

QUANTIFYING THE INTEREST RATE RISK OF BONDS BY SIMULATION

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## ABSTRACT

# QUANTIFYING THE INTEREST RATE RISK OF BONDS BY SIMULATION

Managing interest rate risk of bonds is very important in practice. Therefore, many methods have been developed to measure the interest rate risk. In this thesis, simulation is used to quantify the interest rate risk of bonds. Only government bonds are considered, which are often assumed to have “0” default risk. So, the only relevant risk is the interest rate risk.

In this thesis popular interest rate models are analyzed in a practical framework. Bond pricing, interest rate simulation, parameter estimation and risk simulation principles are explained for six different short-rate models. For each model, the exact method for simulating the short-rate is given; the analytical bond prices of the models and the bond prices obtained by simulation are compared; MLE is used to estimate the model parameters and risk simulations are made including a sensitivity analysis of VaR with respect to the model parameters.

In the last part of this thesis, the risk estimation performance of the different models is compared. The US, German and the Canadian bond market data are used for analysis. The quality of the risk estimates of the models for different maturities is assessed. It is observed that for all models and data that the arbitrage-free and the equilibrium versions lead to the same Value at Risk results. For all implementations we use R, which is a programming language for statistical computing.

## ÖZET

# TAHVİLLERİN FAİZ ORANI RİSKİNİN SİMÜLASYONLA ÖLÇÜLMESİ

Tahvillerin faiz oranı riskini yönetmek, uygulamada çok önemlidir. Bu nedenle, faiz oranı riskini ölçmek için birçok metod geliştirilmiştir. Bu çalışmada, tahvillerin faiz oranı riskini hesaplamak için simülasyon kullanılmıştır. Sadece devlet tahvilleri, batma riski genellikle “0” kabul edilen, incelenmiştir. Dolayısıyla, ilgili tek risk faiz oranı riskidir.

Bu tezde, yaygın faiz haddi modelleri uygulamaya dönük bir çerçevede analiz edilmiştir. Altı farklı kısa dönem faiz haddi modeli için, tahvil fiyatlama, faiz oranı simülasyonu, parametre tahmini ve risk simülasyonu esasları açıklanmıştır. Her model için, kısa dönem faiz oranı simülasyonunun kesin metodu verilmiştir; modellerin analitik tahvil fiyatlarıyla simülasyon ile hesaplanan tahvil fiyatları karşılaştırılmıştır; model parametreleri maksimum olabilirlik tahmin (MLE) yöntemi kullanılarak elde edilmiştir ve risk simülasyonları, riske maruz değerin model parametrelerine duyarlılık analizi ile birlikte, yapılmıştır.

Tezin son bölümünde, farklı modellerin risk tahmin performansı karşılaştırılmıştır. Analizlerde, Amerikan, Alman ve Kanada tahvil piyasası verileri kullanılmıştır. Modellerin risk tahmin kaliteleri, farklı vadeler için değerlendirilmiştir. Modellerin arbitrajı engelleyen ya da denge versiyonlarını kullanmanın risk ölçümünü etkilemediği gösterilmiştir. Tüm uygulamalar için, programlama dili ve istatistiksel hesaplama ortamı olan R kullanılmıştır.

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## LIST OF SYMBOLS/ABBREVIATIONS

$d$	Number of time steps
$dt$	Length of time steps
$dW_t$	Brownian Motion
$F(t; T, S)$	Simply compounded forward rate at time $t$ for the expiry $T$ and maturity $S$
$f(t; T, S)$	Continuously compounded forward rate at time $t$ for the expiry $T$ and maturity $S$
$f(t, T)$	Instantaneous forward rate at time $t$ for maturity $T$
$f^M(t, T)$	Market instantaneous forward rate at time $t$ for maturity $T$
$L(t, T)$	Simply-compounded yield from time $t$ for maturity $T$
$n$	Sample size
$P(t, T)$	Bond price at time $t$ for maturity $T$
$P^M(t, T)$	Market bond price at time $t$ for maturity $T$
$R(t, T)$	Continuously compounded yield from time $t$ for maturity $T$
$r(t)$	Short-rate at time $t$
$r_0$	Initial short-rate
$T$	maturity
$t$	time
$Y^k(t, T)$	K-times-per-year compounded yield from time $t$ for maturity $T$
$\alpha$	Type-I error
$\beta$	Mean-reversion parameter
$\mu$	Long-term mean parameter
$\lambda$	Market price of risk
$\rho$	Correlation coefficient
$\sigma$	Volatility parameter
$\varphi(t)$	Deterministic shift function at time $t$

AF	Arbitrage-free
BDT	Black-Derman-Toy
BP	Bond price
CIR	Cox-Ingersoll-Ross
EURIBOR	Euro Interbank Offered Rate
MLE	Maximum likelihood estimation
VAR	Value-at-risk

# 1. INTRODUCTION

A bond is a financial debt instrument requiring the issuer (borrower) to repay to the lender/investor the amount borrowed plus interest over a specified period of time (maturity). The size of the bond market (also called debt or fixed income market) in 2009 was an estimated of \$82.2 trillion of which the size of the USA bond market debt was \$31.2 trillion according to “Bank for International Settlements” [1]. Bond prices are less volatile than stock prices, which makes them safer investments. However, they also have some risks, such as the default risk and the interest rate risk. In this thesis we will consider the interest rate risk.

Fundamental features of bonds and some classic interest rate risk measures (e.g. duration and convexity) are well explained in a lot of sources [2–6]. However, practitioners need more explicit methods for both pricing and risk measuring. For this reasons, interest rate models (i.e. short rate models) were developed. There are a lot of books on interest rate modeling, such as Brigo and Mercurio [7], Filipovic [8], Cairns [9], Rebonato [10], Tuckman [11] and Shreve [12]. However, these books are very mathematical and thus it is difficult to understand the models not only for practitioners.

The aim of this thesis is to explain and analyze popular interest rate models in a practical, simulation and risk oriented, framework. We also try to answer the questions, how to quantify the interest rate risk of bonds and more importantly which model or models should be used to quantify the risk correctly. These questions, especially the second one, are not discussed much in the literature. Although some papers compare different short-rate models empirically, most of them are pricing, forecasting or calibration oriented rather than risk oriented [13–16]. In this thesis, we use the US, German and the Canadian bond market data to compare the risk estimation performance of the different models.

The thesis is organized as follows. Chapter 2 gives general information about

bonds and risk in bond investments. Fundamentals of interest rate modeling are explained in Chapter 3. Interest rate simulation, parameter estimation, bond pricing and risk simulation principles are explained for the equilibrium and the arbitrage-free models in Chapter 4 and in Chapter 5, respectively. In chapter 6, the risk estimation performance of the different models is compared by using the market data. The last chapter gives our conclusions.

## 2. BONDS

In this chapter bonds are described in general. After that we summarize the different compounding methods that are used for bond pricing. We also explain the notion of forward rates. Finally, risk in bond investments is explained. Lyuu [2] and Brigo & Mercurio [7] are references generally used in this chapter.

### 2.1. Definitions

A bond is a contract between the issuer (borrower) and the bondholder (lender). The issuer promises to pay the bondholder interest. In other words, the issuer of the bond will repay to the lender/investor the amount borrowed (principal) plus interest over a specified period of time (maturity). There are two main types of bonds: government and corporate. We will only consider government bonds as the majority of the bonds in the world and all bonds in Turkey are government bonds. Government bond contracts can be considered as credits given to a government. Basically, governments borrow money to finance their treasury and promise to repay this money plus an interest rate at a fixed time called maturity. Generally bonds of stable governments are considered to have no default risk. Some important definitions about bonds are given below [2].

- The Par-Value (Face Value, Stated Value, Principal Value, Redemption Value or Maturity Value) is the amount (borrowed money+interest) the issuer will pay to the bond holder at maturity.
- Coupon Bonds are bonds for which the issuer promises to pay the investor (bond holder) coupon paying (of the interests) at periodic intervals before maturity. We will only consider zero-coupon bonds, since they are easier to implement.
- Zero-Coupon Bonds do not have any periodic coupon payments. Instead there exists only the single payment at maturity. A zero-coupon bond can be considered as a long-term bank account where the investor withdraws his initial investment + interest at maturity.

- Yield (Yield to Maturity) is the interest rate that bond holders earn for buying the bond. The issuer of the bond will make the interest payment according to the specified yield.

## 2.2. Pricing Zero-Coupon Bonds

The price of any financial instrument is equal to the present value of the expected cash flows of that financial instrument. As zero-coupon bonds have only a single future payment, we only have to discount this payment with the specified yield to obtain the price of the bond. As a convention, year is used as time scale in the bond market. However different compounding methods for interest rates are used. Note that yield is an interest rate itself. So different compounding methods for interest rates define different notions of yield. The three main compounding methods are: simple-compounding, periodic-compounding and continuous compounding. The formulas for the compounding methods are taken from Brigo and Mercurio [7].

### 2.2.1. Yield in Simple-Compounding

Here the yield  $L(t, T)$  is defined as the simply-compounded constant interest rate from time  $t$  for the maturity  $T$ . It is the rate at which the bond price  $P(t, T)$  has to carry interest proportional to the investment time  $T - t$  to produce an amount of one unit at maturity  $T$ . This definition easily leads to the yield formula:

$$L(t, T) = \frac{1 - P(t, T)}{(T - t)P(t, T)} .$$

Equivalently the bond price is

$$P(t, T) = \frac{1}{1 + L(t, T)(T - t)} .$$

At the Istanbul Stock Exchange bond market, simple-compounding is used to obtain bond prices from yields.

### 2.2.2. Yield in Periodic-Compounding

This method is also called  $k$ -times-per-year compounding. Here the yield is defined similar to the section above, the only difference is, that a  $k$ -times-per-year compounding is used. Thus using the same principle as above we equate the bond price plus the total value of the interest paying to the Par-Value (value at maturity) 1 and thus get the equation

$$P(t, T) \left( 1 + \frac{Y^k(t, T)}{k} \right)^{k(T-t)} = 1 .$$

Elementary manipulations of that equation lead to the formula for the bond price

$$P(t, T) = \frac{1}{\left( 1 + \frac{Y^k(t, T)}{k} \right)^{k(T-t)}} ,$$

and to the yield formula:

$$Y^k(t, T) = k \left( \frac{1}{P(t, T)} \right)^{1/(k(T-t))} - k .$$

In the US Treasury bond market and in many European countries semi-annual compounding (ie. periodic compounding with  $k = 2$ ) is used as standard.

### 2.2.3. Yield in Continuous Compounding

Again the principle remains the same but the compounding method is now continuous compounding. Again equating the bond price plus the interest paying to the Par-Value 1 we have

$$P(t, T) e^{R(t, T)(T-t)} = 1 .$$



Elementary manipulations of that equation lead to the formula for the bond price

$$P(t, T) = e^{-R(t, T)(T-t)},$$

and to the yield formula:

$$R(t, T) = -\frac{\ln P(t, T)}{(T-t)}. \quad (2.1)$$

Note that k-times-per year compounding is equivalent to continuous compounding when k approaches infinity.

$$\lim_{k \rightarrow +\infty} \left(1 + \frac{Y}{k}\right)^{k(T-t)} = e^{Y(T-t)}.$$

2.2.3.1. An Example. Consider a zero-coupon bond traded in the IMKB bond market on the 07.09.2009 that matures 611 days later. Its simple yield is 9.73% and has a par-value of 100 units. Since we have the simply-compounded yield, the bond price according to simple-compounding method will be:

$$\begin{aligned} P(0, 611/365) &= \frac{100}{(1 + 0.0973 * 611/365)}, \\ &= 85.994. \end{aligned}$$

So the investor invests 85.994 for buying the bond and her investment becomes 100 in 611 days with simple compounding. Interest payment is  $100 - 85.994 = 14.006$ . Note that yield is the interest rate that equates the bond's price (present value) with its par-value

Corresponding periodic compounded yield, 1-time-per year compounded, will be:

$$\begin{aligned}
 Y^k(t, T) &= k \left( \frac{1}{P(t, T)} \right)^{1/(k(T-t))} - k, \\
 Y^1(0, 611/365) &= 1 * \left( \frac{100}{85.994} \right)^{1/(611/365)} - 1, \\
 &= 0.0943,
 \end{aligned}$$

Note that  $k = 1$  and we used yearly time scale. So the annually compounded yield is 9.43%.

Corresponding continuously-compounded yield is:

$$\begin{aligned}
 R(t, T) &= -\frac{\ln P(t, T)/1}{(T - t)}, \\
 R(0, 611/365) &= -\frac{\ln 85.994/100}{(611/365)}, \\
 &= 0.0901,
 \end{aligned}$$

And the continuously compounded yield is 9.01%. Continuous compounding has many advantages and we will use it in this thesis.

### 2.3. Yield Curve

Also referred as zero-coupon curve or term structure of interest rates, is the plot of yields to maturity for bonds of equal quality differing solely in their time to maturity [2]. Different compounding methods can be used for the yield curve. Continuous-compounding is applicable for arbitrary maturity and is therefore widely used in practice.

Consider 4 US zero-coupon bonds with unit par-value and maturities 6 month, 1 year, 18 month and 2 years. They are traded in the US bond market with the prices: 0.987, 0.962, 0.931, 0.902 respectively. First we calculate the continuous yields, for plotting the yield curve. R-code that calculates the yields (with continuous-compounding) and plots the yield curve is given in Figure 2.1.

```

cont_yield<-function(price,matur){
  yield= -log(price)/matur
  yield
}
c(0.987,0.962,0.931,0.902)
matur= c(0.5,1,1.5,2)
yields=cont_yield(price,matur)
yields
#[1] 0.02617048 0.03874083 0.04766400 0.05157038
plot(matur,yields,"l",main="Sample Yield Curve")
smooth=spline(matur,yields)
points(smooth$x,smooth$y,"l",col="red")

```

Figure 2.1. R-codes for plotting the yield curve

Note that, we use R's spline function to get a smooth curve.

As it can be seen from Figure 2.2, yield curves are usually upward sloping as a longer maturity leads to a higher yield. This is reasonable as longer maturities entail a greater risk for the investor. So, a risk premium is needed by the market, since there is more uncertainty till the end of the investment.

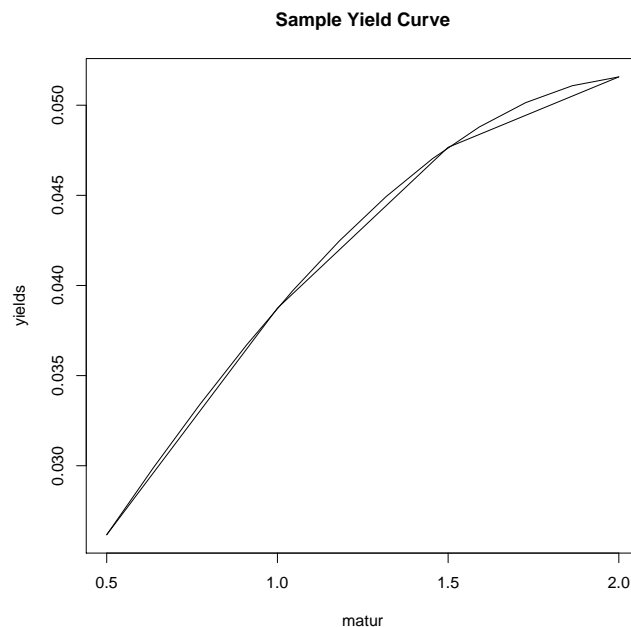


Figure 2.2. A Sample Yield Curve

## 2.4. The Forward Rate

Future interest rates, currently expected by the market are called forward rates. To understand the notion of forward rates, consider the following explanation that is taken from Fabozzi [5]. Suppose there are two investment alternatives,

- Alternative 1 Buy a one-year instrument.
- Alternative 2 Buy a six-month instrument and when it matures re-invest it to another six-month instrument.

With Alternative 1 the investor will realize the one-year yield and this rate is known. In contrast, with alternative 2, the investor will realize the six-month yield, but the six-month yield six months from now is unknown. Therefore, for alternative 2 the rate that will be earned one year from now is not known with certainty.

The future rate currently expected by the market, which makes the two alternatives indifferent is called the forward rate and given as:

$$f(0; 0.5, 1) = \frac{1}{1 - 0.5} \left( \frac{(1 + z_2)^2}{1 + z_1} - 1 \right).$$

where  $z_1$  is the 6-month rate for period  $[0, 0.5]$ ,  $z_2$  is the 6-month rate for period  $[0, 1]$  and  $f$  is the annual forward rate for period  $[0.5, 1]$ . So, if the six-month yield six months from now is less (greater) than  $f$  in the market, then investing in alternative 1 (alternative 2) would be more profitable. If the six-month yield six months from now is the forward rate, then the two investment alternatives are indifferent.

### 2.4.1. Simply-Compounded Forward Rate

More formally, with  $T$  denoting the expiry and  $S$  the maturity, the simply compounded forward interest rate prevailing at time  $t$  is defined by,

$$F(t; T, S) = \frac{1}{(S - T)} \left( \frac{P(t, T)}{P(t, S)} - 1 \right).$$

So, the investor first realizes the yield with maturity  $T$  then at time  $T$  he re-invests his money at forward rate  $F$  between times  $S$  and  $T$ . Note that when  $T = 0$  the forward rate at  $t = 0$  will be the  $S$ -year yield, since  $P(0, 0) = 1$  and

$$\begin{aligned} F(0; 0, S) &= \frac{1}{S} \left( \frac{P(0, 0)}{P(0, S)} - 1 \right), \\ &= \frac{1 - P(0, S)}{SP(0, S)}, \end{aligned}$$

So that  $F(0; 0, S) = L(0, S)$ , since the simply-compounded yield is given as

$$L(t, S) = \frac{1 - P(t, S)}{(S - t)P(t, S)}.$$

### 2.4.2. Continuously-Compounded Forward Rate

The logic is the same as for the simply-compounded forward rate. Again we compare two investment alternatives and the forward rate is the rate that makes these two alternatives indifferent.

Then, with  $T$  denoting the expiry and  $S$  the maturity, the first alternative consists of holding a bond with maturity  $S$ ; the second alternative is holding a bond with maturity  $T$  and at time  $T$  re-investing into a bond with maturity  $S - T$  and yield equal to the forward rate.

More formally, if we invest 1-unit amount at both investments and equate them at time  $S$  we get,

$$\begin{aligned} 1 \exp \{R(t, S)(S - t)\} &= 1 \exp \{R(t, T)(T - t)\} \exp \{f(t; T, S)(S - T)\} \\ R(t, S)(S - T) &= R(t, T)(T - t) + f(t; T, S)(S - T) \\ f(t; T, S) &= \frac{R(t, S)(S - T) - R(t, T)(T - t)}{S - T}. \end{aligned}$$

The continuously-compounded yield is given in Formula 2.1, then the continuously-compounded forward rate will be,

$$\begin{aligned} f(t; T, S) &= \frac{-\ln P(t, S) + \ln P(t, T)}{S - T} \\ f(t; T, S) &= \frac{\ln \{P(t, T)/P(t, S)\}}{S - T}. \end{aligned}$$

Consider the example in the yield curve section above. We have 4 zero-coupon bonds with prices 0.987, 0.962, 0.931, 0.902 and maturities 0.5 year, 1 year, 1.5 year and 2 years respectively. Then, the continuously-compounded forward rates in the intervals  $(0, 0.5)$ ,  $(0.5, 1)$ ,  $(1, 1.5)$  and  $(1.5, 2)$  are given by,

$$\begin{aligned} f(0; 0, 0.5) &= \frac{\ln P(0, 0)/P(0, 0.5)}{0.5 - 0} = 0.02617. \\ f(0; 0.5, 1) &= \frac{\ln P(0, 0.5)/P(0, 1)}{1 - 0.5} = 0.0513. \\ f(0; 1, 1.5) &= \frac{\ln P(0, 1)/P(0, 1.5)}{1.5 - 1} = 0.0655. \\ f(0; 1.5, 2) &= \frac{\ln P(0, 1.5)/P(0, 2)}{2 - 1.5} = 0.0633. \end{aligned}$$

We know from the forward rate definition that investing into 2-year yield is equivalent to first investing into 0.5 year yield and re-investing continuously to  $(0.5, 1)$  year,  $(1, 1.5)$  years and  $(1.5, 2)$  years forward rates. Note that 0.5 year yield is equal to the forward rate  $f(0; 0, 0.5)$ . Therefore, if we discount the unit par-value of the 2 year bond with

forward rates back to time  $t = 0$  we should get the 2-year bond price.

$$P(0, 2) = \exp\{-f(0; 1.5, 2)0.5\} \exp\{-f(0; 1, 1.5)0.5\} \exp\{-f(0; 0.5, 1)0.5\} \\ \exp\{-f(0; 0, 0.5)0.5\} = 0.902.$$

Note that forward rates can be seen as future interest rates currently expected by the market in the “risk-neutral world”. So we can’t expect them to be equal to the market interest rates (short rate) in the “real-world”. The notion of the short rate will be explained in the next chapter.

### 2.4.3. Instantaneous Forward Interest Rate

If we consider the definition of the forward rate  $f(t, T, S)$  above, if  $(S - T)$  gets very short we can obtain a forward rate for very short termed bonds. This is called the instantaneous forward interest rate and defined by the limit

$$f(t, T) = \lim_{S \rightarrow T} f(t, T, S).$$

## 2.5. Risk in Bond Investments

In bond contracts, it is guaranteed that investors will get their principal investment plus a constant interest payment at the maturity of the bond. However, if investors want to sell their bonds *before* maturity, they may be faced with a discounted market price. This is due to the fact that the bond prices are inversely related to the interest rates. As it can be seen from the bond price formula in continuous compounding,

$$P(t, T) = e^{-R(t, T)(T-t)},$$

So, when interest rates go up, bond prices go down, and vice versa. As a result, investors can be exposed to *interest rate risk* if they want to sell their bonds before maturity.

More formally, if we denote the price of a zero-coupon bond at time  $t$  with the maturity  $T$  by  $P(t, T)$  then the initial bond price at time  $t = 0$  is  $P(0, T)$  and the loss at time  $t > 0$  will be,

$$Loss(t) = P(0, T) - P(t, T) \quad (2.2)$$

According to 2.2, a bond investor will pay  $P(0, T)$  to buy the bond and receive  $P(t, T)$  at time  $t$  by selling the bond to the bond market. Therefore, if the interest rate level is higher at time  $t$  than at time 0, then the investor will be exposed to a loss. Note that at time  $t = T$ , the bond investor will receive the par-value of the bond independent from the level of the interest rates.

We should also include the opportunity cost of buying a bond in our loss function for more realistic loss computation. To do that, we can assume that a bond investor will put his money into a risk less bank account at time  $t = 0$  rather than buying a bond. If we assume constant interest rates in the investment period (i.e.  $[0, t]$ ) and use continuous compounding, then the profit from that alternative investment will be,

$$Pbank(t) = P(0, T) e^{rt} - P(0, T) \quad (2.3)$$

where  $r > 0$  ensures that  $Pbank(t) > 0$ .

Finally, we can calculate the profit difference between the two investments as the opportunity cost of investing in a bond. The profit from the bond investment will be



negative of (2.2), then the opportunity cost can be written as,

$$\begin{aligned} Opp(t) &= Pbank(t) - (-Loss(t)) \\ &= P(0, T) e^{rt} - P(t, T) \end{aligned} \tag{2.4}$$

which shows how much better we are doing than a risk less investment. If (2.4) is negative, investing in a bond is more profitable, otherwise investing to a bank account is better.

In order to quantify the risk efficiently, we need a coherent risk measure. In this thesis, we will use the value-at-risk (VAR) method, which is the most popular risk measure in practice. VAR is defined as a percentile of the loss distribution over a fixed time horizon  $\Delta t$  (see, e.g., Glasserman [17]). Then, the 95% VAR is the point  $l_p$  satisfying,

$$1 - F_L(l_p) = P(Loss > l_p) = \alpha \tag{2.5}$$

with  $\alpha = 0.05$ . In other words, the 95% VAR is the 95% quantile of the loss distribution. This means practically that the bond investor will, compared to investing in to a bank account, not lose more than  $l_p$  with 95 percent probability.

In the literature, relatively little attention has been directed to measuring the interest rate risk of bonds by using interest rate models, while default risk is discussed a lot. Numerous articles and books have been published on measuring default risk, especially for corporate bonds, [18–21].

However, banks generally hold government bonds (e.g. almost all of the bond market in Turkey consists of government bonds), which is often, at least for strong economies like US and Germany, assumed to have “0” default risk. The development of the debt crisis in Iceland, Ireland and recently Greece shows that the default of a government is more likely than thought before. In this thesis, we only consider zero-coupon government bonds and assume that they have no default risk.

Measuring interest rate risk is very important for banks. Also, the Basel Committee on Banking Supervision stress the importance of managing interest rate risk for banks [22]. In the market, duration and convexity are the most commonly used risk measures by practitioners for measuring the bond price volatility, interest rate risk, (i.e. the sensitivity of the bond price to changes in the interest rates). Duration and convexity use today's yield curve to measure the risk of changes in interest rate levels. Therefore, interest rate risk is also called yield curve risk in this concept [23]. However, these measures can not quantify the risk of holding a bond portfolio for a certain period, they can be used just for comparing the relative risk of different bonds. They are explained in various sources such as Lyuu [2]; Nawalkha, Gloria and Beliaeva [3]; Tuckman [4] and Fabozzi [5, 6].

In the last decade, some articles have been published on measuring the interest rate and default risk of bonds together by using simulation techniques [24–26]. However, the question, which short rate models should be used to quantify the interest rate risk correctly is not answered in the literature. Answering this question is the main aim of our thesis.

### 3. INTEREST RATE MODELS

Interest rates are highly volatile in today's markets. This calls for stochastic interest rate models to model the movements of the interest rates [2]. Interest rate models are also needed for quantifying the interest rate risk of bonds and other interest rate securities.

The first stochastic interest rate model was proposed by Merton in 1973 [27]. Merton's model was followed by the pioneering model of Vasicek in 1977 [28] and some other classical models, such as Dothan in 1978 [29], Cox-Ingersoll and Ross in 1985 [30], Ho-Lee in 1986 [31], Hull-White Extended Vasicek in 1990 [32], Black-Derman-Toy in 1990 [33] and CIR++ in 2001 [34].

#### 3.1. Fundamentals of Interest Rate Modeling

Brigo and Mercurio [7] state that the notion of interest rate modeling is based on the specific dynamics of the instantaneous interest rate process  $r$ . This process is called the short rate process. In this chapter, most of the formulas were taken from Brigo and Mercurio [7].

##### 3.1.1. The Short Rate

The short rate is the limit of the bond interest rate for maturity tending to zero [8]. Therefore, we can not directly observe the short rate in the market. However, we can think of the short rate as the overnight interest rate of reserve banks, since it has the shortest maturity in the market. It is also possible to use the yield of very short termed bonds (eg. 1 month or 3 months to maturity) as approximate value of the short rate. The short rate changes over time so we write  $r_t$  for the short rate process.

The short rate is also defined as the rate at which the bank account grows [7]. Consider a bank account which holds one unit at time zero. We use  $B(t)$  to denote the

value of the bank account over time and have of course  $B(0) = 1$ . Then, for continuous compounding the value of the bank account at any time can be calculated as:

$$B(t) = \exp \left( \int_0^t r_s ds \right)$$

Note that all definitions of interest rates are equivalent for very short time intervals. Indeed, it is easy to show that the short rate can be written as the limit of all the different rates that were defined above [7].

$$\begin{aligned} r(t) &= \lim_{T \rightarrow t^+} R(t, T) \\ &= \lim_{T \rightarrow t^+} L(t, T) \\ &= \lim_{T \rightarrow t^+} Y(t, T) \\ &= \lim_{T \rightarrow t^+} Y^k(t, T) \text{ for each } k. \end{aligned}$$

It is of greatest importance to have a stochastic model that describes the dynamics of the short rate process  $r_t$ . Since, all fundamental quantities (yields, forward rates and bond prices) are defined by the short rate. In fact most stochastic interest rate models characterize the dynamics of the short rate and are therefore also called short rate models. Interest rate models or term structure models are grouped in to two main categories: Equilibrium models and no-arbitrage models.

### 3.2. Equilibrium Interest Rate Models

Hull [35] states that equilibrium models are based on some assumptions about economic variables and they define a process for the short rate  $r$ . The name equilibrium comes from the fact that these models aim to achieve a balance between the supply of bonds and other interest rate derivatives and the demand of investors [9].

Equilibrium models take only the initial short rate as input and they produce

the current term structure of interest rates as output. Therefore, they are also called endogenous term-structure models [7]. Since their input doesn't take enough information, the bond prices of the equilibrium models differ from the market bond prices. This means that, these models are not arbitrage free, they allow arbitrage between bond prices. In other words, equilibrium models can't be perfectly calibrated to the yield curve.

Note that, equilibrium models are also arbitrage-free in the sense that bond prices produced from these models will not allow arbitrage [36]. However, in the context of interest rate modelling arbitrage-free means that model bond prices are consistent with the market prices. In that sense, equilibrium models are not arbitrage-free. The term structure of these models is inconsistent with the current market term structure.

Merton, Vasicek, Dothan and CIR are the most popular equilibrium models.

### 3.3. No-Arbitrage Interest Rate Models

No-arbitrage or arbitrage-free models use the full information of the term structure. They take the current term structure (i.e. yield curve) as input rather than producing it as output like the equilibrium models. In that sense, arbitrage-free models are exogenous term structure models.

For no-arbitrage models, bond prices of the model are equivalent to the market bond prices (i.e. the model yield curve perfectly fits to the market one). Ho-Lee, Hull-White Extended Vasicek, Hull-White Extended CIR and Black-Derman-Toy are some classical no-arbitrage models.

Properties of the equilibrium and no-arbitrage interest rate models mentioned above are summarized in Table 3.1, where  $N$ ,  $LN$ ,  $NC_\chi^2$ ,  $SNC_\chi^2$  stands for normal, log-normal, non-central  $\chi^2$  and shifted non-central  $\chi^2$  distributions, respectively; "Equil". and "Ar. free" denote equilibrium and arbitrage-free, respectively; and AB stands for analytical bond price [7].

Table 3.1. Summary of one-factor short rate models

Model	Dynamics	r	AB	Type
Merton	$dr_t = \alpha dt + \sigma dW_t$	$N$	Yes	Equil
Vasicek	$dr_t = \beta(\mu - r_t)dt + \sigma dW_t$	$N$	Yes	Equil
Dothan	$dr_t = ar_t dt + \sigma r_t dW_t$	$LN$	Yes	Equil
CIR	$dr_t = \beta(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t$	$NC_\chi^2$	Yes	Equil
Ho-Lee	$dr_t = \mu_t + \sigma dW_t$	$N$	Yes	Ar. free
Hull-White	$dr_t = \beta(\mu_t - r_t)dt + \sigma dW_t$	$N$	Yes	Ar. free
BDT	$d(\ln r_t) = \left(\phi(t) + \frac{\sigma'(t)}{\sigma(t)}\right)dt + \sigma(t)dW_t$	$LN$	No	Ar. free
CIR++	$r_t = x_t + \varphi_t, dx_t = \beta(\mu - x_t)dt + \sigma\sqrt{x_t}dW_t$	$SNC_\chi^2$	Yes	Ar. free

### 3.4. Bond Pricing

As stated in the previous sections, there are various interest rate models in the literature that define the dynamics of the short rate. Bond prices can be directly computed for these models, as it is possible to show in a very general setting the following principle:

The arbitrage free bond price is equal to the expected value of its maturity value (par value) discounted by the short rate process. Therefore, Shreve [12] defines the unit par-value zero-coupon bond price given the information up to time  $t$  for the maturity  $T$  as,

$$P(t, T) = E_t \left\{ \exp \left( - \int_t^T r_s ds \right) \right\}. \quad (3.1)$$

In this formula, continuous-compounding is used and expectation is taken over  $r_t$ ; since,  $r$  is random. This formula is similar to the bond price formula with continuously-compounded yield. But as the short rate is random we have to take the expectation of the integral.

From (3.1) it is clear that whenever we define the distribution of  $\exp\left(-\int_t^T r_s ds\right)$  in terms of the short rate  $r_s$ , we can compute the bond prices and from bond prices all rates can be computed [7].

### 3.5. Risk Quantification

In section 2.5, we have argued why it is best to use VaR of opportunity losses as risk measure. In this section, we will explain our risk quantification methodology. According to the VaR definition in (2.5), the loss distribution is needed for computing the VaR (i.e. 95% quantile value for  $\alpha = 0.05$ ). We will use the short rate models that are explained in Section 3 for simulating the loss distribution. Then, we will demonstrate how the VaR is computed by simulation.

In the opportunity loss formula defined in 2.4, we used constant interest rates. Now, since we have short rate models to simulate the interest rates, we can use these simulated interest rates in the loss formula. Then, the final relative loss formula is,

$$Relloss = 1 - \frac{P(t, T)}{P(0, T) \exp\left\{\left(\int_0^t r_u du\right)\right\}} \quad (3.2)$$

Notice that the integral of the interest rates in continuous compounding is taken from 0 to  $t$ , since the investment period is  $[0, t]$  (i.e. the bond investor will sell his bonds at time  $t$ ).

Our risk quantification methodology is explained in the algorithm presented in Table 3.2.

Note that in the simulations, we use 1 day as the step size and 1000 as the sample size. Since we will use exact simulation algorithms for each model, using large step size is not a problem. However, the integral of the short rate in Formula 3.2 is only approximated. Additionally, variance is very small in simulations, which leads to

Table 3.2. Risk quantification algorithm

<b>Algorithm 1</b> Risk Quantification
1: Calibrate model parameters using Maximum Likelihood Estimation (MLE) from historic market data. 2: <b>if</b> Model==Equilibrium Model <b>then</b> 3: Use shortest maturity bond yield in the market as the short rate and calculate the initial bond price (i.e. $P(0, T)$ ) from model's closed form bond price formula. 4: <b>else if</b> Model==Arbitrage-Free Model <b>then</b> 5: Take directly the bond price in the market as the initial bond price. 6: <b>end if</b> 7: Simulate the $t$ time ahead short rates with $dt = 1 \text{ day}$ and sample size $n = 1000$ . 8: Calculate the $t$ time ahead bond prices (i.e. $P(t, T)$ with maturity $(T - t)$ ) from model's bond price formula. 8: Compute the losses for each simulated bond price according to the loss formula in (3.2). 10: Compute the VaR as 95 percent quantile value of the losses.

narrow confidence intervals. Therefore, we can use small sample size like 1000 which saves computation time.



## 4. EQUILIBRIUM INTEREST RATE MODELS

In Chapter 3, we explained the fundamentals of interest rate modelling and we analyzed the two main types of interest rate models: Equilibrium models and arbitrage-free models. In this chapter, we will explain popular equilibrium models. Namely, the Vasicek, the CIR and the Two Factor Equilibrium Vasicek models.

### 4.1. The Vasicek Model

The Vasicek model [28] is one of the earliest short rate models. It is based upon the idea of mean reverting interest rates. In this section, most of the formulas were taken from Brigo and Mercurio [7].

#### 4.1.1. The Short-Rate Dynamics

Vasicek [28] assumed that the instantaneous short rate,  $r$ , evolves under risk-neutral measure by the following dynamics:

$$dr_t = \beta(\mu - r_t)dt + \sigma dW_t, \quad r(0) = r_0 \quad (4.1)$$

where  $r_0$ ,  $\beta$ ,  $\mu$  and  $\sigma$  are positive constants.

The parameters of the model are  $\beta$  (mean-reversion),  $\mu$  (long-term mean) and  $\sigma$  (volatility of the short rate).  $dW_t$  denotes the standard Brownian motion. The idea of mean-reversion implies that in the long run all trajectories of  $r$  will evolve around a mean level  $\mu$ . This can also be inferred from the dynamics (4.1). Whenever the short rate is below  $\mu$  the drift of the process,  $\mu - r_t$ , is positive; whenever the short rate is above  $\mu$  the drift of the process,  $\mu - r_t$ , is negative. Thus, in every case  $r_t$  is pushed towards the level  $\mu$ .

4.1.1.1. Objective Measure (Real-World) Dynamics. As we noted before the process (4.1) is modelled in the risk-neutral world. However, the relevant distribution in risk management is the distribution under objective measure (real-world) rather than the risk-neutral measure, which is used for pricing (see, e.g. Glasserman [17]).

We can make the change of risk-neutral measure to real-world measure by setting,

$$dW^o(t) = dW(t) + \lambda r(t)dt$$

which leads to,

$$dr_t = [\beta \mu - (\beta + \lambda \sigma)r(t)] dt + \sigma dW_t^o \quad (4.2)$$

where  $\lambda$  is a new parameter that is defined as the market price of risk. According to Brigo and Mercurio [7] market price of risk can be explained as the “excess return with respect to a risk-free investment per unit of risk”.

In (4.2) it is assumed that  $\lambda(t)$  has the functional form,

$$\lambda(t) = \lambda r(t)$$

in the short rate. However, also different choices for  $\lambda$  exists in the literature. Indeed, the market price of risk parameter is usually chosen to be constant (i.e.  $\lambda(t) = \lambda$ ) in the Vasicek model [7]. Moreover, there is no consensus in the literature on the selection of  $\lambda$  (see, e.g. Cheridito and Filipovic and Kimmel [37]). Therefore, we assume  $\lambda = 0$  for all models that we analyze in this thesis. Note that, for  $\lambda = 0$  the two dynamics (i.e. (4.1) and (4.2)) coincide.

#### 4.1.2. Simulating the Short Rate

4.1.2.1. Exact Simulation. For the Vasicek model, given the value  $r_s$  at time  $s$  the exact distribution of  $r_t$  is known. It can be shown that it is normal with mean and

variance given by:

$$r_t \sim N \left( r_s e^{-\beta(t-s)} + \mu (1 - e^{-\beta(t-s)}) , \frac{\sigma^2}{2\beta} [1 - e^{-2\beta(t-s)}] \right)$$

To simulate  $r_t$  at times  $0 = t_0 < t_1 < \dots < t_n$ , we can thus easily use the recursion:

$$r_{t_{i+1}} = e^{-\beta(t_{i+1}-t_i)} r_{t_i} + \mu (1 - e^{-\beta(t_{i+1}-t_i)}) + \sigma \sqrt{\frac{1}{2\beta} (1 - e^{-2\beta(t_{i+1}-t_i)})} Z_{i+1} \quad (4.3)$$

where  $Z$  is a vector of iid. standard normal variates. Note that this simple simulation is the simulation of the exact process. A R-function that simulates the Vasicek process by using exact simulation is given in Appendix A.1.

4.1.2.2. Approximate Euler Scheme Simulation. We want to demonstrate that we can also use the simple Euler Scheme to simulate the process in (4.1) without knowing its exact distribution. We simply replace  $dt$  in the equation (4.1) by short time steps  $\Delta t$  and  $dWt$  by a normal random variate with mean 0 and variance  $t$ . Using this principle the stochastic differential equation for the Vasicek model can be rewritten as

$$r_{t+\Delta t} - r_t = \beta(\mu - r_t)\Delta t + \sigma\sqrt{\Delta t}Z,$$

where  $Z$  denotes a standard normal variate. Thus we get the simple recursion:

$$r_{t+\Delta t} = r_t + \beta(\mu - r_t)\Delta t + \sigma\sqrt{\Delta t}Z. \quad (4.4)$$

This simulation principle is called as the Euler scheme. To balance the size of the simulation error and the discretisation error it is suggested to select the number of time steps  $d$  approximately equal to the square root of the sample size  $n$ . A R-function that simulates the short rate process by using the Euler scheme is given in Appendix A.2.

We can assess the quality of the approximate Euler scheme by comparing it with

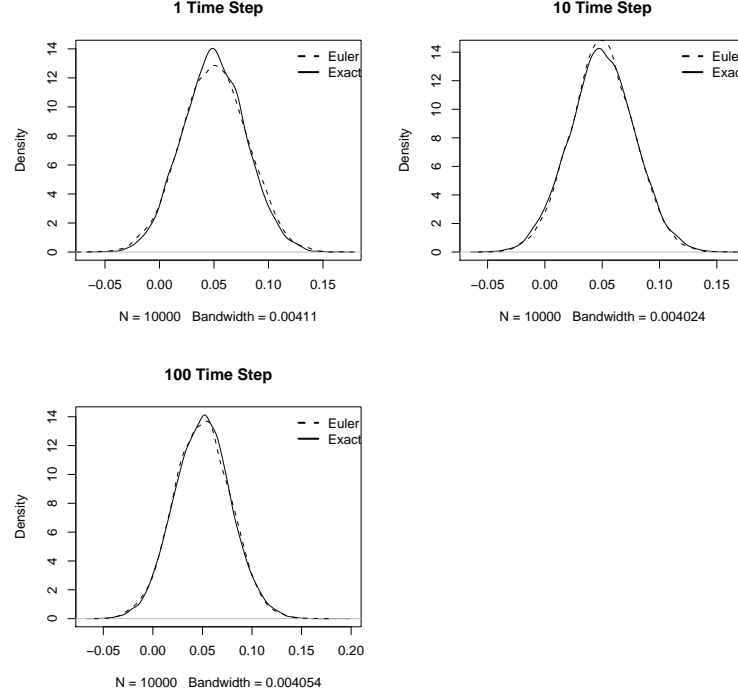


Figure 4.1. Comparison of the approximate and the exact distribution of the short-rate in the Vasicek model.

the exact method. For comparison, density plots of the simulated short rates of both simulation methods will be used. We will use the function `cmtdensity` in R, which can estimate the probability distribution directly from the data. Parameters were set to:  $\mu = 0.052$ ,  $\beta = 0.181$ ,  $\sigma = 0.017$ , which are the parameters of the Vasicek model estimated for US data as we will explain in the next section.

The plots of the densities for using the exact and the approximate simulation and different number of time steps  $d$  are given in Figure 4.1. Note that the two densities are very close for 100 time steps, which shows that the approximation is good and that our simulation algorithms seem to be correct.

Furthermore, four different simulated paths of the short rate are with the same parameters presented in Figure 4.2. Note that the process is mean reverting. The speed of the mean reversion depends on the parameter  $\beta$  of the Vasicek model. We can also see that the short rate can become negative in the Vasicek model. This is a drawback of the model as negative interest rates can not occur.

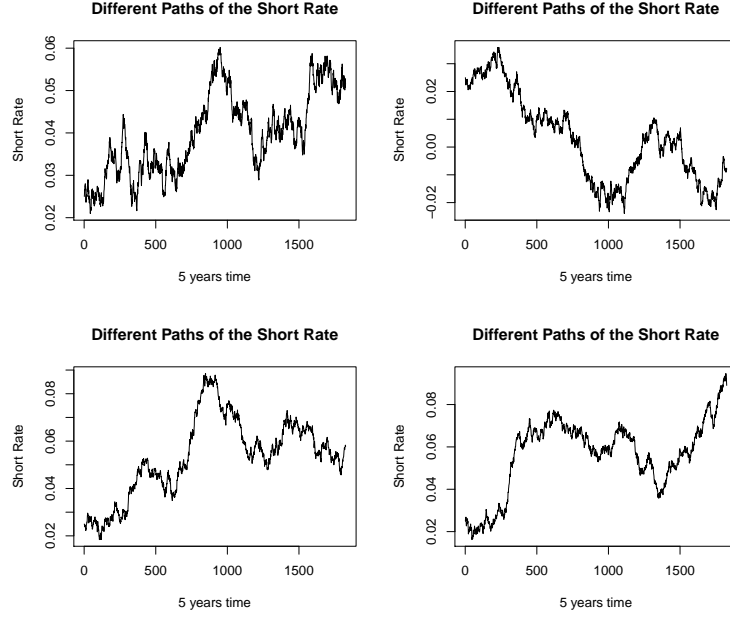


Figure 4.2. Simulated paths of the short rate in the Vasicek model

#### 4.1.3. Maximum Likelihood Estimation of Parameters

Historic short rate data can be used to estimate the parameters of the Vasicek model with maximum likelihood estimation (MLE). As we do not have instantaneous short-rate data in the market, we will use 3 months maturity US monthly yield data from 1952 to 2004 (1952-1999 from [38, 39] and 1999-2004 from [40]) and 3 months period daily “Euro Interbank Offered Rate” (EURIBOR), which is based on the average interest rates at which a panel of more than 50 European banks borrow funds from one another, data from 1999 to 2007 [41] to estimate the standard parameters for the Vasicek model for the US and the European markets. US 3-months maturity yield data and 3 months period EURIBOR interest rate data are shown in Figure 4.3 and Figure 4.4, respectively. Closed-form maximum likelihood estimates for functions of the parameters of the Vasicek model are given in Brigo and Mercurio [7] as:

$$\begin{aligned}\hat{\alpha} &= \frac{n \sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i \sum_{i=1}^n r_{i-1}}{n \sum_{i=1}^n r_{i-1}^2 - (\sum_{i=1}^n r_{i-1})^2} \\ \hat{\mu} &= \frac{\sum_{i=1}^n [r_i - \hat{\alpha} r_{i-1}]}{n(1 - \hat{\alpha})} \\ \hat{V}^2 &= \frac{1}{n} \sum_{i=1}^n [r_i - \hat{\alpha} r_{i-1} - \hat{\mu}(1 - \hat{\alpha})]^2\end{aligned}$$

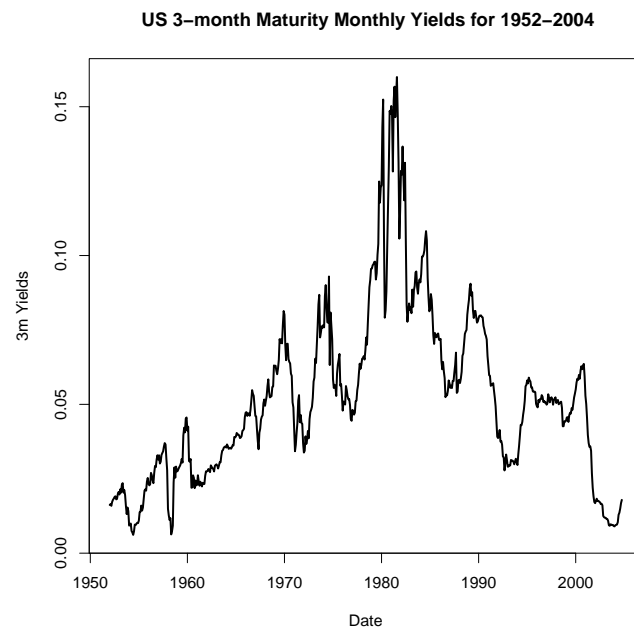


Figure 4.3. US 3-month maturity monthly yields for 1952-2004

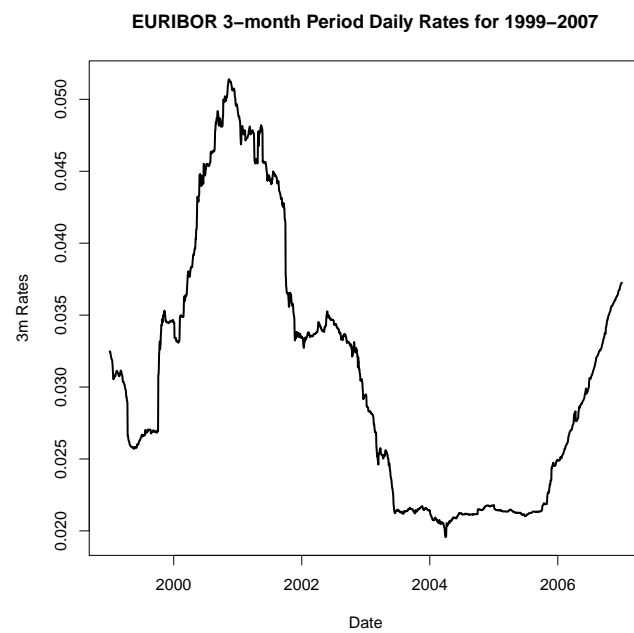


Figure 4.4. EURIBOR 3-month period daily rates for 1999-2007

where  $dt$  is the time step between observed proxies  $r_0, r_1, \dots, r_n$  of  $r$  and,

$$\hat{\beta} = \frac{-\log(\hat{\alpha})}{dt}$$

$$\hat{\sigma}^2 = \frac{2\beta\hat{V}^2}{(1 - e^{-2\hat{\beta}dt})}$$

R-function for calculating the MLE of the parameters of the Vasicek model is given in Appendix A.3. By using that function, we can calculate the MLE's for US and European data as in Figure 4.5. So, for our American bond data with 3-months

```
MLE_Vas_US=Vasicek_MLE(data=USyields$B3month,dt=1/12)
MLE_Vas_US
#[1] 0.05215587 0.18171718 0.01759183
MLE_VAS_EU=Vasicek_MLE(data=EURates$B3M,dt=1/365)
MLE_VAS_EU
#[1] 0.047513000 0.050692963 0.003891468
```

Figure 4.5. R-codes for calculating the MLE's of the Vasicek model for US and European data

time to maturity we obtained as parameter estimates:

$$\hat{\mu} = 0.05215587 \quad \hat{\beta} = 0.18171718 \quad \hat{\sigma} = 0.01759183$$

and for our European interest rate with 3-months period:

$$\hat{\mu} = 0.047513000 \quad \hat{\beta} = 0.050692963 \quad \hat{\sigma} = 0.003891468$$

Note that  $\mu$  estimates are close for both the American and the European data. The estimates are also close to the standard parameter values that are used in the literature (see, e.g. Hull and White [42]) especially for US data.

It is important to mention here that the parameters of the Vasicek model may be not constant over time. If we look at Figure 4.3 it seems obvious that in the early eighties the mean value  $\mu$  and the volatility  $\sigma$  were much higher than in the rest of the

time. So, probably taking for example just the data from 1990 would lead to clearly different parameter estimates. For instance, taking the data between “1980-1” and “1991-1” would lead to the estimates in Figure 4.6.

```
Vasicek_MLE(data=USyields$B3month[337:469],dt=1/12)
#[1] 0.07855993 0.59740640 0.02983012
```

Figure 4.6. R-codes for calculating the MLE’s of the Vasicek model for US data (1980-1990)

#### 4.1.4. Bond Pricing

As mentioned in Section 3 the Vasicek model does not lead to arbitrage free bond prices. Still it is of interest to calculate these prices. For instance, the question of which government bonds are over-priced in the market can be answered by using an equilibrium model like the Vasicek model.

4.1.4.1. Analytical Bond Pricing. It is possible to calculate the expectation of the integral in formula (3.1) for the Vasicek model defined in (4.1). Thus the price of a zero coupon bond when using the Vasicek model as short rate model is given by:

$$P(t, T) = A(t, T)e^{-B(t, T)r_t}, \quad (4.5)$$

where,

$$\begin{aligned} A(t, T) &= \exp \left\{ \left( \mu - \frac{\sigma^2}{2\beta^2} \right) [B(t, T) - T + t] - \frac{\sigma^2}{4\beta} B(t, T)^2 \right\} \\ B(t, T) &= \frac{1}{\beta} [1 - e^{-\beta(T-t)}]. \end{aligned} \quad (4.6)$$

It is easy to see that in the above bond price formula only the current value of the short rate  $r_t$ , the parameters of the Vasicek model and the time to maturity  $T$  are required. Using that bond price formula, it is also easy to calculate the yield curve



implied by the Vasicek model. As the continuously-compounded yield  $R(t)$  is defined by,

$$P(t, T) = e^{-R(t, T)(T-t)}$$

we can simply plug the bond price of the vasicek model into the above equation to get,

$$A(t, T)e^{-B(t, T)r(t)} = e^{-R(t, T)(T-t)}$$

Simple algebra then leads to the yield function

$$R(t, T) = \frac{(r(t)B - \log(A))}{(T - t)} \quad (4.7)$$

R-function for bond price and yield calculations for the Vasicek model is given in Appendix A.4. By using that function, we can calculate analytical bond prices of the Vasicek model as in Figure 4.7

```
para=MLE_Vas_US
VasicekPriceYield(r=0.025,tau=1,Param=para,priceyn=T)
#[1] 0.9730894 price of a 1-year maturity bond
VasicekPriceYield(r=0.025,tau=2,Param=para,priceyn=T)
#[1] 0.943218 price of a 2-year maturity bond
VasicekPriceYield(r=0.025,tau=3,Param=para,priceyn=T)
#[1] 0.9114468 price of a 3-year maturity bond
```

Figure 4.7. R-codes for calculating the analytical bond prices of the Vasicek model

4.1.4.2. Bond Pricing by Simulation. To test our implementation of the bond price formula and our path simulation of the Vasicek model, we try to obtain the bond price by simulation and compare it with the analytical bond price. For calculating the bond price by simulation, we have to use the recursion of the Vasicek model given in (4.3) together with the general bond price formula of (3.1). Of course we evaluate the expectation in that formula by taking simply the average of the results of the  $n$

generated paths of the daily rates. So, the daily discount factor is  $\exp(-r_{daily}/360)$  and we have to multiply all the discount factors of a path. To evaluate that in an efficient way we can simply sum up the rates and divide them by 365 and exponentiate them in the end. The details are given in the short R-code in Figure 4.8 for both the exact

```
res=Vasicek_simul_exact(para=para,r0=0.025,n=1000,dt=1/360,d=360)
y=exp(rowSums(-res)/360)
mean(y)
#[1] 0.973086 1-year Bond Price by exact sim.
1.96*sd(y)/sqrt(1000)
#[1] 0.0005757755 Error Bound
res1=Vasicek_simul_euler(para=para,r0=0.025,n=1000,dt=1/360,d=360)
y1=exp(rowSums(-res1)/360)
mean(y1)
#[1] 0.97313 1-year Bond Price by Euler Scheme sim.
1.96*sd(y1)/sqrt(1000)
#[1] 0.0006027706 Error Bound
```

Figure 4.8. R-codes for obtaining the bond price by simulation for the Vasicek model path simulation and the Euler scheme. Note that  $n = 1000$  paths are enough to obtain a quite precise result. 2 and 3 year bond prices can be calculated similarly.

The analytical bond prices and the bond prices obtained by simulation are compared in Table 4.7. Note that simulation error is small and the analytical prices fall into the confidence interval produced by simulation. Also notice that one error bound is given for both simulations, since it is almost the same for the exact and the approximate method.

Table 4.1. Comparison of bond prices obtained by the analytical formula and the simulation for the Vasicek model

	Analytical BP	Exact Sim. BP	Euler Sim. BP	Error Bound
1-year bond	0.97308	0.97308	0.97327	0.00057
2-year bond	0.94321	0.94312	0.94249	0.00145
3-year bond	0.91144	0.91235	0.91087	0.00246

#### 4.1.5. Vasicek Yield Curve

As we have mentioned above bond prices or equivalently bond yields are fully determined by the Vasicek model, they are only influenced by the current short rate  $r_t$  and by the time to maturity  $T - t$ . Therefore the yield curve of the Vasicek model is just influenced by the current value of the short rate. It is therefore not astonishing that the market yield curve is in many cases not similar to that of the Vasicek model. We demonstrate this fact by comparing the US market yield curve of January 2000 with the yield curve obtained from the Vasicek model. The R-code in Figure 4.9 shows the commands to plot the two yield curves together.

```
maturities=c(0.25,0.5,1,2,5,10)
plot(maturities,USyields["2000-1",2:7],type='l',ylab="yield",
ylim=c(0.03,0.07),xlim=c(0,13),main="January 2000 Yield Curve")
lines(maturities,VasicekPriceYield(r=USyields["2000-1",2],
tau=maturities,Param=para),type="l",lty=2)
legend("topright",legend=c("Vasicek","Market"),
lwd=c(1,1),lty=c(2,1),bty="n")
```

Figure 4.9. R-codes for plotting the yield curve of the Vasicek model

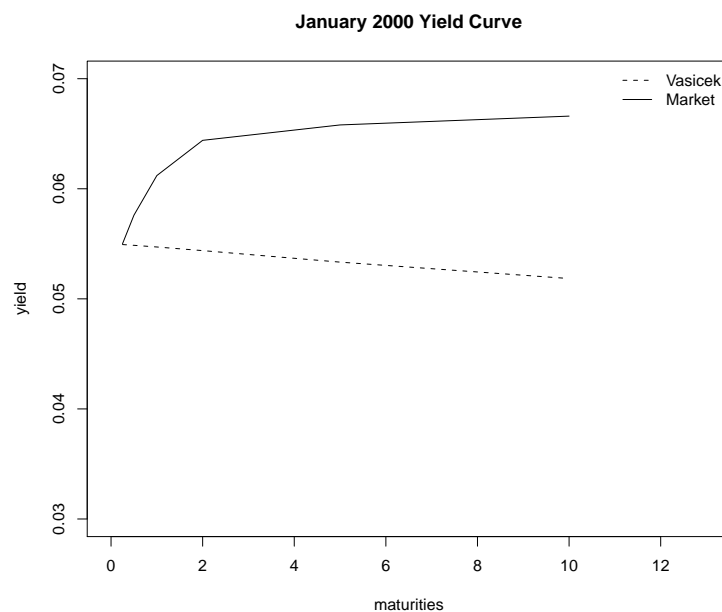


Figure 4.10. Comparison of the yield curves of the Vasicek model and the market

The result of the R-commands is shown as Figure 4.10. Note that in the Vasicek

model the yield of long termed bonds always converges to the parameter  $\mu$  of the model. The yield curve of the Vasicek model is always increasing if the current short rate is below  $\mu$  and it is decreasing when the current short rate is above  $\mu$ . We can see from Figure 4.10 that the yield curve produced by the Vasicek model is very different from the market yield curve. The current short rate value was 5.5 percent, slightly above the long term mean  $\mu = 5.216$  percent we use as parameter for the Vasicek model. Therefore the Vasicek yield curve is flat but slightly decreasing. The yield curve of the US-data on the other hand is increasing with a yield of 6.67 percent for 10 year bonds.

#### 4.1.6. Risk Simulation

In the previous sections, we explained the parameter estimation, short rate simulation and bond pricing for the Vasicek model. In this section, we will illustrate our risk simulation methodology, which is given in algorithm presented in Table 3.2, for the Vasicek model. Note that, real-world and risk-neutral measures are assumed to be the same as explained in Section 4.1.1.1. We will use US data for the risk simulation examples for all models.

Suppose that we were in October 2004, which is the last month in our data for the US bond yields, and we want to quantify the risk of a bond investor, who bought a 1-year bond in October 2004 for a six months period. So the bond investor wants to know his risk, if he sells his bond six months after the investment time.

Firstly, we need to estimate the Vasicek model parameters. We can use the standard parameter estimates (i.e. MLE by using all data) that are found in Section 4.1.3 for risk simulations. These estimates are shown in Figure 4.11.

Note that all data (i.e. 1952-2004 3-month maturity bond yields) were used to estimate the parameters. We also want to estimate the volatility parameter,  $\sigma$ , by using the last 5-years data. Since, considering the recent development of the short rate is more reasonable than using the all data (see, e.g. Bührer [13]). For the  $\mu$  and the  $\beta$  parameter we use again the standard parameters, since they are structural parameters

```

MLE_VAS_US=Vasicek_MLE(data=USyields$B3month,dt=1/12)
MLE_VAS_US
#[1] 0.05215587 0.18171718 0.01759183
Sig_Last5y=Vasicek_MLE(data=USyields$B3month[575:634],dt=1/12)[3]
Sig_Last5y
#[1] 0.007279803

```

Figure 4.11. R-codes for estimating the parameters of the Vasicek model for risk simulations

of the model [13]. So, R-code for calculating the  $\sigma$  estimate from last 5-years data is given also in Figure 4.11. Note that the  $\sigma$  estimate from last 5-years data is only around half of the estimate from all data. This is expected, as we can see from Figure 4.3 that the volatility is smaller for the last years in the data.

Secondly, we calculate the initial 1-year bond price (i.e.  $P(0,1)$ ) from (4.6) by using 3-month yield as the short rate. Then, we simulate the 6-month ahead daily short rates by using the recursion in (4.3) and we calculate the 6-month ahead bond prices (i.e.  $P(0.5,1)$ ) from (4.6). Finally, we compute the losses for each simulated bond price according to the loss formula in (3.2) and then we will compute the VaR as 95 percent quantile value of these losses. Briefly, we will follow the steps in algorithm presented in Table (3.2) in Section 3.5.

Of course, we will use the exact simulation method in risk simulations. R-code for risk simulation using the Vasicek model is given in Appendix A.5.

Then, the VaR of buying a 1-year maturity bond in October 2004 and selling it after 6-months can be calculated by the R-code in Figure 4.12.

We use  $d = 180$ , since we need only the short-rate path up to 6-month. Note that we have to store the short rate path in a matrix for opportunity loss calculation. If we use the volatility parameter that is estimated from last 5-years data, VaR will be estimated as in Figure 4.12. Note that  $VaR2$  is approximately half of the  $VaR1$ , since the  $\sigma$  estimate of the last 5-years data is about half of the standard  $\sigma$  estimate.

```

VaR1=RiskSim_Vasicek(param=MLE_VAS_US,d=180,dt=1/360,T=1,
r0=USyields["2004-10",2])
VaR1
#[1] 0.0141730690 0.0002350664
VaR2=RiskSim_Vasicek(param=c(MLE_VAS_US[1],MLE_VAS_US[2],Sig_Last5y),
d=180,dt=1/360,T=1,r0=USyields["2004-10",2])
VaR2
#[1] 5.946147e-03 8.048868e-05

```

Figure 4.12. R-codes for calculating the VaR for the Vasicek model

The algorithm for calculating the risk of investing in 2, 5 and 10 year bonds is the same. VaR results are summarized in Table 4.2. Notice that VaR is increasing with

Table 4.2. VaR results for the Vasicek model

	MLE-all data	MLE-last 5-year Data for $\sigma$
1-year bond 95% VaR	0.01417	0.00594
2-year bond 95% VaR	0.03028	0.01258
5-year bond 95% VaR	0.06445	0.02643
10-year bond 95% VaR	0.09000	0.03751

increasing maturity. Also note that, the bond investor can lose more than 2.64 percent of his money, compared to investing in a bank account, with 0.05 percent probability if he buys a 5-year maturity bond and sells it after 6-months.

4.1.6.1. Sensitivity Analysis. In this section, we quantify the change in the risk (i.e. VaR) with respect to the change in the parameters (i.e.  $\mu$ ,  $\beta$ ,  $\sigma$ ) of the Vasicek model.

We will again consider a 1-year maturity bond and 6-month risk horizon. In the calculations, we use 3-month maturity bond yield in October 2004 as the initial value of the short rate. We use again the  $\sigma$  estimate calculated from the last 5-year yield data and  $\mu$  and  $\beta$  estimates calculated from the all data. Therefore, the VaR with initial parameters will be *VaR2*, which is calculated in Section 4.1.6. VaR results with respect to  $-30$  percent to  $30$  percent change in the parameters are given in Table 4.3 for the Vasicek model. Note that, in Table 4.3, the fifth column shows the VaR result

for the initial parameter values.

Table 4.3. Sensitivity analysis of the VaR in the Vasicek model

	−%30	−%20	−%10	Initial Values	+%10	+%20	+%30
VaR- $\mu$	−%1.56	−%0.36	−%0.40	0.00594	+%0.4	+%0.48	+%0.81
VaR- $\beta$	+%2.49	+%1.92	+%0.75	0.00594	−%0.32	−%1.27	−%1.53
VaR- $\sigma$	−%30	−%20	−%10	0.00594	+%10	+%20	+%30

Note that, VaR is increasing (decreasing) with increasing (decreasing) long-term mean (i.e.  $\mu$ ) and increasing (decreasing) with decreasing (increasing) mean-reversion (i.e.  $\beta$ ). This is due to the fact that the Vasicek process always pushes the short rate towards the long-term mean value. So, short rate is pulled to a much smaller value when the long-term mean is decreasing and pulled to a much larger value when it is increasing. Moreover, we know that bond prices are inversely related to the interest rates. So, investors will have a higher risk when interest rates go up. This explains why the VaR is increasing with increasing long-term mean.

Mean-reversion parameter shows the speed that the short rate is pushed to the long-term mean. When  $\beta$  increases, short rate will be pushed to the long-term mean faster, which decreases the volatility of the process. Therefore, risk is decreasing with increasing mean-reversion parameter.

Risk is increasing (decreasing) with increasing (decreasing) volatility parameter as expected. Note that, the VaR is mainly influenced by  $\sigma$ , from Table 4.3 we can see that  $\mu$  and  $\beta$  have a very small effect on VaR compared to  $\sigma$ . Therefore, it seems reasonable to estimate the volatility from the last 5-years data and structural parameters (i.e.  $\mu$  and  $\beta$ ) from all data for risk simulations. Parameter sets that were used in the table are given in Figure 4.13.

#	-%30	-%20	-%10	+%10	+%20	+%30
#mu	0.036505	0.041720	0.046935	0.057365	0.062580	0.067795
#beta	0.127190	0.145360	0.163530	0.199870	0.218040	0.236210
#sigma	0.005089	0.005816	0.006543	0.007997	0.008724	0.009451

Figure 4.13. Parameter sets for the sensitivity analysis of VaR for the Vasicek model

## 4.2. The Cox, Ingersoll and Ross (CIR) Model

The CIR model [30] was developed to overcome the problem of possible negative interest rates of the Vasicek model. Therefore, a “square-root” term is introduced in to the diffusion coefficient of the instantaneous short-rate dynamics proposed by Vasicek [28]. The resulting model has been used as a benchmark for many years because of its analytical tractability and the positivity of the instantaneous short rate. In this section most of the formulas were taken from Brigo and Mercurio [7].

### 4.2.1. The Short-Rate Dynamics

The stochastic differential equation describing the CIR model under the risk-neutral measure  $Q$  is

$$dr_t = \beta(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad r(0) = r_0, \quad (4.8)$$

with  $r_0, \beta, \mu, \sigma$  positive constants. Like for the Vasicek model, the parameters of the model are  $\beta$  (mean-reversion),  $\mu$  (long-term mean) and  $\sigma$  (volatility of the short rate).  $dW_t$  denotes the standard Brownian motion. The model is mean-reverting like the Vasicek model.

### 4.2.2. Simulating the Short Rate

4.2.2.1. Exact Simulation. In the CIR model the increments of the short-rate follows a non-central chi-square distribution. More formally, as stated in the Glasserman [17]



the transition density of  $r(t)$  in (4.8) can be written as

$$r(t) = \frac{\sigma^2 (1 - e^{-\beta(t-u)})}{4\beta} \chi_d^2 \left( \frac{4\beta e^{-\beta(t-u)}}{\sigma^2 (1 - e^{-\beta(t-u)})} r(u) \right), \quad t > u, \quad (4.9)$$

where,

$$d = \frac{4\mu\beta}{\sigma^2}. \quad (4.10)$$

This shows that, given  $r(u)$ ,  $r(t)$  is distributed as  $\sigma^2 (1 - e^{-\beta(t-u)}) / (4\beta)$  times a non-central chi-square random variable with  $d$  degrees of freedom and non-centrality parameter

$$\lambda = \frac{4\beta e^{-\beta(t-u)}}{\sigma^2 (1 - e^{-\beta(t-u)})} r(u). \quad (4.11)$$

Thus, to simulate  $r_t$  at times  $0 = t_0 < t_1 < \dots < t_n$ , we can easily use the recursion:

$$r(t_{i+1}) = \frac{\sigma^2 (1 - e^{-\beta(t_{i+1}-t_i)})}{4\beta} \chi_d^2 \left( \frac{4\beta e^{-\beta(t_{i+1}-t_i)}}{\sigma^2 (1 - e^{-\beta(t_{i+1}-t_i)})} r(t_i) \right), \quad (4.12)$$

The R-function that simulates the CIR process by using exact simulation is given in Appendix A.6.

4.2.2.2. Approximate Euler Scheme Simulation. As the exact simulation method requires sampling from the non-central chi-square distribution, we can use again the simple Euler scheme to simulate the process in (4.8) like for the Vasicek model. The Euler discretization of (4.8) suggests simulating  $r(t)$  at times  $0 = t_0 < t_1 < \dots < t_n$  by setting

$$r(t_{i+1}) = r(t_i) + \beta(\mu - r(t_i)) [t_{i+1} - t_i] + \sigma \sqrt{r(t_i)} \sqrt{t_{i+1} - t_i} Z_{i+1}, \quad (4.13)$$

with  $Z_1, \dots, Z_n$  independent  $N(0, 1)$  random variables.

A R-function that simulates the CIR process by using the approximate Euler scheme is given in Appendix A.7. We can again assess the quality of the approximate Euler scheme by comparing the density plots of the simulated short rates of both simulation methods as we have done for the Vasicek model. Parameters were set to:  $\mu = 0.052$ ,  $\beta = 0.128$ ,  $\sigma = 0.066$ , which are standard parameters of the CIR model for US data as we will see in the next section. The plots of the densities for using the exact and the approximate simulation and different number of time steps  $d$  are given in Figure 4.14. Note that densities are again very close for 100 time steps.

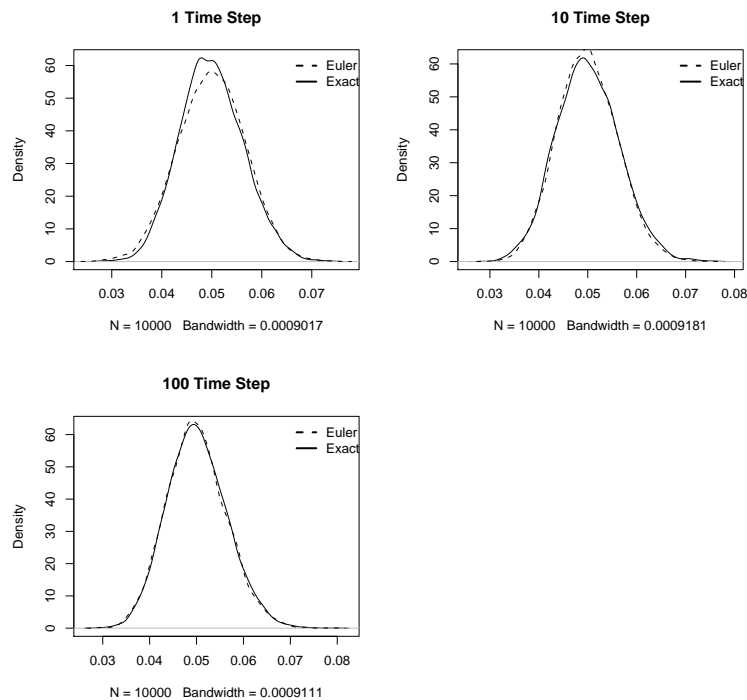


Figure 4.14. Comparison of the approximate and the exact distribution of the short-rate in the CIR model

#### 4.2.3. Maximum Likelihood Estimation of Parameters

We will use again the 3 month maturity US monthly yield data and the 3 month period EURIBOR daily interest rate data to estimate the parameters of the CIR model. However, different to the Vasicek model we do not have closed form formulas for the maximum likelihood estimate for the parameters in the CIR model. This is mainly

due to the non-Gaussian distribution of the short rate in the CIR model. Therefore, we have to use a numerical optimization to obtain the MLE's.

To do that we need the likelihood function of the CIR model. Note that the transition density of the short rate in the CIR model is given in (4.9). From that we can write the cumulative distribution function of  $r(t)$  as

$$P(r(t) \leq y|r(u)) = F_{\chi^2_{(d,\lambda)}} \left( \frac{4\beta y}{\sigma^2 (1 - e^{-\beta(t-u)})} \right), \quad (4.14)$$

with  $d$  as in (4.10) and  $\lambda$  as in (4.11).

Then the probability density function is:

$$P_{r(t)}(y|r(u)) = cp_{\chi^2_{(d,\lambda)}}(cy), \quad (4.15)$$

where  $p_{\chi^2_{(d,\lambda)}}(cy)$  is the density of the non-central chi-square distribution and where,

$$c = \frac{4\beta}{\sigma^2 (1 - e^{-\beta(t-u)})}.$$

From that we can write the likelihood and log-likelihood functions as:

$$L(\mu, \beta, \sigma; y) = \prod_{i=2}^n cp_{\chi^2_{(d,\lambda)}}(cy_i|y_{i-1}),$$

$$l(\mu, \beta, \sigma; y) = \sum_{i=2}^n \log(c) + \sum_{i=2}^n \log \left( p_{\chi^2_{(d,\lambda)}}(cy_i|y_{i-1}) \right),$$

where  $y = r_1, r_2, \dots, r_n$ . Note that  $dt = 1$  month in our data.

As we have found the log-likelihood function of the model we use numerical optimization to find the MLE's. The R-function that calculates the log-likelihood function of the CIR model for a given data set is given in Appendix A.8.

To test our implementation of the MLE method, we can simulate a single short

rate path with arbitrary fixed parameter values and try to obtain these parameter values from simulated data by using the MLE method. R-code that simulates test-data and finds the MLE are given in Figure 4.15. It seems that our implementation

```
test_data=CIR_simul_exact(para=c(0.06,0.1,0.06),r0=0.2,dt=1/12,d=5000,n=1)
MLE_CIR=optim(par=c(0.1,0.1,0.1),fn=CIRloglike,method="L-BFGS-B",
lower=c(0.01,0.01,0.01),upper=c(1,1,1),data=test_data[1,],
times=1/12,test=T)$par
MLE_CIR
#[1] 0.06206643 0.10297200 0.06055140
MLE_CIR_US<-optim(par=c(0.1,0.1,0.1),fn=CIRloglike,method = "L-BFGS-B",
lower=c(0.05,0.05,0.05),upper=c(0.7,0.7,0.7),data=USyields$B3month,
times=USyields[,1])$par
MLE_CIR_US
#[1] 0.05232062 0.12871976 0.06630354
MLE_CIR_EU<-optim(par=c(0.1,0.1,0.1),fn=CIRloglike,method = "L-BFGS-B",
lower=c(0.01,0.01,0.01),upper=c(0.7,0.7,0.7),data=EURates$B3M,
times=EURates[,1])$par
MLE_CIR_EU
#[1] 0.03029181 0.41638803 0.01772860
#MLE_VAS_US [1] 0.05215587 0.18171718 0.01759183
#MLE_VAS_EU [1] 0.04751300 0.05069296 0.00389146
mean(EURates$B3M)
#[1] 0.03082519
mean(USyields$B3month)
#[1] 0.05200569
```

Figure 4.15. R-codes for estimating the parameters of the CIR model

is correct, since the estimated parameter values are quite close to the real values. The estimation precision increases with increasing number of observations.

Now, we can use the MLE method to estimate the parameter values from our real dataset. R-code that implements the MLE method for US and European data is given in Figure 4.15. Note that for the US and the European data, the long-term mean estimate is very close for both models and also is very close to the sample mean.

#### 4.2.4. Bond Pricing

The CIR model is also an equilibrium interest rate model. Therefore, model bond prices will be different than the market prices for the CIR model like for the Vasicek

model.

4.2.4.1. Analytical Bond Pricing. We use the same general principle of bond pricing like for the Vasicek model. Therefore, we have to calculate the expectation of the integral in the formula (3.1) for the CIR model defined in (4.8). It is possible to show that (see, e.g. Brigo and Mercurio [7]), the price at time  $t$  of a zero-coupon bond with maturity  $T$  is

$$p(t, T) = A(t, T)e^{-B(t, T)r(t)},$$

where

$$\begin{aligned} A(t, T) &= \left[ \frac{2h \exp\{(\beta + h)(T - t)/2\}}{2h + (\beta + h)(\exp\{(T - t)h\} - 1)} \right]^{2\beta\mu/\sigma^2}, \\ B(t, T) &= \frac{2(\exp\{(T - t)/h\} - 1)}{2h + (\beta + h)(\exp\{(T - t)/h\} - 1)}, \end{aligned} \quad (4.16)$$

with

$$h = \sqrt{\beta^2 + 2\sigma^2}.$$

From the bond prices we can easily calculate the continuously-compounded yield by using Formula 2.1 like for the Vasicek model. R-function for bond price and yield calculations for the CIR model is given in Appendix A.9.

4.2.4.2. Bond Pricing by Simulation. We can again test our simulation algorithm and bond price formula by trying to obtain a bond price by simulation and comparing it with the analytical price. The principle is the same as for the Vasicek model. The R-code in Figure 4.16 shows the commands to calculate the 1-year maturity bond prices by using the analytical formula of the CIR model and the simulation .

2 and 3 year bond prices can be calculated similarly. Bond prices that are obtained by using the analytical formula and the simulation, both the exact simulation and the

```

param=MLE_CIR_US
CirPriceYield(r=0.05,tau=1,Param=param,priceyn=T)
#[1] 0.951125 #1-year analytical price
#Vasicek 1y Price
#[1] 0.9730894
res=CIR_simul_exact(para=param,r0=0.05,n=1000,dt=1/360,d=360)
y=exp(rowSums(-res)/360)
mean(y)
#[1] 0.9510627 # Simulation (Exact) Price
1.96*sd(y)/sqrt(1000)
#[1] 0.0004696053
res=CIR_simul_euler(para=param,r0=0.05,n=1000,dt=1/360,d=360)
y=exp(rowSums(-res)/360)
mean(y)
#[1] 0.9511621 # Simulation (Euler) Price
1.96*sd(y)/sqrt(1000)
#[1] 0.0004895801

```

Figure 4.16. R-codes for obtaining the bond price by simulation for the CIR model

Euler scheme, are given in Table 4.4. Analytical prices fall into the confidence interval produced by simulations.

Table 4.4. Comparison of bond prices obtained by the analytical formula and the simulation for the CIR model

	Analytical BP	Exact Sim. BP	Euler Sim. BP	Error Bound
1-year bond	0.95112	0.95106	0.95116	0.00046
2-year bond	0.90456	0.90406	0.90465	0.00123
3-year bond	0.86032	0.85879	0.86044	0.00200

#### 4.2.5. CIR Yield Curve

As we have mentioned above, the CIR model is an equilibrium interest rate model. Therefore, we expect that the market yield curve will not be similar to the CIR yield curve. We will again demonstrate this fact by comparing the US market yield curve of January 2000 with the yield curve obtained from the CIR model. We also plot the Vasicek yield curve for comparison. The R-code in Figure 4.17 shows the commands to plot the three yield curves together.

```

param=MLE_CIR_US
maturities=c(0.25,0.5,1,2,5,10)
plot(maturities,USyields["2000-1",2:7],type='l',ylab="yield",lwd=2,
ylim=c(0.03,0.07),xlim=c(0,13),main="January 2000 Yield Curves")
lines(maturities,CirPriceYield(r=USyields["2000-1",2],
tau=maturities,Param=param),type='l',lwd=2,lty=2)
lines(maturities,VasicekPriceYield(r=USyields["2000-1",2],
tau=maturities,Param=MLE_VAS_US),type='l',lwd=2,lty=3)
legend("topright",legend=c("Market","CIR","Vasicek"),
lwd=c(2,2,2),lty=c(1,2,3),bty="n")

```

Figure 4.17. R-codes for plotting the yield curve of the CIR model

From Figure 4.18, we can see that CIR and Vasicek yield curves are almost the same.

#### 4.2.6. Risk Simulation

The risk simulation methodology is same as for the Vasicek model. Note that, it is possible to select  $\lambda = 0$  for the CIR model similar to the Vasicek model. We again calculate the risk of a bond investor, who buys a bond in October 2004 and sells it after 6-months. We use the standard parameter estimates, which were estimated from all US data, and also the volatility estimate calculated from the last 5-years data. If we use all the data (i.e. 1952-2004 3-month maturity bond yields), the standard model parameter estimates using MLE are given in Figure 4.19. Furthermore, the  $\sigma$  estimate calculated from the last 5-years data is also given in Figure 4.19. Note that the  $\sigma$  estimate from last 5-years data is around half of the estimate from all data like for the Vasicek model. R-code for risk simulation using the CIR model is given in Appendix A.10.

By using the risk simulation function, the VaR of buying a 1-year maturity bond in October 2004 and selling it 6-months later can be estimated as in Figure 4.20. If we use the volatility parameter that is estimated from last 5-years data, VaR will be calculated as VaR2 in Figure 4.20. Note that *VaR2* is approximately half of the *VaR1*, since the  $\sigma$  estimate from the last 5-years data is also half of the standard  $\sigma$  estimate. Risk of investing in 2, 5 and 10 year bonds can be calculated similarly. VaR results are

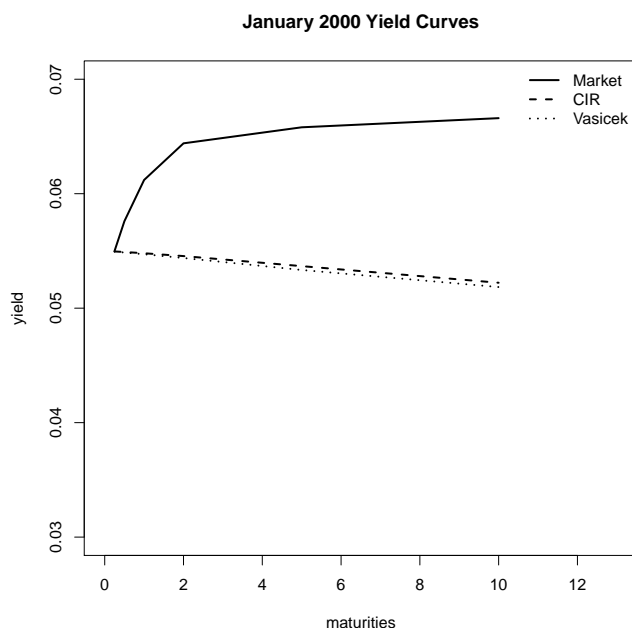


Figure 4.18. Comparison of the yield curves of the CIR and the Vasicek model and the market

```
MLE_CIR_US
#[1] 0.05232062 0.12871976 0.06630354
x<-function(mu,beta,sigma,data,times,test)CIRloglike(param=c(mu,beta,
sigma),data,times)
cirsig_last5y=optim(par=c(0.011),fn=x,method="L-BFGS-B",lower=0.01,
upper=0.90,mu=MLE_CIR_US[1],beta=MLE_CIR_US[2],
data=EURates$B3M[575:634],times=EURates[,1])$par
cirsig_last5y
#[1] 0.03215337
```

Figure 4.19. R-codes for estimating the parameters of the CIR model for risk simulations

```
VaR1=RiskSim_CIR(param=MLE_CIR_US,d=180,dt=1/360,T=1,
r0=USyields["2004-10",2])
VaR1
#[1] 0.0081063178 0.0001364939
VaR2=RiskSim_CIR(param=c(MLE_CIR_US[1],MLE_CIR_US[2],sig_last5y),
d=180,dt=1/360,T=1,r0=USyields["2004-10",2])
VaR2
#[1] 0.0038334866 0.0000630276
```

Figure 4.20. R-codes for calculating the VaR for the CIR model



Table 4.5. VaR results for the CIR model

	MLE-all data	MLE-last 5-year Data for $\sigma$
1-year bond 95% VaR	0.00810	0.00383
2-year bond 95% VaR	0.01748	0.00827
5-year bond 95% VaR	0.03953	0.01880
10-year bond 95% VaR	0.05931	0.02922

summarized in Table 4.5. VaR is increasing with the increasing maturity as expected.

4.2.6.1. Sensitivity Analysis. In this section, we quantify the change in the VaR with respect to the change in the parameters (i.e.  $\mu$ ,  $\beta$ ,  $\sigma$ ) of the CIR model. We will use again a 1-year maturity bond and 6-month risk horizon. Analysis is the same as for the Vasicek model in Section 4.1.6.1. VaR results with respect to  $-30$  percent to  $30$  percent change in the parameters are given in Table 4.6 for the CIR model. Note that

Table 4.6. Sensitivity analysis of the VaR in the CIR model

	−%30	−%20	−%10	Initial Values	+%10	+%20	+%30
VaR- $\mu$	−%2.6	−%2.3	−%1.2	0.00464	+%0.8	+%1.9	+%2.7
VaR- $\beta$	+%2.1	+%1.0	+%0.6	0.00464	−%0.8	−%1.2	−%3.0
VaR- $\sigma$	−%30	−%20	−%10	0.00464	+%9.6	+%20	+%29

long-term mean and mean-reversion parameters affect the VaR in a similar way as for the Vasicek model. In both models, the VaR is mainly influenced by  $\sigma$ . The parameter sets that were used in the calculations are given in Figure 4.21.

#	−%30	−%20	−%10	+%10	+%20	+%30
#mu	0.0366247	0.0418568	0.0470889	0.0575531	0.0627852	0.0680173
#beta	0.0901040	0.1029760	0.1158480	0.1415920	0.1544640	0.1673360
#sigma	0.0225071	0.0257224	0.0289377	0.0353683	0.0385836	0.0417989

Figure 4.21. Parameters sets for the sensitivity of VaR analysis for the CIR model

### 4.3. The Two-Factor Equilibrium Vasicek Model

#### 4.3.1. Motivation

One-factor short rate models have several shortcomings. For instance, they just use the short rate and ignore the information from other rates on the yield curve [2]. Also some yield curve shapes (e.g. inverted shape) can not be generated by using one-factor models such as the Vasicek and the CIR model [7].

Moreover, one-factor models assume a perfect correlation between different rates of the yield curve. At every time instant, interest rates for all maturities are perfectly correlated. This means that a shock to the interest rate curve at time  $t$  is transmitted equally through all maturities [7]. However, this may not be the case in the market (e.g. 30-year yield and 3-month yield may not change in the same direction). For this reasons, multi-factor models are developed. It is shown that two-factor models can explain up to %90 percent variation in the yield curve [43]. In this section we will analyze the Two Factor Equilibrium Vasicek Model (also called Two Factor Vasicek Model or G2 Model [7]) and use it in risk simulation. We use the name *Equilibrium* to make clear that the model is not an arbitrage-free model.

#### 4.3.2. The Short-Rate Dynamics

Brigo and Mercurio [7] state that, for the Two Factor Equilibrium Vasicek Model (Two Factor Vasicek, hereafter), the instantaneous interest rate process under the risk-neutral measure  $Q$  is given by

$$\begin{aligned} r_t &= x_t + y_t, \quad r(0) = x_0 + y_0, \\ dx_t &= \beta_x(\mu_x - x_t)dt + \sigma_x dW_1(t), \quad x(0) = x_0, \\ dy_t &= \beta_y(\mu_y - y_t)dt + \sigma_y dW_2(t), \quad y(0) = y_0, \end{aligned} \tag{4.17}$$

where  $(W_1, W_2)$  is a two-dimensional Brownian motion with instantaneous correlation  $\rho$  as from

$$dW_1(t)dW_2(t) = \rho dt,$$

where  $x_0, y_0, \beta_x, \beta_y, \mu_x, \mu_y, \sigma_x, \sigma_y$  are positive constants, and where  $-1 \leq \rho \leq 1$ .

In the literature, the  $x_t$  and  $y_t$  processes are generally modeled as the short-rate and the long-rate in order to explain the variation of the interest rates better. Note that, the question of which process is modeled as the short rate or the long rate is not important, since both processes are modeled identically.

### 4.3.3. Simulating the Short Rate

4.3.3.1. Exact Simulation. Since the Two Factor Vasicek model is defined as the sum of the two processes, which follow the Vasicek model, we will use the Vasicek recursion in (4.3) to simulate the processes  $x_t$  and  $y_t$  and then we sum them up to get the short rate process  $r_t$ . A R-function that generates the process (4.17) by using exact simulation is given in Appendix A.11.

4.3.3.2. Approximate Euler Scheme Simulation. For demonstration and comparison purposes, we can also use the Euler scheme to simulate the Two Factor Vasicek model. We will use the Euler discretion of Vasicek in (4.4) to simulate the  $x_t$  and  $y_t$  processes. A R-function that generates the process by using approximate Euler scheme simulation is given in Appendix A.12.

Density plots of the simulated short rates of both simulation methods are given in Figure 4.22. Different number of time steps,  $d$ , are used in the simulations for comparison. Parameters were set to the standard short-rate and long-rate parameters of the Two Factor Vasicek model, which will be explained in the next section. Note that densities are again very close for 100 time steps, which shows that the approximation

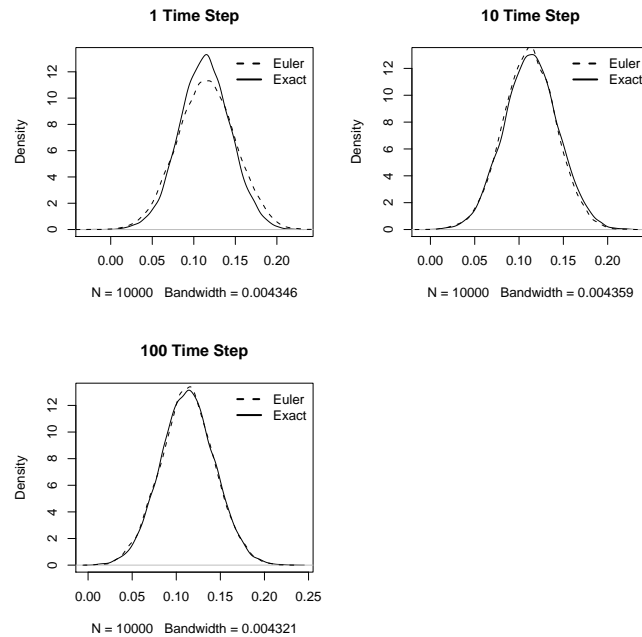


Figure 4.22. Comparison of the approximate and the exact distribution of the short-rate in the Two Factor Equilibrium Vasicek model

is good and that our simulation algorithms seem to be correct.

#### 4.3.4. Selecting the Parameters of the Model

As noted before, the Two Factor Vasicek model is defined as the sum of the short-rate and the long-rate processes, which both follow the Vasicek model. Therefore, we can use the maximum likelihood estimates of the short rate and the long-rate in the Vasicek model as the standard parameter estimates for the Two Factor Vasicek model. 3-month and 10-year maturity bond yields are used to estimate the short-rate and the long-rate, respectively. Then, for the short-rate (say  $x_t$ ) and the long-rate ( $y_t$ ) process of the Two Factor Vasicek model, the standard parameter estimates are given in Figure 4.23.

Note that long rate process has a small mean-reversion speed, which makes it a long-lived factor [4]. For the parameter  $\rho$ , we use the correlation between the historical short and the long rate. In R we can calculate this correlation as in Figure 4.23. Note that rates are highly correlated.

```

ShortR_MLE=Vasicek_MLE(data=USyields$B3month,dt=1/12)
ShortR_MLE
#[1] 0.05215587 0.18171718 0.01759183
LongR_MLE=Vasicek_MLE(data=USyields$B10year,dt=1/12)
LongR_MLE
#[1] 0.06829182 0.08606587 0.01025833
corr=cor(USyields$B3month,USyields$B10year)
corr
#[1] 0.903111

```

Figure 4.23. R-codes for selecting the parameters of the Two Factor Equilibrium Vasicek model

### 4.3.5. Bond Pricing

4.3.5.1. Analytical Bond Pricing. Bond pricing method remains the same for the Two Factor Vasicek model. We have to calculate the expectation of the integral in the formula (3.1) for the Two Factor Vasicek model defined in (4.17). Therefore, it can be shown that the yield at time  $t$  of a zero-coupon bond with maturity  $T$  is given by [44]

$$\begin{aligned}
R(t, T) = & \mu_x + \frac{(1 - e^{\beta_x \tau})}{\beta_x \tau} (x_t - \mu_x) + \mu_y + \frac{(1 - e^{\beta_y \tau})}{\beta_y \tau} (y_t - \mu_y), \\
& - \frac{\sigma_x^2}{2\beta_x^2} \left( 1 + \frac{1 - e^{-2\beta_x \tau}}{2\beta_x \tau} - 2 \frac{1 - e^{-\beta_x \tau}}{\beta_x \tau} \right), \\
& - \frac{\sigma_y^2}{2\beta_y^2} \left( 1 + \frac{1 - e^{-2\beta_y \tau}}{2\beta_y \tau} - 2 \frac{1 - e^{-\beta_y \tau}}{\beta_y \tau} \right), \\
& - \frac{\rho \sigma_x \sigma_y}{\beta_x + \beta_y} \left( 1 - \frac{1 - e^{-\beta_x \tau}}{\beta_x \tau} - \frac{1 - e^{-\beta_y \tau}}{\beta_y \tau} + \frac{1 - e^{-(\beta_x + \beta_y) \tau}}{(\beta_x + \beta_y) \tau} \right), \quad (4.18)
\end{aligned}$$

where  $\tau = (T - t)$ .

Using that yield formula, it is also easy to calculate the bond prices. As the bond price  $P(t, T)$  is defined by,

$$P(t, T) = e^{-R(t, T)(T-t)}$$

R-code for bond price and yield calculations for the Two Factor Vasicek model is in Appendix A.13.

```

param=c(ShortR_MLE,LongR_MLE,corr)
Vas_2fPriceYield(t=0,param=param,x0=USyields["2000-1",2],
y0=USyields["2000-1",7],tau=1,priceyn=T)
#[1] 0.8857535 #1-year analytical bond price
Vas_2fPriceYield(t=0,param=param,x0=USyields["2000-1",2],y0=0,
tau=1,priceyn=T)
#[1] 0.9703777 #1-year price with long_rate=0
VasicekPriceYield(r=0.025,tau=1,Param=MLE_VAS_US,priceyn=T)
#[1] 0.9730894 #1-year Vasicek bond price

```

Figure 4.24. R-codes for calculating the analytical bond prices of the Two Factor Equilibrium Vasicek model

The analytical model bond prices in January 2000 can be calculated by the R-code in Figure 4.24. Note that 1-year bond prices are very close for the Two Factor Vasicek and the Vasicek model, when the initial long-rate is assumed to be 0 in the Two Factor Vasicek model.

4.3.5.2. Bond Pricing by Simulation. To test our simulation algorithm and the bond price formula, we can again try to obtain a bond price by simulation and compare it with the analytical price. The principle remains the same for the Two Factor Vasicek model. January 2000 term structure is used for the initial short and the long-rate. R-code for calculating the 1-year maturity bond price by simulation is given in Figure 4.25. 2 and 3 year maturity bond prices can be calculated similarly. Analytical bond prices and bond prices obtained by simulation, both using the exact and the approximate method, are shown in Table 4.7. Analytical prices fall into the confidence interval produced by simulations.

Table 4.7. Comparison of bond prices obtained by the analytical formula and the simulation for the Two Factor Equilibrium Vasicek model

	Analytical BP	Exact Sim. BP	Euler Sim. BP	Error Bound
1-year bond	0.88575	0.88567	0.88582	0.000858
2-year bond	0.78524	0.78488	0.78521	0.002081
3-year bond	0.69696	0.69373	0.69851	0.003570

```

res=Vas2f_sim_exact(param=param,x0=USyields["2000-1",2],
y0=USyields["2000-1",7],dt=1/360,n=1000,d=360)
y=exp(rowSums(-res$rt)/360)
mean(y)
#[1] 0.8856709 #1-year sim price by exact method
1.96*sd(y)/sqrt(1000)
#[1] 0.000858667 #error bound
res=Vas2f_sim_euler(param=param,x0=USyields["2000-1",2],
y0=USyields["2000-1",7],dt=1/360,n=1000,d=360)
y=exp(rowSums(-res$rt)/360)
mean(y)
#[1] 0.8858207 #1-year sim price by approximate method
1.96*sd(y)/sqrt(1000)
#[1] 0.0009086509 #error bound

```

Figure 4.25. R-codes for obtaining the bond price by simulation for Two Factor Equilibrium Vasicek model

#### 4.3.6. Two Factor Equilibrium Vasicek Yield Curve

As we have mentioned in above various yield curve shapes can be generated with the Two Factor Vasicek model. However, perfect fit to the initial term structure is not possible, since the Two Factor Vasicek model is not an arbitrage-free model.

We can see from Figure 4.27 that different yield curve shapes (e.g. inverted shape) can be generated with the Two Factor Vasicek model. We can also plot the yield curve of the Two Factor Vasicek model in January 2000 with the market yield curve and the yield curve implied by the Vasicek model together for comparison. The R-code in Figure 4.26 shows the commands to plot the yield curves together. From Figure 4.28, we can see that Two Factor Vasicek yield curve fits better to the market yield curve than the Vasicek. Note that for the Two Factor Vasicek model, the short rate is set to the 3 month yield in January 2000 and the long rate is assumed to be zero for a better fit.

```

maturities=c(0.25,0.5,1,2,5,10)
plot(maturities,USyields["2000-1",2:7],type="l",
     ylab="yield", ylim=c(0.03,0.07),xlim=c(0,13),main="January 2000 Yield
     Curves",lwd=2)
lines(maturities,VasicekPriceYield(r=USyields
["2000-1",2],tau=maturities,Param=MLE_VAS_US),type="l",lwd=2,lty=2)
lines(maturities,Vas_2fPriceYield(t=0,param=param,tau=maturities,
x0=USyields["2000-1",2],y0=0,priceyn=F),
type="l",lwd=2,lty=3)
legend("topright",legend=c("Market","Vasicek","2f Vasicek"),
lwd=c(2,2,2),lty=c(1,2,3),bty="n")

```

Figure 4.26. R-codes for plotting the yield curve of the Two Factor Equilibrium Vasicek model

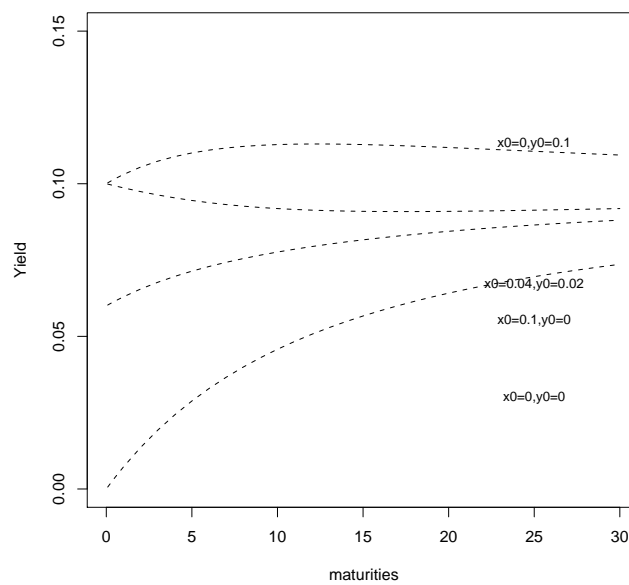


Figure 4.27. Different yield curves of the Two Factor Equilibrium Vasicek model



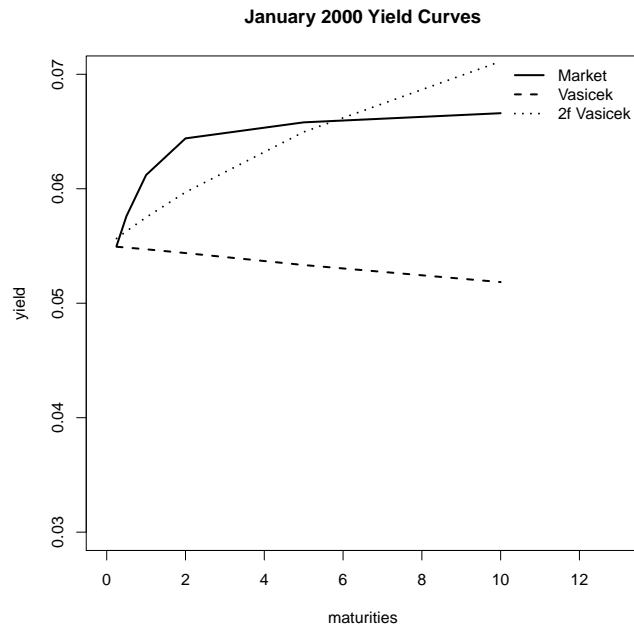


Figure 4.28. Comparison of the yield curves of the Two Factor Equilibrium Vasicek model and the market

#### 4.3.7. Risk Simulation

The risk simulation methodology is the same as for the Vasicek model. We use the standard parameter estimates that are given in Section 4.3.4 as well as the  $\sigma$  estimate that is calculated from last 5-years of the US data. Note that, we estimate two  $\sigma$  values for the short and the long rate. Also, we estimate the correlation between the short and the long rate by using last 5-years data. Then, the standard parameter estimates for the Two Factor Vasicek model are given in Figure 4.29.

The  $\sigma$  estimate of the short and the long rate as well as the correlation estimate using last 5-year data are given also in Figure 4.29. Note that we have used 10-year bond yields to estimate the volatility of the long rate. We will use these estimates for the following risk simulations. R-code for risk simulation for the Two Factor Vasicek model is given in Appendix A.14. So, the VaR of buying a 1-year maturity bond in October 2004 and selling it after 6-months can be estimated by using the r-code given in Figure 4.30.

```

param=c(ShortR_MLE,LongR_MLE,corr)
names(param)=c("mux","betax","sigmax","muy","betay","sigmay","corr")
#      mux      betax      sigmax
# 0.05215587 0.18171718 0.01759183
#      muy      betay      sigmay      corr
# 0.06829182 0.08606587 0.01025833 0.90311101
Short_Sig15y=Vasicek_MLE(data=USyields$B3month[575:634],dt=1/12)[3]
Short_Sig15y
#[1] 0.007279803
Long_Sig15y=Vasicek_MLE(data=USyields$B10year[575:634],dt=1/12)[3]
Long_Sig15y
#[1] 0.008548713
corr15y=cor(USyields$B3month[575:634],USyields$B10year[575:634])
corr15y
#[1] 0.8667879

```

Figure 4.29. R-codes for estimating the parameters of the Two Factor Equilibrium Vasicek model for risk simulations

```

RiskSim_2fVas(x0=USyields["2004-10",2],y0=USyields["2004-10",7],
T=1,param=param,dt=1/360,d=180)
#[1] 0.0226595265 0.0003013651
param15y=c(ShortR_MLE[1:2],Short_Sig15y,LongR_MLE[1:2],
Long_Sig15y,corr15y)
RiskSim_2fVas(x0=USyields["2004-10",2],y0=USyields["2004-10",7],
T=1,param=param15y,dt=1/360,d=180)
#[1] 0.0128004182 0.0002036804

```

Figure 4.30. R-codes for calculating the VaR for the Two Factor Equilibrium Vasicek model

Table 4.8. VaR results for the Two Factor Equilibrium Vasicek model

	MLE-all data	MLE-last 5-year data for $\sigma$ and $\rho$
1-year bond 95% VaR	0.02265	0.01280
2-year bond 95% VaR	0.05062	0.03071
5-year bond 95% VaR	0.11012	0.06739
10-year bond 95% VaR	0.16738	0.10570

Note that initial value of the long rate is taken as the 10-year bond yield in October 2004. If we use the volatility parameters and the correlation parameter that are estimated from last 5-years data, VaR of investing in a 1-year maturity bond can also be calculated as in Figure 4.30. Risk of investing in 2, 5 and 10 year maturity bonds can also be quantified similarly. VaR results are summarized in Table 4.8.

Note that VaR results are much higher than the results of the Vasicek model. This is mainly due to the additive modeling of the short and the long rate in the Two Factor Vasicek model as the risk is increasing with increasing interest rates (i.e. price of the bond at time  $t = 0.5$  year decreases).

4.3.7.1. Sensitivity Analysis. In this section, we quantify the change in the VaR with respect to the change in the parameters (i.e.  $\mu, \beta, \sigma, \rho$ ) of the Two Factor Vasicek model.

1-year maturity bond and 6-month risk horizon will be used. Last 5-year data are used for estimating  $\sigma$  and  $\rho$ . Analysis is the same as for the Vasicek model in Section 4.1.6.1. Of course, we use the exact simulation method like for the Vasicek and the CIR model. VaR results with respect to  $-30$  percent to  $30$  percent change in the parameters are given in Table 4.9 for the Two Factor Vasicek model. For the  $\rho$  the change is limited to  $20$  percent, since  $-1 \leq \rho \leq 1$ .

Note that long-term mean and mean-reversion parameters affect the VaR in the same way as for the Vasicek model, and they have a very small effect on VaR. In both

Table 4.9. Sensitivity analysis of the VaR in the Two Factor Equilibrium Vasicek model

	−%30	−%20	−%10	Initial Val.	+%10	+%20	+%30
VaR- $\mu_x$	−%0.3	−%0.2	−%0.1	0.0128	+%0.4	+%0.64	+%0.7
VaR- $\beta_x$	+%1.1	+%0.78	+%0.50	0.0128	−%0.2	−%0.6	−%1.2
VaR- $\sigma_x$	−%14	−%8.7	−%5.3	0.0128	+%4.5	+%11	+%14
VaR- $\mu_y$	−%0.42	−%0.26	−%0.12	0.0128	+%0.3	+%0.56	+%0.68
VaR- $\beta_y$	+%1.8	+%1.1	+%0.12	0.0128	−%0.11	−%1	−%1.5
VaR- $\sigma_y$	−%14.6	−%10.2	−%5.8	0.0128	+%5	+%10.3	+%15
VaR- $\rho$	−%6.8	−%5.0	−%2.1	0.0128	+%2.4	+%2.7	—

models,  $\sigma$  is the main parameter that affects the VaR. Also notice that the effect of  $\sigma$  is divided equally for the short rate and the long rate and risk is increasing (decreasing) with increasing (decreasing) correlation as the variance of the short rate increases with increasing correlation.

## 5. ARBITRAGE-FREE INTEREST RATE MODELS

In chapter 4 we explained the Vasicek, the CIR and the Two-Factor Equilibrium Vasicek equilibrium models. In this chapter, we will explain popular arbitrage-free interest rate models. Namely, the Hull-White Extended Vasicek (Hull-White, hereafter), CIR++ and Two-Factor Arbitrage Free Vasicek models. In this chapter, most of the formulas were taken from Brigo and Mercurio [7].

### 5.1. The Hull-White Extended Vasicek Model

Hull and White (1990) assumed that the instantaneous short-rate process evolves under the risk-neutral measure according to

$$dr(t) = (\theta(t) - \beta(t)r(t)) dt + \sigma(t)dW(t), \quad (5.1)$$

where  $\theta, \beta$  and  $\sigma$  are deterministic functions of time.

This model can be fitted to the term structure of interest rates and term structure of volatilities. However, fitting the volatilities can be problematic as finding enough and accurate volatility data is hard in the market [7].

Therefore, we consider the following simpler extension of the Vasicek model proposed by Hull and White [45],

$$dr(t) = (\theta(t) - \beta r(t)) dt + \sigma dW(t), \quad (5.2)$$

where  $\beta$  and  $\sigma$  are the constant mean reversion and volatility parameters, respectively. The time-dependent mean parameter,  $\theta$ , is chosen in order to exactly fit the term structure of the interest rates currently observed in the market. Since, the model ensures an exact fit to the current term structure it is an arbitrage-free model. The Hull-White model implies a normal distribution for the short-rate process at each time

point. Moreover, the model is analytically tractable and thus bonds can be priced easily.

### 5.1.1. The Short-Rate Dynamics

It can be shown that the Hull-White process in (5.2) provides an exact fit to the term structure of the interest rates by setting,

$$\theta(t) = \frac{\partial f^M(0, t)}{\partial T} + \beta f^M(0, t) + \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t}), \quad (5.3)$$

where  $f^M(0, T)$  denotes the market instantaneous forward rate at time 0 for the maturity  $T$ , i.e.,

$$f^M(0, T) = -\frac{\partial \ln P^M(0, T)}{\partial T}. \quad (5.4)$$

Integrating (5.2) leads to,

$$\begin{aligned} r(t) &= r(s)e^{-\beta(t-s)} + \int_s^t e^{-\beta(t-u)}\theta(u) du + \sigma \int_s^t e^{-\beta(t-u)}dW(u), \\ &= r(s)e^{-\beta(t-s)} + \alpha(t) - \alpha(s)e^{-\beta(t-s)} + \sigma \int_s^t e^{-\beta(t-u)}dW(u), \end{aligned} \quad (5.5)$$

where

$$\alpha(t) = f^M(0, t) + \frac{\sigma^2}{2\beta^2} (1 - e^{-\beta t})^2. \quad (5.6)$$

### 5.1.2. Simulating the Short Rate

**5.1.2.1. Exact Simulation.** From (5.5), it is easy to show that, given the value  $r_s$  at time  $t$  the distribution of  $r_t$  is normal with mean and variance given by:

$$r_t \sim N \left( r_s e^{-\beta(t-s)} + \alpha(t) - \alpha(s) e^{-\beta(t-s)}, \frac{\sigma^2}{2\beta} [1 - e^{-2\beta(t-s)}] \right).$$

Thus knowing  $\alpha(t)$  is enough to simulate the short rate process in (5.5). A R-function that simulates the Hull-White process by using exact simulation is given in Appendix A.15.

### 5.1.3. Fitting the Instantaneous Forward Rates to the Bond Prices

As it can be seen from (5.5) and (5.6), we need the market instantaneous forward rates to simulate the  $r_t$  process of the Hull-White model. However, we do not have instantaneous forward rate data in the market. Instead of forward rates, generally we have only bond price and maturity data. Therefore, we have to use bond prices to obtain forward rates.

Bond prices can be expressed as the integral of the instantaneous forward rates by using the fact that forward rates are future interest rates, currently expected by the market in the risk neutral world, and using equation (3.1). So, if the time intervals between expiry and maturity dates for the forward rate is infinitesimal we can define the bond price as

$$P(t, T) = \exp \left\{ \left( - \int_t^T f(t, u) du \right) \right\}. \quad (5.7)$$

We can use this equation to obtain forward rates from bond prices. Like for the Vasicek case we again use January 2000 term structure as our dataset, which consists of 6 zero-coupon bonds with 0.25, 0.5, 1, 2, 5, 10 year maturities and 0.055, 0.0576, 0.0612, 0.0644, 0.0658, 0.0666 yields, respectively.

By using only this dataset and (5.7) we can use constant forward rates for the time intervals  $0 - 0.25$ ,  $0.25 - 0.5$ ,  $0.5 - 1$ ,  $1 - 2$ ,  $2 - 5$  and  $5 - 10$  year. However, we need instantaneous forward rates at each time  $t$  to correctly simulate the short rate process. To obtain that, we can use linear interpolation and fit the forward rate function such that we obtain correct bond prices.

Firstly, we assume that the short rate at time  $t = 0$  is the 3-month yield. Then the forward rate at time  $t = 0$  is 0.055, as the initial forward rate is by definition equal to the initial short rate. The forward rate curve is linear and defined by,

$$f_i(t) = a_i t + b_i,$$

where  $t$  denotes time and  $a_i$ ,  $b_i$  are the slope and intercept parameters, respectively. There will be a separate line segment for each time interval (i.e.  $0 - 0.25$ ,  $0.25 - 0.5$ ,  $0.5 - 1$ ,  $1 - 2$ ,  $2 - 5$  and  $5 - 10$ ). Therefore, index  $i$  runs from 1 to 6.

We need to select  $a_i$  and  $b_i$  for each time interval to obtain a fully defined instantaneous forward rate. We will illustrate the calculations for the first 2 intervals.

We know that  $b_1 = 0.055$ , since initial short rate is 0.055, and by using (5.7) we can write

$$\begin{aligned} P(0, 0.25) &= \exp \left\{ \left( - \int_0^{0.25} f(u) du \right) \right\} \\ &= \exp \left\{ \left( - \int_0^{0.25} (a_1 x + 0.055) dx \right) \right\} \\ &= \exp \left\{ \left( - \frac{a_1 0.25^2}{2} - 0.055 * 0.25 \right) \right\}. \end{aligned}$$

From that  $a_1$  is found as,

$$a_1 = -32 (\log(P(0, 0.25)) + 0.055 * 0.25) \simeq 0$$



By finding  $a_1$  and  $b_1$  we have fully specified the forward rate line for the interval  $0 - 0.25$  year. For the second interval (i.e.  $0.25 - 0.5$ ) we have to use again equation (5.7), as we have two unknown parameters  $a_2$  and  $b_2$ .

Note that the starting point of the second line must be the same as the end point of the first line. Denoting with  $t_1$  the end point of the first interval (i.e.  $0.25$ ) we must have

$$\begin{aligned} f_2(t_1) &= f_1(t_1) \\ a_2 t_1 + b_2 &= a_1 t_1 + b_1. \end{aligned}$$

Then,  $b_2$  is  $t_1(a_1 - a_2) + b_1$ . Since, we have found  $t_1, b_1$  and  $a_1$  the only unknown parameter remaining is  $a_2$ . Therefore, we obtain  $a_2$  by solving,

$$\begin{aligned} P(0, 0.5) &= \exp \left\{ \left( - \int_0^{t_2=0.5} (ax + b) dx \right) \right\} \\ &= \exp \left\{ \left( - \int_0^{t_1=0.25} (a_1 x + b_1) dx \right) \right\} \exp \left\{ \left( - \int_{t_1=0.25}^{t_2=0.5} (a_2 x + b_2) dx \right) \right\} \\ &= P(0, 0.25) \exp \left\{ \left( - \frac{a_2(0.5^2 - 0.25^2)}{2} - b_2(0.5 - 0.25) \right) \right\}. \end{aligned}$$

After writing  $b_2$  in terms of  $a_2$  we can find  $a_2$  as  $0.0416$ . Then  $b_2$  is  $t_1(a_1 - a_2) + b_1 = 0.25 * (0 - 0.0416) + 0.055 = 0.0446$ . Since we found  $a_2$  and  $b_2$ , we have fully specified the forward rate for the interval  $0.25 - 0.5$  year. The methodology is the same for other intervals.

After finding the instantaneous forward rates we can find bond prices by using (5.7). The R-function, *HW\_forwPrice*, that calculates the slope and intercept parameters for all intervals and returns the daily instantaneous forward rate and bond price vector is given in Appendix A.28. In the calculations, we have used linear interpolation for the given yields (i.e. January 2000 term structure) and we calculated recursively the forward rates and bond prices for  $[0, 10]$  year. Instantaneous forward rate and the

bond price curves are shown in Figure 5.2 and 5.3, respectively. R-code that plots these curves is given in Figure 5.1.

```
matur=c(0.25,0.5,1,2,5,10)
yield=c(0.0550,0.0576,0.0612,0.0644,0.0658,0.0666)
forward=HW_forwPrice(matur=matur,yield=yield)
prices=HW_forwPrice(matur=matur,yield=yield,forwardyn=F)
i=1:length(forward)
plot(i,forward,"l",main="Instantaneous Forward Rate Curve",
      ylab="Forward Rates",xlab="Time in Days")
plot(i,prices,"l",main="Bond Price Curve",xlab="Time in Days",
      ylab="Bons Prices")
```

Figure 5.1. R-codes for plotting the instantaneous forward rate and the bond price curve for the Hull-White model

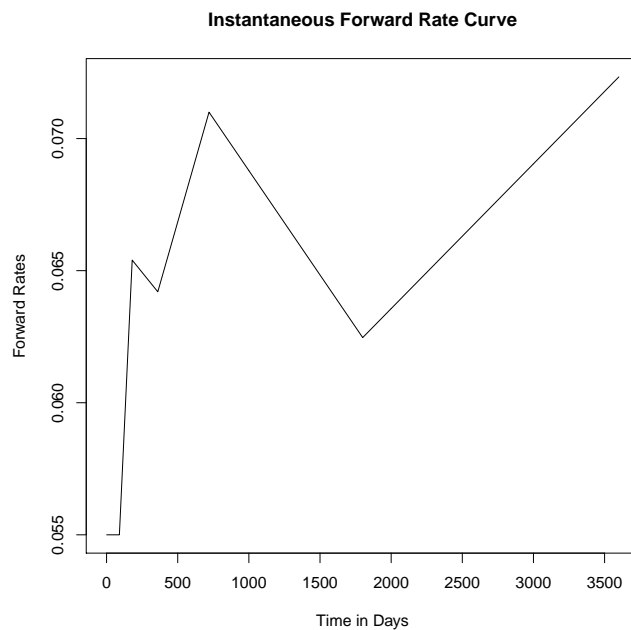


Figure 5.2. Hull-White January 2000 forward rate curve

#### 5.1.4. Selecting the Parameters of the Model

The Hull and White [32] model has the same short rate dynamics as the Vasicek model, only the long-term mean parameter is defined as time-dependent in the Hull-White model. Therefore, for the mean-reversion and volatility parameters of the Hull-White model we will use the maximum likelihood estimates of these parameters in

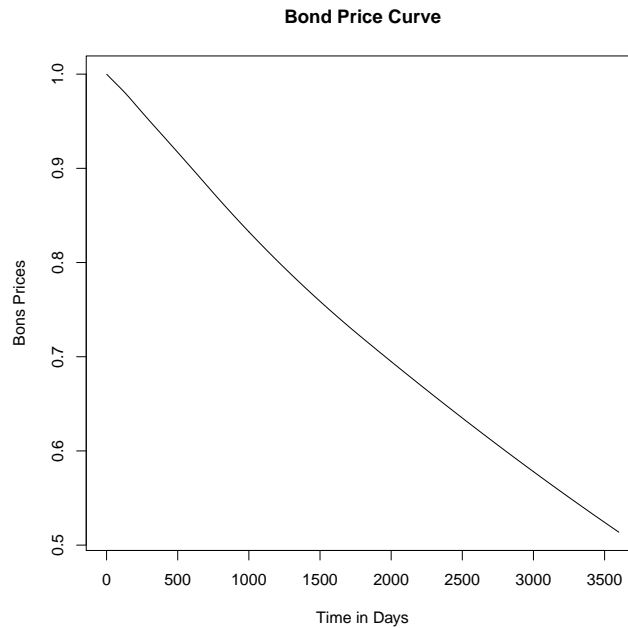


Figure 5.3. Hull-White January 2000 bond price curve

the Vasicek model. Note that, the time-dependent mean parameter of the Hull-White model is defined by the instantaneous forward rates as explained in Section 5.1.3.

### 5.1.5. Bond Pricing

As noted in the previous sections, Hull-White is an arbitrage-free model. Therefore, initial model bond prices are equivalent to the market prices in the Hull-White model. We will demonstrate this principle in the following sections.

5.1.5.1. Analytical Bond Pricing. Bond pricing is the same as for the Vasicek and the CIR model. We will again calculate the expectation of the integral in the formula (3.1) for the Hull-White model defined in (5.2). Therefore, it can be shown that the price of a zero coupon bond in the Hull-White model is given by:

$$P(t, T) = A(t, T)e^{-B(t, T)r_t}, \quad (5.8)$$

where

$$\begin{aligned} A(t, T) &= \frac{P^M(0, T)}{P^M(0, t)} \exp \left\{ B(t, T) f^M(0, t) - \frac{\sigma^2}{4\beta} (1 - e^{-2\beta t}) B(t, T)^2 \right\} \\ B(t, T) &= \frac{1}{\beta} [1 - e^{-\beta(T-t)}]. \end{aligned} \quad (5.9)$$

Note that market bond prices are required for that bond price formula. Thus, the whole term structure is used for calculating bond prices in the Hull-White model. Moreover, when calculating the current bond price, i.e  $t = 0$ , the formula for  $A(t, T)$  reduces to:

$$A(0, T) = P^M(0, T) \exp \{ B(0, T) f^M(0, 0) \}$$

where  $f^M(0, 0)$  is the market instantaneous forward rate at time 0 (i.e. short rate at time 0). The yield calculation is the same as for the Vasicek model. R-code for bond price and yield calculations for general time  $t$  is given in Appendix A.16.

5.1.5.2. Bond Pricing by Simulation. We again test our simulation algorithm and bond price formula by trying to obtain bond prices by simulation and comparing it with the analytical price. The principle is the same as for the Vasicek and the CIR models.

Note that to simulate (5.2) we have to calculate  $\alpha(t)$  values using (5.6). We can use the instantaneous forward rate, which were found in the previous section by fitting them to the bond price data, to calculate the  $\alpha(t)$ .

Moreover, as noted before the Hull-White model always leads to the market price at time 0, since the forward rate at time 0 is equal to the short rate at time 0. We can

prove that easily using the Hull-White bond price Formula 5.9.

$$B(0, T) = \frac{1}{\beta} [1 - e^{-\beta T}]$$

$$A(0, T) = P^M(0, T) \exp \{B(0, T)f^M(0, 0)\} = P^M(0, T) \exp \{B(0, T)r(0)\}$$

$$\begin{aligned} P(0, T) &= A(0, T) \exp(-B(0, T)r(0)) = P^M(0, T) \exp \{B(0, T)r(0)\} \exp(-B(0, T)r(0)) \\ &= P^M(0, T). \end{aligned}$$

Therefore, we will compare the bond prices found by simulation directly with the market bond prices. Again we use January 2000 term structure to test our simulation code. We choose daily simulation steps, since we have daily instantaneous forward rates. R-code for calculating 1 and 2 year maturity bond prices are given in Figure 5.4.

```
param=c(MLE_VAS_US[2],MLE_VAS_US[3])
forward=HW_forwPrice(matur=matur,yield=yield)
R=yield #Market yields in January 2000.
Pm1y=exp(-R[3]*1)
Pm1y
#[1] 0.940635 # 1-year market bond price
res<-HullW_simul_exact(param=param,r0=R[1],n=1000,dt=1/360,
d=360,fm=forward)
y<-exp(rowSums(-res)/360)
mean(y)
#[1] 0.940513 # 1-year price by simulation
1.96*sd(y)/sqrt(1000)
#[1] 0.0001696594 #error bound
Pm2y=exp(-R[4]*2)
Pm2y
#[1] 0.8791498 # 2-year market bond price
res<-HullW_mc(param=param,r0=R[1],n=1000,dt=1/360,d=2*360,fm=forward)
y<-exp(rowSums(-res)/360)
mean(y)
#[1] 0.8788417 # 2-year price by simulation
1.96*sd(y)/sqrt(1000)
#[1] 0.0004328625 #error bound
```

Figure 5.4. R-codes for obtaining the bond price by simulation for the Hull-White model

The principle remains the same for 5 and 10 year maturity bonds. Comparison of market prices and prices obtained by simulation are given in Table 5.1. Note that the simulation error is small for  $n = 1000$  and the market prices fall into the confidence intervals produced by simulation.

Table 5.1. Comparison of market bond prices and the prices obtained by the simulation for the Hull-White model

	Jan. 2000 Market BP	Simulation BP	Error Bound
1-year bond	0.94063	0.94051	0.000169
2-year bond	0.87914	0.87884	0.000432
5-year bond	0.71964	0.71863	0.001166
10-year bond	0.51375	0.51537	0.001815

#### 5.1.6. Hull-White Yield Curve

As we have shown in the previous section, the bond prices of the Hull-White are equivalent to the market bond prices. Therefore, we expect that market and model yield curves are the same. We again demonstrate this fact by comparing the US market yield curve of January 2000 with the yield curve obtained from the Hull-White model. The R-code in Figure 5.5 shows the commands to plot the two yield curves together.

```
maturities=c(0.25,0.5,1,2,5,10)
Pmarket=exp(-maturities*USyields["2000-1",2:7])
plot(maturities,USyields["2000-1",2:7],type='l',ylab="yield",
ylim=c(0.03,0.07),xlim=c(0,13),main="January 2000 Market
and Hull-White Yield Curve",lty=2,lwd=2)
lines(maturities,HwPriceYield(r=USyields["2000-1",2], T=maturities,t=0,
Param=param,Pm0=Pmarket,Pmt=1, fmt=USyields["2000-1",2],priceyn=F),
type='l',lty=4,lwd=2,col="red")
legend("topright",legend=c("Market","Hull-White"),
lwd=c(2,2),lty=c(2,4),col=c("black","red"),bty="n")
```

Figure 5.5. R-codes for plotting the yield curve of the Hull-White model

The result of the R-commands is shown in Figure 5.6. Note that two curves are identical as expected. Also notice that, initial market prices are input to the Hull-White bond price formula.

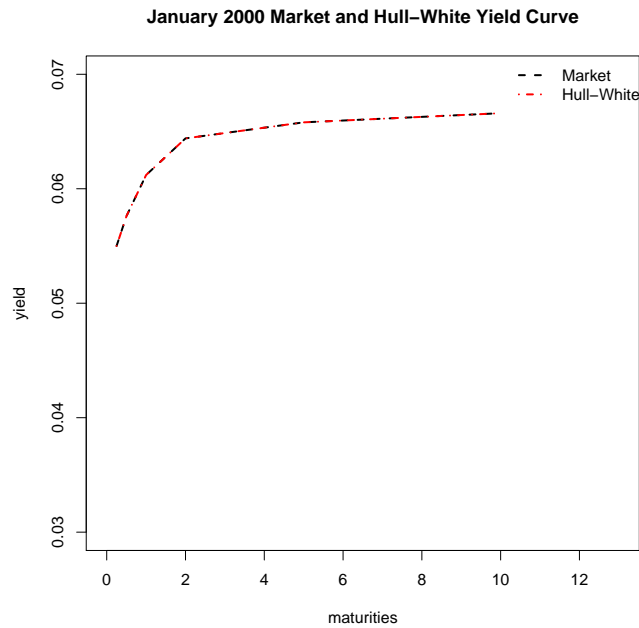


Figure 5.6. Comparison of the yield curves of the Hull-White model and the market

#### 5.1.7. Risk Simulation

The risk simulation methodology is the same as for the Vasicek and the CIR models. As noted in Section 5.1.4, we use the standard parameter estimates of the Vasicek model and also the  $\sigma$  estimate of the Vasicek model that is calculated from last 5-years data. So, the standard mean-reversion and volatility estimates of the Hull-White model are given in Figure 5.7. These are standard parameter estimates for

```
MLE_VAS_US=Vasicek_MLE(data=USyields$B3month,dt=1/12)
MLE_HW=c(MLE_VAS_US[2],MLE_VAS_US[3])
MLE_HW
[1] 0.18171718 0.01759183
Sig_Last5y=Vasicek_MLE(data=USyields$B3month[575:634],dt=1/12)[3]
Sig_Last5y
#[1] 0.007279803
```

Figure 5.7. R-codes for estimating the parameters of the Hull-White model for risk simulations

the Hull-White model. Notice that, Hull and White [42] use  $\beta = 0.1$  and  $\sigma = 0.01$  as standard parameters. The volatility estimate using the last 5-years data is also given in Figure 5.7.

We will use these estimates for the below risk simulations. R-code for risk simulation for the Hull-White model is given in Appendix A.17. Note that we use directly the market price as initial bond price, since Hull-White is an arbitrage-free model. Also notice that, the market prices (i.e.  $P^M(0, T)$  and  $P^M(0, t)$ ) are input to the Hull-White bond price formula. So, the VaR of buying a 1-year maturity bond in October 2004 and selling it after 6-months can be calculated by the R-code in Figure 5.8. Note that

```
USyields2=as.matrix(USyields,rownames.force=NA)
forward=HW_forwPrice(matur=matur,yield=USyields2["2004-10",2:7])
RiskSim_HwVas(param=MLE_HW,r0=USyields["2004-10",2],
Pm=exp(-1*USyields["2004-10",4]),fm=forward,matur=2,
yield6m=USyields["2004-10",3],dt=1/360,d=180)
#[1] 0.0143081401 0.0001484025
RiskSim_HwVas(param=c(MLE_HW[1],Sig_Last5y),r0=USyields["2004-10",2],
Pm=exp(-2*USyields["2004-10",5]),fm=forward,matur=2,
yield6m=USyields["2004-10",3],dt=1/360,d=180)
#[1] 0.0124677411 0.0002103328
```

Figure 5.8. R-codes for calculating the VaR for the Hull-White model

we have calculated the forward rates using the October 2004 term structure. If we use the volatility parameter that is estimated from last 5-years data, the VaR of investing in a 2-year maturity bond is given in Figure 5.8. The risk of investing in 5 and 10 year maturity bonds can be quantified easily using the same principles. Our VaR results are summarized in Table 5.2. Note that the VaR results are very close to the results of the Vasicek model.

Table 5.2. VaR results for the Hull-White model

	MLE-all data	MLE-last 5-year Data for $\sigma$
1-year bond 95% VaR	0.01430	0.00583
2-year bond 95% VaR	0.03002	0.01246
5-year bond 95% VaR	0.06322	0.02657
10-year bond 95% VaR	0.08960	0.03744

**5.1.7.1. Sensitivity Analysis.** In this section, we quantify the change in the VaR with subject to the changes in the parameters (i.e.  $\beta, \sigma$ ) of the Hull-White model.



We will use again a 1-year maturity bond and 6-month risk horizon. Last 5-year data are used for estimating  $\sigma$ . The analysis is the same as for the Vasicek model in Section 4.1.6.1. VaR results with respect to  $-30$  percent to  $30$  percent change in the parameters are given in Table 5.3 for the Hull-White model. Note that the mean-

Table 5.3. Sensitivity analysis of the VaR in the Hull-White model

	−%30	−%20	−%10	Initial Values	+%10	+%20	+%30
VaR- $\beta$	+%3.72	+%2.2	+%0.85	0.0583	−%1.4	−%2.5	−%4.01
VaR- $\sigma$	−%30	−%21	−%10	0.0583	+%10	+%18	+%30

reversion parameter affect the VaR in the same way as the Vasicek model. In both models, the VaR is mainly influenced by  $\sigma$ .

## 5.2. The CIR++ Model

Hull and White [32] proposed an extension of the Cox-Ingersoll and Ross [30] model based on the same idea of considering time dependent coefficients like for the Vasicek model [7]:

$$dr(t) = [\theta(t) - \beta(t)r(t)] dt + \sigma(t)\sqrt{r(t)}dW(t) \quad (5.10)$$

where  $\beta$ ,  $\sigma$  and  $\theta$  are deterministic functions of time. However, this model is not analytically tractable. Indeed, no analytical solution exist for zero-coupon bond prices. The same drawback holds for the simplified dynamics with constant mean reversion and volatility parameters,

$$dr(t) = [\theta(t) - \beta r(t)] dt + \sigma\sqrt{r(t)}dW(t) \quad (5.11)$$

where  $\beta$  and  $\sigma$  are now positive constants, and only  $\theta(t)$ , the time dependent mean parameter is chosen in order to exactly fit the term structure of the interest rates. No analytical expression for  $\theta(t)$  is available in the literature.

A different approach is described by Brigo and Mercurio [34] to extend any time-homogeneous short-rate model, in order to exactly fit the initial term structure. In this approach, a deterministic shift function is added to the short rate process such that “the extended version preserves the analytical tractability of the initial model while exactly fitting the observed term structure of interest rates” [34].

In the case of the Vasicek model, the extension is equivalent to the Hull and White model, which is explained in Section 5.1. In the case of the Cox-Ingersoll-Ross model, however, the extension is analytically more tractable than the model given in (5.10) and (5.11). This extension for the CIR model is called CIR++ model [34].

### 5.2.1. The Short-Rate Dynamics

Brigo and Mercurio [34] assumed that the instantaneous short-rate process evolves under the risk-neutral measure according to,

$$\begin{aligned} dx(t) &= \beta(\mu - x(t))dt + \sigma\sqrt{x(t)}dW(t), \quad x(0) = x_0 \\ r(t) &= x(t) + \varphi(t; \alpha), \end{aligned} \tag{5.12}$$

where  $x_0$ ,  $\beta$ ,  $\mu$  and  $\sigma$  are positive constants and  $\varphi(t; \alpha)$  is the deterministic shift function depending on the parameter vector,  $\alpha$ , and time. The CIR++ model provides an exact fit to the current term structure of the interest rates by setting  $\varphi(t; \alpha) = \varphi^{CIR}(t; \alpha)$  where

$$\begin{aligned} \varphi^{CIR}(t; \alpha) &= f^M(0, t) - f^{CIR}(0, t; \alpha), \\ f^{CIR}(0, t; \alpha) &= \frac{2\beta\theta(\exp\{th\} - 1)}{2h + (\beta + h)(\exp\{th\} - 1)} + x_0 \frac{4h^2 \exp\{th\}}{[2h + (\beta + h)(\exp\{th\} - 1)]^2}, \end{aligned} \tag{5.13}$$

with  $h = \sqrt{\beta^2 + 2\sigma^2}$  and  $f^M(0, t)$  is the market instantaneous forward rate.

### 5.2.2. Simulating the Short Rate

**5.2.2.1. Exact Simulation.** Note that the process (5.12) is equivalent to the original CIR process in (4.8) without the deterministic shift. Thus, we can simulate the CIR++ process (5.12) exactly by simulating the CIR process (4.12) and adding the deterministic shift, which is given in (5.13). The R-function that simulates the CIR++ model by using exact simulation is given in Appendix A.18

**5.2.2.2. Approximate Euler Scheme Simulation.** We can also use the Euler discretization of the CIR model (4.8) to simulate the  $x(t)$  process (5.12), so that we do not have to sample from the non-central chi-square distribution like for the CIR model. Then, we add the deterministic shift to get the CIR++ process. We give a R-function that simulates the CIR++ process by using the approximate Euler scheme in Appendix A.19.

We can again assess the quality of the approximate Euler scheme by comparing the density plots of the simulated short rates of both simulation methods. Parameters were set to:  $\mu = 0.052$ ,  $\beta = 0.128$ ,  $\sigma = 0.066$ , which are standard parameters of the CIR model for US data.

The plots of the densities for using the exact and the approximate simulation and different number of time steps  $d$  are given in Figure 5.9. Note that the two densities are again very close for 100 time steps.

### 5.2.3. Selecting the Parameters of the Model

As we noted before, CIR and CIR++ models have the same dynamics for the short rate, except for the deterministic shift parameter that is added to the short rate process in the CIR++ model. Therefore, for the CIR++ model parameters, we will use the same maximum likelihood estimates of the parameters that were found for the CIR model.

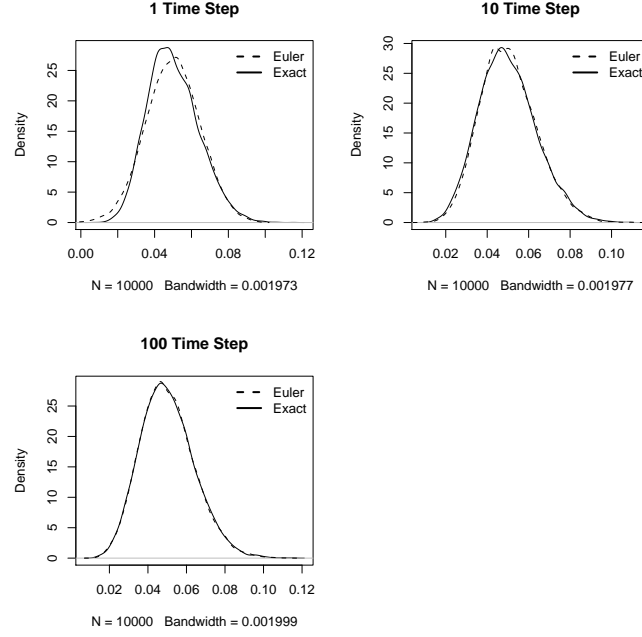


Figure 5.9. Comparison of the approximate and the exact distribution of the short-rate in the CIR++ model

#### 5.2.4. Bond Pricing

As noted before, CIR++ is an arbitrage-free model like the Hull-White model. Therefore, the initial model bond prices are equivalent to the market prices in the CIR++ model. We will demonstrate this principle in the following sections.

**5.2.4.1. Analytical Bond Pricing.** The bond pricing method remains the same for the CIR++ model. We will again calculate the expectation of the integral in the formula (3.1) for the CIR++ model defined in (5.12). Therefore, it is possible to show that the price of a zero-coupon bond at time  $t$  maturing at time  $T$  is given by

$$P(t, T) = \bar{A}(t, T) e^{-B(t, T)r(t)},$$

where

$$\bar{A}(t, T) = \frac{P^M(0, T) A(0, t) \exp\{-B(0, t)x_0\}}{P^M(0, t) A(0, T) \exp\{-B(0, T)x_0\}} A(t, T) e^{B(t, T)\varphi^{CIR}(t; \alpha)},$$

and  $A(t, T)$  and  $B(t, T)$  are defined as in (4.16). After some algebra, one can show that the market bond price at time  $t = 0$  is equal to the model bond price at time  $t = 0$ , which is expected since the CIR++ model is an arbitrage-free model. R-code for bond price and yield calculations for any time  $t$  is given in Appendix A.20.

**5.2.4.2. Bond Pricing by Simulation.** To test our implementation of the bond price formula and our simulation algorithm, we can try to obtain bond prices by simulation and compare them with the market price. The principle is the same as for the Hull-White model. We will compare the bond prices found by simulation directly with the market bond prices, since CIR++ is an arbitrage-free model.

We again use January 2000 term structure to test our simulation code. We choose daily simulation steps, since we have daily instantaneous forward rates. Forward rates that are calculated in Section 5.1.3 will be used. R-code for calculating 1 and 2 year maturity bond prices are given in Figure 5.10. Note that, the model's 1-year bond price is equivalent to the 1-year market bond price. 5 and 10 year bond prices can be calculated similarly. Market prices and the prices obtained by simulation are given in Table 5.4. Note that the simulation error is small for  $n = 1000$  and market prices fall into the confidence intervals produced by simulation.

Table 5.4. Comparison of market bond prices and the prices obtained by the simulation for the CIR++ model

	Market BP	Exact Sim. BP	Euler Sim. BP	Error Bound
1-year bond	0.94063	0.94050	0.94079	0.00050
2-year bond	0.87914	0.87953	0.87955	0.00126
5-year bond	0.71964	0.71744	0.71764	0.00347
10-year bond	0.51375	0.51681	0.50924	0.00535

### 5.2.5. CIR++ Yield Curve

As noted before, CIR++ is an arbitrage-free model. Therefore, we expect that market and model yield curves are the same. We again demonstrate this fact by

```

param=MLE_CIR_US
R=yield #Market yields in January 2000.
Pm1y=exp(-R[3]*1)
Pm1y
#[1] 0.940635 # 1-year market bond price
Ex_CIR_PriceYield(x0=R[1],T=1,t=0,Param=MLE_CIR_US,Pm0=exp(-1*R[3]),
Pmt=1,fmt=R[1])
#[1] 0.940635 # 1-year model price
res<-Ex_CIR_simul_exact(param=param,x0=R[1],n=1000,dt=1/360,d=360,
fm=forward)
y<-exp(rowSums(-res)/360)
mean(y)
#[1] 0.9405054 # 1-year price by exact simulation
1.96*sd(y)/sqrt(1000)
#[1] 0.0005049458#error bound
Pm2y=exp(-R[4]*2)
Pm2y
#[1] 0.8791498 # 2-year market bond price
res<-Ex_CIR_simul_euler(param=param,x0=R[1],n=1000,dt=1/360,d=2*360,
fm=forward)
y<-exp(rowSums(-res)/360)
mean(y)
#[1] 0.8795593 # 2-year price by Euler simulation
1.96*sd(y)/sqrt(1000)
#[1] 0.001334085 #error bound

```

Figure 5.10. R-codes for obtaining the bond price by simulation for the CIR++ model comparing the US market yield curve of January 2000 with the yield curve obtained from the CIR++ model. The R-code in Figure 5.11 shows the commands to plot the two yield curves together.

The result of the R-commands is shown in Figure 5.12. Note that two curves are identical as expected.

### 5.2.6. Risk Simulation

The risk simulation methodology is the same as for the previous models. We use the standard parameter estimates of the CIR model, which are estimated by using all US data, and also the  $\sigma$  estimate that is calculated by using last 5-years US data. The standard mean-reversion and volatility estimates of the CIR++ model are given

```

maturities=c(0.25,0.5,1,2,5,10)
Pmarket=exp(-maturities*USyields["2000-1",2:7])
plot(maturities,USyields["2000-1",2:7],type='l',ylab="yield",
ylim=c(0.03,0.07),xlim=c(0,13),main="January 2000 Market
and CIR++ Yield Curve",lty=2,lwd=2)
lines(maturities,Ex_CIR_PriceYield(x0=USyields["2000-1",2],
T=maturities,t=0,Param=param,Pm0=Pmarket,Pmt=1,
fmt=USyields["2000-1",2],priceyn=F),type='l',lty=4,lwd=2,col="red")
legend("topright",legend=c("Market","Hull-White"),
lwd=c(2,2),lty=c(2,4),col=c("black","red"),bty="n")

```

Figure 5.11. R-codes for plotting the yield curve of the CIR++ model

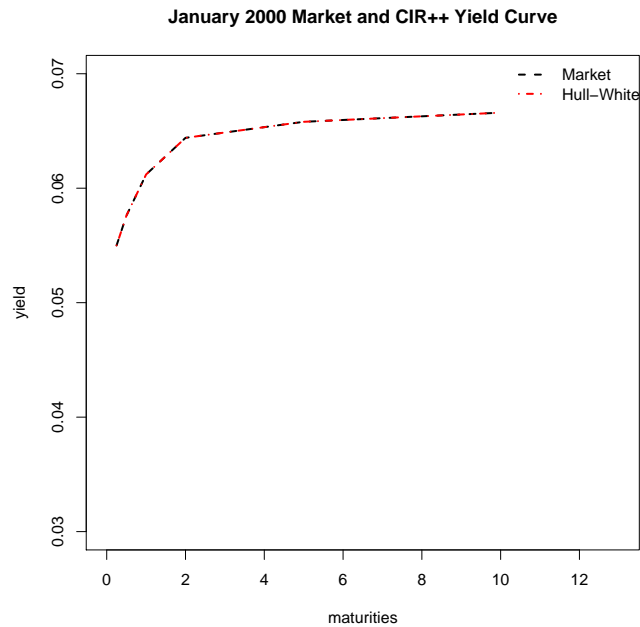


Figure 5.12. Comparison of the yield curves of the CIR++ model and the market

```

MLE_ExCIR=MLE_CIR_US
MLE_ExCIR
#[1] 0.05232062 0.12871976 0.06630354
cirsig_last5y
#[1] 0.03215337
USyields2=as.matrix(USyields,rownames.force=NA)
forward=HW_forwPrice(matur=matur,yield=USyields2["2004-10",2:7])
RiskSim_ExCIR(param=MLE_ExCIR,r0=USyields["2004-10",2],
Pm=exp(-1*USyields["2004-10",4]),fm=forward,matur=1,
yield6m=USyields["2004-10",3],dt=1/360,d=180)
#[1] 0.0080990082 0.0002623806
RiskSim_ExCIR(param=c(MLE_ExCIR[1],MLE_ExCIR[2],cirsig_last5y),
r0=USyields["2004-10",2],Pm=exp(-2*USyields["2004-10",5]),
fm=forward,matur=2,yield6m=USyields["2004-10",3],dt=1/360,d=180)
#[1] 0.0084750073 0.0002412596

```

Figure 5.13. R-codes for calculating the VaR for the CIR++ model

in Figure 5.13. The volatility estimate when using the last 5-years data is also given in Figure 5.13. We use these estimates for the below risk simulations. R-code for the risk simulation for the CIR++ model is given in Appendix A.21.

Note that in the risk simulation function, we use directly the market price as initial bond price. Also notice that the market prices (i.e.  $P^M(0, T)$  and  $P^M(0, t)$ ) are input to the CIR++ bond price formula. So, the VaR of buying a 1-year maturity bond in October 2004 and selling it after 6-months can be estimated using the R-code in Figure 5.13. Note that we have calculated the forward rates using the October 2004 term structure. If we use the volatility parameter that is estimated from the last 5-years data, the VaR of investing in a 2-year maturity bond is can also be calculated by the R-code in Figure 5.13. The risk of investing in 5 and 10 year maturity bonds can also be quantified easily with the same principles. The VaR results are summarized in Table 5.5.

Note that the VaR results are very close to the results of the CIR model. This was to be expected as the underlying dynamics of the short rate is the same for both models and the same parameter estimates were used.



Table 5.5. VaR results for the CIR++ model

	MLE-all data	MLE-last 5-year Data for $\sigma$
1-year bond 95% VaR	0.00809	0.00385
2-year bond 95% VaR	0.01760	0.00847
5-year bond 95% VaR	0.04052	0.01880
10-year bond 95% VaR	0.06074	0.02957

**5.2.6.1. Sensitivity Analysis.** In this section, we quantify the change in the VaR with subject to the change in the parameters (i.e.  $\mu$ ,  $\beta$ ,  $\sigma$ ) of the CIR++ model. We will use again a 1-year maturity bond and 6-month risk horizon. The last 5-year data are used for estimating  $\sigma$ . The analysis here is the same as for the CIR model in Section 4.2.6.1. VaR results with respect to  $-30$  percent to  $30$  percent change in the parameters are given in Table 5.6 for the CIR++ model.

Table 5.6. Sensitivity analysis of the VaR in the CIR++ model

	−%30	−%20	−%10	Initial Values	+%10	+%20	+%30
VaR- $\mu$	−%1.4	−%0.51	−%0.08	0.00385	+%1.7	+%3.6	+%3.8
VaR- $\beta$	+%2.1	+%1	+%0.6	0.00385	−%0.8	−%1.2	−%3.0
VaR- $\sigma$	−%31	−%20	−%10	0.00385	+%8.9	+%20	+%29

Note that the long-term mean and mean-reversion parameters affect the VaR in the same way as for the CIR model. VaR is again mainly influenced by  $\sigma$ .

### 5.3. The Two Factor Arbitrage-Free Vasicek Model

In Section 4.3 we explained the advantages of the two factor models and we analyzed the Two Factor Equilibrium Vasicek model. In this section, we analyze the arbitrage-free version of the Two Factor Vasicek model. Most of the formulas were taken from [7].

### 5.3.1. The Short-Rate Dynamics

The dynamics of the instantaneous-short-rate process under the risk-neutral measure  $Q$  is given by

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0, \quad (5.14)$$

where the processes  $\{x(t) : t \geq 0\}$  and  $\{y(t) : t \geq 0\}$  satisfy

$$\begin{aligned} dx(t) &= -\beta_x x(t)dt + \sigma_x dW_1(t), \quad x(0) = 0, \\ dy(t) &= -\beta_y y(t)dt + \sigma_y dW_2(t), \quad y(0) = 0, \end{aligned} \quad (5.15)$$

where  $(W_1, W_2)$  is a two-dimensional Brownian motion with instantaneous correlation  $\rho$  such that

$$dW_1(t)dW_2(t) = \rho dt.$$

where  $r_0, \beta_x, \beta_y, \sigma_x, \sigma_y$  are positive constants, and we have  $-1 \leq \rho \leq 1$ . The function  $\varphi$  is deterministic and defined in the time interval  $[0, T]$ , with  $T$  denoting the maturity of the bonds. In particular,  $\varphi(0) = r_0$ .

### 5.3.2. Simulating the Short Rate

5.3.2.1. Exact Simulation. Integration of equations (5.15) implies that for each  $s < t$

$$\begin{aligned} r(t) &= x(s)e^{-\beta_x(t-s)} + y(s)e^{-\beta_y(t-s)} \\ &\quad + \sigma_x \int_s^t e^{-\beta_x(t-u)} dW_1(u) + \sigma_y \int_s^t e^{-\beta_y(t-u)} dW_2(u) + \varphi(t), \end{aligned}$$

Then it is easy to show that  $r(t)$  conditional on  $F_s$  is normally distributed with mean and variance given respectively by

$$\begin{aligned} E \{r(t)|F_s\} &= x(s)e^{-\beta_x(t-s)} + y(s)e^{-\beta_y(t-s)} + \varphi(t), \\ Var \{r(t)|F_s\} &= \frac{\sigma_x^2}{2\beta_x} [1 - e^{-2\beta_x(t-s)}] + \frac{\sigma_y^2}{2\beta_y} [1 - e^{-2\beta_y(t-s)}] \\ &\quad + 2\rho \frac{\sigma_x \sigma_y}{\beta_x + \beta_y} [1 - e^{-(\beta_x + \beta_y)(t-s)}]. \end{aligned} \quad (5.16)$$

To simulate  $r_t$  at times  $0 = t_0 < t_1 < \dots < t_n$ , we can thus easily use the following recursions:

$$\begin{aligned} x_{i+1} &= x_i e^{-\beta_x dt} + \sigma_x \sqrt{\frac{1}{2\beta_x} (1 - e^{-2\beta_x dt})} Z_{i+1}^1 \\ y_{i+1} &= y_i e^{-\beta_y dt} + \sigma_y \sqrt{\frac{1}{2\beta_y} (1 - e^{-2\beta_y dt})} Z_{i+1}^2 \\ r_{i+1} &= x_{i+1} + y_{i+1} + \varphi(t), \end{aligned} \quad (5.17)$$

where  $Z^1$  and  $Z^2$  are correlated Brownian motions and  $\varphi$  is a deterministic function defined in the time interval  $[0, T]$  as:

$$\begin{aligned} \varphi(t) &= f^M(0, t) + \frac{\sigma_x^2}{2\beta_x^2} (1 - e^{-\beta_x t})^2 \\ &\quad + \frac{\sigma_y^2}{2\beta_y^2} (1 - e^{-\beta_y t})^2 + \rho \frac{\sigma_x \sigma_y}{\beta_x \beta_y} (1 - e^{-\beta_x t}) (1 - e^{-\beta_y t}). \end{aligned} \quad (5.18)$$

It is necessary to exactly fit the term structure of interest rates being currently observed in the market. Note that market instantaneous forward rates,  $f^M(0, t)$ , are fitted as in Section 5.1.3. A R-function that simulates the Two Factor Arbitrage-Free Vasicek (Two Factor AF Vasicek, hereafter) process by using exact simulation is given in Appendix A.22.

5.3.2.2. Approximate Euler Scheme Simulation. We want to demonstrate that we can also use the simple Euler scheme to simulate the Two Factor AF Vasicek model. We

will use the Euler discretization of Vasicek in (4.4) to simulate the  $x_t$  and  $y_t$  processes. Note that the  $x(t)$  and  $y(t)$  processes are equivalent to the Vasicek short rate process with 0 long-term mean value. Finally, we add the deterministic shift function that is given in (5.18). Therefore, we use the following recursions:

$$\begin{aligned} x_{i+1} &= x_i + \beta_x(-x_i)dt + \sigma_x\sqrt{dt}Z_{i+1}^1 \\ y_{i+1} &= y_i + \beta_y(-y_i)dt + \sigma_y\sqrt{dt}Z_{i+1}^2 \\ r_{i+1} &= x_{i+1} + y_{i+1} + \varphi(t). \end{aligned} \tag{5.19}$$

A R-function that generates the process by using approximate Euler scheme simulation is given in Appendix A.23. Density plots of the simulated short rates of both simulation methods for different number of time steps  $d$  are given in Figure 4.22. Parameters were set to standard short-rate and long-rate parameters of the Two Factor AF Vasicek model, which will be explained in the next section. Note that densities are again very close for  $d = 100$  time steps, which shows that the approximation is good and that our simulation algorithms seem to be correct.

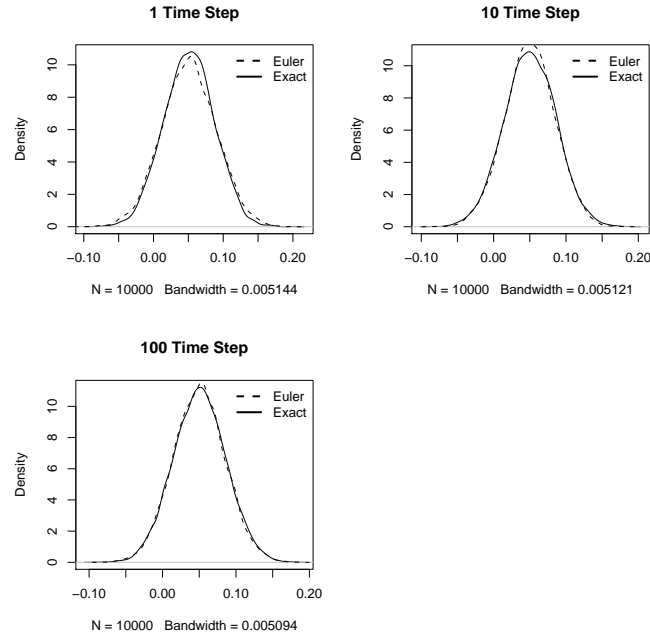


Figure 5.14. Comparison of the approximate and the exact distribution of the short-rate in the Two Factor Arbitrage-Free Vasicek model

### 5.3.3. Selecting the Parameters of the Model

As noted in Section 4.3, the Two Factor Equilibrium Vasicek model is defined as the sum of the short-rate and the long-rate processes, which follow the Vasicek model. Therefore, in section 4.3.4, we note that we use MLE of the parameters of the Vasicek model as standard parameter estimates for the equilibrium version of the Two Factor Vasicek model. For the arbitrage-free version of the model, the principle remains the same. However, as we do not have a mean parameter in the Two Factor AF Vasicek model, we will only use MLE of  $\beta$  and  $\sigma$  of the Vasicek model. 3-month and 10-year maturity bond yields will be used to estimate the short-rate and the long-rate, respectively. Then, for the short-rate (say  $x_t$ ) and the long-rate ( $y_t$ ) process of the Two Factor AF Vasicek model, standard parameter estimates (i.e. MLE by using all data) are given in Figure 5.15. Note that, for the parameter  $\rho$ , we use the correlation

```
AFShortR_MLE=Vasicek_MLE(data=USyields$B3month,dt=1/12)[2:3]
AFShortR_MLE
#[1] 0.18171718 0.01759183
AFLongR_MLE=Vasicek_MLE(data=USyields$B10year,dt=1/12)[2:3]
AFLongR_MLE
#[1] 0.08606587 0.01025833
AFcorr=cor(USyields$B3month,USyields$B10year)
AFcorr
#[1] 0.903111
```

Figure 5.15. R-codes for selecting the parameters of the Two Factor Arbitrage-Free Vasicek model

between the historical short (3-month yield) and the long rate (10-year yield). In R we calculate this correlation by as in Figure 5.15.

### 5.3.4. Bond Pricing

5.3.4.1. Analytical Bond Pricing. Bond pricing method remains the same for the Two Factor AF Vasicek model. We have to calculate the expectation of the integral in the formula (3.1) for the Two Factor Vasicek model defined in (5.15). Therefore, it can be shown that the price at time  $t$  of a zero-coupon bond maturing at time  $T$  and with

unit par-value is

$$\begin{aligned}
 P(t, T) &= \frac{P^M(0, T)}{P^M(0, t)} \exp \{A(t, T)\} \\
 A(t, T) &= \frac{1}{2} [V(t, T) - V(0, T) + V(0, t)] \\
 &\quad - \frac{1 - e^{-\beta_x(T-t)}}{\beta_x} x(t) - \frac{1 - e^{-\beta_y(T-t)}}{\beta_y} y(t),
 \end{aligned} \tag{5.20}$$

where  $P^M(0, T)$  and  $P^M(0, t)$  are market bond prices for maturity  $T$  and  $t$ , respectively and  $V(t, T)$  is given as

$$\begin{aligned}
 V(t, T) &= \frac{\sigma_x^2}{\beta_x^2} \left[ T - t + \frac{2}{\beta_x} e^{-\beta_x(T-t)} - \frac{1}{2\beta_x} e^{-2\beta_x(T-t)} - \frac{3}{2\beta_x} \right] \\
 &\quad + \frac{\sigma_y^2}{\beta_y^2} \left[ T - t + \frac{2}{\beta_y} e^{-\beta_y(T-t)} - \frac{1}{2\beta_y} e^{-2\beta_y(T-t)} - \frac{3}{2\beta_y} \right] \\
 &\quad + 2 \frac{\rho \sigma_x \sigma_y}{\beta_x \beta_y} \left[ T - t + \frac{e^{-\beta_x(T-t)} - 1}{\beta_x} + \frac{e^{-\beta_y(T-t)} - 1}{\beta_y} - \frac{e^{-(\beta_y + \beta_x)(T-t)} - 1}{\beta_x + \beta_y} \right].
 \end{aligned} \tag{5.21}$$

From the bond prices, it is easy to calculate the continuously-compounded yield using (2.1). R-code for bond price and yield calculations for the Two Factor AF Vasicek model is given in Appendix A.24. Then, 1 and 2 year maturity bond prices in January 2000 can be calculated by the R-code in Figure 5.16. Note that market and the model

```

AFparam=c(AFShortR_MLE,AFLongR_MLE,AFcorr)
AF_vas2f_PriceYield(T=1,t=0,Param=AFparam,PmT=exp(-USyields["2000-1",
4]*1),Pmt=1)
#[1] 0.940635 #1-year model bond price
AF_vas2f_PriceYield(T=1,t=0,Param=AFparam,PmT=exp(-USyields["2000-1",
5]*2),Pmt=1,fmt=forward)
#[1] 0.8791498 #2-year model bond price
exp(-USyields["2000-1",4]*1)
#[1] 0.940635 #1-year bond price in the market
exp(-USyields["2000-1",5]*2)
#[1] 0.8791498 #2-year bond price in the market

```

Figure 5.16. R-codes for the Bond Price and Yield Calculations for the Two Factor Arbitrage-Free Vasicek Model

bond prices are exactly the same for the Two Factor AF Vasicek model.

**5.3.4.2. Bond Pricing by Simulation.** We can again test our simulation algorithm and bond price formula by trying to obtain a bond price by simulation and comparing it with the market price. Note that initial model bond prices are equivalent to the market prices, since the Two Factor AF Vasicek is an arbitrage-free model. We again use January 2000 term structure for calculations. Moreover, forward rates that are calculated in Section 5.1.3 will be used. R-code for calculating the 1 year maturity bond price by simulation is given below in Figure 5.17. 2, 5 and 10-year bond prices

```
forward=HW_forwPrice(matur=matur,yield=USyields2["2000-1",2:7])
res=AFVas2f_sim_exact(Param=AFparam,n=1000,d=360,dt=1/360,fm=forward)
y=exp(rowSums(-res$rt)/360)
mean(y)
#[1] 0.9409452 #1-year sim price by exact method
1.96*sd(y)/sqrt(1000)
#[1] 0.000863276 #error bound
res=AFVas2f_sim_euler(Param=AFparam,n=1000,d=360,dt=1/360,fm=forward)
y=exp(rowSums(-res$rt)/360)
mean(y)
#[1] 0.9409747 #1-year sim price by approximate method
1.96*sd(y)/sqrt(1000)
#[1] 0.0008675287 #error bound
exp(-USyields["2000-1",4]*1)
#[1] 0.940635 #1-year market bond price
```

Figure 5.17. R-codes for obtaining the bond price by simulation for the Two Factor Arbitrage-Free Vasicek model

can be calculated similarly. Market bond prices and bond prices obtained by simulation, both using the exact and the approximate method, are compared in Table 5.7. Market bond prices fall into the confidence interval produced by simulations.

### 5.3.5. Two Factor Arbitrage-Free Vasicek Yield Curve

As we have shown in the previous section, Two Factor AF Vasicek model bond prices are equivalent to the market bond prices. Therefore, we expect that market and model yield curves are the same. We again demonstrate this fact by comparing the US

Table 5.7. Comparison of the market bond prices and the prices obtained by the simulation for the Two Factor Arbitrage-Free Vasicek model

	Market BP	Exact Sim. BP	Euler Sim. BP	Error Bound
1-year bond	0.94063	0.94094	0.94097	0.00086
2-year bond	0.87914	0.87868	0.88050	0.00218
5-year bond	0.71964	0.7203	0.71610	0.00613
10-year bond	0.51375	0.51674	0.50984	0.01011

market yield curve of January 2000 with the yield curve obtained from the Two Factor AF Vasicek model. The R-code in Figure 5.18 shows the commands to plot the two yield curves together. The result of the R-commands is shown in Figure 5.19. As

```
maturities=c(0.25,0.5,1,2,5,10)
Pmarket=exp(-maturities*USyields["2000-1",2:7])
plot(maturities,USyields["2000-1",2:7],type='l',ylab="yield",
ylim=c(0.03,0.07),xlim=c(0,13),main="January 2000 Market
and Two Factor AF Vasicek Yield Curve",lty=2,lwd=2)
lines(maturities,AF_vas2f_PriceYield(T=maturities,t=0,
Param=AFparam,PmT=Pmarket,Pmt=1,fmt=USyields["2000-1",2],priceyn=F),
type='l',lty=4,lwd=2,col="red")
legend("topright",legend=c("Market","AF_Vas2f"),
lwd=c(2,2),lty=c(2,4),col=c("black","red"),bty="n")
```

Figure 5.18. R-codes for plotting the yield curve of the Two Factor Arbitrage-Free Vasicek model

expected the two yield curves are exactly the same.

### 5.3.6. Risk Simulation

The risk simulation methodology is the same as for the Two Factor Equilibrium Vasicek model. We use the standard parameter estimates that are given in Section 5.3.3 as well as the  $\sigma$  estimate that is calculated by using last 5-years US data. Note that, we estimate two  $\sigma$  values one for the short and one for the long rate. Also, we estimate the correlation between the short and the long rate by using last 5-years data. Standard parameter estimates for the Two Factor AF Vasicek model are shown in Figure 5.20. Moreover, the  $\sigma$  estimate of the short and the long rate and the correlation estimate



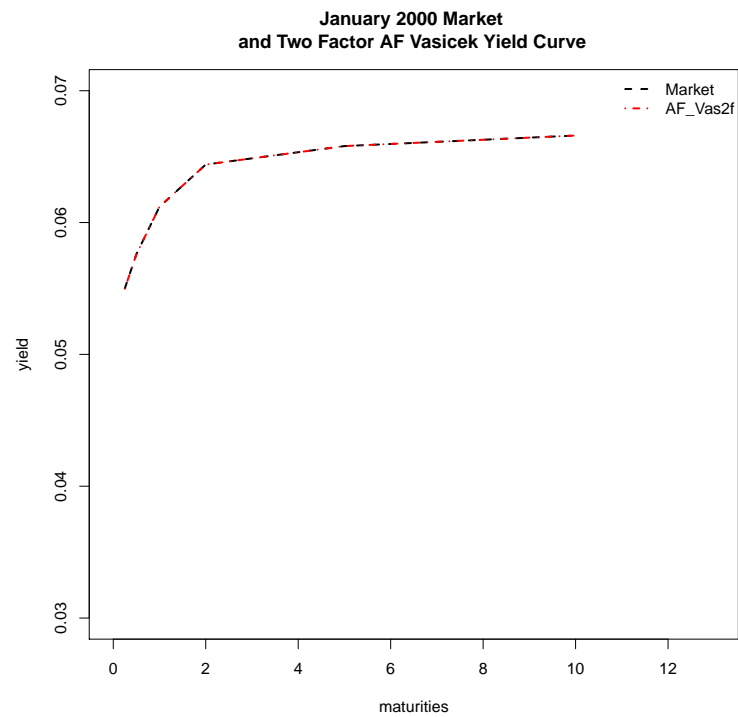


Figure 5.19. Comparison of the yield curves of the Two Factor Arbitrage-Free Vasicek model and the market

```

AFparam=c(AFShortR_MLE,AFLongR_MLE,AFcorr)
names(AFparam)=c("betax","sigmax","betay","sigmay","rho")
AFparam
#      betax      sigmax      betay      sigmay      rho
#0.18171718 0.01759183 0.08606587 0.01025833 0.90311101
AFShort_Sig15y=Vasicek_MLE(data=USyields$B3month[575:634],dt=1/12)[3]
AFShort_Sig15y
#[1] 0.007279803
AFLong_Sig15y=Vasicek_MLE(data=USyields$B10year[575:634],dt=1/12)[3]
AFLong_Sig15y
#[1] 0.008548713
AFCorr15y=cor(USyields$B3month[575:634],USyields$B10year[575:634])
AFCorr15y
#[1] 0.8667879

```

Figure 5.20. R-codes for estimating the parameters of the Two Factor Arbitrage-Free Vasicek model for risk simulations

```

USyields2=as.matrix(USyields,rownames.force=NA)
forward_2004=HW_forwPrice(matur=matur,yield=USyields2["2004-10",2:7])
RiskSim_AF_2fVas(x0=USyields["2004-10",2],y0=USyields["2004-10",7],
T=1,Param=AFparam,Pm=exp(-USyields["2004-10",4]*1),yield6m=USyields
["2004-10",3],fm=forward_2004)
#[1] 0.0223408413 0.0003099921
AFparam15=c(AFparam[1],AFShort_Sig15y,AFparam[2],AFLong_Sig15y,
AFcorr15y)
names(AFparam15)=c("betax","sigmax15y","betay","sigmay15y","rho15y")
AFparam15y
#      betax  sigmax15y      betay  sigmay15y      rho15y
#0.181717182 0.007279803 0.017591831 0.008548713 0.866787878
RiskSim_AF_2fVas(x0=USyields["2004-10",2],y0=USyields["2004-10",7],
T=1,Param=AFparam15y,Pm=exp(-USyields["2004-10",4]*1),yield6m=
USyields["2004-10",3],fm=forward_2004)
#[1] 0.0128580797 0.0001933881

```

Figure 5.21. R-codes for calculating the VaR for the Two Factor Arbitrage-Free Vasicek model

calculated from last 5-years data are also shown in Figure 5.20. Note that we have used 10-year bond yields to estimate the volatility of the long rate. We will use these estimates in the following risk simulations. R-code for risk simulation for the Two Factor AF Vasicek model is given in Appendix A.25. So, R-code for calculating the VaR of buying a 1-year maturity bond in October 2004 and selling it 6-months later can be calculated by the R-code given in Figure 5.21

Note that 1-year maturity bond price and 6-month yield in October 2004 is input to the risk simulation function, since we need the 6-month and 1-year maturity bond prices (i.e.  $P(0, T) = P(0, 1)$  and  $P(0, t) = P(0, 0.5)$ ) for the bond price formula of the Two Factor AF Vasicek Model defined in (5.20). If we use the volatility parameters and the correlation parameter that are estimated from last 5-years data, then the VaR of investing in a 1-year maturity bond calculated as in Figure 5.21. The risk of investing in 2, 5 and 10 year maturity bonds can be quantified similarly. VaR results are summarized in Table 5.8.

Note that the VaR results are very close to the results of the Two Factor Equilibrium Vasicek model as the underlying dynamics of the short rate are the same for

Table 5.8. VaR results for the Two Factor Arbitrage-Free Vasicek model

	MLE-all data	MLE-last 5-year data for $\sigma$ and $\rho$
1-year bond 95% VaR	0.02234	0.01285
2-year bond 95% VaR	0.04831	0.02772
5-year bond 95% VaR	0.10434	0.06683
10-year bond 95% VaR	0.15728	0.11680

both models and same parameter estimates (e.g. for mean-reversion, volatility and correlation) were used.

5.3.6.1. Sensitivity Analysis. In this section, we quantify the change in the VaR with respect to the change in the parameters (i.e.  $\beta$ ,  $\sigma$ ,  $\rho$ ) of the Two Factor AF Vasicek model. 1-year maturity bond and 6-month risk horizon is used. Last 5-year data is used for estimating  $\sigma$  and  $\rho$ . The analysis is the same as for the Two Factor Equilibrium Vasicek model in Section 4.3.7.1. VaR results with respect to  $-30$  percent to  $30$  percent change in the parameters are given in Table 4.9 for the Two Factor AF Vasicek model. Clearly, for  $\rho$  the change is limited to  $20$  percent, since  $-1 \leq \rho \leq 1$ .

Table 5.9. Sensitivity analysis of the VaR in the Two Factor Arbitrage-Free Vasicek model

	−%30	−%20	−%10	Initial Val.	+%10	+%20	+%30
VaR- $\beta_x$	+%0.94	+%0.47	+%0.25	0.0128	−%0.10	−%0.14	−%0.23
VaR- $\sigma_x$	−%13	−%8.4	−%4.0	0.0128	+%4.9	+%9.4	+%14
VaR- $\beta_y$	+%1.1	+%0.70	+%0.42	0.0128	−%0.20	−%0.40	−%0.82
VaR- $\sigma_y$	−%14	−%9.90	−%3.49	0.0128	+%7.5	+%12	+%17
VaR- $\rho$	−%6.3	−%2.4	−%0.8	0.0128	+%0.34	+%4.56	—

Note that the mean-reversion parameters have a very small effect on the VaR. Like for the other models, VaR is mainly influenced by  $\sigma$ . Also notice that effect of  $\sigma$  is divided equally for the short and the long rate. Also risk is increasing (decreasing) with increasing (decreasing) correlation between the rates.

## 6. AN EMPIRICAL COMPARISON OF DIFFERENT MODELS FOR QUANTIFYING THE RISK

### 6.1. Which model should we use?

In Section 2.5 we generally explained the risk of investing in bonds. Our risk quantification methodology is given in 3.5 and we made risk simulations for the Vasicek, CIR, Two-Factor Equilibrium Vasicek, The Hull-White Extended Vasicek, The CIR++ and the Two-Factor Arbitrage Free Vasicek model in Sections 4.1.6, 4.2.6, 4.3.7, 5.1.7, 5.2.6 and 5.3.6, respectively. In the following sections, we will try to assess the quality of the risk estimates of these models by using historic market data. To do that, we will analyze the models risk estimation performance for different maturities.

### 6.2. Historic Risk Assessment of the Quality of the VaR Estimates

We will assess the quality of the different short rate models VaR estimates by comparing the VaR results with the historical losses in the market. For computing the market loss, we will use US monthly yield data from 1952 to 2004 (1952-1999 from [38, 39] and 1999-2004 from [40]) , German monthly yield data from 1973 to 2009 [46] and Canadian monthly yield data from 1978-2010 [47]. The plot for US, German and Canadian 3-month maturity bond yield is given in Figure 6.1. As it can be seen from that plot, yield movements are similar for the US, German and Canadian bonds. However, the volatility of the Canadian 3-month yield seems higher than the US and the German bonds. We also have monthly yield data for 1, 2, 5 and 10 year maturity bonds for US and German data. For Canadian data, we have monthly yield data for 2, 5 and 10 year maturity bonds. Our methodology for historic risk assessment is as follows:

Firstly, the historical loss in the market is calculated for 6 month risk horizon (i.e. the historical loss of buying a  $T$  year maturity bond and selling it 6 months later).

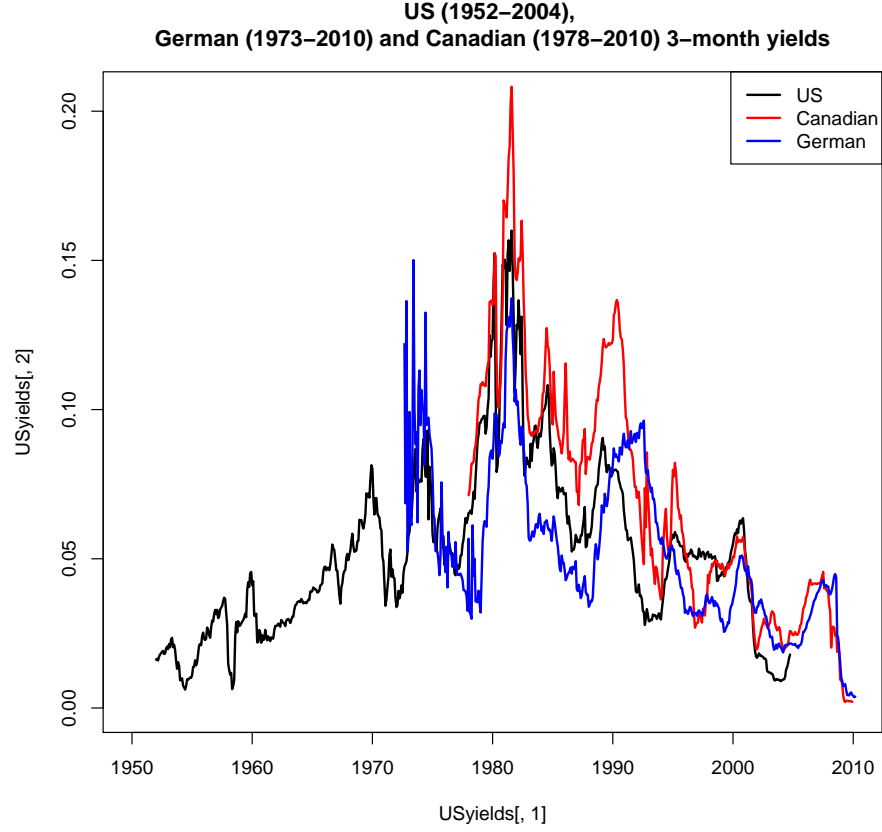


Figure 6.1. 3-month maturity bond yield for the US, German and the Canadian data

Moreover, investing in a 6 month maturity bond will be considered as the opportunity cost of buying a  $T$  year bond. We directly use the historical 6 month yield data in the market for opportunity loss calculation, which means that we do not need to simulate the interest rates. So, historical losses are computed by using the relative opportunity loss formula defined in (3.2) with the 6-month yield as interest rate. The final formula, therefore is:

$$Rel\_Histloss(t_i, T, 0.5) = 1 - \frac{P(t_i + 0.5, t_i + T)}{P(t_i, t_i + T) \exp \{ (Y(t_i, t_i + 0.5) ((t_i + 0.5) - t_i)) \}} \quad (6.1)$$

where  $Y$  is the 6-month yield from time  $t_i$  for maturity  $(t_i + 0.5)$ ,  $P(t_i + 0.5, t_i + T)$  is the bond price at time  $(t_i + 0.5)$  for maturity  $(T + t_i)$  and  $P(t_i, t_i + T)$  is the initial bond price at time  $t_i$  for maturity  $(T + t_i)$  where  $t_i$  runs from the start date of risk simulations to the end date and the risk horizon is 0.5 year. The  $(T + t_i) - (t_i + 0.5) = (T - 0.5)$  year maturity bond price (i.e.  $P(t_i + 0.5, t_i + T)$ ) will be computed by interpolation

using the method explained in Section 5.1.3. We can say that when  $Rel.Histloss$  (6.1) is negative, investing in a  $T$  year maturity bond is more profitable, otherwise investing in a 6 month bond is better. R-code for historic loss calculations is given in Appendix A.26.

Secondly, VaR estimates using interest rate models are computed from the risk simulations. The risk simulation methodology follows our risk quantification algorithm presented in Table 3.2 in Section 3.5, with the following principles and assumptions. The risk horizon is 6 month. Sample size of  $n = 1000$  is enough, as the confidence interval is narrow as we observed in risk simulations. Therefore, a very small number of historical loss results will be in the confidence interval of the VaR estimates. Parameters of the models are estimated as summarized in Table 6.1, where “2f-Vas”, “2fAF-Vas” and “Vas” stands for Two Factor Equilibrium Vasicek, Two Factor Arbitrage Free Vasicek and Vasicek model, respectively. This estimation procedure follows the procedure for our risk simulations for different models in Sections 4.1.6, 4.2.6, 4.3.7, 5.1.7, 5.2.6 and 5.3.6. Note that, 5-year yield data are used to estimate  $\sigma$  and  $\rho$  for Table 6.1. Summary of parameter estimation methods and estimation periods for the different models

	$\mu$	$\beta$	$\sigma$	$\rho$
Vas	MLE-all data	MLE-all data	MLE-5 year data	-
CIR	MLE-all data	MLE-all data	MLE-5 year data	-
2f-Vas	Same as Vas	Same as Vas	Same as Vas	5-year data
Hull-White	-	Same as Vas	Same as Vas	-
CIR++	Same as CIR	Same as CIR	Same as CIR	-
2fAF-Vas	-	Same as Vas	Same as Vas	5-year data

each simulation date (i.e. if we make a risk simulation for January 2000, we will use 1995-1999 data). Therefore, start and end dates for the risk simulations for the US, German and the Canadian data will be 1957-2004, 1978-2009.5 and 1983-2009 respectively (i.e. the data of the first 5 years are only used for calibration). As the interval between two risk simulations we use 6-month, which is equal to the risk horizon so

```

US=pbinom(1:10,95,0.05)
US
#[1] 0.04590857 0.14054465 0.29495090 0.48186372 0.66090652 0.80225611
#[7] 0.89684343 0.95160450 0.97946540 0.99207613
DE=pbinom(1:10,64,0.05)
DE
#[1] 0.1639212 0.3734743 0.6014093 0.7843570 0.8999030 0.9597031
#[7] 0.9857814 0.9955607 0.9987633 0.9996904
CA=pbinom(1:10,53,0.05)
CA
#[1] 0.2499942 0.5018159 0.7271300 0.8753630 0.9518200
#[6] 0.9840124 0.9953887 0.9988315 0.9997375 0.9999473

```

Figure 6.2. R-codes for calculating the binomial probabilities for the US, German and the Canadian data

that the historic risk observations are independent. So, two simulations will be made in each year except for the last year, which leads to total of 95 simulations for the US data, 64 for the German data and 53 for the Canadian data. In the risk simulations 1, 2, 5 and 10 year bonds are used for the US and the German bonds and 2, 5 and 10 year bonds are used for the Canadian bonds . VaR is computed by using the 95% quantile value of the relative losses, which are calculated using (3.2). Note that historical losses will be calculated according to (6.1) for the same dates.

Finally, since we will compute the 95% VaR, we hope that around 5 percent of the historical losses will be larger than the VaR estimates. More formally, we have a straightforward binomial hypothesis. We test,

$$H_0 : p = 0.05, \quad (6.2)$$

against the two sided alternative. Then, we can find our acceptance region by calculating the binomial probabilities for the US, German and the Canadian data. The results for US, German and Canadian data are given in Figure 6.2. So, for two-sided  $\alpha \leq 0.1$  our acceptance region is  $[2, 8]$  for the US data. Then, for two-sided  $\alpha \leq 0.1$  our acceptance region is  $[1, 6]$  for the German data. Finally, our acceptance region is  $[1, 5]$  for the Canadian data. Therefore, in our analysis we will assess the quality of the VaR

estimates according to this criteria, by considering which model is under-estimating or over-estimating the risk for different maturities. The R-function, *Risk\_Sim*, that implements historical risk simulation algorithm is given in Appendix A.27.

### 6.3. Historical Risk Simulation Results

The results for the historical losses and historical risk simulations for 1-year maturity bonds are given in Table 6.2. There, the fifth column shows the number of times historical losses greater than the VaR estimates were observed. We can see from the fifth column that for the US data, the Vasicek, Hull-White and the CIR models are in the acceptance region for the null hypothesis (6.2) and thus they are better at risk estimation than the other models, which are over-estimating the risk. For the German data, the Vasicek, Hull-White, CIR and CIR++ models are slightly better at risk estimation than the two-factor models that are over-estimating the risk.

Table 6.2. Historic risk assessment results for 1-year maturity bonds

	Total Sim	mean(risk)	histloss $\geq$ VaR
US Data			
Vasicek	95	0.0122	3
Hull-White	95	0.0121	3
CIR	95	0.0128	2
CIR++	95	0.0128	1
Vas2f Equil.	95	0.0189	0
Vas2f AF.	95	0.0189	0
German Data			
Vasicek	64	0.0104	1
Hull-White	64	0.0104	1
CIR	64	0.0122	1
CIR++	64	0.0122	1
Vas2f Equil.	64	0.0162	0
Vas2f AF.	64	0.0157	0



The results for 2-year maturity bonds are given in Table 6.3. Note that for the 2-year bonds, VaR estimates of the models are higher than the 1-year bond, since longer maturity bonds have greater risk. Also notice that, the arbitrage-free and the equilibrium version of the models can be grouped in three pairs: the “Vasicek–Hull-White” pair, the “CIR–CIR++” pair and the “Two Factor Equilibrium Vasicek–Two Factor Arbitrage Free (AF) Vasicek” pair, according to their VaR results and the comparison with the historical losses. CIR and CIR++ models are better at risk estimation than the other models for both the US and the German data. On the other hand, the Vasicek and the Hull-White models results are at the upper border of the acceptance region of the null hypothesis for the US data. However, for the German data they are under-estimating the risk. Two factor models are over-estimating the risk for the US data but are at the lower border of the acceptance region for the German data. For the Canadian data, all models are correctly estimating the risk. Note that, VaR is higher for Canadian data than the US and the German data.

The results for the 5-year and the 10-year bonds are shown in Table 6.4 and Table 6.5, respectively. For the US data, all models are under-estimating the risk for 5 year maturity except for the two-factor models, which are over-estimating the risk. For the German data, CIR and CIR++ models are acceptable at risk estimation for 5-year maturity. Moreover, the Vasicek and the Hull-White models under-estimate the risk for 5 and 10 year maturities for both the German and the US data. Two-factor models are best at risk estimation for 10-year maturity for both the US and the German data, although they seem to slightly under-estimate the risk for the US data. For the Canadian data, all models are again correctly estimating the risk for 5 and 10 year maturities. Also note that, once again models’ VaR estimates are higher for the Canadian data than the US and the German data for 5 and 10 year maturities.

We can elaborate further on risk simulation results by considering the plots of models’ VaR estimates and the historical market losses. The plots of VaR estimation and historical loss results for the US data for 1, 2, 5 and 10-year maturity bonds are shown in Figure 6.3, 6.4, 6.5 and 6.6, respectively, for the German data they are given in Figure 6.3, 6.4, 6.5 and 6.6, respectively and for the Canadian data plots are given

Table 6.3. Historic risk assessment results for 2-year maturity bonds

	Total Sim	mean(risk)	histloss $\geq$ VaR
US Data			
Vasicek	95	0.0256	8
Hull-White	95	0.0256	8
CIR	95	0.0276	5
CIR++	95	0.0275	5
Vas2f Equil.	95	0.0401	1
Vas2f AF.	95	0.0403	1
German Data			
Vasicek	64	0.0194	8
Hull-White	64	0.0194	8
CIR	64	0.0253	4
CIR++	64	0.0253	4
Vas2f Equil.	64	0.0322	1
Vas2f AF.	64	0.0315	1
Canadian Data			
Vasicek	53	0.0328	2
Hull-White	53	0.0326	2
CIR	53	0.0318	3
CIR++	53	0.0319	3
Vas2f Equil.	53	0.0491	2
Vas2f AF.	53	0.0492	1

in Figure 6.11, 6.12 and 6.13, respectively.

To sum up, we can make the following conclusions by considering all of the results and the plots for 1, 2, 5 and 10-year maturity bonds.

- For the German data, CIR and CIR++ models' VaR results are better than for Vasicek and Hull-White for all of the maturities, particularly for 5 and 10 year

Table 6.4. Historic risk assessment results for 5-year maturity bonds

	Total Sim	mean(risk)	histloss $\geq$ VaR
US Data			
Vasicek	95	0.0537	17
Hull-White	95	0.0535	17
CIR	95	0.0606	10
CIR++	95	0.0603	11
Vas2f Equil.	95	0.0874	1
Vas2f AF.	95	0.0878	1
German Data			
Vasicek	64	0.0304	11
Hull-White	64	0.0304	10
CIR	64	0.0495	6
CIR++	64	0.0497	6
Vas2f Equil.	64	0.0627	2
Vas2f AF.	64	0.0621	2
Canadian Data			
Vasicek	53	0.0779	2
Hull-White	53	0.0775	2
CIR	53	0.0778	1
CIR++	53	0.0777	1
Vas2f Equil.	53	0.1161	1
Vas2f AF.	53	0.1163	1

bonds, for which Vasicek and Hull-White models are clearly under-estimating the risk.

- For the US data, all models are under-estimating the risk for 5 and 10 year maturities except for the two-factor models, which are over-estimating the risk for 5-year maturity and correctly estimating the risk for 10-year maturity.
- Only two-factor models correctly estimating the risk for 10-year maturity for both the US and the German data.

Table 6.5. Historic risk assessment results for 10-year maturity bonds

	Total Sim	mean(risk)	histloss $\geq$ VaR
US Data			
Vasicek	95	0.0761	17
Hull-White	95	0.0759	17
CIR	95	0.0899	13
CIR++	95	0.0899	13
Vas2f Equil.	95	0.1330	8
Vas2f AF.	95	0.1330	8
German Data			
Vasicek	64	0.0337	14
Hull-White	64	0.0337	14
CIR	64	0.0649	7
CIR++	64	0.0652	8
Vas2f Equil.	64	0.0899	3
Vas2f AF.	64	0.0895	3
Canadian Data			
Vasicek	53	0.1309	2
Hull-White	53	0.1300	2
CIR	53	0.1325	1
CIR++	53	0.1322	1
Vas2f Equil.	53	0.1938	2
Vas2f AF.	53	0.1946	2

- For the Canadian data, all models are correctly estimate the risk for all maturities. A reason seems to be that although historical loss results are similar for the Canadian, US and the German bonds, the models' VaR results are higher for the Canadian data than the US and the German data. Since, 3-month interest rates are more volatile for the Canadian data.
- For the US and the German data, the risk estimation quality is decreasing (i.e. they under-estimate the risk) with increasing maturity for all models except for

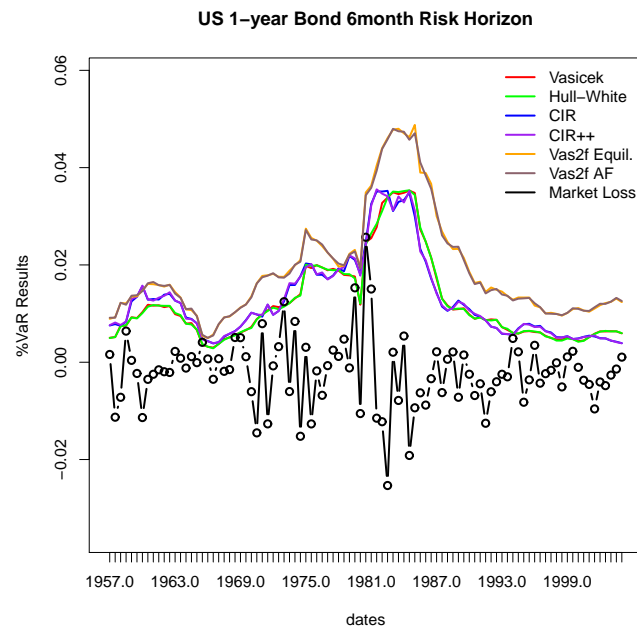


Figure 6.3. VaR and historical loss results for 1-year maturity bonds for the US data

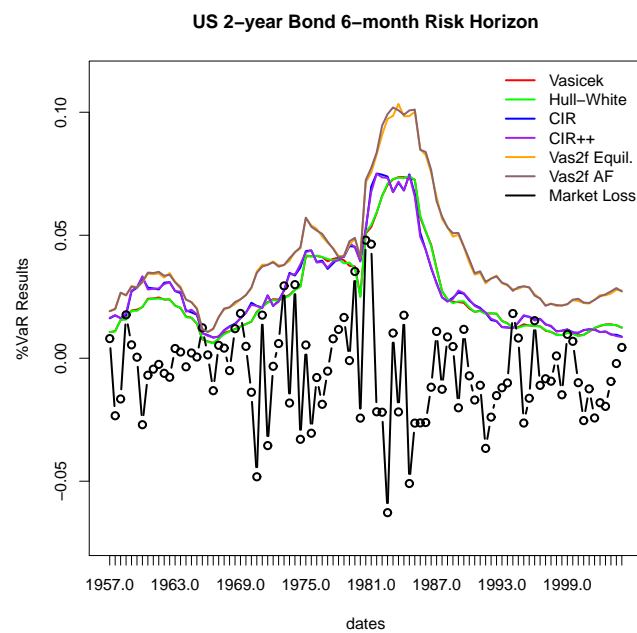


Figure 6.4. VaR and historical loss results for 2-year maturity bonds for the US data

the two factor models.

After considering these results, we can conclude that:

- With increasing maturity VaR is increasing for all models.

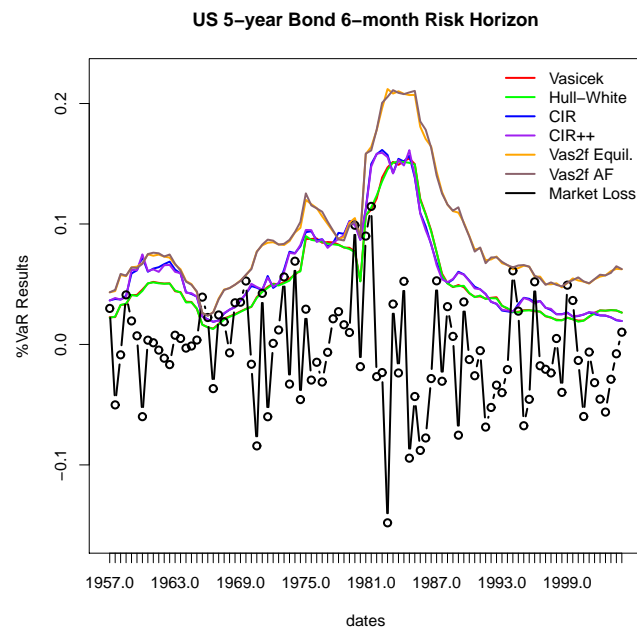


Figure 6.5. VaR and historical loss results for 5-year maturity bonds for the US data

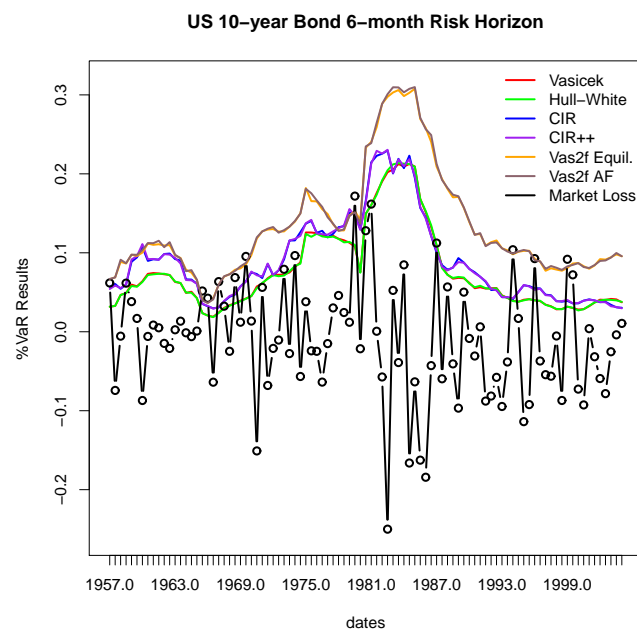


Figure 6.6. VaR and historical loss results for 10-year maturity bonds for the US data

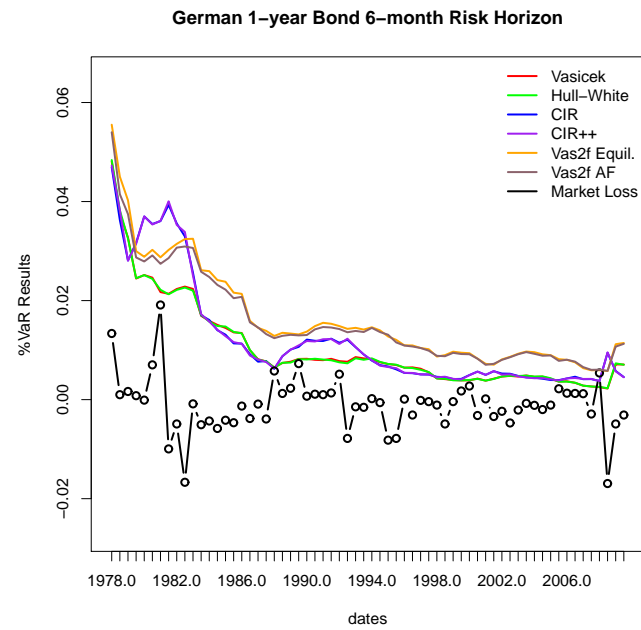


Figure 6.7. VaR and historical loss results for 1-year maturity bonds for the German data

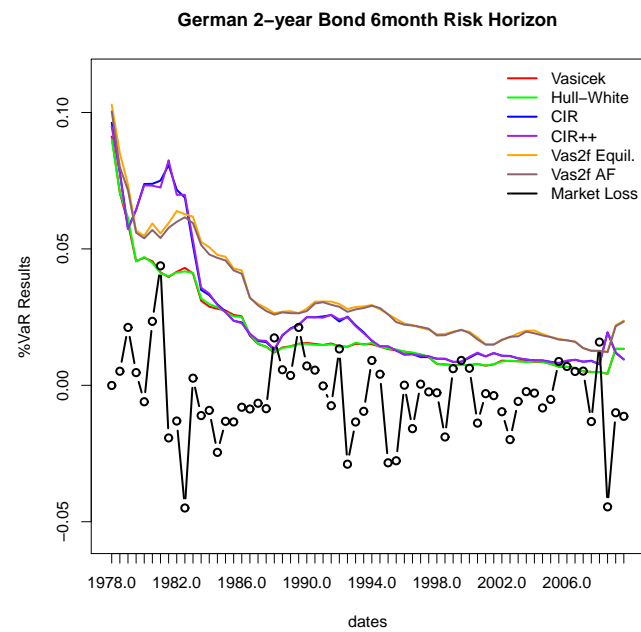


Figure 6.8. VaR and historical loss results for 2-year maturity bonds for the German data

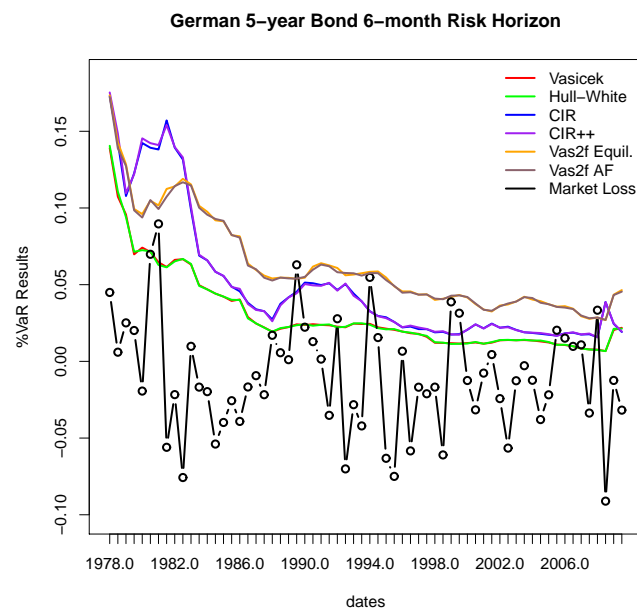


Figure 6.9. VaR and historical loss results for 5-year maturity bonds for the German data

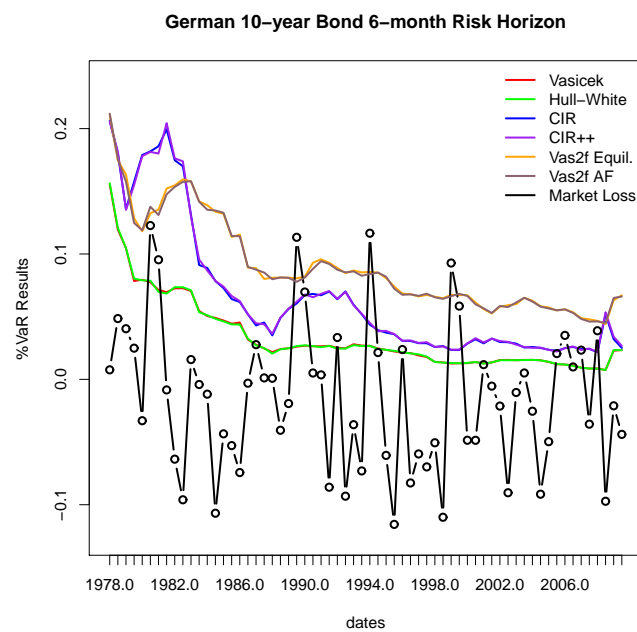


Figure 6.10. VaR and historical loss results for 10-year maturity bonds for the German data



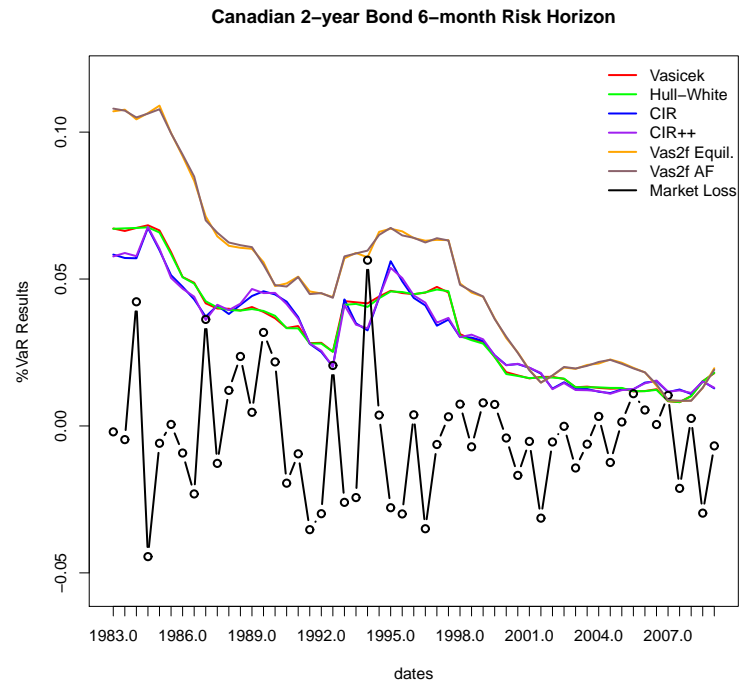


Figure 6.11. VaR and historical loss results for 2-year maturity bonds for the Canadian data

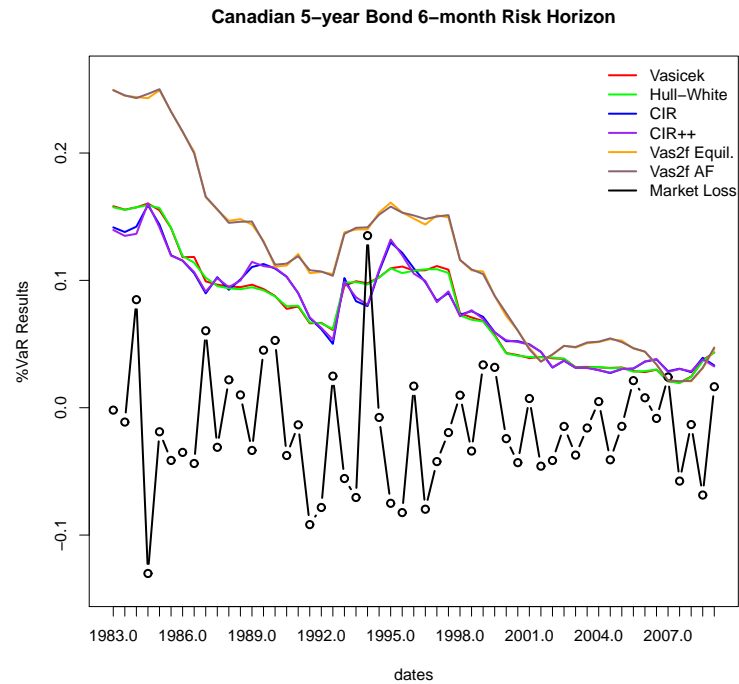


Figure 6.12. VaR and historical loss results for 5-year maturity bonds for the Canadian data

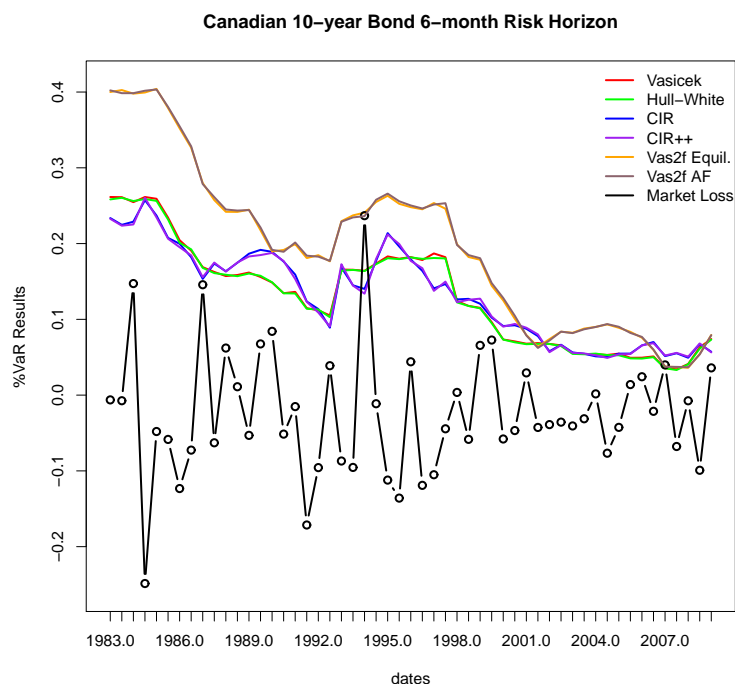


Figure 6.13. VaR and historical loss results for 10-year maturity bonds for the Canadian data

- With increasing maturity historical losses are also increasing.
- Two factor models produce greater VaR than other models, since they model the short and long rate together by summing them up (i.e. risk increases with increasing interest rates).
- Our three equilibrium-arbitrage free model pairs lead to very similar results for all maturities for both the US, German and the Canadian data. Note that in the plots, their results are so close that we can only see three lines. Therefore, we can say that for these models using the arbitrage-free or the equilibrium version does not influence risk quantification. A reason seems to be that the parameter  $\sigma$ , which has a main influence on risk, is estimated in the same way for the equilibrium and the arbitrage-free versions of the models.
- For all maturities except for 1-year, CIR and CIR++ models are better at risk estimation than Vasicek and Hull-White except for the Canadian data, for which they produce similar VaR results. This difference is more clear in the German data. This is reasonable, since we know that the short-rate can become negative in the Vasicek process. Especially for a small initial value like 3-month yield, this

can lead to unrealistic short-rate paths.

- VaR is highest in 1984.5 (i.e. July 1984) for all maturities for the US data. This is due to the highly volatile interest rates in the early 1980's, leading to large volatility estimates.
- VaR is highest in late 1970's and early 1980's for all maturities for the German data. Note that German 3-month yield is more volatile than the US 3-month yield in mid 1970's, which leads to high VaR estimates.
- For the Canadian data, VaR is again highest in early 1980's for all maturities. Since, 3-month rates are highly volatile in late 1970's in Canada.
- Volatility of the losses in the market increases with increasing maturity for all data.
- Two factor models are better at risk estimation for longer maturities, particularly for 10 year maturity as historical losses are increasing with increasing maturity.

## 7. CONCLUSIONS

In this study, we worked on the Vasicek, CIR and the Two Factor Vasicek equilibrium models and also on their arbitrage free counterparts called Hull-White, CIR++ and Two Factor Arbitrage-Free Vasicek model. We explained and illustrated the short-rate simulation, parameter estimation, bond pricing and risk simulation for all of the models. Note that for the two factor models, we used an additive modeling framework (i.e. the short-rate and the long-rate are added together).

We used MLE for estimating the model parameters. Note that for the Vasicek model closed-form formulas exist for maximum likelihood estimators. We also used the MLE of the Vasicek model for the parameter estimates of the Hull-White, Two Factor Equilibrium Vasicek and the Two Factor Arbitrage-Free Vasicek models. For the CIR model we have used R's *optim* function to find the MLE numerically and we used the MLE of the CIR model for the parameter estimates of the CIR++ model.

Short-rate simulation and bond pricing is not so difficult, as for all of the models the exact distribution of  $r_t$  is known and analytical formulas exist for bond pricing. Therefore, risk simulation is easy and fast as small sample sizes are enough to obtain quite precise results.

In the last chapter of the thesis, we made an empirical comparison of the models for quantifying the interest rate risk. We used the US, German and the Canadian bond market data to test six short-rate models' risk quantification performances. We can summarize our results as follows:

- For the three equilibrium-arbitrage free model pairs, the VaR results are almost the same for all maturities for both the US, German and the Canadian data. So, we conclude that for all three models using the arbitrage-free or the equilibrium version does not influence the risk quantification.
- For 1 and 2 year maturities, Vasicek-Hull White and CIR-CIR++ pairs' VaR

estimates are acceptable for the US and the German data.

- For 5 and 10 year maturities, two factor models are better at risk estimation for the US and the German data.
- For the US and the German data, the Vasicek and the Hull-White models produce more optimistic VaR estimates than the CIR and the CIR++ models. Two factor models lead to the highest risk estimates.
- All models are correctly estimating the risk for all maturities for the Canadian data.
- For the Canadian data, Vasicek-Hull White and CIR-CIR++ models' VaR estimates are similar. However, two factor models again lead to the highest risk estimates like for the US and the German data.

Which short rate model should be used to quantify bond risk? As they are clearly easier to implement we suggest to use one of the three equilibrium models we have tried. It also makes sense to use all three and to interpret the three results as an interval holding the true VaR. However, also note that model's risk estimation performance varies for countries (e.g. 10-year bond risk in Germany is clearly above the VaR results for three dates). Therefore, it is reasonable to use different models for different countries to estimate the risk .

## APPENDIX A: R CODES

### A.1. R Codes for the Exact Simulation of the Vasicek Model

```
Vasicek_simul_exact<- function(para,r0,n,dt,d){
# exact simulation of n paths of the short rate process
# Short rate path is returned as a matrix
# para ... parameters of the Vasicek model
# r0 ... starting value of the interest rate process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
mu=para[1]
beta=para[2]
sigma= para[3]
r=matrix(nrow=n,ncol=d)
r[,1]=r0
  for(i in 1:(d-1))
    r[,i+1]= r[,i]*exp(-beta*dt) + mu*(1-exp(-beta*dt))+
      sigma*sqrt((1-exp(-2*beta*dt))/(2*beta))*rnorm(n)
r
}
```

### A.2. R Codes for the Approximate Simulation of the Vasicek Model

```
Vasicek_simul_euler <- function(para,r0,n,dt,d,exact=T){
# approximate simulation of n paths of the short rate process
# Short rate path is returned as a matrix
# para ... parameters of the Vasicek model
# r0 ... starting value of the short rate process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
mu=para[1]
beta=para[2]
sigma= para[3]
r=matrix(nrow=n,ncol=d)
r[,1]=r0
  for(i in 1:(d-1))
    r[,i+1]= r[,i] + beta*(mu-r[,i])*dt+ sigma*sqrt(dt)*rnorm(n)
r
}
```

### A.3. R Codes for the MLE of the Parameters of the Vasicek Model

```
Vasicek_MLE<-function(data,dt = 1/12){
#returns a vector with the estimates for the parameters mu, beta and
#sigma of the Vasicek model
#data ... vector of short rate data
#dt ... time interval (in years) between the data points given in data
N = length(data)
rate <- data[2:N]
lagrate<- data[1:(N-1)]
alpha<-(N*sum(rate*lagrate) - sum(rate)*sum(lagrate))/
(N*sum(lagrate^2)- sum(lagrate)^2)
beta_hat = -log(alpha)/dt
mu_hat = sum(rate-alpha*lagrate) / (N*(1-alpha))
v2hat<-sum((rate-lagrate*alpha-mu_hat*(1-alpha))^2)/N
sigma_hat<-sqrt(2*beta_hat*v2hat/(1-alpha^2))
c(mu_hat,beta_hat,sigma_hat)
}
```

### A.4. R Codes for the Bond Price and the Yield Calculations for the Vasicek Model

```
VasicekPriceYield<-function(r,tau=1,Param,priceyn=F){
# r ... r(t) current value of short rate
# tau ... (T-t), time to maturity
# Param ... vector holding the parameters of the Vasicek model
mu =Param[1]
beta =Param[2]
sigma =Param[3]
B=(1-exp(-beta*tau))/beta
A=((B-tau)*(beta^2*mu-0.5*sigma^2)/beta^2 - sigma^2*B^2/(4*beta))
if(priceyn){
  if(tau==0) return(1) #price is par-value(1) at maturity
  else return(exp(A-B*r))
}else return((r*B-A)/tau)
}
```

### A.5. R Codes for Risk Simulation for the Vasicek Model

```
RiskSim_Vasicek<-function(param,d=180,dt=1/360,r0,T,n=1000,m=10){
# evaluates the VaR value for 6m risk horizon.
# r0... initial value of the short rate
```

```

# T... maturity of the bond
# n... simulation size
# m... outer loop for calculating a conf. interval for VaR
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
Var6m=0
Init.Price=VasicekPriceYield(r=r0,tau=T,Param=param,priceyn=1)
for(i in 1:m)
{
  res=Vasicek_simul_exact(para=param,r0=r0,n=n,dt=dt,d=d)
  Sim.Price6m=VasicekPriceYield(r=res[,d],tau=T-1/2,
  Param=param,priceyn=T)
  Loss6m= 1-Sim.Price6m/(Init.Price*exp(rowSums(res)*dt))
  Var6m[i]= quantile(Loss6m,0.95)
}
return(c(mean(Var6m),1.96*sd(Var6m)/sqrt(m)))
}

```

## A.6. R Codes for the Exact Simulation of the CIR Model

```

CIR_simul_exact <- function(para,r0,n,dt,d,exact=T){
# exact simulation of n paths of the short rate process
# para ... parameters of the CIR model
# r0 ... starting value of the interest rate process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
# c... multiplying term for the chi-square distribution
# df... degree of freedom
# ncp... non-centrality parameter
mu=para[1]
beta=para[2]
sigma= para[3]
c= sigma^2*(1-exp(-beta*dt))/(4*beta)
df= 4*mu*beta/sigma^2
r=matrix(nrow=n,ncol=d)
r[,1]=r0
  for(i in 1:(d-1)){
    ncp= r[,i]*exp(-beta*dt)/c
    r[,i+1] = c*rchisq(n,df=df,ncp=ncp)
  }
r
}

```



## A.7. R Codes for the Approximate Simulation of the CIR Model

```

CIR_simul_euler<- function(para,r0,n,dt,d){
# approximate simulation of n paths of the short rate process
# para ... parameters of the CIR model
# r0 ... starting value of the interest rate process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
# c... multiplying term for the chi-square distribution
# df... degree of freedom
# ncp... non-centrality parameter
mu=para[1]
beta=para[2]
sigma= para[3]
c= sigma^2*(1-exp(-beta*dt))/(4*beta)
df= 4*mu*beta/sigma^2
r=matrix(nrow=n,ncol=d)
r[,1]=r0
  for(i in 1:(d-1))
    r[,i+1]=r[,i]+beta*(mu-r[,i])*dt+sigma*sqrt(r[,i]*dt)*rnorm(n)
r
}

```

## A.8. R Codes for Calculating the Log-Likelihood Function of the CIR Model

```

CIRloglike<-function(param,data,times,test=F,addsign=T){
# CIR log-likelihood Function
# para ... parameters of the CIR model
# dt ... time interval (in years) between the data points given in data
# c... multiplying term for the chi-square distribution
# df... degree of freedom
# ncp... non-centrality parameter
mu=param[1]
beta=param[2]
sigma=param[3]
N<-length(data)
  if(test==T)
    dt=times
  else
    dt<-diff(times,1)
rate=data[1:(N-1)]
lagrate=data[2:N]

```

```

ncp= rate*((4*beta*exp(-beta*dt)) / (sigma^2*(1-exp(-beta*dt))))
d= 4*mu*beta/sigma^2
c= 4*beta/(sigma^2*(1-exp(-beta*dt)))
res<- sum(dchisq(c*lagrate,df=d,ncp=ncp,log=TRUE)+log(c))
  if(addsign)
    return(-res)
  else
    return(res)
}

```

### A.9. R Codes for the Bond Price and Yield Calculations for the CIR Model

```

CirPriceYield<- function(r,tau,Param,priceyn=F){
# r ... r(t) current value of short rate
# tau ... (T-t), time to maturity
# Param ... vector holding the parameters of the CIR model
mu=Param[1]
beta=Param[2]
sigma2<-Param[3]^2
h=sqrt(beta^2+2*sigma2)
B= 2*(exp(h*tau)-1) / (2*h+(beta+h)*(exp(tau*h)-1))
A= ((2*h*exp((beta+h)*(tau)/2)) / (2*h+(beta+h)*
(exp(tau*h)-1)))^(2*beta*mu/sigma2)
  if(priceyn){
    if(tau==0) return(1) #price is par-value(1) at maturity
    else return(A*exp(-B*r))
  }
  else return((r*B-log(A))/tau)
}

```

### A.10. R Codes for Risk Simulation for the CIR Model

```

RiskSim_CIR<-function(param,r0,T,n=1000,m=10,d=180,dt=1/360){
# evaluates the VaR value for 6m risk horizon.
# r0... initial value of the short rate
# T... maturity of the bond
# n... simulation size
# m... outer loop for calculating a conf. interval for VaR
# dt ... length of time step (in years)
# d ... number of time steps that should be generated

```

```

Var6m=0
Init.Price=CirPriceYield(r=r0,tau=T,Param=param,priceyn=1)
for(i in 1:m)
{
  res=CIR_simul_exact(para=param,r0=r0,n=n,dt=dt,d=d,exact=T)
  Sim.Price6m=CirPriceYield(r=res[,d],tau=T-1/2,Param=param,priceyn=T)
  Loss6m= 1-Sim.Price6m/(Init.Price*exp(rowSums(res)*dt))
  Var6m[i]= quantile(Loss6m,0.95)
}
return(c(mean(Var6m),1.96*sd(Var6m)/sqrt(m)))
}

```

## A.11. R Codes for the Exact Simulation of the Two Factor Equilibrium Vasicek Model

```

Vas2f_sim_exact<-function(param,x0,y0,dt,n,d){
# exact simulation of n paths of the short rate process
# Short rate path and x(t),y(t) processes returned as a matrix
# param ... parameters of the 2-factor Vasicek model
# x0 ... starting value of the x (short rate) process
# y0 ... starting value of the y (long rate) process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
mu_x   =param[1]
beta_x =param[2]
sigma_x=param[3]
mu_y   =param[4]
beta_y =param[5]
sigma_y=param[6]
rho    =param[7]
x=matrix(nrow=n,ncol=d)
y=matrix(nrow=n,ncol=d)
r=matrix(nrow=n,ncol=d)
x[,1]=x0
y[,1]=y0
r[,1]=x0+y0
R<-matrix(c(1 ,rho,rho,1),nrow=2)# Correlation Matrix
L<- t(chol(R))# Cholesky decomposition
for(i in 1:(d-1))
{
  Z<- matrix(rnorm(n*2),nrow=2)
  Z<- L%*%Z ## Correlated random variates
  Z2=t(Z)

```

```

    x[,i+1]=x[,i]*exp(-beta_x*dt)+mu_x*(1-exp(-beta_x*dt))+
    sigma_x*sqrt((1-exp(-2*beta_x*dt))/(2*beta_x))*Z2[,1]
    y[,i+1]=y[,i]*exp(-beta_y*dt)+mu_y*(1-exp(-beta_y*dt))+
    sigma_y*sqrt((1-exp(-2*beta_y*dt))/(2*beta_y))*Z2[,2]
    r[,i+1]=x[,i+1]+y[,i+1]
  }
  return(list(rt=r,xt=x,yt=y))
}

```

## A.12. R Codes for the Approximate Simulation of the Two Factor Equilibrium Vasicek Model

```

Vas2f_sim_euler<-function(param,x0,y0,dt,n,d){
# approximate simulation of n paths of the short rate process
# Short rate path and x(t),y(t) processes returned as a matrix
# param ... parameters of the 2-factor Vasicek model
# x0 ... starting value of the x process
# y0 ... starting value of the y process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
mu_x  =param[1]
beta_x =param[2]
sigma_x=param[3]
mu_y  =param[4]
beta_y =param[5]
sigma_y=param[6]
rho    =param[7]
x=matrix(nrow=n,ncol=d)
y=matrix(nrow=n,ncol=d)
r=matrix(nrow=n,ncol=d)
x[,1]=x0
y[,1]=y0
r[,1]=x0+y0
R<-matrix(c(1 ,rho,rho,1),nrow=2)# Correlation Matrix
L<- t(chol(R))# Cholesky decomposition
for(i in 1:(d-1))
{
  Z<- matrix(rnorm(n*2),nrow=2)
  Z<- L%*%Z ## Correlated random variates
  Z2=t(Z)
  x[,i+1]=x[,i]+beta_x*(mu_x-x[,i])*dt+sigma_x*sqrt(dt)*Z2[,1]
  y[,i+1]=y[,i]+beta_y*(mu_y-y[,i])*dt+sigma_y*sqrt(dt)*Z2[,2]
}
}

```

```

    r[,i+1]=x[,i+1]+y[,i+1]
  }
  return(list(rt=r,xt=x,yt=y))
}

```

### A.13. R Codes for the Bond Price and the Yield Calculations for the Two Factor Equilibrium Vasicek Model

```

Vas_2fPriceYield<-function(t,param,x0,y0,x,y,tau,priceyn=F,d){
# param ... vector holding the parameters of the 2f Vasicek model
# x0 ... initial value of short rate
# y0 ... initial value of the long rate
# x ... short rate process x(t)
# y ... long rate process y(t)
# d ... number of time steps in the simulations
# tau ... (T-t), time to maturity
# t ... time
mu_x  =param[1]
beta_x =param[2]
sigma_x=param[3]
mu_y  =param[4]
beta_y =param[5]
sigma_y=param[6]
rho   =param[7]
yield=mu_x+(1-exp(-beta_x*tau))/(beta_x*tau)*((if(t==0){x0}else
{x[,d]})-mu_x)+mu_y+(1-exp(-beta_y*tau))/(beta_y*tau)*((if(t==0)
{y0}else{y[,d]})-mu_y)-sigma_x^2/(2*beta_x^2)*(1+(1-exp(-2*beta_x*
tau))/(2*beta_x*tau)-2*(1-exp(-beta_x*tau))/(beta_x*tau))-
sigma_y^2/(2*beta_y^2)*(1+(1-exp(-2*beta_y*tau))/(2*beta_y*tau)-
2*(1-exp(-beta_y*tau))/(beta_y*tau))-(rho*sigma_x*sigma_y)/
(beta_x*beta_y)*(1-(1-exp(-beta_x*tau))/(beta_x*tau)-
(1-exp(-beta_y*tau))/(beta_y*tau)+(1-exp(-(beta_x+beta_y)*tau))/
((beta_x+beta_y)*tau))
price=exp(-yield*tau)
  if(priceyn)
    if(tau==0) return(1) #price is par-value(1) at maturity
    else return(price)
  else
    return(yield)
}

```

## A.14. R Codes for Risk Simulation for the Two Factor Equilibrium Vasicek Model

```

RiskSim_2fVas<-function(x0,y0,T,param,n=1000,m=10,d=180,dt=1/360){
#evaluates the VaR value for 6m horizon.
#r0... initial value of the short rate
#T... maturity of the bond
#n... simulation size
#m... outer loop for calculating a conf. interval for VaR
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
Var6m=0
Init.Price=Vas_2fPriceYield(t=0,tau=T-0,param=param,x0,y0,priceyn=T)
for(i in 1:m)
{
  res=Vas2f_sim_exact(param=param,n=n,dt=dt,d=d,x0=x0,y0=y0)
  Sim.Price6m=Vas_2fPriceYield(t=1/2,tau=T-1/2,param=param,
  x=res$xt,y=res$yt,priceyn=T,d=d)
  Loss6m= 1-Sim.Price6m/(Init.Price*exp(rowSums(res$rt)*dt))
  Var6m[i]= quantile(Loss6m,0.95)
}
return(c(mean(Var6m),1.96*sd(Var6m)/sqrt(m)))
}

```

## A.15. R Codes for the Exact Simulation of the Hull-White Model

```

HullW_simul_exact <- function(param,r0,n,dt,d,fm,exact=T){
# exact simulation of n paths of the short rate process
# Short rate path is returned as a matrix
# para ... parameters of the Hull-White model
# r0 ... starting value of the short rate process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
beta = param[1]
sigma = param[2]
r=matrix(nrow=n,ncol=d)
r[,1]=r0
alphanat=0
alphanat[1]= fm[1]
  for(i in 1:(d-1)){
    alphanat[i+1]= fm[i] + sigma^2/(2*beta^2)*(1-exp(-beta*dt*i))^2
    r[,i+1]= r[,i]*exp(-beta*dt) + alphanat[i+1]-alphanat[i]*exp(-beta*dt) +

```

```

    sqrt(sigma^2/(2*beta)*(1-exp(-2*beta*dt)))*rnorm(n)
  }
r
}

```

## A.16. R Codes for the Bond Price and the Yield Calculations for the Hull-White Model

```

HwPriceYield<-function(r,T,t,Param,priceyn=T,Pm0,Pmt,fmt){
# r ... r(t) current value of short rate
# T ... maturity
# t ... time
# Param ... vector holding the parameters of the Vasicek model
# Pm0 ... initial market bond price for maturity T
# Pmt ... initial market bond price for maturity t
# fmt ... initial forward rate at time t
beta    = Param[1]
sigma   = Param[2]
B = 1/beta*(1-exp(-beta*(T-t)))
A = Pm0/Pmt*exp(B*fmt-sigma^2/(4*beta)*(1-exp(-2*beta*t))*B^2)
if(priceyn){
  if((T-t)==0) return(1) #price is par-value(1) at maturity
  else return(A*exp(-B*r))
}
else return((r*B-log(A))/(T-t))
}

```

## A.17. R Codes for Risk Simulation for the Hull-White Model

```

RiskSim_HwVas<-function(param,r0,n=1000,m=30,Pm,fm,matur,yield6m,d=180,dt=1/360){
#evaluates the VaR value for 6m risk horizon for the Hull-White model.
#r0... initial value of the short rate
#T... maturity of the bond
#n... simulation size
#m... outer loop for calculating a conf. interval for VaR
#dt ... length of time step (in years)
#d ... number of time steps that should be generated
Var6m=0
for(i in 1:m)
{

```

```

    res=HullW_simul_exact(para=param,r0=r0,n=n,dt=dt,d=d,fm=fm)
    Sim.Price6m=HwPriceYield(r=res[,d],T=matur,t=1/2,Param=param,
    Pm0=Pm,Pmt=exp(-yield6m*0.5),fmt=fm[d])
    Loss6m= 1-Sim.Price6m/(Pm*exp(rowSums(res)*dt))
    Var6m[i]= quantile(Loss6m,0.95)
  }
return(c(mean(Var6m),1.96*sd(Var6m)/sqrt(m)))
}

```

## A.18. R Codes for the Exact Simulation of the CIR++ Model

```

Ex_CIR_simul_exact<-function(param,x0,n,dt,d,fm){
# exact simulation n paths of the short rate process and
# returns them as a matrix
# para ... parameters of the CIR++ model
# r0 ... starting value of the short rate process
# n ... number of paths that are generated
# dt ... length of time step (in days)
# d ... number of time steps that should be generated
# fm... market instantaneous forward rate vector
# c... multiplying term of the chi-square
# df... degree of freedom
# ncp... non-centrality parameter
mu=param[1]
beta=param[2]
sigma= param[3]
c= sigma^2*(1-exp(-beta*dt)) / (4*beta)
df= 4*mu*beta/sigma^2
x=matrix(nrow=n,ncol=d)
r=matrix(nrow=n,ncol=d)
x[,1]=x0
h=sqrt(beta^2+2*sigma^2)
fCIR=0
fCIR[1]=x0
phiCIR=0
phiCIR[1]=fm[1]-fCIR[1]
r[,1]=x0+phiCIR[1]
  for(i in 1:(d-1)){
    fCIR[i+1]=(2*beta*mu*(exp(i*dt*h)-1))/(2*h+(beta+h)*(exp(i*dt*h)-1))+
    (x0*4*h^2*exp(i*dt*h))/(2*h+(beta+h)*(exp(i*dt*h)-1))^2
    phiCIR[i+1]=fm[i+1]-fCIR[i+1]
    ncp= x[,i]*exp(-beta*dt)/c
    x[,i+1] = c*rchisq(n,df=df,ncp=ncp)
  }
}

```



```

    r[,i+1] = x[,i+1]+phiCIR[i+1]
  }
r
}

```

## A.19. R Codes for the Approximate Simulation of the CIR++ Model

```

Ex_CIR_simul_euler<-function(param,x0,n,dt,d,fm){
# exact simulation n paths of the short rate process and
# returns them as a matrix
# para ... parameters of the CIR++ model
# r0 ... starting value of the short rate process
# n ... number of paths that are generated
# dt ... length of time step (in days)
# d ... number of time steps that should be generated
# fm... market instantaneous forward rate vector
# c... multiplying term of the chi-square
# df... degree of freedom
# ncp... non-centrality parameter
mu=param[1]
beta=param[2]
sigma= param[3]
c= sigma^2*(1-exp(-beta*dt)) / (4*beta)
df= 4*mu*beta/sigma^2
x=matrix(nrow=n,ncol=d)
r=matrix(nrow=n,ncol=d)
x[,1]=x0
h=sqrt(beta^2+2*sigma^2)
fCIR=0
fCIR[1]=x0
phiCIR=0
phiCIR[1]=fm[1]-fCIR[1]
r[,1]=x0+phiCIR[1]
  for(i in 1:(d-1)){
    fCIR[i+1]=(2*beta*mu*(exp(i*dt*h)-1))/(2*h+(beta+h)*(exp(i*dt*h)-1))+
      (x0*4*h^2*exp(i*dt*h))/(2*h+(beta+h)*(exp(i*dt*h)-1))^2
    phiCIR[i+1]=fm[i+1]-fCIR[i+1]
    ncp= x[,i]*exp(-beta*dt)/c
    x[,i+1]=x[,i]+beta*(mu-r[,i])*dt+sigma*sqrt(r[,i]*dt)*rnorm(n)
    r[,i+1] = x[,i+1]+phiCIR[i+1]
  }
r
}

```

## A.20. R Codes for the Bond Price and the Yield Calculations for the CIR++ Model

```

Ex_CIR_PriceYield<-function(r,x0,T,t,Param,priceyn=T,Pm0,Pmt,fmt){
# x0 ... initial value for the x process
# r ... short rate process with shift added
# T ... maturity
# t ... time
# Param ... vector holding the parameters of the Vasicek model
# Pm0 ... initial market bond price for maturity T
# Pmt ... initial market bond price for maturity t
# fmt ... initial forward rate at time t
mu=Param[1]
beta=Param[2]
sigma2<-Param[3]^2
h=sqrt(beta^2+2*sigma2)
B<-function(t,T) 2*(exp(h*(T-t))-1) / (2*h+(beta+h)*(exp((T-t)*h)-1))
A<-function(t,T) ((2*h*exp((beta+h)*(T-t)/2))/(2*h+(beta+h)*
(exp((T-t)*h)-1)))^(2*beta*mu/sigma2)
fCIRt=(2*beta*mu*(exp(t*h)-1))/(2*h+(beta+h)*(exp(t*h)-1))+
(x0*4*h^2*exp(t*h))/(2*h+(beta+h)*(exp(t*h)-1))^2
phiCIRt=fmt-fCIRt
A_bar=Pm0*A(0,t)*exp(-B(0,t)*x0)/(Pmt*A(0,T)*exp(-B(0,T)*x0))*
A(t,T)*exp(B(t,T)*phiCIRt)
#if t=0 add the shift to the x process
#else simulated r values will already be shifted
      if(t==0)
        r=x0+phiCIRt
      else
        r=r
price=A_bar*exp(-B(t,T)*r)
yield=-log(price)/(T-t)
if(priceyn){
  if((T-t)==0) return(1) #price is par-value(1) at maturity
  else return(price)
}else return(yield)
}

```

## A.21. R Codes for Risk Simulation for the CIR++ Model

```

RiskSim_ExCIR<-function(param,r0,n=1000,m=10,Pm,fm,matur,yield6m,d=180,dt=1/360){
#evaluates the VaR value for 6m risk horizon for the CIR++ model.
#time...simulation horizon
#r0... initial value of the short rate

```

```

#T... maturity of the bond
#n... simulation size
#m... inner loop for calculating a conf. interval for VaR
#dt ... length of time step (in years)
#d ... number of time steps that should be generated
Var6m=0
for(i in 1:m)
{
  res=Ex_CIR_simul_exact(param=param,x0=r0,n=n,dt=dt,d=d,fm=fm)
  Sim.Price6m=Ex_CIR_PriceYield(r=res[,d],x0=r0,T=matur,t=1/2,
  Param=param,Pm0=Pm,Pmt=exp(-yield6m*0.5),fmt=fm[d])
  Loss6m= 1-Sim.Price6m/(Pm*exp(rowSums(res)*dt))
  Var6m[i]= quantile(Loss6m,0.95)
}
return(c(mean(Var6m),1.96*sd(Var6m)/sqrt(m)))
}

```

## A.22. R Codes for the Exact Simulation of the Two Factor Arbitrage-Free Vasicek Model

```

AFVas2f_sim_exact<-function(Param,n,d,dt,fm){
# exact simulation of n paths of the short rate process
# Short rate path and x,y processes are returned as a matrix
# Param ... parameters of the 2f AF Vasicek model
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
betax=Param[1]
sigmax=Param[2]
betay=Param[3]
sigmay=Param[4]
rho=Param[5]
x=matrix(nrow=n,ncol=d)
y=matrix(nrow=n,ncol=d)
r=matrix(nrow=n,ncol=d)
R<-matrix(c(1,rho,rho,1),nrow=2)# Correlation Matrix
L<- t(chol(R)) # Cholesky decomposition
x[,1]=0
y[,1]=0
phi=0
phi[1]=fm[1]
r[,1]=phi[1]
  for(i in 1:(d-1)){
    Z<- matrix(rnorm(n*2),nrow=2)

```

```

Z<- L%*%Z ## Correlated random variates
Z2=t(Z)
x[,i+1]=x[,i]*exp(-betax*dt)+sqrt(sigmax^2/(2*betax)*
(1-exp(-2*betax*dt)))*Z2[,1]
y[,i+1]=y[,i]*exp(-betay*dt)+sqrt(sigmay^2/(2*betay)*
(1-exp(-2*betay*dt)))*Z2[,2]
phi[i+1]=fm[i+1]+sigmax^2/(2*betax^2)*(1-exp(-betax*i*dt))^2+
sigmay^2/(2*betay^2)*(1-exp(-betay*i*dt))^2+rho*sigmax*sigmay/
(betax*betay)*(1-exp(-betax*i*dt))*(1-exp(-betay*i*dt))
r[,i+1]=x[,i+1]+y[,i+1]+phi[i+1]
}
return(list(rt=r,Xt=x,Yt=y))
}

```

### A.23. R Codes for the Approximate Simulation of the Two Factor Arbitrage-Free Vasicek Model

```

AFVas2f_sim_euler<-function(Param,dt,n,d,fm){
# approximate simulation of n paths of the short rate process
# Short rate path and x(t),y(t) processes returned as a matrix
# param ... parameters of the 2-factor AF Vasicek model
# x0 ... starting value of the x process
# y0 ... starting value of the y process
# n ... number of paths that are generated
# dt ... length of time step (in years)
# d ... number of time steps that should be generated
# fm ... daily forward rates
betax=Param[1]
sigmax=Param[2]
betay=Param[3]
sigmay=Param[4]
rho=Param[5]
x=matrix(nrow=n,ncol=d)
y=matrix(nrow=n,ncol=d)
r=matrix(nrow=n,ncol=d)
x[,1]=0
y[,1]=0
phi=0
phi[1]=fm[1]
r[,1]=phi[1]
R<-matrix(c(1 ,rho,rho,1),nrow=2)# Correlation Matrix
L<- t(chol(R))# Cholesky decomposition
for(i in 1:(d-1))

```

```

{
  Z<- matrix(rnorm(n*2),nrow=2)
  Z<- L%*%Z ## Correlated random variates
  Z2=t(Z)
  x[,i+1]=x[,i]+betax*(-x[,i])*dt+sigmax*sqrt(dt)*Z2[,1]
  y[,i+1]=y[,i]+betay*(-y[,i])*dt+sigmay*sqrt(dt)*Z2[,2]
  phi[i+1]=fm[i+1]+sigmax^2/(2*betax^2)*(1-exp(-betax*i*dt))^2+
  sigmay^2/(2*betay^2)*(1-exp(-betay*i*dt))^2+rho*sigmax*sigmay/
  (betax*betay)*(1-exp(-betax*i*dt))*(1-exp(-betay*i*dt))
  r[,i+1]=x[,i+1]+y[,i+1]+phi[i+1]
}
return(list(rt=r,xt=x,yt=y))
}

```

## A.24. R Codes for the Bond Price and the Yield Calculations for the Two Factor Arbitrage-Free Vasicek Model

```

AF_vas2f_PriceYield<-function(x,y,T,t,Param,PmT,Pmt,fmt,d,priceyn=T){
# x ... x(t) (short-rate) process
# y ... y(t) (long-rate) process
# T ... maturity
# t ... time
# Param ... vector holding the parameters of the Vasicek model
# PmT ... initial market bond price for maturity T
# Pmt ... initial market bond price for maturity t
# fmt ... initial forward rate at time t
# d ... time step used in the simulation.
betax=Param[1]
sigmax=Param[2]
betay=Param[3]
sigmay=Param[4]
rho=Param[5]
V<-function(t,T){
  sigmax^2/betax^2*(T-t+2/betax*exp(-betax*(T-t))-1/(2*betax)*
  exp(-2*betax*(T-t))-3/(2*betax))+sigmay^2/betay^2*(T-t+2/betay*
  exp(-betay*(T-t))-1/(2*betay)* exp(-2*betay*(T-t))-3/(2*betay))+
  2*rho*sigmax*sigmay/(betax*betay)*(T-t+(exp(-betax*(T-t))-1)/
  betax+(exp(-betay*(T-t))-1)/betay-(exp(-(betax+betay)*(T-t))-1)/
  (betax+betay))}
#if t=0 x(0)=0 and y(0)=0 else x[,t]=x[,d],y[,t]=y[,6]
#x[,d] and y[,d] will be found from simulations.
At=1/2*(V(t,T)-V(0,T)+V(0,t))-(1-exp(-betax*(T-t)))/betax*(if(t==0){0}
else{x[,d]})-(1-exp(-betay*(T-t)))/betay*(if(t==0){0}else{y[,d]})

```

```

Price=PmT/Pmt*exp(At)
yield=-log(Price)/(T-t)
  if(priceyn)
    return(Price)
  else
    return(yield)
}

```

## A.25. R Codes for Risk Simulation for the Two Factor Arbitrage-Free Vasicek Model

```

RiskSim_AF_2fVas<-function(x0,y0,T,Param,n=1000,m=30,d=180,dt=1/360,Pm,yield6m,fm){
#evaluates the VaR value for 6m horizon.
#x0 ... initial value of the short rate
#y0 ... initial value of the long rate
#T ... maturity of the bond
#n ... simulation size
#m ... outer loop for calculating a conf. interval for VaR
#Pm ... initial market price for maturity T
#Pmt ... initial market price for maturity t
#yield6m ... initial 6-month yield
#fm ... forward rate vector for the interval [0,10] year
Var6m=0
for(i in 1:m){
  res=AFVas2f_sim_exact(Param=Param,n=n,d=d,dt=dt,fm=fm)
  Sim.Price6m=AF_vas2f_PriceYield(x=res$Xt,y=res$Yt,T=T,t=1/2,
  Param=Param,PmT=Pm,Pmt=exp(-yield6m*0.5),fmt=fmt,d=d)
  Loss6m= 1-Sim.Price6m/(Pm*exp(rowSums(res$rt)*dt))
  Var6m[i]= quantile(Loss6m,0.95)
}
return(c(mean(Var6m),1.96*sd(Var6m)/sqrt(m)))
}

```

## A.26. R Codes for Historic Loss Calculations

```

Historic_loss<-function(Pm_buy,Pm_sell,opp_cost,time,sim_total){
#Pm_buy ... historical prices of bonds at simulation dates.
#Pm_sell ... historical prices of bonds after risk_horizon.
#time ... time of holding the bond (i.e. 6-month).
#opp_cost ... 6-month yield at simulation dates
#sim_total ... total risk simulations

```

```

hist=0
hist[1]=1-(Pm_sell[1+1]/(Pm_buy[1]*exp(opp_cost[1]*time)))
for(i in 1:(sim_total-1))
hist[i+1]=1-(Pm_sell[i+2]/(Pm_buy[i+1]*exp(opp_cost[i+1]*time)))
return(hist)
}

```

## A.27. R Codes for Historical Risk Simulations

```

Risk_Sim<-function(func,...,data,str_date=1957,end_date=2004,
period=2,model="vasicek",sim=T,bond="1-year",forward,datas){
#str_date ... start date of risk simulations.
#end_date ... end date of risk simulations.
#period ... number of simulations in a year.
#sim_total ... total number of simulations
#results ... matrix for storing %VaR results.
#1st column for simulation dates, 2nd for mean VaR results
#3rd for error bounds.
#yields ... yields at simulation dates.
#r0 ... initial short rate (3-month) at simulation dates.
#L0 ... initial long rate (10-year) at simulation dates.
#Pm ... Market Prices at simulation dates.
#forward .. daily forward rates at simulation dates.
#rt_MLE_vas ... MLE for short rate in the Vasicek model.
#lt_sigma_hat ... MLE for long rate in the Vasicek model.
#rt_sigma_hat ... MLE of sigma of the short rate, estimated from
#last 5-years data for each simulation date.
#For CIR and CIR++, numerical optimization is made seperately for
#the US and the German data.
#Three dots (...) is used for assigning further arguments to the risk
#simulation functions of different models.
#First 5 year data is only used for calibration (i.e. risk simulations
#start after 60 months).
data2      =as.matrix(data,rownames.force=NA)
d          =length(data2[1,])
sim_total  =(end_date-str_date)*period+1
results    =matrix(0,nrow=sim_total,ncol=3)
yields     =matrix(0,nrow=sim_total+1,ncol=(d-1))
results[1,1]=data[61,1]
yields[1,]  = data2[61,2:d]
yields[sim_total+1,] = data2[(sim_total/period)*12+61,2:d]
for(i in 1:(sim_total-1))
{
  results[i+1,1] =data[(12/period)*i+61,1]

```

```

        yields[i+1,] =data2[(12/period)*i+61,2:d]
    }
r0=yields[,1]
if(datas=="CA"){
    Pm6m    =exp(-yields[,2]*0.5)
    Pm1y    =0
    Pm2y    =exp(-yields[,3]*2)
    Pm5y    =exp(-yields[,4]*5)
    Pm10y   =exp(-yields[,5]*10)
    yield6m =yields[,2]
    LO      =yields[,6]
}
if(datas=="US")
{
    Pm6m    =exp(-yields[,2]*0.5)
    Pm1y    =exp(-yields[,3]*1)
    Pm2y    =exp(-yields[,4]*2)
    Pm5y    =exp(-yields[,5]*5)
    Pm10y   =exp(-yields[,6]*10)
    yield6m =yields[,2]
    LO      =yields[,5]
}
if(datas=="DE")
{
    Pm6m    =exp(-yields[,1]*0.5)
    Pm1y    =exp(-yields[,2]*1)
    Pm2y    =exp(-yields[,3]*2)
    Pm5y    =exp(-yields[,7]*5)
    Pm10y   =exp(-yields[,d-1]*10)
    LO      =yields[,d-1]
    yield6m =yields[,1]
}
if(bond=="1-year")
Pm=Pm1y
if(bond=="2-year")
Pm=Pm2y
if(bond=="5-year")
Pm=Pm5y
if(bond=="10-year")
Pm=Pm10y
rt_MLE_vas=Vasicek_MLE(data=data[,2])
lt_MLE_vas=Vasicek_MLE(data=data[,d])
rt_sigma_hat=rep(0,sim_total)
for(i in 1:(sim_total))
    rt_sigma_hat[i]=Vasicek_MLE(data=data[((12/period)*(i-1)+1):
((12/period)*(i-1)+60),2])[3]

```



```

lt_sigma_hat=rep(0,sim_total)
for(i in 1:(sim_total))
  lt_sigma_hat[i]=Vasicek_MLE(data=data[((12/period)*(i-1)+1):
    ((12/period)*(i-1)+60),d])[3]
corr=rep(0,sim_total)
for(i in 1:(sim_total))
  corr[i]=cor(data[((12/period)*(i-1)+1):((12/period)*(i-1)+61),2],
    data[((12/period)*(i-1)+1):((12/period)*(i-1)+60),d])
if(model=="vasicek"&sim==T)
{
  for(i in 1:(sim_total))
    results[i,2:3]=func(param=c(rt_MLE_vas[1],rt_MLE_vas[2],
      rt_sigma_hat[i]),r0=r0[i],...)
}
if((model=="cir" || model=="cir++")&sim==T)
{
  sigma_hat=rep(0,sim_total)
  MLE_cir=optim(par=c(0.001,0.001,0.018),fn=CIRloglike,
    method = "L-BFGS-B",lower=c(0.001,0.05,0.018),upper=c(0.9,2,0.9),
    data=data[,2],times=1/12,test=T)$par
  x<-function(mu,beta,sigma,data,times,test)CIRloglike(param=
    c(mu,beta,sigma),data,times,test)
  if(datas=="US" || datas=="CA")
  {
    for(i in 1:(sim_total))
      sigma_hat[i]=optim(par=c(0.011),fn=x,method="L-BFGS-B",
        lower=0.017,upper=0.90,mu=MLE_cir[1],beta=MLE_cir[2],
        data=data[((12/period)*(i-1)+1):((12/period)*(i-1)+62),2],
        times=1/12,test=T)$par
  }
  if(datas=="DE")
  {
    for(i in 1:(sim_total))
      sigma_hat[i]=optim(par=c(0.01),fn=x,method="L-BFGS-B",
        lower=0.0199,upper=0.90,mu=MLE_cir[1],beta=MLE_cir[2],
        data=data[((12/period)*(i-1)+1):((12/period)*(i-1)+63),2],
        times=1/12,test=T)$par
  }
}
if(model=="cir"){
  for(i in 1:(sim_total))
    results[i,2:3]=func(param=c(MLE_cir[1],MLE_cir[2],sigma_hat[i]),
      r0=r0[i],...)
}
if(model=="cir++"){
  for(i in 1:(sim_total))

```

```

    results[i,2:3]=func(param=c(MLE_cir[1],MLE_cir[2],sigma_hat[i]),
    yield6m=yield6m[i],    r0=r0[i],Pm=Pm[i],fm=forward[,i],...)
  }
if(model=="Hull-White"&sim==T)
{
  for(i in 1:(sim_total))
    results[i,2:3]=func(param=c(rt_MLE_vas[2],rt_sigma_hat[i]),
    yield6m=yield6m[i],r0=r0[i],Pm=Pm[i],fm=forward[,i],...)
}
if(model=="Vas_2f"&sim==T)
{
  for(i in 1:(sim_total))
    results[i,2:3]=func(param=c(rt_MLE_vas[1],rt_MLE_vas[2],
    rt_sigma_hat[i],lt_MLE_vas[1],lt_MLE_vas[2],lt_sigma_hat[i],
    corr[i]),x0=r0[i],y0=L0[i],...)
}
if(model=="AF_Vas_2f"&sim==T)
{
  for(i in 1:(sim_total))
    results[i,2:3]=func(Param=c(rt_MLE_vas[2],rt_sigma_hat[i],
    lt_MLE_vas2[2],lt_sigma_hat[i],corr[i]),x0=r0[i],y0=L0[i],
    Pm=Pm[i],fm=forward[,i],yield6m=yield6m[i],...)
}
if(sim)
return(results)
else
return(list(Pm6m=Pm6m,Pm1y=Pm1y,Pm2y=Pm2y,Pm5y=Pm5y,
Pm10y=Pm10y,yields=yields))
}

```

## A.28. R Codes for Fitting Instantaneous Forward Rates to the Bond Prices

```

HW_forwPrice<-function(matur,yield,data="US",forwardyn=T){
#Uses linear interpolation to find daily forward rates for each time x.
#f(x)=ax+b...forward rate line for 0 to longest maturity.
#Finds all a and b values for each interval in the maturity vector.
#Uses  $p(0,T)=\exp(-\int_0^T f(x) dx)$  to find daily bond prices for each time x .
#matur...bond maturities
#yield...bond yields
#USyields ... 0.25,0.5,1,2,5,10 year maturities
#Germanyyields ... 0.5,1,2,3,4,5,6,7,8,9,10 year maturities
#Canadianyyields ... 0.25,0.5,2,5,10 year maturities
R=yield

```

```

price=c(1,exp(-R*matur))
a=0;b=0
b[1]=R[1]
a[1]=uniroot(function(a) log(price[2])+a*matur[1]^2/2+b[1]*matur[1],c(-1,1))[[1]]
  for(i in 2:length(matur)){
    a[i]=uniroot(function(y) log(price[i+1]/price[i])+y/2*(matur[i]^2-matur[i-1]^2)+
      (matur[i-1]*(a[i-1]-y)+b[i-1])*(matur[i]-matur[i-1]),c(-1,1))[[1]]
    b[i]=matur[i-1]*(a[i-1]-a[i])+b[i-1]
  }
# 1year=360 days
fwd=0
prices=0
  for(x in 1:(360*matur[length(matur)])){
if(x<=matur[1]*360)
  {
    fwd[x]=a[1]*x/360+b[1]
    prices[x]=exp(-a[1]*(x/360)^2/2-b[1]*x/360)
  }
if(x>matur[1]*360&x<=matur[2]*360){
  {
    fwd[x]=a[2]*x/360+b[2]
    prices[x]=price[2]*exp(-a[2]*(x/360)^2/2-b[2]*x/360+a[2]*matur[1]^2/2+
      b[2]*matur[1])
  }
}
if(x>matur[2]*360&x<=matur[3]*360){
  {
    fwd[x]=a[3]*x/360+b[3]
    prices[x]=price[3]*exp(-a[3]*(x/360)^2/2-b[3]*x/360+a[3]*matur[2]^2/2+
      b[3]*matur[2])
  }
}
if(x>matur[3]*360&x<=matur[4]*360){
  {
    fwd[x]=a[4]*x/360+b[4]
    prices[x]= price[4]*exp(-a[4]*(x/360)^2/2-b[4]*x/360+a[4]*matur[3]^2/2+
      b[4]*matur[3])
  }
}
if(x>matur[4]*360&x<=matur[5]*360){
  {
    fwd[x]=a[5]*x/360+b[5]
    prices[x]= price[5]*exp(-a[5]*(x/360)^2/2-b[5]*x/360+a[5]*matur[4]^2/2+
      b[5]*matur[4])
  }
}
}

```

```

if(data!="CA")
{
if(x>matur[5]*360&x<=matur[6]*360){
    fwd[x]=a[6]*x/360+b[6]
    prices[x]= price[6]*exp(-a[6]*(x/360)^2/2-b[6]*x/360+a[6]*matur[5]^2/2+
    b[6]*matur[5])
}
}
if(data=="DE")
{
if(x>matur[6]*360&x<=matur[7]*360)
{
    fwd[x]=a[7]*x/360+b[7]
    prices[x]= price[7]*exp(-a[7]*(x/360)^2/2-b[7]*x/360+a[7]*matur[6]^2/2+
    b[7]*matur[6])
}
if(x>matur[7]*360&x<=matur[8]*360)
{
    fwd[x]=a[8]*x/360+b[8]
    prices[x]= price[8]*exp(-a[8]*(x/360)^2/2-b[8]*x/360+a[8]*matur[7]^2/2+
    b[8]*matur[7])
}
if(x>matur[8]*360&x<=matur[9]*360)
{
    fwd[x]=a[9]*x/360+b[9]
    prices[x]= price[9]*exp(-a[9]*(x/360)^2/2-b[9]*x/360+a[9]*matur[8]^2/2+
    b[9]*matur[8])
}
if(x>matur[9]*360&x<=matur[10]*360)
{
    fwd[x]=a[10]*x/360+b[10]
    prices[x]= price[10]*exp(-a[10]*(x/360)^2/2-b[10]*x/360+a[10]*matur[9]^2/2+
    b[10]*matur[9])
}
if(x>matur[10]*360&x<=matur[11]*360)
{
    fwd[x]=a[11]*x/360+b[11]
    prices[x]= price[11]*exp(-a[11]*(x/360)^2/2-b[11]*x/360+a[11]*matur[10]^2/2+
    b[11]*matur[10])
}
}
}
if(forwardyn)
return(fwd)
else
return(prices)
}

```

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