QUANTIFYING THE RISK OF PORTFOLIOS CONTAINING STOCKS AND COMMODITIES

by

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ABSTRACT

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In this study, we used copula method in order to model the multivariate return distributions of stock portfolios and, in this manner, to implement this model for risk measure evaluations in practice. Copulas are used to describe the dependence between random variables thus, are enjoyed to model the marginals separately and to represent the dependence structure between them. We also modeled the multivariate return distributions of stock portfolios diversified with commodities, precious metals, crude oil etc. and fitted a set of copulas to the joint return data. With this aim, we selected 20 stocks from New York Stock Exchange, gold and crude oil and constructed stock portfolios, stock portfolios with gold, stock portfolios with crude oil and stock portfolios with gold and crude oil in order to analyze whether the copula method fits the multivariate return distributions of selected portfolios. In order to check the validity of the models, we implemented daily and weekly back testing using 20 different α values. We found that t distribution and generalized hyperbolic distributions are very nice models for modeling individual financial instruments returns and the t copula is the best copula to represent the dependence structure between financial instruments returns. We used this model to calculate the risks of portfolios and observed that adding gold decreases the risk of portfolios where crude oil behaves like an ordinary stock.

ÖZET

HİSSE SENEDİ VE EMTİA İÇEREN PORTFÖYLERİN RİSK ÖLÇÜMÜ

Bu çalışmada, hisse senedi portföylerinin çok değişkenli dağılımlarını modellemek ve böylelikle pratikte risk hesaplamaları yapabilmek için kopulaları kullandık. Kopulalar rassal değişkenler arasındaki bağımlılık yapısını tanımlayabilmek için kullanılır, böylece bileşenlerin ayrı olarak modellenmesinde ve aralarındaki bağımlılık yapısının temsil edilmesinde yararlanılır. Aynı zamanda emtia, kıymetli metaller, ham petrol vs. ile çeşitlendirilmiş hisse senedi portföylerinin çok değişkenli dağılımlarını modelledik ve birleşik getiri verisine bir kopulalar kümesini oturttuk. New York Hisse Senedi Borsası'ndan 20 adet hisse senedi, altın ve ham petrol seçtik ve kopula metodunun seçilmiş portföylerin çok değişkenli dağılımlarına uygun olup olmadığını analiz etmek için hisse senedi portföyü, altın ile hisse senedi portföyü, ham petrol ile hisse senedi portföyü ve altın ve ham petrol ile hisse senedi portföyü oluşturduk. Modellerin uygunluğunu kontrol etmek için 20 değişik α değeri kullanılarak günlük ve haftalık geriye dönük test uyguladık. t ve genelleştirilmiş hiperbolik dağılımların tek finansal araç getirilerini modellemek için çok cazip modeller olduğunu ve t kopulanın finansal araç getirilerinin aralarındaki bağımlılık yapısın en iyi temsil eden kopula olduğunu gördük. Bu modeli portföylerin risklerini hesaplamak için kullandık ve ham petrol olağan bir hisse senedi gibi davranırken, altının portföy riskini azalttığını gözlemledik.

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LIST OF SYMBOLS

I	Identity matrix
l	Value at Risk
M	Comonotonicity copula
R	Daily log returns
S	Value of portfolio
w_i	Fraction of the total value of the portfolio invested into financial instrument i
W	Countermonotonicity copula
α	Confidence level
Δ	Time horizon
λ	Coefficient of tail dependence
μ	Mean
Π	Independence copula
ρ	Correlation matrix
σ	Standard deviation
Σ	Covariance matrix

LIST OF ACRONYMS/ABBREVIATIONS

AIC	Akaike's Information Criterion
C	Copula Function
CDF	Cumulative Distribution Function
CLT	Central Limit Theorem
CVaR	Conditional Value at Risk
DF	Distribution Function
EIA	Energy Information Administration
ES	Expected Shortfall
GHD	Generalized Hyperbolic Distribution
IFM	Inference Functions for Margins
IID	Independent and Identically Distributed
LR	Likelihood Ratio
MLE	Maximum Likelihood Estimation
NIG	Normal Inverse Gaussian Distribution
NYSE	New York Stock Exchange
POF	Proportion of Failures
Q-Q	Quantile-Quantile Plots
RVS	Random Variables
TUFF	Time Until First Failure
VaR	Value at Risk
VG	Variance Gamma Distribution
WTI	West Texas Intermediate

1. INTRODUCTION

Most references define risk as "the potential that an action or activity (including the choice of inaction) will lead to a loss (an outcome)". The risk that originates with events thousands of miles away that have nothing to do with the domestic market is the result of increasingly global markets. Information is available instantaneously, which means that change, and subsequent market reactions, occur very quickly. Therefore, understanding risk has become an important part of the financial decisions. Especially in the investment world, risk is considered inseparable from the performance.

A central issue in modern risk management is the selection of the risk measure. There are different approaches to measure risk; such as *Value-at-Risk (VaR)* and *Conditional Value-at-Risk (CVaR)* which is also known as *Expected Shortfall*. Although VaR is not a coherent risk measure like CVaR, it is probably the most widely used risk measure in financial applications. VaR attempts to answer the question,"How much money might I lose?" based on probabilities and within parameters set by the risk manager [1].

A portfolio is an investment in several different financial instruments at the same time. The risk of a portfolio can be calculated by three classical approaches; variance - covariance method, historical simulation and Monte Carlo simulation. Variance - covariance method is based on analytical estimation of the volatility of asset returns and of the correlations between these asset price movements which is the basic parametric approach for portfolio risk calculation. Historical simulation in VaR analysis is a procedure for predicting VaR by simulating or constructing the cumulative distribution function of assets returns over time and is easy to implement and reduces the risk measure estimation problem to a one dimensional problem. However, it turned out that these approaches have clear drawbacks and estimate inaccurate risks. The Monte Carlo method is a rather general name for any approach to risk measurement that involves the simulation of an explicit parametric model for risk-factor changes [2].

The usual Monte Carlo method generates random market scenarios assuming that the risk factors follow a multivariate normal distribution; however the asset returns have shown that they are far from the normal distribution because of having fat tails and high kurtosis. To be able to model the dependence between return series adequately, the use of *copulas* is considered.

One relevant aspect which is frequently left out from simulation models is the dependence relationship among the processes under consideration. That omission can have a serious impact on the results of risk assessment and, consequently, on the conclusions drawn from them [3]. The idea of copulas was introduced by Sklar [4]. A copula is a multivariate distribution function defined on the n-dimensional unit hypercube $[0,1]^n$ with uniformly distributed marginals. The main advantage of copula functions is that they enable us to tackle the problem of specification of marginal univariate distributions separately from the specification of market comovement and dependence. Therefore, the idea of copulas has attracted attention for risk calculations to overcome these problems and to be an alternative to the mentioned approaches. The most useful copula models for dimensions higher than three are the Gauss and the t copula. The Gauss copula does not have tail dependence while t copula has both lower and upper tail dependence. [2]. Thus, it is more appropriate to use the t copula for model fitting.

The first objective of this study is therefore to model the multivariate return distributions of stock portfolios by the concept of copula and use this model for risk measure evaluations in practice utilizing the statistical software R and its useful add-on packages. The second aim is to observe whether this model fits the multivariate return distributions of stock portfolios diversified with commodities, precious metals, crude oil etc. and to analyze whether adding these financial instruments reduces the risk.

The thesis is organized as follows: The basic concepts of risk measuring are explained with the formal definitions of VaR and CVaR in Chapter 2. The classical risk calculation methods are also explained shortly. The copula method is introduced in

Chapter 3 by giving the essential definitions. Simulation from copulas and calculating risks by Monte Carlo simulation is explained in Chapter 4. Moreover, in Chapter 5, back testing concept and methods are introduced. We introduce the required data, selected models and the all steps of the risk quantification in Chapter 5. Finally, in Chapter 6, we compare the models and present the back testing and risk calculations results.

2. BASIC CONCEPTS OF RISK MEASURING

In this section, basic concepts of risk measuring will be explained briefly. First of all, the concept of financial risk will be given. In the second part, we will talk about risk measures by giving the explanation of Value-at-Risk and conditional Value-at-Risk. In the last part, we will give further information about standard methods for portfolio risk calculation.

2.1. The Concept of Financial Risk

A risk is a potential problem, a situation that, if it materializes, may adversely affect the project. For financial risks, we might arrive at a definition such as “any event or action that may adversely affect an organizations ability to achieve its objectives and execute its strategies” or, alternatively, “the quantifiable likelihood of loss or less-than-expected returns” [2]. Understanding risk is an important step in determining how to manage, since eliminating it is not always possible and desirable.

According to A.Horcher [1], financial risk management is a process to deal with the uncertainties resulting from financial markets. The process of financial risk management is an ongoing one. Strategies need to be implemented and refined as the market and requirements change. In general, the process can be summarized as follows:

- Identify and prioritize key financial risks.
- Determine an appropriate level of risk tolerance.
- Implement risk management strategy in accordance with policy.
- Measure, report, monitor, and refine as needed.

For many years, the riskiness of an asset was assessed based only on the variability of its returns. In contrast, modern portfolio theory considers not only an asset’s riskiness, but also its contribution to the overall riskiness of the portfolio to which it is added. Organizations may have an opportunity to reduce risk as a result of risk diversification [1].

The financial industry considers several types of risks. The most apparent types of financial risks that an organization faces are the major factor risks such as; foreign exchange risk, interest rate risk, commodity price risk and equity price risk. Other important financial risks can be exemplified as credit risk, operational risk, liquidity risk and systemic risk. The interaction of several risks that are defined above can alter the potential impact to an organization.

2.2. Risk Measures

The risk of a financial position can be measured with four different approaches; the notional-amount approach, factor-sensitivity measures, risk measures based on the loss distribution, risk measures based on scenarios. Most modern measures of the risk in a portfolio are statistical quantities describing the conditional or unconditional loss distribution of the portfolio over some predetermined horizon Δ [2]. Among these approaches, we will focus on the "risk measures based on the loss distribution" in which the variance, the Value-at-Risk and the expected shortfall are included.

2.2.1. Value-at-Risk

Value-at-Risk (VaR) has become a key tool for risk management and it has been a widely accepted risk measure since the 1990s. VaR attempts to answer the question, "How much money might I lose?" based on probabilities and within parameters set by the risk manager [1].

Given some confidence level $\alpha \in (0, 1)$. The VaR of a portfolio at the confidence level α is given by the smallest number l such that the probability that the loss L

exceeds l is no larger than $(1-\alpha)$. Formally,

$$VaR_\alpha = inf\,\{l \in \mathbb{R} : P(L > l) \leq 1-\alpha\} \tag{2.1}$$

As it can seen from the definition, VaR is thus simply a *quantile* of the loss distribution. Typical values for α are 0.95 or 0.99 regarding various time horizons. It is also possible to say that VaR_α is the loss described by the $1-\alpha$ quantile of the return distribution.

Despite its conceptual simplicity, ease of computation and ready applicability, VaR has been charged to have several conceptual problems. Among others, Artzner *et al.* [5, 6], have mentioned the following shortcomings:

- VaR measures only percentiles of profit-loss distributions and disregards any loss beyond the VaR level; namely, tail risk.
- VaR is not coherent, since it is not sub-additive.

Sub-additivity is an important property of incremental risk. That is, the sum of the incremental risks of the positions in a portfolio equals the total risk of the portfolio. This property has important applications in the allocation of risk to different units, where the goal is to keep the sum of the risks equal to the total risk [7].

To remedy these shortcomings inherent in VaR that are mentioned above, Artzner *et al.* [5] have proposed the use of the expected shortfall.

2.2.2. Conditional Value-at-Risk

Conditional Value at Risk (CVaR), or alternatively Expected shortfall (ES) is a second risk measure approach which is closely related to VaR. It is preferred by risk managers because of being sub-additive which assures its coherence as a risk measure despite of VaR. Nevertheless, VaR is still popular in the financial world mainly because of having an intuitive interpretation.

CVaR is the expected amount of loss of a portfolio given that it has exceeded the VaR in some investment horizon under a given confidence level [8]. Formally;

$$CVaR_\alpha = E(L \mid L > VaR_\alpha) \tag{2.2}$$

Obviously $CVaR_\alpha$ depends only on the distribution of L and obviously $ES_\alpha \geq VaR_\alpha$. For continuous loss distributions an even more intuitive expression can be derived which shows that expected shortfall can be interpreted as the expected loss that is incurred in the event that VaR is exceeded [2].

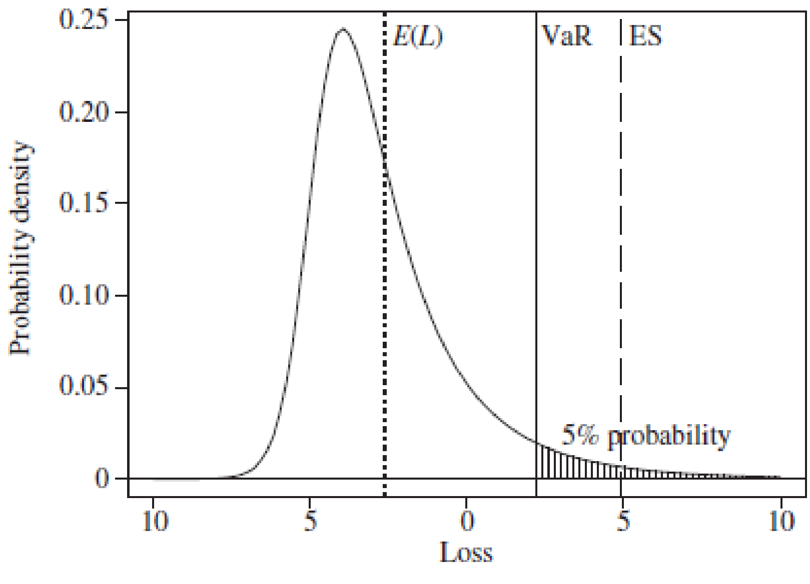

Figure 2.1. An example of a loss distribution with the mean loss and risk measures $VaR_\alpha = 0.05$, $ES_\alpha = 0.05$.

2.2.3. VaR and ES under normal distribution

According to Yamai and Yoshiba [9], when the profit-loss distribution is normal, VaR and ES give essentially the same information. Both VaR and ES are scalar multiples of the standard deviation. Therefore, VaR provides the same information on tail loss as does expected shortfall.

On the other hand, VaR may have tail risk if the profit-loss distribution is not normal. Non-normality of the profit-loss distribution is caused by non-linearity of the portfolio position or non-normality of the underlying asset prices.

2.3. Standard Methods for Portfolio Risk Calculation

A portfolio is an investment in several stocks at the same time. The idea of holding a portfolio is reducing the risk according to the well known principle: “Do not put all eggs into a single basket”. This section includes the problem of estimating risk measures for the loss distribution of a loss $L_\Delta = S_0 - S_\Delta = S_0(1 - r_l(\Delta))$ where $r_l(\Delta)$ is the return for time horizon Δ and S_0 is the value of the portfolio at time 0.

The returns of financial assets can be calculated by arithmetic returns, geometric returns and log returns. In the following sections, we discuss some standard methods used in the financial industry for measuring the portfolio risk concentrating on log returns and risk measures VaR and ES.

2.3.1. Variance-Covariance Method (Mean-Variance Method)

This method is also called “Approximate Multi Normal Model” and based on analytical estimation of the volatility of asset returns and of the correlations between these asset price movements. It is the basic parametric approach for portfolio risk calculation. The reason why this method is called “Approximate Multi Normal Model” is that the portfolio log return is calculated by summing the weighted log returns of the individual assets although this summing can be done only for the arithmetic returns. Thus the log returns are approximated as the arithmetic returns since they are very close to each other around zero [8].

The model is given by Equations 2.3 to 2.6. The random variable for the return distribution of asset i is X_i with parameters μ_i and σ_i. Its relative amount is w_i which contains the fractions of the total value of the portfolio invested into stock i.

The random variable for the return distribution of the portfolio with d stocks is X_p and it is the weighted sum of the asset returns. X_p is assumed to have a multi normal distribution with mean vector μ^T and covariance matrix Σ. The diagonals of Σ are the variances of marginal distributions where the non-diagonal elements are $Cov(X_i, X_j) = \rho_{i,j}\sigma_i\sigma_j$. Thus, the parameters of the model; means, variances and correlations, can be easily estimated using historical data.

$$X_i = N(\mu_i, \sigma_i^2), i = 1, \ldots, d \tag{2.3}$$

$$X_p \approx \sum_{i=1}^{d} w_i X_i = X^T w \Rightarrow X_p \sim N(\mu_p, \sigma_p^2), \sum_{i=1}^{d} w_i = 1 \tag{2.4}$$

$$\mu_p = \sum_{i}^{d} w_i \mu_i = w^T \mu, \sigma_p^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} w_i w_j \rho_{ij} \sigma_i \sigma_j = w^T \Sigma w \tag{2.5}$$

Using the definition of the random variate L and the formulas for μ_p and σ_p, we can easily get a simple approximation formula for the VaR [10]:

$$VaR_\alpha \approx S_0(1 - e^{\mu_p \Delta t + z_{1-\alpha}\sigma_p \sqrt{\Delta t}}) \tag{2.6}$$

where $z_{1-\alpha}$ is the $1 - \alpha$ quantile of the standard normal distribution.

The variance-covariance method offers a simple analytical solution to the risk-measurement problem but this convenience is achieved at the cost of two crude simplifying assumptions. First, linearization may not always offer a good approximation of the relationship between the true loss distribution and the risk-factor changes, second, the assumption of normality is unlikely to be realistic for the distribution of the risk-factor changes, certainly for daily data and probably also for weekly and even monthly data.

2.3.2. Historical Simulation

Historical simulation in VaR analysis is a procedure for predicting VaR by simulating or constructing the cumulative distribution function of assets returns over time. Unlike other VaR methods, historical simulation does not require any assumption about the distributions of asset returns. This method assumes that the log returns are independent and identically distributed (i.i.d.) sequences. For validating this assumption, it must be checked whether the distribution of the X_{i+1} values are influenced by the X_i values. The log returns of a portfolio can be simulated by randomly selecting the log returns of one day of the historical data, and then the corresponding risk measure can be calculated.

This method is easy to implement and reduces the risk measure estimation problem to a one dimensional problem. No statistical estimation of the multivariate distribution of X is necessary, and no assumptions about the dependence structure of risk factor changes are made [2]. Because of having more extremes in the tails, risk estimates of the historical simulation are expected to be more accurate than the multi normal model. However, the results of historic simulation are very unstable due to the small number of data in the tails. This problem gets even more severe when considering more stocks [10]. The success of this method is dependent on our ability to collect sufficient quantities of relevant, synchronized data for all risk factors. Whenever there are gaps in the risk-factor history, or whenever new risk factors are introduced into the modeling, there may be problems to fill the gaps and complete the historical record. These problems will tend to reduce the effective value of n and mean that empirical estimates of VaR and expected shortfall have very poor accuracy [2].

2.3.3. Monte Carlo Method

The Monte Carlo method is a rather general name for any approach to risk measurement that involves the simulation of an explicit parametric model for risk factor changes. As such, the method can be either conditional or unconditional depending on whether the model adopted is a dynamic time series model for risk factor changes or a static distributional model [2].

The Monte Carlo method is used to calculate the expected value or in other words the mean of a certain random variate. The result of the single run of a simulation is one realization x of the output random variate X. To estimate the mean value of that output random variate; the average of all generated variates X_i for $i = 1, \ldots, n$ can be simply used. For that sample average;

$$\bar{X} = \frac{1}{n} \sum_{i}^{n} X_i \tag{2.7}$$

The sample mean can be used as estimator for μ

$$\widehat{\mu} = \bar{X} \tag{2.8}$$

$\bar{X}$ is the best unbiased estimate for μ_X. $E(\bar{X}) = \mu_X$ and $Var(\bar{X}) = \sigma^2/n$ guarantee that the simulation leads close to a correct result. The main problem is the size of the error. By the central limit theorem (CLT) for random samples from a population with mean μ_X and finite variance σ_X^2, the sample mean distribution converges to the normal distribution. Thus the following result can be used:

$$P(|\mu_X - \bar{X}| > F_N^{-1}(1 - \frac{\alpha}{2}) \cdot \frac{s}{\sqrt{n}}) = \alpha \tag{2.9}$$

where $F_N^{-1}(.)$ denotes the inverse cumulative distribution function (CDF) of the standard normal distribution. It is obvious that this error bound based on the convergence result of the CLT need not be close to correct for small sample sizes [10]. This method will be introduced in more detail in the next chapters.

3. COPULA

In the previous chapter, we have seen some classical approaches for risk estimation. We have also realized that they have clear drawbacks. The idea of copulas attracted more attention by being an alternative to the mentioned approaches.

To be able to model the interdependence between return series in an adequate way one might consider the use of copulas [11]. Copulas have become an important tool in finance with various applications, e.g., risk management, derivative pricing, portfolio management, etc. A copula is a multivariate distribution with uniform marginals. The idea is that a copula just describes the relation between random variates without including its marginal distribution [12].

A great advantage of the copula model is the separate modeling of the dependence and the marginal behavior of the univariate series. Another great advantage is that the marginal distributions do not have to be similar to each other so that each marginal distribution can be modeled separately [11]. These advantages of copulas are useful in risk management, where we very often have a much better idea about the marginal behavior of individual risk factors than we have about their dependence structure. The copula approach allows us to combine our more developed marginal models with a variety of possible dependence models and to investigate the sensitivity of risk to the dependence specification. Since the copulas, we present are easily simulated, they lend themselves in particular to Monte Carlo studies of risk [2].

3.1. Basic Definitions

An d dimensional copula is a distribution function on $[0,1]^d$ with standard uniform marginal distributions. A function $C : [0,1]^d \to [0,1]$ is an *d-copula* (d-dimensional copula) if it enjoys the following properties:

- $C(u_1, \ldots, u_d)$ is increasing in each component u_i
- $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i \in \{1, \ldots, d\}, u_i \in [0, 1]$
- For all $(a_i, \ldots, a_d), (b_i, \ldots, b_d) \in [0, 1]^d$ with $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\ldots+i_d} C(u_{1i_1}, \ldots, u_{di_d}) \geq 0, \tag{3.1}$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \ldots, d\}$.

The first property is required for any multivariate density functions while the second property is required for uniform marginal distributions. The third property, called the rectangle property, ensures that if the random vector $(U_1, \ldots, U_d)$ has CDF C then, $P(a_1 \leq U_1 \leq b_1, \ldots, a_d \leq U_d \leq b_d)$ is non-negative. If a function fulfills these properties, then it is a copula.

A multivariate distribution function is constructed by choosing a copula and some marginals, then structuring it in the right way. The importance of copulas in the study of multivariate distribution function (dfs) is summarized by the following elegant theorem.

Theorem 3.1 (Sklar's Theorem, 1959). *Let F be a joint distribution function with margins $F_1, \ldots, F_n$. Then there exists a copula* C $: [0, 1]^n \to [0, 1]$ *such that, for all $x_1, \ldots, x_d$ in $\mathbb{R} = [-\infty, \infty]$,*

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_n(x_d)) \tag{3.2}$$

If the margins are continuous, then C *is unique; otherwise C is uniquely determined on Ran $F_1\times$ Ran $F_2 \times \ldots \times$ Ran F_d, where Ran $F_i = F_i(\bar{\mathbb{R}})$ denotes the range of F_i. Conversely, if* C *is a copula and F_1,...,F_d are univariate distribution functions, then the function* F *defined above is a joint distribution function with margins F_1,...,F_d.*

3.1.1. Copula of F (Frey, McNeil, Nyfeler, 2001)

We extract a unique copula C from a multivariate dfs F with continuous margins $F_d, \ldots, F_d$ by calculating

$$C(u_1, \ldots, u_n) = F(F^{-1}(u_1), \ldots, F^{-n}(u_d)) \tag{3.3}$$

where $F_1^{-1}, \ldots, F_d^{-1}$ are (generalized) inverses of $F_1, \ldots, F_d$. We call C the copula of F, or of any random vector with distribution function F.

3.1.2. Fréchet-Hoeffding Bounds for Joint Distribution Functions

According to Nelsen (2006) [13], Fréchet-Hoeffding Bounds are universal bounds for copulas, i.e., for any copula C and for all $u, v \in [0, 1]$,

$$W(u, v) = max(u + v - 1, 0) \leq C(u, v) \leq min(u, v) = M(u, v) \tag{3.4}$$

As a consequence of Sklar's theorem, if X and Y are random variables with a joint distribution function H and margins F and G, respectively, then for all x, y in $\overline{\mathbb{R}}$,

$$max(F(x) + G(y) - 1, 0) \leq H(x, y) \leq min(F(x), G(y)) \tag{3.5}$$

Because M and W are copulas, the above bounds are joint distribution functions and are called the *Fréchet-Hoeffding* bounds for joint distribution functions H with margins F and G.

3.1.3. Examples of Copulas

There is a number of examples provided in this section including fundamental copulas, implicit copulas, explicit copulas and survival copulas. Before going further, giving the properties of some multivariate distributions will be informative to better understand the examples of copulas.

Multivariate Normal Distribution

The d-dimensional random vector $X = (X_1, ..., X_d)$ is said to have a (non-singular) multivariate Normal distribution with mean vector μ and positive definite matrix Σ, denoted $X \sim N_d(\mu, \Sigma)$, if its density is given by;

$$f(x) = \frac{1}{(2\Pi)^{N/2}|\Sigma|^{1/2}} \cdot exp\left(-\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{2}\right) \tag{3.6}$$

Multivariate Student's t Distribution

Particularly in finance and risk management, Student's t distribution has been used instead of the normal distribution, because of its fat tail behavior, which can be applied to capture financial extreme events [14].

A d-dimensional random vector $X = (X_1, ..., X_d)$ is said to have a (non-singular) multivariate Student's t distribution with mean vector μ, positive definite matrix Σ and with v degrees of freedom, denoted $X \sim t_d(\mu, \Sigma, v)$, if its density is given by;

$$f(x) = \frac{\Gamma(\frac{v+d}{2})}{\Gamma(\frac{v}{2})(\pi v)^{d/2}|\Sigma|^{1/2}} \cdot exp\left(1 + \frac{(x-\mu)'\Sigma(x-\mu)}{v}\right)^{-\frac{v+d}{2}} \tag{3.7}$$

The multivariate Student's t distribution belongs to the class of multivariate normal variance mixtures and has the representation

$$X_d = \mu + \sqrt{W}Z \tag{3.8}$$

where $Z \sim N_d(0, \Sigma)$ and W is independent of Z and satisfies $v/W \sim \chi^2_v$, equivalently W has an inverse gamma distribution $W \sim IG(v/2, v/2)$ [11].

Multivariate Generalized Hyperbolic distributions

The random vector X is said to have a multivariate generalized hyperbolic distribution (GHD) if

$$X = \mu + W\gamma + \sqrt{W}AZ \tag{3.9}$$

where

(i) $Z \sim N_k(0, I_k)$
(ii) $A \in \mathbb{R}^{d\times k}$
(iii) $\mu, \gamma \in \mathbb{R}^d$
(iv) $W \geq 0$ is a scalar-valued random variable which is independent of Z and has a Generalized Inverse Gaussian distribution, written $GIG(\lambda, \chi, \varphi)$.

The parameters of a GHD distribution given by the above definition admit the following interpretation:

- λ, χ, φ determine the shape of the distribution. That is, how much weight is assigned to the tails and to the center. In general, the larger those parameters the closer the distribution is to the normal distribution.
- μ is the location parameter.
- $\Sigma = AA'$ is the dispersion parameter.
- γ is the skewness parameter. If $\gamma = 0$ then the distribution is symmetric around μ.

From the definition of GHD, we can observe that the conditional distribution of $X|W = w$ is normal;

$$X|W = w \sim N_d(\mu + w\gamma, w\Sigma) \tag{3.10}$$

where $\Sigma = AA'$. Because of this, it is also called normal mean-variance mixture distribution.

The expected value and the variance are given by

$$E[X] = \mu + E[W]\gamma \tag{3.11}$$

$$Var(X) = E[W]\Sigma + Var(W)\gamma\gamma' \tag{3.12}$$

when the mixture variable W has finite variance $Var(W)$ [15].

Since the conditional distribution of X given W is Gaussian with mean $\mu + W\gamma$ and variance $W\Sigma$ the GH density can be found by mixing $X|W$ with respect to W.

$$f(x) = c \cdot \frac{K_{\lambda-d/2}\left(\sqrt{(\chi + (x-\mu)'\Sigma^{-1}(x-\mu))(\varphi + \gamma'\Sigma^{-1}\gamma)}\right) e^{((x-\mu)'\Sigma^{-1}\gamma)}}{\left(\sqrt{(\chi + (x-\mu)'\Sigma^{-1}(x-\mu))(\varphi + \gamma'\Sigma^{-1}\gamma)}\right)^{d/2-\lambda}} \tag{3.13}$$

where the normalizing constant c is given by,

$$c = \frac{(\sqrt{\chi\varphi})^{-\lambda}\varphi^{\lambda}(\varphi + \gamma'\Sigma^{-1}\gamma)^{d/2-\lambda}}{(2\pi)^{d/2}|\Sigma|^{1/2}K_{\lambda}(\sqrt{\chi\varphi})} \tag{3.14}$$

The GHD contains several special cases known under special names.

- If $\lambda = \frac{d+1}{2}$ the name generalized is dropped and we have a multivariate *hyperbolic (hyp)* distribution. The univariate margins are still GH distributed. Inversely, when $\lambda = 1$ we get a multivariate GHD with hyperbolic margins.
- If $\lambda = -\frac{1}{2}$ the distribution is called *Normal Inverse Gaussian (NIG).*
- If $\chi = 0$ and $\lambda > 0$ one gets a limiting case which is known amongst others as *Variance Gamma (VG)* distribution.

- If $\varphi = 0$ and $\lambda < 0$ the *generalized hyperbolic Student's t* distribution is obtained (called simply Student's t).

There are several alternative parameterizations for the GH distribution. One of the mostly used representation is the $(\lambda, \chi, \varphi, \mu, \Sigma, \gamma)$ parametrization which is obtained as the normal mean-variance mixture distribution when $W \sim GIG(\lambda, \chi, \varphi)$. However, this parametrization has an identification problem. Indeed, the distributions $GHD_d(\lambda, \chi, \varphi, \mu, \Sigma, \gamma)$ and $GHD_d(\lambda, \chi/k, k\varphi, \mu, k\Sigma, k\gamma)$ are identical for any $k > 0$. Therefore, an identifying problem occurs when we start to fit the parameters of a GH distribution to data. This problem may be solved by introducing a suitable constraint.

When the GHD was introduced in Barndorff-Nielsen (1977), the $(\lambda, \alpha, \mu, \Delta, \delta, \beta)$ parametrization for the multivariate case was used:

$$\begin{aligned} f_x(x) = & \frac{(\alpha^2 - \beta'\Delta\beta)^{\lambda/2}}{(2\pi)^(d/2)\sqrt{|\Delta|\alpha^{\lambda-d/2}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta'\delta\beta})}} \\ & \times \frac{K_{\lambda-d/2}(\alpha\sqrt{\delta^2 + (x-\mu)'\Delta^{-1}(x-\mu)})e^{\beta'(x-\mu)}}{(\sqrt{\delta^2 + (x-\mu)'\Delta^{-1}(x-\mu)})^{d/2-\lambda}} \end{aligned} \tag{3.15}$$

Similar to the $(\lambda, \chi, \varphi, \mu, \Sigma, \gamma)$ parametrization, there is an identification problem which can be solved by constraining the determinant of Δ to 1 [15]. The univariate case of the above expression is the most widely used parametrization of the GHD in literature.

The GHD is closed under linear transformations which seems a useful property for portfolio management. If $X \sim GHD_d(\lambda, \chi, \varphi, \mu, \Sigma, \gamma)$ and $Y = BX + b$ where $B \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$, then $Y \sim GHD_k(\lambda, \chi, \varphi, B\mu + b, B\Sigma B', B\gamma)$ which means that the linear transformations of GHD still remain in the GHD class. Thus, if we set $B = w^T = (w_1, w_2, \ldots, w_d)$ and $b = 0$, then the portfolio $y = w^T X$ is a one-dimensional

GHD:

$$y \sim GHD_1(\lambda, \chi, \varphi, w^T\mu, w^T\Sigma w, w^T\gamma) \tag{3.16}$$

3.1.3.1. Fundamental copulas. These copulas represent a number of important special dependence structures. The *independence copula* is

$$\Pi(u_1, \dots, u_d) = \prod_{i=1}^{d} u_i \tag{3.17}$$

Random variables with continuous distributions are independent if and only if their dependence structure is given by (3.6).

The *comonotonicity* copula is the Fréchet upper bound copula which represents perfect dependence

$$M(u_1, \dots, u_d) = min\{u_1, \dots, u_d\} \tag{3.18}$$

Observe that this special copula is the joint df of the random vector $(U, \dots, U)$, where $U \sim U(0, 1)$.

The *countermonotonicity* copula is the two-dimensional Fréchet lower bound copula

$$W(u_1, u_2) = max(u_1 + u_2 - 1, 0) \tag{3.19}$$

This copula is the joint df of the random vector $(U, 1 - U)$, where $U \sim U(0, 1)$.

3.1.3.2. Implicit copulas. If $X \sim N_d(\mu, \Sigma)$ is a Gaussian random vector, then its copula is a so-called *Gauss copula*, or alternatively *Normal copula*. The Gauss copula is perhaps the most popular elliptical copula in applications. For a given correlation

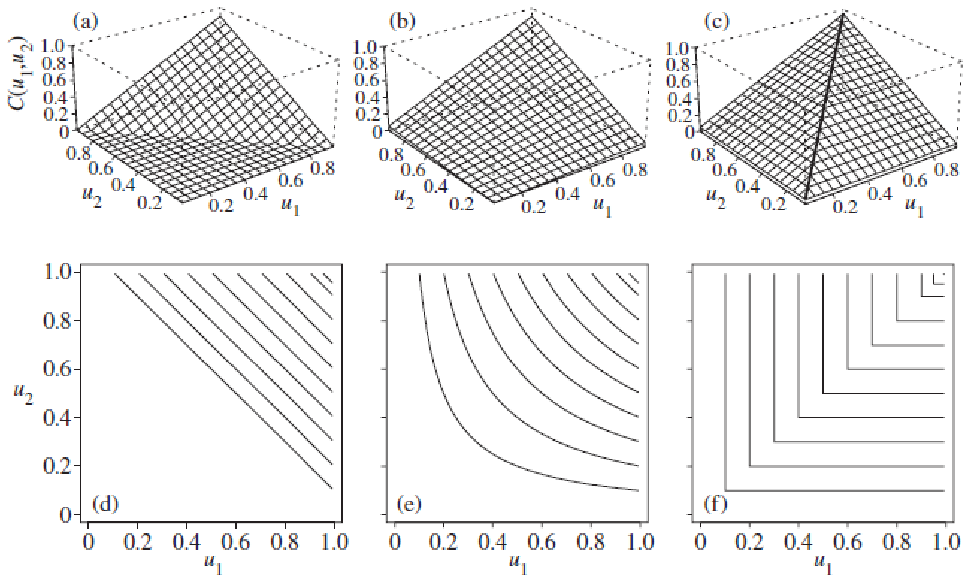

Figure 3.1. Distribution function plots of three fundamental copulas: (a),(d) countermonotonicity, (b),(e) independence and (c),(f) comonotonicity.

matrix $\Sigma \in \mathbb{R}^{dxd}$, the Gauss copula with parameter matrix Σ can be written as,

$$C_{\Sigma}^{Gauss}(u) = \Phi_{\Sigma}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)) \tag{3.20}$$

where Φ^{-1} is the inverse cumulative distribution function of a standard normal and Φ_{Σ} is the joint cumulative distribution function of a multivariate normal distribution with mean vector zero and covariance matrix equal to the correlation matrix Σ. The density can be written as;

$$C_{\Sigma}^{Gauss}(u) = \frac{1}{\sqrt{det\Sigma}} exp\left(-\frac{1}{2}\begin{pmatrix}\Phi^{-1}(u_1)\\ .\\ .\\ .\\ \Phi^{-1}(u_d)\end{pmatrix}^T \cdot (\Sigma^{-1} - \mathbf{I}) \cdot \begin{pmatrix}\Phi^{-1}(u_1)\\ .\\ .\\ .\\ \Phi^{-1}(u_d)\end{pmatrix}\right) \tag{3.21}$$

where $\mathbf{I}$ is the identity matrix [16].

Note that both the independence and comonotonicity copulas are special cases of the Gauss copula. If $\Sigma = I_d$, we obtain the independence copula; if $\Sigma = J_d$, the $d \times d$ matrix consisting entirely of ones, then we obtain comonotonicity. Also, for $d = 2$ and $\Sigma = -1$ the Gauss copula is equal to the countermonotonicity copula.

The $t-$ *copula* is another popular model which is derived in a similar way as the Gauss copula. It is an elliptical copula derived from the multivariate t-distribution. According to *Sklar's Theorem*, the $t-$ *copula* of the random vector $u \in [0,1]^d$ can be expressed as;

$$C^t_{v,\rho}(u) = t^d_{v,\rho}(t_v^{-1}(u_1), \ldots, t_v^{-1}(u_n))$$

where $\rho_{i,j} = \Sigma_{i,j}/\sqrt{\Sigma_{i,i}\Sigma_{i,j}}$ with $i, j \in 1, \ldots, n$. $t^n_{v,\rho}(.)$ denotes the distribution function F, t_v^{-1} represents the inverse of the marginal t-distribution function F_i^{-1} and v corresponds to the degree of freedom [12]. For estimation purposes it is useful to note that the density of the $t-$ *copula* has the form;

$$c^t_{v,\rho}(u_1, \ldots, u_d) = \frac{1}{\sqrt{|\rho|}} \frac{\Gamma(\frac{v+d}{2})[\Gamma(\frac{v}{2})]^{d-1}}{[\Gamma(\frac{v+1}{2})]^d} \frac{\prod_{k=1}^d (1 + \frac{y_k^2}{v})^{\frac{v+1}{2}}}{(1 + \frac{y'\rho^{-1}y}{v})^{\frac{v+d}{2}}} \tag{3.22}$$

As in the case of the Gauss copula, if $\rho = J_d$ then we obtain comonotonicity. However, in contrast to the Gauss copula, if $\rho = I_d$ we do not obtain the independence copula (assuming $v < \infty$) since uncorrelated multivariate t-distributed random variables (rvs) are not independent.

Since the t-distribution tends to normal distribution when v goes to infinity, the t-copula also tends to the normal copula as $v \to +\infty$ [17].

$$v \to +\infty \Rightarrow sup_{u \in [0,1]^n} |C^t_{v,\rho}(u) - C^{Gauss}_{\Sigma}(u)| \to 0 \tag{3.23}$$

<u>3.1.3.3. Explicit copulas.</u> While the Gauss and t-copulas are implied by well-known multivariate dfs and do not have a simple closed forms, we can write down a number

of copulas which do have simple closed forms. An example is the bivariate *Gumbel copula*:

$$C_\theta^{Gu}(u_1, u_2) = e^{-((-lnu_1))^\theta + (-lnu_2))^\theta)^{\frac{1}{\theta}}}, 1 \leq \theta \leq \infty \tag{3.24}$$

A further example is the bivariate *Clayton copula*:

$$C_\theta^{CI}(u_1, u_2) = (u_1^\theta + u_1^\theta - 1)^{\frac{-1}{\theta}}, 0 \leq \theta \leq \infty \tag{3.25}$$

<u>3.1.3.4. Survival copulas.</u> Let X be a random vector with multivariate survival function $\bar{F}$, marginal dfs $F_1, \ldots, F_d$ and marginal survival functions $\bar{F}_1, \ldots, \bar{F}_d$ i.e. $\bar{F}_i = 1 - \bar{F}_i$. We have the identity

$$\bar{F}(x_1, \ldots, x_d) = \hat{C}(F_1(x_1), \ldots, F_d(x_d)) \tag{3.26}$$

for a copula $\hat{C}$, which is known as a survival copula. In general, the term *survival copula of a copula* C will be used to denote the df of $1 - U$ when U has df C.

3.2. Dependence Measures

There exist different methods to measure the dependence between random variables. In the following section, three kinds of dependence will be explained; the usual Pearson linear correlation and copula based dependence measures; rank correlation and the coefficients of tail dependence. All these dependences measures yield a scalar measurement for a pair of random variables (X_1, X_2) although the nature and properties of the measure are different in each case [2].

3.2.1. Linear Correlation

Correlation can refer to any departure of two or more random variables from independence, but technically it refers to any of several more specialized types of relationship between mean values. There are several correlation coefficients, often denoted ρ or r, measuring the degree of correlation. The most common of these is the *Pearson correlation coefficient*, which is sensitive only to a linear relationship between two variables (which may exist even if one is a nonlinear function of the other) [18].

Given two rvs X_1 and X_2, the linear correlation coefficient is defined as:

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}} \tag{3.27}$$

where Cov denotes covariance and Var denotes the variance. The Pearson correlation is defined only if both variances are nonzero. If X_1 and X_2 are independent, then $\rho(X_1, X_2) = 0$, but it should be well known to all users of correlation that the converse is false: the uncorrelatedness of X_1 and X_2 does not in general imply their independence.

The correlation coefficient takes values between -1 and 1, namely, $\rho \in [-1, 1]$. If $|\rho(X_1, X_2)| = 1$, then X_1 and X_2 are perfectly linearly dependent, meaning that $X_2 = \alpha + \beta X_1$ almost surely for some $\alpha \in \mathbb{R}$ and $\beta \neq 0$, with $\beta > 0$ for positive linear dependence and $\beta < 0$ for negative linear dependence.

Another important remark is that correlation is only defined when the variances of X_1 and X_2 are finite. This restriction to finite variance models is not ideal for a dependence measure and can cause problems when we work with heavy-tailed distributions [2]. Since the financial asset returns are not normal and have heavy tails, using the linear correlation coefficient can cause problems. Thus, it is necessary to search for other dependence measures.

3.2.2. Rank Correlation

The main problem in financial management is comparing the probability that the prices of two or more assets rise (or fall) together with the probability that one of the assets rises (falls) while the other one falls (rises). If they move in the same direction without regarding up or down, this is called *concordance*. Mathematically, two points in $\mathbb{R}^2$ denoted by (x_1, x_2) and $(\tilde{x_1}, \tilde{x_2})$, are said concordant if $(x_1, \tilde{x_1})(x_2, \tilde{x_2}) > 0$ and to be discordant if $(x_1, \tilde{x_1})(x_2, \tilde{x_2}) < 0$. The rank correlation measures; *Kendall's tau* and *Spearman's rho* deal with measuring the concordance of rvs. The reason for looking at rank correlations in this thesis is that they can be used to calibrate copulas to empirical data.

<u>3.2.2.1. Kendall's tau.</u> Let (X_1, X_2) and $(\tilde{X_1}, \tilde{X_2})$ be two independent pairs of random variables from F, then Kendall's rank correlation is given by [19];

$$\rho_\tau(X_1, X_2) = P((X_1 - \tilde{X_1})(X_2 - \tilde{X_2}) > 0] - P[(X_1 - \tilde{X_1})(X_2 - \tilde{X_2}) < 0) \qquad (3.28)$$

If X_2 tends to increase with X_1, then we expect the probability of concordance to be high relative to the probability of discordance; if X_2 tends to decrease with increasing X_1, then we expect the opposite.

For continuous random variables, Kendall's tau can be rewritten as:

$$\rho_\tau(X_1, X_2) = 2P((X_1 - \tilde{X_1})(X_2 - \tilde{X_2}) > 0) - 1 \qquad (3.29)$$

From this equation, it can be seen that Kendall's tau varies between -1 and 1.

<u>3.2.2.2. Spearman's rho.</u> It is simply the linear correlation of the probability transformed rvs, which for continuous rvs is the linear correlation of their unique copula. Let X_1 and X_2 be rvs with marginal distribution functions F_1 and F_2 and joint distribution

function F. Spearman's rank correlation is defined by

$$\rho_S(X_1, X_2) = \rho(F_1(X_1), F_2(X_2)), \tag{3.30}$$

where ρ is the usual linear correlation [19].

Spearman's rho matrix for the general multivariate random vector X is given by $\rho_S(X) = \rho(F_1(X_1), \ldots, F_2(X_d))$ and must again be positive semidefinite.

Considering two random variables X_1 and X_2 with marginal distributions F_1 and F_2, Spearman's rho equals:

$$\rho_S = \rho(F_1(X_1), F_2(X_2)) = \frac{Cov(F_1(X_1), F_2(X_2))}{\sqrt{Var(F_1(X_1))Var(F_2(X_2))}} \tag{3.31}$$

Kendall's tau and Spearman's rho have the followings properties in common:

- They are both symmetric dependence measures taking values in the interval [-1,1]
- They give the value zero for independent rvs, although a rank correlation of 0 does not necessarily imply independence.
- It can be shown that they take the value 1 when X_1 and X_2 are comonotonic and the value -1 when they are countermonotonic.

3.2.3. Coefficient of Tail Dependence

If we are particularly concerned with extreme values an asymptotic measure of *tail dependence* can be defined for pairs of random variables X_1 and X_2. If the marginal distributions of these random variables are continuous then this dependence measure is also a function of their copula, and is thus invariant under strictly increasing transformations.

Let X_1 and X_2 be random variables with distribution functions F_1 and F_2. By definition, the *coefficient of (upper) tail dependence* of X_1 and X_2 is

$$\lambda_U = \lim_{\alpha \to 1^-} P(X_2 > F_2^{-1}(\alpha) | X_1 > F_1^{-1}\alpha) \tag{3.32}$$

provided a limit $\lambda \in [0, 1]$ exists. λ_U can also be interpreted in terms of VaR with the probability level α. If $\lambda \in (0, 1]$, X_1 and X_2 are said to be *asymptotically dependent* (in the upper tail) and if $\lambda = 0$ they are *asymptotically independent* [19].

Analogously, the *coefficient of lower tail dependence* is

$$\lambda_L = \lim_{\alpha \to 0^+} P(X_2 < F_2^{-1}(\alpha) | X_1 < F_1^{-1}\alpha). \tag{3.33}$$

If F_1 and F_2 are continuous dfs, then we get simple expressions for λ_L and λ_U in terms of the unique copula C of the bivariate distribution. Using elementary conditional probability, we have the lower tail dependence;

$$\lambda_L = \lim_{\alpha \to 0^+} \frac{P[X_2 < F_2^{-1}(\alpha), X_1 < F_1^{-1}\alpha]}{P(X_1 < F_1^{-1}\alpha)} = \lim_{\alpha \to 0^+} \frac{C(\alpha, \alpha)}{\alpha} \tag{3.34}$$

For upper tail dependence we obtain;

$$\lambda_U = \lim_{\alpha \to 1^-} \frac{\hat{C}(1-\alpha, 1-\alpha)}{1-\alpha)} = \lim_{\alpha \to 0^+} \frac{\hat{C}(\alpha, \alpha)}{\alpha} \tag{3.35}$$

where $\hat{C}$ is the survival copula of C. For radially symmetric copulas we must have $\lambda_L = \lambda_U$, since $C = \hat{C}$ for such copulas [2].

The Gauss copula is asymptotically independent in both tails. To evaluate the tail-dependence coefficient for the Gauss copula C_Σ^{Gauss}, let $(X_1, X_2) := (\phi^{-1}(U_1), \phi^{-1}(U_2))$, so that (X_1, X_2) has a bivariate normal distribution with standard margins and correlation ρ.

$$\lambda = 2 \lim_{\alpha \to 0^+} P(\phi^{-1}(U_2) < \phi^{-1}(\alpha) | \phi^{-1}(U_1) < \phi^{-1}(\alpha)) = 2 \lim_{\alpha \to 0^+} P(X_2 < x | X_1 = x) \tag{3.36}$$

Using the fact that $X_2 | X_1 = x \sim N(\rho x, 1 - \rho^2)$, it can be calculated that

$$\lambda = 2 \lim_{\alpha \to -\infty} \Phi(x\sqrt{1-\rho}/\sqrt{1+\rho}) = 0 \tag{3.37}$$

provided $\rho < 1$. Regardless of how high a correlation we choose, if we go far enough into the tail, extreme events appear to occur independently in each margin.

To evaluate the tail dependence coefficient for the t-copula $C^t_{\upsilon,\rho}$, let $(X_1, X_2) := (t_\upsilon^{-1}(U_1), t_\upsilon^{-1}(U_2))$ where t_υ denotes the df of a univariate t distribution with υ degrees of freedom. Thus $(X_1, X_2) \sim t_2(\upsilon, 0, P)$, where P is a correlation matrix with off-diagonal element ρ. By calculating the conditional density from the joint and marginal densities of a bivariate t distribution, it may be verified that, conditional on $X_1 = x$,

$$\left(\frac{\upsilon+1}{\upsilon+x^2}\right)^{1/2} \frac{X_2 - \rho x}{\sqrt{1-\rho^2}} \sim t_{\upsilon+1} \tag{3.38}$$

Using an argument similar in the Gauss copula, we find that

$$\lambda = 2t_{\upsilon+1}\left(-\sqrt{\frac{(\upsilon+1)(1-\rho)}{1+\rho}}\right) \tag{3.39}$$

Provided that $\rho > -1$, the copula of the bivariate t-distribution is asymptotically dependent in both the upper and the lower tail [2].

3.3. Fitting Copulas to Data

Copulas represent a powerful tool for tackling the problem of how to describe a joint distribution by letting the researcher deal separately with the needs of marginal and joint distribution modeling. Thus, one can choose for each data series the marginal distribution that best fits the sample, and afterward put everything together using a copula function with some desirable properties [20].

We assume that we have data vectors $X_1, \ldots, X_n$ with identical distribution function F, describing financial risk factor returns; we write $X_t = (X_{t,1}, \ldots, X_{t,d})'$ for an individual data vector and $X = (X_1, \ldots, X_d)'$ for a generic random vector with df F. We assume further that this df F has continuous margins $F_1, \ldots, F_d$ and thus, by Sklar's Theorem, a unique representation $F(x) = C(F_1(x_1), \ldots, F_d(x_d))$ [2].

Finding a good multivariate model that describes both marginal behavior and dependence structure effectively is a difficult issue especially in higher dimensions. Since the copula approach to multivariate models facilitates this approach and allows us to consider the issue of whether tail dependence appears to be present in our data.

From a statistical point of view, a copula function is basically a very simple expression of a multivariate model and, as for most multivariate statistical models, much of the classical statistical inference theory is not applicable. There are various methods for estimating the parameters θ of a parametric copula C_θ. For instance, *method-of-moments procedure using sample rank correlation estimates* is a simple method which has the advantage that marginal distributions do not need to be estimated, and consequently inference about the copula is in a sense "margin-free". However, the main method that can be applied is *maximum likelihood estimation (MLE).*

3.3.1. Maximum Likelihood Method

While estimating margins and copula in one single optimization, splitting the modeling into two steps can yield more insight and allow a more detailed analysis of the different model components. In the first step, general approaches of estimating margins and constructing a pseudo-sample of observations from the copula will be described briefly.

Let $\widehat{F}_1, \ldots, \widehat{F}_d$ denote estimates of the marginal dfs. The pseudo-sample from the copula consists of the vectors $\widehat{U}_1, \ldots, \widehat{U}_n$ where

$$\widehat{U}_t = (U_{t,1}, \ldots, U_{t,d}) = (\widehat{F}_1(X_{t,1}), \ldots, \widehat{F}_d(X_{t,d}))' \tag{3.40}$$

Observe that, even if the original data vectors $X_1, \ldots, X_n$ are iid, the pseudo-sample data are generally dependent, because the marginal estimates $\widehat{F}_i$ will in most cases be constructed from all of the original data vectors through the univariate samples $X_{1,i}, \ldots, X_{n,i}$. Possible methods for obtaining the marginal estimate $\widehat{F}_i$ include the following.

(i) *Parametric estimation:* We choose an appropriate parametric model for the data in question and fit it by ML: for financial risk factor return data we might consider the generalized hyperbolic distribution, or one of its special cases such as Student t or normal inverse Gaussian. In our study, we have made the experiments using generalized hyperbolic and Student t distribution.

(ii) *Non-parametric estimation with variant of empirical df:* We could estimate F_j using

$$F^*_{i,n}(x) = \frac{1}{n+1} \sum_{t=1}^{n} I_{\{X_{t,i} \leq x\}} \tag{3.41}$$

Let C_θ denote a parametric copula, where θ is the vector of parameters to be estimated. The MLE is obtained by maximizing,

$$lnL(\theta; \widehat{U}_1, \ldots, \widehat{U}_n) = \sum_{t=1}^{n} lnc_\theta(\widehat{U}_t) \tag{3.42}$$

with respect to θ, where c_θ denotes the copula density and $\widehat{U}_t$ denotes a pseudo-observation from the copula.

Obviously the statistical quality of the estimates of the copula parameters depends very much on the quality of the estimates of the marginal distributions used in the formation of the pseudo-sample from the copula.

When margins are estimated parametrically, inference about the copula using (3.42) amounts to what has been termed the *inference-functions for margins (IFM).* When margins are estimated non-parametrically, the estimates of the copula parameters may be regarded as semi parametric and the approach has been labeled pseudo-maximum likelihood [2].

The MLE is generally found by numerical maximization of the resulting log-likelihood with respect to the parameters. The ML method could be very computationally intensive, especially in the case of a high dimension, because it is necessary to estimate jointly the parameters of the marginal distributions and the parameters of the dependence structure represented by the copula.

3.3.2. Inference Functions for Margins (IFM)

The log-likelihood function that is estimated is composed into two positive terms: one term involving the copula density and its parameters, and one term involving the margins and all parameters of the copula density. For that reason, Joe and Xu (1996) proposed that these set of parameters should be estimated in two steps:

(i) As a first step, they estimate the margins' parameters θ_1 by performing the estimation of the univariate marginal distributions:

$$\widehat{\theta_1} = ArgMax_{\theta_1} \sum_{t=1}^{T} \sum_{j=1}^{n} lnf_j(x_{jt}; \theta_1) \tag{3.43}$$

(ii) As a second step, given θ_1, they perform the estimation of the copula parameter θ_2.

$$\widehat{\theta_2} = ArgMax_{\theta_2} \sum_{t=1}^{T} lnC(F_1(x_{1t}), \ldots, F_n(x_{nt}); \theta_1, \widehat{\theta_1}) \tag{3.44}$$

This method is called inference for the margins or IFM. The IFM estimator is defined as the vector:

$$\widehat{\theta}_{IFM} = (\widehat{\theta}_1, \widehat{\theta}_2)' \tag{3.45}$$

We call l the entire log-likelihood function, l_j the log-likelihood of the jth marginal, and l_c the log-likelihood for the copula itself. Hence, the IFM estimator is the solution of:

$$(\frac{\partial l_1}{\partial \theta_{11}}, \frac{\partial l_2}{\partial \theta_{12}}, \ldots, \frac{\partial l_n}{\partial \theta_{1n}}, \frac{\partial l_c}{\partial \theta_2}) = 0' \tag{3.46}$$

while the MLE comes from solving

$$(\frac{\partial l}{\partial \theta_{11}}, \frac{\partial l}{\partial \theta_{12}}, \ldots, \frac{\partial l}{\partial \theta_{1n}}, \frac{\partial l}{\partial \theta_2}) = 0' \tag{3.47}$$

so, the equivalence of the two estimators, in general, does not hold. It is simple to see that the IFM estimator provides a good starting point for obtaining an exact MLE [20].

For estimating the normal and t copula, we can use the corresponding densities that we have explained before. In the case of the *Gauss copula*, the log-likelihood becomes;

$$lnL(\Sigma; \widehat{U}_1, \ldots, \widehat{U}_n) = \sum_{t=1}^{n} lnf_\Sigma(\Phi^{-1}(\widehat{U}_{t,1}), \ldots, \Phi^{-1}(\widehat{U}_{t,d})) - \sum_{t=1}^{n}\sum_{j=1}^{d} ln\phi(\Phi^{-1}(\widehat{U}_{t,j})), \tag{3.48}$$

where f_Σ is used to denote the joint density of a random vector with $N_d(0, \Sigma)$ distribution. It is clear that the second term is not relevant in the maximization with respect to P, and the MLE is given by

$$\widehat{P} = ArgMax_{\Sigma \in \rho} \sum_{t=1}^{n} lnf_\Sigma(Y_t) \tag{3.49}$$

where $Y_{t,j} = \Phi^{-1}(\widehat{U}_{t,j})$ for $j = 1, \ldots, d$ and ρ denotes the set of all possible linear correlation matrices. To perform this maximization in practice, we can search over the set of unrestricted lower-triangular matrices with ones on the diagonal. This search is feasible in low dimensions but very slow in high dimensions, since the number of parameters is $O(d^2)$. Instead of maximizing over ρ, we maximize over the set of all covariance matrices. This maximization problem has the analytical solution $\widehat{\Sigma} = (1/n)\Sigma_{t=1}^{n} Y_t Y_t'$, which is the MLE of the covariance matrix Σ for iid normal data with $N_d(0, \Sigma)$ distribution. In practice, $\widehat{\Sigma}$ is likely to be close to a correlation matrix. As an approximate solution to the original problem we could take the correlation matrix [2].

In the case of the t *copula*, the log-likelihood becomes;

$$lnL(v, P; \widehat{U}_1, \ldots, \widehat{U}_n) = \sum_{t=1}^{n} lng_{v,P}(t_v^{-1}(\widehat{U}_{t,1}), \ldots, t_v^{-1}(\widehat{U}_{t,d})) - \sum_{t=1}^{n}\sum_{j=1}^{d} lng_v(t_v^{-1}(\widehat{U}_{t,j})), \tag{3.50}$$

where $g_{v,P}$ denotes the joint density of a random vector with $t_d(v, 0, P)$ distribution, P is a linear correlation matrix, g_v is the density of a univariate $t_1(v, 0, 1)$ distribution, and t_v^{-1} is the corresponding quantile function.

In relatively low dimensions, we could search over the set of correlation matrices P and degrees of freedom parameter v for a global maximum. For higher dimensional cases it would be easier to estimate P using Kendall's tau estimates and to estimate the single parameter v by maximum likelihood.

4. MONTE CARLO SIMULATION IN FINANCE

Monte Carlo methods are used in finance to value and analyze instruments, portfolios and investments by simulating the various sources of uncertainty affecting their value, and then determining their average value over the range of resultant outcomes by the help of stochastic asset models [21]. This method helps us seeing all the possible outcomes of our decisions and assess the impact of risk, allowing for better decision making under uncertainty.

Monte Carlo methods are based on the analogy between probability and volume. The mathematics of measure formalizes the intuitive notion of probability, associating an event with a set of outcomes and defining the probability of the event to be its volume or measure relative to that of a universe of possible outcomes. Monte Carlo uses this identity in reverse, calculating the volume of a set by interpreting the volume as probability. In the simplest case, this means sampling randomly from a universe of possible outcomes and taking the fraction of random draws that fall in a given set as an estimate converges to the correct value as the number of draws increases. The central limit theorem provides information about the likely magnitude of the error in the estimate after a finite number of draws.

A small step takes us from volumes to integrals. Consider the problem of estimating the integral of a function f over the unit interval. We may represent the integral;

$$\alpha = \int_0^1 f(x)dx \tag{4.1}$$

as an expectation $E[f(U)]$ with U uniformly distributed between 0 and 1. Suppose we have $U_1, U_2, \ldots$ independently and uniformly from [0,1]. Evaluating the function f at

n of these random points and averaging the results produces the Monte Carlo estimate

$$\widehat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i) \tag{4.2}$$

If f is integrable over [0,1] then, by the strong law of large numbers, $\widehat{\alpha}_n \to \alpha$ with probability 1 as $n \to \infty$ then the error $\widehat{\alpha}_n - \alpha$ in the Monte Carlo estimate is approximately normally distributed with mean 0 and standard deviation $\sigma_f/\sqrt{n}$, the quality of this approximation is improving with increasing n. The parameter σ_f would typically be unknown in a setting in which α is unknown, but it can be estimated using the sample standard deviation

$$s_f = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(f(U_i) - \widehat{\alpha}_n)^2} \tag{4.3}$$

Hence, in this estimate, we obtain also a measure of the error from the functions $f(U_1), \ldots, f(U_n)$ [22].

4.1. Calculating VaR by Monte Carlo Simulation

Estimating loss probabilities and VaR by simulation is conceptually simple and can be illustrated by the following algorithm:

- For each of n independent replications
  - (i) generate a vector of ΔS
  - (ii) revalue portfolio and compute loss $V(S,t) - V(S+\Delta S, t+\Delta t)$
- Estimate $P(L > x)$ using

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1\{L_i > x\}} \tag{4.4}$$

where L_i is the loss on the ith replication.

The simulation algorithm above estimates the loss probabilities $P(L > x)$ rather than VaR [22].

Let $\widehat{F}_{L,n}$ denote the empirical distribution of portfolio losses based on n simulated replications,

$$\widehat{F}_{L,n}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{L_i \leq x\}. \tag{4.5}$$

A simple estimate of the VaR at probability p (e.g., $p = 0.01$) is the empirical quantile

$$\widehat{x}_p = \widehat{F}_{L,n}^{-1}(1-p) \tag{4.6}$$

with the inverse of constant function $\widehat{F}_{L,n}$. Applying piecewise linear interpolation to $\widehat{F}_{L,n}$ before taking the inverse generally produces more accurate quantile estimates.

Under minimal conditions, the empirical quantile $\widehat{x}_p$ converges to the true quantile x_p with probability 1 as $n \to \infty$ [22].

4.2. Simulation from Copulas

Simulation of random variables with particular marginals and various dependence structures is an important practical application of copulas. If we can generate a vector X with the df F, we can transform each component with its own marginal df to obtain a vector $U = (U_1, \ldots, U_d)' = (F_1(X_1), \ldots, F_d(X_d))'$ with df C, the copula of F [2]. We will focus on Gauss and t copula because they are the most widely known and applied copulas. The formulas of these copulas are not simple, but the generation is very easy.

Simulation Methods for Gauss Copula

(i) Simulate n independent standard normal random variables $z = (z_1, \ldots, z_n)$,

(ii) Find the Cholesky decomposition L of ρ such that, $\rho = LL^T$, where ρ is the correlation matrix and L is a lower triangular matrix,

(iii) Set $y = Lz$

(iv) Set $u_i = \Phi(y_i)$ with $i = 1, \ldots, n$ and where Φ denotes the univariate standard normal distribution function,

(v) $(u_1, \ldots, u_n)' = (F_1(x_1), \ldots, F_n(x_n))'$ where F_i denotes the ith margin.

Simulation Methods for t Copula

(i) Simulate n independent standard normal random variables $z = (z_1, \ldots, z_n)$,

(ii) Find the Cholesky decomposition L of ρ such that, $\rho = LL^T$, where ρ is the correlation matrix and L is a lower triangular matrix,

(iii) Set $y = Lz$

(iv) Simulate a random variate s, which is independent of z, from χ^2 with v degrees of freedom,

(v) Set $t = \frac{\sqrt{v}}{\sqrt{s}} y$

(vi) Set $u_i = T_v(t_i)$ with $i = 1, \ldots, n$ and where T_v denotes the univariate t distribution function with v degrees of freedom,

(vii) $(u_1, \ldots, u_n)' = (F_1(x_1), \ldots, F_n(x_n))'$ where F_i denotes the ith margin.

5. BACK TESTING VALUE-AT-RISK

Up to now, we have considered various methods for simulating risk measures at a time t for the distribution of losses in the next period. When this procedure is continually implemented over time we have the opportunity to monitor the performance of methods and compare their relative performance. This process of monitoring is known as *backtesting*.

Financial risk model evaluation or back testing is a key part of the internal model's approach to market risk management as laid out by the Basel Committee on Banking Supervision (1996). VaR models are useful only if they predict future risks accurately. In order to evaluate the quality of the estimates, the models should always be back tested with appropriate methods. Facts about back testing VaR is presented here following Nieppola, 2009.

Back testing is a statistical procedure where actual profits and losses are systematically compared to corresponding VaR estimates. For example, if the confidence level used for calculating daily VaR is 99%, we expect an exception to occur once in every 100 days on average.

In the back testing process we could statistically examine whether the frequency of exceptions over some specified time interval is in line with the selected confidence level. These types of tests are known as tests of *unconditional coverage*. However, a good VaR model not only produces the "correct" amount of exceptions but also exceptions that are evenly spread over time i.e. are independent of each other. Clustering of exceptions indicates that the model does not accurately capture the changes in market volatility and correlations. Tests of *conditional coverage* therefore examine also conditioning, or time variation, in the data. In this section, we will explain different methods for back testing a VaR model.

5.1. Unconditional Coverage

The most common test of a VaR model is to count the number of VaR exceptions, when portfolio losses exceed VaR estimates. If the number of exceptions is less than the selected confidence level would indicate, the system overestimates risk. On the contrary, too many exceptions signal underestimation of risk. Denoting the number of exceptions as x and the total number of observations as T, we may define the failure rate as x/T. If a confidence level of α is used, we have a null hypothesis that the frequency of tail losses is equal to $p = 1 - \alpha$. Assuming that the model is accurate, the observed failure rate x/T should act as an unbiased measure of p [23].

Each trading outcome either produces a VaR violation or not. The number of exceptions x follows a binomial probability distribution. As the number of observations increase, the binomial distribution can be approximated by a normal distribution. By utilizing this binomial distribution we can examine the accuracy of the VaR model. However, when making a statistical back test that either accepts or rejects a null hypothesis (of the model being "correct"), there is a trade off between two types of errors. Type 1 error refers to the possibility of rejecting a correct model and type 2 error to the possibility of not rejecting an incorrect model. A statistically powerful test would efficiently keep low both probabilities.

Figure 5.1 describes an accurate model, where $p = 1\%$. The probability of committing a type 1 error (rejecting a correct model), is 10.8%. On contrary, Figure 5.1 presents an inaccurate model, where $p = 3\%$. The probability for accepting an inaccurate model, i.e. committing a type 2 error is 12.8%.

5.1.1. Kupiec Tests

5.1.1.1. POF test. Kupiec's test, also known as the POF-test (proportion of failures), is the most widely known test based of failure rates has been suggested by Kupiec (1995). This test measures whether the number of exceptions is consistent with the

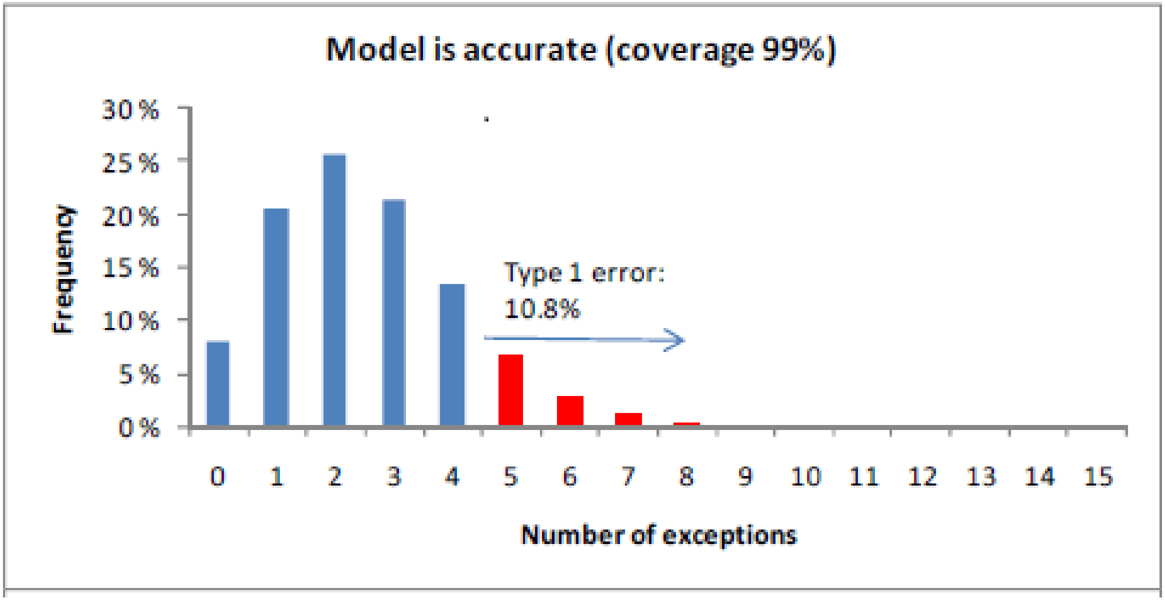

Figure 5.1. Type 1 Error.

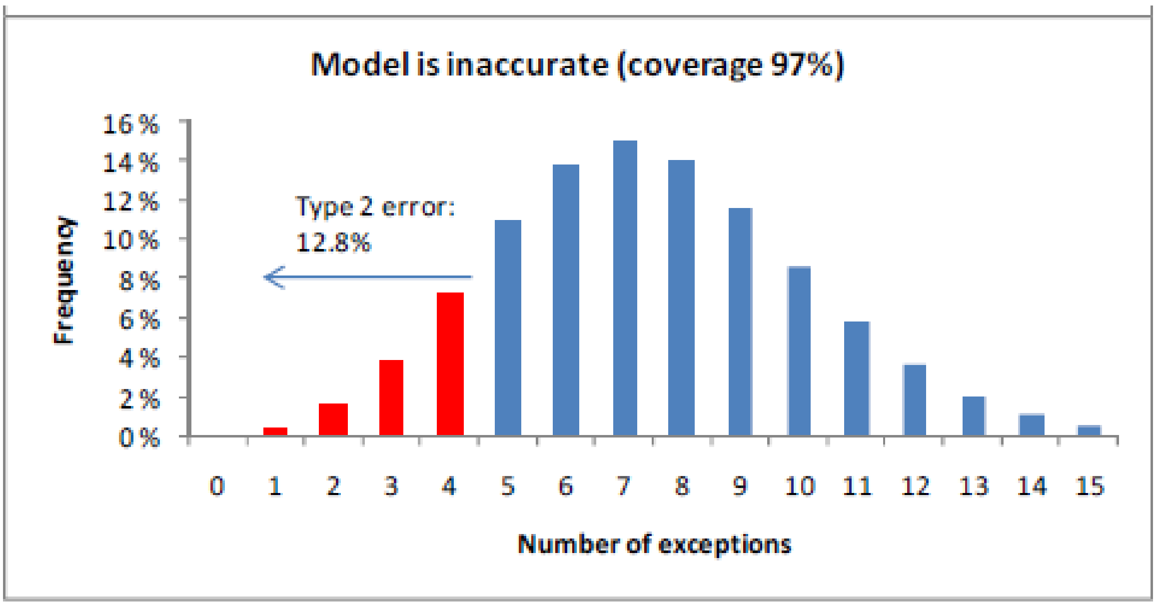

Figure 5.2. Type 2 Error.

confidence level.

Under null hypothesis that the model is "correct", the number of exceptions follows the binomial distribution. The only information required to implement a POF-test is the number of observations (T), number of exceptions (x) and the confidence level (c). The null hypothesis for the POF-test is

$$H_0 = p = \widehat{p} = \frac{x}{T} \tag{5.1}$$

The idea is to find out whether the observed failure rate $\widehat{p}$ is significantly different from p, the failure rate suggested by the confidence level. According to Kupiec (1995), the POF-test is best conducted as a likelihood-ratio (LR) test. The test statistic takes the form,

$$LR_{POF} = -2ln(\frac{(1-p)^{T-x}p^x}{[1-(\frac{x}{T})]^{T-x}(\frac{x}{T})^x}). \tag{5.2}$$

Under the null hypothesis that the model is correct, LR_{POF} is asymptotically χ^2 distributed with one degree of freedom. If the value of the LR_{POF} exceeds the critical value of the χ^2 (see Appendix 1 for the critical values), the null hypothesis will be rejected and the model is deemed as inaccurate. The confidence level for any test should be selected to balance between type 1 and type 2 errors. A level of this magnitude implies that the model will be rejected only if the evidence against it is fairly strong.

Kupiec's POF-test is hampered by two shortcomings. First, the test is statistically weak with sample sizes consistent with current regulatory framework. Secondly, POF-test considers only the frequency of losses and not the time when they occur. As a result, it may fail to reject a model that produces clustered exceptions.

5.1.1.2. TUFF test. Kupiec (1995) has also suggested another type of back test, namely the TUFF-test (time until first failure). This test measures the time t it takes for the first exception to occur and it is based on similar assumptions as the POF-test. The test statistic is the likelihood-ratio statistic;

$$LR_{TUFF} = -2ln(\frac{p(1-p)^{t-1}}{(\frac{1}{t})(1-\frac{1}{t})^{t-1}}) \tag{5.3}$$

LR_{TUFF} is also distributed as a χ^2 with one degree of freedom. If the test statistic falls below the critical value the model is accepted, and if not, the model is rejected. The problem with the TUFF-test is that the test has low power in identifying bad VaR models.

Due to the severe lack of power, there is hardly any reason to use TUFF-test in model back testing especially when there are more powerful methods available. The TUFF-test is best used only before the POF-test when there is no larger set of data available. The test also provides a useful framework for testing independence of exceptions in the mixed Kupiec's test by Haas (2001).

5.2. Conditional Coverage

The Basel framework and unconditional coverage tests focus only on the number of exceptions. In theory, however, we would expect these exceptions to be evenly spread over time. Good VaR models are capable of reacting to changing volatility and correlations in a way that exceptions occur independently of each other, whereas bad models tend to produce a sequence of consecutive exceptions [24]. Tests of conditional coverage try to not only examine the frequency of VaR violations but also the time when they occur.

5.2.1. Christoffersen's Interval Forecast Test

The most widely known test of conditional coverage has been proposed by Christoffersen (1998). He uses the same log-likelihood testing framework as Kupiec, but extends the test to include also a separate statistic for independence of exceptions. In addition to the correct rate of coverage, his test examines whether the probability of an exception on any day depends on the outcome of the previous day.

The test is carried out by first defining an indicator variable that gets a value of 1 if VaR is exceeded and value of 0 if VaR is not exceeded. Then define n_{ij} as the number of days when condition j occurred assuming that condition i occurred on the previous day. In addition, let π_i represent the probability of observing an exception conditional on state i on the previous day:

$$\pi_0 = \frac{n_{01}}{n_{00}+n_{01}}, \pi_1 = \frac{n_{11}}{n_{10}+n_{11}}, \pi = \frac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}} \tag{5.4}$$

If the model is accurate, then an exception today should not depend on whether or not an exception occurred on the previous day. In other words, under the null hypothesis the probabilities π_0 and π_1 should be the equal. The relevant test statistic for independence of exceptions is a likelihood-ratio:

$$LR_{ind} = -2ln(\frac{(1-\pi)^{n_{00}+n_{10}}\pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n}_{00}\pi_0^{n_{01}}(1-\pi_1)^{n}_{10}\pi^{n}_{1\,11}}) \tag{5.5}$$

By combining this independence statistic with Kupiec's POF-test we obtain a joint test that examines both properties of a good VaR model, the correct failure rate and independence of exceptions, i.e. conditional coverage:

$$LR_{cc} = LR_{POF} + LR_{ind} \tag{5.6}$$

LR_{cc} is also χ^2 distributed, but in this case with two degrees of freedom since there are two separate LR-statistics in the test. If the value of the LR_{cc} -statistic is lower than the critical value of the χ^2 distribution, the model passes the test. Higher values lead to rejection of the model.

5.2.2. Mixed Kupiec-Test

Christoffersen's interval forecast test is a useful back test in studying independence of VaR violations but unfortunately it is unable to capture dependence in all forms because it considers only the dependence of observations between two successive days. It is possible that the likelihood of VaR violation today does not depend on the violation yesterday but on a violation occurred, for instance, a week ago [25].

Haas (2001) argues that the interval forecast test by Christoffersen is too weak to produce feasible results and proposes a mixed Kupiec test which measures the time between exceptions instead of observing only whether an exception today depends on the outcome of the previous day. Thus, the test is potentially able to capture more general forms of dependence. The test statistic for each exception takes the form,

$$LR_i = -2ln(\frac{p(1-p)_i^t - 1}{(\frac{1}{t_i})(1-\frac{1}{t_i})^{t_i-1}}) \tag{5.7}$$

where t_i is the time between exceptions i and $i-1$. For the first exception the test statistic is computed as a normal TUFF-test. Having calculated the LR-statistics for each exception, we receive a test for independence where the null hypothesis is that the exceptions are independent from each other. With n exceptions, the test statistic for independence is

$$LR_{ind} = \sum_{i=2}^{n}[-2ln(\frac{p(1-p)_i^t - 1}{(\frac{1}{t_i})(1-\frac{1}{t_i})^{t_i-1}})] - 2ln(\frac{p(1-p)^{t-1}}{(\frac{1}{t})(1-\frac{1}{t})^{t-1}}) \tag{5.8}$$

which is a χ^2 distributed with n degrees of freedom. Similar to the Christoffersen's framework, the independence test can be combined with the POF-test to obtain a mixed test for independence and coverage, namely the mixed Kupiec test:

$$LR_{mix} = LR_{POF} + LR_{ind} \tag{5.9}$$

The LR_{mix} -statistic is χ^2 distributed with $n+1$ degrees of freedom. Just like with other likelihood-ratio tests, the statistic is compared to the critical values of χ^2 distribution. If the test statistic is lower, the model is accepted, and if not, the model is rejected.

6. RISK QUANTIFICATION PROBLEM

In our study, different copulas with various marginal distributions were fitted to a dataset of stock prices together with the commodity prices. Our main issue is to compare the fit of the models and monitor the performance of our model by implementing back testing to the stock portfolios especially when gold and crude oil are added.

A model is developed in order to estimate the VaR and CVaR which are the selected risk measures we selected for this study on various portfolios. The parameter estimation of this model was performed utilizing the two step estimation procedure. The estimated parameters were used for the portfolio risk calculation with Monte Carlo simulation. In the end, back testing was performed for checking the validity of this model. In the following sections, all these steps will be clarified.

6.1. Required Data

Regarding the financial world, gold and crude oil are considered as being the most attractive commodities. We therefore used them together with the stocks for constructing portfolios.

We used daily gold prices which are obtained from http://www.gold.org/, "World Gold Council", with the currency of US dollars and daily crude oil prices traded in West Texas Intermediate (WTI) are derived from "U.S. Energy Information Administration (EIA)", http://www.eia.gov/, with the currency of US dollars per barrel. The stock prices are traded in New York Stock Exchange (NYSE) obtained from http://finance.yahoo.com/ and the dataset includes the adjusted closing prices of 20 different stocks. All data were observed between 01/01/2000 – 31/12/2011, therefore, each stock and commodity consist of 3019 data points.

The stocks were selected from different industries in order to minimize the correlation between them and to construct a diversified portfolio. The selected stocks, their corresponding sectors and industries are given in Table 6.1.

Table 6.1. Stocks from NYSE.

Symbol	**Company Name**	**Sector**	**Industry**
AAPL	Apple Inc.	Technology	Personal Computers
ABT	Abbott Laboratories	Health Care	Drug Manufacturers - Major
BA	The Boeing Company	Industrial Goods	Aerospace - Defense Products
BBY	Best Buy Co. Inc.	Services	Electronics Stores
BP	BP plc	Basic Materials	Major Integrated Oil & Gas
C	Citigroup, Inc.	Financial	Money Center Banks
CMS	CMS Energy Corp.	Utilities	Electric Utilities
DIS	Walt Disney Co.	Services	Entertainment - Diversified
F	Ford Motor Co.	Consumer Goods	Auto Manufacturers - Major
GE	General Electric Company	Industrial Goods	Diversified Machinery
HD	The Home Depot, Inc.	Services	Home Improvement Stores
K	Kellogg Company	Consumer Goods	Processed & Packaged Goods
KO	The Coca-Cola Company	Consumer Goods	Beverages - Soft Drinks
MCD	McDonald's Corp.	Services	Restaurants
MMM	3M Co.	Conglomerates	Conglomerates
MO	Altria Group Inc.	Consumer Goods	Cigarettes
PG	Procter & Gamble Co.	Consumer Goods	Personal Products
TOL	Toll Brothers Inc.	Industrial Goods	Residential Construction
UNP	Union Pacific Corporation	Services	Railroads
WMT	Wal-Mart Stores Inc.	Services	Discount, Variety Stores

To be able to do a stochastic simulation, the daily log-returns are calculated using the daily adjusted closing prices. After transformation, we have 3018 daily log returns which are assumed to be independent identically distributed (iid) and follow a normal distribution with mean expectation μ_d and variance σ_d;

$$R_i = log(S_i) - log(S_i - 1), i = 1, \dots, 3018. \tag{6.1}$$

Simulated log returns can simply be summed to find the future price of a financial instrument.

6.2. Most Relevant Models

As we have mentioned, different models were fitted to the stock portfolios diversified with gold and crude oil. Computations of risk measures requires a realistic modeling of such distributions. The main classical and adequate model for risk calculation is the multinormal model especially for the lower level risks. However, it is not a good model because of its thin tails for more extreme levels. Copulas, which are presented in detail in Chapter 3, are better models for risk estimation because of their ability to model the dependence structure of multivariate variables in finance. Hence, we have a wide range of distributions for the marginals and very different structures for the dependence between them. From recent studies, we know that the most suitable fitting seems to be the Gaussian and t copula with t and GHD marginals. Thus, in this study, we will analyze the following models in detail:

- Multinormal Model
- Normal copula with t marginals
- t copula with t marginals
- t copula with GHD marginals

6.3. "Two Step" Estimation Procedure

For estimating the parameters, "IFM Method", introduced in Chapter 3.3, is implemented. The log-likelihood function is composed of copula density's parameters and margins' parameters . These parameters are estimated in two steps. As a first step, the margins parameters must be fitted by performing the estimation of univariate marginal distributions using MLE. In financial world, three kind of marginal distributions, normal, t and GHD, are used mostly. However, it is also known that financial data are far from normal distribution because of being fat tailed and having high kurtosis. In our study, we elaborated on these mentioned marginals for our dataset; especially gold and crude oil are examined because of the lack of statistical analysis in the literature.

Before going further, we considered the issue of testing whether the daily log returns of gold and crude oil are from a normal, t or generalized hyperbolic distribution. This can be assessed graphically with a quantile-quantile plot (Q-Q) against a chosen distribution which shows the relationship between empirical quantiles of the data and theoretical quantiles of a reference distribution. The Q-Q plots of the gold and crude oil's returns against normal, t and GHD are given in Appendix B.

From the Q-Q plots, we can observe that the inverted "S shaped curve of gold and crude oil points suggests that the empirical quantiles of the data tend to be larger than the corresponding quantiles of a normal distribution, indicating that the normal distribution is a poor model for these returns. On the other side, Q-Q plots indicate that the fitted t and GHD models are capable to explain the extreme returns in the tails. It is obvious that GHD can capture the high kurtosis and fat tails of gold and crude oil adequately.

The best fitting marginal distributions for stocks and commodities were determined by looking at the corresponding AIC values, which is a measure of the relative goodness of fit of a statistical model. As it can be noticed from the Table 6.2, the lowest AIC values were found for the t-distribution or the GHD for all stocks and commodities, namely they model the marginals much better than the normal distribution. In most cases both distributions can be used for fitting the marginals, since the t-distribution values are very close to GHD values.

Though normal distribution is not an adequate model for the selected stocks and commodities, the marginal parameters are estimated regarding the normal, t and GH distributions since the multinormal model is very popular and therefore also part of our study. The parameters of the normal and the t distributions were estimated by the help of the *fit all function* [26] implementing the maximum likelihood estimation and given

Table 6.2. AIC values for normal, t and GHD.

Stock	**Normal**($n_{par} = 2$)	**t**($n_{par} = 3$)	**GHD**($n_{par} = 5$)
AAPL	-12310.06	**-13206.71**	-13196.01
ABT	-16272.66	**-16819.70**	-16816.36
BA	-14699.82	**-15064.78**	-15064.22
BBY	-12225.75	**-13278.03**	-13274.31
BP	-15212.37	**-15905.18**	-15905.16
C	-11370.65	-13616.52	**-13629.31**
CMS	-14207.04	-15597.09	**-15598.95**
DIS	-14468.46	-15075.67	**-15075.83**
F	-12367.61	**-13189.06**	-13187.71
GE	-14515.65	-15352.23	**-15368.83**
HD	-14202.47	-14870.06	**-14876.10**
K	-16782.45	-17625.19	**-17633.57**
KO	-16846.61	-17589.16	**-17598.36**
MCD	-16132.50	**-16608.25**	-16607.44
MMM	-16267.61	-16800.95	**-16807.09**
MO	-15877.20	-16884.42	**-16884.73**
PG	-16583.80	**-17981.05**	-17977.89
TOL	-12868.49	-13049.59	**-13054.43**
UNP	-15124.39	-15514.36	**-15524.04**
WMT	-16095.67	-16689.54	**-16692.74**
GOLD	-18226.42	-18794.06	**-18907.18**
OIL	-13446.81	**-13848.38**	-13847.55

in Table 6.3 and Table 6.4 respectively. Regarding the "alpha/delta" parametrization, the parameters of the fitted GHD to the daily log returns are estimated using the *ghyp* package of the statistical software R and given in Table 6.5. All these parameters were estimated using the full 12 years data.

Considering the most appropriate marginal distribution, the data were transformed into uniform variates using the CDF transformation of the corresponding marginal distribution. Thus the first step of the IFM method is finished.

As second step, the parameters of the copula were estimated using the parameters that were found in the first step. The copulas are fitted to the transformed data by the help of *copula* package of the statistical software R and its built-in functions.

The correlation matrix, ρ, was estimated as a parameter of the normal copula for a given correlation matrix. In practice, the covariance matrix Σ is likely to be close to being a correlation matrix. As starting value for the correlation matrix, the "Pearson" correlation matrix is used since it is an approximate solution to the original problem.

As t copula parameters, correlation matrix and degrees of freedom were estimated. In relatively low dimensions, we search over the set of correlation matrices ρ and degrees of freedom parameter v for a global maximum. For higher dimensional work, it would be easier to estimate ρ using Kendall's tau or Spearman's rho estimates. In our study, Spearman's rho was used as rank correlation.

The copula parameters for the mixed portfolios and their log likelihood results will be discussed in the next chapter.

6.4. Quantifying VaR and CVaR

In this section, generating tail-loss probabilities and calculating portfolio risk by Monte Carlo Simulation will be described. Before going further, we have to point

Table 6.3. Fitted normal distributions to the stocks returns.

Stock	μ	σ
AAPL	0.000885	0.031460
ABT	0.000281	0.016317
BA	0.000278	0.021174
BBY	0.000004	0.031902
BP	0.000046	0.019451
C	-0.000783	0.036758
CMS	0.000022	0.022976
DIS	0.000119	0.022002
F	-0.000242	0.031161
GE	-0.000223	0.021830
HD	-0.000073	0.022993
K	0.000290	0.014996
KO	0.000166	0.014837
MCD	0.000393	0.016701
MMM	0.000276	0.016331
MO	0.000782	0.017422
PG	0.000166	0.015497
TOL	0.000495	0.028680
UNP	0.000595	0.019736
WMT	0.000018	0.016803
GOLD	0.000551	0.011805
OIL	0.000447	0.026060

Table 6.4. Fitted t distributions to the stocks returns.

Stock	μ	σ	**df**
AAPL	0.001060	0.020691	3.90
ABT	0.000258	0.011141	3.63
BA	0.000528	0.015447	4.13
BBY	0.000237	0.018879	3.04
BP	0.000529	0.012816	3.50
C	-0.000258	0.013769	1.78
CMS	0.000794	0.012166	2.64
DIS	-0.000057	0.014405	3.28
F	-0.000837	0.019616	3.25
GE	-0.000069	0.012455	2.54
HD	-0.000231	0.014702	3.16
K	0.000288	0.008619	2.59
KO	0.000238	0.008943	2.79
MCD	0.000512	0.011534	3.63
MMM	0.000276	0.010724	3.20
MO	0.001041	0.009882	2.67
PG	0.000314	0.008254	2.68
TOL	-0.000011	0.022971	5.43
UNP	0.000649	0.013712	3.53
WMT	-0.000104	0.010812	3.10
GOLD	0.000703	0.007575	3.04
OIL	0.001019	0.019012	4.22

Table 6.5. Fitted GHD to the stocks returns.

Stocks	λ	α	δ	β	μ
AAPL	-1.633161	11.14573	0.037181	-1.703187	0.001611
ABT	-1.606070	17.08683	0.019971	0.277913	0.000207
BA	-1.425549	27.04979	0.026923	-2.027053	0.001193
BBY	-1.499823	2.38044	0.032694	-0.529363	0.000508
BP	-1.651586	12.28029	0.023363	-3.398277	0.001509
C	-0.602828	8.52771	0.014492	-0.533197	-0.000091
CMS	-1.331923	2.72759	0.019891	-2.573416	0.001388
DIS	-1.086661	22.39618	0.021585	0.943293	-0.000323
F	-1.602705	3.26482	0.035139	1.456542	-0.001662
GE	-0.409664	29.99757	0.012550	-0.433088	-0.000018
HD	-0.486645	33.32347	0.016772	0.878580	-0.000518
K	-0.748361	32.44286	0.010743	0.078281	0.000272
KO	-0.599642	42.01769	0.010457	-0.755636	0.000337
MCD	-1.358695	27.46028	0.019261	-1.564898	0.000781
MMM	-0.582691	44.63036	0.013046	0.005270	0.000275
MO	-1.131092	15.13351	0.014771	-1.722396	0.001304
PG	-1.263029	9.21894	0.013051	-1.159886	0.000430
TOL	1.007387	58.94036	0.019060	4.067157	-0.002884
UNP	-0.325920	46.91598	0.015683	0.230980	0.000504
WMT	-0.955118	31.14005	0.015362	0.896576	-0.000238
GOLD	0.488401	76.21868	0.000000	-5.290373	0.000000
OIL	-1.652227	17.79451	0.035426	-2.190063	0.002192

out that we consider portfolios with d equally weighted financial instruments where d ranges from 2 to 12.

6.4.1. Generating Portfolio Return Using Naive Simulation

For one model, we assume that the log returns of d financial instruments over a time horizon T follow a normal copula with a correlation matrix $\rho \in \mathbb{R}^{d\times d}$ where L denotes the (lower triangular) Cholesky decomposition of ρ satisfying $LL' = \rho$.

To generate a random return vector from the normal copula, it is well known that we start with a vector Z of d iid. standard normal variates which is then transformed into correlated normal vector by $Y = LZ$. The log return vector $S = (S_1, S_2, \ldots, S_d)$ is the result of the component-wise transform

$$S_j = F_j^{-1}(\Phi(Y_j)) \tag{6.2}$$

where $\Phi(.)$ denotes the CDF of standard normal distribution and $F_j(.)$ the CDF of the marginal distribution of the return of the jth financial instrument.

The portfolio return is a function of the random input vector Z and random variate Y which depends on the fixed parameter ρ and on the CDFs of the marginal distributions. Then the return function is given by;

$$R(Z, Y) = \sum_{j=1}^{d} w_j e^{F_j^{-1}(\Phi(Y_j))}. \tag{6.3}$$

As another model, we assume that the log returns of d financial instruments over a time horizon T follow a t copula with v degrees of freedom and its dependence structure is described by the positive definite matrix ρ where L denotes the (lower triangular) Cholesky decomposition of ρ satisfying $LL' = \rho$.

For generating the random return vector from the t copula we again start with a vector Z of d iid. standard normal variates which is then transformed into the correlated normal vector by $Y = LZ$ as we have discussed while simulating log return vector using normal copula. We generate a random variate K from χ^2 with v degrees of freedom which is independent of Z. Thus we obtain, the vector T from the multivariate t distribution calculating $T = Y/\sqrt{K/v}$. The log-return vector $S = (S_1, S_2, \ldots, S_d)$ is then the result of the component-wise transform

$$S_j = F_j^{-1}(G_v(T_j)) \tag{6.4}$$

where $G_v(.)$ denotes the CDF of the t distribution with v degrees of freedom and $F_j(.)$ the CDF of the marginal distribution of the return of the jth financial instrument.

The portfolio return is a function of the random input vector Z and random variate T which depends on the fixed parameter ρ and v and on the CDFs of the marginal distributions. Then the return function is given as in Equation 6.5;

$$R(Z,T) = \sum_{j=1}^{d} w_j e^{F_j^{-1}(G_v(T_j))}. \tag{6.5}$$

At this point, we have to mention that the evaluation of the inverse CDFs $F^{-1}(.)$ of the marginals is a difficult numerical task. In order to simplify this task, the “Runuran” package (Leydold and Hörmann, 2008) [27] of the statistical software R is used. It is about 40 times faster than the built-in R function `qt()` and 10,000 times faster compared to the quantile function of the *ghyp* package.

6.4.2. Calculating VaR and CVaR

After generating portfolio return, the portfolio is revalued and its loss is computed;

$$L = V(S,t) - V(S + \Delta S, t + \Delta t). \tag{6.6}$$

In this manner, we can easily use L for estimating the VaR by calculating the "1-α" *quantile* of loss distribution. Thus we have the property;

$$P(L > VaR(1-\alpha)) = \alpha. \tag{6.7}$$

Likewise, we have computed the CVaR as,

$$CVaR(1-\alpha) = E(L|L > VaR(1-\alpha)). \tag{6.8}$$

To sum up; by implementing selected models, we calculated VaR and CVaR of the stock portfolios diversified with gold and crude oil using different α values for daily and weekly time horizons. This model is implemented with $n = 1000$ inner repetitions for generating d-dimensional log returns and with $m = 100$ outer repetitions for estimating the standard error of the estimated risk measures.

6.5. Back Testing

As we have mentioned before, back testing is one of the most important part of our study because it gives us the chance to monitor the correctness of our estimated risk measures. By the help of back testing, we can statistically examine whether the frequency of exceptions over a time horizon is in line with the selected confidence level.

In our study, we utilized *the proportion of failures (POF)* test which is the one of the *unconditional coverage* method of back testing. This method requires the total number of observations n, number of exceptions x and confidence level $1-\alpha$. We used various portfolios constructed of eleven years dataset for determining the number of exceptions by counting the portfolio losses that exceeded the VaR estimates. The daily and weekly VaRs were estimated and the comparison with the portfolio loss was done iteratively.

The null hypothesis for the POF test is

$$H_0 = p = \widehat{p} = \frac{x}{n}. \tag{6.9}$$

The test statistic which is conducted as a likelihood ratio (LR) test takes the form;

$$LR_{POF} = -2ln\frac{(1-p)^{n-x}p^x}{[1-(\frac{x}{n})]^{n-x}(\frac{x}{n})^x}. \tag{6.10}$$

Under the null hypothesis that the model is correct, LR_{POF} is asymptotically χ^2 distributed with one degree of freedom. If the value of the LR_{POF} exceeds the critical value of the χ^2, the null hypothesis will be rejected and the model is considered to be "inaccurate".

6.6. R Code Outputs of the Selected Models

The R code for the selected models consists of three main parts. In the first part of the code, the parameters of copula model with marginal parameters are estimated by the help of MLE using the function `ParameterEst`().

The `ParameterEst`() function includes four different parts which estimate the parameters of the normal copula with normal marginals, the normal copula with t marginals, the t copula with t marginals and the t copula with GHD marginals. After estimating the marginal parameters, the copula parameters are obtained as explained in Chapter 6.3.. In Appendix C, R code for fitting t-copula with t-distributed marginals are given as an example of the parameter estimation part of the copula model.

In the second part of the code, we calculate the VaR and CVaR of the given equally weighted portfolio using the selected copula model for different α values and time horizons with `risk`() command. For calculating the risk of the portfolio, we generate log returns using the estimated parameters in the first part of the model. An R example of calculating risk measures is given in Appendix C.

As the last part of our code, daily back testing is implemented using the command `Backtesting`(). The daily back testing code is given in Appendix C.

6.6.1. Use of the R Codes

Before talking about the outputs of the model, we will present the inputs of the `ParameterEst`(`data`, `copula`, `marginals`) function. `data` is assumed to be log returns of the portfolio, `copula` is the copula type and `marginals` is the marginal type of the selected model. Here, it is important to choose the copula and marginals type regarding the desired models.

The outputs of the function depends on the chosen model. For t marginals, we estimate *location*, *scale* and *degrees of freedom* parameters, while for ghyp marginals, we estimate *lambda*, *alpha*, *delta*, *beta* and *mu* parameters. The estimated parameter of the normal copula is *correlation matrix* where as for the t copula parameters, we also estimate the *degrees of freedom* of the copula together with *correlation matrix*.

As an example, we import the 12 years stock data of “Apple Inc.” and “Abbott Laboratories” and call the `ParameterEst`() for “t copula - t marginals” model. The results are as in Table 6.6:

Table 6.6. Parameter Estimations of 2 Stocks Portfolio.

Marginal Parameters		
	aapl	abt
location	0.00106	0.00026
scale	0.02069	0.01114
degrees of freedom	3.89522	3.63113
Copula Parameters		
correlation matrix	aapl	abt
	1.00000	0.20552
	0.20552	1.00000
degrees of freedom	4.306492	

For calculating the portfolio risk; we use `n` as the inner repetitions, `m` as the outer repetitions, `w` as the fractions of the total value of the portfolio invested into financial instruments, `alpha` as the quantile values and `T` as the time horizon. It is important to use the time horizon correctly. For estimating the daily risk, we use $T = 1/258$.

As outputs, we obtain VaR and CVaR for various α values. If we want to estimate the standard errors of the corresponding risk measures, we call the function using `risk(data, n, m, w, alpha, T, copula, marginals, error = TRUE)`. The default command for “error” is `FALSE`.

The results for `risk(data, n, m, w, alpha, T, copula, marginals, error = TRUE)` is given in Table 6.7.

The `BackTesting(data, n, m, w, alpha, T, copula, marginals)` function is used for implementing the back testing with the inputs that were mentioned before. We can see whether the selected model is accurate or inaccurate as in Table 6.8.

Table 6.7. VaR and CVaR Results of 2 Stocks Portfolio.

	0.999	**0.990**	**0.950**	**0.900**
VaR	0.07808	0.04627	0.02614	0.01879
ES	0.10090	0.06287	0.03943	0.03076
	0.999	**0.990**	**0.950**	**0.900**
VaR Error	0.00266	0.00076	0.00024	0.00016
ES Error	0.00596	0.00146	0.00049	0.00030

Table 6.8. Daily Back Testing Results of 2 Stocks Portfolio.

α	**Accuracy**
0.999	the model is accurate
0.990	the model is accurate
0.950	the model is accurate
0.900	the model is accurate

7. RESULTS FOR COPULA FITTING

In this section, the results of our study will be given sequentially. Firstly, we will talk about the selected portfolios and their dependence structures. In the second part, we will compare the fits of different models to the stock portfolios with gold and crude oil. Then, back testing results will be given. In the last part, we will discuss whether adding gold and crude oil affects the portfolio risk.

7.1. Selected Portfolios and Their Dependence Structure

The correlation matrix of the daily log returns for 12 years is given in Tables 7.1 and 7.2 for analyzing the affects of the dependence between stocks and commodities to the risk estimation. From the correlation matrix, we can see that correlations exists between stocks even if they belong to different industries. In this study, we will elaborate on the data for 2010 - 2011 period, thus the correlation matrix of the daily log returns for last two years period is also given in Appendix D.

More to the point, we can determine that the minimum correlation exists between gold and the other stocks. Analyzing the values, we can say that gold is uncorrelated with stocks and even has a negative correlation with some stocks. The correlation between crude oil and the stocks is relatively low with respect to the correlations of the stocks between each other.

We can examine that the minimum correlation is -0.096 between GOLD and WMT whereas the maximum is 0.583 between C and GE.

For assessing the stability, we compared the correlations of all eleven years and that of the last two years. The most considerable point is the dependence between stocks and crude oil. Regarding the last two years data, the minimum correlation between crude oil and stocks is 0.151 and the maximum is 0.426, while for the eleven

Table 7.1. Correlation Matrix of the Daily Log-Returns for 12 Years.

	AAPL	ABT	BA	BBY	BP	C	CMS	DIS	F	GE	HD
AAPL	1.00	0.13	0.25	0.29	0.23	0.28	0.18	0.32	0.25	0.36	0.28
ABT	0.13	1.00	0.31	0.17	0.27	0.27	0.30	0.27	0.21	0.32	0.25
BA	0.25	0.31	1.00	0.31	0.38	0.39	0.30	0.46	0.37	0.49	0.42
BBY	0.29	0.17	0.31	1.00	0.22	0.33	0.20	0.40	0.32	0.39	0.52
BP	0.23	0.27	0.38	0.22	1.00	0.36	0.32	0.38	0.29	0.42	0.31
C	0.28	0.27	0.39	0.33	0.36	1.00	0.29	0.42	0.41	0.58	0.42
CMS	0.18	0.30	0.30	0.20	0.32	0.29	1.00	0.33	0.28	0.32	0.26
DIS	0.32	0.27	0.46	0.40	0.38	0.42	0.33	1.00	0.39	0.53	0.46
F	0.25	0.21	0.37	0.32	0.29	0.41	0.28	0.39	1.00	0.44	0.37
GE	0.36	0.32	0.49	0.39	0.42	0.58	0.32	0.53	0.44	1.00	0.50
HD	0.28	0.25	0.42	0.52	0.31	0.42	0.26	0.46	0.37	0.50	1.00
K	0.12	0.34	0.27	0.20	0.26	0.25	0.26	0.28	0.22	0.32	0.25
KO	0.18	0.33	0.33	0.19	0.31	0.27	0.27	0.32	0.26	0.34	0.32
MCD	0.20	0.25	0.34	0.28	0.28	0.27	0.23	0.35	0.27	0.35	0.37
MMM	0.29	0.34	0.48	0.36	0.42	0.42	0.31	0.47	0.39	0.56	0.46
MO	0.12	0.26	0.25	0.09	0.27	0.20	0.23	0.24	0.20	0.25	0.21
PG	0.13	0.35	0.30	0.19	0.23	0.27	0.23	0.27	0.23	0.34	0.27
TOL	0.28	0.21	0.38	0.37	0.32	0.44	0.26	0.40	0.36	0.46	0.49
UNP	0.28	0.26	0.45	0.34	0.41	0.42	0.28	0.44	0.39	0.48	0.42
WMT	0.24	0.29	0.35	0.39	0.25	0.30	0.21	0.36	0.27	0.41	0.57
GOLD	-0.02	-0.03	-0.07	-0.05	0.07	-0.05	-0.01	-0.06	-0.06	-0.04	-0.08
OIL	0.07	-0.01	0.12	0.04	0.28	0.11	0.09	0.12	0.08	0.10	0.01

Table 7.2. Correlation Matrix of the Daily Log-Returns for 12 Years Data (cont.).

	K	KO	MCD	MMM	MO	PG	TOL	UNP	WMT	GOLD	OIL
AAPL	0.12	0.18	0.20	0.29	0.12	0.13	0.28	0.28	0.24	-0.02	0.07
ABT	0.34	0.33	0.25	0.34	0.26	0.35	0.21	0.26	0.29	-0.03	-0.01
BA	0.27	0.33	0.34	0.48	0.25	0.30	0.38	0.45	0.35	-0.07	0.12
BBY	0.20	0.19	0.28	0.36	0.09	0.19	0.37	0.34	0.39	-0.05	0.04
BP	0.26	0.31	0.28	0.42	0.27	0.23	0.32	0.41	0.25	0.07	0.28
C	0.25	0.27	0.27	0.42	0.20	0.27	0.44	0.42	0.30	-0.05	0.11
CMS	0.26	0.27	0.23	0.31	0.23	0.23	0.26	0.28	0.21	-0.01	0.09
DIS	0.28	0.32	0.35	0.47	0.24	0.27	0.40	0.44	0.36	-0.06	0.12
F	0.22	0.26	0.27	0.39	0.20	0.23	0.36	0.39	0.27	-0.06	0.08
GE	0.32	0.34	0.35	0.56	0.25	0.34	0.46	0.48	0.41	-0.04	0.10
HD	0.25	0.32	0.37	0.46	0.21	0.27	0.49	0.42	0.57	-0.08	0.01
K	1.00	0.41	0.26	0.35	0.30	0.37	0.19	0.28	0.29	-0.04	0.04
KO	0.41	1.00	0.32	0.40	0.30	0.42	0.23	0.29	0.33	-0.01	0.06
MCD	0.26	0.32	1.00	0.35	0.26	0.32	0.31	0.32	0.35	-0.06	0.04
MMM	0.35	0.40	0.35	1.00	0.31	0.41	0.42	0.49	0.42	-0.07	0.08
MO	0.30	0.30	0.26	0.31	1.00	0.30	0.21	0.23	0.22	-0.03	0.04
PG	0.37	0.42	0.32	0.41	0.30	1.00	0.22	0.29	0.34	-0.05	0.03
TOL	0.19	0.23	0.31	0.42	0.21	0.22	1.00	0.43	0.34	-0.04	0.07
UNP	0.28	0.29	0.32	0.49	0.23	0.29	0.43	1.00	0.34	-0.01	0.12
WMT	0.29	0.33	0.35	0.42	0.22	0.34	0.34	0.34	1.00	-0.10	-0.05
GOLD	-0.04	-0.01	-0.06	-0.07	-0.03	-0.05	-0.04	-0.01	-0.10	1.00	0.15
OIL	0.04	0.06	0.04	0.08	0.04	0.03	0.07	0.12	-0.05	0.15	1.00

years data, the minimum correlation is -0.046 and the maximum is 0.280. This means that in the last years, crude oil has lost the property of diversifying the portfolio considering its higher dependence between other stocks.

At this point, we considered whether 2008 economic crisis affects the crude oil prices and checked the dependence of crude oil between other financial instruments for two years period before and after the year 2008. We can see from Table 7.3 that negative correlation exists between crude oil and other stocks before year 2008. After the crisis dependence has increased considerably. However, the dependence between gold and crude oil has a different behavior. We can almost say that crude oil and gold are independent during 2002 - 2003 period where as the dependence increased to 0.210 during the period 2005 - 2006 and reached 0.280 in the period 2009 - 2010. Since the standard deviations have not changed during these periods, the affect of crude oil to the portfolio risk depends on the correlation between other stocks and gold. In the following sections, we will also analyze the risk of portfolios diversified with crude oil for different periods.

Firstly, the risk calculation was performed for five portfolios consisting of different stocks. These portfolio are selected considering medium wealth investors. They consist of a small number of stocks as this reduces the transaction cost. For analyzing the affects of gold and crude oil, the portfolios are then diversified by adding gold and crude oil separately and by adding gold and crude oil together to the mentioned portfolios.

(i) AAPL - ABT
(ii) BBY - KO - TOL
(iii) BA - K - PG - UNP
(iv) C - CMS - F - MO - WMT
(v) AAPL - BP - DIS - GE - HD - K - MCD - MMM - PG - TOL

Table 7.3. Dependence Between Crude Oil and Other Financial Instruments for Different Periods.

	2002 - 2003	**2005 - 2006**	**2009 - 2010**
AAPL	-0.064	0.009	0.289
ABT	-0.092	-0.163	0.202
BA	-0.069	-0.064	0.410
BBY	0.030	-0.113	0.254
BP	-0.001	0.423	0.309
C	-0.069	-0.111	0.206
CMS	-0.028	0.048	0.376
DIS	-0.038	-0.104	0.383
F	-0.024	-0.109	0.254
GE	-0.043	-0.205	0.310
HD	-0.074	-0.126	0.314
K	0.016	-0.084	0.212
KO	0.011	-0.085	0.229
MCD	-0.086	-0.059	0.244
MMM	-0.074	-0.022	0.366
MO	-0.007	-0.058	0.150
PG	-0.065	-0.125	0.258
TOL	-0.061	0.053	0.293
UNP	-0.096	-0.050	0.403
WMT	-0.123	-0.148	0.102
GOLD	0.082	0.210	0.228

7.2. Comparing the Fits of Different Models

The estimated copula parameters, corresponding log likelihood and AIC values for 20 portfolios using 2010 - 2011 period data of the mentioned portfolios are given in Appendix E. The lowest AIC values can be observed for t copula with t marginals model. Nevertheless, the AIC values of t copula with t marginals and t copula with GHD marginals are not far from each other, thus it is appropriate to use either of them.

Adding stocks to the portfolio increases the degrees of freedom of the t copula, namely, the distribution of the portfolios approximates to normal distribution. In this manner, the AIC values of normal copula approaches the AIC values of the t copula, but still the t copula with t and GHD marginals fits better.

After adding gold to the stock portfolios, we examined that the estimated degrees of freedom of t copula decreased in comparison with the estimated degrees of freedom of the portfolios with 10 stocks. However, crude oil has no decreasing effect on the degrees of freedom of the mentioned stock portfolios.

7.3. Back Testing Results

We implemented daily back testing for each portfolio and weekly back testing for two stocks portfolio and two stocks portfolio diversified with gold and crude oil using 20 different α values in order to check the validity of the selected models. The main point is carrying out the back testing method for stock portfolios diversified with gold and crude oil since these are not considered in the literature.

The results show us that the models that we have constructed are accurate. We have also realized that the t copula with t and GHD marginals are the most accurate models among selected models. From the results of the sum of the exceptions that are given in Appendix F, we can conclude that the multivariate normal model is not an accurate model comparing with the copula models.

7.4. Comments on Portfolio Risks

We estimated the VaR and CVaR for 2012-01-01 using two years data with 20 different α values for daily and weekly time horizons. From the results which are given in Appendix G, we can observe whether gold and crude oil have any affect on the risk of stock portfolios.

For all selected portfolios, gold has a decreasing effect on the risk while crude oil has no notable affect. This means that adding gold to the stock portfolios decreases the risk. On the other hand, crude oil behaves much like an ordinary stock. However, diversifying stock portfolios with both gold and crude oil decreases the risk by the help of gold.

All the risk calculations were done using 2010 - 2011 period. Nevertheless, at this point, we have also calculated the risk of the portfolios for 2004-01-01 and 2007-01-01 for analyzing the affect of dependence structure between stocks and commodities. For these periods, we can observe a serious decrease of the risk by adding each gold and crude oil to the stock portfolios. But the most impressing decrease is provided by diversifying stock portfolios with gold and crude oil at the same time.

8. CONCLUSION

In this study, we considered to construct the multivariate return distributions of stock portfolios using copula model and to evaluate this model in practice for risk measuring. Moreover, we used the copula model to observe its suitability for the multivariate return distributions of portfolios constructed of stocks, gold and crude oil and to analyze whether adding these financial instruments to stock portfolios reduces the risk.

Since the empirical distributions of asset returns have fatter tails and higher kurtosis than the normal distribution, the usual Monte Carlo method which generates asset returns assuming that the risk factors follow a multivariate normal distribution is not realistic. Besides, the dependence between the asset returns is assumed to be linear in the multinormal model and it does not take into account tail dependence. The variance-covariance method is a weak method as assuming the normality and linearization which may not always offer a good approximation of the relationship between the true loss distribution and the risk-factor changes. Historical simulation has the serious absence that there are not enough data in the empirical tails. To estimate an accurate risk, these clear drawbacks of classical approaches can be solved considering the copula method.

Copulas are extremely useful concepts which tackle the problem of specification of marginal univariate distributions separately and represent the dependence structure between them. Copulas help in the understanding of dependence at a deeper level and facilitate a bottom-up approach to multivariate model building where we very often have a much better idea about the marginal behavior of individual risk factors than we do about their dependence structure. In the first part of our study, we fitted different copulas to our dataset consisting of daily stock returns from NYSE to model the return distributions of stock portfolios. Looking at the Q-Q plots, we found that the fitted t and GHD models are also capable to explain the extreme returns in the tails of gold and crude oil. The portfolios of two, three, four, five and ten stocks are constructed considering medium wealth investors and the portfolios are then diversified by adding gold and crude oil separately and by adding gold and crude oil together to the portfolios using the dataset of 2010 - 2011 period. Among different models, the t-copula is found to be the best fitting copula according to the log-likelihood and AIC values. Thus it is appropriate to use the t-copula with t and GHD marginals for representing the portfolio return distributions.

We implemented daily and weekly back testing for each portfolio using 20 different α values in order to check the validity of the constructed models. From the results we observed that the models are accurate for stock portfolios and also for stock portfolios diversified with gold and crude oil. Especially, looking at the results of the sum of the exceptions, we can say that the multivariate normal model is not an accurate model comparing with the copula models. As we have concluded before, the t copula model with t and GHD marginals are the most accurate model.

We simulated price return scenarios for the mentioned portfolios with d equally weighted financial instruments from the fitted copulas with Monte Carlo method and calculated the VaR and CVaR at 20 different per cent level ranges from 99.00% to 99.90% for daily and weekly time horizons. For all portfolios, we observed the effects of gold and crude oil to the risk measure regarding 2010 - 2011 period. Adding gold to the stock portfolios decreases the risk where crude oil behaves like an ordinary stock. However, diversifying stock portfolios with both gold and crude oil decreases the risk by the help of gold.

This study can be enjoyed in several ways. First of all, we learned that the multivariate return distributions of stock portfolios and stock portfolios diversified with gold and crude oil can be modeled adequately with t copula approach for risk measure evaluations in practice. The validity of this model is also checked by back testing method. On the other hand, we observed that the gold can be used for diversifying the stock portfolios regarding the reducing effect on the risk measure.

APPENDIX A: CHI-SQUARE DISTRIBUTION TABLE

Table A.1. Critical Values for the Chi-Squared Distribution.

	α									
v	**0.995**	**0.990**	**0.975**	**0.950**	**0.900**	**0.100**	**0.050**	**0.025**	**0.010**	**0.005**
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.647	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.041	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.170
150	109.142	112.668	117.985	122.692	128.275	172.581	179.581	185.800	193.207	198.360
200	152.241	156.432	162.728	168.279	174.835	226.021	233.994	241.058	249.445	255.264

APPENDIX B: Q-Q PLOTS OF DAILY LOG-RETURNS WITH THE FITTED T DISTRIBUTION AND GHD

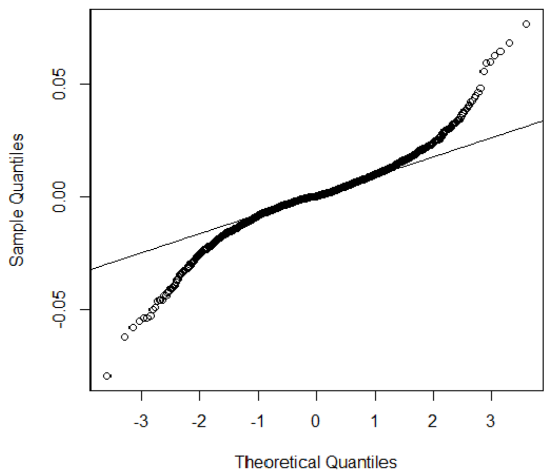

Figure B.1. Q-Q plot against normal distribution for the log-returns of GOLD.

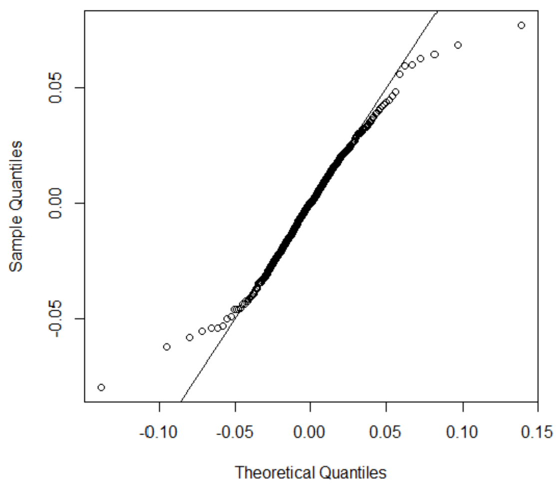

Figure B.2. Q-Q plot against t distribution for the log-returns of GOLD.

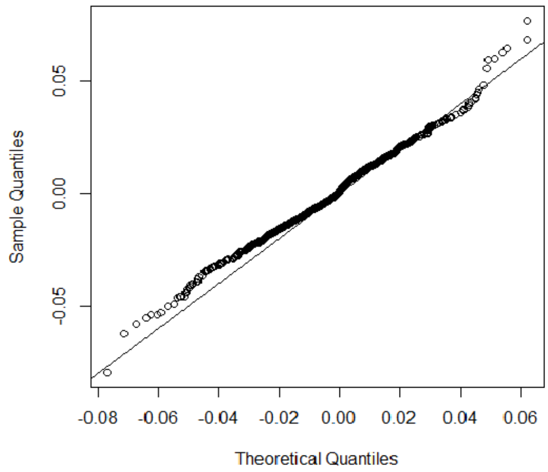

Figure B.3. Q-Q plot against GHD for the log-returns of GOLD.

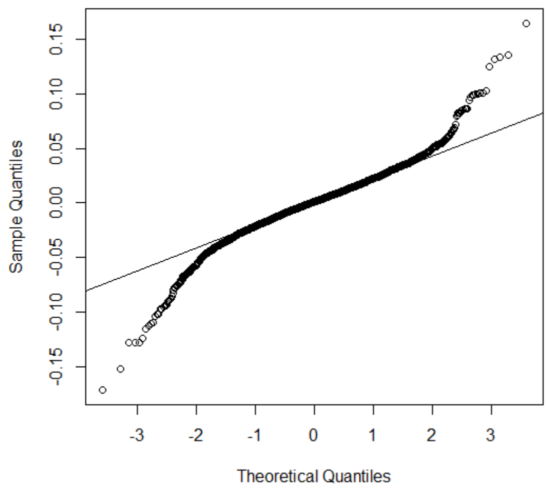

Figure B.4. Q-Q plot against normal distribution for the log-returns of CRUDE OIL.

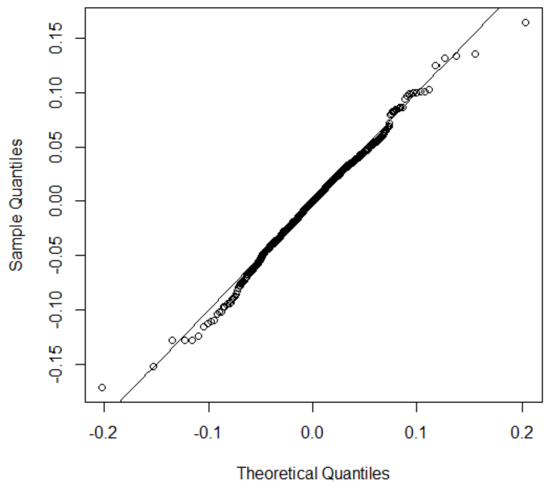

Figure B.5. Q-Q plot against t distribution for the log-returns of CRUDE OIL.

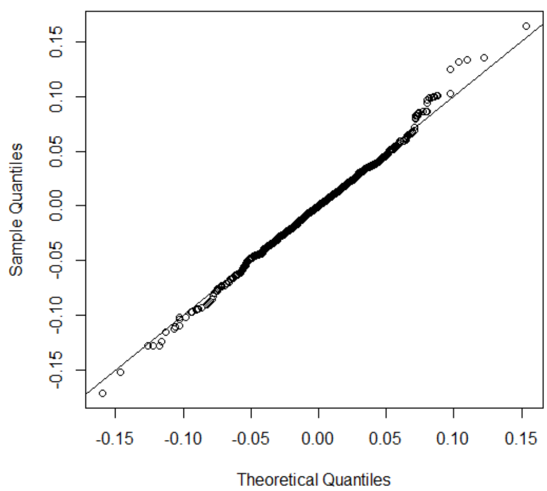

Figure B.6. Q-Q plot against GHD for the log-returns of CRUDE OIL.

APPENDIX C: R CODES

Fitting t-distributed marginals:

```
for(i in 1:dim){
# estimates the parameters of marginals using MLEt function
# assigns the estimated parameters as "tpar"
# obtains the uniform marginals by taking the CDF of t distributed marginals
tpar[i,1] <- MLEt(data[,i],npar=3)[2]
tpar[i,2] <- MLEt(data[,i],npar=3)[3]
tpar[i,3] <- MLEt(data[,i],npar=3)[4]
# density of t distribution with location and scale parameters
cdf[,i] <- pt3(data[,i],tpar[i,1],tpar[i,2],tpar[i,3])
}
```

Fitting t copula:

```
findMLEtcopula <- function (cdf){
# estimates the parameters of t-copula
# starting value of the correlation matrix is estimated by "Spearman" method
# cdf ... the cumulative distribution function of the marginals

dim <- length(cdf[1,])
R <- cor(cdf,method="spearman")

# assigning the row-wise elements of correlation matrix
# as parameter to be used in copula function
par <- cormatrix2vector(R,dim)

# constructs a t copula object
# and fits the copula parameters using maximum likelihood estimation
t.cop <- tCopula(par,dim=dim,dispstr="un",df=4)
fit.ml <- fitCopula(t.cop, cdf, method="ml", start=c(rep(0,length(par)),dim))
estimates <- fit.ml@estimate
}
```

Simulating d dimensional log-returns using t copula with t marginals:

```
rtcopulat <- function(n, parameters) {
# generates t copula with t marginals using correlation matrix and dof of
# t copula and dof, location and scale parameters of the marginals
# returns a d*n matrix holding the random vectors where d is the portfolio size
# and n is the sample size
# calculates the quantile function of the uniform distribution using Runuran
# package in order to have a fast inversion algorithm
# n...  sample size
# data ... daily log-returns of assets

# generating t-distributed vector
# see http://en.wikipedia.org/wiki/Student's_t-distribution
L <- t(chol(parameters$R))
z <- matrix(rnorm(n*parameters$dim),ncol=n,nrow=parameters$dim)
Z <- L%*%z
chi <- rchisq(n,parameters$dof)
tdistvec <- t(Z)/sqrt(chi/parameters$dof)
cdf <- pt(tdistvec,df=parameters$dof)
res <- matrix(nrow=n,ncol=parameters$dim)
for(i in 1:parameters$dim){
obj <- udt(df=parameters$df[i])
gen <- pinvd.new(obj)
res[,i] <- uq(gen,cdf[,i])*parameters$scale[i] + parameters$location[i]
}
res
}
```

Calculating the VaR and CVaR of the given portfolio:

```
risk <- function(data, n=10^3, m=100, w, alpha, T, copula=c("normal","t"),
marginals=c("normal","t","ghyp"),error=FALSE){
# estimates the VaR and ES of the given data using chosen copula and marginals
# data ... daily log-returns of financial instruments
# n ... inner repetitions
```

```
# m ... outer repetitions
# w ... weights of financial instruments
# alpha ... confidence level
# T ... time horizon
# copula ... copula type
# marginals ... marginals type

VaR <- matrix(nrow=m,ncol=length(alpha))
ES <- matrix(nrow=m,ncol=length(alpha))
TH <- round(258*T)
parameters <- ParameterEst(data,copula,marginals)

for(i in 1:m){
logreturnmatrix <- 0

if (copula=="normal"  && marginals=="normal" )
for(j in 1:TH) logreturnmatrix<-logreturnmatrix+rnormalcopulanormal(n,parameters)
if (copula=="normal"  && marginals=="t" )
4for(j in 1:TH) logreturnmatrix<-logreturnmatrix+rnormalcopulat(n,parameters)
if (copula=="t"  && marginals=="t" )
4 for(j in 1:TH) logreturnmatrix<-logreturnmatrix+rtcopulat(n,parameters)
if (copula=="t"  && marginals=="ghyp" )
for(j in 1:TH) logreturnmatrix<-logreturnmatrix+tcopulaGhyp(n,parameters)

loss <- 1-exp(logreturnmatrix)%*%w
for(k in 1:length(alpha)){
VaR[i,k] <- quantile(loss,1-alpha[k])
ES[i,k] <- mean(loss[loss>VaR[i,k]])
}
}

res <- matrix(ncol=length(alpha),nrow=2,
dimnames=list(c("VaR","ES"),c((1-alpha))))
er <- matrix(ncol=length(alpha),nrow=2,
dimnames=list(c("VaR Error","ES Error"),c((1-alpha))))

for(j in 1:length(alpha)){
```

```
res[1,j] <- mean(VaR[,j])
res[2,j] <- mean(ES[,j])
er[1,j] <- qnorm(0.95)*sd(VaR[,j])/sqrt(m)
er[2,j] <- qnorm(0.95)*sd(ES[,j])/sqrt(m)
}

print(res)
if(error==TRUE) print(er)
}
```

Daily back testing:

```
Backtesting <- function(data, n=10^3, m=2, w,alpha ,T,copula=c("normal","t"),
marginals=c("normal","t","ghyp")){
# daily backtesting for being sure of the accuracy of our VaR model
# data ... daily log-returns of assets
# n...  inner repetitions
# m ... outer repetitions for error calculations
# w ... weight vector
# alpha ... confidence level

data <- as.matrix(data)
d <- round((length(data[,1])-500))
# number of days that will be checked
sum <- matrix(0,ncol=length(alpha))
LR <- matrix(0,ncol=length(alpha))
# likelihood ratio for checking the validity of the method
p <- matrix(0,ncol=length(alpha))
chisq <- matrix(0,ncol=length(alpha))

for(j in 1:length(alpha)){
chisq[,j] <- qchisq(1-alpha[j],df=1)
}

LossData <- 1 - exp(data)%*%w

for(i in 1:d){
```

```
dt <- data.frame(data[i:(500+(i-1)),])
VaR <- riskBT(dt,n,m,w,alpha,T,copula,marginals)[1,]

for(j in 1:length(alpha)){
if (LossData[500+i] > VaR[j]) sum[,j] <- sum[,j] + 1
}
}
#print(sum)

for(i in 1:length(alpha)){
p[,i] <- sum[,i]/d
LR[,i] <- -2*log((((1-p[,i])^(d-sum[,i]))*(p[,i]^sum[,i]))
/(((1-sum[,i]/d)^(d-sum[,i]))*((sum[,i]/d)^sum[,i])))
if(LR[,i] > chisq[i])
print("the model is inaccurate") else
print("the model is accurate")
}
}
```

APPENDIX D: CORRELATION MATRIX OF 2010 - 2011 PERIOD

Table D.1. Correlation Matrix of the Daily Log-Returns for 2 Years.

	AAPL	ABT	BA	BBY	BP	C	CMS	DIS	F	GE	HD
AAPL	1.00	0.39	0.55	0.38	0.36	0.48	0.50	0.52	0.55	0.54	0.49
ABT	0.39	1.00	0.55	0.32	0.45	0.51	0.58	0.57	0.47	0.59	0.51
BA	0.55	0.55	1.00	0.43	0.43	0.63	0.59	0.71	0.63	0.70	0.61
BBY	0.38	0.32	0.43	1.00	0.29	0.46	0.39	0.43	0.45	0.41	0.49
BP	0.36	0.45	0.43	0.29	1.00	0.42	0.44	0.45	0.44	0.50	0.37
C	0.48	0.51	0.63	0.46	0.42	1.00	0.58	0.67	0.63	0.70	0.56
CMS	0.50	0.58	0.59	0.39	0.44	0.58	1.00	0.61	0.53	0.63	0.55
DIS	0.52	0.57	0.71	0.43	0.45	0.67	0.61	1.00	0.63	0.72	0.61
F	0.55	0.47	0.63	0.45	0.44	0.63	0.53	0.63	1.00	0.65	0.59
GE	0.54	0.59	0.70	0.41	0.50	0.70	0.63	0.72	0.65	1.00	0.62
HD	0.49	0.51	0.61	0.49	0.37	0.56	0.55	0.61	0.59	0.62	1.00
K	0.30	0.42	0.44	0.27	0.20	0.35	0.40	0.44	0.33	0.44	0.37
KO	0.44	0.60	0.58	0.39	0.44	0.50	0.60	0.58	0.45	0.61	0.53
MCD	0.46	0.51	0.56	0.35	0.38	0.47	0.53	0.54	0.49	0.54	0.56
MMM	0.56	0.59	0.69	0.44	0.50	0.64	0.62	0.72	0.63	0.73	0.60
MO	0.43	0.60	0.59	0.35	0.42	0.48	0.59	0.55	0.46	0.60	0.49
PG	0.41	0.56	0.55	0.31	0.37	0.48	0.57	0.56	0.45	0.59	0.52
TOL	0.47	0.43	0.59	0.45	0.41	0.58	0.49	0.60	0.56	0.62	0.62
UNP	0.55	0.54	0.70	0.43	0.46	0.64	0.58	0.70	0.65	0.72	0.62
WMT	0.35	0.49	0.46	0.35	0.33	0.44	0.50	0.51	0.46	0.51	0.56
GOLD	0.08	0.03	0.07	-0.03	0.03	0.00	0.08	0.04	0.02	0.01	-0.02
OIL	0.33	0.33	0.40	0.24	0.30	0.39	0.38	0.38	0.33	0.41	0.31

Table D.2. Correlation Matrix of the Daily Log-Returns for 2 Years (cont.).

	K	KO	MCD	MMM	MO	PG	TOL	UNP	WMT	GOLD	OIL
AAPL	0.30	0.44	0.46	0.56	0.43	0.41	0.47	0.55	0.35	0.08	0.33
ABT	0.42	0.60	0.51	0.59	0.60	0.56	0.43	0.54	0.49	0.03	0.33
BA	0.44	0.58	0.56	0.69	0.59	0.55	0.59	0.70	0.46	0.07	0.40
BBY	0.27	0.39	0.35	0.44	0.35	0.31	0.45	0.43	0.35	-0.03	0.24
BP	0.20	0.44	0.38	0.50	0.42	0.37	0.41	0.46	0.33	0.03	0.30
C	0.35	0.50	0.47	0.64	0.48	0.48	0.58	0.64	0.44	0.00	0.39
CMS	0.40	0.60	0.53	0.62	0.59	0.57	0.49	0.58	0.50	0.08	0.38
DIS	0.44	0.58	0.54	0.72	0.55	0.56	0.60	0.70	0.51	0.04	0.38
F	0.33	0.45	0.49	0.63	0.46	0.45	0.56	0.65	0.46	0.02	0.33
GE	0.44	0.61	0.54	0.73	0.60	0.59	0.62	0.72	0.51	0.01	0.41
HD	0.37	0.53	0.56	0.60	0.49	0.52	0.62	0.62	0.56	-0.02	0.31
K	1.00	0.46	0.40	0.41	0.45	0.47	0.32	0.40	0.40	0.00	0.22
KO	0.46	1.00	0.59	0.61	0.60	0.63	0.44	0.53	0.53	0.06	0.37
MCD	0.40	0.59	1.00	0.58	0.54	0.53	0.49	0.57	0.52	0.06	0.28
MMM	0.41	0.61	0.58	1.00	0.59	0.58	0.60	0.71	0.53	0.06	0.43
MO	0.45	0.60	0.54	0.59	1.00	0.60	0.45	0.55	0.50	0.08	0.32
PG	0.47	0.63	0.53	0.58	0.60	1.00	0.46	0.53	0.54	0.04	0.28
TOL	0.32	0.44	0.49	0.60	0.45	0.46	1.00	0.57	0.43	-0.02	0.36
UNP	0.40	0.53	0.57	0.71	0.55	0.53	0.57	1.00	0.47	0.05	0.43
WMT	0.40	0.53	0.52	0.53	0.50	0.54	0.43	0.47	1.00	0.01	0.15
GOLD	0.00	0.06	0.06	0.06	0.08	0.04	-0.02	0.05	0.01	1.00	0.21
OIL	0.22	0.37	0.28	0.43	0.32	0.28	0.36	0.43	0.15	0.21	1.00

APPENDIX E: COPULA FITTING RESULTS FOR PORTFOLIOS WITH GOLD AND CRUDE OIL

Llh: Log-likelihood Value

E.1. Copula Fitting Results for Stock Portfolios

Table E.1. Results of Copula Fitting for Portfolio 1.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	0.365	35.985	-69.970
t	t	0.309 / v=2.723	56.049	-108.099
t	GHD	0.310 / v=2.866	53.382	-102.765

Table E.2. Results of Copula Fitting for Portfolio 2.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	131.550	-261.099
t	t	ρ_{t-t} / v=6.950	142.365	-280.730
t	GHD	ρ_{t-GHD} / v=8.050	139.435	-274.869

Table E.3. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 2.

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm-t}	0.378	0.487	0.430
ρ_{t-t}	0.375	0.494	0.412
ρ_{t-GHD}	0.376	0.495	0.416

Table E.4. Results of Copula Fitting for Portfolio 3.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	364.455	-726.910
t	t	ρ_{t-t} / v=7.797	387.747	-771.493
t	GHD	ρ_{t-GHD} / v=8.301	382.575	-761.150

Table E.5. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 3.

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}
ρ_{norm-t}	0.482	0.541	0.681	0.530	0.431	0.521
ρ_{t-t}	0.498	0.545	0.682	0.542	0.448	0.525
ρ_{t-GHD}	0.497	0.546	0.680	0.541	0.447	0.525

Table E.6. Results of Copula Fitting for Portfolio 4.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	438.446	-874.893
t	t	ρ_{t-t} / v=7.525	464.321	-924.642
t	GHD	ρ_{t-GHD} / v=8.195	459.151	-914.302

Table E.7. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 4.

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_{23}	ρ_{24}	ρ_{25}	ρ_{34}	ρ_{35}	ρ_{45}
ρ_{norm-t}	0.548	0.627	0.430	0.392	0.512	0.557	0.479	0.440	0.438	0.467
ρ_{t-t}	0.551	0.636	0.421	0.398	0.496	0.540	0.477	0.428	0.447	0.444
ρ_{t-GHD}	0.554	0.632	0.419	0.400	0.495	0.540	0.479	0.426	0.445	0.443

Table E.8. Results of Copula Fitting for Portfolio 5.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	1,366.422	-2,730.844
t	t	ρ_{t-t} / v=11.310	1,430.153	-2,856.306
t	GHD	ρ_{t-GHD} / v=12.230	1,413.468	-2,822.936

ρ_{norm-t} = (0.405, 0.507, 0.525, 0.470, 0.299 ,0.428, 0.553, 0.392, 0.473,0.500, 0.565, 0.416, 0.273 , 0.411 ,0.571, 0.401,0.449,0.714,0.608,0.474, 0.533, 0.722, 0.543 ,0.593, 0.615, 0.475 ,0.529 ,0.736 ,0.581, 0.604,0.401 ,0.550 ,0.598, 0.514, 0.615,0.419 ,0.459, 0.530, 0.358, 0.572 ,0.522, 0.482,0.569, 0.593,0.443)

ρ_{t-t} = (0.429, 0.504, 0.534, 0.458, 0.291, 0.421, 0.547, 0.389, 0.464, 0.530, 0.594, 0.451,0.310, 0.432, 0.603, 0.410, 0.468, 0.722, 0.609, 0.486, 0.544, 0.730, 0.553, 0.585, 0.622, 0.483, 0.541, 0.750, 0.598, 0.602, 0.412, 0.552, 0.603, 0.524, 0.601, 0.420, 0.483, 0.532, 0.360, 0.579, 0.527, 0.473, 0.578, 0.591, 0.434)

ρ_{t-GHD} = (0.425, 0.502, 0.530 ,0.459, 0.289 ,0.419, 0.546, 0.388, 0.463, 0.524, 0.588 , 0.449, 0.305 ,0.429 ,0.596, 0.405 ,0.466 ,0.718, 0.611, 0.485, 0.542, 0.727 ,0.553, 0.586, 0.623, 0.480 ,0.537, 0.744 ,0.595, 0.601, 0.415, 0.553 ,0.602, 0.526, 0.603, 0.420, 0.477, 0.530, 0.361,0.577 ,0.527, 0.472,0.575, 0.589, 0.434)

E.2. Copula Fitting Results for Stock Portfolios with Gold

Table E.9. Results of Copula Fitting for Portfolio 1 with Gold.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	37.695	-73.390
t	t	ρ_{t-t} / v=6.101	52.344	-100.688
t	GHD	ρ_{t-GHD} / v=6.599	49.814	-95.628

Table E.10. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 1 with Gold.

	$\rho_{1,Gold}$	$\rho_{2,Gold}$
ρ_{norm-t}	0.077	0.055
ρ_{t-t}	0.082	0.068
ρ_{t-ghd}	0.078	0.063

Table E.11. Results of Copula Fitting for Portfolio 2 with Gold.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	133.790	-265.580
t	t	ρ_{t-t} / v=7.2069	152.804	-301.607
t	GHD	ρ_{t-GHD} / v=8.0340	149.275	-294.549

Table E.12. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 2 with Gold.

	$\rho_{1,Gold}$	$\rho_{2,Gold}$	$\rho_{3,Gold}$
ρ_{norm-t}	-0.005	0.083	0.004
ρ_{t-t}	0.015	0.105	0.024
ρ_{t-GHD}	0.013	0.101	0.024

Table E.13. Results of Copula Fitting for Portfolio 3 with Gold.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	442.279	-882.557
t	t	ρ_{t-t} / v=8.663	469.273	-934.545
t	GHD	ρ_{t-GHD} / v=9.440	464.073	-924.146

Table E.14. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 3 with Gold.

	$\rho_{1,Gold}$	$\rho_{2,Gold}$	$\rho_{3,Gold}$	$\rho_{4,Gold}$
ρ_{norm-t}	0.083	0.011	0.035	0.059
ρ_{t-t}	0.073	0.032	0.049	0.052
ρ_{t-GHD}	0.067	0.031	0.049	0.049

Table E.15. Results of Copula Fitting for Portfolio 4 with Gold.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	502.314	-1,002.628
t	t	ρ_{t-t} / v=8.112	536.330	-1,068.660
t	GHD	ρ_{t-GHD} / v=9.210	528.440	-1,052.881

Table E.16. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 4 with Gold.

	$\rho_{1,Gold}$	$\rho_{2,Gold}$	$\rho_{3,Gold}$	$\rho_{4,Gold}$	$\rho_{5,Gold}$
ρ_{norm-t}	0.025	0.091	0.007	0.092	0.022
ρ_{t-t}	0.061	0.083	0.014	0.095	0.028
ρ_{t-GHD}	0.057	0.082	0.009	0.093	0.027

Table E.17. Results of Copula Fitting for Portfolio 5 with Gold.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	442.279	-882.557
t	t	ρ_{t-t} / v=8.663	469.273	-934.545
t	GHD	ρ_{t-GHD} / v=9.440	464.073	-924.146

Table E.18. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 5 with Gold.

	$\rho_{1,Gold}$	$\rho_{2,Gold}$	$\rho_{3,Gold}$	$\rho_{4,Gold}$	$\rho_{5,Gold}$
ρ_{norm-t}	0.077	0.058	0.047	0.021	-0.020
ρ_{t-t}	0.063	0.067	0.038	0.014	-0.023
ρ_{t-GHD}	0.059	0.064	0.039	0.012	-0.024

Table E.19. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 5 with Gold (cont.).

	$\rho_{6,Gold}$	$\rho_{7,Gold}$	$\rho_{8,Gold}$	$\rho_{9,Gold}$	$\rho_{10,Gold}$
ρ_{norm-t}	0.011	0.066	0.078	0.035	0.004
ρ_{t-t}	0.010	0.050	0.075	0.034	-0.007
ρ_{t-GHD}	0.010	0.050	0.072	0.036	-0.006

E.3. Copula Fitting Results for Stock Portfolios with Crude Oil

Table E.20. Results of Copula Fitting for Portfolio 1 with Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	80.707	-159.413
t	t	ρ_{t-t} / v=5.279	101.012	-198.024
t	GHD	ρ_{t-GHD} / v=5.865	96.718	-189.435

Table E.21. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 1 with Crude Oil.

	$\rho_{1,Oil}$	$\rho_{2,Oil}$
ρ_{norm-t}	0.333	0.333
ρ_{t-t}	0.331	0.320
ρ_{t-ghd}	0.334	0.323

Table E.22. Results of Copula Fitting for Portfolio 2 with Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	187.064	-372.128
t	t	ρ_{t-t} / v=7.699	208.633	-413.266
t	GHD	ρ_{t-GHD} / v=8.501	204.477	-404.953

Table E.23. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 2 with Crude Oil.

	$\rho_{1,Oil}$	$\rho_{2,Oil}$	$\rho_{3,Oil}$
ρ_{norm-t}	0.276	0.376	0.369
ρ_{t-t}	0.292	0.390	0.381
ρ_{t-GHD}	0.293	0.390	0.384

Table E.24. Results of Copula Fitting for Portfolio 3 with Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	425.867	-849.733
t	t	ρ_{t-t} / v=7.942	459.892	-915.783
t	GHD	ρ_{t-GHD} / v=8.603	453.025	-902.050

Table E.25. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 3 with Crude Oil.

	$\rho_{1,Oil}$	$\rho_{2,Oil}$	$\rho_{3,Oil}$	$\rho_{4,Oil}$
ρ_{norm-t}	0.421	0.239	0.282	0.431
ρ_{t-t}	0.430	0.263	0.297	0.441
ρ_{t-GHD}	0.431	0.263	0.298	0.442

Table E.26. Results of Copula Fitting for Portfolio 4 with Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	502.314	-1,002.628
t	t	ρ_{t-t} / v=8.112	536.330	-1,068.660
t	GHD	ρ_{t-GHD} / v=9.209	528.440	-1,052.881

Table E.27. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 4 with Crude Oil.

	$\rho_{1,Oil}$	$\rho_{2,Oil}$	$\rho_{3,Oil}$	$\rho_{4,Oil}$	$\rho_{5,Oil}$
ρ_{norm-t}	0.412	0.379	0.357	0.309	0.164
ρ_{t-t}	0.424	0.374	0.388	0.313	0.197
ρ_{t-GHD}	0.424	0.377	0.386	0.314	0.196

Table E.28. Results of Copula Fitting for Portfolio 5 with Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	ρ_{norm-t}	1,437.997	-2,873.994
t	t	ρ_{t-t} / v=11.242	1,513.583	-3,023.166
t	GHD	ρ_{t-GHD} / v=12.197	1,495.148	-2,986.296

Table E.29. ρ_{norm}, ρ_{t-t} and ρ_{t-GHD} for Portfolio 5 with Crude Oil.

	$\rho_{1,Oil}$	$\rho_{2,Oil}$	$\rho_{3,Oil}$	$\rho_{4,Oil}$	$\rho_{5,Oil}$	$\rho_{6,Oil}$	$\rho_{7,Oil}$	$\rho_{8,Oil}$	$\rho_{9,Oil}$	$\rho_{10,Oil}$
ρ_{norm-t}	0.333	0.385	0.391	0.420	0.323	0.239	0.303	0.452	0.282	0.369
ρ_{t-t}	0.352	0.432	0.405	0.437	0.329	0.263	0.318	0.476	0.306	0.370
ρ_{t-GHD}	0.350	0.430	0.402	0.435	0.330	0.261	0.318	0.473	0.304	0.371

E.4. Copula Fitting Results for Stock Portfolios with Gold and Crude Oil

In this section, we will just state the dependence between gold and crude oil as the correlation parameter since the other correlations between stocks and commodities are estimated in the previous sections.

Table E.30. Results of Copula Fitting for Portfolio 1 with Gold and Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	0.215	92.711	-183.421
t	t	0.240 / v=6.894	118.208	-232.415
t	GHD	0.235 / v=7.651	112.765	-221.529

Table E.31. Results of Copula Fitting for Portfolio 2 with Gold and Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	0.215	201.452	-400.903
t	t	0.235 / v=7.457	234.945	-465.891
t	GHD	0.232 / v=8.191	229.210	-454.420

Table E.32. Results of Copula Fitting for Portfolio 3 with Gold and Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	0.215	438.561	-875.122
t	t	0.217 / v=8.291	477.999	-951.997
t	GHD	0.211 / v=9.065	469.439	-934.878

Table E.33. Results of Copula Fitting for Portfolio 4 with Gold and Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	0.215	517.539	-1033.079
t	t	0.236 / v=8.626	557.711	-1111.422
t	GHD	0.233 / v=9.748	548.939	-1093.879

Table E.34. Results of Copula Fitting for Portfolio 5 with Gold and Crude Oil.

Copula	Marginals	Parameter(s)	Llh	AIC
Normal	t	0.215	1454.544	-2907.088
t	t	0.214 / v=11.778	1531.305	-3058.610
t	GHD	0.209 / v=12.888	1511.607	-3019.214

APPENDIX F: BACK TESTING EXCEPTION RESULTS

Table F.1. Back Testing Exception Results.

		α							
Portfolio	**Model**	**0.001**	**0.002**	**0.003**	**0.004**	**0.005**	**0.006**	**0.007**	**0.008**
st2	Multivariate Normal	14	19	21	23	25	26	27	29
	normal - t	5	9	12	13	18	20	22	23
	t - t	5	9	10	13	18	19	20	21
	t - ghyp	5	9	10	12	16	19	20	20
st2 + gold	Multivariate Normal	15	19	20	21	24	25	28	30
	normal - t	4	8	14	16	19	21	23	25
	t - t	3	8	12	16	19	20	22	23
	t - ghyp	2	8	11	15	18	20	21	24
st2 + oil	Multivariate Normal	21	25	27	29	34	34	35	36
	normal - t	10	15	22	23	28	29	31	35
	t - t	8	14	21	23	26	27	31	33
	t - ghyp	8	13	21	22	24	27	30	31
st2 + gold + oil	Multivariate Normal	21	26	27	28	31	33	35	35
	normal - t	11	14	21	22	28	29	30	34
	t - t	8	14	21	23	26	27	31	33
	t - ghyp	7	13	21	23	25	28	32	33

APPENDIX G: VaR AND CVaR RESULTS

Table G.1. Daily $VaR_{0.999}$ and $CVaR_{0.999}$ of AAPL-ABT with Gold and Crude Oil.

Method	Stock Portfolios			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.03306	0.000541	0.03600	0.000740
Normal Copula with t Marginals	0.04038	0.000943	0.04896	0.002119
t Copula with t Marginals	0.04334	0.001155	0.05409	0.002358
t Copula with GHD Marginals	0.04370	0.001070	0.05052	0.001674
	Stock Portfolios with Gold			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.02549	0.000496	0.02746	0.000614
Normal Copula with t Marginals	0.03026	0.000654	0.03689	0.001446
t Copula with t Marginals	0.03319	0.000900	0.04218	0.001959
t Copula with GHD Marginals	0.03365	0.000835	0.03958	0.001317
	Stock Portfolios with Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.03470	0.000593	0.03765	0.000725
Normal Copula with t Marginals	0.04068	0.000906	0.04855	0.001552
t Copula with t Marginals	0.04453	0.001028	0.05452	0.002226
t Copula with GHD Marginals	0.04602	0.001242	0.05645	0.001940
	Stock Portfolios with Gold and Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.02845	0.000426	0.03076	0.000607
Normal Copula with t Marginals	0.03392	0.000752	0.04083	0.001886
t Copula with t Marginals	0.03729	0.001074	0.04545	0.002334
t Copula with GHD Marginals	0.03652	0.000758	0.04379	0.001522

Table G.2. Weekly $VaR_{0.999}$ and $CVaR_{0.999}$ of AAPL-ABT with Gold and Crude Oil.

Method	Stock Portfolios			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.07097	0.001030	0.07707	0.001543
Normal Copula with t Marginals	0.07275	0.001225	0.08136	0.001804
t Copula with t Marginals	0.07607	0.001306	0.09099	0.002724
t Copula with GHD Marginals	0.07488	0.001562	0.08412	0.002057
	Stock Portfolios with Gold			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.05567	0.000850	0.06074	0.001328
Normal Copula with t Marginals	0.05421	0.000820	0.06107	0.001528
t Copula with t Marginals	0.05783	0.001363	0.06704	0.002426
t Copula with GHD Marginals	0.05827	0.000916	0.06658	0.001471
	Stock Portfolios with Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.07609	0.001090	0.08348	0.001722
Normal Copula with t Marginals	0.07358	0.001233	0.08327	0.002119
t Copula with t Marginals	0.07960	0.001462	0.09116	0.002750
t Copula with GHD Marginals	0.07818	0.001309	0.08808	0.002058
	Stock Portfolios with Gold and Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.06229	0.000838	0.06828	0.001313
Normal Copula with t Marginals	0.06089	0.000931	0.06903	0.001669
t Copula with t Marginals	0.06579	0.001299	0.07620	0.002143
t Copula with GHD Marginals	0.06628	0.001077	0.07631	0.002060

Table G.3. Weekly $VaR_{0.99}$ and $CVaR_{0.99}$ of AAPL-ABT with Gold and Crude Oil.

Method	Stock Portfolios			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.02577	0.000250	0.02957	0.000297
Normal Copula with t Marginals	0.02673	0.000373	0.03357	0.000565
t Copula with t Marginals	0.02745	0.000380	0.03625	0.000609
t Copula with GHD Marginals	0.02830	0.000363	0.03597	0.000503
	Stock Portfolios with Gold			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.02001	0.000200	0.02290	0.000249
Normal Copula with t Marginals	0.02030	0.000242	0.02545	0.000379
t Copula with t Marginals	0.02090	0.000275	0.02753	0.000509
t Copula with GHD Marginals	0.02170	0.000220	0.02755	0.000393
	Stock Portfolios with Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.02713	0.000261	0.03099	0.00029
Normal Copula with t Marginals	0.02808	0.000331	0.03463	0.0005006
t Copula with t Marginals	0.02879	0.000361	0.03708	0.0005497
t Copula with GHD Marginals	0.02959	0.000360	0.03732	0.0005913
	Stock Portfolios with Gold and Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.02232	0.000201	0.02561	0.000250
Normal Copula with t Marginals	0.02287	0.000299	0.02828	0.000416
t Copula with t Marginals	0.02352	0.000292	0.03047	0.000577
t Copula with GHD Marginals	0.02475	0.000306	0.03116	0.000497

Table G.4. Weekly $VaR_{0.99}$ and $CVaR_{0.99}$ of AAPL-ABT with Gold and Crude Oil.

Method	Stock Portfolios			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.05637	0.0005715	0.06425	0.0006747
Normal Copula with t Marginals	0.05368	0.0005191	0.06370	0.0006704
t Copula with t Marginals	0.05330	0.0005861	0.06528	0.0008093
t Copula with GHD Marginals	0.05473	0.0006032	0.06508	0.0008694
	Stock Portfolios with Gold			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.04333	0.0003944	0.04931	0.0005229
Normal Copula with t Marginals	0.03962	0.0003129	0.04717	0.0004387
t Copula with t Marginals	0.04052	0.0004144	0.04960	0.0006889
t Copula with GHD Marginals	0.04295	0.0004532	0.05103	0.0005839
	Stock Portfolios with Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.05908	0.0005845	0.06783	0.0007448
Normal Copula with t Marginals	0.05530	0.0005173	0.06516	0.0007655
t Copula with t Marginals	0.05675	0.0005972	0.06852	0.0008429
t Copula with GHD Marginals	0.05854	0.0005357	0.06900	0.0007714
	Stock Portfolios with Gold and Crude Oil			
	VaR		CVaR	
	Result	SE	Result	SE
Exact Multinormal Model	0.04905	0.0004575	0.05623	0.0005856
Normal Copula with t Marginals	0.04513	0.0003923	0.05359	0.0005726
t Copula with t Marginals	0.04629	0.0004674	0.05645	0.0007022
t Copula with GHD Marginals	0.04858	0.0004842	0.05780	0.0006165

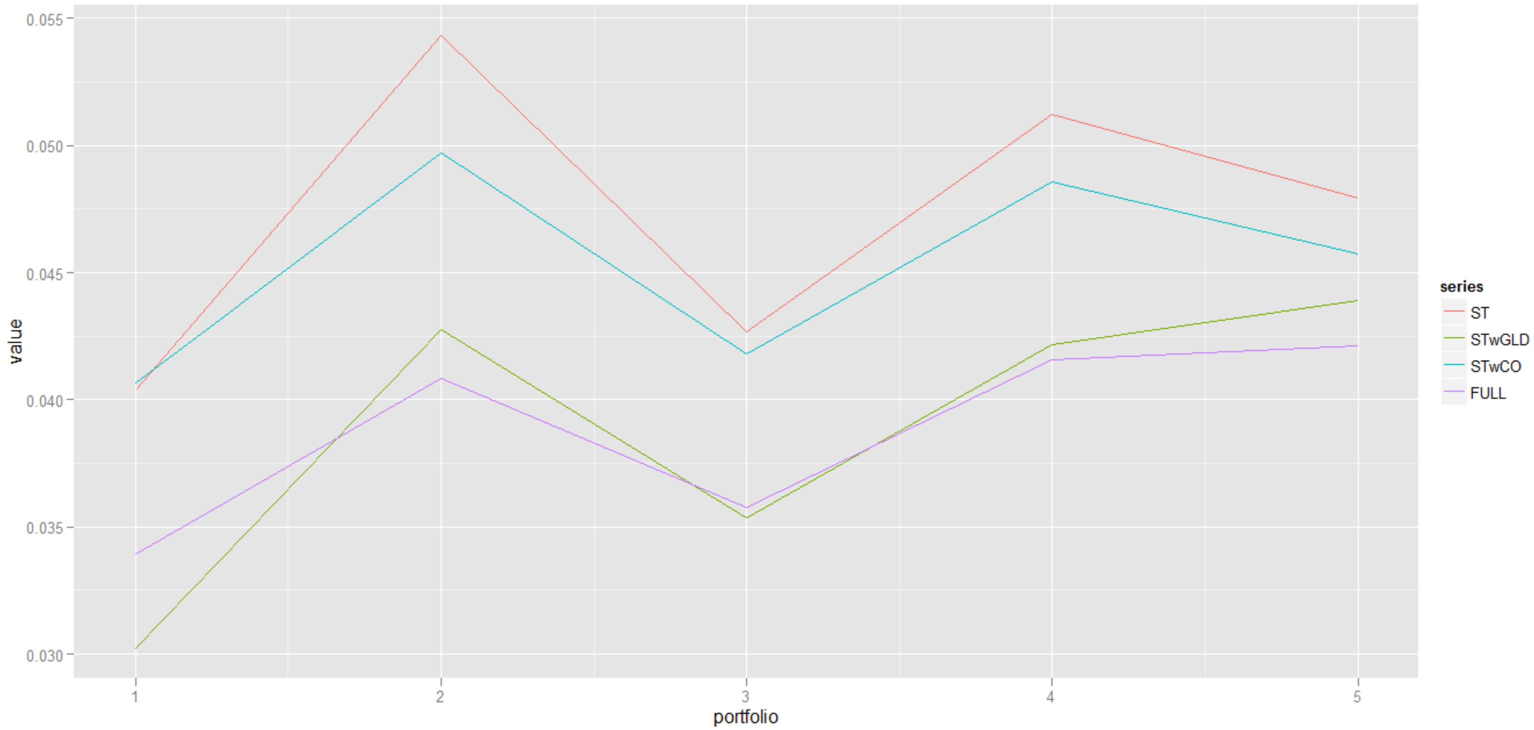

Figure G.1. Daily $VaR_{0.999}$ of the portfolios computed with normal copula t marginals model.

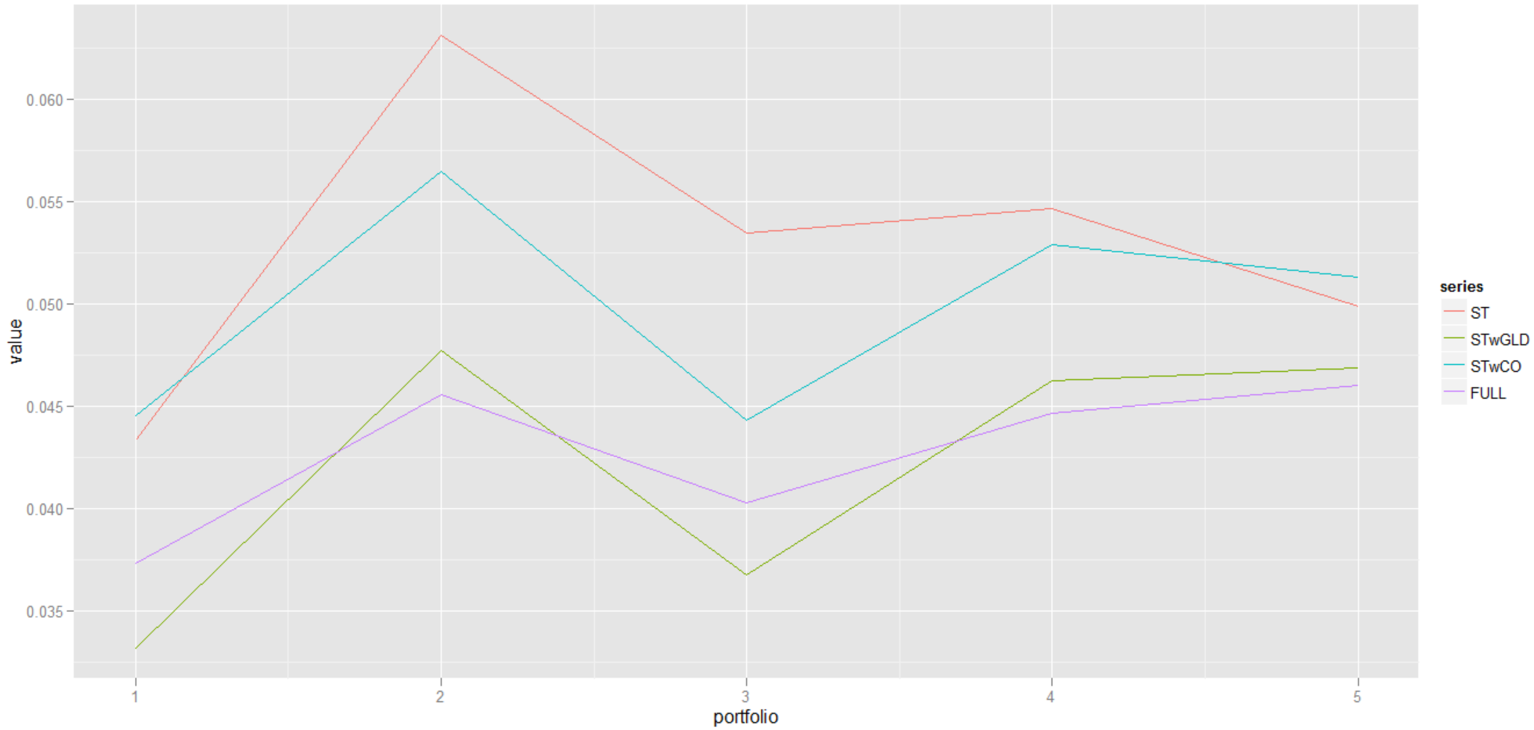

Figure G.2. Daily $VaR_{0.999}$ of the portfolios computed with t copula t marginals model.

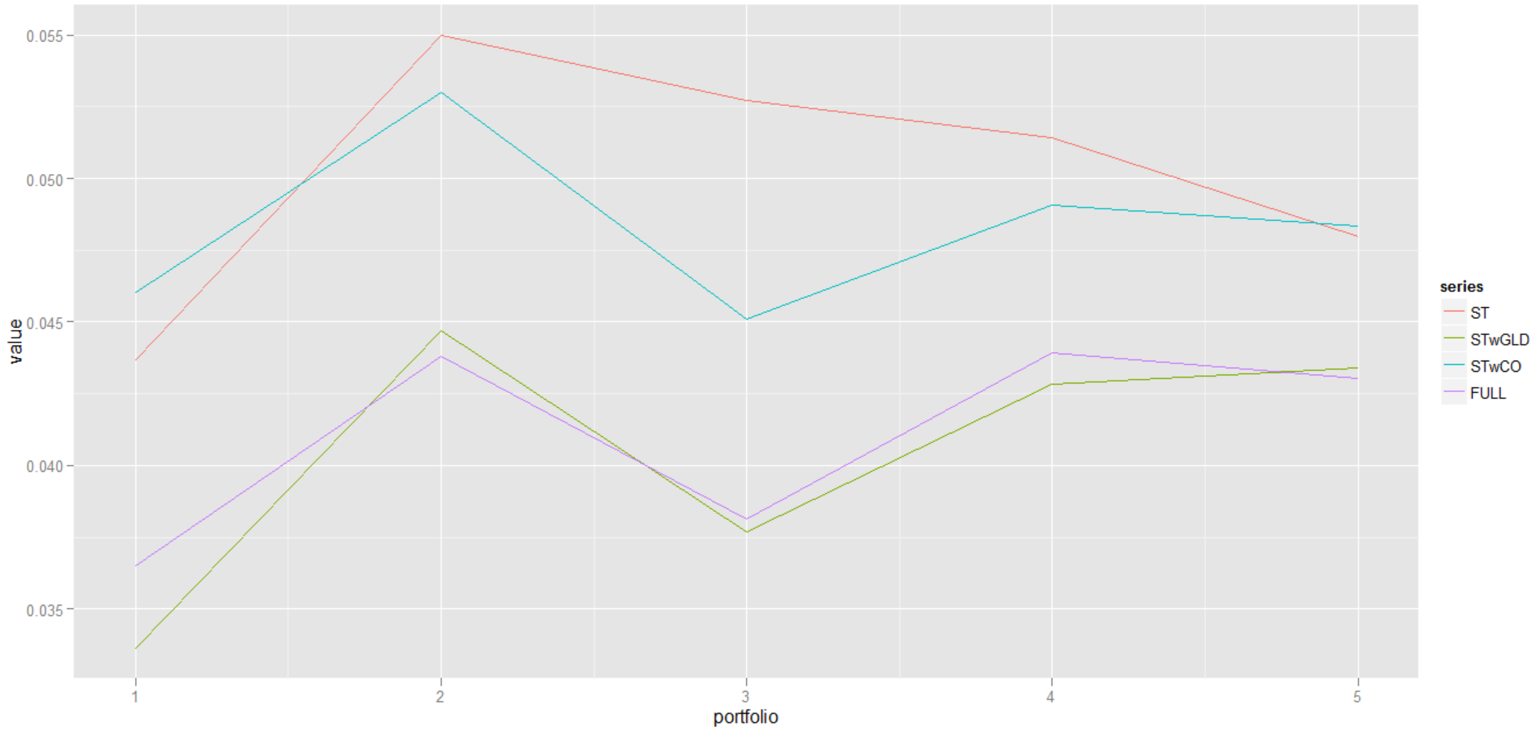

Figure G.3. Daily $VaR_{0.999}$ of the portfolios computed with t copula ghyp marginals model.

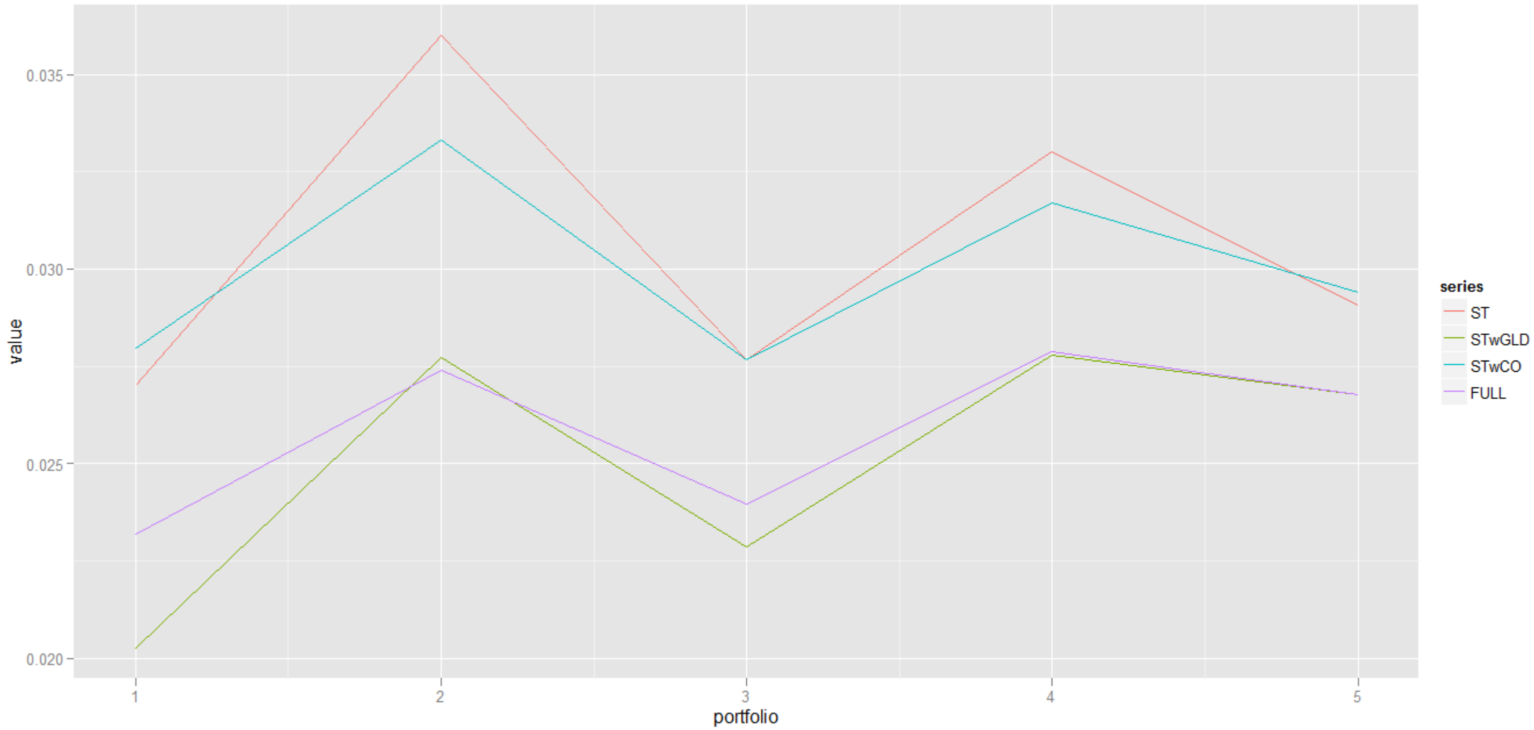

Figure G.4. Daily $VaR_{0.99}$ of the portfolios computed with normal copula t marginals model.

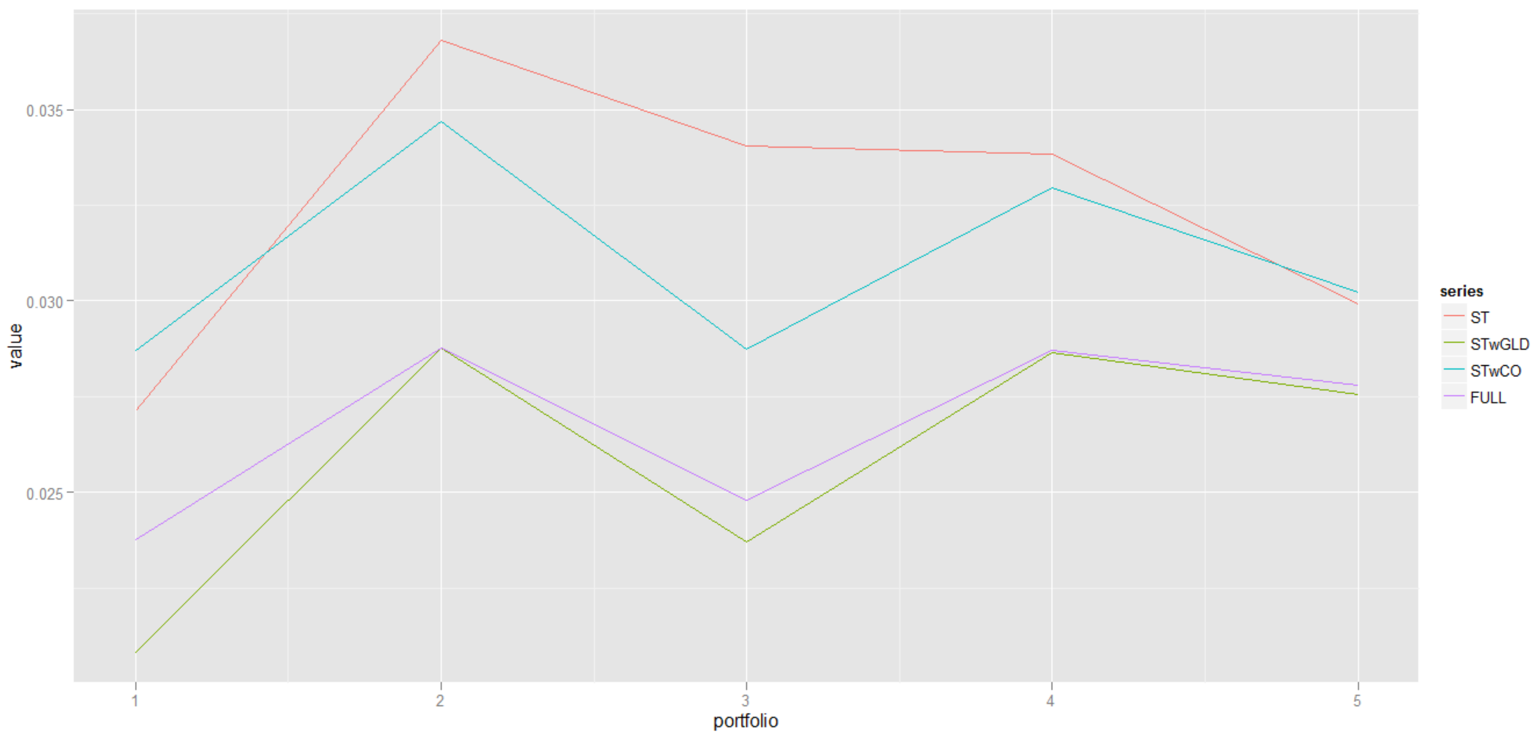

Figure G.5. Daily $VaR_{0.99}$ of the portfolios computed with t copula t marginals model.

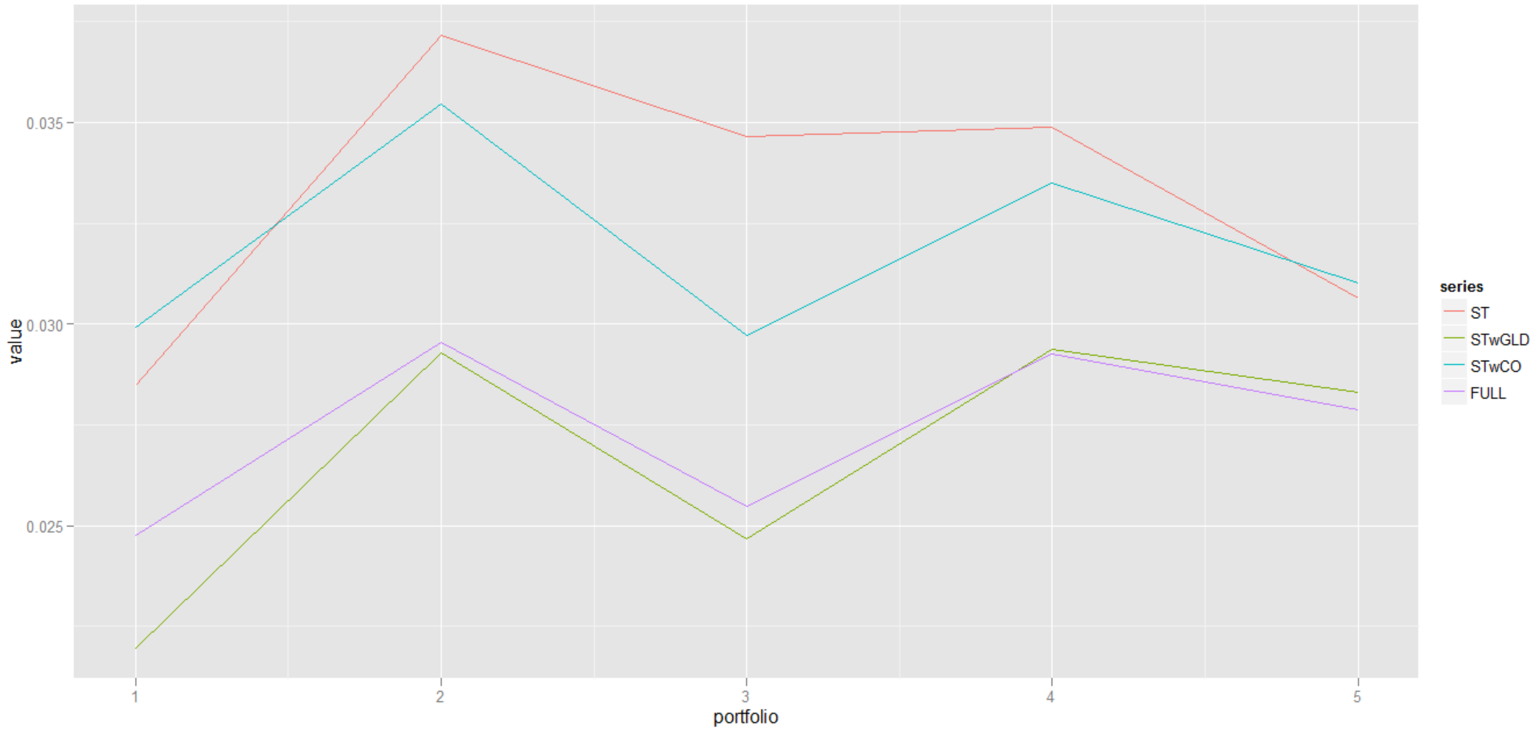

Figure G.6. Daily $VaR_{0.99}$ of the portfolios computed with t copula ghyp marginals model.

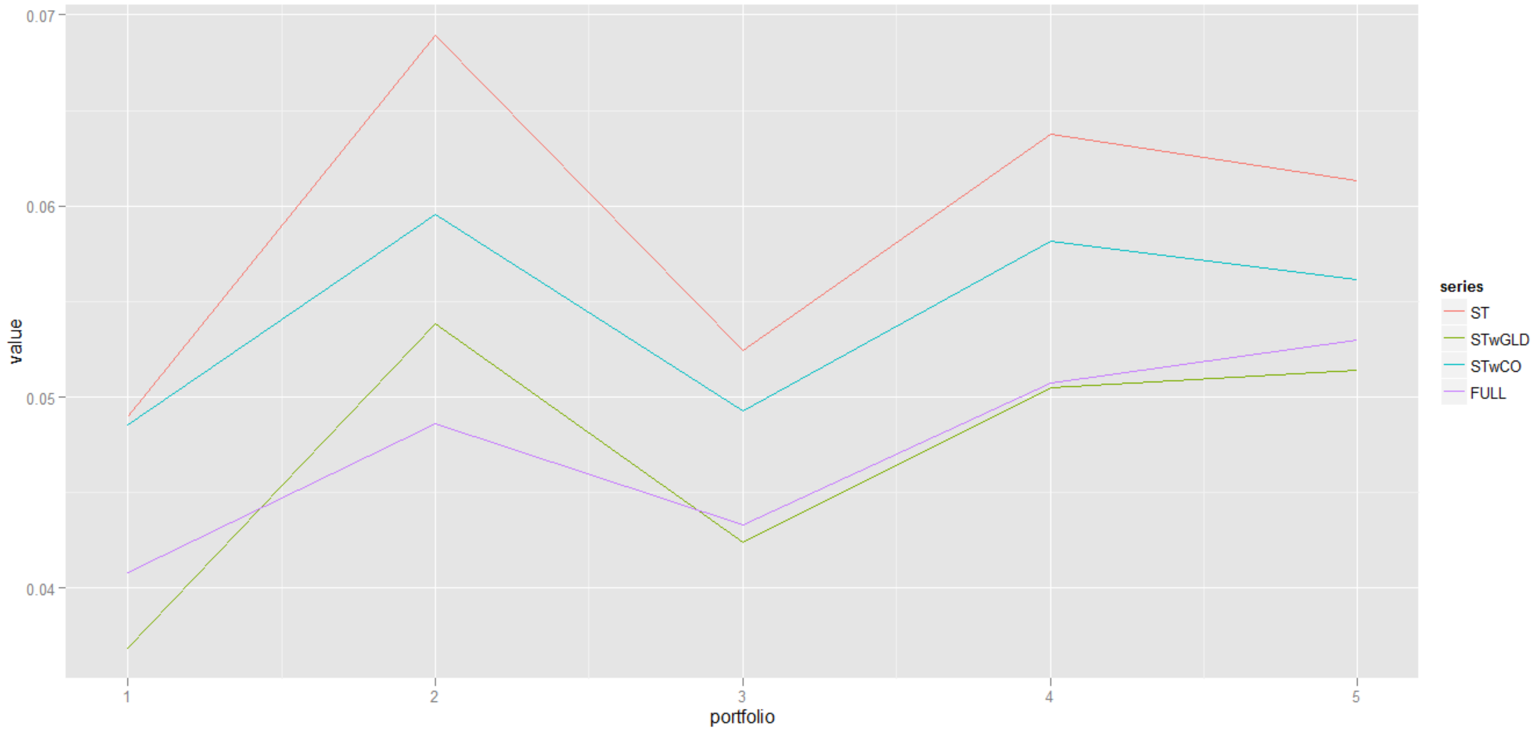

Figure G.7. Daily $CVaR_{0.999}$ of the portfolios computed with normal copula t marginals model.

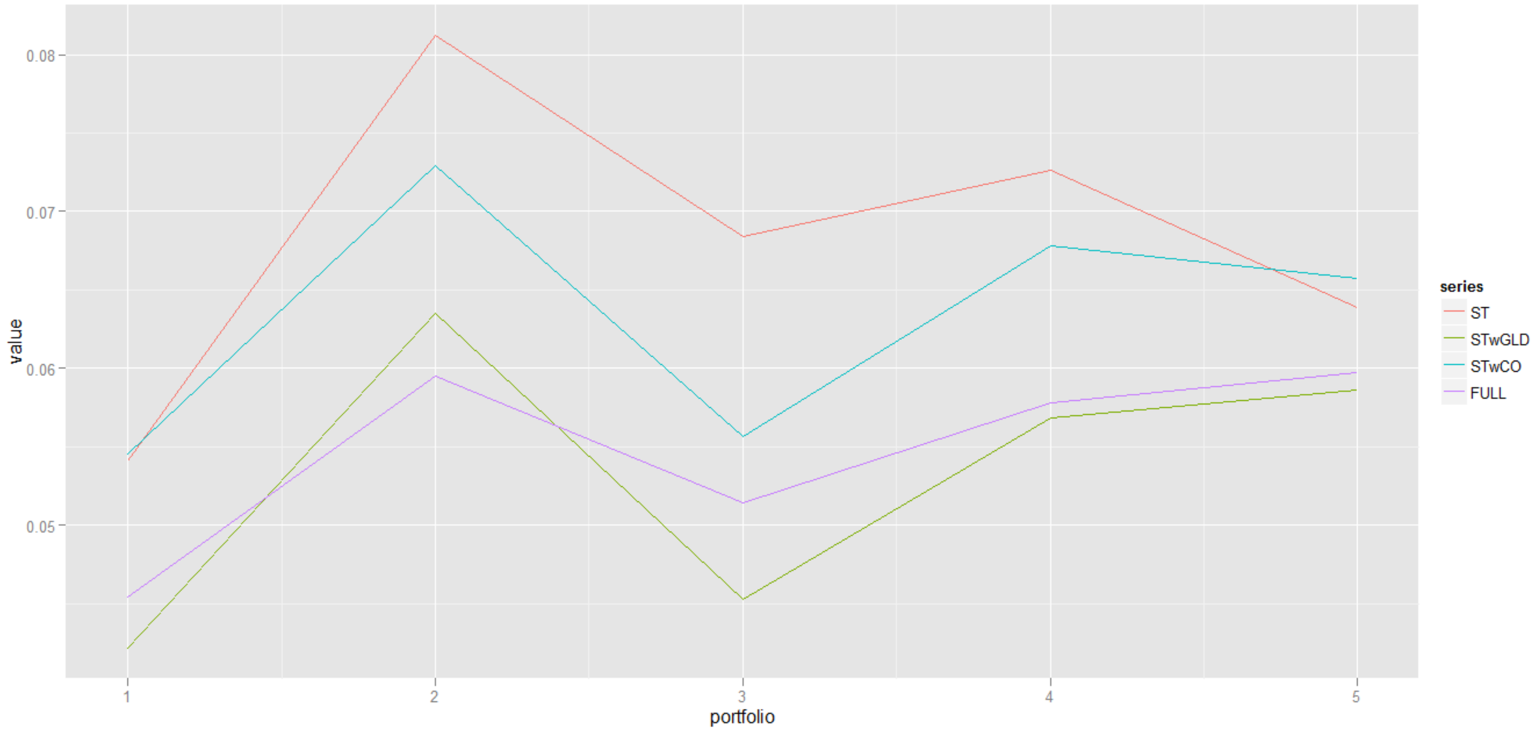

Figure G.8. Daily $CVaR_{0.999}$ of the portfolios computed with t copula t marginals model.

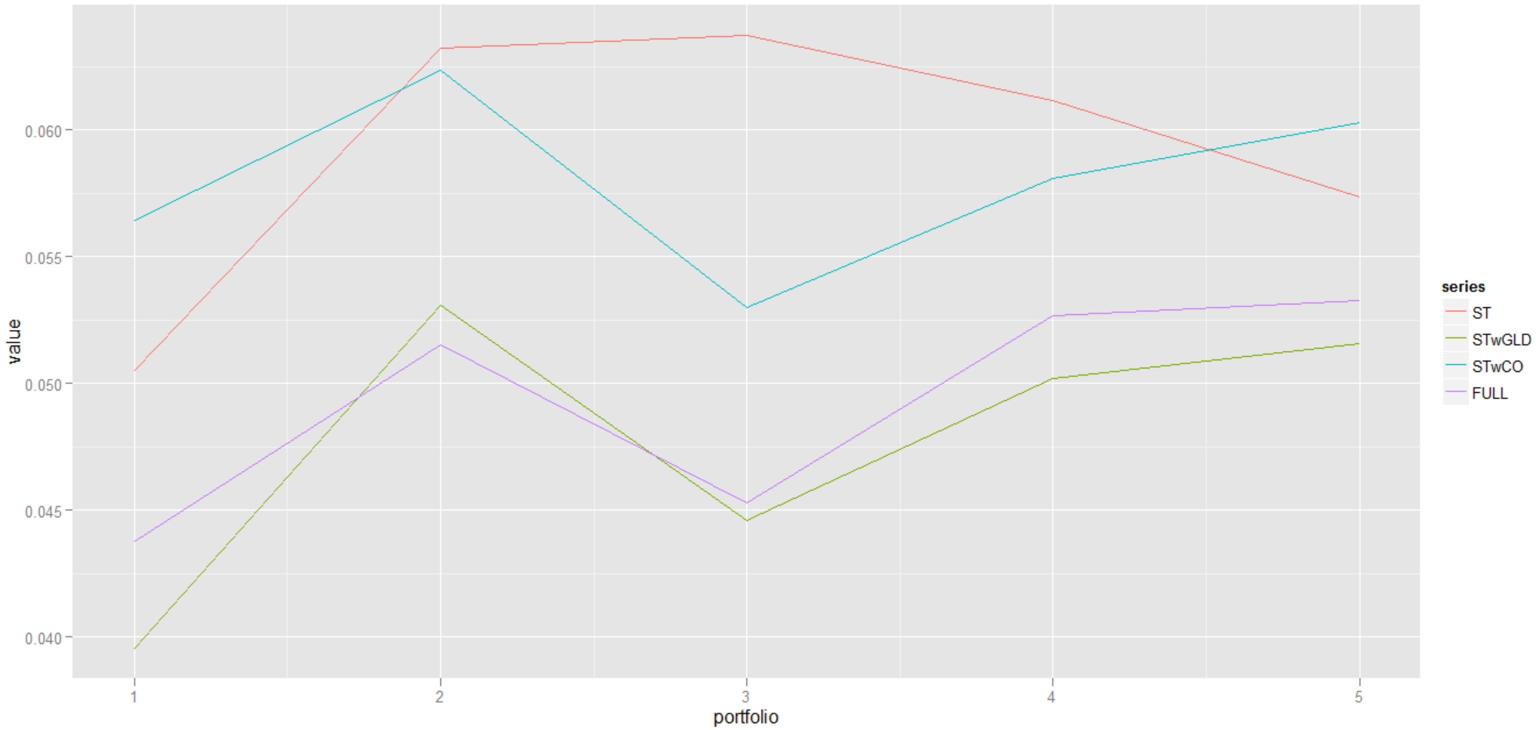

Figure G.9. Daily $CVaR_{0.999}$ of the portfolios computed with t copula ghyp marginals model.

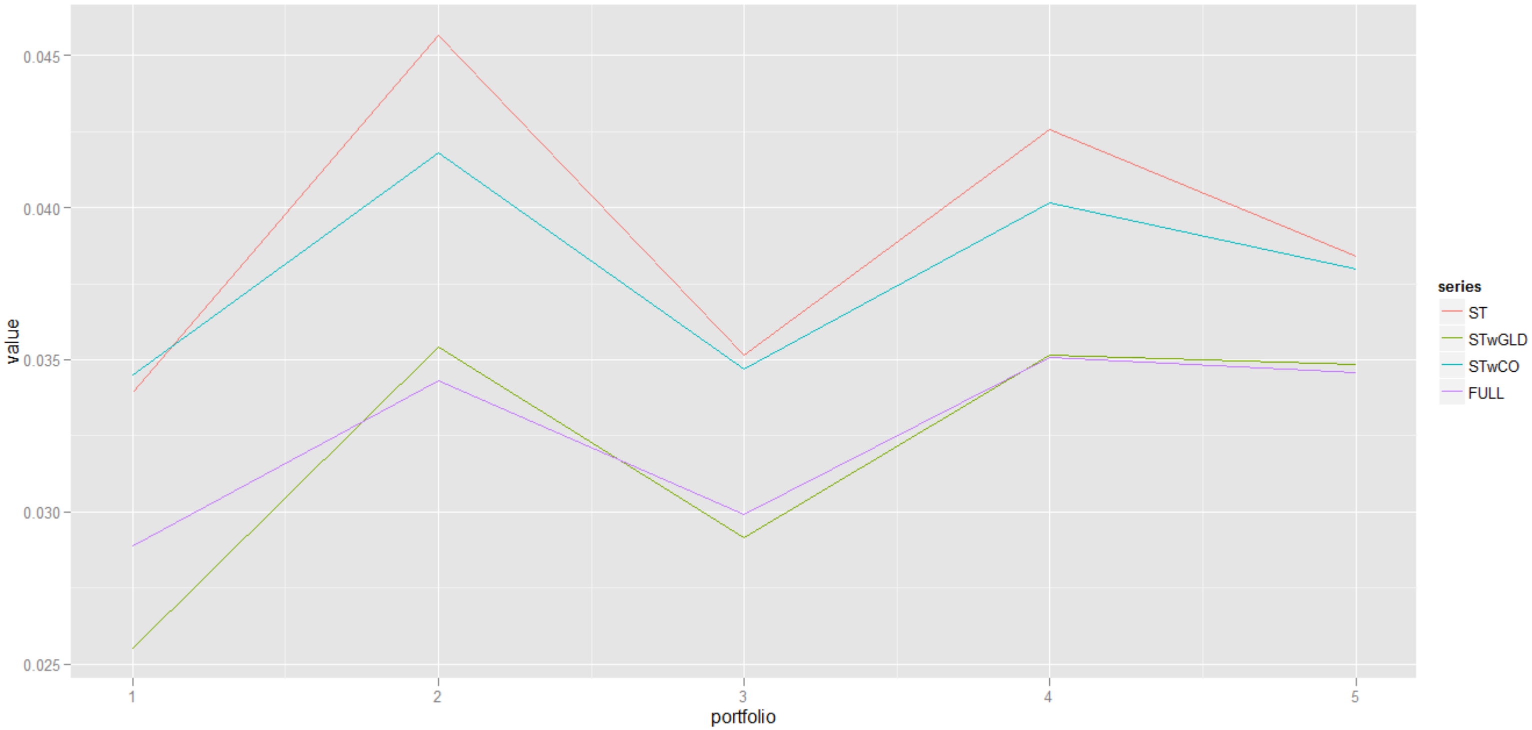

Figure G.10. Daily $CVaR_{0.99}$ of the portfolios computed with normal copula t marginals model.

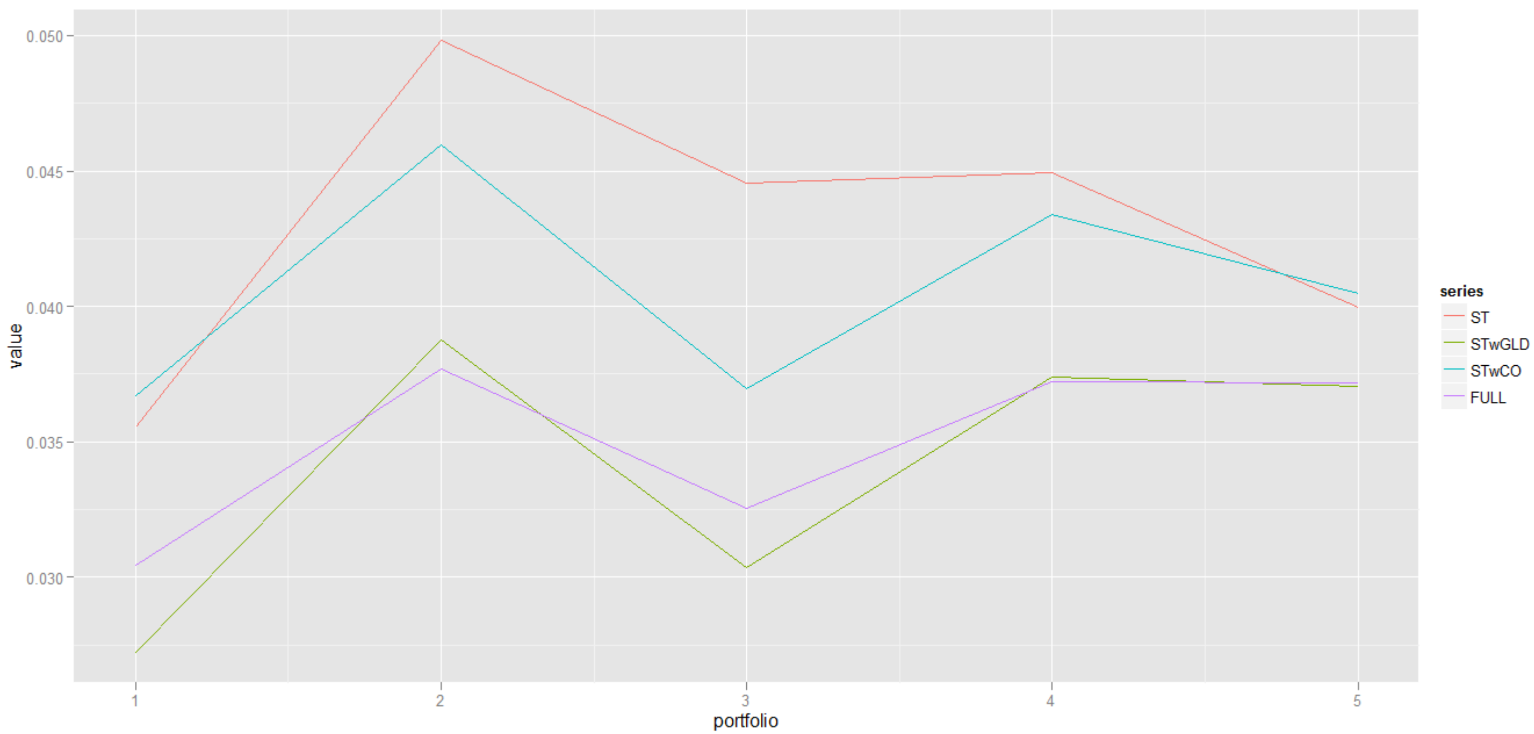

Figure G.11. Daily $CVaR_{0.99}$ of the portfolios computed with t copula t marginals model.

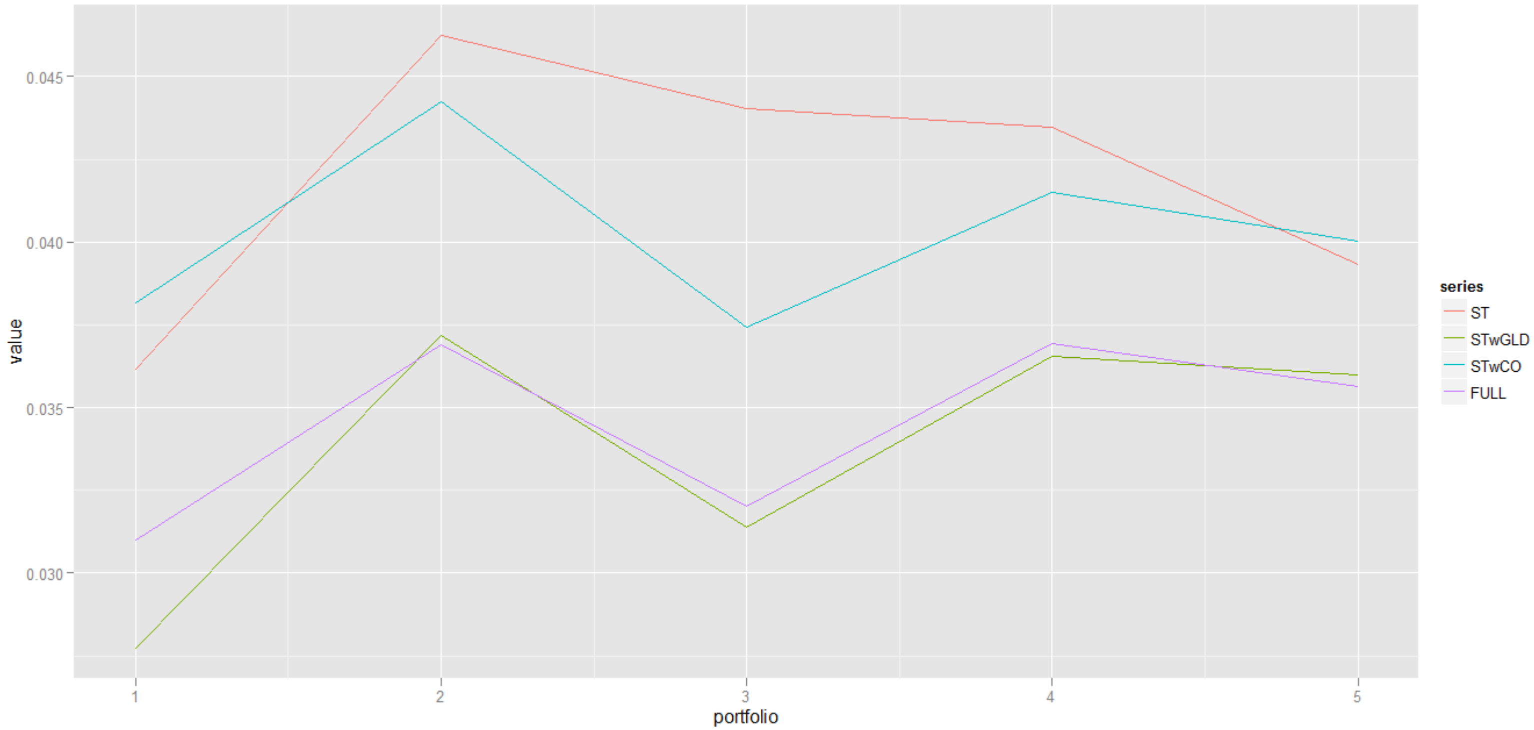

Figure G.12. Daily $CVaR_{0.99}$ of the portfolios computed with t copula ghyp marginals model.

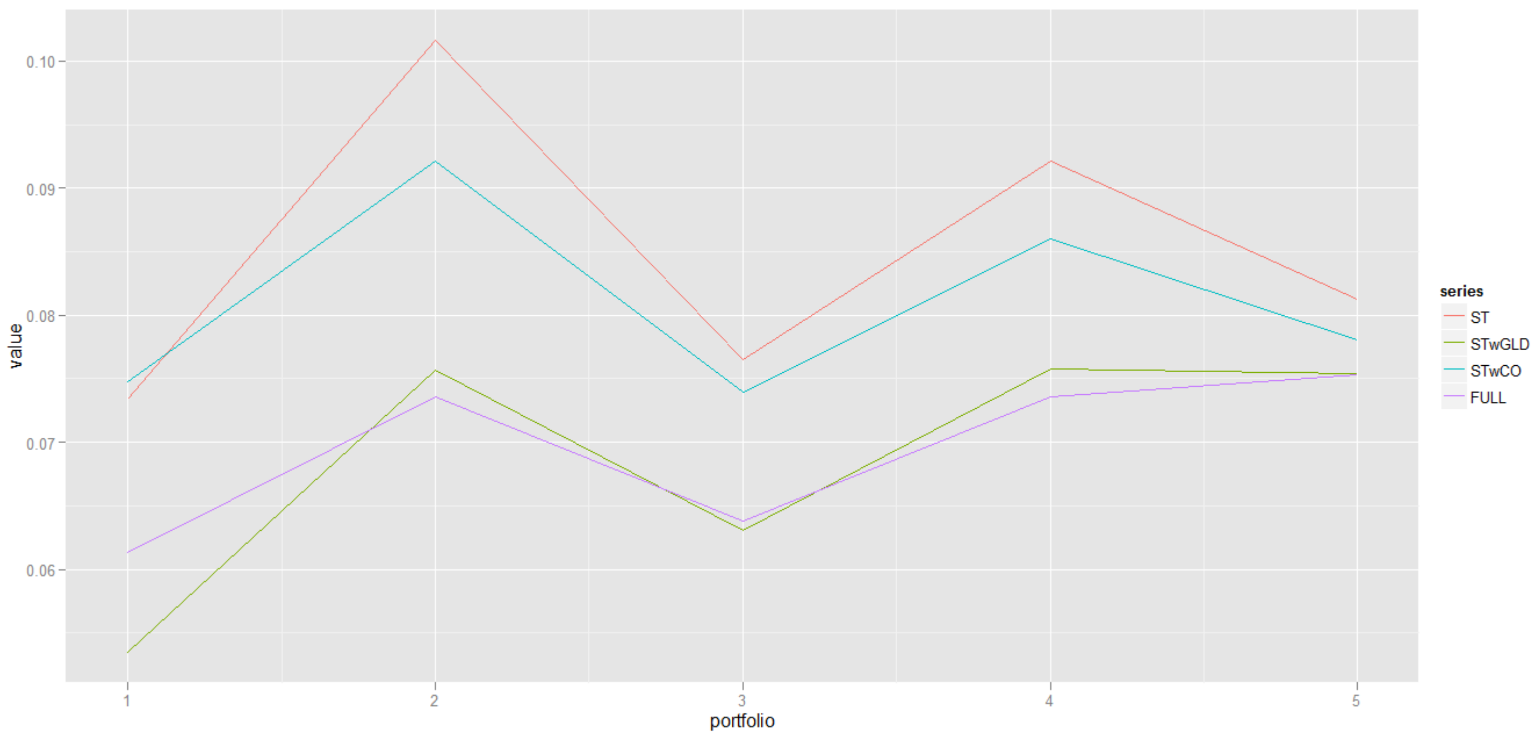

Figure G.13. Weekly $VaR_{0.999}$ of the portfolios computed with normal copula t marginals model.

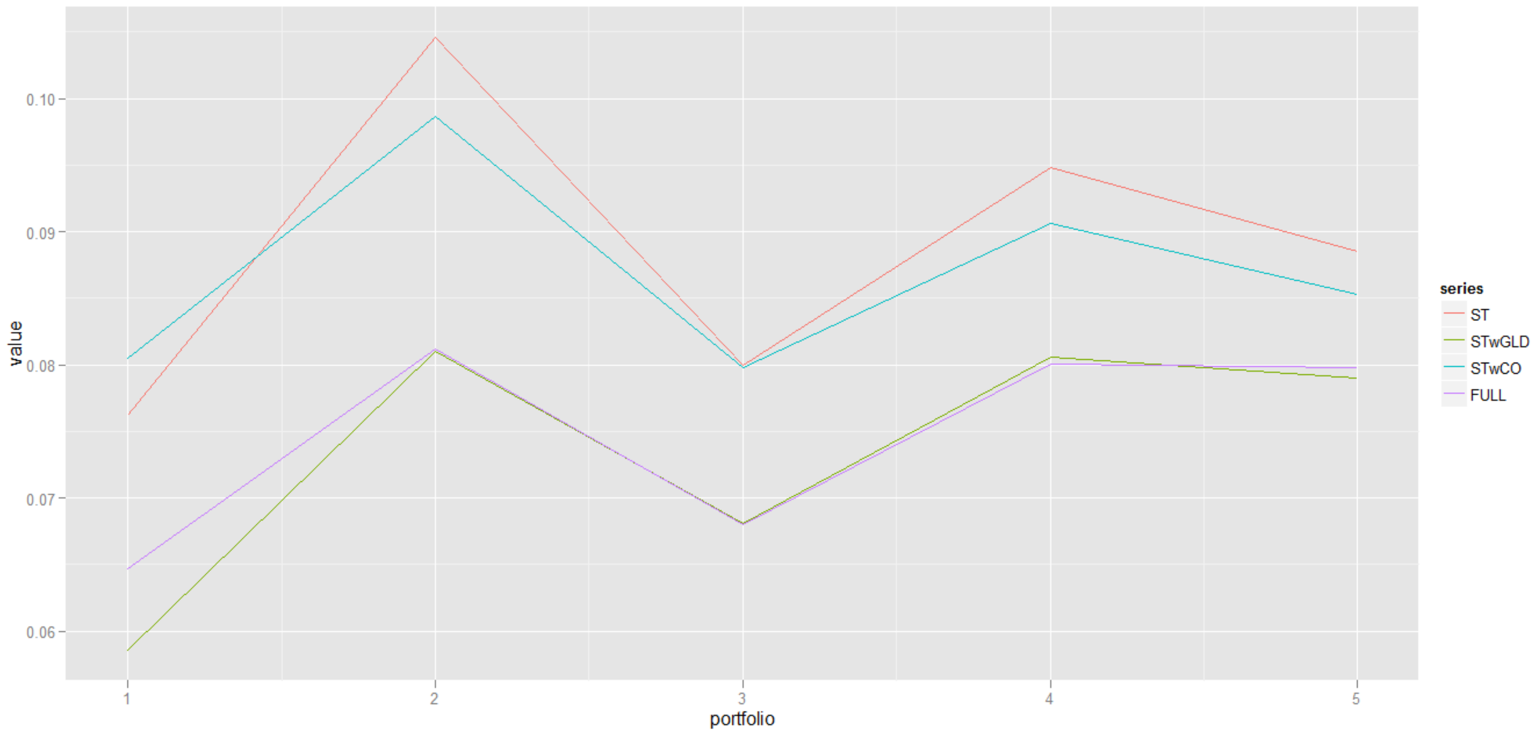

Figure G.14. Weekly $VaR_{0.999}$ of the portfolios computed with t copula t marginals model.

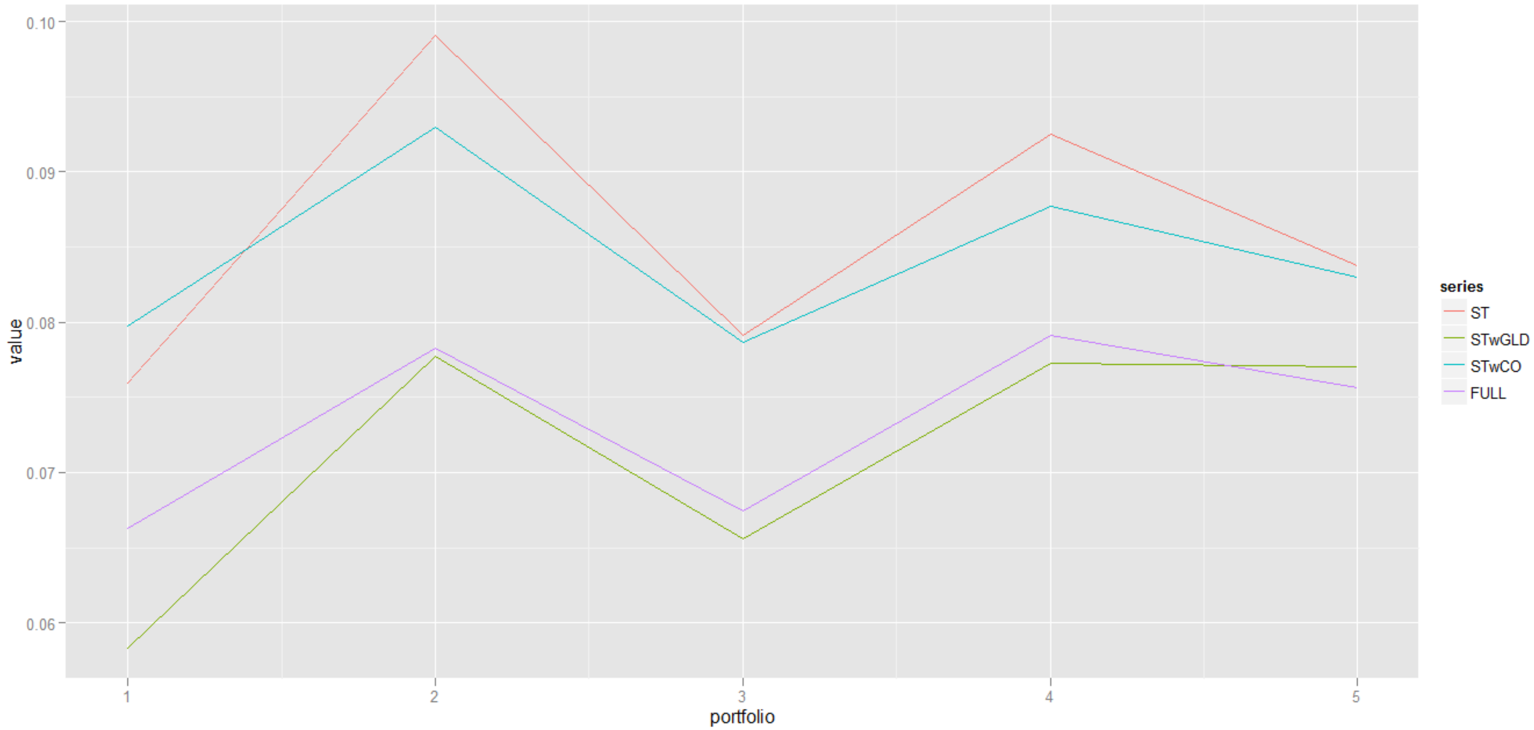

Figure G.15. Weekly $VaR_{0.999}$ of the portfolios computed with t copula ghyp marginals model.

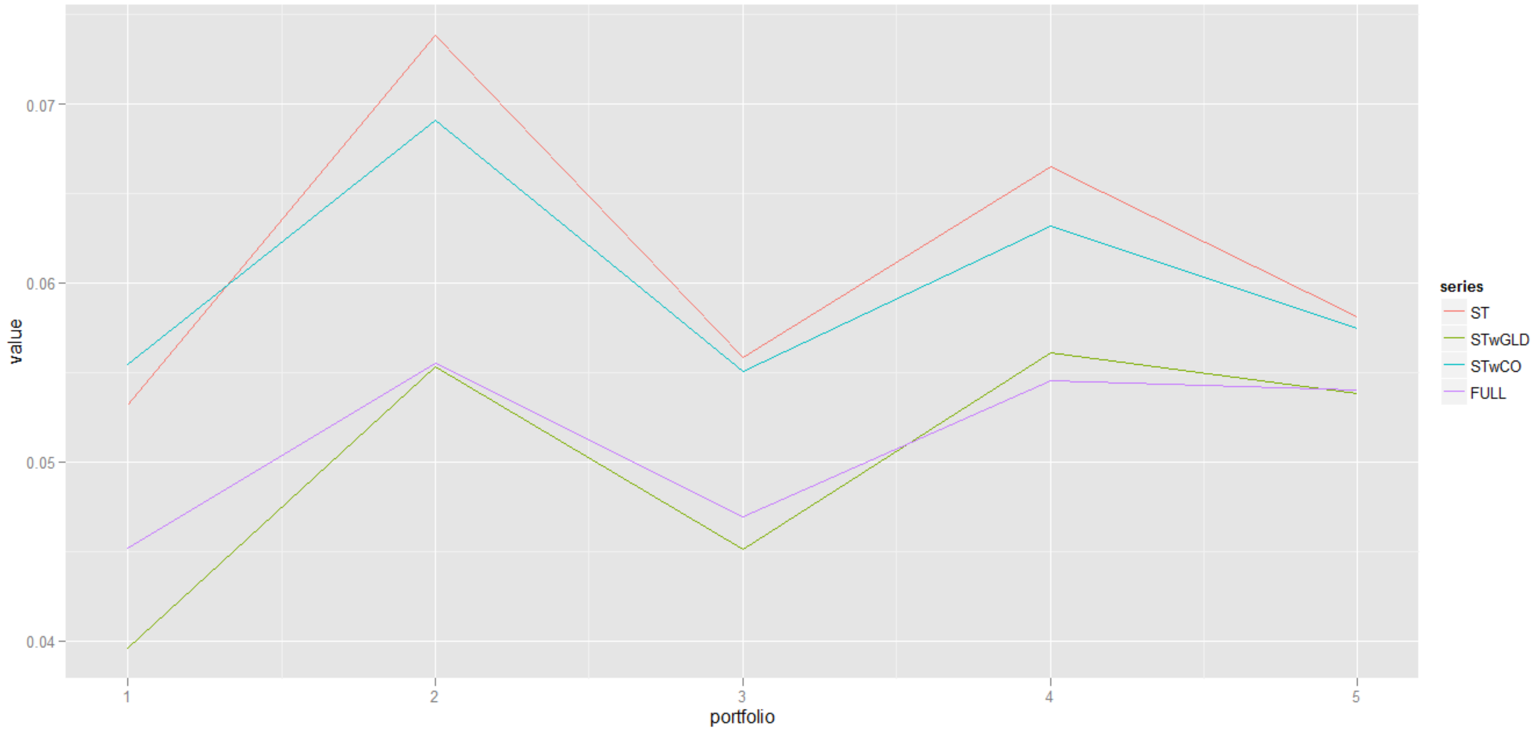

Figure G.16. Weekly $VaR_{0.99}$ of the portfolios computed with normal copula t marginals model.

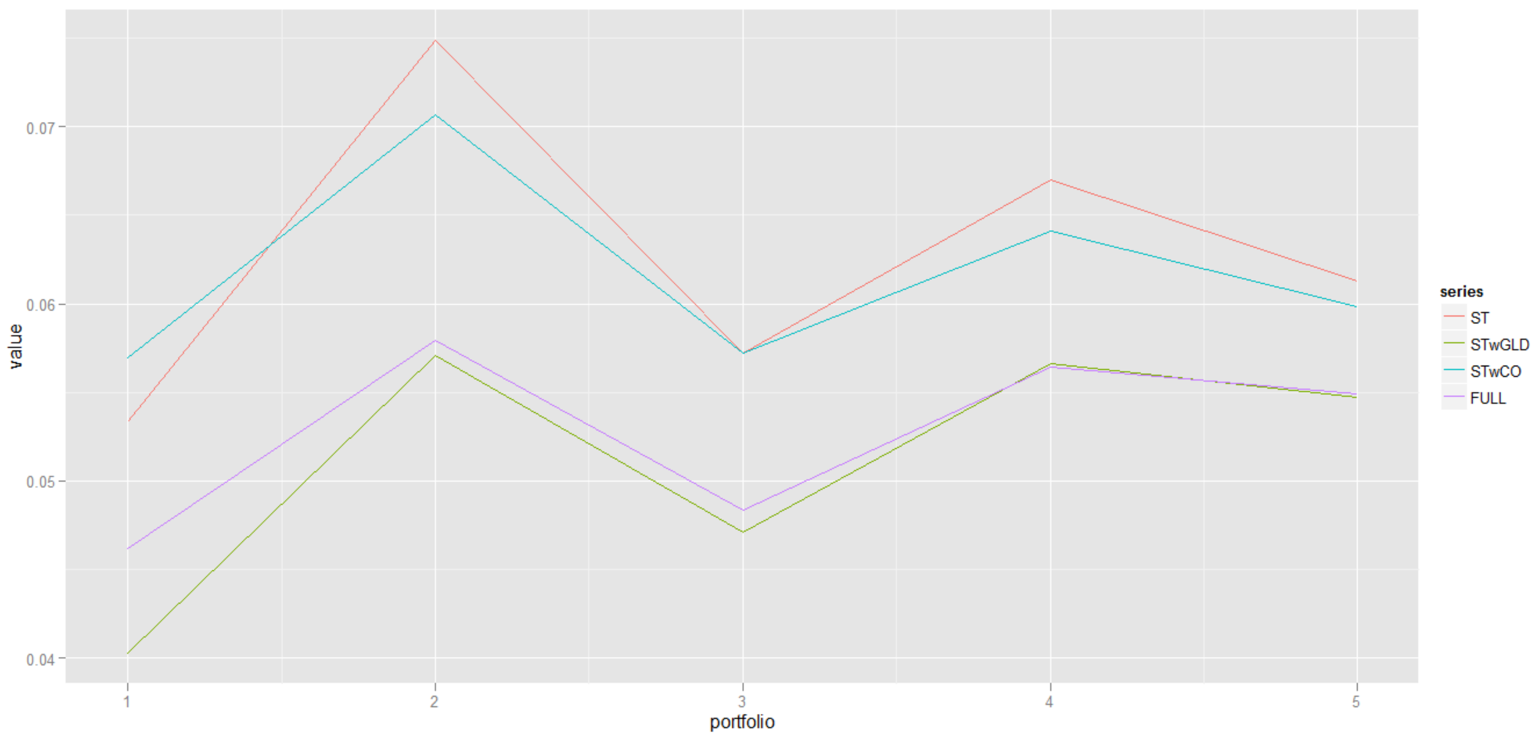

Figure G.17. Weekly $VaR_{0.99}$ of the portfolios computed with t copula t marginals model.

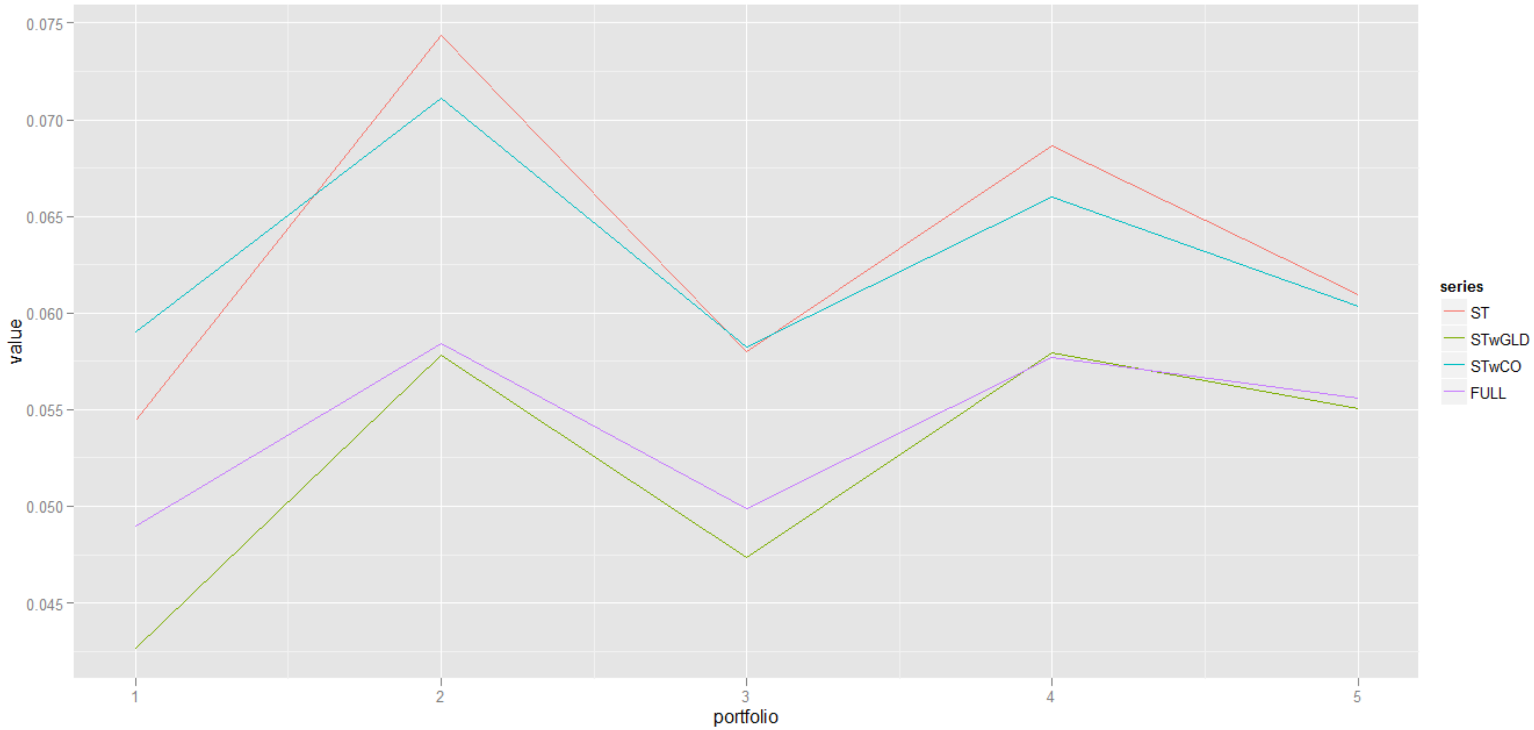

Figure G.18. Weekly $VaR_{0.99}$ of the portfolios computed with t copula ghyp marginals model.

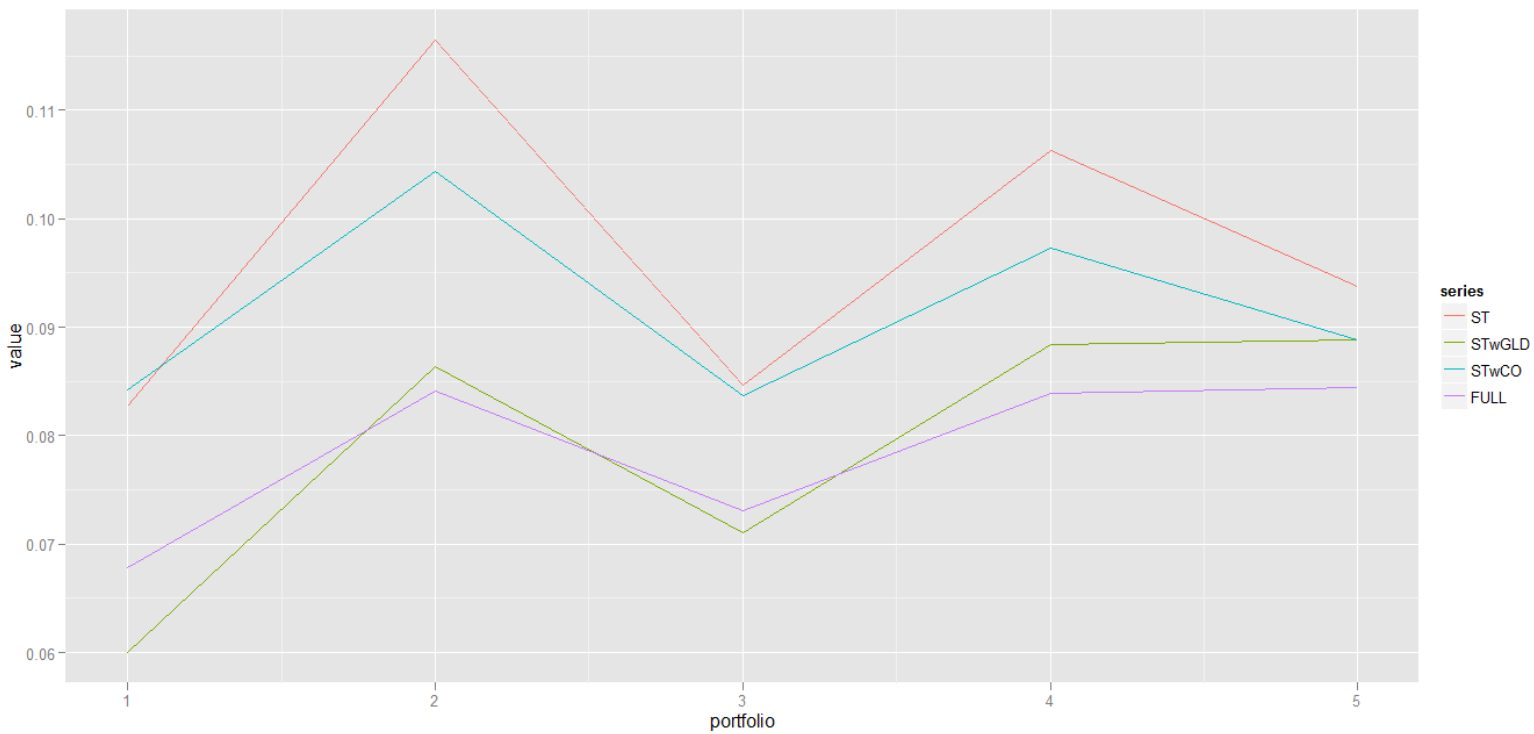

Figure G.19. Weekly $CVaR_{0.999}$ of the portfolios computed with normal copula t marginals model.

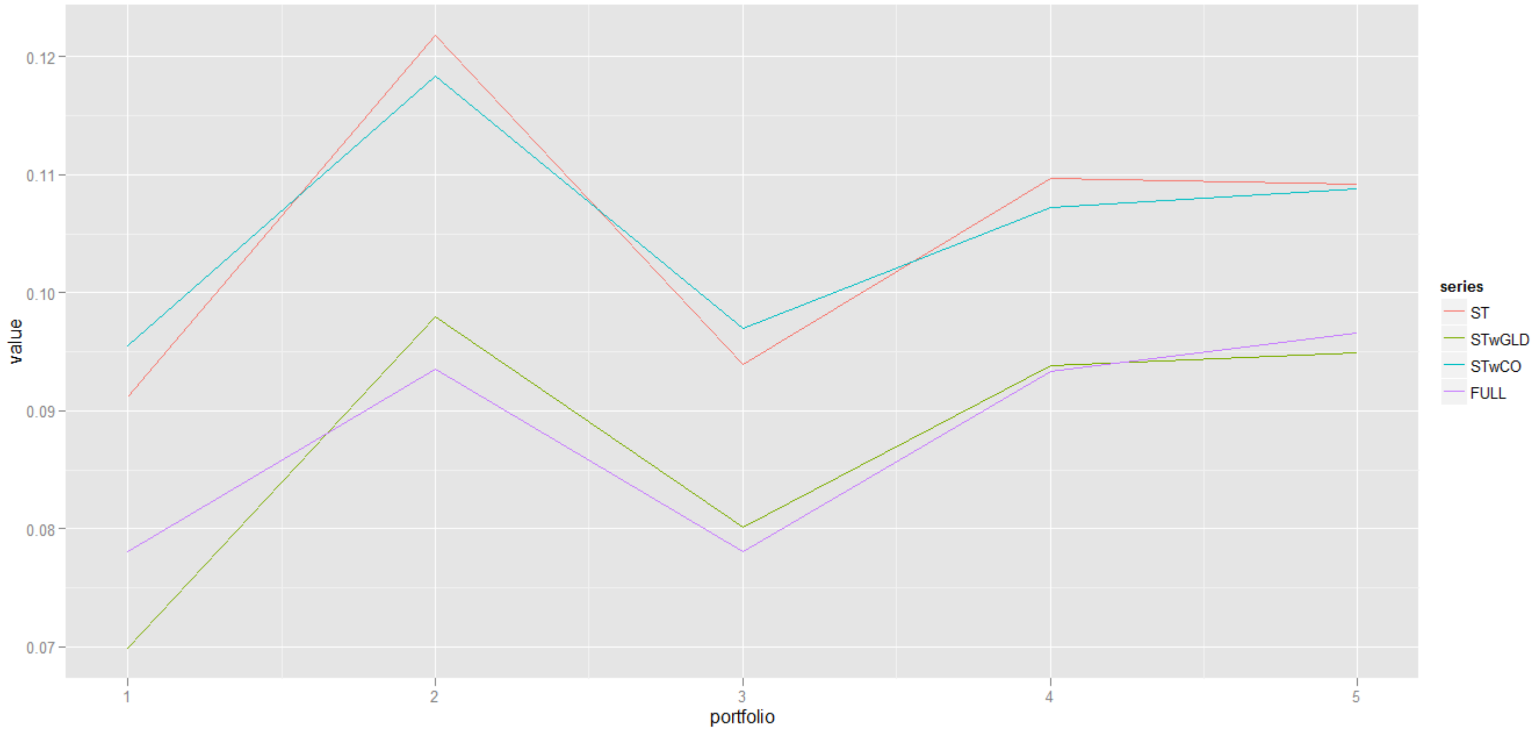

Figure G.20. Weekly $CVaR_{0.999}$ of the portfolios computed with t copula t marginals model.

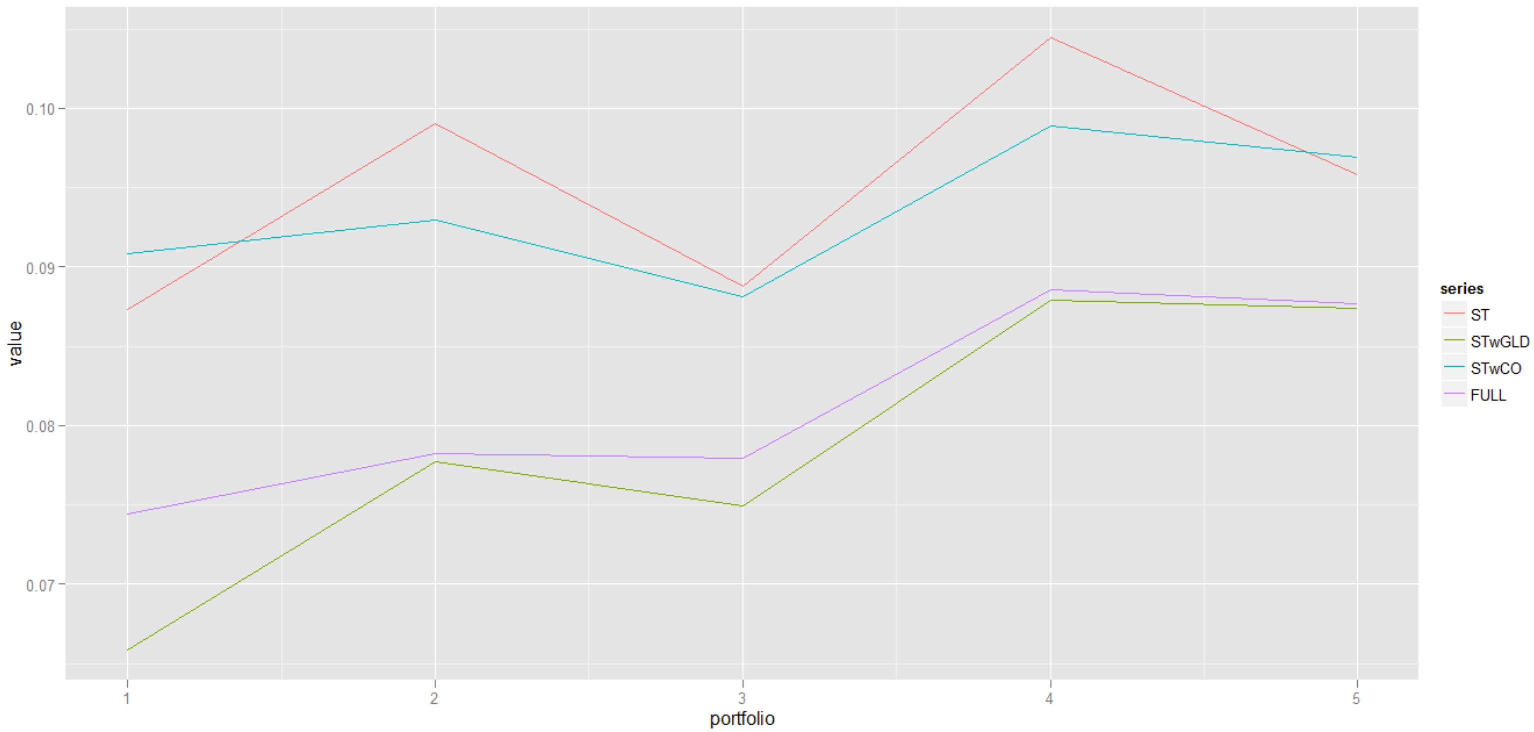

Figure G.21. Weekly $CVaR_{0.999}$ of the portfolios computed with t copula ghyp marginals model.

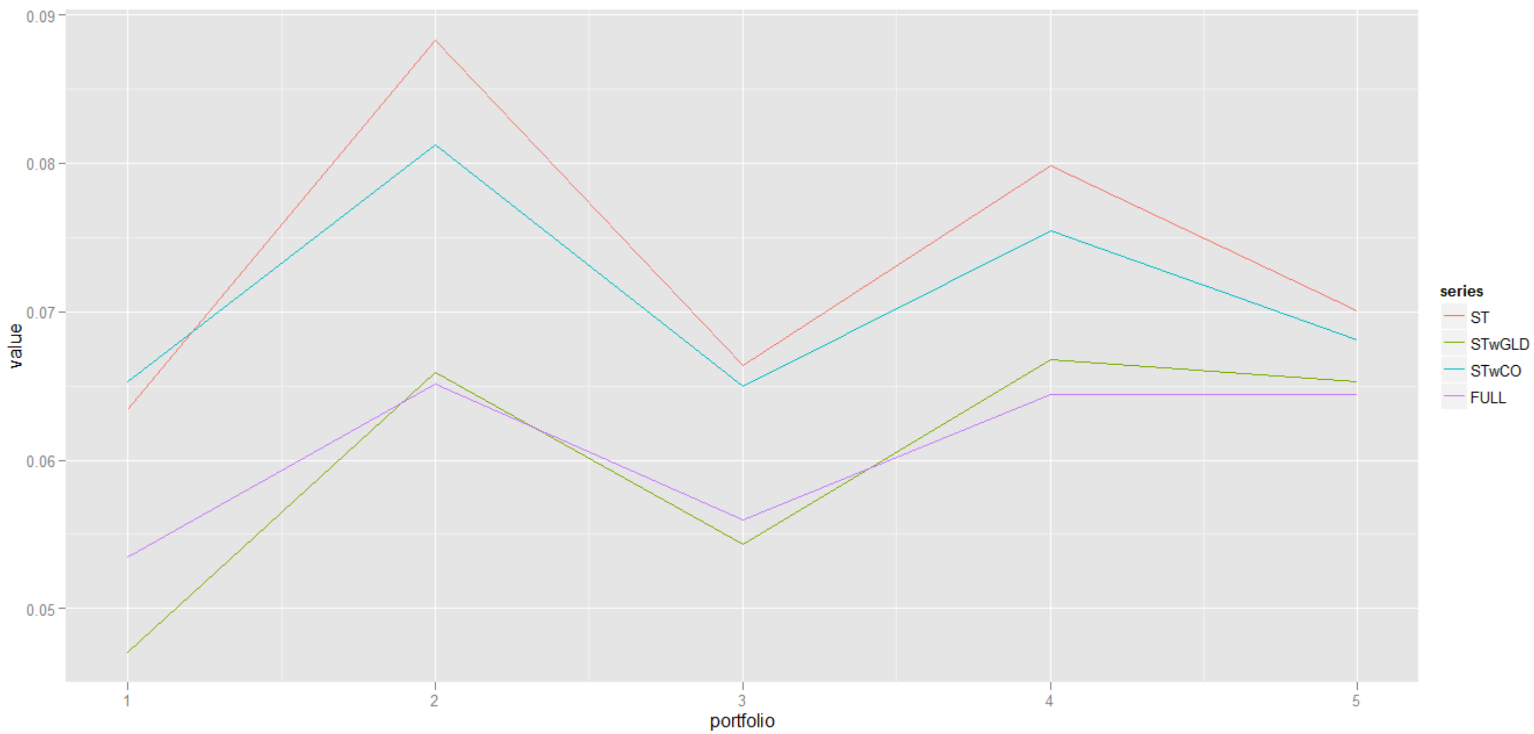

Figure G.22. Weekly $CVaR_{0.99}$ of the portfolios computed with normal copula t marginals model.

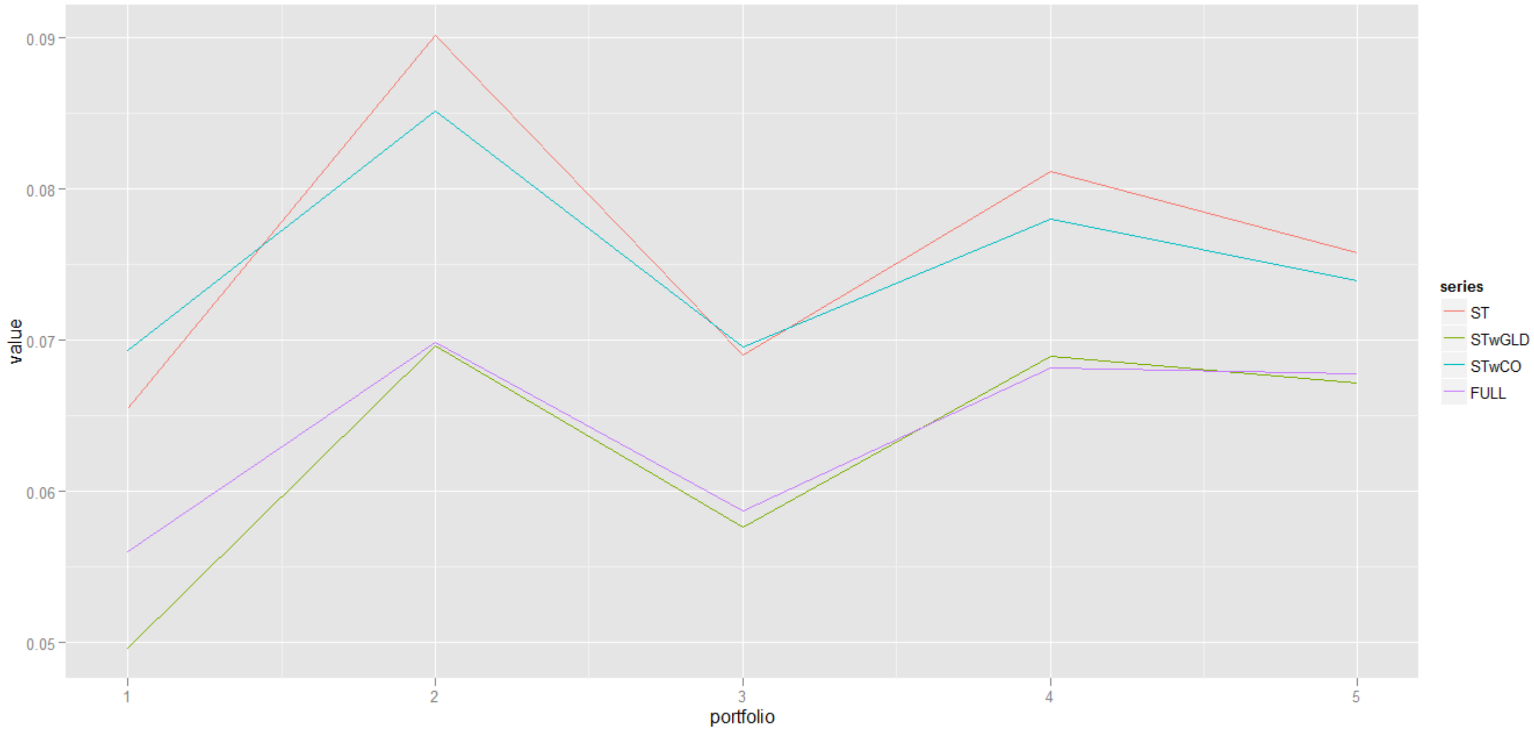

Figure G.23. Weekly $CVaR_{0.99}$ of the portfolios computed with t copula t marginals model.

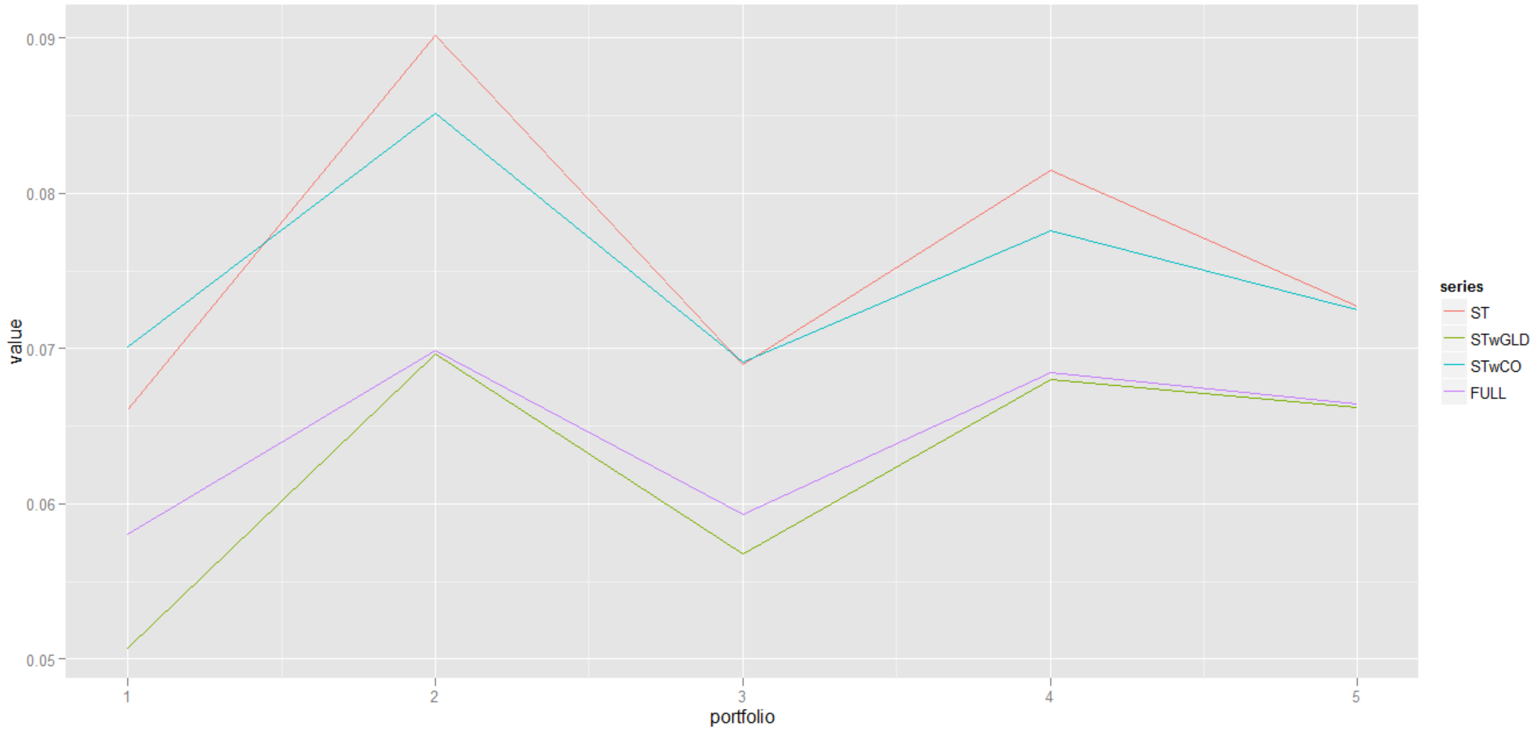

Figure G.24. Weekly $CVaR_{0.99}$ of the portfolios computed with t copula ghyp marginals model.

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