i on the current solution.

Fix $\mathbf{A}$, $\mathbf{M}$ to the values recorded in the current solution and $\mathbf{X}$ in the upper level problem.

If the new solution is not tabu active

By applying the gradient ascent algorithm obtain $\mathbf{Q}$ variables.

Fix the firm's variables in the competitor's problem and solve it using the BB algorithm with NLP relaxation.

Get the values for $\mathbf{A}$ and $\mathbf{M}$.

Return to the firm's objective function, fix $\mathbf{Q}$, $\mathbf{X}$, $\mathbf{A}$, and $\mathbf{M}$ and calculate the objective function value *Obj* of the firm.

If *Obj*> *Obj_Best_Neigh* then **do**

Set *Obj_Best_Neigh*:= *Obj* and $\mathbf{Q}_{best}$:=$\mathbf{Q}$, $\mathbf{X}_{best}$:=$\mathbf{X}$.

If Obj>Obj^* then **do**

Set this neighbor as the incumbent, and update Obj^*:=Obj

and $\mathbf{Q}^*$:=$\mathbf{Q}$, $\mathbf{X}^*$:=$\mathbf{X}$.

Set *num_nonimp_iter*:=0.

End **If**

End **If**

Put the newly generated solution into the tabu list, but only

the location values $\mathbf{X}$ of the firm and the attractiveness levels

$\mathbf{A}$, and $\mathbf{M}$ of the competitor.

End **If**

num_neigh:=*num_neigh*+1.

End **While**

End **For**

Set the best neighboring solution as the current solution and use

the attractiveness levels $\mathbf{A}$ and $\mathbf{M}$ of the competitor in the next iteration

for all moves.

num_nonimp_iter:=*num_nonimp_iter*+1.

num_iter:=*num_iter*+1.

End **While**

Return the incumbent as a favorable solution to the given instance.

REFERENCES

Aboolian, R., O. Berman, and D. Krass, 2007a, "Competitive Facility Location and Design Problem", *European Journal of Operational Research*, No.182, pp. 40–62.

Aboolian, R., O. Berman, and D. Krass, 2007b, "Competitive Facility Location Model with Concave Demand", *European Journal of Operational Research*, No.181, pp. 598–619.

Achabal, D. D., W. L. Gorr, and W. L. Mahajan, 1982, "MULTILOC: A multiple store location decision model", *Journal of Retailing*, No.2, pp. 5–25.

Adjiman, C. S., I.P. Androulakis, and C.A. Floudas, 1997, "Global optimization of MINLP problems in process synthesis and design", *Computers and Chemical Engineering*, Vol. 21, pp. 445–450.

Adjiman, C. S., S. Dallwig, C.A. Floudas, A. Neumaier, 1998a, "A global optimization method, αBB, for general twice-differentiable constrained NLPs-I. Theoretical advances", *Computers and Chemical Engineering*, Vol. 22, No.9, pp. 1137–1158.

Adjiman, C. S., I.P. Androulakis, and C.A. Floudas, 1998b, "A global optimization method, αBB, for general twice-differentiable constrained NLPs-II. Implementation and computational results", *Computers and Chemical Engineering*, Vol. 22, No.9, pp. 1159–1179.

Adjiman, C. S., I.P. Androulakis, and C.A. Floudas, 2000, "Global optimization of mixed-integer nonlinear problems", *AIChE*, Vol. 46, pp. 176–248.

Alpaydın, E., 2004, *Introduction to Machine Learning*, The MIT Press, Cambridge, Massachusetts.

Androulakis, I. P., C. D. Maranas, and C. A. Floudas, 1995, "αBB: A global optimization method for general constrained nonconvex problems", *Journal of Global Optimization*, Vol. 7, pp. 337–363.

Balcan, M.-F., A. Blum, and K. Yang, 2004, "Co-Training and Expansion: Towards Bridging Theory and Practice", *Proceedings of Neural Information Processing System (NIPS 2004).*

Bard, J. F., 1998, *Practical Bilevel Optimization Algorithms and Applications*, Kluwer Academic Publishers, Dordrecht.

Beasley, J. E., 1993a, "Lagrangean heuristics for location problems", *European Journal of Operational Research*, Vol. 65, pp. 383–399.

Beasley, J. E., 1993b, *Lagrangean relaxation. In: Reeves CR (Ed). Modern Heuristic Techniques for Combinatorial Problems*, Halsted Press, New York.

Benati, S. and P. Hansen, 2002, "The maximum capture problem with random utilities: Problem formulation and algorithms", *European Journal of Operational Research*, No.143, pp. 518–530.

Berman, O. and D. Krass, 1998, "Flow intercepting spatial interaction model: a new approach to optimal location of competitive facilities", *Location Science*, Vol.6, pp. 41–65.

Berman, O. and D. Krass, 2002, "Locating multiple competitive facilities: spatial interaction models with variable expenditures", *Annals of Operational Research*, No.111, pp. 197–225.

Berman, O., D. Krass, and Z. Drezner, 2003, "The gradual covering decay location problem on a network", *European Journal of Operational Research*, No.151, pp. 474–480.

Berman, O., Z. Drezner, D. Krass, G. O. Wesolowsky, 2009, "The variable radius covering problem", *European Journal of Operational Research*, Vol.196, pp. 516–525.

Bertsekas, D. P., 1995, *Nonlinear Programming*, Athena Scientific, Boston.

Bezdek, J. C., 1981, *Pattern recognition with fuzzy objective function algorithms*, Kluwer Academic Publishers, New York, Plenum.

Bezdek, J. C. and N. R. Pal, 1995, "Two Soft Relatives of Learning Vector Quantization", *Neural Networks*, Vol. 8, No.5, pp. 729–743.

Bhadury, J., H. A. Eiselt, and J. H. Jaramillo, 2003, "An alternating heuristic for medianoid and centroid problems in the plane", *Computers and Operations Research*, Vol. 30, pp. 553–565.

Blum, A. and T. Mitchell, 1998, "Combining Labeled and Unlabeled Data with Co-Training", *Proceedings of the 1998 Conference on Computational Learning Theory*.

Boots, B. and R. South, 1997, "Modeling retail trade areas using higher-order, multiplicatively weighted Voronoi diagrams", *Journal of Retailing*, Vol. 73, pp. 519–536.

Chapelle, E., B. Schölkopf and A. Zien, 2006, *Semi-Supervised Learning*, The MIT Press, Cambridge, Massachusetts.

Colson, B., P. Marcotte, and G. Savard, 2007, "An overview of bilevel optimization", *Annals of Operations Research*, Vol. 153, pp. 235–256.

Daskin, M. S., 1995, *Network and Discrete Location Models, Algorithms, and Applications*, Willey, New York.

Daskin, M. S., L. V. Synder, and R. T. Berger, 2003, "Facility Location in Supply Chain Design" *Working Paper*, No. 03-010.

Dempe, S., 2003, "Annotated bibliography on bilevel programming and mathematical programs with equilibrium constraints", *Optimization*, Vol. 52, No.3, pp. 333–359.

Drezner, Z., 1982, "Competitive location strategies for two facilities", *Regional Science and Urban Economics*, Vol. 12, pp. 485–493.

Drezner, T. and Z. Drezner, 1994, "Locating a single new facility among existing, unequally attractive facilities", *Journal of Regional Science*, No.2, pp. 237–252.

Drezner, Z.(Ed.), 1995, *Facility Location: A Survey of Applications and Methods*, Springer, New York.

Drezner, T. and Z. Drezner, 1998, "Facility location in anticipation of future competition", *Location Science*, Vol. 6, pp. 155–173.

Drezner, T. and Z. Drezner, 2002, "Validating the gravity-based competitive location model using inferred attractiveness", *Annals of Operations Research*, No.111, pp. 227–237.

Drezner, T., Z. Drezner, and S. Salhi, 2002, "Solving the multiple competitive facility location problem", *European Journal of Operational Research*, No.142, pp. 138–151.

Drezner, T. and Z. Drezner, 2004, "Finding the optimal solution to the Huff based competitive location model", *Computational Management Science*, No.2, pp. 193–208.

Drezner, T. and Z. Drezner, 2006, "Multiple Facilities Location in the Plane using the Gravity Model", *Geographical Analysis*, No.38, pp. 391–406.

Drezner, T. and Z. Drezner, 2008, "Lost demand in a competitive environment", *Journal of the Operational Research Society*, No.3, pp. 362–371.

Edmunds, T. A. and J.F. Bard, 1992, " An algorithm for the mixed-integer nonlinear bilevel programming problem", *Annals of Operations Research*, Vol. 34, pp. 149–162.

Fernandez, J., B. Pelegrín, F. Plastria, B. Tóth, 2004, "Solving a Huff-like competitive location and design model for profit maximization in the plane", *Mathematical Programming*, No.2, pp. 247–265.

Fernandez, J., B. Pelegrín, F. Plastria, B. Tóth, 2007, "Planar location and design of a new facility with inner and outer competition: an interval lexicographical-like solution procedure", *Networks and Spatial Economics*, No.7, pp. 19–44.

Fischer, K., 2002, "Sequential discrete p-facility models for competitive location planning", *Annals of Operations Research*, Vol. 111(1–4), pp. 253–270.

Floudas, C. A., 2000, "Deterministic Global Optimization: Theory, Methods, and Applications", Kluwer Academic Publishers.

Glover, F., M. Laguna, and R. Martí, 2007, *Principles of Tabu Search. In: Handbook on Approximation Algorithms and Metaheuristics, Gonzalez T (ed).*, Chapman & Hall/CRC, Boca Raton.

Grossmann, I. E. and C. A. Floudas, 1987, "Active constraint strategy for flexibility analysis in chemical processes", *Computers and Chemical Engineering*, Vol. 11, pp. 675–693.

Grossmann, I. E., J. Viswanathan, A. Vecchietti, R. Raman, E. Kalvelagen E, 2004, "DICOPT Solver Manual", http://www.gams.com/dd/docs/solvers/dicopt.pdf.

Guta, B., 2003, *Subgradient Optimization Methods in Integer Programming with an Application to a Radiation Therapy*, Ph.D. Thesis, University of Kaiserslautern.

Gümüş, Z. H. and C.A. Floudas, 2005, "Global optimization of mixed-integer bilevel programming problems", *Computational Management Science*, Vol. 2, pp. 181–212.

Hall, R. W. (Ed.), 1999, *Handbook of Transportation Science*, Kluwer Academic Publishers, Dordrecht.

Held, M., P. Wolfe, and H. P. Crowder, 1974, "Validation of subgradient optimization", *Mathematical Programming*, Vol. 6, pp. 62–88.

Hotelling, H., 1929, "Stability in competition", *Economic Journal*, No.39, pp. 41–57.

Huff, D. L., 1964, "Defining and estimating a trade area", *Journal of Marketing*, No.28, pp. 34–38.

Huff, D. L., 1966, "A Programmed Solution for Approximating an Optimum Retail Location", *Land Economics*, No.42, pp. 293–303.

ILOG CPLEX 11.0 User's Manual, 2007.

Jan, R.-H. and M.-S. Chern, 1994, "Nonlinear integer bilevel programming", *European Journal of Operational Research*, Vol. 72, pp. 574–587.

Karasakal, O. and E. K. Karasakal, 2004, "A maximal covering location model in the presence of partial coverage", *Computers and Operations Research*, No.31, pp. 1515–1526.

Korpela, J. and A. Lehmusvaara, 1999, "A customer oriented approach to warehouse network evaluation and design", *International Journal of Production Economics*, Vol.59, pp. 135–146.

Korpela, J., A. Lehmusvaara, and M. Tuominen, 2000, "Customer service based design of the supply chain", *International Journal of Production Economics*, Vol.69, pp. 193–204.

Küçükaydın, H., N. Aras, and İ.K. Altınel, 2010a, "A discrete competitive facility location problem with variable attractiveness", *Journal of the Operational Research Society*, doi: 10.1057/jors.2010.136.

Küçükaydın, H., N. Aras, and İ.K. Altınel, 2010b, "A Hybrid Tabu Search Heuristic for a Bilevel Competitive Facility Location Model", *Proceedings of the 7th International Workshop on Hybrid Metaheuristics, HM2010*, Springer Lecture Notes in Computer Science, Vol. 6373/2010.

Küçükaydın, H., N. Aras, and İ.K. Altınel, 2011a, "Competitive facility location problem with attractiveness adjustment of the follower: A bilevel programming model and its solution", *European Journal of Operational Research*, Vol. 208, No.3, pp. 206–220.

Küçükaydın, H., N. Aras, and İ.K. Altınel, 2012, "A Leader-Follower Game in Competitive Facility Location", *Computers and Operations Research*, Vol. 39, pp. 437–448.

Küçükaydın, H., N. Aras, and İ.K. Altınel, 2011b, "Çiftdüzeyli Bir Rekabetçi Tesis Yer Seçimi Problemi İçin Tabu Arama Sezgiseli", *Turkish Journal of Industrial Engineering*, to appear.

Labbé, M. and S.L. Hakimi, 1991, "Market and locational equilibrium for two competitors", *Operations Research*, Vol. 39, No.5, pp 749–756.

Laguna, M. and R. Martí, 2003, *Scatter Search: Methodology and Implementations in C*, Kluwer Academic Publishers, Norwell, Massachusetts.

Lederer, P. J. and A.P. Hurter,Jr., 1986, "Competition of firms: Discriminatory pricing and location", *Econometrica*, Vol. 54 , No.3, pp. 623–640.

Lederer, P. J., 1986, "Duopoly competition in networks", *Annals of Operations Research*, Vol. 6, No.4, pp. 99–109.

Lederer, P. J. and J.-F. Thisse, 1990, "Competitive location on networks under delivered pricing", *Operations Research Letters*, No.9, pp. 147–153.

Louveaux, F. V., 1993, "Stochastic Location Analysis", *Location Science*, Vol. 1, No.2, pp. 127–154.

Love, R. F., J. G. Morris, and G. O. Weselowsky, 1988, *Facility Location Models and Methods*, North-Holland, Amsterdam.

Melo, M. T., Nickel, S., and F. Saldanha-da-Gama, 2009, “Facility location and supply chain management - A review”, *European Journal of Operational Research*, Vol. 196, pp. 401–412.

Miller, T. C., T. L. Friesz, and R. L. Tobin, 1996, *Equilibrium facility location on networks*, Springer Verlag, New York, NY.

Moore, J. T. and J. F. Bard, 1990, “The mixed-integer linear bilevel programming problem”, *Operations Research*, Vol. 38, No.5, pp. 911–921.

Murtagh, B. A., M. A. Saunders, W. Murray, P. E. Gill, 2010, “MINOS Solver Manual”, http://www.gams.com/dd/docs/solvers/minos.pdf.

Nakanishi, M. and L. G. Cooper, 1974, “Parameter Estimate for Multiplicative Interactive Choice Model: Least Squares Approach”, *Journal of Marketing Research*, No.11, pp. 303–311.

Nemhauser, G. L. and L. A. Wolsey, 1998, *Integer and Combinatorial Optimization*, Willey, New York.

Pérez, M. D. G. and B. Pelegrín, 2003, “All Stackelberg location equilibria in the Hotelling’s duopoly model on a tree with parametric prices”, *Annals of Operations Research*, Vol. 122 (1–4), pp. 177–192.

Pérez, M. D. G., P.F. Hernández, and B. Pelegrín, 2004, “On price competition in location-price models with spatially separated markets”, *Sociedad de Estadística e Investigación Operativa Top*, Vol. 12, No.2, pp. 351–374.

Plastria, F. and E. Carrizosa, 2004, “Optimal location and design of a competitive facility”, *Mathematical Programming*, No.2, pp. 247–265.

Plastria, F. and L. Vanhaverbeke, 2008, “Discrete models for competitive location with foresight”, *Computers and Operations Research*, Vol. 35, No.3, pp. 683–700.

Press, W. H., B. P. Flannery, S. A. Teukolsky, W. T. Vetterling, 1986, *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, New York.

Reilly, W. J., 1931, *The Law of Retail Gravitation*, Knickerbocker Press, New York, NY.

Rhim, H., T.H. Ho, and U.S. Karmarkar, 2003, "Competitive location, production, and market selection", *European Journal of Operational Research*, Vol.149, No.1, pp. 211–228.

Rosenthal, R. E., 2010, *GAMS – A User's Guide*.

Sáiz, M. E., E. M. T. Hendrix, J. Fernández, B. Pelegrín, 2009, "On a branch-and-bound approach for a Huff-like Stackelberg location problem". *OR Spectrum*, Vol. 31, No.3, pp. 679–705.

Sarkar, J., B. Gupta, and D. Pal, 1997, "Location equilibrium for cournot oligopoly in spatially separated markets", *Journal of Regional Science*, Vol. 37, No.2, pp. 195–212.

Serra, D. and C. ReVelle, 1993, "Market capture by two competitors: The pre-emptive location problem", *Economics Working Paper Series 39*.

Serra, D. and C. ReVelle, 1999, "Competitive location and pricing on networks", *Economics and Business Working Papers Series 219*, pp. 1–42.

Serra, D. and R. Colomé, 2001,"Consumer choice and optimal locations models: Formulations and heuristics", *Papers in Regional Science*, Vol. 80, pp. 439–464.

Synder, L., 2006, "Facility location under uncertainty: a review", *IIE Transactions*, Vol. 38, pp. 537–544.

Steiner, W. J., 2010, "A Stackelberg-Nash model for new product design", *OR Spectrum* Vol. 32, pp. 21–48.

Tóth, B., J. Fernandez, B. Pelegrín, F. Plastria, 2009, "Sequential versus simultaneous approach in the location and design of two new facilities using planar Huff-like models", *Computers and Operations Research*, No.36, pp. 1393–1405.

von Stackelberg, H., 1934, *Marktform und Gleichgewicht*, Springer Verlag, Vienna.

Waltz, R. A and T. D. Plantenga, 2009, *Knitro User's Manual Version 6.0 Ziena Optimization Inc.*.

Wen, U. P. and Y. H. Yang, 1990, "Algorithms for solving the mixed integer two-level linear programming problem", *Computers and Operations Research*, Vol. 17, No.2, pp. 133–142.

Wendell, R. E. and R. D. McKelvey, 1981, "New perspectives in competitive location theory", *European Journal of Operational Research*, Vol.6, pp. 174–182.