SUPPLY DISRUPTION IN MULTISTAGE PRODUCTION SYSTEMS

by

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ABSTRACT

SUPPLY DISRUPTION IN MULTISTAGE PRODUCTION SYSTEMS

This study aims to analyze multistage production systems under supply disruption. First part of the study examine an N stage serial system under supply disruption risk. It is assumed that the unreliable supplier is unique and known beforehand. It is assumed that the disrupted supplier can not place a new order during disruption however it can continue to deliver its on hand inventory. An approximation approach for an N stage serial system is also developed. A numerical analysis is presented and the reorder intervals that are integer multiples of the reorder intervals of their successors are obtained for the exact model of a two-stage serial system. In the approximate model, power-of-two policy is applied to determine the reorder intervals. Furthermore, the error due to applying the approximate model and the error due to using the power-of-two policy are computed. It is shown that the approximation provides good performance when the reorder interval of the unreliable supplier is large enough. Second part of the study deals with the assembly systems that are due to supply disruption. The examined assembly system is a simple system that consists of the stage that forms the finished product and its direct predecessors. It is assumed that the unreliable supplier is unique and can be one of the direct predecessors of the root stage. It is also assumed that the proportion of the reorder intervals of the direct predecessors of the root stage to each other and to the root stage must be integer. The model is analyzed by taking into consideration two cases since the average cost of the system changes according to the relation between reorder intervals. An approximation approach is developed by approximating the probability of disruption. A three-stage assembly system is examined in the numerical analysis part.

ÖZET

ÇOK AŞAMALI ÜRETİM SISTEMLERİNDE TEDARİK ENGELİ

Bu çalışma tedarik engeli riski altındaki birden çok aşamalı üretim sistemlerini analiz etmeyi amaçlamaktadır. Çalışmanın ilk kısmı tedarik engeli riski altındaki N aşamalı bir seri üretim sistemini incelemektedir. Güvenilir olmayan tedarikçinin tek ve önceden bilindiği varsayılmıştır. Engellenen tedarikçinin tedarik engeli süresince yeni bir sipariş veremediği ancak elinde olan ürünleri bir sonraki aşamaya temin etmeye devam edebildiği varsayılmıştır. Ayrıca N aşamalı seri üretim sistemi için yaklaşıkk bir model geliştirilmiştir. Sayısal bir analiz sunulmuş ve iki aşamalı üretim sisteminin tam modeli için kendilerinden sonra gelen aşamaların sipariş aralıklarının tam sayı katı olan sipariş aralıkları elde edilmiştir. Tahmini modelde, sipariş aralıklarını belirlemek için ikinin katı politikası uygulanmıştır. Ayrıca tahmini modeli ve ikinin katı politikasını kullanmaktan ileri gelen hatalar hesaplanmıştır. Güvenilir olmayan tedarikçinin sipariş aralığı yeterince büyük olduğunda tahmini modelin iyi bir performans sağladığı gösterilmiştir. Çalışmanın ikinci kısmı tedarik engeli riski altında olan montaj hatlarını ele almaktadır. İncelenen montaj hattı nihai ürünü oluşturan kök aşama ve bu aşamadan doğrudan önce gelen aşamalardan oluşmaktadır. Güvenilir olmayan tedarikçinin tek ve kök aşamadan önce gelen aşamalardan biri olduğu varsayılmıştır. Ayrıca aşamaların sipariş aralıklarının birbirine oranının tam sayı olduğu varsayılmıştır. Sistemin ortalama maliyeti sipariş aralıkları arasındaki ilişkiye göre değiştiğinden model iki durum göz önüne alınarak analiz edilmiştir. Yaklaşık bir engel olasılığı kullanılarak tahmini bir model geliştirilmiştir. Çalışmanın sayısal analiz kısmında üç aşamalı bir montaj hattı incelenmiştir. Elde edilen sonuçlara göre tahmini model ve birbirinin tam sayı katı olan sipariş aralıkları kullanmanın modelin ortalama maliyeti üzerinde anlamlı bir etkisinin olmadığı gösterilmiştir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

A	Rate matrix
$AC_A()$	Average approximate cost of the serial/assembly system per unit time
$AC_E()$	Average exact cost of the serial/assembly system per unit time
$AC_{E,j}()$	Average exact cost of stage j per unit time
$A()$	Arc set corresponding to the directed graph of serial/assembly system
C	Node set in the algorithm for finding the optimal ordered partition for a serial system
c	Constant
C_1	Set of stages of case 1 in the assembly system
C_2	Set of stages of case 2 in the assembly system
$C_{i,EOQ}()$	The classical economic reorder interval problem of stage i
C_{EOQ}	Sum of the classical economic reorder interval problems through stage 1 to N-1
$CC_E()$	Total fixed and holding cost of the exact serial system in a cycle
$CC_A()$	Total fixed and holding cost of the approximate serial system in a cycle
CT	Cycle time
c_s	Shortage cost per lost sale
D	Constant parameter for demand
DS	Dry state
EC_0	Average cost of the exact model of the serial system per uit time when power-of-two policy is not used
EC_1	Average cost of the exact model of the serial system per unit time when the reorder intervals of power-of-two policy are used
EC_2	Average cost of the approximate model of the serial system per unit time when the reorder intervals of power-of-two policy are used

G	Directed graph that represents the serial system
G_k	Subgraph of G
H	Hessian matrix of the objective function of the serial system
h_i	Echelon holding cost of stage i
h'_i	Conventional holding cost of stage i
HC_j	Total holding cost of stage j in a cycle
k	Constant
K_i	Fixed ordering cost of stage i per order
k_j	The proportion of the reorder interval of stage j to the reorder interval of stage 0
l_i	The smallest nonnegative integer for the reorder interval of stage i in the serial system when power-of-two policy is used
m_j	The proportion of the reorder interval of stage j to the reorder interval of stage N
n_i	The smallest nonnegative integer for the reorder interval of stage i in the serial system
s_j	The proportion of the reorder interval of stage N to the reorder interval of stage j
N	Number of stage in a serial or in an assembly system
$N()$	Node set corresponding to the directed graph of serial/assembly system
P_i	Set of all the predecessors of stage i including itself
$P_{WS,DS}()$	Probability of being in the dry state at a given time given that process is in the wet state at time 0
PAE	Percentage approximation error in the serial system
$PEPO2$	Percentage error due to applying power-of-two policy in the serial system
$Q_{i,j}$	Probability of making a transition into state j given that process is in state i
Q	Transition matrix
V	Set of the roots of all clusters except for the one containing the final product in the minimum violators algorithm
r	Set of roots of the clusters that solve the constraint (4.3)

S	State space of Markov Chain
SQ	Set shows the sequence of nodes in the algorithm for finding the optimal ordered partition for a serial system
S_j	Set of all the successors of stage j including itself
T_i	Reorder interval of stage i
T_L	Base planning period
T_i^I	Reorder interval of stage i which is integer multiple of other stages in the assembly system
T_i^o	Reorder interval of stage i which is integer multiple of its direct successor
T_i^p	Reorder interval of stage i which is power-of-two multiple of the base planning period
X	An exponentially distributed product
$X()$	State of the Markov Chain
WS	Wet state
β	Probability of supply disruption
β_0	Approximate probability of supply disruption
λ	Wet period rate
μ	Dry period rate
$\sigma_i()$	Node that precedes stage i in the sequence SQ

1. INTRODUCTION

A few decades ago, manufacturers are used to be concerned mostly about the uncertainty in demand. They also used to assume that supply will be delivered at exact quantity at the required time. However, since random conditions have vital effects on the performance of production systems, we can no longer rely on this assumption. Sources of supply uncertainties, such as breakdowns, strikes, economic and political crises, etc., effect optimal inventory management policies. Ignoring the risk of supply may result in unmet demands, shortage costs and loss of goodwill. One way of dealing with this problem might be holding additional inventory. However, holding too much inventory might be costly if disruptions are not frequent and holding cost is high. Other strategies might be ordering from another supplier if it is cheaper or simply accepting the risk of supply. Therefore, in order to mitigate the impacts of supply uncertainties, managers should make significant changes in their inventory plans.

There exists various types of supply uncertainties. For the case of the yield uncertainty, suppliers can provide only some of the ordered quantity that changes according to the distribution of the yield rate. The reason of the yield uncertainty may be the defective items that have random amounts. In some cases, yield uncertainty depends on the order quantity. On the other hand, supply disruption has a form of all-or-nothing type. Some disruptions destroy the on hand inventory of the supplier while others only prevent the supplier from delivering inventory. Sometimes uncertainty is due to the capacity constraints when demand is greater than the capacity of the supplier, in that case only a part of the demand can be processed. Machine breakdowns, maintenance, reworked items are the possible reasons of the uncertain capacity.

This thesis is concerned with supply disruption in multistage production and inventory systems. A multistage production system needs more than one stage to produce a finished product. In the first part of the study, we analyze serial systems under supply disruption with constant demand. In a serial system, each stage has only one predecessor and one successor, except root stage. The root stage is the only stage that does not have a predecessor and we assume that disruptions occur only at the root stage at a random time and last for a random length of time. The unreliable supplier can not place any order until the end of disruption. We assume that disruptions do not destroy the on hand inventory of the unreliable supplier, it only prevents the supplier from placing new orders during disruption period. The wet period is the length of time in which supply is available while the dry period is the length of time in which supply is unavailable. Both of these periods are assumed to be exponentially distributed. We model the problem in terms of reorder intervals instead of order quantities. A reorder interval is the time between two consecutive orders. Both nested and stationary policies are assumed to solve the problem. A stationary policy requires that a reorder interval is constant, it does not change over time. A nested policy requires that if a stage of the serial system places an order, then the successor of that stage must place an order simultaneously. Therefore, the reorder interval of a stage's predecessor is greater than or equal to the reorder interval of that stage. We assume that each stage has a fixed ordering cost as well as a cost for holding inventory. According to our model, if there is no on hand inventory during disruption, then unmet demands are assumed to be lost and a shortage cost is paid for each unit of them. Our goal is to determine the reorder interval of each stage that minimize the expected cost of the serial system.

In the second part of the study, we examine assembly systems with disruption in supply. An assembly system consists of several stages that assemble a finished product from a set of parts. Each part can be assembled from another set of parts since each stage has a unique successor but several predecessors. We consider a simple assembly system that consists of only the stage in which the semi-finished products take the form of a finished product and its direct predecessors. The unreliable supplier is unique and known beforehand. It can be one of the stages of the assembly system that does not have a predecessor. Other suppliers have the knowledge of the availability status of the unreliable supplier. As in the problem with serial systems, disruptions only prevent the unreliable supplier from placing new orders during the dry period. Moreover, although the unreliable supplier can not place any order during disruption, all of the suppliers including the disrupted one can continue to deliver their on hand inventories until they have none which means even if the unreliable supplier is disrupted, if it still has on hand inventory, then other suppliers keep on giving orders. On the other hand, since other suppliers know the amount of inventory the disrupted supplier has, once its inventory level hits zero, they stop giving orders till the end of the disruption. Another important assumption is that the proportion of the reorder interval of the disrupted supplier to the reorder intervals of other suppliers must be integer. We also assume both nested and stationary policies. Each stage has a fixed ordering cost per order and a holding cost per unit. Stockouts are assumed to be lost and have a shortage cost per lost sale. We try to determine the reorder intervals that minimize the expected cost of the system. Because of the nature of the problem, we examine the problem in two cases. Case 1 assumes that the reorder interval of the unreliable supplier is greater than or equal to the reorder intervals of other suppliers whereas in case 2, it is less than others except the root stage.

We make a number of contributions in this study. We analyze multistage production systems and supply disruptions jointly. We develop an approximate model for the case of the serial system reorder interval problem. We prove that the objective function of the approximate model is unimodal under mild conditions. For practical reasons, we assume that the proportion of the reorder interval of a stage to the reorder interval of its successor must be an integer. Besides, we also show how to determine a reorder interval which is power-of-two multiple of the base planning period in a serial system. For the numerical examples, we compute the reorder intervals that are integer multiples of their successors' reorder intervals for the exact model and we also compute the reorder intervals that are power-of-two multiples of the base planning period for the approximate model. We measure the error due to using the power-of-two policy and the error due to using the approximate model. For the assembly system, we show both the exact and the approximate models under supply disruption. For the numerical analysis part, we examine a simple assembly system that consists of three stages. We obtain the reorder intervals that are integer multiples of each other for the approximate and the exact model and measure the errors due to using the integer reorder intervals and applying the approximate model.

The remainder of this thesis is organized as follows. In Chapter 2, a review of the relevant literature is provided. Chapter 3 deals with the problem of serial systems with supply disruption and results of the experiments are reported. The assembly system with supply disruption is analyzed and experiments are presented in Chapter 4. The last chapter is conclusion for the thesis.

2. LITERATURE REVIEW

In this thesis, we deal with multistage production and inventory problems under supply disruption. Since multistage production systems and supply disruption are our main concerns in this study, we focus on the problems of modeling multistage production and inventory systems as well as the problems of supply uncertainties in the literature.

There is a vast literature on supply uncertainty which can be classified into three categories as yield uncertainty, capacity uncertainty and disruption in supply. Mostly, these problems are examined in either single stage or multistage production systems. We begin this chapter by analyzing papers of single stage production systems with supply uncertainty.

2.1. Single Stage Production Systems with Supply Uncertainty

[1] aims to extend the classical economic order quantity to the case in which the quantity received is a random variable. Two cases are taken into consideration; first one assumes that the standard deviation of the amount received is independent from the quantity ordered while the second one assumes that it is proportional to the quantity ordered. The cost penalties are also showed if EOQ is used instead of the proposed cost functions for both cases. [1] concludes the study by emphasizing that the order quantity which minimizes cost function depends on only the standard deviation of the amount received.

[2] consider a periodic review inventory system with randomness in yield. It is proved that for a single period, randomness in demand does not affect the order size regardless of the yield model and the optimal policy does not have an order up to point. [2] further show that the function values of finite horizon converge to that of infinite horizon.

Unlike the other papers, [3] study a periodic review production system with uncertain demand by analyzing both random yields and variable capacity jointly for the finite horizon and infinite horizon problems by generalizing the study of [2]. A stochastically proportional yield model is used. For the single period case, it is showed that the reorder point and the optimal planned production quantity are independent of the random capacity but depend on the yield rate. However they are effected by the random capacity in the multiperiod problem case and the optimal policy is an order-up-to type. It is also proven that the solution of the finite horizon problem converges to that of the infinite-horizon problem.

[4] examine a periodic review inventory model which deals with both random yield and random environment for a single, multiple and infinite periods. It is assumed that demand is a random variable and because of the randomness in capacity, yield is also random and unmet demands are backordered. Contrary to the expectations, random environment does not make the structure of the ordering policy more difficult since base stock is stil an optimal policy that depends on the environment. [5] give a detailed review of lot sizing problem with random yields in the literature and divide the problem of the randomness in yield into groups such as the model with the creation of good units is a Bernoulli process, stochastically proportional yield model, model with the distribution of good units change with batch size, and an approach that assumes the yield uncertainty as a result of the random capacity.

[6] model a periodic review inventory problem with uncertain demand and uncertain capacity for single, multiple and infinite periods. The randomness in capacity has no influence on the optimal policy for the model of single period while in the models of multiple period and infinite-horizon, it depends on the distribution of the capacity as in the study of [3]. However in the setting of the infinite horizon, the optimal order policy is an extended myopic policies.

[7] considers a periodic review inventory system with stochastic demand and variable capacity. A procedure for optimal base stock levels by constructing an analogy between the class of base stock production/inventory policies that operate under

demand /capacity uncertainty, and the G/G/1 queues is proposed.

[8] studies a non-stationary periodic review inventory model with uncertainty in both demand and capacity and for the case of infinite horizon problem. [8] develops upper and lower bounds of the optimal order up-to levels that converge as the planning horizons get longer. The differences between the upper and lower bounds of the optimal policies decrease as the length of planning horizons becomes longer.

[9] prove that although one thinks that the positive inventory is a buffer against supply disruption, zero inventory is an optimal policy when probability of paying no inventory cost is large enough. [9] show that with this condition, it is more advantageous to maintain zero inventory.

[10] analyze a model which considers supply disruptions in classical inventory models. An EOQ type inventory model is considered in which supply is only available during an interval of random length and then unavailable for another interval of random length. The decision maker knows the availability status of the supplier and a zero-inventory ordering policy is assumed. The amount of demand which can not be satisfied during dry period is assumed to be lost. Renewal reward theory is applied to construct the objective function and to determine the optimal order quantities. Two different cases are considered and numerical analysis are presented with the interpretation of the results. Both wet and dry periods are exponentially distributed in the first case while only the wet period is exponentially distributed and the dry period is deterministic in the second one. According to the numerical examples, the total cost is lower but the ordered quantity is higher in the latter case. The main reason is that the shortage cost is smaller in the case with the deterministic dry period and as the ordered quantity increases, the shortage cost decreases. Therefore a larger value of ordered quantity is needed to obtain the minimum value of the objective function. As a result, [10] think that as the constant value of the dry period approaches to zero, the model of the latter case will converge to the classical EOQ model. [11] corrected their cost function and proved that it is unimodal.

[12] also try to determine order quantities and reorder points of an inventory model when there exists any uncertainty. [12] analyze the inventory model for single and multiple suppliers which can be on and off at different times and they apply renewal reward theorem to construct the cost function. When all suppliers are available, the objective function of the model reduces to the classical EOQ model. [13] analyze a stochastic inventory problem with deterministic demand in which the manufacturer works with two different suppliers who are unreliable. These two suppliers might be disrupted or not for random durations. The manufacturer can obtain inventory from either of them when the reorder level drops to r if one of them is available. The aim of the manufacturer is to determine the order quantity and the reorder point to minimize its total cost when there is risk of supply disruption. Erlang distribution is used for on periods while off periods have a general distribution. Renewal reward theorem is applied to build the long run average cost per time. [13] consider two different cases. The first one is a $E_2[E_2]/M[M]$ model, both suppliers have Erlang distribution and its numerical solution is presented. The optimal order quantities of their model and EOQ model are compared and it is found that the optimal order quantity is always greater than that of the EOQ model. The second case assumes large values for the order quantity in order to use their limiting probabilities. However for the order quantites that do not have large numbers, this assumption provides poor results. an alternative inventory policy is also considered in which the decision variables are depend on the number of available suppliers.

[14] indicate that in order to minimize the cost function, the manufacturer must determine the order quantity and the reorder point when there is risk of supply disruption. Therefore [14] develop closed form approximate solutions for the globally optimal order quantity/inventory reorder point, the optimal order quantity given an inventory reorder point and the optimal inventory reorder point given an order quantity. It is proven that the close form results are upper bounds to the exact cost for nonnegative reorder point values. Numerical results are also presented to verify the results.

[15] consider a continuous review inventory model with a single retailer and a single supplier. Their study differs from the others in two aspects; firstly, it deals with

both supplier and retailer disruption. Secondly, if disruptions occur at the retailer, then they destroy all the inventory at the retailer. On the other hand a disruption at the supplier only prevents the supplier from providing inventory. [15] also indicate that the costs savings of the proposed model outweigh the choice of using EOQ model and they show that the disruption at the retailer has a greater affect on the cost function than that of the supplier disruption. A tight approximation is also presented for the cost function.

[16] analyzes a continuous review inventory system with deterministic demand under supply disruption. [16] extends the study of [10] by presenting a tight approximation. The on and off periods are exponentially distributed as in the paper of [10] and orders are not placed unless there is no inventory on hand. Besides, there is no lead time for orders and unmet demands are lost. It is assumed that wet periods are longer than dry periods on average and it is cheaper to satisfy demands instead of paying shortage costs for every unit of it. [16] approximates the cost function of [11] by assuming that the system approaches steady state quickly enough that when there is no inventory we can ignore the transient nature of the system at this moment. The approximation is reliable when λ and μ have higher values. It is proven that the cost function of the approximate model is convex and its optimal cost value is greater than the optimal cost value of the EOQ model. Besides, the optimal order quantity is greater than that of the EOQ model. It shows that cost of applying EOQ model instead of EOQD, EOQ with disruption model, can be large when there exists uncertainty. Cost function of EOQD converges to the cost function of EOQ when λ gets comparatively smaller than μ. [16] shows that the optimal cost function and order quantity are greater than that of the exact cost function and presents the percentage error due to using approximate model instead of the exact model. It is also proven that the power-of-two policy can be at most 6% worse than the overall policy.

Periodic review inventory systems with supply disruption also take place in the literature. [17] consider a single item, periodic review inventory system with deterministic demand and uncertainty in supply. Supply follows a bernoulli process and on and off probabilities are assumed to be nonstationary. [17] show that the optimal ordering policy is of order-up-to type and develop an algorithm to obtain the optimal order-up-to levels. [18] model a periodic review, single item, deterministic demand inventory model with supply uncertainty. Their study extends the paper of [17] by considering that supply is either fully available, partially available or completely unavailable.

Apart from these studies, the cases in which both supply and demand are random are examined as well. [19] examines the (Q,r) inventory system with an unreliable supplier. Demand is assumed to be poisson and on and off periods are exponentially distributed. Firstly, the zero lead time problem with allowable number of outstanding orders is analyzed and found that the reorder point is set to 0 by the approximation. Secondly, the problem with a constant lead time with at most one number of outstanding orders is analyzed. Q is obtained from the EOQ model and then reorder point is found. It is found that although the approximation of the zero lead time problem is good when number of cases are large, the approximation of the constant lead time problem is not very good.

[20] extends the study of [10] by allowing random demand and random lead time in a continuous review stochastic inventory problem. It is assumed that on periods are E_k random variables while off periods have general distribution. The decision variables are the order quantity and the reorder point when there is risk of supply disruption. The objective function is constructed using renewal reward theorem and analysis of the cost function is presented. In the second part of the study, it is assumed that the order cost is quite large while the holding cost is relatively small. This assumption ensures large values for the order quantity of the system and simplifies the analysis of the cost function. It is proven that the cost function is convex if mild conditions are assumed.

[21] model a (s,S) type inventory system with stochastic demand under supply disruption. Their study differs from others since it deals with partial backorders, some proportion of the unmet demand is backordered while the remaining ones are assumed to be lost. [21] examine the impacts of average on/off periods and the proportion of backordered products on optimal s and S to manage an optimal inventory policy. It is found that s^* and S^* are decreasing in μ and increasing in λ. It is also indicated

that an increase in the probability that a stockout becomes a backorder, makes the expected cost per demand arriving during stockout increase, as well.

[22] studies an exact cost minimization model for a (s,Q) inventory system with stochastically distributed demand and lead time as an extension to the paper of [19]. Unmet demands are assumed to be lost and on and off periods are exponentially distributed. The retailer can obtain its orders that are placed before disruption regardless of the availability status of the supplier. The outstanding orders can be at most one. Numerical illustration is also presented to give more insight. [23] models a similar system to [22] by analyzing a continuous review (s,Q) type inventory system in which lead time is hyperexponentially distributed and poisson demand is assumed, and the maximum number of outstanding orders at any time is limited to one. Alternating renewal process is used to model the availability of the supplier. [24] study a continuous review inventory system with random demand, random lead time and supply disruption. Their study differs from others as it assumes that when supplier is off, it also stops processing an outstanding order, so it is not same with active processing. This variability in lead time has important influence on an inventory system, since as it increases, it makes the average cost and the risk of stockout increase, too.

[25] considers a periodic review inventory system with backorder. A supply chain model with two suppliers is examined in which one of them is unreliable while other one is reliable but more expensive and both of them are capacity constrained. [25] points out that the duration and the frequency of a disruption has a significant impact on the optimal inventory management. For instance, if the capacity of the reliable supplier is constrained when it is infinite at the unreliable supplier, the manufacturer may want to mitigate the effects of the disruption by holding additional inventory. On the other hand, if the reliable supplier has volume flexibility, then it is found that sourcing mitigation is optimal when disruptions are rare.

[26] examine a single period inventory system with a firm and two suppliers as in the study of [25]; first supplier has both recurrent (random yield) and disruption risks while the other one is perfectly reliable but more expensive. Unlike the study

of [25], both recurrent and disruption uncertainties are unresolved when an order is placed with the first supplier. It is assumed that the firm orders from the unreliable one when there is no supply risk, however when demand can not be met, it reserves from the reliable one. The aim of the study is to determine the order quantities of the both suppliers under different conditions. [26] observe that when uncertainty in supply is the result of the disruption, it is optimal to order more from the reliable supplier and less from the unreliable one. On the other hand when the increase in risk of supply is due to the recurrent uncertainty, unreliable supplier should be preferred instead of the reliable one.

[27] model an inventory problem with supply disruptions, phase type times to on and off periods, and random demand. [27] analyze both ZIO and non-ZIO policies and the robustness of a time-dependent ordering policy under these conditions and conclude that time dependent policies are more advantageous than non-real-time policies.

[28] aim to model a multiperiod supply chain design problem with one supplier and multiple retailers and facilities. The supplier is reliable whereas the facilities that transport products to retailers are unreliable. Since each facility differs from the others, their reliabilities are also different. Therefore they must find the number of opened facilities and decide which one should be preferred and which inventory policy must be applied. They develop a solution algorithm for the nonlinear optimization problem and prove its convergence.

[29] analyze a One Warehouse Multiple Retailer system under supply disruption with zero lead time and present numerical analysis. They consider two different inventory systems; centralized and decentralized. Inventory is stocked at the warehouse in the centralized system while it is stocked at the retailers in the decentralized system. The aim of considering two inventory systems is to analyze the risk pooling and the risk diversification effect. Disruption occurs only at the warehouse in the centralized system while it occurs only at the retailers in the decentralized system. They analyze the cases in which demand is deterministic but supply is disrupted and demand is random but supply is deterministic. When demand is deterministic and supply is due to

the disruption, decentralized system is optimal for a risk averse inventory policy since decentralized system mitigates the disruption risk. On the other hand when supply is deterministic and demand is uncertain, it is optimal to follow the centralized system policy since expected cost of the decentralized system equals to n times expected cost of the centralized system. When both supply and demand is uncertain, the decentralized system is optimal for longer disruptions while centralized system is optimal for shorter and less frequent disruptions. For a risk averse strategy is is more advantageous to prefer decentralized system.

2.2. Multistage Production Systems with Supply Uncertainty

[30] examine an N-stage serial production system with stochastic yield rates in each stage which are independent of the input size. The output of a stage equals to the product of its yield rate and the input value. The manufacturer has the overage cost for the overage quantity, the shortage cost for the unsatisfied demands and the production cost. Therefore decision variables of the model are the input sizes for all the stages of the serial system. It is assumed that the input can not exceed the output and two conditions are proposed for the solution of the model. First one indicates that it must be less expensive to dispose an item at stage i than to process it and dispose it at the next stage. Second one indicates that manufacturing the product is profitable rather than losing them. [30] present an optimal policy which defines a relationship between the output of a stage and a critical number of that stage. [30] also consider the case of positive inventory and provide numerical analysis.

[31] consider a multistage production system with yield uncertainty and present numerical examples to provide more insight. They analyze the problem in two cases: single production run and multi-production runs. They assume that at the beginning of each stage, they have an amount of input and after processing the semi-finished items, the amount of input can be greater than the output, it can be less than the output or it can be equal to the output. Therefore the model has three choices to make. Before beginning the next production stage, it must process all the non-defective units of the previous stage if the output is equal to the required input value; it must reduce

the output value by disposal of some units if the amount of output is greater than the required input and finally it must purchase additional semi-finished units if the amount of output is less than the required input. According to the model, the manufacturer has two critical numbers defining the structure of the optimal policy. If the amount of output is less than or equal to the lower limit, then the amount of input equals to the lower limit; if it is between the upper and the lower limit, then it equals to the amount of input; if it is greater than or equal to the upper limit, then the amount of input equals to the upper limit. The model indicates that if there is available and free supply of raw material, then the procurement cost is zero and if the procured semi-finished units are limited, then the input value of the next stage is the minimum of the limited procured semi-finished units and the difference of the lower limit and the output (if the lower limit is greater than the output). For the case of multi-production run, the optimal policy is not simple so an approximate approach is developed to find good control rules.

[32] extends the study of [31] by analyzing multiple production runs in a serial system with binomial yields at each stage. It is assumed that the number of production runs is limited. Two cases are examined separately; first one assumes that there is no set-up costs for additional production runs and the second one assumes that there exists set-up costs for additional production runs. At each stage, there exists non-defective items distributed binomially. The manufacturer has to pay a shortage cost and an overage cost for the unsatisfied demands and the redundant finished products. Therefore the optimal amounts of available semi-finished products for all of the stages are crucial to determine. [32] indicates that if the amount of finished products are not sufficient for the customer demand, then the manufacturer has to process an optimal batch to meet the unsatisfied demand. Because of the complexity of the model, a decomposition approach is developed to solve the model and numerical illustration is also presented.

[33] consider an assembly system with random demand and supply uncertainty. It is assumed that a finished product is assembled using two components that are delivered from two different and unreliable suppliers. The demand for the finished

product is random. It is also assumed that there exists initial inventory levels for the finished product and its components before beginning processing and the initial inventory of component 1 is greater than or equal to the initial inventory of component 2. The decision variables are the amount of the finished product and its components to satisfy the customer demand. [33] consider a release level constraint that ensures with a given probability there will be sufficient amount of components to assemble the finished product. They develop an approximate cost function since the exact cost function is not easy to solve. In order to measure the error due to applying the approximate cost function, a sensitivity analysis is conducted and it is seen that the error is not very large even for the large values of the release factor levels. [33] analyze the joint supplier case which offers that two different components can be obtained from a joint supplier as a set. If delivering the components is more costly than the individual suppliers, then manufacturer stops ordering from the joint supplier. If the cost of joint suppliers is higher than the joint supplier is used. [33] consider the multi-period case and conclude that it might be optimal to order extra components for the next periods.

2.3. Multistage Production Systems with Deterministic Supply

[34] analyze multistage production systems by incorporating power-of-two policy into the model. They examine serial, assembly, distribution and other general systems with constant demand and propose algorithms. They determine optimal reorder intervals instead of order quantity for each stage by showing the reasons that it is easier to analyze multistage production systems in terms of reorder intervals to get rid of various constraints and there is no need to adjust the reorder intervals when demand patterns change. Both nested and stationary policies are assumed although they are not optimal for all production systems so nonnested policies are examined also. Echelon inventory method is implemented instead of a conventional inventory approach for computing annual holding cost of the system. Although they have the same computational result, it is easier to use the echelon inventory approach in multistage production systems. The main problem is to find the optimal order partition that ensures nestedness condition. A new bill of material network is created that consists of subgraphs. A node set in a

subgraph is defined as a cluster and all stages in a cluster has the same reorder interval. After determining the optimal partition, it is easier to find the reorder intervals of each stage. Because of the practical issues, a power-of-two policy is applied for finding them. It assumes that there exists a base planning period, a minimum reorder interval, and all reorder intervals are powers of two multiples of it. It is proven that the average cost of this type of solution can not be more than 6% higher than the average cost of any other optimal policy. They further examine models of a general system structure and observe the effect of constraints on available time for setup and production.

This study can be considered as an extension to the studies of [34], [10] and [16]. These papers provide a foundation for our work. However we provide new insights since we consider multistage production systems and risk of supply disruption jointly. We also provide an approximation model for serial systems under supply disruption.

3. SERIAL SYSTEMS UNDER SUPPLY DISRUPTION

A serial system is composed of consecutive stages that convert a raw material into finished consumer goods. A depiction of a serial system with n stages is presented in Figure 3.1. As can be seen in Figure 3.1, each stage has a successor except for stage 1. Stage 1 is the final stage in which a semi-finished good is transformed into a finished product. A serial system consists of a node set and an arc set denoted as $N(G)$ and $A(G)$, respectively, where G indicates the directed graph that represents the serial production system. Nodes in the node set imply the production stages while arcs show the order of the production stages.

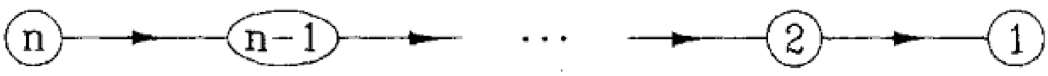

Figure 3.1. Example of A Serial System.

3.1. Problem Definition

This study considers a serial production system with an unreliable supplier who may be disrupted randomly. There can be only one unreliable supplier and it is stage N.

We assume that when stage N is disrupted, it stops placing new orders during disruption. We also assume that disruption does not destroy the on hand inventory of the disrupted supplier. Therefore, if the unreliable supplier has inventory when it is disrupted, it can still deliver its on hand inventory regardless of the disruption until it has none. However, when it consumes all the inventory on hand, since it can not give a new order, the serial system can not complete the production process and consequently it can not satisfy the demand of the customer until the end of the disruption.

Figure 3.2 shows an example of the disruption process that we consider in this study. We see that the first disruption begins at time A and ends at time B. The performance of the production system is not affected by the disruption since the unreliable supplier has still inventory when disruption ends at time B. We should notice that there is a possibility that a disruption may last so short that the unreliable supplier has still on hand inventory at the end of the dry period. Therefore, it manages to deliver inventory regardless of the disruption and gets rid of paying shortage cost as in this example. On the other hand, if we examine the second disruption that begins at time C, we see that disruption has a crucial impact on the performance of the system. We see that stage N has still inventory on hand when it is disrupted. Therefore it can satisfy the demand of stage N-1 until it has no inventory. However, when it consumes all of it, it can not satisfy the demand of stage N-1 since it can not place orders during disruption. Therefore, as we can see in Figure 3.2, when stage N is out of stock at the second disruption, stage N-1 has to wait until the end of the disruption to place a new order. As a result, the production system has to pay shortage cost for each unit of the unmet demand that occurs during the disruption period. Furthermore, we assume

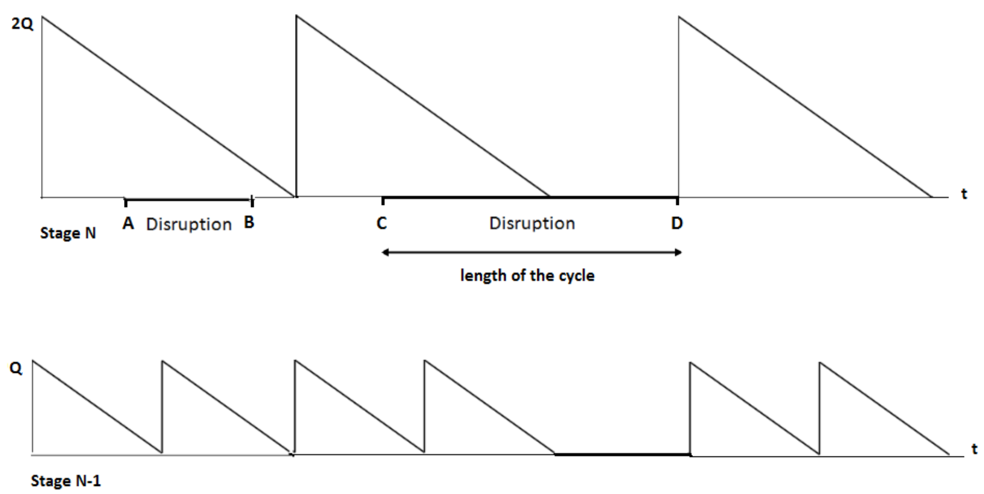

Figure 3.2. Disruption in a Serial System.

both a nested and a stationary policy while analyzing serial systems. A nested policy demands that whenever stage i places an order, stages from 1 to $i-1$, the successors of stage i, must place an order, too. A nested policy makes work-in-process inventory as small as possible thus it decreases the holding cost of the system. A stationary policy demands that the reorder interval of stage i can not change over time.

We use echelon stock method to compute the holding cost of the system per cycle. According to this approach, the holding cost of stage i equals to the product of its echelon holding cost and echelon stock. An echelon holding cost takes into consideration only the cost of the inventory at stage i by subtracting the conventional holding cost of its predecessor from its conventional holding cost ($h_i = h'_i - h'_{i+1}$ where h'_i is the conventional holding cost of stage i). On the other hand, an echelon stock for stage i equals to the sum of the inventories from stage 1 through stage i. At the end, both methods have the same result however it is easier to implement the echelon stock method in multistage systems.

We also assume that demand is constant and denoted as D units per year. The amount of demand can be different for each stage since stage i can need several semi-finished items produced at stage $i+1$. However, we assume that the amount of demand is the same for all the stages. The fixed ordering cost is K_i per order and the echelon holding cost is h_i per unit per year for stage i. Supply is only available during an interval of random length called the wet period and then unavailable for another interval of random length called the dry period. Both wet and dry periods are exponentially distributed with rates λ and μ, respectively. Unmet demands are assumed to be lost with a shortage cost of c_s per lost sale. Orders are placed when inventory level hits zero and there is no lead time for orders. We assume that parameters of the model are nonnegative.

As in the study of [34], we model our problem in terms of reorder intervals rather than order quantities because by doing so, we get rid of complex constraints that make our model more difficult to solve. T_i is the reorder interval of stage i. However, since a reorder interval can take any value, it is difficult to use it in practice. Therefore, we assume that the proportion of the reorder interval of stage i to the reorder interval of its successor must be integer.

3.2. The Exact Model for A Two-Stage Serial System under Supply Disruption

We analyze a two-stage serial system which is due to supply disruption. We assume that disruption occurs only at stage 2. T_1 and T_2 are the target reorder intervals of stages 1 and 2.

Proposition 3.1. *The probability of disruption in an order cycle of stage 2 is called as β:*

$$\beta(T_2) = \frac{\lambda}{\lambda + \mu}(1 - e^{-(\lambda+\mu)T_2}).$$

Proof. We show the proof as it was explained in [35] .

Let $X(t)$ be the state of our system at time t. $X(t)$ is a Markov chain with state space S. $S = \{WS, DS\}$ with WS corresponding to the wet state and DS corresponding to the dry state of the system. $Q_{i,j}$ is the probability of making a transition into state j given that the process is in state i. Suppose that $Q = \{Q_{i,j}\}$ where $i, j \in S$. Then Q is a transition matrix with zero diagonal entries. A is the rate matrix which is obtained by writing the holding parameters of the states on the main diagonal with a minus sign. Row sum of A matrix equals to zero. The holding parameters of our system are λ and μ for wet and dry states, respectively. We want to obtain $P_{WS,DS}(T_2) = P\{X(T_2) = DS|X(0) = WS\}$, the probability of being in the dry state at time T_2 given that the process is in the wet state at time 0.

First, we need to form the transition matrix and the rate matrix of the system.

The transition matrix is:

$$Q = \begin{array}{c c} & \begin{array}{cc} WS & DS \end{array} \\ \begin{array}{c} WS \\ DS \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

. The rate matrix is:

$$A = \begin{array}{c c} & \begin{array}{cc} WS & DS \end{array} \\ \begin{array}{c} WS \\ DS \end{array} & \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \end{array}$$

. The backward equation for a Markov process is:

$$P'(t) = A\ P(t).$$

By using the backward equation, we obtain:

$$P'_{WS,WS}(T_2) = \lambda(P_{DS,WS}(T_2) - P_{WS,WS}(T_2))$$
$$P'_{DS,WS}(T_2) = \mu(P_{WS,WS}(T_2) - P_{DS,WS}(T_2)).$$

If we multiply $P'_{WS,WS}(T_2)$ and $P'_{DS,WS}(T_2)$ with μ and λ, respectively, and then take the sum of them, we get:

$$\mu P'_{WS,WS}(T_2) + \lambda P'_{DS,WS}(T_2) = 0$$

by integration, we obtain:

$$\mu P_{WS,WS}(T_2) + \lambda P_{DS,WS}(T_2) = c$$

for some constant c. We know that

$$P(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we set $T_2 = 0$ to obtain:

$$\begin{aligned} \mu P_{WS,WS}(0) + \lambda P_{DS,WS}(0) &= \mu * 1 + \lambda * 0 \\ c &= \mu \\ \mu P_{WS,WS}(0) + \lambda P_{DS,WS}(0) &= \mu, \end{aligned}$$

if we solve it for $P_{WS,DS}(T_2)$ and plug it into $P'_{WS,WS}(T_2)$, we get:

$$P'_{WS,WS}(T_2) = \mu - (\lambda + \mu) P_{WS,WS}(T_2).$$

Now, let us define $g(T_2) = P_{WS,WS}(T_2) - \frac{\mu}{\lambda+\mu}$, then we find that:

$$\begin{aligned} g'(T2) &= -(\lambda + \mu) g(T_2) \\ \frac{g'(T_2)}{g(T_2)} &= -(\lambda + \mu) \end{aligned}$$

by taking integration of both sides, we obtain:

$$\begin{aligned} \log g(T_2) &= -(\lambda + \mu) T_2 + c_0 \\ g(T_2) &= e^{-(\lambda+\mu) T_2} k \end{aligned}$$

for some constant k. Thus,

$$P_{WS,WS}(T_2) = e^{-(\lambda+\mu) T_2} k + \frac{\mu}{\lambda + \mu}.$$

For $T_2 = 0$,

$$\begin{aligned} P_{WS,WS}(0) = 1 = k + \frac{\mu}{\lambda+\mu} \\ k = \frac{\lambda}{\lambda+\mu} \\ P_{WS,WS}(T_2) = \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)T_2} + \frac{\mu}{\lambda+\mu}. \end{aligned}$$

Since the row sum of a transition matrix equals to 1, we finally find that:

$$\begin{aligned} P_{WS,DS}(T_2) + P_{WS,WS}(T_2) = 1 \\ P_{WS,DS}(T_2) = 1 - P_{WS,WS}(T_2) \\ P_{WS,DS}(T_2) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)T_2}). \end{aligned}$$

□

Proposition 3.2. *We define the cycle time of the system as the length of time between two consecutive orders at stage 2 and it is given by*

$$\begin{aligned} E[CT] &= T_2 + E[Duration\ of\ a\ dry\ period|X(T_2) = DS] \\ &= T_2 + \beta(T_2)\frac{1}{\mu}. \end{aligned}$$

Proof. We show the proof as in [36] that explains the memoryless property of the exponential distribution.

The exponential distribution has an important property which indicates that if X is an exponentially distributed product that has lasted for t hours then we say that the remaining life of X is independent of the amount of time it has already exhausted. This is called as the memoryless property of the exponential distribution. Therefore by the memoryless property, the rate of the distribution is constant.

Let us say that we have an item that has lasted for t hours, then the probability that it does not survive for an additional time s is:

$$\begin{aligned} P\{X \in (t, t+s) | X > t\} &= \frac{P\{X \in (t, t+s), X > t\}}{P\{X > t\}} \\ &= \frac{P\{X \in (t, t+s)\}}{P\{X > t\}} \\ &\approx \frac{f(t)dt}{1 - F(t)} = r(t)dt. \end{aligned}$$

If X is distributed exponentially, then by the memoryless property,

$$\begin{aligned} r(t) &= \frac{f(t)}{1 - F(t)} \\ &= \frac{\mu e^{-\mu t}}{e^{-\mu t}} \\ &= \mu. \end{aligned}$$

Therefore the expected lifetime of X is constant and equals to $\frac{1}{\mu}$. This is same for our model, too. It does not matter whether the supplier is disrupted for two hours or ten hours, because the remaining duration of the dry period is independent of the amount of time that the supplier is disrupted. Therefore the expected duration of the dry period is equal to $\frac{1}{\mu}$ so the expected cycle time is :

$$E[CT] = T_2 + \beta(T_2)\frac{1}{\mu}.$$

□

The total fixed and holding cost of stage 2 per cycle is:

$$CC_E(T_2) = K_2 + \frac{1}{2}T_2^2 h_2 D.$$

Since we use echelon stock method and assume that demand is deterministic, the echelon holding cost of stage 2 is $\frac{1}{2}T_2^2 h_2 D$ and the shape of the on hand inventory is the saw-toothed curve as in Figure 3.2.

The total fixed and holding cost of stage 1 per cycle is:

$$CC_E(T_1) = \frac{T_2}{T_1}K_1 + \frac{1}{2}T_2T_1h_1D.$$

We note that since a nested policy is assumed in our model, it is required that if stage 2 gives an order, then stage 1 must give an order, too. Therefore it ensures that T_2 must be greater than or equal to T_1 and stage 1 orders and holds inventory $(\frac{T_2}{T_1})$ times per cycle.

The expected cycle cost of the system is:

$$ECC_E[T_1, T_2] = K_2 + \frac{T_2}{T_1}K_1 + \frac{1}{2}T_2^2h_2D + \frac{1}{2}T_2T_1h_1D + Dc_s\beta(T_2)\frac{1}{\mu}.$$

Average cost of the system per unit time is determined by dividing the expected cycle cost of the system to the expected cycle time:

$$AC_E(T_1, T_2) = \frac{K_2 + (\frac{T_2}{T_1})K_1 + \frac{1}{2}T_2^2Dh_2 + \frac{1}{2}T_2T_1Dh_1 + Dc_s\beta(T_2)\frac{1}{\mu}}{T_2 + \beta(T_2)\frac{1}{\mu}}.$$

Proposition 3.3. *$E[CT]$ is concave in T_2.*

Proof.

$$\frac{\partial^2 E[CT]}{\partial T_2^2} = -\frac{\lambda}{\mu}(\lambda + \mu)e^{-(\lambda+\mu)T_2} < 0.$$

Since it is negative, we conclude that $E[CT]$ is concave in T_2. □

Proposition 3.4. *$ECC_E[T_1, T_2]$ is concave in T_2 for $0 < T_2 < \hat{T}_2$ where,*

$$\hat{T}_2 := \frac{1}{\lambda + \mu} \ln \frac{c_s\lambda(\lambda + \mu)}{h_2\mu}.$$

Proof.

$$\frac{\partial^2 ECC_E[T_1,T_2]}{\partial T_2^2} = Dh_2 - \frac{Dc_s\lambda(\lambda+\mu)e^{-(\lambda+\mu)T_2}}{\mu}.$$

In order to say that $ECC_E[T_1,T_2]$ is concave in T_2 for $0 < T_2 < \hat{T}_2$, the second derivative of $ECC_E[T_1,T_2]$ with respect to T_2 must be less than zero.

$$\frac{\partial^2 ECC_E[T_1,T_2]}{\partial T_2^2} < 0.$$

Thus,

$$\begin{aligned}
\frac{Dc_s\lambda(\lambda+\mu)e^{-(\lambda+\mu)T_2}}{\mu} &> Dh_2 \\
e^{-(\lambda+\mu)T_2} &> \frac{h_2\mu}{c_s\lambda(\lambda+\mu)} \\
T_2 &< -\frac{1}{\lambda+\mu}\ln\frac{h_2\mu}{c_s\lambda(\lambda+\mu)} \\
\hat{T}_2 &:= \frac{1}{\lambda+\mu}\ln\frac{c_s\lambda(\lambda+\mu)}{h_2\mu}.
\end{aligned}$$

□

Proposition 3.5. *$AC_E(T_1,T_2)$ attains its minimum at T_2=T_2*, where T_2* solves the following equation:*

$$\begin{aligned}
\frac{\partial AC_E(T_1,T_2)}{\partial T_2} &= \left(\frac{K_1}{T_1}+\frac{T_1Dh_1}{2}-Dc_s\right)\left(\frac{1}{\lambda+\mu}-\frac{1}{\lambda+\mu}e^{-(\lambda+\mu)T_2}-T_2e^{-(\lambda+\mu)T_2}\right)(\frac{\lambda}{\mu}) \\
&+ \left(DT_2h_2\frac{\lambda}{\mu}\right)\left(\frac{1}{\lambda+\mu}-\frac{1}{\lambda+\mu}e^{-(\lambda+\mu)T_2}-\frac{T_2}{2}e^{-(\lambda+\mu)T_2}\right) \\
&- K_2\left(1+\frac{\lambda}{\mu}e^{-(\lambda+\mu)T_2}\right).
\end{aligned}$$

The proposed reorder interval model is:

$$\min \quad AC_E(T_1^o, T_2^o) \tag{3.1}$$

s. t.

$$T_2^o \geq T_1^o \geq 0 \tag{3.2}$$

$$T_2^o = nT_1^o \qquad n \in \{1, 2, \ldots\} \tag{3.3}$$

The aim of the model is to minimize the average cost of the system per unit time. T_2^o is the reorder interval of stage 2 that is integer multiple of T_1^o, the reorder interval of stage 1. In order to understand the model clearly, first let us consider the relaxed model which does not contain the constraint (3.3). So as to satisfy the constraints (3.1) and (3.2), the network of the serial system is partitioned into the subgraphs that contain the sets of stages that have the same reorder interval to obtain the minimum average cost of the system. After determining the subgraphs and the reorder intervals, we convert these reorder intervals to the ones that are integer multiples of each other. The constraint (3.3) satisfies this need.

3.2.1. The Behavior of the Cost Function

In order to understand the behavior of the objective function according to the change in the parameters of the model, we need to make some numerical analysis. Therefore, we take a set of values for each parameter and determine the reorder intervals and the average cost of the system for each value of the selected parameter. This process is repeated for all the parameters of the model. The following example is only a small part of the sensitivity analysis, further insight is given in the numerical analysis part of the study. We use two different values of the fixed cost of stage 2. We want to know how the fixed cost of stage 2 effects the reorder intervals and the cost function of the system. The cost parameters of Table 3.1 are $K_1 = 100$, $K_2 = 25$, $h_1 = 1$, $h_2 = 0.25$, $D = 50$, $cs = 0.5$, $\lambda = 1$, $\mu = 1$ and the cost parameters of Table 3.2 are $K_1 = 100$, $K_2 = 400$, $h_1 = 1$, $h_2 = 0.25$, $D = 50$, $c_s = 0.5$, $\lambda = 1$, $\mu = 1$. According to Table 3.1, (2,2) gives the minimum of the cost function. However, when we increase

Table 3.1. Example of A Two Stage Serial System with Disruption.

	$\mathbf{T_2^o}$									
$\mathbf{T_1^o}$	1	2	3	4	5	6	7	8	9	10
1	116.6	125.3	133.9	141.6	148.8	155.7	162.5	169.1	175.6	182.1
2		105.2		119.4		132.6		145.5		158.3
3			119.6			140.3			159.8	
4				141.6				169.1		
5					167					201.1
6						194.2				
7							222.5			
8								251.4		
9									280.9	
10										310.7

Table 3.2. Example of A Two Stage Serial System with Disruption.

	$\mathbf{T_2^o}$									
$\mathbf{T_1^o}$	1	2	3	4	5	6	7	8	9	10
1	378.4	275.9	241.1	225	217	213.4	212.5	213.2	215.1	217.8
2		255.8		202.7		190.3		189.7		194
3			226.8			198			199.3	
4				225				213.2		
5					235.2					236.9
6						251.9				
7							272.5			
8								295.5		
9									320.3	
10										346.4

the fixed cost of stage 2 from 25 to 400, (2,8) becomes the new minimizing point. It means that when we increase the fixed cost (ordering cost) of stage 2, model chooses to increase the reorder interval to minimize the cost function. It makes sense because as the frequency of orders increases, the total fixed ordering cost will be larger which contributes to the expected cost function as well. Therefore, a greater reorder interval is more advantageous in this case.

3.3. An Approximate Model for An N Stage Serial System under Supply Disruption

We consider a supply disruption problem in an N stage serial system with constant demand. Disruption occurs at stage N. As in the two-stage serial system problem, we assume both a nested and a stationary policy and we use echelon stock method. The fixed ordering cost is K_i per order and the holding cost is h_i per unit per year for stage i. Supply is available for a random length called the wet period and unavailable for a random length called the dry period. Both wet and dry periods are exponentially distributed with rates λ and μ, respectively. Unmet demands are assumed to be lost with a shortage cost of c_s per lost sale and all model parameters are nonnegative. We model the problem in terms of reorder intervals as in the study of [34]. T_i is the reorder interval of stage i.

We approximate the cost function as in the study of [16]. [16] assumes that when the inventory level is zero, the system reaches steady state quickly enough that we can ignore the transient nature of the system.

We know that β is:

$$\beta(T_N) = \frac{\lambda}{\lambda + \mu}(1 - e^{-(\lambda+\mu)T_N}).$$

We approximate the objective function by omitting $(1-e^{-(\lambda+\mu)T_N})$ from $\beta(T_N)$,

$$\beta_0 = \frac{\lambda}{(\lambda+\mu)}.$$

This approximation is expected to work well when T_N is sufficiently large.

We define the expected cycle time of the system as the length of time between two consecutive orders at stage N:

$$\begin{aligned} E[CT] &= T_N + E[Duration\ of\ a\ dry\ period|X(T_N) = DS] \\ &= T_N + \beta_0 \frac{1}{\mu}. \end{aligned}$$

The total fixed and holding cost of stage i per cycle is:

$$CC_A(T_i) = \frac{T_N}{T_i}K_i + \frac{1}{2}T_N T_i h_i D, \qquad i \neq N.$$

The total fixed and holding cost of stage N per cycle is:

$$CC_A(T_N) = K_N + \frac{1}{2}T_N^2 D h_N.$$

We divide the expected cycle cost to the expected cycle time so as to obtain the average cost of the system per unit time. The approximate average cost of an N stage serial system is:

$$AC_A(T_1,\ldots,T_N) = \frac{K_N + T_N\sum_{i=1}^{N-1}\frac{K_i}{T_i} + \frac{1}{2}T_N^2 D h_N + DT_N\sum_{i=1}^{N-1}\frac{1}{2}T_i h_i + Dc_s\beta_0\frac{1}{\mu}}{T_N + \beta_0\frac{1}{\mu}}.$$

We know that the classical economic reorder interval problem is:

$$C_{i,EOQ}(T_i) = \frac{K_i}{T_i} + \frac{1}{2}D h_i T_i,$$

we define C_{EOQ} as:

$$C_{EOQ} = \sum_{i=1}^{N-1} C_{i,EOQ}(T_i).$$

Then the objective function becomes:

$$AC_A(T_1, \ldots, T_N) = \frac{K_N + \frac{1}{2}T_N^2 h_N D + T_N C_{EOQ} + Dc_s\beta_0\frac{1}{\mu}}{T_N + \beta_0\frac{1}{\mu}}.$$

In order to have an understanding of the cost function's shape and check the joint convexity of the function, we need to take the first and second derivatives of it with respect to T_i.

$$\frac{\partial AC_A(T_1, \ldots, T_N)}{\partial T_i} = \frac{T_N}{T_N + \beta_0\frac{1}{\mu}} C'_{i,EOQ}(T_i), \qquad i \neq N.$$

$$\frac{\partial AC_A(T_1, \ldots, T_N)}{\partial T_N} = \frac{T_N^2\frac{1}{2}Dh_N + T_N Dh_N\beta_0\frac{1}{\mu} - (K_N + Dc_s\beta_0\frac{1}{\mu} - \beta_0\frac{1}{\mu}C_{EOQ})}{(T_N + \beta_0\frac{1}{\mu})^2}.$$

$$\frac{\partial^2 AC_A(T_1, \ldots, T_N)}{\partial T_i \partial T_j} = 0, \qquad i \neq j \neq N.$$

$$\frac{\partial^2 AC_A(T_1, \ldots, T_N)}{\partial T_i^2} = \frac{T_N}{T_N + \beta_0\frac{1}{\mu}} C''_{i,EOQ}(T_i), \qquad i = 1, \ldots, N-1.$$

$$\frac{\partial^2 AC_A(T_1, \ldots, T_N)}{\partial T_N^2} = \frac{\beta_0^2(\frac{1}{\mu})^2 Dh_N + 2K_N + 2\beta_0\frac{1}{\mu}(Dc_s - C_{EOQ})}{(T_N + \beta_0\frac{1}{\mu})^3}.$$

$$\frac{\partial^2 AC_A(T_1, \ldots, T_N)}{\partial T_i \partial T_N} = C'_{i,EOQ}(T_i)\frac{\beta_0\frac{1}{\mu}}{(T_N + \beta_0\frac{1}{\mu})^2} \qquad i \neq N.$$

We know that a function is convex if and only if its hessian matrix is positive (semi) definite. Therefore, we need to check its hessian matrix;

$$H = \begin{bmatrix} \frac{T_N}{T_N+\beta_0\frac{1}{\mu}}C''_{1,EOQ}(T_1) & \cdots & 0 & \cdots & C'_{1,EOQ}(T_1)\frac{\beta_0\frac{1}{\mu}}{(T_N+\beta_0\frac{1}{\mu})^2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C'_{1,EOQ}(T_1)(\frac{\beta_0\frac{1}{\mu}}{(T_N+\beta_0\frac{1}{\mu})^2}) & \cdots & 0 & \cdots & \frac{(\beta_0\frac{1}{\mu})^2 Dh_N+2K_N+2\beta_0\frac{1}{\mu}(Dc_s-C_{EOQ})}{(T_N+\beta_0\frac{1}{\mu})^3} \end{bmatrix}.$$

As we can see, $\left[\frac{(\beta_0\frac{1}{\mu})^2 Dh_N+2K_N+2\beta_0\frac{1}{\mu}(Dc_s-C_{EOQ})}{(T_N+\beta_0\frac{1}{\mu})^3}\right]$ might be negative since Dc_s might be less than C_{EOQ} so it violates the positive (semi) definiteness condition.

Therefore we assume that,

$$Dc_s > \sum_{i=1}^{N-1} \sqrt{2K_i Dh_i}.$$

Proof. We know that,

$$C_{EOQ} = \sum_{i=1}^{N-1} C_{i,EOQ}(T_i)$$

and

$$C_{i,EOQ}(T_i) = \frac{K_i}{T_i} + \frac{1}{2}Dh_i T_i.$$

When we solve the derivative of $C_{i,EOQ}(T_i)$ with respect to T_i, we find that:

$$T_i^* = \sqrt{\frac{2K_i}{Dh_i}},$$

if we plug T_i^* into $C_{i,EOQ}(T_i)$, we obtain,

$$\begin{aligned} C_{i,EOQ}(T_i^*) &= K_i\sqrt{\frac{Dh_i}{K_i}} + \frac{1}{2}Dh_i\sqrt{\frac{K_i}{Dh_i}} \\ &= \sqrt{\frac{1}{2}K_iDh_i} + \frac{1}{2}\sqrt{2K_iDh_i} \\ &= \sqrt{2K_iDh_i}. \end{aligned}$$

Therefore,

$$C_{EOQ}^* = \sum_{i=1}^{N-1} \sqrt{2K_iDh_i}$$

and

$$Dc_s > \sum_{i=1}^{N-1} \sqrt{2K_iDh_i}.$$

□

This assumption ensures that the retailer will not be able to choose to lose demands instead of serving them since the total shortage cost is more costly.

We solve the first derivative of the average cost per unit time with respect to T_i in order to obtain T_i^*:

$$\frac{\partial AC_A(T_1,\ldots,T_N)}{\partial T_i} = \frac{T_N}{T_N + \beta_0\frac{1}{\mu}}C'_{i,EOQ}(T_i), \qquad i \neq N$$

$$T_i^* = \sqrt{\frac{2K_i}{Dh_i}}, \qquad i = 1,\ldots,N-1.$$

So as to determine T_N^*, we need to use T_i^* at the first derivative of the average cost with respect to T_N.

$$\frac{\partial AC_A(T_1, ..., T_N)}{\partial T_N} = \frac{T_N^2 \frac{1}{2} Dh_N + T_N Dh_N \beta_0 \frac{1}{\mu} - (K_N + Dc_s \beta_0 \frac{1}{\mu} - \beta_0 \frac{1}{\mu} C^*_{EOQ})}{(T_N + \beta_0 \frac{1}{\mu})^2} \tag{3.4}$$

where

$$C^*_{EOQ} = \sum_{i=1}^{N-1} C_{i,EOQ}(T_i^*).$$

When we make the numerator of equation (3.4) equal to zero, it takes the form of $aT_N^2 + bT_N - c^* = 0$ in which $a = \frac{1}{2} Dh_N$, $b = Dh_N \beta_0 \frac{1}{\mu}$ and $c^* = K_N + Dcs\beta_0 \frac{1}{\mu} - \beta_0 \frac{1}{\mu} C^*_{EOQ}$. Thus it has two roots:

$$T_N = \frac{-b + \sqrt{b^2 + 4ac^*}}{2a}$$
$$T_N = \frac{-b - \sqrt{b^2 + 4ac^*}}{2a}.$$

As we can see, the first one is positive while the second one is negative. We use the positive one in our model since we assume that all reorder intervals must be greater than or equal to zero.

$$T_N^* = \frac{-b + \sqrt{b^2 + 4ac^*}}{2a}, \quad c^* > 0, a > 0 \ \ (if\ c^* < 0,\ then\ T_N^* = 0).$$

Now, we can form the hessian matrix. When we use the unique values of the reorder intervals in the hessian matrix, we see that its diagonal entries are positive while the non-diagonal entries are zero. Therefore, we conclude that the hessian matrix is positive definite. We know that if the hessian is positive definite at a certain point, then the function attains a local minimum at this point. Therefore $(T_1^*, \ldots, T_N^*)$ is the unique minimizor of the average cost function of the approximate model which also leads us to the conclusion of the unimodality of it.

Thus the optimal solution for the reorder intervals that are integer multiples of the reorder intervals of their successors is:

$$AC_A((n_i+1)T^o_{i-1}) - AC_A(n_iT^o_{i-1}) \geq 0, \qquad \forall i = 2, \ldots, N$$

in which $n_iT^o_{i-1} = T^o_i$.

Thus it enables us to minimize the average cost per unit time by finding the smallest nonnegative integer n_i for each stage.

$$n_i \geq \frac{\sqrt{D^2(T^o_{i-1})^2h_i^2 + 8K_iDh_i} - DT^o_{i-1}h_i}{2Dh_iT^o_{i-1}}, \qquad i = 2, ..., N-1$$

which reduces to

$$n_i \geq \frac{T^*_i\sqrt{4 + \frac{(T^o_{i-1})^2}{(T^*_i)^2}} - T^o_{i-1}}{2T^o_{i-1}}, \qquad i = 2, ..., N-1$$

and for the last stage

$$n_N \geq \frac{\sqrt{b^2 + 4ac + a^2(T^o_{N-1})^2} - b - aT^o_{N-1}}{2aT^o_{N-1}}$$

which reduces to

$$n_N \geq \frac{\sqrt{2aT^*_N + b + a^2(T^o_{N-1})^2} - b - aT^o_{N-1}}{2aT^o_{N-1}}.$$

The proposed reorder interval model is:

$$\min \quad AC_A(T^o_1, \ldots, T^o_N) \tag{3.5}$$

s. t.

$$T^o_i = n_iT^o_{i-1} \qquad n_i \in \{1, 2, \ldots\} \tag{3.6}$$

$$T^o_i \geq T^o_{i-1} \geq 0 \tag{3.7}$$

Another way of determining a reorder interval is to make it equal to a power-of-two multiple of T_L, the minimum reorder interval:

$$T_i^p = 2^{l_i} T_L.$$

The optimal powers of two solution is:

$$AC_A(2^{l_i+1}T_L) - AC_A(2^{l_i}T_L) \geq 0, \qquad \forall i = 1, \ldots, N.$$

It enables us to minimize the expected cost per cycle by finding the smallest nonnegative integer l_i.

$$2^{l_i} \geq \sqrt{\frac{K_i}{Dh_i} \frac{1}{T_L}}, \qquad i = 1, ..., N-1$$

which reduces to

$$2^{l_i} \geq \frac{T_i^*}{\sqrt{2}T_L}, \qquad i = 1, ..., N-1$$

and for the last stage

$$2^{l_N} \geq \frac{2\sqrt{(\frac{9b^2}{4} + 8ac^*)} - 3b}{8aT_L}$$

which reduces to

$$2^{l_N} \geq \frac{2\sqrt{\frac{9b^2}{4} + 8(T_N^*)^2 a^2 + 8(T_N^*)ab} - 3b}{8aT_L}.$$

If we apply the power-of-two policy, the proposed reorder interval model becomes:

$$\min \quad AC_A(T_1^p, \ldots, T_N^p) \tag{3.8}$$

s. t.

$$T_i^p = 2^{l_i} T_L \qquad l_i \in \{0, 1, \ldots\} \tag{3.9}$$

$$T_i^p \geq T_{i-1}^p \geq 0 \tag{3.10}$$

3.3.1. An Algorithm for Finding the Optimal Reorder Intervals of An N Stage Serial System

We use the algorithm in [34] to find the optimal clusters and the reorder intervals of an N stage serial system. We apply the power-of-two policy to obtain the reorder intervals in the numerical illustration. Therefore, the algorithm for finding the reorder intervals involves only the power-of-two policy.

Now, let us define C as a node set.

$$C^i \leftarrow \{i\}, \qquad i = 1, \ldots, N.$$

We redefine the total fixed and the holding cost of each stage per cycle as follows:

$$CC_A(T(C^i)) = \frac{K(C^i)}{T(C^i)} + \frac{DT(C^i)h(C^i)}{2}, \qquad i = 1, \ldots, N-1.$$

$$CC_A(T(C^N)) = K(C^N) + \frac{DT(C^N)^2 h(C^N)}{2}.$$

The average cost per unit time is:

$$AC_A(T(C^1), \ldots, T(C^N)) = \frac{T(C^N)\sum_{i=1}^{N} CC_A(T(C^i)) + Dc_s\beta_0\frac{1}{\mu}}{T(C^N) + \beta_0\frac{1}{\mu}}.$$

The reorder interval model for an N stage serial system becomes:

$$\min \quad AC_A(T^p(C^1), \ldots, T^p(C^N)) \tag{3.11}$$

s. t.

$$T^p(C^i) = 2^{l_i} T_L \qquad l_i \in \{0, 1, \ldots\} \tag{3.12}$$

$$T^p(C^i) \geq T^p(C^{i-1}) \geq 0 \tag{3.13}$$

The algorithm has three steps to follow. First, let us consider the relaxed problem that does not contain the constraint (3.12). We determine the optimal clusters by partitioning the network of the serial system into connected subgraphs in the first step. In the second step, we solved the relaxed model and find the reorder intervals that satisfy the constraints (3.12) and (3.11). In the final step, we satisfy the constraint (3.13) by converting these reorder intervals into the ones that are powers of two multiples of T_L.

Algorithm

- Step 1: Find the clusters.
  (i) Set $C^i \leftarrow \{i\}$ and $\sigma(i) \leftarrow i-1$ for all $1 \leq i \leq N$ and $SQ \leftarrow \{1, 2, \ldots, N\}$. Set $j \leftarrow 2$. $\sigma(i)$ is the node that precedes i in the sequence SQ.
  (ii) If $T^*(C^j) \geq T^*(C^{[\sigma(j)]})$, go to step 1.d, otherwise collapse $C^{[\sigma(j)]}$ into C^j, $C^j \leftarrow C^{[\sigma(j)]} \cup C^j$, $\sigma(j) \leftarrow \sigma(\sigma(j))$ and $SQ \leftarrow SQ \backslash \{\sigma(j)\}$, thus the expected cost per cycle and the average cost per unit time becomes:

$$\tilde{CC}_A(T(i)) = \frac{\sum_{j \in C^i} K_j}{T(i)} + \frac{DT(i) \sum_{j \in C^i} h_j}{2}, \qquad i = 1, \ldots, N.$$

$$\begin{aligned} AC_A(T(1), \ldots, T(N)) &= \frac{T(N)\tilde{CC}_A(T(N))}{T(N) + \beta_0 \frac{1}{\mu}} \\ &+ \frac{T(N) \sum_{i \in SQ, i \neq N} \tilde{CC}_A(T(i)) + Dc_s \beta_0 \frac{1}{\mu}}{T(N) + \beta_0 \frac{1}{\mu}}. \end{aligned}$$

We note that $T(N) = T(C^N)$ when there is no collapse in the Nth node.

(iii) If $\sigma(j) > 0$, go to (ii).

(iv) Set $j \leftarrow j+1$, if $j \leq N$, go to step (ii).

(v) Re-index the clusters $\{C^i : i \in SQ\}$ so that $SQ = \{1, .., N\}$ and if $j \in C^i$, $k \in C^l$ and $j < k$ then $i < l$. Comment: $\{C^k : k \in SQ\}$ are the clusters, the optimal partition is $\{G_k : k \in SQ\}$ where G_k is the subgraph of G.

- Step 2: Find the solution of the relaxation model. For each cluster C^k, $k \in SQ$, set,

$$T^*(k) = T^*(C^k) = \sqrt{\frac{2\sum_{i \in C^k} K_i}{D\sum_{i \in C^k} h_i}}, \qquad k \neq N$$

and for the last node:

$$T^*(N) = T^*(C^N) = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 + 4\tilde{a}\tilde{c^*}}}{2\tilde{a}}$$

where $\tilde{a} = \frac{1}{2}D\sum_{i \in C^N} h_i$, $\tilde{b} = D\beta_0\frac{1}{\mu}\sum_{i \in C^N} h_i$ and $\tilde{c^*} = \sum_{i \in C^N} K_i + Dcs\beta_0\frac{1}{\mu} - \beta_0\frac{1}{\mu}C^*_{EOQ}$. ($C^*_{EOQ} = \sum_{k=1}^{N-1}\frac{\sum_{i \in C^k} K_i}{T^*(k)} + \frac{D\sum_{i \in C^k} h_i T^*(k)}{2}$). For each $i \in C^k$ set $T^*_i = T^*(k)$

- Step 2: Find the solution of reorder interval model. For each $i \in C^k$ set $T^p_i = 2^{l_i}T_L$ where

$$2^{l_i} \geq \frac{T^*_i}{\sqrt{2}T_L} \geq 2^{l_i - 1}, \qquad k \neq N.$$

For the Nth node:

$$2^{l_N} \geq \frac{2\sqrt{(\frac{9b^2}{4} + 8((T^*_N)^2a^2 + 8(T^*_N)ab)} - 3b}{8a} \geq 2^{l_N - 1}.$$

3.4. Numerical Analysis

In this part of the study, we try to give more insight by applying numerical analysis. We aim to observe the change at the reorder intervals of the production stages as well as the change at the cost function according to the parameters of the model. We also measure the error due to using the approximate model and the power-of-two policy.

3.4.1. Sensitivity Analysis of the Exact Model for A Two-Stage Serial System

Our goal is to understand the behavior of the average cost of the exact model for a two-stage serial system according to the change at the parameters of the model. Although the total number of experiments that are conducted is 400, we present only one experiment for each parameter. Table 3.3 shows the test values of each parameter. We begin our numerical analysis by examining the impact of the fixed cost of stage

Table 3.3. Parameter Set.

Parameter	Symbol	Tested Values
Fixed Cost of the Last Stage	K_2	25, 50, 100, 200, 400
Holding Cost of the Last Stage	h_2	0.25, 0.5, 1, 1.5, 2
Demand	D	50, 200
Shortage Cost	c_s	0.5,10
Wet period Rate	λ	0.1, 1
Dry Period Rate	μ	1, 4

2. We know that it has a direct effect on the reorder interval of stage 2. We use five different values of the fixed cost of stage 2 to show the change more clearly. K_2 takes the values of 25, 50, 100, 200, 400 while other parameters are set as; D=50, h_1=1, h_2=0.25, c_s=10, K_1=100, λ=1, μ=1. According to the given results, the change in K_2 does not affect the reorder interval of stage 2 until it is set to 200. When it takes the values of 200 and 400, the reorder intervals become (2,8) and (2,10). It means that when the fixed cost per order for an item gets large enough, the model chooses to order less frequently by increasing the reorder interval of the item. Besides, the relation between the average cost per unit time and K_2 is linear. As K_2 gets larger, the average cost of the system gets larger also.

Table 3.4. Behavior of the Model with Respect to K_2.

K_2	Reorder Intervals	Expected Cost
25	(2,6)	169.230
50	(2,6)	173.076
100	(2,6)	180.769
200	(2,8)	194.117
400	(2,10)	216.666

Table 3.5 illustrates the impact of the holding cost of stage 2 on the average cost per unit time of the serial system. Again we know that it has a direct effect on the reorder interval of stage 2, as well. According to Table 3.5 h_2 takes the values of 0.25, 0.5, 1, 1.5, 2 while D=50, h_1=1, c_s=10, K_1=100, K_2=25, λ=1, μ=1. As expected, there is a linear relation between the average cost per unit time and the holding cost of stage 2. Therefore, when $h_2 > 1$, model chooses to order more frequently to decrease the duration of holding inventory in a cycle in order to reduce the total holding cost of the system. It makes sense because the model knows that it should decrease the order quantity to decrease the total holding cost of the system per cycle. The situation is pretty same with the case of the holding cost of stage 1, as well. As h_1 takes larger values, the reorder interval of stage 1 decreases since the model chooses to decrease the order quantity of stage 1 to decrease the total holding cost of the serial system. Table

Table 3.5. Behavior of the Model with Respect to h_2.

h_2	Reorder Intervals	Expected Cost
0.25	(2,6)	169.230
0.5	(2,4)	194.43
1	(2,2)	229.007
1.5	(2,2)	249.080
2	(2,2)	269.154

3.6 demonstrates the impact of demand on the reorder intervals when h_2=0.25, h_1=1, c_s=10, K_1=100, K_2=25, λ=1, μ=1. We see that when demand is greater than 50, it

Table 3.6. Behavior of the Model with Respect to D.

D	Reorder Intervals	Expected Cost
50	(2,6)	169.230
200	(1,6)	480.765

is cheaper to order more frequently for stage 1.

Table 3.7 shows the relation between λ and the reorder intervals when h_2=0.25, h_1=1, c_s=10, K_1=100, K_2=25, D=50, μ=1. As λ increases, the expected duration of

Table 3.7. Behavior of the Model with Respect to λ.

λ	Reorder Intervals	Expected Cost
0.1	(2,4)	139.346
1	(2,6)	169.230

the wet period decreases. Let us consider the case in which λ is comparatively higher than μ, as $\lambda \to \infty$, $\beta \to 1$, the probability of disruption gets higher resulting in a higher expected shortage cost and a higher average cost per unit time. Therefore, in order to mitigate the affect of disruption, the model chooses to hold more inventory in a cycle by increasing the reorder interval of the unreliable supplier. We should not forget that increasing the reorder interval of stage 2 means increasing the order quantity of stage 2 as well. By stocking more order quantity the model tries to get rid of paying extra shortage cost due to the risk of supply disruption.

Table 4.3 illustrates the relationship between μ and the reorder intervals when h_2=0.25, h_1=1, c_s=10, K_1=100, K_2=100, D=50, λ=1. As μ gets higher, the expected duration of the dry period decreases as well as the probability of disruption. Since disruption risk is smaller, the model does not have to order more so it can decrease the reorder interval of stage 2 as we can see in the results. Moreover, since disruption risk gets smaller, the average cost per unit time decreases as well, as μ takes larger values. Table 3.9 illustrates the effect of the shortage cost on the average cost per unit time

Table 3.8. Behavior of the Model with Respect to μ.

μ	Reorder Intervals	Expected Cost
1	(2,6)	180.769
4	(2,4)	154.32

and the reorder intervals when h_2=0.25, h_1=1, K_1=100, K_2=25, D=50, λ=1, μ=1. We see that as c_s gets higher, the effect of disruption will be more destructive since the expected shortage cost will be higher. Consequently, by increasing T_2, model aims to hold more inventory in a cycle to mitigate this effect.

Table 3.9. Behavior of the Model with Respect to c_s.

c_s	Reorder Intervals	Expected Cost
0.5	(2,2)	105.29
10	(2,6)	169.23

3.4.2. Approximation Error

We examine two models so as to analyze the percentage error due to the approximation of β. The first one is the exact model for a two-stage serial system under supply disruption and the second one is an approximate model for a two-stage serial system under supply disruption.

EC_0 is the average cost of the exact model per unit time when we use the reorder intervals of the approximate model that are power-of-two multiples of T_L.

$$EC_0 = AC_E(T_1^p, T_2^p).$$

We must notice that T_1^p, T_2^p are the reorder intervals that are found at the approximate model as the power-of-two multiples of T_L. We compute EC_0 by plugging these values into the exact model for the two-stage serial system. On the other hand, EC_2 is the average cost of the approximate model per unit time when we use the reorder intervals that are powers of two. These reorder intervals satisfy the constraints (3.8)- (3.10).

$$EC_2 = AC_A(T_1^p, T_2^p).$$

We determine the percentage error by

$$PAE = \frac{EC_0 - EC_2}{EC_0}.$$

Table 3.10 shows the average and the maximum values of the percentage approximation error. We use the same data set as we have used in the sensitivity analysis for a two-stage serial system. At the first row of Table 3.10, K_2, λ and μ are equal to 25, 1 and 1, respectively, while h_2, c_s and D take different values at the data set of Table 3.3. Therefore, the first row shows the average and maximum values of the percentage approximation error of this data set when K_2, λ and μ are equal to 25, 1 and 1, respectively. The same process is repeated for the different values of K_2, λ and μ. The overall average percentage error is 0.080479. According to the results, the gap between the two approaches gets closer when λ and μ get larger and it gets farther when they get smaller. Besides, we know that the increase in K_2 causes T_2 to take a higher value. As a result it also contributes to decreasing the gap between the two approaches as it makes T_2 greater. Therefore, we conclude that the approximation provides good performance when T_2, λ and μ are sufficiently large.

3.4.3. Error Due to the Power-of-Two Policy

In this part of the study, we want to measure the percentage error due to applying the power-of-two policy. EC_1 is the average cost of the exact model per unit time when we use the reorder intervals that are not powers of two.

$$EC_1 = AC_A(T_1, T_2).$$

Thus, its reorder interval model is

$$\begin{aligned} \min \quad & AC_A(T_1, T_2) && (3.14) \\ \text{s. t.} \quad & \\ & T_2 \geq T_1 \geq 0 && (3.15) \end{aligned}$$

It is the relaxed model that we consider for the approximate model in Section 3.3.1.

The formulation that we use to measure the error is:

$$PEPO2 = \frac{EC_0 - EC_1}{EC_1}.$$

The average percentage error is 0.001 which is so small.

Table 3.10. The Percentage Approximation Error.

K_2	λ	μ	Average Error %	Max Error %
25	1	1	0.023	0.037
25	1	4	0	0
25	0.1	1	0.015	0.024
25	0.1	4	0	0
50	1	1	0.025	0.038
50	1	4	0	0
50	0.1	1	0.013	0.024
50	0.1	4	0	0
100	1	1	0.02	0.039
100	1	4	0	0
100	0.1	1	0.005	0.019
100	0.1	4	0	0
200	1	1	0.013	0.038
200	1	4	0	0
200	0.1	1	0.002	0.004
200	0.1	4	0	0
400	1	1	0,007	0.037
400	1	4	0	0
400	0.1	1	0	0.004
400	0.1	4	0	0

4. ASSEMBLY SYSTEMS UNDER SUPPLY DISRUPTION

As in the serial systems, assembly systems have several stages that aim to create a finished product. An assembly system consists of a node set and an arc set denoted as $N(G)$ and $A(G)$, respectively, in which G implies the directed graph that represents the production system. Nodes in the node set imply the assembly stages while arcs show the order of the assembly stages. In fact, a serial system is a special type of an assembly system, difference is that in an assembly system, several semi-finished items are used to assemble a finished product and many other semi-finished items can be used to assemble those semi-finished items. All stages have only one successor whereas the predecessor of a stage might be more than one. The only stage that does not have a successor is the root stage which satisfies the external demand.

4.1. Problem Definition

We consider a supply disruption problem in an assembly system with an unreliable supplier. The unreliable supplier is unique and known beforehand. It can be one of the stages of the assembly system that does not have a predecessor.

We assume both a nested and a stationary policy which is also optimal for assembly systems. Demand is constant and denoted as D units per year. We assume that the finished product is assembled using one unit of the each semi-finished items which also means if $j \in P_i$, P_i is the set of all the predecessors of part i including itself, then one unit of j is consumed for each unit of i ordered. For stage i, the fixed ordering cost is K_i per order and the echelon holding cost is h_i per unit per year. The echelon holding cost of part i is the difference of the conventional holding cost of part i, h'_i, and the sum of the holding costs of its direct predecessors. The echelon inventory of part i is the sum of the inventories of its all successors and its inventory as well. As before, both wet and dry periods are exponentially distributed with rates λ and μ, respectively. Stockouts are assumed to be lost with a shortage cost of c_s per lost sale. Orders are placed when inventory level hits zero and there is no lead time for orders.

As in the serial systems, we model our problem in terms of reorder intervals. T_i is the reorder interval of stage i. We assume that all the parameters of the model are nonnegative. We try to determine the reorder intervals for each stage that minimize the average cost of the assembly system per unit time.

Figure 4.1 shows the structure of the assembly system that we examine in our study: As it can be seen, it consists of the root stage that satisfies the external demand

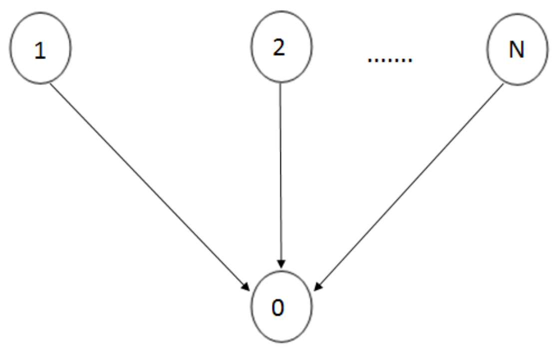

Figure 4.1. The Assembly System That is Examined in This Study.

and all the stages that are direct predecessors of the root stage. Although it seems a comparatively simple structure, we analyze it in two cases since our objective function changes according to the relation between reorder intervals. Without loss of generality, we assume that N is the unreliable supplier which may have disruption in supply.

4.2. Case 1: $\mathbf{T_N \geq T_j}$, $\quad \mathbf{j \neq N}$

The relationship between T_N, the reorder interval of the unreliable supplier, and other reorder intervals has a significant effect in the model. Let us assume the case in which $T_N \geq T_j$ for some j. As before, other suppliers know whether the unreliable supplier is in the wet period or not and they are also informed about the amount of inventory it holds in the dry period. Therefore, since disruption does not destroy the inventory of the unreliable supplier, if it has inventory when disruption begins, it can continue to deliver its on hand inventory until it has none. Consequently, other suppliers can carry on placing orders as well. On the other hand, if the unreliable supplier is in the dry period and it has no inventory on hand, then other suppliers stop giving orders until the end of the dry period. Because, if they continue to place orders, there will be a mismatch in the number of the semi-finished items and the final product will not be produced. We also assume that the proportion of T_N to T_j, $\forall j = 0, \ldots, N-1$, must be integer.

In order to make the model more understandable, we explain it through the following figures. Let us assume that Figure 4.2 shows the assembly system that we consider. As we can see, the finished product is assembled using part 1 and part 2. Suppose, in Figure 4.3, disruption occurs at stage 2 and T_0, T_1 and T_2 are such

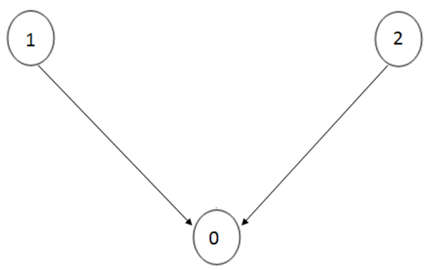

Figure 4.2. Example of an Assembly System

that $T_2 > T_1$ and $T_1 = T_0$. Also stage 0 and stage 1 place an order of size Q every time inventory level hits zero while stage 2 places an order of size 2Q. When a supply disruption occurs in the system, the disruption process of the assembly system will be like as in Figure 4.3. According to Figure 4.3, the first disruption occurs at time A, however it does not effect the assembly process of the system since it ends before the unreliable supplier consumes its on hand inventory. On the other hand, the second disruption that occurs at time C lasts longer than the first one and ends after the unreliable supplier consumes its on hand inventory. As we can see, when stage 2 consumes all the inventory it has in the dry period, stage 1 stops giving new orders and wait for the end of the disruption. Because even if it places a new order, stage 0 will not be able to complete the assembly process since there will be a mismatch in the number of the semi-finished items. Therefore, although stage 1 orders and delivers its inventory, stage 0 has to stock it with a higher holding cost than stage 1 since $h'_0 > h'_1$. So that the second disruption affects the performance of the assembly system and results in lost sales. Now, we can describe the model for case 1. Disruption occurs

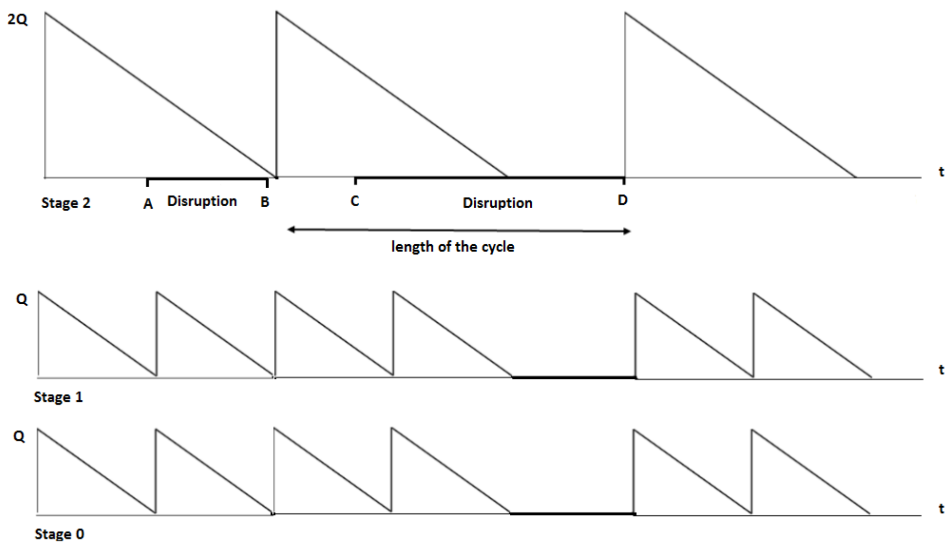

Figure 4.3. Disruption in An Assembly System According to Case 1.

at stage N and C_1 is the set of stages for which the reorder interval of a stage is less than or equal to T_N.

$$C_1 = \{j \in \{1, 2, ..., N\} : T_j \leq T_N\}.$$

The probability that there will be a disruption at stage N whenever its inventory level hits zero is:

$$\beta(T_N) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)T_N}).$$

The proportion of T_N to T_j is:

$$s_j = \frac{T_N}{T_j}, \qquad j \in C_1.$$

As we have emphasized before s_j must be integer.

We define the cycle time of the system as the length of time between two consecutive orders at stage N:

$$E[CT] = T_N + \beta(T_N)\frac{1}{\mu}.$$

The fixed cost of stage j per cycle is:

$$FC_j = s_j K_j, \qquad j \in C_1.$$

The holding cost of stage j per cycle is:

$$HC_j = \frac{1}{2} D h_j T_j^2 s_j, \qquad j \in C_1.$$

Stage j orders and holds inventory s_j times per cycle since $T_N \geq T_j$. Therefore, s_j must be multiplied with the fixed cost and the holding cost of stage j while computing the cycle cost of it.

The average exact cost of stage j per unit time is:

$$AC_{E_j}(T_j, T_N) = \frac{K_j s_j + \frac{1}{2} D h_j T_j^2 s_j}{T_N + \beta(T_N)\frac{1}{\mu}}, \qquad j \in C_1.$$

We know that the average cost of the classical economic reorder interval problem is:

$$C_{j,EOQ}(T_j) = \frac{K_j}{T_j} + \frac{1}{2}Dh_jT_j.$$

Then the average per unit time exact cost of stage j becomes:

$$AC_{E,j}(T_j, T_N) = \frac{T_N C_{j,EOQ}(T_j)}{T_N + \beta(T_N)\frac{1}{\mu}}, \qquad j \in C_1.$$

The average per unit time exact cost of stage 0 is:

$$AC_{E,0}(T_0, T_N) = \frac{T_N C_{0,EOQ}(T_0) + Dc_s\beta(T_N)\frac{1}{\mu}}{T_N + \beta(T_N)\frac{1}{\mu}}.$$

4.3. Case 2: $\mathbf{T_N < T_j}$ for some $\mathbf{j \neq N}$

Case 2 is a little more complicated than Case 1. Before explaining the reason, let us first describe the model. As before, disruption occurs at stage N and other suppliers know whether it is in the wet period or not. If it is in the dry period and it has no inventory on hand, then other suppliers stop giving orders until the end of the dry period.

We know that stage N may or may not consume its inventory during the dry period. Because there is a possibility that the disruption period may last so short that the assembly process is not even effected if stage N has still inventory on hand during disruption. However if disruption lasts longer enough to make stage N consume all the inventory it has, then stage j will still have inventory on hand since $T_j > T_N$.

Let us explain Case 2 in detail through Figure 4.2 and Figure 4.4. As in Case 1, assume that Figure 4.2 shows the assembly system of Case 2 and Figure 4.4 shows the disruption process of the system according to Case 2. Stage 2 represents the unreliable supplier. When disruption occurs, we see that stage 1 has still inventory on hand although stage 2 has none. However, contrary to Case 1, stage 1 does not deliver its semi-finished items to stage 0 since there will be a mismatch in the number of the semi-finished items of stage 2 and stage 1. If it does, stage 0 can not complete its assembly process and holds the inventory until the end of the dry period in a higher holding cost than stage 1. ($h'_0 > h'_1$). Therefore, stage 1 prefers to hold them instead of stage 0 to reduce the total holding cost of the system. When the disruption ends, it will deliver its on hand inventory to stage 0 and place a new order. Now, we can

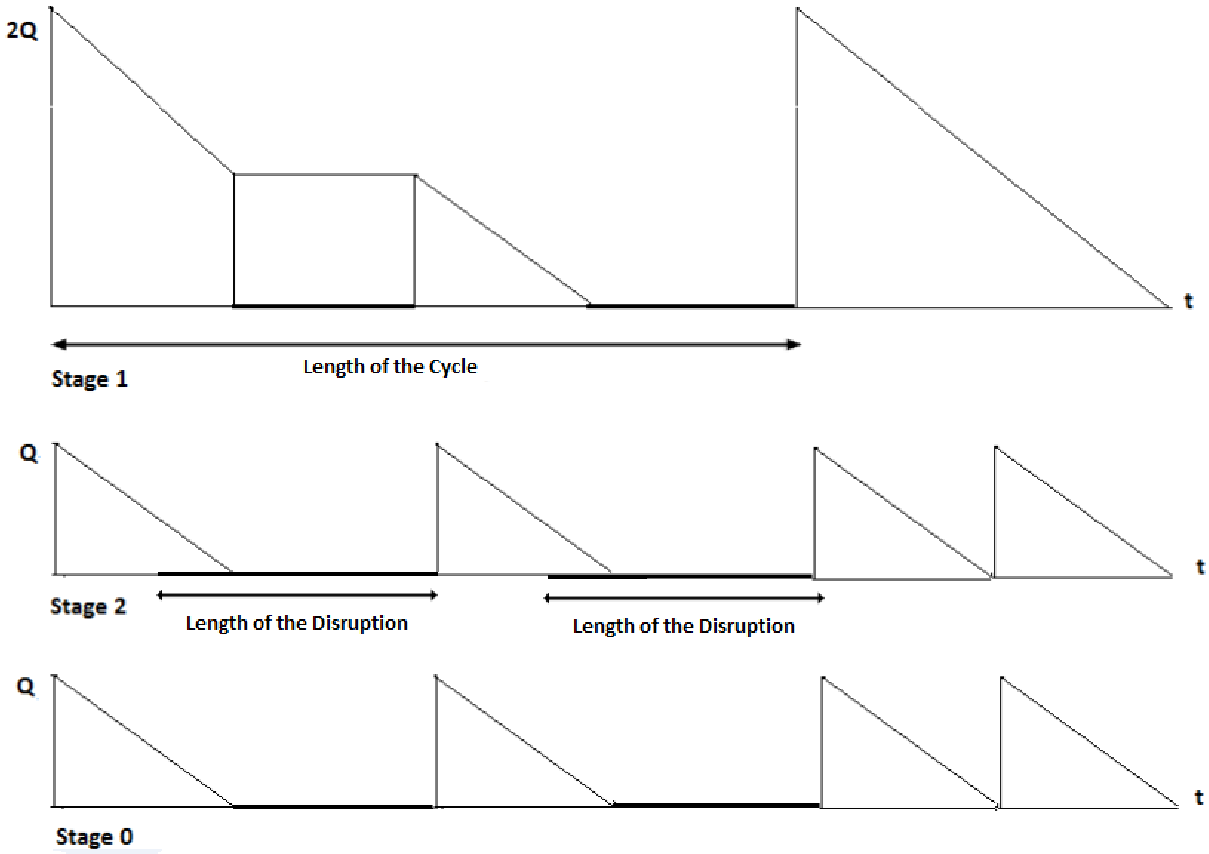

Figure 4.4. Disruption in An Assembly System According to Case 2.

describe the model for an N stage assembly system. Disruption occurs at stage N. Then C_2 is the set of stages for which the reorder interval of a stage is greater than T_N.

$$C_2 = \{j \in \{1, 2, ..., N\} : T_j > T_N\}.$$

The proportion of the reorder interval of stage j to the reorder interval of stage N is:

$$m_j = \frac{T_j}{T_N}, \qquad j \in C_2.$$

A cycle consists of m_j subcycles. In subcycle i ($i = 1, 2, \ldots, m_j$), the inventory level starts at $(DT_j - (i-1)DT_N)$ and decreases to $(DT_j - iDT_N)$ at rate D. A disruption occurs in subcycle i with probability $\beta(T_N)$, and it lasts for an average duration of $\frac{1}{\mu}$. The inventory remaining in subcycle i, $(DT_j - iDT_N)$, is carried through this duration. Therefore, the inventory holding cost of subcycle i is given by

$$h_j \frac{1}{2} T_N \Big((DT_j - (i-1)DT_N) + (DT_j - iDT_N) \Big) + \beta(T_N)(DT_j - iDT_N)\frac{1}{\mu}.$$

As there are m_j such subcycles, the total inventory holding cost is:

$$E[HC_j] = h_j \sum_{i=1}^{m_j} \frac{1}{2} T_N \Big((DT_j - (i-1)DT_N) + (DT_j - iDT_N) \Big) + \beta(T_N)(DT_j - iDT_N)\frac{1}{\mu}.$$

We reduce the expected holding cost of stage j per cycle to a simple form:

$$\begin{aligned} E[HC_j] &= h_j \sum_{i=1}^{m_j} \frac{1}{2} T_N \Big((DT_j - (i-1)DT_N) + (DT_j - iDT_N) \Big) + \beta(T_N)(DT_j - iDT_N)\frac{1}{\mu} \\ &= h_j \sum_{i=1}^{m_j} \Big(T_N \frac{2(DT_j - iDT_N) + DT_N}{2} + \beta(T_N)(DT_j - iDT_N)\frac{1}{\mu} \Big), \end{aligned}$$

let's extract Dh_j from the bracket,

$$\begin{aligned} E[HC_j] &= Dh_j \sum_{i=1}^{m_j} \Big(T_N(T_j - iT_N) + \frac{T_N^2}{2} + \beta(T_N)\frac{1}{\mu}(T_j - iT_N) \Big) \\ &= Dh_j \sum_{i=1}^{m_j} \Big((T_j - iT_N)(T_N + \beta(T_N)\frac{1}{\mu}) + \frac{T_N^2}{2} \Big) \\ &= Dh_j T_N (T_N + \beta(T_N)\frac{1}{\mu}) \sum_{i=1}^{m_j} \Big((m_j - i) + Dh_j m_j \frac{T_N^2}{2} \Big). \end{aligned}$$

Since

$$\sum_{k=0}^{m_j-1} = \frac{m_j(m_j-1)}{2},$$

$$\begin{aligned}
E[HC_j] &= \frac{m_j(m_j-1)}{2} Dh_j T_N (T_N + \beta(T_N)\frac{1}{\mu}) + Dh_j m_j \frac{T_N^2}{2} \\
&= \frac{Dh_j}{2}\left(T_N^2 m_j^2 + \beta(T_N)\frac{1}{\mu} T_N (m_j^2 - m_j)\right) \\
&= \frac{Dh_j}{2} T_j^2 + \frac{Dh_j}{2}\beta(T_N)\frac{1}{\mu} T_j (\frac{T_j}{T_N} - 1).
\end{aligned}$$

The expected cycle time is the length of time until node j has no inventory on hand and the unreliable supplier must be in the wet period, as well.

$$\begin{aligned}
E[CT] &= \sum_{i=1}^{m_j} T_N + E[\mathit{Disruption\ Duration\ in\ a\ replenishment\ cycle}] \qquad (4.1) \\
&= m_j T_N + \sum_{i=1}^{m_j} E[\mathit{Disruption\ Duration\ in\ } i^{th} \mathit{\ subcycle}] \\
&= m_j T_N + \sum_{i=1}^{m_j} \beta(T_N)\frac{1}{\mu} \\
&= T_j + \frac{T_j}{T_N}\beta(T_N)\frac{1}{\mu}.
\end{aligned}$$

Note that $\beta(T_N)$ is the probability that a disruption occurs in the i^{th} subcycle, $i = 1, 2, \ldots, m_j$, and $\frac{1}{\mu}$ is the expected disruption length when such a disruption occurs.

We should notice that in an assembly system, some reorder intervals may belong to the set of Case 1 while some of them may belong to the set of Case 2. Therefore, their holding and fixed costs must be computed according to the rules of the set they belong to and the total average cost of the assembly system must be computed by taking sum of the average costs of each stage.

Thus the average exact cost of the assembly system per unit time is:

$$\begin{aligned} AC_E(T_0,\ldots,T_N) &= \frac{T_N\sum_{j\in C_1\cup\{0\}} C_{j,EOQ}(T_j) + Dc_s\beta(T_N)\frac{1}{\mu}}{T_N+\beta(T_N)\frac{1}{\mu}} \\ &+ \sum_{j\in C_2} \frac{K_j + \frac{1}{2}Dh_j\left(T_j^2 + \beta(T_N)\frac{1}{\mu}\frac{1}{T_N}T_j(T_j-T_N)\right) + Dc_s\beta(T_N)\frac{1}{\mu}}{\frac{T_j}{T_N}(T_N+\beta(T_N)\frac{1}{\mu})} \end{aligned}$$

which reduces to

$$AC_E(T_0,\ldots,T_N) = \frac{T_N\sum_{j=0}^{N} C_{j,EOQ}(T_j) + D\beta(T_N)\frac{1}{\mu}(c_s + \sum_{j\in C_2}\frac{1}{2}h_j(T_j-T_N))}{T_N+\beta(T_N)\frac{1}{\mu}}.$$

The proposed reorder interval model of the assembly system is:

$$\min \quad AC_E(T_0^I,\ldots,T_N^I) \tag{4.2}$$

s. t.

$$T_j^I \geq T_0^I \qquad \forall j\in\{1,\ldots,N\} \tag{4.3}$$

$$T_j^I \leq T_N^I \qquad \forall j\in C_1 \tag{4.4}$$

$$T_j^I > T_N^I \qquad \forall j\in C_2 \tag{4.5}$$

$$T_N^I = s_jT_j^I \qquad \forall j\in C_1,\ n_j\in\{1,2,\ldots\} \tag{4.6}$$

$$T_j^I = m_jT_N^I \qquad \forall j\in C_2,\ m_j\in\{2,3,\ldots\} \tag{4.7}$$

$$T_j^I = k_jT_0^I \qquad \forall j\in\{1,\ldots,N\},\ k_j\in\{1,2,3,\ldots\} \tag{4.8}$$

The objective of the model is to minimize the average cost of the assembly system per unit time. T_j^I is the decision variable of the model that is integer multiple of its successor. The constraint (4.3) ensures that all reorder intervals must be greater than or equal to T_0^I. The constraint (4.4) ensures that if j belongs to C_1, then T_j^I must be less than or equal to T_N^I. The constraint (4.5) assures that if j belongs to C_2, then T_j^I must be greater than T_N^I. The constraint (4.6) provides that if j belongs to C_1, then T_N^I equals to the product of T_j^I and s_j, which must be an integer. The constraint (4.7) assures that if j belongs to C_2, then T_j^I equals to the product of the T_N^I and m_j. m_j must be an integer as well and it must be greater than 1 since $T_j^I > T_N^I$ according to

Case 2. The constraint (4.8) ensures that the proportion of T_j^I to T_0^I must be integer.

4.3.1. The Approximate Model for A Two Level Assembly System

We consider a simple assembly system which is the same with the one in Figure 4.2. The final product is assembled at stage 0 by using the parts of stage 1 and stage 2. We assume that disruption occurs at stage 2. We build a new objective function by approximating β. β_0 is the new probability of disruption in the approximate model.

$$\beta_0 = \frac{\lambda}{(\lambda+\mu)}.$$

The approximate average cost per unit time is:

$$AC_A(T_0, T_1, T_2) \;=\; \begin{cases} \frac{T_2 \sum_{j=0}^{N} C_{j,EOQ}(T_j) + D\beta_0 \frac{1}{\mu} c_s}{T_2 + \beta_0 \frac{1}{\mu}} & \text{for} \quad T_1 \le T_2 \\ \\ \frac{T_2 \sum_{j=0}^{N} C_{j,EOQ}(T_j) + D\beta_0 \frac{1}{\mu}(c_s + \frac{h_1}{2}(T_1 - T_2))}{T_2 + \beta_0 \frac{1}{\mu}} & \text{for} \quad T_1 > T_2 \end{cases}$$

And the proposed reorder interval model is:

$$\min \quad AC_A(T_0^I, T_1^I, T_2^I) \tag{4.9}$$

s. t.

$$T_j^I \ge T_0^I \qquad \forall j \in \{1,2\} \tag{4.10}$$

$$T_1^I \le T_2^I \qquad 1 \in C_1 \tag{4.11}$$

$$T_1^I > T_2^I \qquad 1 \in C_2 \tag{4.12}$$

$$T_2^I = s_1 T_1^I \qquad 1 \in C_1,\ n_1 \in \{1,2,\dots\} \tag{4.13}$$

$$T_1^I = m_1 T_2^I \qquad 1 \in C_2,\ m_1 \in \{2,3,\dots\} \tag{4.14}$$

$$T_j^I = k_j T_0^I \qquad \forall j \in \{1,2\},\ k_j \in \{1,2,3,\dots\} \tag{4.15}$$

$$T_j^I \in \mathbb{N} \qquad \forall j \tag{4.16}$$

Now, we need to determine the reorder intervals that enable us to obtain the minimum objective function value. The average cost of the assembly system per unit time is determined according to the relation between T_1 and T_2. If $T_1 \leq T_2$, then we conclude that T_1 and T_2 belong to C_1. On the other hand, if $T_1 > T_2$, then they belong to C_2. Therefore the average cost per unit time is computed according to the case T_1 and T_2 belong to. In order to determine optimal reorder intervals,we use the fmincon function of the MATLAB. Fmincon is useful function of MATLAB that finds the local minimum T_0, T_1 and T_2 values of the model that satisfies the nestedness property of the model. Therefore we do not need to check whether they satisfy the constraint (4.3) or not.

After obtaining the optimal clusters, the final task to be done is to convert them into the reorder intervals that are integer multiples of each other in order to satisfy the constraints (4.6), (4.7) and (4.8).

An Approximate Method for Integer Reorder Intervals

Now, firstly let us assume that T_2 is the biggest one among the others.

We observe that,

- if $\frac{T_2}{T0} > 1.5$, then T_2 and T_0 will not be taking the same integer value. Then we need to check the relation between T1 and other reorder intervals.
  - (i) If $\frac{T_2}{T_1} \leq 1.5$ and $\frac{T_1}{T_0} \leq 1.5$, then the reorder intervals whose proportion is the smallest will be taking the same integer value.
  - (ii) If $\frac{T_2}{T_1} > 1.5$ and $\frac{T_1}{T_0} \leq 1.5$, then T_1 and T_0 will be taking the same integer value.
  - (iii) If $\frac{T_2}{T_1} \leq 1.5$ and $\frac{T_1}{T_0} > 1.5$, then this time, T_1 and T_2 will be taking the same integer value.

Secondly, we assume that T_1 is the biggest one among the others under the same assumptions of the first case.

- if $\frac{T_1}{T0} > 1.5$, then reorder intervals of stages 0 and 1 will not be taking the same integer value. Therefore, we need to check the relation between T_2 and other reorder intervals.
  - (i) If $\frac{T_1}{T_2} \leq 1.5$ and $\frac{T_2}{T_0} \leq 1.5$, then the reorder intervals whose proportion is the smallest will be taking the same integer value.
  - (ii) If $\frac{T_1}{T_2} > 1.5$ and $\frac{T_2}{T_0} \leq 1.5$, then T_2 and T_0 will be taking the same integer value.
  - (iii) If $\frac{T_1}{T_2} \leq 1.5$ and $\frac{T_2}{T_0} > 1.5$, then this time, T_1 and T_2 will be taking the same integer value.

- If $\frac{T_1}{T_2}$, $\frac{T_1}{T_0}$ and $\frac{T_2}{T_0}$ are greater than 1.5, then all reorder intervals will be taking different integer values.

- If $\frac{T_1}{T_2}$, $\frac{T_1}{T_0}$ and $\frac{T_2}{T_0}$ are less than or equal to 1.5, then all of them will have the same integer value.

Let us assume that we have two optimal clusters. In order to find out whether their reorder intervals will be taking the same integer value or not, we should examine their proportion. If it is less than or equal to 1.5, then we conclude that they will be having the same integer value; if not, they will have different integer values.

Now, let us assume that we have only one optimal cluster, then we need to examine the integer values that are close to the optimal reorder value and pick up the one that gives the smallest average cost. The same approach is valid for the others as well, after deciding which ones have the same integer value, we need to analyze the integer values that are close to the optimal reorder values and choose the ones that provide the minimum average cost.

4.3.2. Numerical Analysis

In this part of the study, we provide numerical examples to examine the model in detail so that we can fully understand the behavior of the model. We analyze a three-stage assembly system as in Figure 4.2. Stage 0 creates the finished product by supplying the semi-finished items from stage 1 and stage 2. We assume that stage 2 is due to the risk of supply disruption.

Table 4.1 shows the parameters of the model. Each parameter has a wide range

Table 4.1. Parameter Set.

Parameter	Symbol	Tested Values
Fixed Cost of Stage 0	K_0	100
Fixed Cost of Stage 1	K_1	50, 200, 400
Fixed Cost of Stage 2	K_2	50, 100, 400
Holding Cost of Stage 0	h_0	0.2
Holding Cost of Stage 1	h_1	0.2
Holding Cost of Stage 2	h_2	0.2
Demand	D	10, 100
Shortage Cost	c_s	5, 10
Wet period Rate	λ	1
Dry Period Rate	μ	0.2, 0.5, 1

of values that enable us to analyze the model clearly. First, let us examine the reorder intervals that are computed according to the classical economic reorder interval problem. We know that the solution of the problem is

$$T_i = \sqrt{\frac{2K_i}{Dh_i}}.$$

For the given parameter values, their proportions to each other equal to

$$\frac{T_1}{T_0} \in \{0.7, 1.41, 2\}.$$

$$\frac{T_2}{T_0} \in \{0.7, 1, 2\}.$$

They show us the range of values k_1 and k_2 may take in the model. As we can see, the gap between T_0, T_1 and T_2 is quite large. This variety among the reorder intervals is a considerable contribution to the numerical analysis.

We should also notice that the values of the fixed costs of the stages have an important role in determining k_1 and k_2 since they are completely related with T_0, T_1 and T_2. In order to justify our claim, let us first analyze Table 4.2 that shows the relation between K_1 and the reorder intervals. The parameters of the model are set as $K_2 = 50$, $D = 10$, $c_s = 5$, $h_0 = h_1 = h_2 = 0.2$, $\lambda = 1$ and $\mu = 0.2$. As we can see, when K_1 equal to 50, T_1 takes the value of 8 however when we increase the value of K_1 to 400, T_1 increases as well and take the value of 16. The model prefers to order less frequently by decreasing the order quantity of the fixed cost of stage 1 since the fixed cost is larger now. Besides there is a linear relation between the average cost of the system and K_1. The situation is pretty same with the cases of K_0 and K_2, therefore we do not provide the numerical examples of these cases. Now, let us also examine

Table 4.2. Behavior of the Model with Respect to K_1.

K_1	T_0, T_1, T_2	Expected Cost
50	8, 8, 8	49.34
400	8, 16, 8	69.67

$\beta(T_2)$ and β_0.

$$\beta(T_2) \in \{0.49, \ldots, 0.83\}.$$

$$\beta_0 \in \{0.5, \ldots, 0.83\}.$$

As it can be seen, the minimum value for $\beta(T_2)$ is 0.49 while it is 0.5 for β_0 for the given parameter values. Both values are high enough to examine the effect of disruption in the assembly system. If they took smaller values, since the risk of disruption would be smaller, it would be very difficult to fully understand the impact of disruption. Consequently, it is essential for us to prefer higher $\beta(T_2)$ and β_0 values in the model.

We should notice that since the rates of the wet and the dry periods form β and β_0, they have an important role in the model. Therefore, let us analyze Table 4.3 so as to show the impact of μ on β_0 and the reorder interval of the unreliable supplier. The model parameters are set as $K_2 = 50$, $K_1 = 50$, $D = 100$, $h_0 = h_1 = h_2 = 0.2$, $\lambda = 1$ and $c_s = 5$. We increase the value of μ from 0.2 to 1 in order to decrease β_0. This sudden increase makes T_2 take on a smaller value since the probability of disruption is smaller now. Therefore, from these results we can conclude that when the disruption probability is smaller, model prefers to hold less inventory by decreasing the reorder interval of the unreliable supplier. Finally, we examine the range of values that the

Table 4.3. Behavior of the Model with Respect to μ.

μ	T_0, T_1, T_2	Expected Cost
0.2	3, 3, 9	298.73
1	3, 3, 6	203.07

total shortage cost take in the numerical analysis.

$$Dc_s\beta(T_2)\frac{1}{\mu} \in \{25, \ldots, 4165\}.$$

$$Dc_s\beta_0\frac{1}{\mu} \in \{25, \ldots, 4167\}.$$

It is obvious that the range of values the total shortage cost take is quite wide. It takes larger values as μ gets smaller and c_s and D get larger. It makes sense because as μ decreases, the duration of the dry period increases. Therefore, it is more difficult to mitigate the cost of disruption when μ takes smaller values and c_s and D take larger values. Consequently, the model prefers a higher reorder interval especially for stage 2 in order to decrease the total shortage cost.

Let us explain it more clearly by giving a small example. We want the effect of disruption to be more destructive. According to our claim, the increase of the shortage cost per lost sale should result in a higher T_2. Table 4.4 shows how the change of c_s effects the model. We set the parameters as $K_2 = 50$, $K_1 = 50$, $D = 10$, $h_0 = h_1 = h_2 = 0.2$, $\lambda = 1$ and $\mu = 0.2$. As it is expected, the increase of c_s cause T_2 to take on a higher value and it also makes the average cost per unit time increase as well. Now, by using these parameter values, we obtain the reorder interval of the each

Table 4.4. Behavior of the Model with Respect to c_s.

c_s	T_0, T_1, T_2	Expected Cost
5	8, 8, 8	49.34
10	8, 8, 16	63.40

stage and check whether they satisfy the nestedness policy. If they do not, then we determine the optimal clusters of the assembly system by using the minimum violators algorithm. Afterwards, in order to find the integer reorder intervals that are multiples of each other, we examine each possible combination of integer reorder intervals and select the ones that give the minimum objective function. We should recall that this selection process is also a guideline for us to propose the round off method. Therefore, apart from this selection process, we also obtain the reorder intervals that are determined according to the round off method. Finally, we measure the errors due to applying the round off method and using the approximate beta so as to measure accuracy of the approximation method and the round off method.

4.3.3. Approximation Error

In order to find out the measure of the error due to approximating the average cost of the assembly system, we aim to compute the percentage error by using AC_0 and AC_2. AC_0 is the average cost of the exact model per unit time with integer reorder intervals.

$$AC_0 = AC_E(T_0^{I*}, T_1^{I*}, T_2^{I*}).$$

AC_1 is the average cost of the exact model when we we use the integer reorder intervals of the approximate cost.

$$AC_1 = AC_E(T_0^I, T_1^I, T_2^I).$$

We determine the percentage error by

$$PAE = \frac{AC_1 - AC_0}{AC_0}.$$

According to the results, the average percentage error turns out to be 0 which means approximation provides same integer reorder intervals with the exact cost function. Therefore, we conclude that approximation provides good results when T_2, λ and μ are reasonably high and it does not have a significant impact on the average cost of the assembly system.

4.3.4. Error Due to Using Reorder Intervals of Round off Method

In this section, we want to measure the error due to applying the method that we propose to obtain the integer reorder intervals that are multiples of each other. AC_2 is the average cost of the exact model with the reorder intervals of the round off method.

$$AC_2 = AC_E(T_0^I, T_1^I, T_2^I).$$

We determine the percentage error by

$$PER = \frac{AC_2 - AC_1}{AC_1}.$$

The overall average percentage error is 0 which indicates that the round off method is good at estimating the integer reorder intervals that are multiples of each other. Therefore we can conclude that on average it provides accurate results for the model. We should notice that PER provides the error between the cost of the exact model when we use the reorder intervals of the approximate model and the cost of exact model when we apply the round off method. Furthermore, we can also measure the error between the cost of applying the round off method and the cost of exact model with the optimal integer reorder intervals so that we can measure the total error.

$$PTE = \frac{AC_2 - AC_0}{AC_0}.$$

Since there is no difference between the integer reorder intervals that we obtain from the exact and approximate models, the average percentage error is still 0.

5. CONCLUSIONS

In the first part of the study, a model for serial systems is proposed to cope with the risk of supply disruption. Firstly, a two-stage serial system is analyzed, the proposed approach is to set up a model to mitigate the cost of disruption by determining reorder intervals that give the minimum objective function. For practical reasons, only integer reorder intervals that are integer multiples of each other are taken into consideration. Therefore, in the numerical analysis, after examining all the integer reorder intervals, the ones that minimize the objective function are picked up. Moreover, so as to figure out the effects of the parameters on the behavior of the model, a sensitivity analysis is conducted. According to the results, fixed costs of the serial system stages cause reorder intervals to take on higher values since the model needs to prefer ordering less frequently to get rid of higher ordering costs. On the other hand, holding costs have an inverse impact on the reorder intervals since greater reorder intervals mean greater total holding costs in a cycle. Moreover, the dry period rate and the wet period rate have direct relation with the reorder interval of the disrupted supplier. The increase of the dry period rate makes the probability of disruption smaller, consequently since the model thinks that risk is smaller now, it can order less frequently by holding less inventory. The situation is pretty same with the wet period rate. As the wet period rate gets comparatively higher than the dry period rate, the probability of disruption increases. As a result, model prefers to hold more inventory to mitigate the effect of disruption. Besides, as expected, the increase of the shortage cost per lost sale cause the model to provide a larger reorder interval of the unreliable supplier in case of the disruption.

Secondly, an approximation approach is developed for an N stage serial system by approximating the probability of disruption. It is shown that the objective function is unimodal under mild conditions. Therefore it provides unique minimum reorder intervals. The algorithm in the study of [34] is used to obtain the optimal clusters and the reorder intervals that satisfy the condition of the nestedness policy. A power-of-two policy is also applied to obtain the reorder intervals that are powers-of-two multiples of the base planning period. A numerical analysis is conducted to measure the error due to the approximation approach and the power-of-two policy. It is shown that as long as the reorder interval of the unreliable supplier is high enough, the approximation provides good results. In fact, the overall average percentage error turns out to be only 0.08. It is also shown that the average percentage error of the power-of-two policy is not significant as well.

In the second part of the study, assembly systems due to the risk of disruption are examined. The essential property of the model is to assume that the reorder intervals are integer multiples of each other. It is shown that the model changes according to the relation among the reorder intervals that are predecessors of the root stage therefore it is needed to be examined by taking into two different cases. Moreover, the minimum violators algorithm is applied to obtain the optimal clusters and the reorder intervals that satisfy the condition of the nestedness policy. An approximation approach is also developed by approximating the probability of disruption. For the part of numerical analysis, a three-stage assembly system is examined and a method called round off is developed to convert the reorder intervals that are integer multiples of each other. Besides, the errors due to conducting the approximate model and using the reorder intervals that are integer multiples of each other are measured as well. It is shown that for the given parameter values the gap between the exact and the approximate model equals to zero which proves that approximation works well when the reorder interval of the unreliable supplier is high enough. Moreover, the percentage error due to applying the round off method also turns out to be zero, it means that the proposed method is accurate enough to apply in the model.

For further studies, this study can be extended by analyzing supply disruption in a distribution system which is also a multistage production systems. Besides, it can be assumed that the number of unreliable supplier is more than one and they can be disrupted independently.

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