EDGE-EXTREMAL GRAPHS UNDER DEGREE AND MATCHING NUMBER RESTRICTIONS

by

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ABSTRACT

EDGE-EXTREMAL GRAPHS UNDER DEGREE AND MATCHING NUMBER RESTRICTIONS

A graph with an upper bound on its matching number but without a bound on its maximum degree, or a graph with an upper bound on its maximum degree but without a bound on its matching number would have infinitely many edges. In order to limit the maximum number of edges of a graph to a finite number, bounds on both maximum degree and matching number are needed. The edge-extremal problem deals with maximizing the number of edges of a graph under restrictions on its maximum degree and matching number. This type of problems are generally studied in the field of Extremal Graph Theory whose main concern is to find extremal graphs that satisfy a certain property. The answer to the edge-extremal problem is known for arbitrary graphs [1]. It is interesting to solve the edge-extremal problem when imposed some structure on the given graphs since the maximum number of edges may change upon narrowing the graph class. The answer when the graphs belong to some chosen graph classes is provided by a recent master thesis [2]. The problem has been answered in that thesis for bipartite graphs, split graphs, disjoint union of split graphs and unit interval graphs. It is observed that star graphs seem to play a central role in the bound on edges. Some open questions have been therefore posed concerning how allowing or disallowing stars affects the bound on the number of edges. In this thesis we provide, to the best of our knowledge, the first results of the edge-extremal problem in claw-free graphs. We find an answer to the change in edge-extremal instances for general graphs when we do not allow claws, which is a special star graph. For this purpose, we develop several claw-free graph constructions and we find the number of edges of an edge-extremal claw-free graph, not only by giving the number itself but also by providing an edge-extremal claw-free graph for each possible case.

ÖZET

MAKSİMUM DERECESİ VE EŞLEME SAYISI SINIRLI, KENAR SAYISI EN ÇOK OLAN ÇİZGELER

Eşleme sayısı üstten sınırlı fakat maksimum derecesi sınırlı olmayan bir çizgenin ya da maksimum derecesi üstten sınırlı fakat eşleme sayısı sınırlı olmayan bir çizgenin sonsuz sayıda kenarı olabilir. Bir çizgenin maksimum kenar sayısının sonlu bir sayı olabilmesi için hem maksimum derecesi hem de eşleme sayısının üstten sınırlanması gerekir. Uç kenar problemi, maksimum derecesi ve eşleme sayısı sınırlı bir çizgenin kenar sayısını maksimize etmek ile ilgilenir. Bu tip problemler genelde, ana konusu belli özellikleri sağlayan uç çizgeleri bulmak olan Uç Çizge Teorisi alanında çalışılır. Uç kenar probleminin çözümü genel çizgeler sınıfı için bilinmektedir [1]. Uç kenar problemini, çizgelere belli yapısal özellikler eklendiğinde çözmeye çalışmak ilginçtir çünkü maksimum kenar sayısı çizge sınıfı daraltıldığında değişebilir. Çizgeler seçilen bir çizge sınıfına ait iken uç kenar probleminin çözümü yakın zamanda yapılmış bir yüksek lisans tezinde sunulmuştur [2]. Orada problem, bipartit çizgeler, split çizgeler, split çizgelerin ayrık birleşimi ve birim aralık çizgeler sınıfları için çözülmüştür. Yıldız çizgelerin, kenar sayısındaki sınırlarda çok önemli bir rol oynadığı görülmüştür. Bu sebeple, yıldız çizgelere izin verip vermemenin kenar sayısını nasıl etkileyeceğiyle ilgili bazı açık sorular sorulmuştur. Bu tezde, uç kenar probleminin pençesiz çizgelerdeki ilk sonuçlarını sunuyoruz. Özel bir yıldız çizge olan pençeye izin verilmediğinde uç kenar probleminin genel çizgelerdeki sonucunun nasıl değiştiğine bir cevap buluyoruz. Bu amaçla, birkaç pençesiz çizge yapısı geliştiriyoruz ve kenar sayısı en çok olan pençesiz çizgelerin kenar sayısını buluyoruz. Sadece sayıyı vermekle kalmıyor, aynı zamanda her olası durum için kenar sayısı en çok olan pençesiz çizgeler sunuyoruz.

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LIST OF SYMBOLS

C_n	A cycle on n vertices
$E(G)$	Edge set of graph G
$\overline{G}$	Complement of graph G
K_n	Complete graph on n vertices
$K_{\|U\|,\|W\|}$	Complete bipartite graph with partitions of size $\|U\|$ and $\|W\|$
$L(G)$	Line graph of graph G
n	Number of vertices
$N(v)$	Neighborhood of vertex v
P_n	Path graph on n vertices
$R(i,j)$	Ramsey number
$V(G)$	Vertex set of graph G
$\alpha(G)$	Independence number of graph G
$\delta(G)$	Minimum degree in graph G
$\Delta(G)$	Maximum degree in graph G
$\nu(G)$	Matching number of G
$\omega(G)$	Clique number of graph G

LIST OF ACRONYMS/ABBREVIATIONS

$\mathcal{CF}$	Class of claw-free graphs
$\mathcal{GEN}$	Class of general graphs
$\mathcal{K}_p$	Class of complete graphs on p vertices
$\mathcal{M}_{\mathcal{CF}}(i,j)$	All graphs G (with no isolated vertex) from $\mathcal{CF}$ satisfying $\Delta(G) < i$ and $\nu(G) < j$
$\mathcal{M}_{\mathcal{GEN}}(i,j)$	All graphs G (with no isolated vertex) from $\mathcal{GEN}$ satisfying $\Delta(G) < i$ and $\nu(G) < j$
$\mathcal{M}_{\mathcal{UNIT}}(i,j)$	All graphs G (with no isolated vertex) from $\mathcal{UNIT}$ satisfying $\Delta(G) < i$ and $\nu(G) < j$
$\mathcal{R}_{p,r}$	Class of graphs that are r-regular on p vertices
$\mathcal{R}'_{p,r}$	Class of graphs that have p vertices where $p-1$ of them have degree r and one of them is of degree $r-1$ and where p and r are odd
$\mathcal{UNIT}$	Class of unit interval graphs

1. INTRODUCTION

Graph theory is becoming more and more important as it is being effectively used in fields as varied as computer science, linguistics, genomics, communication networks, algorithms, computation, and operations research in order to model many types of relations and processes in physical, biological, social and information systems. Many practical problems can be represented by graphs and the methods in graph theory have also been used to prove fundamental results in pure mathematics. In this sense, the problems defined on graphs may arise from practical situations or may provide mathematical insight. A class of problems that might fit both of these characterizations are extremal problems on graphs. These problems seek to determine how large or small some parameter of a graph can be, while satisfying some set of conditions. In this thesis we will ask how many edges a graph can have under restrictions on its maximum degree and matching number. We will call this the *edge-extremal problem.* The answer to this problem is known for arbitrary graphs. We will find the corresponding answer when the graphs belong to some chosen graph class. A graph class is a collection of graphs sharing some common property. We are interested in the study of how extremal values change when we impose some structure on the given graphs. If the maximum number of edges changes upon narrowing the graph class, we might be able to say which structural features of the class allow the solution.

A *claw* is the complete bipartite graph $K_{1,3}$, that is, a star graph with three edges, three leaves, and one central vertex. A *claw-free graph* is a graph in which no induced subgraph is a claw. Figure 1.1 shows the claw graph, a claw-free graph, and a graph with an induced claw. In this thesis, we will try to find an answer to the edge-extremal problem in the class of claw-free graphs. The motivation in choosing claw-free graphs as the graph class to analyze is that the result for general graphs suggests that the edge-extremal graph contains star graphs in most cases. We are wondering how are the edge-extremal instances for general graphs affected if we do not allow claws, which

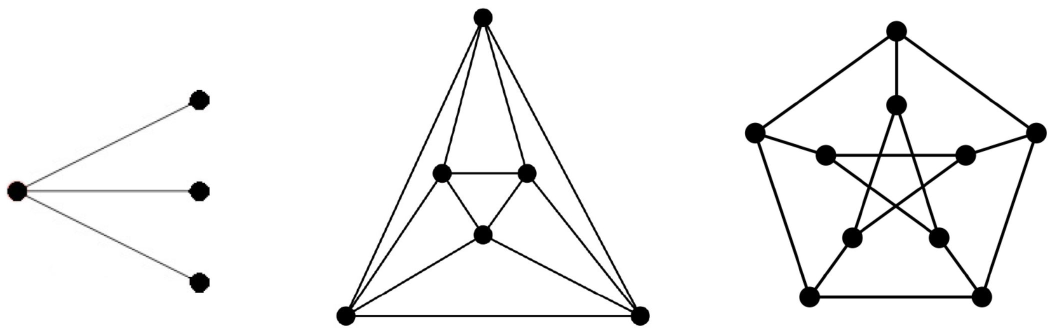


Figure 1.1. Claw graph, a claw-free graph, and a graph with an induced claw

is a special star graph.

Another motivation to solve the edge-extremal problem is its relation to the well-known hard problem of finding Ramsey numbers on graphs. Although this approach to the edge-extremal problem is not in the scope of this thesis, and therefore will not be analyzed in detail, we find this relation worthwhile to mention. We briefly comment on this relation in the thesis.

2. EXTREMAL GRAPH THEORY

2.1. Literature Review

Extremal Graph Theory studies extremal (maximal or minimal) graphs which satisfy a certain property. Generally speaking, it concerns any problem that tries to find the relation between graph invariants such as order, size, girth or minimum degree and a graph property like being Hamiltonian, containing a perfect matching, or containing an odd cycle. This theory started with the following question: "What is the minimum size of a graph with a given order to ensure that it contains a triangle as a subgraph?" One equivalent way to consider this question is: "What is the maximum size of a graph with a given order such that it does not contain a triangle as a subgraph?" This problem was solved by Mantel in 1907 [3]. In 1941 [4], the famous Hungarian mathematician Paul Turan gave a much more general answer to this question by answering the following problem: What is the maximum number of edges of a graph with a given order n such that it does not contain the complete graph K_r as a subgraph? Some more examples of questions studied in extremal graph theory would be: Which acyclic graphs on n vertices have the maximum number of edges? What is the minimum number of edges of a graph of order n and connectivity k? The main problem of this thesis is of this nature. Here we seek to maximize the number of edges, given constraints on maximum degree and matching number. We will consider this problem for claw-free graphs.

General extremal problem in extremal graph theory is defined as follows: Given a family $\mathcal{L}$ of forbidden subgraphs, find those graphs G which contain no graph in $\mathcal{L}$ and have the maximum number of edges. In 1941 [4], Turán proved his theorem determining those graphs of order n, not containing the complete graph K_k of order k, and extremal with respect to size, that is, with as many edges as possible. The Erdős–Stone theorem extends Turán's theorem by bounding the number of edges in a graph that does not have a fixed Turán graph as a subgraph. Via this theorem, similar bounds in extremal graph theory can be proven for any excluded subgraph, depending on the chromatic number of the subgraph. It is named after Paul Erdős and Arthur

Stone, who proved it in 1946 [5], and it has been described as the "fundamental theorem of extremal graph theory". Another crucial year for the extremal graph theory was 1975 when Szemerédi proved his result concerning arithmetic progressions in subsets of the integers [6]. Anyone wishing to find further literature on general extremal graph theory is recommended to read, among others, Bollobás' book [7], or the survey [8].

In a more general sense, a graph is extremal with respect to some parameter if it has the maximum (or minimum) value for this parameter as a function of other fixed parameters in the graph. Extremal graphs from this point of view, instead of forbidden subgraphs, are also widely studied in the literature. One example, which is closely related to this thesis, would be to maximize the number of edges of a graph while putting some upper bounds on its maximum degree and its maximum matching. This problem dates back to 1970's. The exact bound for the general graphs, with arbitrary upper bounds on the maximum degree and matching number, was first obtained in [9]. In [1], the authors give a different proof of the same result. Their proof is more structural in approach as opposed to the methods in [9]. Therefore, the answer to this problem is known for arbitrary graphs. The corresponding answer when the graphs belong to some chosen graph classes is also provided by a recent master thesis [2]. The thesis have looked at edge-extremal graphs with bounded degree and matching number on specific graph classes. They have answered this question for bipartite graphs, split graphs, disjoint union of split graphs and unit interval graphs. They observed that i-stars seems to play a central role in the bound on edges. They have therefore posed some open questions asking whether allowing or disallowing stars would make any change on the number of edges of an edge-extremal graph. This is exactly where we take over the responsibility.

In this thesis we will provide, to the best of our knowledge, the first results of edge-extremal problem in claw-free graphs. We will find an answer to the change in edge-extremal instances for general graphs when we do not allow claws, which is a special star graph. This is new to the literature and will contribute, in particular to the edge-extremal problem, in a complementary manner.

2.2. Overview of the Thesis

In Chapter 3, some preliminary work is presented. It starts with basic graph theoretical definitions which will be used in the thesis. Then, we continue with claw-free graphs and related results that concern our subject. In the same chapter, we introduce the edge-extremal problem on general graphs and we present the relation of this problem with Ramsey numbers. In Chapter 4, after defining some graph classes that will be frequently used onwards, we analyze the result of edge-extremal problem on general graphs in a more structural way using previously defined graph classes. Next, we start to solve the edge-extremal problem on claw-free graphs, first on a special case and then on some elementary cases. Before presenting the main result of this thesis, we give claw-free constructions of two important graphs. Then, we give and prove our main result on the edge-extremal problem in claw-free graphs. We close the chapter with some examples illustrating different possible cases. Finally, Chapter 5 concludes the thesis with some final remarks and open questions about the topic.

3. PRELIMINARIES

In this chapter we give most of the definitions and notation required throughout the thesis. Additional notation and definitions are presented when they are needed. We start with the basic definitions of graph theory and then we give some results on claw-free graphs. We make an observation and prove an important lemma about claw-free graphs. Later, we discuss the edge-extremal problem on general graphs. In the last section of this chapter, we present the relation between the edge-extremal problem and Ramsey numbers on graphs.

3.1. Graph Theoretical Definitions

A graph G is a finite nonempty set of vertices V and edges E. We may emphasize that V or E is the set of vertices or edges in G by writing $V(G)$ or $E(G)$, respectively. In Figure 3.1, each vertex is represented by a circle and each edge is represented by a line segment. Edges are named by their start and end points. For instance, an edge between vertices u and v is named uv. We say u is adjacent to v if there is an edge (uv) between vertices u and v.

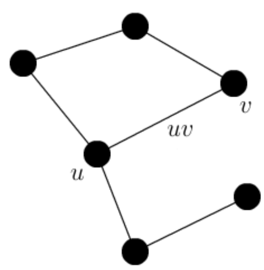


Figure 3.1. An example of a graph representation

In a given graph $G = (V(G), E(G))$, the number of vertices in $V(G)$ is called the *order* of G and the number of edges in $E(G)$ is called the *size* of G. The *neighborhood* of a vertex v is the set of all adjacent vertices to v and denoted by $N(v)$. The number of edges incident with a vertex v is called the *degree* of v and is denoted by $deg\ v$. The maximum and minimum degrees in G are denoted respectively by $\Delta(G)$ and $\delta(G)$. A vertex adjacent to all other vertices of G except itself is *universal* in G.

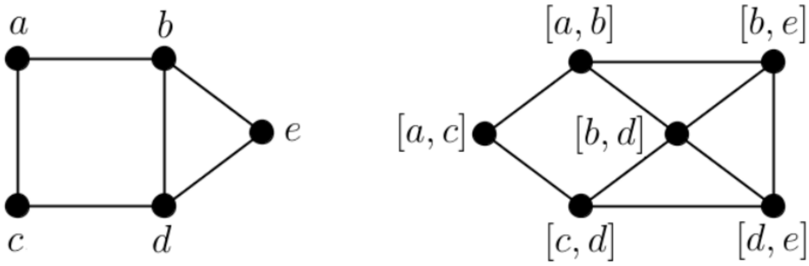
A *regular* graph is a graph where each vertex has the same number of neighbors, that is, every vertex has the same degree. A regular graph with vertices of degree r is called a r-*regular* graph. A graph G is called r-*almost-regular* if $\Delta(G) - \delta(G) < r$. Thus, regular graphs are 1-almost-regular. In this thesis, we will call a graph G *almost regular* if $\Delta(G) - \delta(G) = 1$.

G' is a *subgraph* of G, if vertices and edges of G' form subsets of the vertices and edges of G. H is an *induced subgraph* of G if it has exactly the edges that appear in G over the same vertex set. The *complement* of G, denoted by $\overline{G}$, is the graph on $V(G)$ where two vertices are adjacent if and only if they are not adjacent in G.

For every graph G, the *line graph* of G, denoted $L(G)$, is the graph with vertex set $E(G)$, where there is an edge between two vertices $e, e' \in E(G)$ if and only if the edges e and e' are incident in G. In other words, the line graph of a graph G is obtained by associating a vertex with each edge of the graph G and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. A graph is a *line graph* if it is the line graph of some graph.

In Figure 3.2, graph G contains six edges, which means that $L(G)$ contains six vertices. The vertices $[a, b]$ and $[a, c]$ are linked by an edge in $L(G)$ because the corresponding edges in G have the vertex a in common. However, there is no edge linking the vertices $[a, c]$ and $[b, e]$ in $L(G)$ because those two edges in G have no ends in common.

An isomorphism of graphs G and H is a bijection between the vertex sets of G and H, $f : V(G) \to V(H)$, such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. If an isomorphism exists between two graphs, then the graphs are called *isomorphic* and denoted as $G \simeq H$.



(a) Graph G (b) Its line graph $L(G)$

Figure 3.2. An example of a line graph

A *stable set* or an *independent set* is a set of vertices in a graph, no two of which are adjacent. That is, it is a set S of vertices such that for every two vertices in S, there is no edge connecting the two. A *maximal independent set* is an independent set that is not a subset of any other independent set. In other words, adding any other vertex to a maximal independent set forces the set to contain an edge. The stability number (also called independence number) of graph G, denoted $\alpha(G)$, is the size of the largest stable set of G. In Figure 3.3, the set $\{v_4, v_6\}$ is an independent set but not maximal since we can also add the vertex v_1 to this set without violating the independence property. The set $\{v_2, v_6\}$ is a maximal independent set and the set $\{v_1, v_4, v_6\}$ is a maximum independent set and therefore $\alpha(G) = 3$.

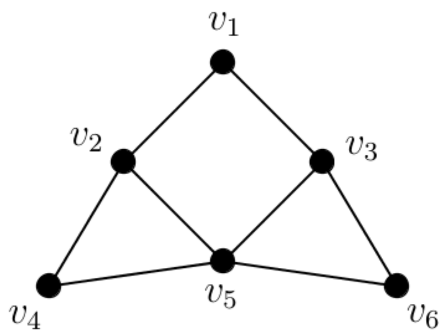


Figure 3.3. Independent set, maximal independent set, independence number

A *complete graph* is a graph in which every pair of distinct vertices is connected by an edge. The complete graph on n vertices is denoted by K_n. The complete graph K_n has $n(n-1)/2$ edges and the degree of each vertex in K_n is $n-1$.

A *clique* in an undirected graph $G = (V, E)$ is a subset of the vertex set $C \subseteq V$, such that every two distinct vertices in C are adjacent. This is equivalent to saying that the subgraph induced by C is complete. A *maximal clique* is a clique that cannot be extended by adding one more vertex, that is, a clique which is not included in the vertex set of a larger clique. The clique number of graph G, denoted $\omega(G)$, is the size of the largest clique in G.

Two edges are said to be *adjacent* if they share a common endpoint. Edges that are not adjacent are *independent*. A *matching* of a graph G is a subset $M \subseteq E(G)$ of pairwise independent edges. We denote by $V(M)$ the set of endpoints of M. A vertex v of G is *saturated* by M if $v \in V(M)$ and *exposed* by M otherwise. A matching M is *maximal* in G if no other matching of G contains M. A matching is a *maximum* matching of G if it is a matching of maximum cardinality. The size of a largest matching in G is called its matching number, denoted by $\nu(G)$. A matching is a *perfect* matching of G if $V(M) = V(G)$, that is, a perfect matching is a matching which matches all vertices of the graph. If $G \setminus u$ has a perfect matching for all $u \in V(G)$, G is *factor-critical.*

A sequence of distinct vertices $v_1, v_2, v_3, ..., v_{i-1}, v_i$ with $v_{j-1}v_j \in E$, for $2 \leq j \leq i$ is a *path* in G. A path on n vertices is denoted by P_n. The *length* of a path is the number of edges it contains. If the first and last vertices in a path are the same, it is a *cycle.* A cycle on n vertices is denoted by C_n. An edge between non-consecutive vertices of a cycle is a *chord.* A cycle with no chord is an *induced cycle.*

In an undirected graph G, two vertices u and v are called *connected* if G contains a path from u to v. Otherwise, they are called *disconnected.* A graph is said to be *connected* if every pair of vertices in the graph is connected. A graph that is not connected is *disconnected.*

A *connected component* of an undirected graph is a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the supergraph. A vertex with no incident edges is itself a connected component. A graph that is itself connected has exactly one connected component, consisting of the whole graph.

A vertex in an undirected connected graph is a *cut vertex* if removing it (and edges through it) disconnects the graph. Specifically, a cut vertex is any vertex whose removal increases the number of connected components.

A graph is *bipartite* if its vertices can be partitioned into two sets U and W such that every edge is between a vertex in U and a vertex in W. Equivalently, one may say that its vertices can be colored black or white such that each edge is between vertices of different colors. A bipartite graph with partition (U, W) where every vertex of U is adjacent to every vertex of W, is called a *complete bipartite* graph, denoted by $K_{|U|,|W|}$. In Figure 3.4, the graph on the left is a bipartite graph while the graph on the right is the complete bipartite graph $K_{4,3}$. The complete bipartite graph $K_{1,i-1}$ is called an i-star and the graph $K_{1,3}$ is called a *claw*, that is, a claw is a star graph with three edges, three leaves, and one central vertex. A *claw-free graph* is a graph in which no induced subgraph is a claw.

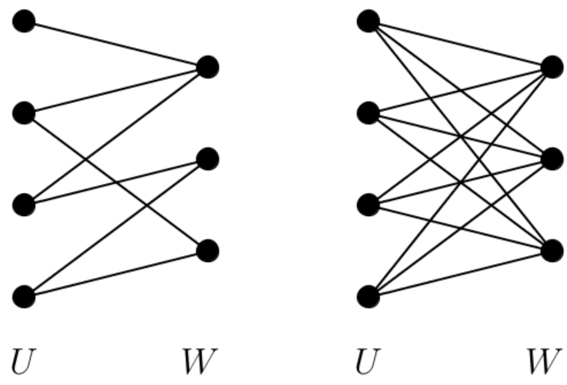


Figure 3.4. Bipartite graph and complete bipartite graph

3.2. Claw-free Graphs

As previously defined, a claw-free graph is a graph that does not have a claw as an induced subgraph. To put it another way, a claw-free graph is a graph in which the neighborhood of any vertex is the complement of a triangle-free graph. In order for a vertex to be claw-center, its neighborhood should contain an independent set of size 3, which means a triangle in the complement of its neighborhood. As a result, a graph G is claw-free if and only if the complement of the neighborhood of any vertex in G is triangle-free.

One should note that the property of being claw-free is hereditary. In other words, claw-freeness is a property that is closed under induced subgraphs. If G is claw-free, then so must every induced subgraph of G. In particular, when we remove one vertex (with all the edges incident to it) from a claw-free graph, the resulting graph is also claw-free.

Claw-free graphs have been a subject of interest of many authors in the recent years. They were initially studied as a generalization of line graphs, and gained further motivation through three key findings about them: the fact that all claw-free connected graphs of even order have perfect matchings [10], the discovery of polynomial time algorithms for finding maximum independent sets in claw-free graphs [11], and the characterization of claw-free perfect graphs [12]. Now, they are the subject of hundreds of mathematical research papers and several surveys.

In terms of the recognition of claw-free graphs, it is obvious that claw-freeness in a given graph can be tested in polynomial time with complexity at most $O(n^4)$, since it is sufficient to test each 4-tuple of vertices to determine whether they induce a claw. More efficiently but less easily, one can also test whether a graph is claw-free by checking, for each vertex of the graph, that the complement graph of its neighbors does not contain a triangle.

In terms of the structure of claw-free graphs, Chudnovsky and Seymour [13] give a series of papers in which they prove a structure theory for claw-free graphs. In this series of papers, they give a structural description of all claw-free graphs. They prove that every claw-free graph either belongs to one of a few basic classes, or admits a decomposition in a useful way [14].

The Observation 3.2 is a simple but important one since we will make use of it in our proofs many times. In order to easily prove this observation, we use the following well-known (attributed to a 1916 paper by Dénes Kőnig) theorem.

Theorem 3.1. *A bipartite graph contains no odd cycles.*

Proof. Suppose that $G(V = A \cup B, E)$ is bipartite. Assume for contradiction that there exists a cycle $v_1, v_2, ..., v_k, v_1$ in G with k odd. Without loss of generality we can assume that $v_1 \in A$. Using the fact that G is bipartite, a simple induction argument suffices to show that $v_i \in A$ for i odd and $v_i \in B$ for i even. But then $v_k v_1 \in E$ is an edge with both endpoints in A, which contradicts the fact that G is bipartite. Therefore a bipartite graph G has no odd cycles. □

In fact, any graph that contains no odd cycles is necessarily bipartite, as well. This we will not prove, but is one way of characterization for bipartite graphs.

Observation 3.2. *If the neighborhood of a vertex v can be partitioned into two cliques, then v is not a claw-center.*

Proof. If the neighborhood of a vertex v can be partitioned into two cliques, the complement of the neighborhood of v is a bipartite graph. Since, by Theorem 3.1, bipartite graphs do not contain triangles, v is not a claw-center. □

The following two theorems from the literature help us to prove Lemma 3.5 which will later be used in the proof of the main result of this thesis.

Theorem 3.3. *(Theorem 2.1 in [15]) Every nontrivial connected graph contains at least two vertices that are not cut vertices.*

Theorem 3.4. *([10]) Every claw-free connected graph with an even number of vertices has a perfect matching.*

Lemma 3.5. *Let G be a connected claw-free graph with matching number $\nu(G) = \nu$ and let n denote the number of vertices of G. Then, $n \leq 2\nu + 1$.*

Proof. Let G be a connected claw-free graph with matching number $\nu(G) = \nu$. Assume for a contradiction that $n > 2\nu + 1$.

If n is even, then by Theorem 3.4, G has a perfect matching and in that case $\nu(G) = \frac{n}{2} > \nu + \frac{1}{2} > \nu$. This is a contradiction to $\nu(G) = \nu$.

If n is odd, then remove a non-cut vertex v from G. By Theorem 3.3, there exists such a non-cut vertex in G. Since v is not a cut vertex, the graph $G - \{v\}$ is connected. As claw-freeness is a hereditary property, the graph $G - \{v\}$ is claw-free. Also the graph $G - \{v\}$ is of even order $n - 1$. Therefore, by Theorem 3.4, $G - \{v\}$ has a perfect matching of size $\frac{n-1}{2} > \nu$. This is a contradiction to $\nu(G) = \nu$. $\square$

3.3. Edge-extremal Problem on General Graphs

In this section we summarize the main problem studied in this thesis, the edge-extremal problem. This is a problem where we ask how many edges a graph can have at most under restrictions on its maximum degree and matching number. All graphs considered in this problem are finite, simple, and undirected. Of course if the number of vertices of the graph is bounded, then the maximum number of edges it can have is given immediately. Here we are interested in the number of edges without restricting or knowing the number of vertices. The solution of the edge-extremal problem on general graphs is already known from [1], and we give a brief description of this result here. For a more detailed version, we direct the reader to this paper.

Consider a graph G with an upper bound $j \geq 1$ on its matching number, but without a bound on its maximum degree or number of vertices. Whatever j is, the maximum number of edges that G can have would be infinite since the star graph $K_{1,i}$ has matching number 1 and i edges without a bound on i. This is shown in Figure 3.5.

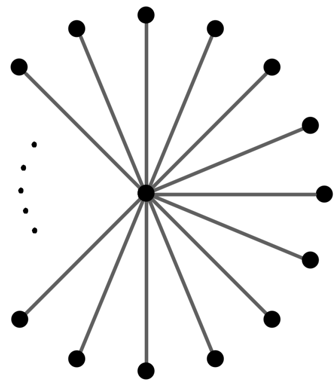


Figure 3.5. A graph with no bound on its maximum degree

Now consider a graph G with an upper bound $i \geq 1$ on its maximum degree, but without a bound on its matching number or number of vertices. Whatever i is, the maximum number of edges that G can have would be infinite since the disjoint union of j copies of K_2 graphs has maximum degree 1 and matching number j without a bound on j. This is shown in Figure 3.6.

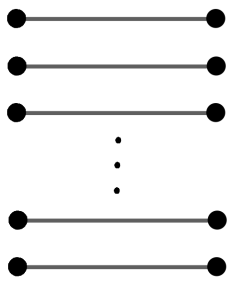


Figure 3.6. A graph with no bound on its matching number

Therefore, in order to limit the size of a graph to a finite number, bounds on both maximum degree and matching number are needed.

Let $\mathcal{F}$ denote a graph class. $\mathcal{M}_{\mathcal{F}}(i,j)$ denotes all graphs G (with no isolated vertex) from $\mathcal{F}$ satisfying $\Delta(G) < i$ and $\nu(G) < j$, where $\Delta(G)$ and $\nu(G)$ denote the maximum degree and the matching number of G, respectively. The edge-extremal problem is the following: Given i, j and $\mathcal{F}$, what is the maximum number of edges of a graph in $\mathcal{M}_{\mathcal{F}}(i,j)$ can have? A graph achieving the maximum number of edges is called *edge-extremal* in $\mathcal{M}_{\mathcal{F}}(i,j)$. Let $\mathcal{CF}$ denote the class of claw-free graphs. In this thesis, we attempt to determine the maximum number of edges of a graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$ for a given i, j.

Let $\mathcal{GEN}$ denote the class of general graphs. Recall that a graph G is *factor-critical* if $G \setminus u$ has a perfect matching for all $u \in V(G)$ and that an *i-star* is the complete bipartite graph $K_{1,i-1}$. From [1], it follows that edge-extremal graphs G in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ have two types of connected components: factor-critical components and i-stars. Further calculations in [1] gives an upper bound for the maximum number of edges of a graph $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$. Without going into the details of the calculation, the number of edges in an edge-extremal graph G is given by

$$|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$$

The solution obtained depends on i being odd or even. For odd i, G is a disjoint union of K_i and i-stars, where the number of K_i is as large as possible. In this case, the number of edges in edge-extremal G is given by

$$|E(G)| = (i-1)(j-1) + (\frac{i-1}{2}) \left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor$$

To give an example, an edge-extremal instance in $\mathcal{M}_{\mathcal{GEN}}(5,6)$ has 24 edges and consists of two K_5 and one 5-star. This graph is shown in Figure 3.7.

In the case where i is even, the factor-critical components are created by the following process: Remove a maximum matching from K_i, introduce a new vertex v

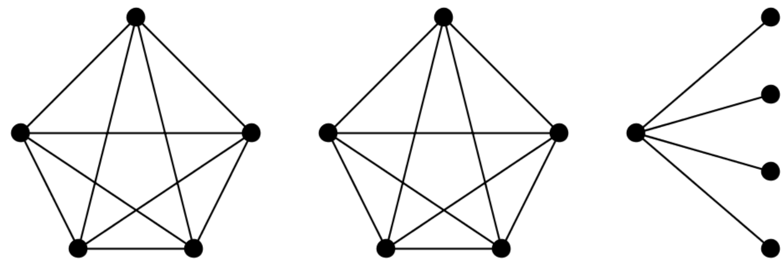


Figure 3.7. An edge-extremal instance in $\mathcal{M}_{\mathcal{GEN}}(5,6)$

and add an edge from v to any of the $i-1$ vertices in the modified graph. We refer to this modified K_i as $R'_{i+1,i-1}$ since it has $i+1$ vertices where every vertex is of degree $i-1$, except one which has degree $i-2$. This notation will be more clear in the following chapter. An edge-extremal instance for even i is a disjoint union of copies of $R'_{i+1,i-1}$ and i-stars, where the number of copies of $R'_{i+1,i-1}$ is as large as possible. In this case, the number of edges in edge-extremal G is given by

$$|E(G)| = (i-1)(j-1) + (\frac{i}{2}-1)\left\lfloor \frac{j-1}{\frac{i}{2}} \right\rfloor$$

To give an example, an edge-extremal instance in $\mathcal{M}_{\mathcal{GEN}}(4,6)$ has 17 edges and consists of two copies of $R'_{5,3}$ and one 4-star. This graph is shown in Figure 3.8.

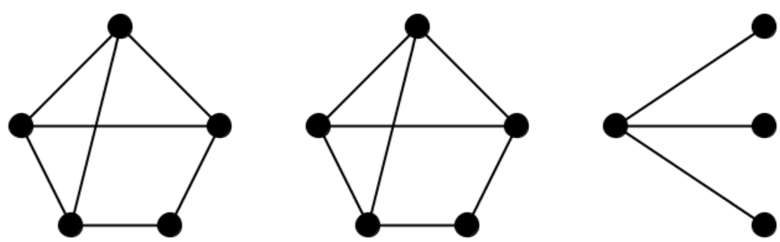


Figure 3.8. An edge-extremal instance in $\mathcal{M}_{\mathcal{GEN}}(4,6)$

This was a brief summary of the results obtained for general graphs by Balachandran and Khare [1]. Consequently, edge-extremal graphs contain star graphs as induced subgraphs in most cases. This gives rise to the question: how would edge-extremal graphs look if star graphs were not allowed as induced subgraphs? This is exactly the question we are seeking the answer of in this thesis. As claws are the smallest star graphs, studying edge-extremal claw-free graphs will give us the answer to this question.

Consider the following example: What is the maximum number of edges of a graph in $\mathcal{M}_{\mathcal{GEN}}(5,2)$? The result for general graphs implies that the answer is 4 and this graph is a 5-star. The answer to the same question for $\mathcal{M}_{\mathcal{CF}}(5,2)$ cannot be more than 4 since every claw-free graph is also in the class of general graphs. Can it be 4, i.e. is there a graph in $\mathcal{M}_{\mathcal{CF}}(5,2)$ which has 4 edges? The answer is no. The edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(5,2)$ has 3 edges and it is a K_3. The edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(5,2)$ and in $\mathcal{M}_{\mathcal{CF}}(5,2)$ are shown in Figure 3.9.

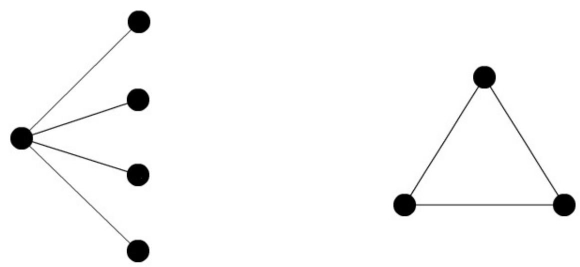


Figure 3.9. Edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(5,2)$ and in $\mathcal{M}_{\mathcal{CF}}(5,2)$

This is the smallest example where the claw-free condition decreases the maximum number of edges. We observe that the maximum number of edges changes in case we narrow the graph class and we want to identify exactly the cases where this change occurs in claw-free graphs.

3.4. Relation to Ramsey Numbers

Ramsey theory typically investigates the questions of the form: "how many elements of some structure must there be to guarantee that a particular property will hold?". The classical problem in Ramsey theory is the party problem, which asks the minimum number of guests $R(i,j)$ that must be invited so that at least i will know each other or at least j will not know each other. Here, $R(i,j)$ is called a *Ramsey* *number*. In the language of graph theory, Ramsey number is defined as follows:

Definition 3.6. *The Ramsey number $R(i, j)$ is the smallest $n \in \mathbb{N}$ such that in any edge coloring of K_n by two colors, blue and red, there exists a blue copy of K_i or a red copy of K_j.*

The following definition is equivalent to the above definition of Ramsey number.

Definition 3.7. *The Ramsey number $R(i, j)$ is the smallest $n \in \mathbb{N}$ such that any graph on n vertices contains a clique of size i or an independent set of size j.*

The question that we first need to ask must be about the existence of Ramsey numbers and the answer lies in the Ramsey's theorem which guarantees the existence of these numbers. It provides an upper bound for all the Ramsey numbers and therefore it can be deduced that the smallest numbers always exist.

Theorem 3.8. *(Ramsey's theorem). [16] For any $i, j \geq 1$, $R(i, j) \leq \binom{i+j-2}{j-1}$.*

The calculation of Ramsey numbers is a difficult problem, and no general method is known. In fact, there are very few numbers i and j for which we know the exact value of $R(i, j)$. For others, we generally have lower and upper bounds for value of $R(i, j)$. However, there is a vast gap between the best known lower and upper bounds. Faced with such difficulty, it is natural to restrict the set of considered graphs.

The reason why we mention Ramsey numbers in this thesis will be more clear with the following well-known observation.

Observation 3.9. *Let G be a graph, let $L(G)$ be the line graph of G, and let $i \geq 4$ and $j \geq 1$ be two integers. Then G has a vertex of degree at least i if and only if $L(G)$ has a clique of size i. Moreover, G has a matching of size j if and only if $L(G)$ has an independent set of size j.*

Proof. Let $L(G)$ be the line graph of G. Edges that meet in a vertex in G are mutually adjacent vertices in $L(G)$ and vice versa. Moreover, independent edges in G, a matching, are independent vertices in $L(G)$ and vice versa. □

With the above observation, it is now clear to see that the problem we are trying to solve in this thesis can actually be formulated in terms of determining Ramsey numbers on line graphs. Consider the edge-extremal problem on general graphs. We are looking for the maximum number of edges in graphs G, with $\Delta(G) < i$ and $\nu(G) < j$. Let $L(G)$ be the line graph of G. By limiting how many edges can meet in a single vertex in G, we are limiting how many vertices can be mutually adjacent in $L(G)$, which is the same as limiting the size of the largest clique in $L(G)$. Similarly, by limiting the matching number of G, we are limiting the size of the largest independent set in $L(G)$. Also, maximizing edges in G is the same as maximizing vertices in $L(G)$. Therefore, the edge-extremal graphs G with $\Delta(G) < i$ and $\nu(G) < j$ correspond to the line graphs with the largest possible number of vertices that do not contain K_i or $\overline{K_j}$ as an induced subgraph. This is exactly $R(i, j) - 1$ for $L(G)$.

In this thesis, by solving the edge-extremal problem for claw-free graphs, we are actually finding the Ramsey numbers for the line graphs of claw-free graphs. The number of edges of an edge-extremal graph $G \in \mathcal{M}_{\mathcal{CF}}(i, j)$ is 1 less than the Ramsey number of the line graphs of claw-free graphs.

4. EDGE-EXTREMAL PROBLEM ON CLAW-FREE GRAPHS

4.1. Some Graph Classes

In this first section of Chapter 4, we present three graph classes that will be frequently used in edge-extremal claw-free graphs. We briefly define these graph classes, state the number of edges and give examples of graphs that they contain. While the first two classes consist of well-known graphs of graph theory, the third one is newly defined by us.

$\mathcal{K}_p$: This is the class of complete graphs on p vertices. This class has only one element which is the complete graph on p vertices, K_p. There is nothing special about the construction of K_p since it is a graph in which every pair of distinct vertices is connected by an edge. We simply connect each of p vertices to every other $p-1$ vertices. K_p, the unique graph of this class, has $\frac{p(p-1)}{2}$ edges. $K_6 \in \mathcal{K}_6$ and $K_8 \in \mathcal{K}_8$ which have respectively 15 and 28 edges, are shown in Figure 4.1.

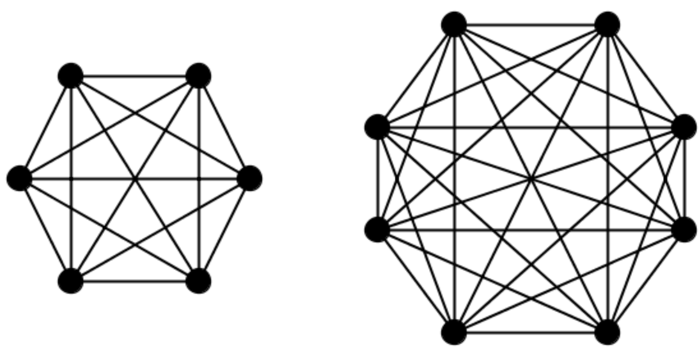


Figure 4.1. The complete graphs K_6 and K_8

$\mathcal{R}_{p,r}$: This is the class of graphs that are r-regular on p vertices. It is well known that the necessary and sufficient conditions for an r-regular graph of order p to exist are that $p \geq r+1$ and that the product $p \cdot r$ is even. The graphs from this class have $\frac{pr}{2}$ edges. Note that the class $\mathcal{R}_{p,p-1}$ is exactly the class $\mathcal{K}_p$. Therefore $\mathcal{K}_p$ is a subclass of

$\mathcal{R}_{p,r}$. Two non-isomorphic elements of $\mathcal{R}_{8,4}$ which have 16 edges are shown in Figure 4.2. They are not isomorphic because the graph on the left does not contain a claw as an induced subgraph while the one on the right does.

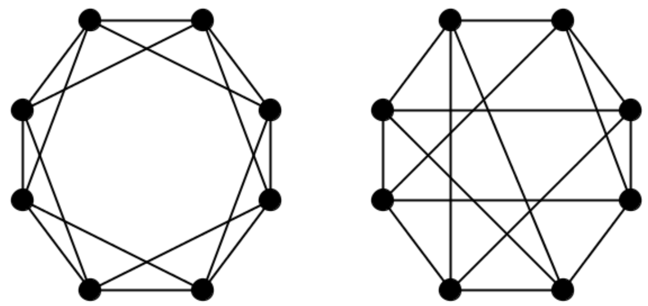


Figure 4.2. Two non-isomorphic elements of $\mathcal{R}_{8,4}$

$\mathcal{R}'_{p,r}$: This is the class of graphs that have p vertices where $p-1$ of them have degree r and one of them is of degree $r-1$ and where p and r are odd. The graphs from this class have $\frac{(p-1)r+(r-1)}{2}$ edges. The graphs of this class is almost regular since the difference of maximum and minimum degrees in these graphs is 1. Moreover, there is only one vertex of minimum degree. Two non-isomorphic elements of $\mathcal{R}'_{7,3}$ which have 10 edges are shown in Figure 4.3. They are not isomorphic because the graph on the left does not contain a claw as an induced subgraph while the one on the right does.

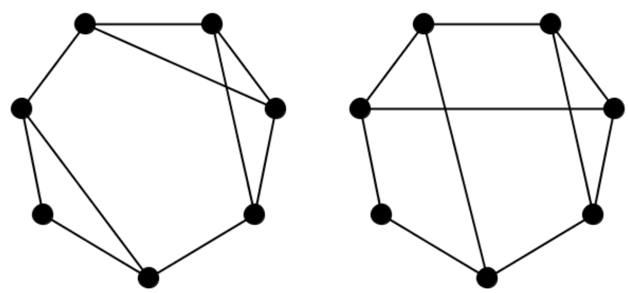


Figure 4.3. Two non-isomorphic elements of $\mathcal{R}'_{7,3}$

4.2. Structural Analysis of Edge-extremal Graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$

We previously made an introduction to the edge-extremal problem on general graphs in Section 3.3. In this section, we want to analyze the edge-extremal graphs of general graphs in a more structural way, rather than solely concentrating on their number of edges. In other words, we would like to show how one can obtain an edge-extremal graph of $\mathcal{M}_{\mathcal{GEN}}(i,j)$ for a given i and j.

We deduce from [1] that an edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$ can be obtained by r copies of i-stars and q copies of

$$\begin{cases} K_i & \text{if i is odd} \\ R'_{i+1,i-1} & \text{if i is even} \end{cases}$$

where q and r are respectively the quotient and the remainder of the division of $j-1$ by $\lceil \frac{i-1}{2} \rceil$. In other words, $j-1 = q\left\lceil \frac{i-1}{2} \right\rceil + r$, where q is as large as possible. Consequently,

$$q = \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor \quad \textit{and} \quad r = j-1-\left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor \left\lceil \frac{i-1}{2} \right\rceil$$

- If i is odd, the number of edges of an edge-extremal graph G is

$$\begin{aligned} |E(G)| &= (i-1)(j-1) + (\frac{i-1}{2}) \left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor \\ &= (i-1)(q\frac{i-1}{2} + r) + (\frac{i-1}{2})q \\ &= q\frac{i(i-1)}{2} + r(i-1) \end{aligned}$$

This can be obtained by r copies of i-stars and q copies of K_i because this graph has $\Delta = i-1 < i$ and $\nu = q(\frac{i-1}{2}) + r = j-1 < j$ (each i-star has $\nu = 1$ and each K_i has $\nu = \frac{i-1}{2}$).

- If i is even, the number of edges of an edge-extremal graph G is

$$\begin{aligned}|E(G)| &= (i-1)(j-1) + (\frac{i}{2}-1)\left\lfloor \frac{j-1}{\frac{i}{2}} \right\rfloor \\ &= (i-1)(q\frac{i}{2}+r) + (\frac{i-2}{2})q \\ &= q\frac{i(i-1)+i-2}{2} + r(i-1)\end{aligned}$$

This can be obtained by r copies of i-stars and q copies of $R'_{i+1,i-1}$ because this graph has $\Delta = i-1 < i$ and $\nu = q(\frac{i}{2}) + r = j-1 < j$ (each i-star has $\nu = 1$ and each $R'_{i+1,i-1}$ has $\nu = \frac{i}{2}$).

4.3. A Special Case

We previously stated that an edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$ can be obtained by the union of r copies of i-star and q copies of K_i if i is odd or q copies of $R'_{i+1,i-1}$ if i is even, where q and r are respectively the quotient and the remainder of the division of $j-1$ by $\lceil \frac{i-1}{2} \rceil$. As observed, i-stars play an important role in edge-extremal graphs in the class of general graphs. This motivated us to study the edge-extremal problem on graphs that do not contain i-stars, which are exactly the claw-free graphs, for $i \geq 3$.

Before going into the detailed analysis of claw-free edge-extremal graphs, we would like to mention the special case where the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ do not contain any i-star, i.e. where $r = 0$. The importance of this case is clear for us: the edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$ consists of q copies of K_i if i is odd or q copies of $R'_{i+1,i-1}$ if i is even. Accordingly, if both K_i and $R'_{i+1,i-1}$ are claw-free graphs, then from here we can conclude that the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ and in $\mathcal{M}_{\mathcal{CF}}(i,j)$ coincide when $r = 0$.

Note that r is the remainder of the division of $j-1$ by $\lceil \frac{i-1}{2} \rceil$. Hence, $r = 0$ indicates that $\lceil \frac{i-1}{2} \rceil$ divides $j-1$.

Lemma 4.1. *If $\lceil \frac{i-1}{2} \rceil$ divides $j-1$, then the unique edge-extremal graph G in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is the disjoint union of*

$$\begin{cases} \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \textit{ copies of } K_i & \textit{if } i \textit{ is odd} \\ \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \textit{ copies of } R'_{i+1,i-1} & \textit{if } i \textit{ is even} \end{cases}$$

and $|E(G)| = (i-1)(j-1) + \lfloor \frac{i-1}{2} \rfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil}$.

Proof. All we need to show is that the graphs K_i and $R'_{i+1,i-1}$ are claw-free. It is clear that the complete graph K_i is claw-free since an induced claw requires an independent set of vertices of size 3 whereas the independence number of K_i is 1.

Now, recall that in Section 3.3, we constructed $R'_{i+1,i-1}$ as follows: Remove a maximum matching from K_i, introduce a new vertex v and add an edge from v to any of the $i-1$ vertices in the modified graph. This simple construction yields a graph on $i+1$ vertices where each vertex has degree $i-1$, except one vertex which is of degree $i-2$. One can easily check and see that the independence number of $R'_{i+1,i-1}$, constructed as above, is 2. Therefore, it is claw-free.

For the uniqueness of G, we know from [1] that when $\lceil \frac{i-1}{2} \rceil$ divides $j-1$, there are no i-stars in any edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$. It is also said in [1] that any edge-extremal graph in this case has $\frac{j-1}{\lceil \frac{i-1}{2} \rceil}$ components with $\nu = \lceil \frac{i-1}{2} \rceil$ for each. Since G has no isolated vertices, it follows that G is the unique edge-extremal graph in this case. □

A summary of this special case with examples is as follows: When $\lceil \frac{i-1}{2} \rceil$ divides $j-1$, i.e. when $r = 0$, there are no i-stars in any edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$ [1]. This results in an edge-extremal graph which is also claw-free.

- i is odd

  The edge-extremal graph consists of q copies of K_i. This graph is claw-free. An example would be $i = 7$, $j = 10$. In this case, we have $r = 0$ and $q = 3$. The edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(7, 10)$ is the disjoint union of 3 K_7 and has 63 edges. This edge-extremal graph is shown in Figure 4.4.

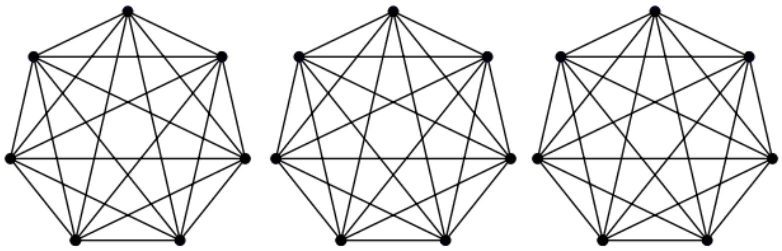


Figure 4.4. The graph 3 K_7

- i is even

  The edge-extremal graph consists of q copies of $R'_{i+1,i-1}$. This graph is claw-free. An example would be $i = 6$, $j = 13$. In this case, we have $r = 0$ and $q = 4$. The edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(6, 13)$ is the disjoint union of 4 copies of $R'_{7,5}$ and has 68 edges. This edge-extremal graph is shown in Figure 4.5.

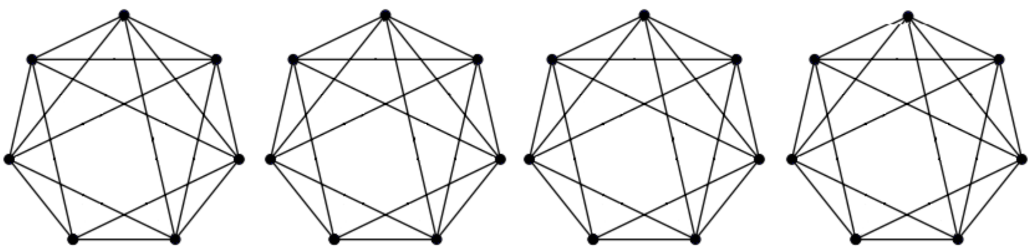


Figure 4.5. The graph 4 $R'_{7,5}$

4.4. Some Elementary Cases

On our way to the final result, we find it useful to have a discussion of some elementary cases of edge-extremal claw-free graphs. This kind of analysis will help us understand better where we may get stuck, as well as rule out some cases to analyze.

Elementary cases that we want to consider in this section are the cases where $i = 2$, $i = 3$, and $j = 2$.

Recall that the authors in [1] give an upper bound for the maximum number of edges of a graph $G \in \mathcal{M}_{\mathcal{GEN}}(i, j)$. According to their result, the number of edges in an edge-extremal graph G is given by

$$|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$$

- $i = 2$

The result for general graphs when $i = 2$ suggests that an edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i, j)$ has $|E(G)| = j - 1$ edges. This edge-extremal graph can simply be obtained by $j - 1$ independent copies of K_2, where each K_2 has matching number 1 (so in total the matching number is $j - 1 < j$) and each vertex has degree $1 < i = 2$. Notice that this graph is claw-free. Therefore the edge-extremal graph $G \in \mathcal{M}_{\mathcal{CF}}(2, j)$ is the disjoint union of $j - 1$ copies of K_2.

- $i = 3$

The result for general graphs when $i = 3$ suggests that an edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i, j)$ has $|E(G)| = 3(j - 1)$ edges. This edge-extremal graph can only be obtained by $j - 1$ independent copies of K_3, where each K_3 has matching number 1 (so in total the matching number is $j - 1 < j$) and each vertex has degree $2 < i = 3$. Notice that this graph is claw-free. Therefore the edge-extremal graph $G \in \mathcal{M}_{\mathcal{CF}}(3, j)$ is the disjoint union of $j - 1$ copies of K_3.

- $j = 2$

The result for general graphs when $j = 2$ and $i \geq 4$ suggests that the edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i, j)$ has $|E(G)| = i - 1$ edges, for both i odd and i even.

This edge-extremal graph can uniquely be obtained by an i-star, where the matching number is 1 and the maximum degree is $i-1$. However this graph is not claw-free! An i-star where $i \geq 4$ contains a claw as an induced subgraph.

We want to find an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$ when $j = 2$ and $i \geq 4$. Without any calculation, we observe that the claw-free graph with matching number at most 1 and that has the maximum number of edges is K_3. Even though the maximum degree is allowed to be more than 2, we realize that the matching number becomes a bottleneck and prevents the use of maximum degree at its maximum allowed level. Therefore, when $i \geq 4$, no matter how large i is, the only edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i,2)$ is K_3.

4.5. Construction of $R_{p,r}$: A Claw-free Regular Graph

We defined the graph class $\mathcal{R}_{p,r}$ in Section 4.1 as the class of graphs that are r-regular on p vertices. In this section, we will first give a construction for a subclass of $\mathcal{R}_{p,r}$, which consists of graphs that are r-regular on p vertices where r is even. For a given p and r, with r even, there may be several non-isomorphic graphs in $\mathcal{R}_{p,r}$. We will denote the particular graph obtained by our construction as $R_{p,r}$. Secondly, we will claim and prove that the construction that we suggest for a given p and r forms a claw-free graph, i.e. $R_{p,r}$ is claw-free.

We construct $R_{p,r}$ with $r = 2k$ where $k \in \mathbb{Z}^+$ as follows: Put all p vertices around a circle and connect each vertex by an edge to its k nearest neighbors on both side on the circle. A more formal description of this construction would be:

Graph: $R_{p,r}$ with $r = 2k$ where $k \in \mathbb{Z}^+$
Vertex set: $\{v_i : i \in \{1,2,3,....,p\}\}$
Edge set: Every v_i is connected to $v_{i-k}, v_{i-k+1}, ..., v_{i-1}, v_{i+1}, ..., v_{i+k-1}, v_{i+k}$ where indices are of modulo p.

Let us consider the following example shown in Figure 4.6 in order to understand the suggested construction better. It shows the graph $R_{11,4}$ that has 11 vertices, each of which is of degree 4. The vertices $v_1, v_2, ..., v_{11}$ are ordered in the clockwise direction and each vertex is connected to its 2 nearest neighbors on both side on the circle.

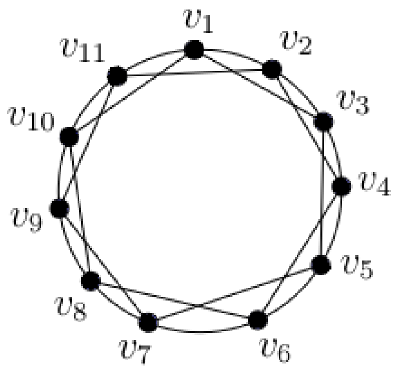


Figure 4.6. The graph $R_{11,4}$

Lemma 4.2. *$R_{p,r}$ is claw-free.*

Proof. By construction, we observe that the neighborhood of any vertex in $R_{p,r}$ can be partitioned into two cliques. The neighborhood of a vertex on each side, right and left, on the circle forms a clique of size k. By Observation 3.2, we conclude that no vertex of $R_{p,r}$ is a claw-center and therefore $R_{p,r}$ is claw-free. □

4.6. Construction of $R'_{p,r}$: A Claw-free Almost Regular Graph

The graphs from the class $\mathcal{R}'_{p,r}$ defined in Section 4.1 have the following features: They have p vertices; $p-1$ of them have degree r and one of them is of degree $r-1$ where p and r are odd. We said that the graphs of this class are almost regular since the difference of maximum and minimum degrees in these graphs is 1. Moreover, there is only one vertex of minimum degree.

In this section, we will first give a construction for a subclass of $\mathcal{R}'_{p,r}$ where we only put the additional condition $r+2 \leq p \leq 2r+1$. There may be several non-isomorphic graphs belonging to this subclass. We will denote the particular graph that is obtained by our construction as $R'_{p,r}$. Secondly, we will claim and prove that the construction that we suggest for a given p and r forms a claw-free graph, i.e. $R'_{p,r}$ is claw-free.

The construction of $R'_{p,r}$ is more complicated than the construction of $R_{p,r}$. In fact, in order to construct $R'_{p,r}$, we first construct $R_{p,r-1}$ and add some additional edges in a systematic way in order to reach the desired vertex degrees while keeping the property of being claw-free. Let us describe the construction as follows:

Graph: $R'_{p,r}$
Vertex set: $\{v_i : i \in \{1, 2, 3, ..., p\}\}$
Edge set: $E \cup E'$ where E is the edge set of the graph $R_{p,r-1}$ and E' is the edge set of size $\frac{p-1}{2}$ that we specify as follows:

Let the greatest common divisor of p and $r+1$ be m; we show it as $\gcd(p, r+1) = m$. For each ℓ, $1 \leq \ell \leq m$, we define the sequence $s_\ell = \{\{\ell + t(\frac{r+1}{2})\}\ (mod\, p)\}$ where $0 \leq t \leq \frac{p}{m} - 1$. That is to say, we generate the following sequences;

$$s_1 = \{1,\ \ (1 + (\tfrac{r+1}{2})),\ \ (1 + 2(\tfrac{r+1}{2})),\ \ (1 + 3(\tfrac{r+1}{2})), ..., (1 + (\tfrac{p}{m} - 1)(\tfrac{r+1}{2}))\}$$

$$s_2 = \{2,\ \ (2 + (\tfrac{r+1}{2})),\ \ (2 + 2(\tfrac{r+1}{2})),\ \ (2 + 3(\tfrac{r+1}{2})), ..., (2 + (\tfrac{p}{m} - 1)(\tfrac{r+1}{2}))\}$$

.

.

.

$$s_m = \{m,\ \ (m + (\tfrac{r+1}{2})),\ \ (m + 2(\tfrac{r+1}{2})),\ \ (m + 3(\tfrac{r+1}{2})), ..., (m + (\tfrac{p}{m} - 1)(\tfrac{r+1}{2}))\}$$

where the elements of these sequences are all in modulo p.

The important properties of these sequences are:

P1: There are in total p elements. Each sequence has $\frac{p}{m}$ elements because $0 \leq t \leq \frac{p}{m} - 1$. Since there are m sequences each having $\frac{p}{m}$ elements, there are in total p elements.

P2: The elements in a sequence are equivalent modulo m. This follows from the fact that $\frac{r+1}{2}$ is a multiple of m. We said that $\gcd(p, r+1) = m$. First of all, m is odd since p is odd. Say $r + 1 = mk$ for some k, then k must be even since $r + 1$ is even but m is odd. Put $k = 2a$, then $r + 1 = 2ma$, and $\frac{r+1}{2} = ma$. Therefore, $\frac{r+1}{2}$ is a multiple of m. Since in a sequence in order to get the next element, we only add $\frac{r+1}{2}$, and since $\frac{r+1}{2}$ is a multiple of m, we conclude that the elements in a sequence are equivalent modulo m.

P3: The sequences are disjoint, i.e. there is no common element of any two different sequences ($s_i \cap s_j = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, ..., m\}$). This is a simple observation that follows from P2. Seeing that elements in a sequence are equivalent modulo m, and also that each sequence s_ℓ starts with number ℓ where $1 \leq \ell \leq m$, we simply deduce that two elements from two different sequences cannot be the same because the first elements of the sequences are not equivalent modulo m.

Since each of the elements in the sequences are different and since there are in total p elements all written in modulo p, we conclude that the elements are exactly the numbers from 0 to $p - 1$. If we replace the number 0 with p, we get numbers from 1 to p. In fact, these numbers represent the indices of the vertices of the graph.

Also note that the order of the elements in the sequences is extremely important. This order will determine between which pairs of vertices we will add an edge.

Having said these properties, now it is time to specify the edges that will be added. We form a long sequence using the sequences s_ℓ for $1 \leq \ell \leq m$ by writing side by side the elements of these sequences as

$$\{\underbrace{1, ..., (1 + (\frac{p}{m} - 1)(\frac{r+1}{2}))}_{s_1}, \underbrace{.......... ....}_{s_2}, ..., \underbrace{m, ..., (m + (\frac{p}{m} - 1)(\frac{r+1}{2}))}_{s_m}\}$$

This is indeed a permutation of the numbers from 1 to p. These numbers are the indices of the vertices. We then add edges between two consecutive elements of this long sequence, i.e. we add an edge between first and second element, then another edge between third and fourth element, etc. There are p elements and p is odd. Therefore, the last edge that will be added is the one between $(p-2)^{th}$ and $(p-1)^{th}$ element and the p^{th} element will not be connected to any vertex in this sequence. In total, $\frac{p-1}{2}$ edges will be added and these edges are the elements of E'. Figure 4.7 shows how we add edges between the vertices with indices in the long sequence.

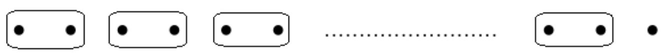


Figure 4.7. The edges of E'

Since each sequence s_ℓ has $\frac{p}{m}$ elements, which is an odd number, the last element of s_1 will be connected to the first element of s_2. Similarly, the last element of s_3 will be connected to the first element of s_4, etc. Figure 4.8 shows on the sequences s_ℓ, $1 \leq \ell \leq m$, the way we add edges between the vertices.

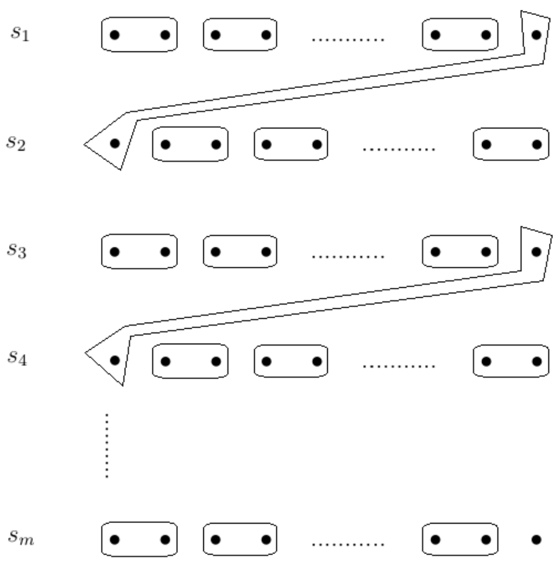


Figure 4.8. How do we add the edges of E'?

Before going any further, we would like to give an example in order to make the construction that is described above easier to understand. Suppose that we want to construct the graph $R'_{15,11}$. We first construct the graph $R_{15,10}$ as described in Section 4.5 and we call E the edge set of this graph. Figure 4.9 shows the graph $R_{15,10}$ where the vertices are ordered in the clockwise direction around the circle and each vertex is connected to its 5 nearest neighbors on both side on the circle.

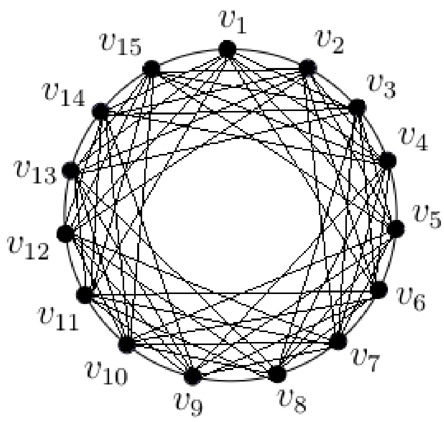


Figure 4.9. The graph $R_{15,10}$

The second step of the construction is to determine the edge set E'. We have $m = (p, r + 1) = (15, 12) = 3$. We form the sequences s_ℓ for $1 \leq \ell \leq 3$ as previously defined, and then the long sequence s.

$s_1 = \{1, 7, 13, 4, 10\}$
$s_2 = \{2, 8, 14, 5, 11\}$
$s_3 = \{3, 9, 15, 6, 12\}$

$$s = \{1, 7, 13, 4, 10, 2, 8, 14, 5, 11, 3, 9, 15, 6, 12\}$$

We add edges on the graph $R_{15,10}$ between the vertices $\{v_1, v_7\}$, $\{v_{13}, v_4\}$, $\{v_{10}, v_2\}$, $\{v_8, v_{14}\}$, $\{v_5, v_{11}\}$, $\{v_3, v_9\}$, and $\{v_{15}, v_6\}$. This is the edge set E' and Figure 4.10 shows the graph $R'_{15,11}$ where each vertex has degree 11, except one, v_{12}, which is of degree 10. The edges of E' are shown in bold.

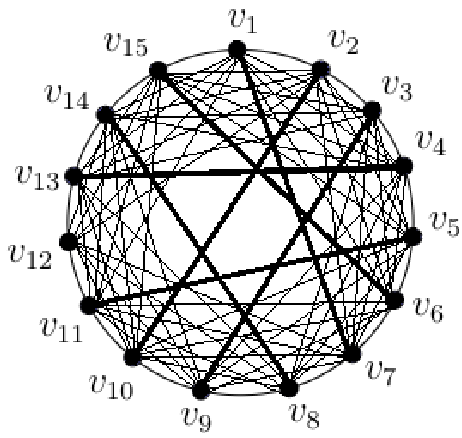


Figure 4.10. The graph $R'_{15,11}$

Another example with less edges in order to clearly see the degrees of the vertices would be the graph $R'_{11,5}$. After properly constructing the graph $R_{11,4}$, we add edges according to the sequence

$$s = s_1 = \{1, 4, 7, 10, 2, 5, 8, 11, 3, 6, 9\}$$

In this example there is only one sequence since $m = (p, r + 1) = (11, 6) = 1$. The graph $R'_{11,5}$ is shown in Figure 4.11, where the bold edges represent the edges of E'.

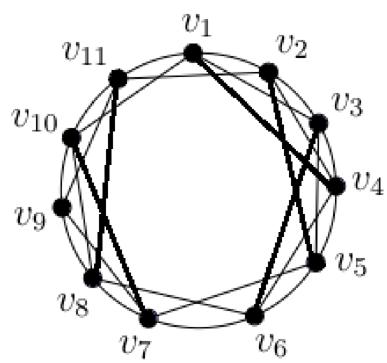


Figure 4.11. The graph $R'_{11,5}$

Claim 4.3. *E and E' are disjoint edge sets.*

Proof. E is the edge set of the graph $R_{p,r-1}$ where every vertex is connected to its nearest $\frac{r-1}{2}$ neighbors on both side on the circle whereas E' is comprised of the edges that are between the vertices of distance either $\frac{r+1}{2}$ (if added between two vertices with indices in the same sequence) or $\frac{r+3}{2}$ (if added between two vertices with indices from two consecutive sequences). $\square$

By the edges in E, the vertices have degree $r-1$. When we also add the edges of E', each vertex becomes of degree r, except the one that is not connected to any vertex in the long sequence and it is of degree $r-1$.

There may be several other and possibly easier methods of constructing a graph with p vertices where $p-1$ of them have degree r and one of them is of degree $r-1$ and where p and r are odd. However, our construction is special in the way that it generates claw-free graphs.

Lemma 4.4. *$R'_{p,r}$ is claw-free.*

Proof. While constructing the graph $R'_{p,r}$, we first construct the graph $R_{p,r-1}$. In the latter one, all vertices around the circle, ordered in the clockwise direction, are similar; they are all connected to their nearest $\frac{r-1}{2}$ neighbors on both side, right and left, on the circle.

However, the addition of the edge set E' creates two different types of vertices since there are two types of edges in E': one is the edge added between two elements of the same sequence, and the other is the edge added between two elements of two consecutive sequences. Accordingly, Type 1 vertices are the ones whose both right and left neighbors form a clique, and Type 2 vertices are the ones that are connected to their nearest $(\frac{r+3}{2})^{th}$ neighbor instead of $(\frac{r+1}{2})^{th}$. The sets of Type 1 and Type 2 vertices are respectively denoted as U_1 and U_2 and we have $V(R'_{p,r}) = U_1 \cup U_2$. Note that U_2 can possibly be an empty set. Figure 4.12 and Figure 4.13 shows how Type 1 and Type 2 vertices come up with addition of E' to the graph $R_{p,r-1}$. The edges of E' are shown in bold.

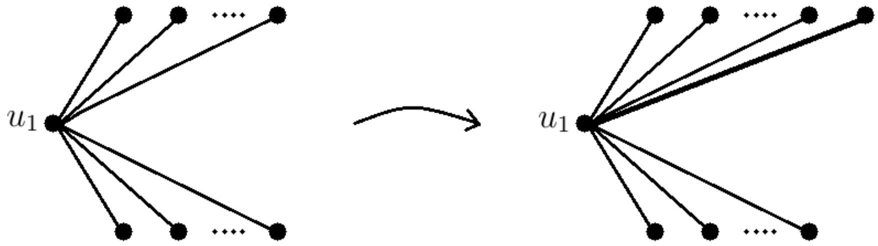


Figure 4.12. Type 1 vertex u_1 in $R'_{p,r}$

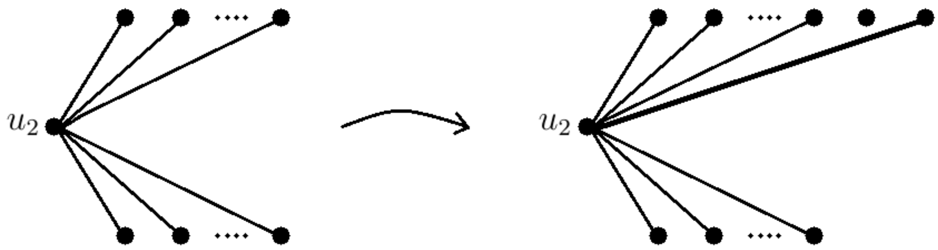


Figure 4.13. Type 2 vertex u_2 in $R'_{p,r}$

The vertices in U_1 are not claw-centers because the neighborhood of any vertex in U_1 can be partitioned into two cliques. The neighborhood of a vertex on each side, right and left, on the circle forms cliques of sizes $\frac{r+1}{2}$ or $\frac{r-1}{2}$. By Observation 3.2, we conclude that no vertex of U_1 is a claw-center.

As a matter of fact, the vertices in U_2 are not claw-centers either. In Figure 4.14, where the indices of the vertices should be considered in modulo p, v_t is a vertex of the set U_2. The vertices v_t and $v_{t+\frac{r+1}{2}}$ have indices in the same sequence and v_t is connected to $v_{t+\frac{r+3}{2}}$ whose index belongs to the next sequence. Therefore $v_{t+\frac{r+3}{2}}$ is also a vertex of U_2. Furthermore, the vertices v_{t+1}, $v_{t+\frac{r+3}{2}}$, and $v_{t-\frac{r-1}{2}}$ have all indices in the same sequence. Now, the important observation is that each sequence has at most 1 vertex connected to a vertex from another sequence. Hence, v_{t+1} is a vertex of U_1 since $v_{t+\frac{r+3}{2}} \in U_2$. As an element of U_1, v_{t+1} has 2 options for being connected, $v_{t+\frac{r+3}{2}}$

and $v_{t-\frac{r-1}{2}}$. Since $v_{t+\frac{r+3}{2}}$ is connected to v_t, completing its degree to r, we conclude that v_{t+1} is connected to $v_{t-\frac{r-1}{2}}$.

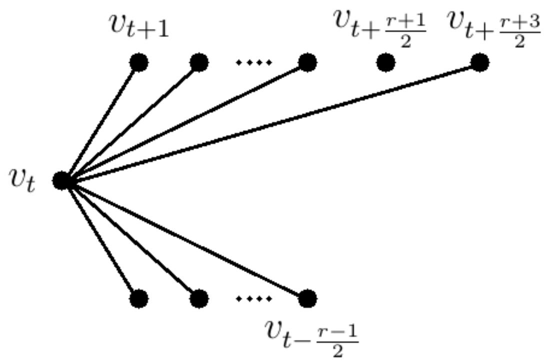


Figure 4.14. Type 2 vertices in $R'_{p,r}$ are not claw-centers

The aim of this brief analysis is to be able to show that the neighborhood of v_t can be partitioned into two cliques. Now, since v_{t+1} is connected to $v_{t-\frac{r-1}{2}}$, the right neighbors of v_t with v_{t+1} form a clique of size $\frac{r+1}{2}$ and its other neighbors form a clique of size $\frac{r-1}{2}$. By Observation 3.2, we conclude that no vertex of U_2 is a claw-center.

There is actually a Type 3 vertex which is connected to no vertices in the long sequence. There is only one such vertex in each $R'_{p,r}$. It is of degree $r-1$ and its neighborhood on each side, right and left, on the circle forms a clique of size $\frac{r-1}{2}$. By Observation 3.2, we conclude that it is not a claw-center.

Since there is no claw-center, the graph $R'_{p,r}$ is claw-free. $\square$

Back to our previous examples, for the graph $R'_{15,11}$ we had $s_1 = \{1, 7, 13, 4, 10\}$, $s_2 = \{2, 8, 14, 5, 11\}$, and $s_3 = \{3, 9, 15, 6, 12\}$. Therefore

$$U_1 = \{v_1, v_7, v_{13}, v_4, v_8, v_{14}, v_5, v_{11}, v_3, v_9, v_{15}, v_6, v_{12}\}, \quad U_2 = \{v_{10}, v_2\}$$

and for the graph $R'_{11,5}$, we have

$$U_1 = \{v_1, v_4, v_7, v_{10}, v_2, v_5, v_8, v_{11}, v_3, v_6, v_9\}, \quad U_2 = \emptyset$$

One can easily check that the neighborhood of every vertex in these graphs can be partitioned into two cliques in the way shown above.

4.7. The Main Result

In this section, we solve the edge-extremal problem on claw-free graphs. The edge-extremal problem simply asks for the *number* of edges of an edge-extremal graph for a given i and j on the given graph class. How edge-extremal graphs themselves look like is not a concern of this problem initially. In this thesis we actually describe the edge-extremal claw-free graphs themselves, in addition to giving their number of edges.

Before we proceed to the main result, we prove the following lemma since it will be used in the proof of the main theorem. Although it is a well-known fact, we prove it here for the sake of completeness.

Lemma 4.5. *If $a_i > 0$ for $1 \leq i \leq n$ and $n \geq 2$, then $\sum_{i=1}^{n} a_i^2 < (\sum_{i=1}^{n} a_i)^2$.*

Proof. In order to prove this statement, we will use induction on n. The statement is true for $n = 2$: $(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2 > a_1^2 + a_2^2$ since $a_1, a_2 > 0$.

Now for the inductive step assume that $(a_1^2 + a_2^2 + ... + a_n^2) < (a_1 + a_2 + ... + a_n)^2$

is true. Then,

$$\begin{aligned}
a_1^2 + a_2^2 + ... + a_n^2 + a_{n+1}^2 &= (a_1^2 + a_2^2 + ... + a_n^2) + a_{n+1}^2 \\
&< (a_1 + a_2 + ... + a_n)^2 + a_{n+1}^2 \qquad \textit{(using the assumption)} \\
&= y^2 + a_{n+1}^2 \qquad \textit{(rewriting } y = a_1 + a_2 + ... + a_n\textit{)} \\
&< (y + a_{n+1})^2 \qquad \textit{(using } a_1^2 + a_2^2 < (a_1 + a_2)^2\textit{)} \\
&= (a_1 + a_2 + ... + a_n + a_{n+1})^2 \qquad \textit{(plug back for } y\textit{)}
\end{aligned}$$

We showed that $(a_1^2 + a_2^2 + ... + a_n^2) < (a_1 + a_2 + ... + a_n)^2$ implies

$$(a_1^2 + a_2^2 + ... + a_n^2 + a_{n+1}^2) < (a_1 + a_2 + ... + a_n + a_{n+1})^2$$

This concludes the induction. □

Our main result in this thesis is the following:

Theorem 4.6. *(i) If $i \geq 2j$, then the edge-extremal graph G in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is unique and $G \simeq K_{2j-1}$ with $|E(G)| = (2j-1)(j-1)$.*

(ii) If $i < 2j$, then an edge-extremal graph G in $\mathcal{M}_{\mathcal{CF}}(i,j)$ can be obtained by

$$\begin{cases} q-1 \text{ copies of } K_i \text{ and one } R_{i+2r,i-1} & \text{if } i \text{ is odd} \\ q-1 \text{ copies of } R'_{i+1,i-1} \text{ and one } R'_{i+2r+1,i-1} & \text{if } i \text{ is even} \end{cases}$$

where q and r are respectively the quotient and the remainder of the division of $j-1$ by $\lceil \frac{i-1}{2} \rceil$, and in this case $|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$.

Proof. (i) $i \geq 2j$

We argue that the edge-extremal graph $G \in \mathcal{M}_{\mathcal{CF}}(i,j)$ is K_{2j-1}. In order to prove this statement, we have to show two things:

- $K_{2j-1} \in \mathcal{M}_{\mathcal{CF}}(i,j)$

  The matching number of K_{2j-1} is $j-1<j$, the maximum degree in K_{2j-1} is $2j-2 \le i-2 < i$, and K_{2j-1} is a claw-free graph since it is a complete graph. Therefore $K_{2j-1} \in \mathcal{M}_{\mathcal{CF}}(i,j)$.

- K_{2j-1} is the edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$

  When $i \ge 2j$, if the edge-extremal graph G in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is connected, then by Lemma 3.5, the maximum number of vertices that G can have is $2j-1$. Also because $2j \le i$, we have $2j-1 \le i-1$. Hence each vertex can have degree $2j-2$, without violating the maximum degree bound. This yields K_{2j-1}.

  Now assume that, for $i \ge 2j$, the edge-extremal graph G in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is disconnected having $k \ge 2$ connected components $G_1, G_2, \ldots.., G_k$. Let the matching number of G_ℓ be j_ℓ for $1 \le \ell \le k$. We know that $j_\ell \ge 1$ and $\sum_{\ell=1}^{k} j_\ell = j-1$. We also know by Lemma 3.5 that each G_ℓ has at most $2j_\ell+1$ vertices, i.e. each G_ℓ has at most $\frac{(2j_\ell+1)2j_\ell}{2} = (2j_\ell+1)j_\ell$ edges. Then the maximum number of edges of G:

$$
\begin{aligned}
\sum_{\ell=1}^{k}(2j_\ell+1)j_\ell &= \sum_{\ell=1}^{k} j_\ell + 2\sum_{\ell=1}^{k} j_\ell^2 \\
&< \sum_{\ell=1}^{k} j_\ell + 2\Big(\sum_{\ell=1}^{k} j_\ell\Big)^2 \qquad (\textit{due to Lemma } 4.5) \\
&= j-1+2(j-1)^2 \qquad (\textit{since } \sum_{\ell=1}^{k} j_\ell = j-1) \\
&= \frac{(2j-1)(2j-2)}{2} \\
&= \textit{number of edges of } K_{2j-1}
\end{aligned}
$$

  This means that the maximum number of edges that the disconnected G can have is less than the number of edges of K_{2j-1}. This concludes the proof that when $i \ge 2j$, the unique edge-extremal graph G in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is K_{2j-1}

and $|E(G)| = (2j-1)(j-1)$.

(ii) $i < 2j$

First of all, if $i < 2j$, then $\lceil \frac{i-1}{2} \rceil \leq j-1$. When we divide $j-1$ by $\lceil \frac{i-1}{2} \rceil$, the quotient will be greater than or equal to 1, i.e. $q \geq 1$. It is important to note the fact $q \geq 1$ in order to make the term '$q-1$ copies of K_i' or '$q-1$ copies of $R'_{i+1,i-1}$' meaningful.

Now, we again have to show two things:

- The suggested graphs are in $\mathcal{M}_{\mathcal{CF}}(i,j)$

  Showing that they are in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is equivalent to show that they are in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ and that they are claw-free graphs.

  The suggested graphs are

$$\begin{cases} q-1 \ copies \ of \ K_i \ and \ one \ R_{i+2r,i-1} & \text{if i is odd} \\ q-1 \ copies \ of \ R'_{i+1,i-1} \ and \ one \ R'_{i+2r+1,i-1} & \text{if i is even} \end{cases}$$

  In the case where i is odd, K_i is a complete graph on i vertices with each vertex having degree $i-1$ and matching number $\frac{i-1}{2}$. $R_{i+2r,i-1}$ is an $(i-1)$-regular graph on $i+2r$ vertices with each vertex having degree $i-1$ and matching number $\frac{i+2r-1}{2}$. We have $\Delta = i-1 < i$ and $\nu = (q-1)(\frac{i-1}{2}) + \frac{i+2r-1}{2} = q(\frac{i-1}{2}) + r = j-1 < j$. Therefore the graph suggested for the case i odd is in $\mathcal{M}_{\mathcal{GEN}}(i,j)$. Also, it is claw-free because it is the union of the copies of a complete graph, which is claw-free, and the regular graph $R_{i+2r,i-1}$, which is shown to be claw-free in Lemma 4.2. We conclude that the suggested graph is in $\mathcal{M}_{\mathcal{CF}}(i,j)$.

  In the case where i is even, the graph $R'_{i+1,i-1}$ has $\Delta = i-1$ and $\nu = \frac{i}{2}$. The graph $R'_{i+2r+1,i-1}$ has $\Delta = i-1$ and $\nu = \frac{i+2r}{2}$. Therefore $q-1$ copies of $R'_{i+1,i-1}$ and one $R'_{i+2r+1,i-1}$ has $\Delta = i-1 < i$ and $\nu = (q-1)\frac{i}{2} + \frac{i+2r}{2} = q\frac{i}{2} + r = j-1 < j$. Hence it is in $\mathcal{M}_{\mathcal{GEN}}(i,j)$. Furthermore, it is claw-free because in Lemma 4.4, we showed that the graph $R'_{p,r}$

is claw-free for p and r odd and where the condition $r+2 \leq p \leq 2r+1$ is satisfied, which are true for both $R'_{i+1,i-1}$ and $R'_{i+2r+1,i-1}$ since in the second graph, we have $r < \frac{i}{2}$ due to the division rule.

- The suggested graphs are edge-extremal in $\mathcal{M}_{\mathcal{CF}}(i,j)$

  We will show that the suggested graphs have the same number of edges as the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$. Remember that the number of edges in an edge-extremal $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$ is given by

$$|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$$

  In the case where i is odd, the number of edges in an edge-extremal $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$ is

$$|E(G)| = (i-1)(j-1) + (\frac{i-1}{2}) \left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor$$

  The graph that we suggest as an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$ for i odd is $q-1$ copies of K_i and one $R_{i+2r,i-1}$. The number of edges of this graph is

$$\begin{aligned}
\frac{(i+2r)(i-1)}{2} + (q-1)\frac{i(i-1)}{2} &= r(i-1) + \frac{i(i-1)}{2} + (q-1)\frac{i(i-1)}{2} \\
&= r(i-1) + q\frac{i(i-1)}{2} \\
&= r(i-1) + q\frac{(i-1+1)(i-1)}{2} \\
&= r(i-1) + q(i-1)(\frac{i-1}{2}) + q(\frac{i-1}{2}) \\
&= (i-1)\left[q(\frac{i-1}{2}) + r\right] + q(\frac{i-1}{2}) \\
&= (i-1)(j-1) + (\frac{i-1}{2}) \left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor
\end{aligned}$$

  where the last equation holds as $q(\frac{i-1}{2}) + r = j-1$ and $q = \left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor$ for i odd.

In the case where i is even, the number of edges in an edge-extremal $G \in \mathcal{M}_{\mathcal{GEN}}(i,j)$ is

$$|E(G)| = (i-1)(j-1) + (\frac{i}{2} - 1)\left\lfloor \frac{j-1}{\frac{i}{2}} \right\rfloor$$

The graph that we suggest as an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$ for i even is $q-1$ copies of $R'_{i+1,i-1}$ and one $R'_{i+2r+1,i-1}$. The number of edges of this graph is

$$\begin{aligned}
&\frac{(i+2r)(i-1)+(i-2)}{2} + (q-1)\frac{i(i-1)+(i-2)}{2} \\
= \quad & (q-1)\frac{i(i-1)}{2} + \frac{(i+2r)(i-1)}{2} + q(\frac{i-2}{2}) \\
= \quad & (i-1)\left[(q-1)\frac{i}{2} + \frac{i+2r}{2}\right] + q(\frac{i-2}{2}) \\
= \quad & (i-1)(q\frac{i}{2} + r) + q(\frac{i-2}{2}) \\
= \quad & (i-1)(j-1) + (\frac{i}{2} - 1)\left\lfloor \frac{j-1}{\frac{i}{2}} \right\rfloor
\end{aligned}$$

where the last equation holds as $q(\frac{i}{2}) + r = j-1$ and $q = \left\lfloor \frac{j-1}{\frac{i}{2}} \right\rfloor$ for i even. We here showed that the graphs that we suggest as edge-extremal graphs in $\mathcal{M}_{\mathcal{CF}}(i,j)$ have the same number of edges as edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$. This is equivalent to say that they reach the upper bound. Therefore the suggested graphs are edge-extremal in $\mathcal{M}_{\mathcal{CF}}(i,j)$.

□

There is also a corollary to this theorem in order to underline the importance of claw-free condition.

Corollary 4.7. *If $i \geq 2j$, the edge difference between the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i,j)$ and in $\mathcal{M}_{\mathcal{CF}}(i,j)$ is $(j-1)(i-2j)$.*

Proof. If $i \geq 2j$, then $\lceil \frac{i-1}{2} \rceil > j - 1$. This gives $q = 0$ and $r = j - 1$.

When $i \geq 2j$, the edge-extremal graph $G \in \mathcal{M}_{\mathcal{GEN}}(i, j)$ has $|E(G)| = (i-1)(j-1)$ edges. This can be obtained by $j - 1$ copies of i-star. We also showed that the edge-extremal graph $G \in \mathcal{M}_{\mathcal{CF}}(i, j)$ is K_{2j-1}.

Then the edge difference between the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i, j)$ and in $\mathcal{M}_{\mathcal{CF}}(i, j)$ is

$$(i-1)(j-1) - \frac{(2j-1)(2j-2)}{2} = (i-1)(j-1) - (2j-1)(j-1) = (j-1)(i-2j)$$

$\square$

Observe that if $i = 2j$, the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i, j)$ and in $\mathcal{M}_{\mathcal{CF}}(i, j)$ have the same number of edges. However, as the difference $i - 2j$ gets larger, the edge difference between the edge-extremal graphs in $\mathcal{M}_{\mathcal{GEN}}(i, j)$ and in $\mathcal{M}_{\mathcal{CF}}(i, j)$ gets larger. This, in a sense, shows how i-stars play a central role in edge-extremal graphs for the class of general graphs.

4.8. Examples

In this section, we give examples of edge-extremal claw-free graphs for each possible case. There are 3 possible cases based on our main result:

- $i \geq 2j$

  Consider an edge-extremal instance G from the graph family $\mathcal{M}_{\mathcal{CF}}(12, 5)$. Since $i \geq 2j$, G is unique and $G \simeq K_9$. It is shown in Figure 4.15 and

$$|E(G)| = (2j-1)(j-1) = (2 \cdot 5 - 1)(5 - 1) = 36$$

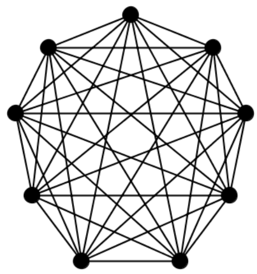


Figure 4.15. An edge-extremal instance in $\mathcal{M}_{\mathcal{CF}}(12, 5)$

- $i < 2j$ and i is odd

  Consider the graph family $\mathcal{M}_{\mathcal{CF}}(7, 9)$. We have $i = 7$, $j = 9$, $q = 2$, and $r = 2$. Since $i < 2j$ and i is odd, an edge-extremal instance G is $K_7 \cup R_{11,6}$ where the graph $R_{11,6}$ will be constructed in the way we described in Section 4.5. The graph G is shown in Figure 4.16 and

$$|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor = (7-1)(9-1) + \left\lfloor \frac{7-1}{2} \right\rfloor \left\lfloor \frac{9-1}{\lceil \frac{7-1}{2} \rceil} \right\rfloor = 54$$

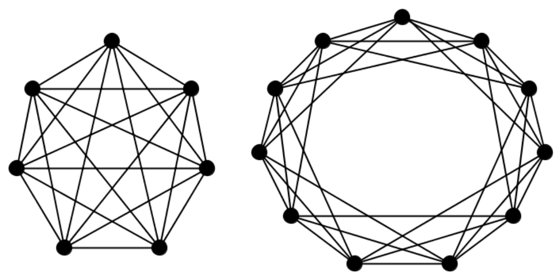


Figure 4.16. An edge-extremal instance in $\mathcal{M}_{\mathcal{CF}}(7, 9)$

- $i < 2j$ and i is even

  Consider the graph family $\mathcal{M}_{\mathcal{CF}}(6, 11)$. We have $i = 6$, $j = 11$, $q = 3$, and $r = 1$. Since $i < 2j$ and i is even, an edge-extremal instance G is $(2\, R'_{7,5}) \cup R'_{9,5}$ where the graphs $R'_{7,5}$ and $R'_{9,5}$ will be constructed in the way we described in Section 4.6. The graph G is shown in Figure 4.17 and

$$|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor = (6-1)(11-1) + \left\lfloor \frac{6-1}{2} \right\rfloor \left\lfloor \frac{11-1}{\lceil \frac{6-1}{2} \rceil} \right\rfloor = 56$$

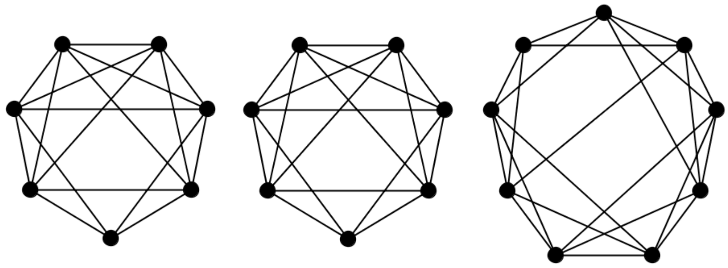


Figure 4.17. An edge-extremal instance in $\mathcal{M}_{\mathcal{CF}}(6, 11)$

5. CONCLUSION

In this chapter, we will point out the results of the thesis, give some general remarks about the work done and state some open questions for which finding answers would be valuable for future work.

5.1. Summary

In this thesis, we have looked at edge-extremal graphs with bounded degree and matching number on the class of claw-free graphs. In particular, we let $\mathcal{CF}$ be the class of claw-free graphs and $\mathcal{M}_{\mathcal{CF}}(i,j)$ all graphs G (with no isolated vertex) from $\mathcal{CF}$ satisfying $\Delta(G) < i$ and $\nu(G) < j$. We tried to find an answer for: What is the maximum number of edges a graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$ can achieve, for a given i and j. This is equivalent to asking for the Ramsey number of line graphs of claw-free graphs. We managed to solve this problem and achieved our primary purpose.

We explained the solution of the edge-extremal problem on general graphs $\mathcal{GEN}$, which is due to Balachandran and Khare [1]. This solution has been used in this thesis as an upper bound for the number of edges of an edge-extremal claw-free graph because every claw-free graph is also a member of $\mathcal{GEN}$. The edge-extremal graphs contain claws in most cases; we tried to find claw-free constructions with the same number of edges as the general case. It was possible for the case $i \leq 2j$ while it was not when $i > 2j$.

We developed a claw-free construction for r-regular graphs $R_{p,r}$ when r is even. As a by product, we actually created a claw-free construction for odd r too. However, we did not present it here since it would not serve our purpose: When r is odd, the number of vertices p must be even in order $R_{p,r}$ to exist, but the connected components of edge-extremal graphs have all odd number of vertices since increasing the number of vertices by 1 to make it odd does not increase the matching number while increasing the number of edges.

We developed a claw-free construction of the graph $R'_{p,r}$, where p and r odd. This graph is not, and cannot be, r-regular since p is odd. However, it is the closest graph to an r-regular graph on p vertices because each vertex has degree r, except one which is of degree $r-1$. We proposed a procedure to construct claw-free $R'_{p,r}$ for each given p and r, where p and r are odd and where $r+2 \leq p \leq 2r+1$ holds.

Using the above constructions, we were able to provide an edge-extremal claw-free graph for each possible case. For a given i and j, we find the maximum number of edges a graph in $\mathcal{M}_{\mathcal{CF}}(i,j)$ can achieve by constructing an edge-extremal claw-free graph.

5.2. Final Comments

Graphs are a mathematical way of representing connections or relationships between objects. They are very useful in modelling and solving real world problems since they are used in designing, representing and planning the use of networks. Graphs are also used to represent various problems in coding, telecommunications and parallel programming.

Study of graph problems on particular graph classes, regardless of their hardness in the general case, forms an important area of graph theory. In fact, investigating graph problems on particular graph classes often leads, for instance, to the discovery of new and very useful structural properties and characterizations of the graph class in question. These new approaches were then used to produce new and more efficient algorithms for specific practical problems related to this graph class.

In this thesis, we solved the edge-extremal problem on claw-free graphs even though its solution on general graphs was already known. We believed that narrowing the graph class would result in a decrease in the number of edges of an edge-extremal graph, and we were right. Forbidding the claw graph significantly reduced the number of edges for most of the cases. On our way to the main result, we have built claw-free constructions of some graphs.

During the attempts at proving claims that we made, most of the time, we have found counter-examples to our conjectures. Each time, we tried to understand the points that we missed and made new claims. One of our most important assets was that we had an upper bound for the maximum number of edges for the general case. Thus, if we could find a claw-free graph whose number of edges reaches that bound, we could immediately conclude that it is edge-extremal. This point of view led us to search for graphs in $\mathcal{M}_{\mathcal{CF}}(i,j)$ having that number of edges. This also meant proving that they are claw-free, which required to develop some construction methods. Moreover, it was not possible for some cases to reach that upper bound. In those cases, we tried to understand the reason behind this lack of edges, and once found, the proof has come by itself.

We felt incredibly satisfied while bunching all the subcases together and ultimately finding our main result. We are hoping to have achieved our secondary goal, which is to draw the reader's interest and to provide interesting insight on the subject.

5.3. Open Problems

This study on the thesis has revealed some interesting questions and arises several interesting future research directions:

- In this thesis, we have looked for a solution for the number of edges of an edge-extremal graph in claw-free graphs family. We have forbidden the edge-extremal graphs to contain claw-graph as an induced subgraph. The claw-graph, i.e. 4-star, is the smallest nontrivial star graph (3-star is P_3 and 2-star is P_2). It would be interesting to investigate larger star graphs and to find a formula in terms of the size of the star graph. Can we find an explicit formula in terms of i, j, and k, for the maximum number of edges that a graph G can have where $\Delta(G) < i$, $\nu(G) < j$ and where G does not contain k-star as an induced subgraph?

- We have forbidden the claw-graph because the result for general graphs suggests that an edge-extremal graph contains star graphs in most cases and the claw-graph is the smallest nontrivial star graph. The result for general graphs also suggests that in most cases an edge-extremal graphs contains cliques K_i or nearly-clique components $R'_{i+1,i-1}$. The smallest nontrivial clique is K_3, which is the triangle graph. How are the edge-extremal instances for general graphs affected if we do not allow triangles, instead of claws? Solving the edge-extremal problem in triangle-free graphs seems to be at least as interesting as in claw-free graphs. Currently, we are working on this problem.

- What if we forbid k-clique? Can we find an explicit formula in terms of i, j, and k, for the maximum number of edges that a graph G can have where $\Delta(G) < i$, $\nu(G) < j$ and where G does not contain k-clique as an induced subgraph, that is $\omega(G) < k$?

- Another interesting study would be to solve the edge-extremal problem on chordal graphs. Finding a solution to this problem was previously attempted in [2]. However, despite significant efforts put on this problem, the authors state that they could not solve it within the limited time frame of a master thesis.

- The problem is open also for interval graphs, which is a subclass of chordal graphs. However, we think that solving the edge-extremal problem in interval graphs is not easier than solving it in chordal graphs. Also, an interval graph is a unit interval graph if and only if it is claw-free [17] and the edge-extremal problem has been solved for unit interval graphs in [2]. When we combine their results with ours, we see that putting the interval condition on claw-free graphs significantly reduces the number of edges of an edge-extremal graph. For instance, for an edge-extremal instance G from the graph family $\mathcal{M}_{\mathcal{CF}}(10, 7)$ the number of edges is $|E(G)| = 58$, whereas for an edge-extremal instance G from the graph family $\mathcal{M}_{\mathcal{UNIT}}(10, 7)$ the number of edges is $|E(G)| = 48$.

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