MISSION BASED COMPONENT TEST PLANS

by

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ABSTRACT

MISSION BASED COMPONENT TEST PLANS

One of the widely used methods of system reliability test plans is based on the idea of computing the component test plans that ensure prespecified reliability levels for those components. Another widely used approach suggests the determination of the system test plans which guarantee a certain reliability level for the system. Since the main concern is the reliability of the system, the latter approach can be thought as more advantageous over the former one. However, testing a system as a whole can be very costly, very difficult or even impossible, which makes the first approach more appropriate to use. The idea of combining the advantageous points of both approachs gave birth of another method called as system based component test plans.

One can find many works in the literature that applies the system based component testing idea with respect to different system topologies. However, realibility definition employed in all of these studies assumes a prespecified amount of fixed time during which the object whose reliability is the concern, works without a failure. This reliability definition is invalid for systems designed to perform sequence of missions, which are possibly in random order and have possibly random durations. In this thesis, a new method which we call as mission based component test plans is proposed as a means for the determination of the optimum component test plans for series systems and serial connection of redundant subsystems.

ÖZET

GÖREV TABANLI BİLEŞEN SINAMI

Dizge güvenilirlik sınamlarının planlanmasında sıkça uygulanan yaklaşımlardan biri, bileşenlerin sağlaması arzulanan güvenilirlik düzeyleri belirlendikten sonra bu düzeyleri güven altına alan bileşen sınam sürelerinin hesaplanması düşüncesine dayanır. Diğer bir yaklaşım ise dizgenin bütün olarak sınanması ve arzulanan dizge güvenilirlik düzeyini sağlayan dizge sınam sürelerinin belirlenmesidir. Amacın bir bütün olarak dizgenin güvenilirliği hakkında yargıya varmak olduğu düşünülürse, ikinci yaklaşımın birinciye göre daha üstün olduğu düşünülebilir. Ancak dizgeyi bütün olarak denemek çok maliyetli veya çok zor olabilir, hatta bazı durumlarda olurlu olmayabilir. Böyle durumlarda da birinci yaklaşım ikinciye üstünlük sağlar. Bu iki yaklaşımın olumlu taraflarını bir araya getirme düşüncesi dizge tabanlı bileşen sınamı adı altında yazına geçmiş yeni bir yöntemin doğuşuna olanak sağladı.

Konuyla ilgili yazında dizge tabanlı bileşen sınamı düşüncesini değişik dizgeler için ele alan birçok çalışma bulunmaktadır. Fakat bu çalışmaların tamamında bir nesnenin güvenilirlik tanımı nesnenin belirli bir süre boyunca bozulmadan çalışma olasılığı olarak kabul edilmiştir. Bu güvenilirlik tanımı sıraları ve süreleri rasgele bir dizi görevi gerçekleştirmek için tasarlanan aygıtlar için geçerli olmamaktadır. Bu çalışmada görev tabanlı bileşen sınamı adını verdiğimiz yeni bir kavram, güvenilirlik tanımları görev tabanlı olarak belirlenen dizgelerin eniyi bileşen sınam sürelerini hesaplama yöntemi olarak sunulmakta ve sıralı dizgeler ile seri bağlanmış koşut alt dizgeler için bileşen sınamları tasarlanmaktadır.

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LIST OF SYMBOLS/ABBREVIATIONS

∂A	Boundary of a set $A \subset \mathbb{R}^n$
B_h^{-1}	Inverse of basis matrix of the dual problem $D^{(h)}(m)$
c_j	Test cost for component j
$c_j(i)$	Test cost for component j at mission i
$C(i)$	Set of required components during mission i
$D(i)$	Duration of mission i
D	A convex subset of $\mathbb{R}^n$
$D(m)$	Dual linear program for a given m
$D^{(h)}(m)$	Dual linear program for a given m at iteration h
F	A compact subset of $\mathbb{R}^n$
F_I	Feasible solution index set associated with $\rho(R_0)$
F_{II}	Feasible solution index set associated with $\rho(R_1)$
f_I^h	Optimum solution of type I problem at iteration h, new column generated from $\rho(R_0)$
f_{II}^h	Optimum solution of type II problem at iteration h, new column generated from $\rho(R_1)$
g	Reverse convex constraint
G	A convex subset of $\mathbb{R}^n$ defined by the area $g(x) \leq 0$
H_0	Null hypothesis
H_1	Alternative hypothesis
$J(u)$	Set of indices of hyperplanes binding at vertex u
k	Number of components
k_n	Total magnitudes of $C(i)$'s for all $i \in M$
K	Set of components
L	Laplace transform
LI	Lifetime of the system
l_i	Magnitude of $C(i)$
$l_i(x)$	Hyperplane cutting polytope $P^{(i)}$ at iteration i
m	Maximum number of allowable component failures

m^*	Optimum maximum number of allowable component failures
M	Set of missions
n	Number of missions
$n_j(i)$	Number of components of the j^{th} subsystem during mission i
N	Total number of failures
$N_j(i)$	Total number of failures of component j during mission i
$\mathbb{N}$	Natural numbers
$N(u)$	Set of neighbouring vertices of vertex u
P	Transition matrix of the markov renewal process lying behind the markovian mission environment
$\tilde{P}$	Transformed transition matrix
$P^{(i)}$	Polytope obtained at iteration i
$P_1(m)$	Mission based component test problem with a sequence of missions for a given m
$P_1'(m)$	Semi-infinite linear programming model obtained for a given m
$P_2(m)$	Primal linear program for a given m
$P_2^{(h)}(m)$	Primal linear program at iteration h for a given m
R_s	System reliability level
$R(\lambda)$	System reliability as a function of component failure rates λ
R_0	Unacceptable reliability level
R_1	Acceptable reliability level
$\mathbb{R}$	Real line
$\mathbb{R}^{k\times n}$	$k \times n$ real vector space
$\mathbb{R}_+^{k\times n}$	$k \times n$ nonnegative real vector space
$\mathbb{R}_+^{k_n}$	k_n nonnegative real vector space
T_n	Completion time of mission n
t_j	Test time of component j
$t_j(i)$	Test time of component j at mission i
$t_{j,m}(i)$	Test time of component j at mission i for a given m
$t^*_{j,m}(i)$	Optimum test time of component j at mission i for a given m
$t^*_j(i)$	Optimum test time of component j at mission i

u	Upperbound vector on component failure rates
$u_j(i)$	Upperbound for failure rate of component j during mission i
$V^{(i)}$	Set of vertices of polytope $P^{(i)}$
V^+	Set of vertices lying above the cut $l_i(x)$
V^-	Set of vertices lying below the cut $l_i(x)$
$w^*(m)$	Optimum dual solution of $D^{(h)}(m)$
z^*	Minimum total test cost
$z^*(m)$	Minimum total test cost for a given m
$z^*_{I,h}(m)$	Optimum objective value of type I problem for a given m at iteration h
$z^*_{II,h}(m)$	Optimum objective value of type II problem for a given m at iteration h
$z^*_{P_2^{(h)}}(m)$	Optimum objective value of P_2 for a given m at iteration h
$z^*_{D^{(h)}}(m)$	Optimum objective value of D for a given m at iteration h
z^*_{CDC}	Optimum objective value of CDC problem
z^*_{DCP}	Optimum objective value of DCP problem
α	A given upperbound on consumer risk
β	A given upperbound on producer risk
δ, π	Dual variables of $P_2(m)$
Γ	Prior information set
ϵ	A very small quantity
λ	Failure rate vector
$\lambda_j(i)$	Failure rate of component j at mission i
$\lambda_{\gamma,m}$	Poisson parameter for which $\psi(\lambda_{\gamma,m}) = \gamma$
Λ_I	$\{\lambda \in \mathbb{R}^{k_n} : R(\lambda) \leq R_0\}$
Λ_{II}	$\{\lambda \in \mathbb{R}^{k_n} : R(\lambda) \geq R_1\}$
$\rho(R_0)$	Set of feasible component failure rates for which system rejection is correct
$\rho(R_1)$	Set of feasible component failure rates for which system acceptance is correct
$\mu(i)$	Rate of the exponential distributon that duration of mission i follows from

Ξ	Set of polytopes covering F
$\psi_m(\gamma)$	Cumulative Poisson distribution with parameter γ
CDC	Canonical D.C.
D.C.	Difference of two convex functions
DCP	D.C. problem
LRCP	Linear reverse convex programming problem
SQP	Sequential quadratic programming

1. INTRODUCTION

System is a collection of objects that are organized to perform a predetermined mission. The smallest unit of the system is called a component. A system can be a subsystem of a larger system. Systems are designed to perform certain missions. In order to accomplish the mission, the system should be working until the end of the mission. Definition of system reliability depends on the structure of the situation at hand. If there is only one mission to be accomplished or if the system is required to perform for a predetermined amount of time, namely mission time, reliability is defined as probability of failure free operation of the system during the mission time. However, if there is a sequence of missions that are possibly in random order and have possibly random durations, a different reliability definition is required which is referred in this work. In such a case, the probability that an object to fulfill the requirement(s) that are put on it until the end of the last mission is defined to be the (mission) reliability of that object. To ensure a high (mission) reliability for the system, the system as a whole or its components and subsystems should be tested. In other words, they should be operated under different real working conditions, like different temperatures, different pressures, etc. This testing procedure is called as reliability testing. Reliability testing process ends after a predetermined number of failure occurences or a time limitation for the test is exceeded. Test times for each component and subsystem should be carefully determined so that they reflect the real system reliability level. Therefore, test plans are essential for complex systems in this sense because they ensure the system to function with certain level of reliability.

An efficient test plan should serve three main objectives: (1) it should guarantee a predetermined system reliability level, (2) it should be capable to identify the problems that must be removed before the system true mission begins, (3) it should have the minimum possible total test cost. A widely used and incorrect approach in the design of test plans is to assign certain level of reliabilities to the components and to determine the number of component tests that assure the component reliabilities with a certain level of confidence. Another approach suggests testing the system as a whole and basing the test plans on the desired value of system reliability. Although the second approach seems to be more accurate, component tests have following four advantages over system tests: (1) they are more economical, (2) they enable the collection of more information about the components, (3) it is possible to test each component seperately and independently, (4) they are more timely in providing information about system reliability. However, system owners may feel uncomfortable if all the reliability test solely depends only on the reliability information at component level. In fact, a combination of the system and component level reliability testing approachs would help.

The desire of combining the more accurate structure of system tests with the advantages of component tests has led to a well known approach in reliability testing: *system based component tests* (Altınel 1990). This approach basically suggests hypothesis test for system reliability based on independent experimentation over components. In other words, it says, experiment with the components and accept or reject the system based upon the component test information. However, for all the studies based on system based component test approach up to today, reliability has been defined as the probability of failure free operation of the system for a prespecified amount of fixed mission time. This approach is inappropriate for many real life cases such as for devices designed to perform missions that consist of possibly random sequence of phases that have possibly random durations. For instance, consider a bus which is supposed to leave station A to go to the station D. Moreover, let it be scheduled to stop at stations B and C and get some passenger from B and C before arriving station D. As soon as the bus enters a station suppose that it is maintained to replace failed components with new ones and it becomes fully operational. In such a situation, the bus actually has a deterministic sequence of missions to accomplish. The first mission finishes when bus arrives station B, and second mission is accomplished by the time the bus enters station C. Finally, the last mission of the bus is to arrive station D leaving station C. Durations of the missions are also supposed to follow some probability distributions. The bus should be working until it arrives its final destination D. Moreover, it should be working through the missions. The reliability definition required for such a bus deviates from the conventional fixed time based reliability definitions. Therefore, the reliability is defined as the probability for the bus to be operational until the end of the last mission. This example actually illustrates the cases where we need a mission based reliability definition. There is a sequence of missions to be accomplished and the system should be functional until the end of the last mission. The sequence of missions can also be in a Markovian order. For instance, the bus can be supposed to visit the stations, but the order of the visits can be upto the environmental conditions that follows a probability distribution. In such a situation, although the sequence of the missions is not deterministic the reliability is still defined as the probability of being functional until the end of the last mission. In such a case, stochastic failure and optimization models that have been effectively used within the system based component testing framework can no longer be applicable. Therefore, in this thesis a new approach named as *mission based component testing* is proposed as a solution procedure for the type of reliability ensurance problems in which reliability of an object is defined as the probability for the object to fulfill the requirements that are put on it during all missions and where missions are possibly in random order and have possibly random durations. This approach yields some stochastic models and optimization results that are useful in the determination of optimum component test plans for devices designed to perform a sequence of missions. Similar to the system based component test approach, a hypothesis test for the system reliability based on independent experimentation over components is envisaged. Therefore, mission based component test approach is a special type of system based component test method.

This thesis is organized into seven sections. After the introduction, in the second section, the literature review consisting historical background behind the system based component testing approach is introduced while third section gives the general formulation of the semi-infinite linear programming model that is exploited in the optimum component test time finding process. In section four, the reliability models for which the mission based component testing method is proposed, are presented. Later on, for the reliability models given in the forth section, fifth section states the solution procedure for both the semi-infinite programming model obtained in the third section and the sub-problems which exist in the body of the general problem. Sixth section consists of numerical examples for the reliability models submitted in the forth section.

Finally, concluding remarks and future research directions are provided in the seventh section.

2. SYSTEM BASED COMPONENT TESTING

System based component testing problem was first addressed by Gal (1974). In his paper, he assumes that component failures follow from exponential distribution and he studies a situation in which a prespecified unacceptable reliability level, R_0, is to be demonstrated at a specified confidence level $1-\alpha$ which is known as the consumer's risk in the literature. He provides exact analytic solutions for series systems and systems with serial connection of redundant subsystems. Moreover, he gives approximate solutions for parallel systems. Denoting the system reliability which is a function of individual component failure rates, as R_s, the cost of testing component j per unit time as c_j and test time of component j as t_j, he proposes a general solution procedure in order to obtain the optimum component test times of components which minimize the total test cost for k components

$$\sum_{j=1}^{k} c_j t_j \tag{2.1}$$

whilst satisfying the probability requirement

$$P(\textit{accept the system};\ R_s \leq R_0) \leq \alpha. \tag{2.2}$$

Here, the system is accepted if and only if there are no component failures during the tests. Gal's formulation is criticized to have two drawbacks which are its rigidity as no component failures are permitted and the fact that the producer's risk is not taken into consideration. It was Mazumdar (1977) who first consulted this missing case by extending the Gal's formulation such that a certain acceptable reliability level R_1 is to be satisfied at a confidence level $1-\beta$. In mathematical words, the constraint demonstrating the producer's risk as

$$P(\textit{reject the system};\ R_s \geq R_1) \leq \beta \tag{2.3}$$

is added to the original formulation of Gal given by (2.1) and (2.2). Constraints (2.2) and (2.3) actually put upper limits on type I and type II errors of the classical hypothesis testing problem in which the null hypothesis states that the system is unaceptable, (i.e., $H_0 : R_s \leq R_0$) while the alternative hypothesis claims that it is acceptable, (i.e., $H_0 : R_s \geq R_1$). Here, R_0 and R_1 clearly stands for unacceptable and acceptable reliability levels for the system respectively.

In addition, Mazumdar adopted a new system acceptance criteria such that each component is allowed to fail during its test and assumed to be replaced with a new identical component simultaneously when a failure occurs, and the system is accepted if and only if total number of component failures do not exceed a predefined integer m which is to be found by an iterative procedure that enables feasibility with respect to (2) and (3), and leads least possible total cost. This rule is referred to as the sum rule in the sequel. It should be noted that this is a general form of the Gal's acceptance method since Gal considered only the case where $m = 0$. Previously in the literature, Gnedenko *et al.* (1969) have used the sum of component failures as a means for providing system reliability confidence limits. This is referred as the M-method by Gertsbakh (1989). In addition, Easterling *et al.* (1991) have provided a justification for using the sum rule for a series system. Based on the same ideas Mazumdar (1980) generalized his model and solution procedures for a serial connection of redundant subsystems in 1980. Likewise his previous studies, exponential lifetimes are assumed for components, and he provides an algorithmic method in order to find optimum m, i.e., m^*, and the corresponding optimum test times t_{j,m^*} for components, which minimizes (2.1) subject to (2.2) and (2.3).

While dealing with series system, both Gal (1974) and Mazumdar (1977) show that the optimum component test times were independent of component test costs c_j and interestingly enough, they are identical. Same results are obtained for also serial connection of redundant subsystems (Mazumdar 1980). At a first glance, this result can be justified with the fact that reliability of the systems with subsystems (or components) connected in a serial manner depends on the weakest subsystem of the series because its failure will cause whole system to stop. However, in practice it might be difficult to convince the managers to schedule the component tests such that each and every component is tested for the same amount of time independent of the cost of the testing.

None of these works considers the situation in which prior information about the component failure rates are obtained before testing. However, most of the time system managers have prior information on component reliabilities due to historical experiences. The first study where prior information is considered is due to Altınel (1992). In his work the prior information is taken in the form of upperbounds on component failure rates and it is assumed that this information is obtained at no cost. He treats the system based component test problem as a mathematical programming problem. The availability of such prior information affects the optimum component test times and over all test costs. He obtains optimum component test times based on this formulation, and demonstrate that they are no longer equal. Moreover, the use of such prior information also leads to considerable reductions in total test costs. His approach, which is explained in detail in his earlier work Altınel (1990), is important not only because it is the first mathematical programming view of the problem, but also leads to solution procedures for more general cases (Altınel, 1994; Altınel and Özekici, 1997; Altınel and Özekici, 1998). In (Altınel and Özekici, 1997) authors relax the assumption of exponential failure rates, and in (Altınel and Özekici, 1998) they study the case of dependent component failures using the concepts developed by Çınlar and Özekici (1987).

An unrealistic assumption that is often made in the literature is that the devices for which the reliability testing is being made are designed to perform a given mission during which the failure rate of the components of the device remain constant. There are however, many devices in real life which are designed for missions consisting of phases such that the set of the components that are vital to carry the mission may change depending on the specific requirements of the cases (Somani *et al.*, 1992). Failure rates of the device components may also change from phase to phase although they remain constant within the phases because each phase may impose different conditions like different pressure, temperature, etc. on the components. Mura and Bondavalli

(1999) provide an application example for this situation in which a spacecraft involved in and where the mission consists of four basic phases: launch, hibernation, planet and scientific observation. Phased-mission systems were initially introduced by Esary and Ziehms (1975) and an abundant literature has accumulated since then. The order of the phases may be deterministic or stochastic as Kim and Park (1994) suggests and components can either be repairable or non-repairable. Alam and Al-Saggaf (1986) considers a phased-mission system where the repair activity begins as soon as a failure occurs. Under the light of phased-system framework, Altınel *et al.* (2001) have generalized Altınel's early works for a dynamic component testing of series system with redundant subsystems depending on an earlier work of Feyzioğlu (1998). In this work, each component is assumed to have exponentially distributed lifetime in each environment it is operated. However, authors manage to show that the system's overall lifetime does not necessarily follow from exponential distribution, as it is assumed in all the available literature on testing serial connection of redundant subsystems. They also claim that their formulation is strong enough to represent any system with redundant subsystems where component failure rate functions of the components are piecewise constant. Altınel *et al.* (2001) further extent this case with component testing of repairable systems in multistage missions in 2001. Moreover, based on the phased-mission reasoning and the mathematical model point of view derived in the previous study of Altınel (1990), Altınel *et al.* (2002) also carried the research for the systems with random missions. Likewise, Feyzioğlu *et al.* (2003) overtakes the systems with phased missions depending on an earlier study of Feyzioğlu (2003). A study from Feyzioğlu *et al.* (2006) summarizes the system based component testing problem for different system and mission types and provides numerical examples for all system types.

Reliability definition employed in all the above studies envisage failure free operation of the system during a predefined amount of time, namely the mission time. This makes optimization tools of the solution procedures that are applied for many system based component testing studies, inappropriate for the cases in which reliability of an object is defined as the probability of the object to fulfill the requirements that are put on it until the end of the last mission where missions are possibly in random order and have possibly random durations. Çekyay (2007) provides reliability functions for several different systems with this reliability definition. A new method named as *mission based component testing* will be developed in the following sections of this thesis in order to find the optimum component test times for several systems where this new reliability definition is employed.

3. PROBLEM FORMULATION

In order to explain the problem of minimizing the total test cost while making hypothesis tests on the system reliability level based on independent experimentation over the system components, let us concentrate on a system composed of k components which is designed to perform n missions. Denote $K = \{1, ..., k\}$ as the set of components and $M = \{1, ..., n\}$ as the set of missions the system has to accomplish. At that point, one can claim that some of the components of the system may not be vital to carry mission i. Hence, the set $C(i)$ can be defined as the set of components required for completion of mission i. $C(i)$ is a subset of K for all $i \in M$. If all the components are required for all missions, $C(i)$ can be taken as K for all $i \in M$. Besides, let missions have random sequence and random durations, and define $c_j(i)$ and $t_j(i)$ as nonnegative unit cost of testing component j in mission i and testing time of component j in mission i respectively. Let R_0 and R_1 be the system reliability levels that are required to be demonstrated at specified confidence levels $1 - \alpha$ and $1 - \beta$ respectively. In addition, suppose the system reliability function is denoted as R_s. It is assumed that R_0, R_1, α and β are chosen from open interval $(0, 1)$ such that $\alpha + \beta < 1$ and $R_0 < R_1$. Components are also assumed to have exponential life times with constant failure rates within the missions which is denoted by $\lambda_j(i)$, and they are tested with replacement. Component failures are assumed to be mutually independent. System reliability level R_s can be thought as dependent on the reliability levels of the components. Therefore, it is possible to denote the system reliability as a function of component failure rate vector λ as $R_s(\lambda)$ or simply as $R(\lambda)$. If sum rule is employed as a means for accepting or rejecting the system, and m denote the allowable total number of failures of components, the system acceptance probability can be stated as in the following equality;

$$P[\textit{accept the system}] \equiv P\left[\sum_{i \in M}\sum_{j \in C(i)} N_j(i) \leq m\right] \tag{3.1}$$

where $N_j(i)$ is the number of component j failures during its test for mission i, which takes $t_{j,m}(i)$ time units for a given m. It can be observed that since each component j has an exponential life with failure rate $\lambda_j(i)$ in mission i, components are tested with replacement and component failures are assumed to be mutually indepedent, failures form a Poisson process. To be more specific, $N_j(i)$ has a Poisson distribution with parameter $\lambda_j(i)t_{j,m}(i)$. Consequently, total failures $N = \sum_{i\in M}\sum_{j\in C(i)} N_j(i)$ has a Poisson distribution with parameter $\sum_{i\in M}\sum_{j\in C(i)} \lambda_j(i)t_{j,m}(i)$.

Under the acceptance terminology given above and concerning a sequence of missions, finding the optimum component test times problem stated with (2.1), (2.2) and (2.3) can be expressed as the following optimization problem;

$$\min \sum_{i\in M}\sum_{j\in C(i)} c_j(i)t_{j,m}(i) \tag{3.2}$$

s.t.

$$P(\sum_{i\in M}\sum_{j\in C(i)} N_j(i) \leq m; R_s \leq R_0) \leq \alpha \tag{3.3}$$

$$P(\sum_{i\in M}\sum_{j\in C(i)} N_j(i) > m; R_s \geq R_1) \leq \beta \tag{3.4}$$

On the other hand, there may be prior information about the component failure rates due to previous experiences. This information is assumed to be obtained at no cost and it can also be corporated into the above model. Let prior information on component failure rates be denoted by Γ which is assumed to be a nonempty and compact subset of nonnegative real numbers. Γ can be in the form of upper bounds; e.g. $\Gamma = \{\lambda_j(i) \in \mathbb{R}_+^{k_n} : \lambda_j(i) \leq u_j(i) \quad j \in C(i),\ i \in M\}$, where k_n is taken as total magnitudes of $C(i)$'s which is the set of components required in mission i. In other words, $k_n = \sum_{i\in M}|C(i)|$. Thus, upper bounds on component failure rates are at hand before testing the system component wise. Let us define two new sets as $\Lambda_I \equiv \{\lambda_j(i) \in \mathbb{R}_+^{k_n} : R(\lambda) \leq R_0\}$ and $\Lambda_{II} \equiv \{\lambda_j(i) \in \mathbb{R}_+^{k_n} : R(\lambda) \geq R_1\}$. Clearly these two sets define the k_n dimensional component failure rates that are feasible with respect to system reliability constraints (3.3) and (3.4). If these sets are combined with the prior information set Γ, one can obtain the final feasible failure rate vectors with respect to both system reliability requirements and prior information on failure rates. For instance, let $\rho(R_0) \equiv \Lambda_I \cap \Gamma$ and $\rho(R_1) \equiv \Lambda_{II} \cap \Gamma$. Consequently,

$$\rho(R_0) \equiv \{\lambda_j(i) \in \mathbb{R}_+^{k_n} : R(\lambda) \leq R_0, \quad 0 \leq \lambda_j(i) \leq u_j(i), \quad j \in C(i), i \in M\} \tag{3.5}$$

and

$$\rho(R_1) \equiv \{\lambda_j(i) \in \mathbb{R}_+^{k_n} : R(\lambda) \geq R_1, \quad 0 \leq \lambda_j(i) \leq u_j(i), \quad j \in C(i), i \in M\} \tag{3.6}$$

hold where the prior information Γ is given as

$$\Gamma = \{\lambda_j(i) \in \mathbb{R}_+^{k_n} : \lambda_j(i) \leq u_j(i), \quad j \in C(i), i \in M\}. \tag{3.7}$$

It should be noted that, if there is no prior information on component failure rates, namely nothing is known about the failure rates a priori, the only restriction is due to the system reliability contraints. Formally speaking, $\Gamma = \mathbb{R}_+^{k_n}$ implying that any failure rate vector from positive real orthant is feasible with respect to prior information. In such a case, the final feasible regions of failure rates are $\rho(R_0) \equiv \Lambda_I$ and $\rho(R_1) \equiv \Lambda_{II}$.

Assuming that $\rho(R_0)$ and $\rho(R_1)$ are nonempty, it can be said that more than one feasible failure rate vector thus more than one value for the system acceptance probability $P\left[\sum_{i \in M}\sum_{j \in C(i)} N_j(i) \leq m\right]$ can be found. Then, the probability constraints (3.3) and (3.4) are surely satisfied for all feasible λ vectors if they are modified as follows:

$$\max_{\lambda \in \rho(R_0)} P\left[\sum_{i \in M}\sum_{j \in C(i)} N_j(i) \leq m\right] \leq \alpha \tag{3.8}$$

$$\min_{\lambda \in \rho(R_1)} P\left[\sum_{i \in M}\sum_{j \in C(i)} N_j(i) \leq m\right] \geq 1-\beta \tag{3.9}$$

Suppose that Y is a random variable that has a Poisson distribution with parameter y. Define $\lambda_{\gamma,m}$ to be the value of y for which $P[Y \leq m] = \gamma$ and let $\psi_m(y)$ be a function of y that stands for $P[Y \leq m]$. In other words, $\psi_m(y) = \sum_{k=0}^{m} \frac{e^{-y}y^k}{k!}$. By definition $\psi_m(\lambda_{\gamma,m}) = \gamma$ holds. Because N has a Poisson distribution with parameter $\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i)$, $P\left[\sum_{i \in M}\sum_{j \in C(i)} N_j(i) \leq m\right] = \psi_m(\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i))$. Then, under the light of new terminology, (3.8) and (3.9) inequalities become:

$$\underset{\lambda \in \rho(R_0)}{max}\, \psi_m\left(\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i)\right) \leq \alpha, \tag{3.10}$$

and

$$\min_{\lambda \in \rho(R_1)} \psi_m\left(\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i)\right) \geq 1-\beta. \tag{3.11}$$

Let us consider $\psi_m\left(\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i)\right) \leq \alpha$ and $\psi_m\left(\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i)\right) \geq 1-\beta$. Moreover, assume $t_{j,m}(i)$'s are nonnegative coefficients of $\lambda_{j,m}(i)$'s for all $j \in C(i), i \in M$. Then inequalities are equivalent to $\psi_m\left(\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i)\right) \leq \psi_m(\lambda_{\alpha,m})$ and $\psi_m\left(\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i)\right) \geq \psi_m(\lambda_{1-\beta,m})$ because it has previously set that $\psi_m(\lambda_{\alpha,m}) = \alpha$, and by the same reasoning $\psi_m(\lambda_{1-\beta,m}) = 1-\beta$ holds. In addition, $\psi_m(y)$ is a strictly decreasing and continuous function of y for nonnegative values of y. It is also invertible with respect to y. Therefore, by taking the inverse of both sides the first inequality becomes $\sum_{i \in M}\sum_{j \in C(i)} \lambda_j(i)t_{j,m}(i) \geq \lambda_{\alpha,m}$ for nonnegative λ vectors. Similarly, the other inequality becomes $\sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \leq \lambda_{1-\beta,m}$. Thus, the inequalities (3.10) and (3.11) result respectively in the constraints which are numbered with (3.13) and (3.14) of the mathematical programming problem $P_1(m)$ for which the formulation is given below.

$P_1(m)$:

$$z^*(m) = \quad min \sum_{i \in M} \sum_{j \in C(i)} c_j(i) t_{j,m}(i) \tag{3.12}$$

$$s.t.$$

$$min\{\sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) : \lambda \in \rho(R_0)\} \geq \lambda_{\alpha,m} \tag{3.13}$$

$$max\{\sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) : \lambda \in \rho(R_1)\} \leq \lambda_{1-\beta,m} \tag{3.14}$$

$$t_{j,m}(i) \geq 0 \qquad j \in C(i),\ i \in M \tag{3.15}$$

As it can be seen (3.13) is a minimization and (3.14) is a maximization problem in λ now. This change is due to the inversion of $\psi_m(y)$ with respect to y, and the desire of forcing the constraints (2.2) and (2.3) to hold for all feasible component failure rate vectors. Thus, for all the component failure rate vectors in $\rho(R_0)$ the constraint $\sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \geq \lambda_{\alpha,m}$ is forced to hold. Similarly, $\sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \leq \lambda_{1-\beta,m}$ must be satisfied for all λ in $\rho(R_1)$. This formulation is based on the assumption that $\rho(R_0)$ and $\rho(R_1)$ are nonempty. Otherwise, the formulation becomes infeasible.

An optimal solution of $P_1(m)$ is denoted as $t^*_{j,m}(i)$ $j \in C(i)$, $i \in M$. These are the minimum cost component test times for a given value of m, and $z^*(m)$ is the associated total test cost. As a result, the minimum total test cost is $z^* = z^*(m^*) = min\{z^*(m) : m \in \mathbb{N}\}$ and it is obtained by solving $P_1(m)$ parametrically with respect to m. Then the optimal component test times, which are referred to as $t^*_j(i)$ $j \in C(i)$, $i \in M$ form a solution of $P_1(m^*)$. Note that $t^*_{j,m^*}(i) = t^*_j(i)$ for any component j in the mission i by definition.

The inequalities (3.13) and (3.14) are referred as type I and type II inequalities. One may also realize that left hand sides of these inequalities consist of optimization problems one of which is a minimization and the other is a maximization in λ over $\rho(R_0)$ and $\rho(R_1)$ respectively. These optimization problems are also named as type I and type II problems. As it can be observed, type I and type II problems in their given forms take the test times of components in each mission as the coefficient of their decision variables which are component failure rates in each mission. Type I and type II problems have linear objective functions but shapes of their feasible regions depend heavily on the structure of the system reliability function $R(\lambda)$. Consequently, based on $R(\lambda)$, type I and type II problems may lie within the range of linear programming, convex programming, reverse convex programming, difference of convex (d.c.) programming and global optimization subjects. In their general form, type I and type II problems are nonconvex optimization problems with linear objective functions. Solvability of the type I and type II problems affects also the solvability of the general model.

As stated before, once the component test times in each mission are determined, type I and type II problems become two optimization problems having these component test times as coefficients in their objective functions for the component failure rates in each mission. However, since the optimum test times for each component in each mission is not known, and they are the actual decision variables of the developed model $P_1(m)$, an iterative algorithm is needed which initially finds optimum component test results of $P_1(m)$. Later on, two optimum component failure rate vectors complying with the constraints of the type I and the type II problems respectively, should be obtained by the algorithm. This algorithmic solution procedure which is initially developed by Altınel (1990) is explained in detail in the fifth section. Before preeceding, an equivalent form of $P_1(m)$ given below which is denoted as $P_1'(m)$ may help for a better understanding of the model.

$P_1'(m)$:

$$z^*(m) = \quad min\sum_{i\in M}\sum_{j\in C(i)} c_j(i)t_{j,m}(i) \tag{3.16}$$

$s.t.$

$$\sum_{i\in M}\sum_{j\in C(i)} \lambda_j(i)t_{j,m}(i) \geq \lambda_{\alpha,m} \qquad \lambda \in \rho(R_0) \tag{3.17}$$

$$\sum_{i\in M}\sum_{j\in C(i)} \lambda_j(i)t_{j,m}(i) \leq \lambda_{1-\beta,m} \qquad \lambda \in \rho(R_1) \tag{3.18}$$

$$t_{j,m}(i) \geq 0 \qquad j \in C(i),\ i \in M \tag{3.19}$$

This is a semi-infinite linear programming model, because it has infinitely many constraints, and finitely many, k_n, variables. The set of constraints (3.17) and (3.18) describe two cones each of which consists of infinitely many inequalities. In other words, the feasible set of $P_1'(m)$ or equivalently $P_1(m)$, is the intersection of these two cones described by infinitely many linear inequalities and the positive orthant.

$P_1'(m)$ or equivalently $P_1(m)$ does not have to be feasible for all m values. Feasibility with respect to m value means that it is possible to find optimum component test times for the problem $P_1(m)$. It is possible for some m values that the two cones described by (3.17) and (3.18) do not intersect. Values of the R_0 and R_1 as well as shape defined by the prior information set plays important roles on the feasibility of the m value. A sufficient condition for the feasibility is $\frac{M_{II}}{M_I} < 1$, where $M_I = \min_{\lambda\in R_0}\{\sum_{i\in M}\sum_{j\in C(i)} \lambda_j(i)\}$ and $M_{II} = \max_{\lambda\in R_1}\{\sum_{i\in M}\sum_{j\in C(i)} \lambda_j(i)\}$. Namely, the $P_1(m)$ is feasible for all the values of R_0 and R_1 leading the $M_{II} < M_I$ result. This sufficient condition is derived from the fact that $\left\{\frac{\lambda_{1-\beta,m}}{\lambda_{\alpha,m}}\right\}_{m=0}^{\infty}$ is a strictly increasing sequence converging to 1 under the assumptions of α,$\beta > 0$, and $\alpha+\beta < 1$ (Altınel, 1990). Details regarding proofs and examples for the sufficient condition is provided in (Altınel, 1994). Another interesting result is that the feasibility of the problem with respect to m has a stable nature. Namely, once $P_1(m)$ has a solution then $P_1(m')$ has also a solution for any $m' > m$, (Altınel, 1994).

As previously stated, if no prior information is available, namely the only restriction on the component failure rates is due to the system reliability constraints and the sum rule is used as a means for accepting or rejecting a system, the existence of a test plan is not guaranteed for every system topology. For instance, no test plan exists for parallel system under stated conditions. This is because type II problem becomes unbounded for any realization of the parameters like α, β, R_0 and R_1 which makes the general model infeasible for any value of m (Altınel, 1990). However, this drawback disappears if the prior information on the component failure rates obtained as a nonempty and compact subset of real numbers, simple upperbounds on component failure rates for example. The prior knowledge set limits the value of the objective function of the type II problem in this case, and does not allow any failure rate to be arbitrarily large.

4. RELIABILITY MODELS

This section contains the reliability models for the cases where there is a sequence of missions that has possibly random durations and the sequence of the missions is also possibly random. The probability of the completion of first n missions is employed as the reliability function of the models. For the derivations of the reliability models, we refer to Çekyay (2007). We only present the final reliability formulations and results. As a reminder, we should point out that we only consider series systems and serial connection of redundant subsystems with deterministic sequence of missions and series systems with Markovian sequence of missions within the scope of this thesis.

4.1. Mission Reliability for Deterministic Sequence of Missions

Suppose the sequence of the missions is deterministic. This assumption makes it easier to handle the problems. Note that, by saying deterministic sequence of missions it is not meant that duration of mission i, which is denoted by $D(i)$, is deterministic for all $i \in M$. Duration of mission i still follows a probability distribution for all $i \in M$. For convenience, we will be assuming that $D(i)$'s follow exponential distribution throughout the study.

4.1.1. Series System

By deterministic sequence of missions, it is meant that the sequence of the missions is fixed and known. Since series system is considered, all the components needed in mission i are supposed to operate without failure. Therefore, if the set of the components that are required to perform without failure during mission i is denoted by $C(i)$ and the lifetime of the system as LI, probability of completion of the mission i is given by following formula (Çekyay, 2007),

$$P_i\{LI > D(i)|D(i)\} = e^{-\sum\limits_{j \in C(i)} lambda_j(i)D(i)} \tag{4.1}$$

As stated in Çekyay (2007), by referring memoryless property, mission based reliability $R(\lambda)$ of a system having a finite number of components connected in series can be obtained as

$$R(\lambda) = P_i\{LI > T_n\} = E\left[\prod_{i\in M} e^{-\sum\limits_{j\in C(i)} \lambda_j(i)D(i)}\right] = \prod_{i\in M} L_i\left(\sum_{j\in C(i)} \lambda_j(i)\right). \tag{4.2}$$

where T_n stands for the completion time of the mission n, and $L_i(\alpha) = E\left[\exp\{-\alpha D(i)\}\right]$ is the Laplace transform of $D(i)$.

Following lemma enables the formulation of type I and type II problems. This is because shapes of the feasible regions $\rho(R_0)$ and $\rho(R_1)$ of type I and type II problems mostly depends on the reliability function $R(\lambda)$. Proof of Lemma 4.1 is given in Appendix A. Before giving the lemma, define $L_i^k(c_i^T\alpha_i) = E\left[e^{-(c_i^T\alpha_i)D(i)}D(i)^k\right]$ as Laplace transform of $c_i^T\alpha_i$ depending on natural number k which is power of $D(i)$ in the expectation.

Lemma 4.1 $R = \prod_{i=1}^{n} L_i^k(c_i^T\alpha_i)$, *multiplication of the Laplace transforms, is a convex function of* $\alpha_i = (\alpha_{i1}, ..., \alpha_{il_i}) \in \mathbb{R}_+^{l_i}$, *for all* $i = 1, ..., n$, *and for all* $c_i = (c_{i1}, ..., c_{il_i}) \in \mathbb{R}_+^{l_i}$, $k \in \mathbb{N}$, *when* $D(i)$ *is distributed exponentially with rate* $\mu(i)$.

To have a more compact form of (4.2), we again refer to Çekyay (2007), and obtain the following form of the reliability function for series systems;

$$R(\lambda) = \prod_{i\in M}\left(\frac{\mu(i)}{\mu(i) + \sum\limits_{j\in C(i)} \lambda_j(i)}\right). \tag{4.3}$$

Due to Lemma 4.1, type I problem arising for the series system with deterministic sequence of missions, becomes an easy to solve convex programming problem. This is because the reliability function (4.3) is a convex function of the failure rates. Then the constraint $R(\lambda) \leq R_0$ existing in the body of type I problem, defines a convex set. Then, the intersection of the area defined by system reliability constraint $R(\lambda) \leq R_0$, and the prior information Γ, which is the feasible region type I problem stated in equation (3.5), is a convex set. We are to minimize a linear function over a convex region, which is a relatively easy task to accomplish. The solution method employed for the solution of type I problem is given in chapter 5 in detail.

On the other hand, type II problem becomes a reverse convex optimization problem due to the $R(\lambda) \geq R_1$ constraint of type II problem. This problem is also known as a d.c. programming problem in the canonical form (Horst *et al.*, 2000). Many new methods has been developed in recent years for the solution of the problems having linear objective functions and feasible regions that is intersection of a convex set and a region defined by a reverse convex constraint (Tuy, 1995; Horst *et al.*, 2000; Horst and Tuy, 1996). The chosen method for the solution of type II problem is also explained in detail in chapter 5.

4.1.2. Serial Connection of Redundant Subsystems

In this model, system includes serial connection of redundant subsystems instead of single components. By "redundant subsystem" it is meant a system with components that are connected in a parallel. A subsystem of such a system works until all of its components fail and the whole system fails as soon as one of its subsystems fail. Referring again Çekyay (2007) reliability of such a system can be expressed as:

$$R(\lambda) = \prod_{i=1}^{n} R_i(\lambda) = \prod_{i=1}^{n} E\left[\prod_{j \epsilon C(i)} (1 - (1 - e^{-\lambda_j(i) D(i)})^{n_j(i)})\right] \tag{4.4}$$

where $n_j(i)$ stands for the number of components of the subsystem j during mission i, and $C(i)$ for set of the subsystems that are required to stay working during mission i. We refer the reader to Çekyay (2007) for details of the derivation of this formulation and mission based reliability functions of many different systems in general. Following lemma is also due to Çekyay (2007), and it helps the analysis of the reliability function (4.4).

Lemma 4.2 *(Çekyay (2007)) For any* $f_0, f_1, ..., f_k \in \mathbb{R}$ *and* $k \geq 0$

$$\prod_{j=0}^{k}(1 - f_j) = \sum_{t_0=0}^{1}...\sum_{t_k=0}^{1}(-1)^{t_0+...+t_k} f_0^{t_0}...f_k^{t_k}. \tag{4.5}$$

By using Lemma 4.2, following form of the $R_i(\lambda)$, which is equal to the probability of completion of mission i, is obtained as,

$$R_i(\lambda) = \sum_{t_1=0}^{1}...\sum_{t_{l_i}=0}^{1}(-1)^{t_1+...+t_{l_i}} E[f_1^{t_1}...f_{l_i}^{t_{l_i}}] \tag{4.6}$$

where l_i is the magnitude of $C(i)$ which is the set of subsystems that are required to operate without failure during mission i. Note that $f_j = (1 - e^{-\lambda_j(i)D(i)})^{n_j(i)}$ holds.

At that point, one can make use of the mathematical equations $f_j = (1 - e^{-\lambda_j(i)D(i)})^{n_j(i)} = \sum_{r_j=0}^{n_j(i)} \binom{n_j(i)}{r_j}(-1)^{r_j} e^{-r_j\lambda_j(i)D(i)}$. Then it becomes possible to get rid of the expectation term that exists in the body of equation (4.6). For instance, while for $t_{s_1}, ..., t_{s_\alpha} = 1$ and all others are zero, following equation concerning the expectation can be obtained;

$$E[f_{s_1}...f_{s_\alpha}] = \sum_{r_{s_1}=0}^{n_{s_1}(i)}...\sum_{r_{s\alpha}=0}^{n_{s_\alpha}(i)} \binom{n_{s_1}(i)}{r_{s_1}}...\binom{n_{s_\alpha}(i)}{r_{s\alpha}}(-1)^{r_{s_1}+...+r_{s_\alpha}} L_i(r_{s_1}\lambda_{s_1}(i)+...+r_{s_\alpha}\lambda_{s_\alpha}(i)) \tag{4.7}$$

Due to Lemma 4.1, Laplace transformation term in the body of the expectations are convex functions of λ, because the term is also a special form of the multiplication of the Laplace transforms. Because of the coefficient of Laplace terms can be negative and positive depending on the power of -1 existing in equation (4.7), $R_i(\lambda)$ becomes a linear combination of convex functions. Because some terms have negative coefficients $R_i(\lambda)$ is clearly a d.c. (difference of convex) function. Note that the summation of convex functions is convex. We also know that the overall system reliability function $R(\lambda)$ is the multiplication of the $R_i(\lambda)$'s. Then, $R(\lambda)$ is a function that is obtained by multiplying d.c. functions. However, it is also a known fact that multiplication of d.c. functions is itself a d.c. function (Horst and Tuy, 1996). Therefore, $R(\lambda)$ is a d.c. function as well.

Consequently, type I problem becomes a d.c. programming problem with a linear objective function and a feasible region defined by a d.c. set. Similar to type I problem, type II problem is also a d.c. programming problem. This is because the constraint $R(\lambda) \geq R_1$ can be replaced with $-R(\lambda) \leq -R_1$, and $-R(\lambda)$ is also a d.c. function (Horst and Tuy, 1996). Therefore, both of the type I and type II problems of the serial connection of redundant subsystems are both d.c. optimization problems. Many converging algorithms have been offered for the solution of d.c. optimization problems. The reader is referred to Tuy (1995), Horst and Tuy (1996), Horst *et al.* (2000), and references therein. The method we choose is explained in Chapter 5.

4.2. Mission Reliability for a Series System with a Markovian Sequence of Missions

For systems such that the order in which the missions will proceed is Markovian, the reliability functions are significantly different than their equivalent ones with deterministic sequences. If the mission is accepted as the state of the underlying Markov process, and the transition matrix of the Markov process lying behind the system is denoted by P, then transition rates are stated as $\{\mu(i);\ i \in M\}$ where M denotes the set of missions (states for this case).

Before proceeding to the mission reliability analysis of such systems, there is a need for new terminology definitions. First of all, let $\tilde{P}$ be a new transition matrix, which is necessary for the reliability formulations of the systems with Markovian missions. The terms of the new Markovian transition matrix $\tilde{P}$ are obtained by multiplicating the corresponding term of the transition matrix P and the probability of completing the mission corresponding to the row on hand. To be more specific, $\tilde{P}(i,x)$ equals multiplication of $P(i,x)$ by the probability of completing the mission i for all $x \in M$ (Çekyay, 2007). To have a consistent transition matrix with sum of the rows equal to 1, a new column added to the end of the matrix, such that $\tilde{P}(i,\Delta) = 1 - \sum_{x \in M} \tilde{P}(i,x)$ where Δ states for the added column. On the other hand, probability of completing mission i is simply equal to their equivalent forms with deterministic missions.

Making use of new terminology, for systems with Markovian sequence of missions, the general reliability function form depending on the initial state i is given as follows:

$$R_i(\lambda) = \sum_{x \in M} \tilde{P}^n(i,x), \tag{4.8}$$

by Çekyay (2007).

By referring equation (4.8), and the reliability formulation for mission i given in (4.3), one can easily construct the transformed matrix $\tilde{P}$ and determines the reliability function for series systems with Markovian sequence of missions by using it. For instance for all $i, x \in M$,

$$\tilde{P}(i,x) = P(i,x)L_i(\sum_{j \in M} \lambda_j(i)) = \frac{P(i,x)\mu(i)}{\mu(i) + \sum_{j=1}^{n} \lambda_j(i)} \tag{4.9}$$

holds, where L stands for the Laplace transform, and duration of mission i follows from an exponential distribution with rate $\mu(i)$. Elements of $\tilde{P}$ is then equal to the multiplication of a constant term with a function which is known to be convex due to Lemma 4.1. One can easily obtain also that the terms of $\tilde{P}^n$ are multiplications of the Laplace of linear terms and constants. Then, the mission reliability of the system given by (4.8) contains summation of the positive linear combination of the multiplications of the Laplace transforms of linear terms for a series system. However, we know from Lemma 4.1 that multiplication of the Laplace transformation of linear terms is convex. Hence, summation of the positive linear combinations of the multiplication of the Laplace transforms of linear terms is also convex. Therefore, $R_i(\lambda)$ turns out to be a convex function of the failure rates for the series system with Markovian missions, as in its deterministic case.

Similar to what we have for series systems with deterministic sequence of missions, we have a convex optimization problem for type I, and a reverse convex, or canonical d.c. programming problem for type II problem. Thus, the solution methods which are explained in a detailed manner in Chapter 5 employed for the deterministic case remains still valid for the Markovian case for series systems.

5. SOLUTION PROCEDURE

In this section, we provide solution procedures for the general semi-infinite linear program given by equations (3.16)-(3.19) and subproblems that exist in the body of the general model. We consider series systems with both deterministic and Markovian sequence of missions and serial connection of redundant subsystems with deterministic sequence of missions.

5.1. Semi-Infinite Linear Program

Let F_I and F_{II} be two index sets including some feasible failure rate vectors from $\rho(R_0)$ and $\rho(R_1)$. In other words, any failure rate vector with an index from F_I is feasible with respect to type I problem and any failure rate vector with an index from F_{II} is feasible with respect to type II problem. Therefore, the labeled failure rate vector f_g^I belongs to $\rho(R_0)$ if $g \in F_I$ and it belongs to $\rho(R_1)$ if $g \in F_{II}$. Let $P_2(m)$ be the following primal linear program and $D(m)$ its dual.

$P_2(m)$:

$$z^*_{P_2}(m) = \quad min \sum_{i \in M} \sum_{j \in C(i)} c_j(i) t_{j,m}(i) \tag{5.1}$$

$s.t.$

$$\sum_{i \in M} \sum_{j \in C(i)} f^I_{gj}(i) t_{j,m}(i) \geq \lambda_{\alpha,m} \qquad g \in F_I \qquad \text{(Dual variable } \pi_g) \tag{5.2}$$

$$\sum_{i \in M} \sum_{j \in C(i)} f^{II}_{gj}(i) t_{j,m}(i) \leq \lambda_{1-\beta,m} \qquad g \in F_{II} \qquad \text{(Dual variable } \delta_g) \tag{5.3}$$

$$t_{j,m}(i) \geq 0 \qquad j \in C(i),\ i \in M \tag{5.4}$$

$D(m)$:

$$z^*_D(m) = \max \lambda_{\alpha,m} \sum_{g \in F_I} \pi_g - \lambda_{1-\beta,m} \sum_{g \in F_{II}} \delta_g \tag{5.5}$$

s.t.

$$\sum_{g \in F_I} f^I_{gj}(i)\pi_g - \sum_{g \in F_{II}} f^{II}_{gj}(i)\delta_g \leq c_j(i) \qquad j \in C(i),\ i \in M \tag{5.6}$$

$$\pi_g \geq 0 \quad g \in F_I \tag{5.7}$$

$$\delta_g \geq 0 \quad g \in F_{II} \tag{5.8}$$

If F_I and F_{II} are obtained as finite and if they contain the component test times which solves $P_1(m)$ to optimality, then $P_2(m)$ or its dual $D(m)$ can be solved to compute optimum test times instead of solving $P_1(m)$given by (3.12)-(3.15).

The algorithmic idea which is quite simple depends on this argument and combines two well known methods which are cutting plane and column generation methods. Algorithm starts with empty F_I and F_{II}, or equivalently unconstrained $P_2(m)$, and continues to generate new columns until a solution that is arbitrarily close to optimal solution is obtained. In other words, in each step new failure rate vectors from $\rho(R_0)$ and $\rho(R_1)$ added to F_I and F_{II} until a failure rate vector that is approximately close to the failure rate vector leading to optimal component test times in (3.12). Because the addition of a failure rate vector to F_I or/and F_{II} increases the constraint number of $P_2(m)$, $P_2(m)$ can have a very large constraint set. Thus, it is preferable to solve $D(m)$ by using revised simplex method, which can increase the efficiency. Recall that $c_j(i) \geq 0$ for all $j \in C(i)$, $i \in M$. Then by letting s_j denote the slack variable for the row j of $D(m)$ it can be seen that $s_j = c_j$ for all j, $\pi_g = 0$ for any index $g \in F_I$ and $\delta_g = 0$ for any index $g \in F_{II}$ is a basic feasible solution for any given value of m. In other words, $D(m)$ is feasible for any m, and $k_n \times k_n$ identity matrix can be a starting basis for any given value of m.

Since we have a minimization problem at hand, the simplex algorithm stops if and only if $z_j - c_j \geq 0$ for all nonbasic columns of $D(m)$, or equivalently $min\{z_j - c_j : \, for \; every \; nonbasic \; j\} \geq 0$. Note that $z_j - c_j$ is the reduced cost associated with the j^{th} nonbasic variable namely j^{th} nonbasic column.

Assume that F_I and F_{II} are bounded. Since nonbasic columns are either from F_I or from F_{II}, and since $c_j = \lambda_{\alpha,m}$ for all j in F_I and $c_j = -\lambda_{1-\beta,m}$ for all j in F_{II}, by denoting an optimal dual solution of $D(m)$ by $w^*_{j,m}(i)$, and by using the fact that $w^*_{j,m}(i) = t^*_{j,m}(i)$ for all $j \in C(i), i \in M$, stopping condition can be equivalently written as;

$$\left(\min_{g \in F_I} \sum_{i \in M} \sum_{j \in C(i)} t^*_{j,m}(i) f^I_{gj}(i) \geq \lambda_{\alpha,m} \; and \; \max_{g \in F_{II}} \sum_{i \in M} \sum_{j \in C(i)} t^*_{j,m}(i) f^{II}_{gj}(i) \leq \lambda_{1-\beta,m} \right). \quad (5.9)$$

Therefore, in order not to include only the nonbasic columns from F_I and F_{II} but all possible nonbasic columns which are to be generated from the two feasible failure rate sets $\rho(R_0)$ and $\rho(R_1)$, stopping condition can be modified to;

$$\left(\min_{\lambda \in \rho(R_0)} \sum_{i \in M} \sum_{j \in C(i)} t^*_{j,m}(i) \lambda_j(i) \geq \lambda_{\alpha,m} \; and \; \max_{\lambda \in \rho(R_1)} \sum_{i \in M} \sum_{j \in C(i)} t^*_{j,m}(i) \lambda_j(i) \leq \lambda_{1-\beta,m} \right). \quad (5.10)$$

As can be observed this stopping criterion forces the result of the solution procedure to comply with the system reliability constraints (3.13) and (3.14) of the original formulation $P_1(m)$.

According to that stopping condition, one has to solve two optimization problems with respect to failure rates, whose coefficients of the decision variables in the objective function are the component test times which currently solve $P_2(m)$ to optimality.

This optimization procedure is illustrated in Figure 5.1 below. We define $P_2^{(h)}(m)$ and $D^{(h)}(m)$ as linear program $P_2(m)$ and its dual $D(m)$ at the iteration h of the algorithm, and $z^*_{P_2^{(h)}}(m)$ and $z^*_{D^{(h)}}(m)$ as their objective values. Denote also the optimum objective values of the type I and type II problems at iteration h as $z^*_{I,h}(m)$ and $z^*_{II,h}(m)$ and their objective coefficients as;($w^*_{1,m}(1)$, $w^*_{2,m}(1)$,..., $w^*_{l_1,m}(1)$; $w^*_{1,m}(2)$, $w^*_{2,m}(2)$, ..., $w^*_{l_2,m}(2)$; ... ; $w^*_{1,m}(n)$, $w^*_{2,m}(n)$, ..., $w^*_{l_n,m}(n)$), which is an optimal dual solution of $D(m)$ or equivalently the component times; ($t^*_{1,m}(1)$, $t^*_{2,m}(1)$, ..., $t^*_{l_1,m}(1)$; $t^*_{1,m}(2)$, $t^*_{2,m}(2)$, ..., $t^*_{l_2,m}(2)$; ...; $t^*_{1,m}(n)$, $t^*_{2,m}(n)$, ..., $t^*_{l_n,m}(n)$) that solve the $P_2(m)$ to optimality. Here, l_i stands for the magnitude of the set $C(i)$, e.g., $l_i = |C(i)|$. Finally we let f^I_h and f^{II}_h be the optimal solutions of type I and type II problems. They are the new columns generated at iteration h in order to update B_h^{-1}, which denotes the inverse of the basic matrix B_h of $D^{(h)}(m)$. Then, based on the above definitions,

$$\begin{aligned}
z^*_{I,h}(m) &= \min_{\lambda\in\rho(R_0)} \sum_{i\in M}\sum_{j\in C(i)} w^*_{hj,m}(i)\lambda_j(i) \\
&= \min_{\lambda\in\rho(R_0)} \sum_{i\in M}\sum_{j\in C(i)} t^*_{hj,m}(i)\lambda_j(i) \\
&= \sum_{i\in M}\sum_{j\in C(i)} t^*_{hj,m}(i) f^I_{hj}(i)
\end{aligned}$$

and

$$\begin{aligned}
z^*_{II,h}(m) &= \max_{\lambda\in\rho(R_1)} \sum_{i\in M}\sum_{j\in C(i)} w^*_{hj,m}(i)\lambda_j(i) \\
&= \max_{\lambda\in\rho(R_1)} \sum_{i\in M}\sum_{j\in C(i)} t^*_{hj,m}(i)\lambda_j(i) \\
&= \sum_{i\in M}\sum_{j\in C(i)} t^*_{hj,m}(i) f^{II}_{hj}(i)
\end{aligned}$$

At iteration h, if $D^{(h)}(m)$ is bounded and the condition $z^*_{I,h}(m) \geq \lambda_{\alpha,m}$ and $z^*_{II,h}(m) \leq \lambda_{1-\beta,m}$ holds then $t^*_{j,m}(i)$ for all $j \in C(i)$, $i \in M$ is an optimal solution of $P_2(m)$ and $P_1(m)$; thus, the algorithm stops with $z^*_{D^{(h)}}(m) = z^*_{P_2^{(h)}}(m) = z^*(m)$. If $D^{(h)}(m)$ is bounded and either $z^*_{I,h}(m) < \lambda_{\alpha,m}$ or $z^*_{II,h}(m) > \lambda_{1-\beta,m}$, or both are true, then either $\sum_{i\in M}\sum_{j\in C(i)} t^*_{hj,m}(i) f^I_{hj}(i) \geq \lambda_{\alpha,m}$ or $\sum_{i\in M}\sum_{j\in C(i)} t^*_{hj,m}(i) f^{II}_{hj}(i) \leq \lambda_{1-\beta,m}$, or both are violated, and the basis inverse B_h^{-1} is updated by pivoting on f^I_h or on f^{II}_h, General Column Generation Algorithm Input: R_0, R_1, $\lambda_{\alpha,m}$, $\lambda_{1-\beta,m}$, c,u, ε

Output: $t^*_{j,m}(i)$ $j \in C(i)$, $i \in M$ or "INFEASIBLE m" message

begin

1. $w^*_{1j,m}(i) \leftarrow 0$ $j \in C(i)$, $i \in M$, $B_1^{-1} \leftarrow I_{(k_n \times k_n)}$, $z^*(m) \leftarrow 0$, $h \leftarrow 1$;
2. $z^*_{I,h}(m) \leftarrow \min\limits_{\lambda \in \rho(R_0)} \sum\limits_{i \in M} \sum\limits_{j \in C(i)} w^*_{hj,m}(i)\lambda_j(i)$ And call the optimum solution f_h^I ;
3. $z^*_{II,h}(m) \leftarrow \max\limits_{\lambda \in \rho(R_1)} \sum\limits_{i \in M} \sum\limits_{j \in C(i)} w^*_{hj,m}(i)\lambda_j(i)$ And call the optimum solution f_h^{II} ;
4. if ($z^*_{I,h}(m) \geq \lambda_{\alpha,m}$ and $z^*_{II,h}(m) \leq \lambda_{1-\beta,m}$) then STOP, $w^*_{hj,m}(i)$ $j \in C(i)$, $i \in M$ are the optimum test times and $z^*_{D^{(h)}}(m)$ is the minimum total test cost for this value of m ;
5. else

    begin

        update B_h^{-1} with f_h^I and f_h^{II} as two new columns;

        update dual solution $w^*_{hj,m}(i)$ $j \in C(i)$, $i \in M$;

        if $D^{(h)}(m)$ is UNBOUNDED then STOP, and output "INFEASIBLE m" message

        else $h \leftarrow h + 1$ go to 2 ;

    end;

end;

Figure 5.1. General Column Generation Algorithm

or on both. Consequently, it is possible to see that if General Column Generation Algorithm stops in finitely many iterations either the infeasibility of $P_1(m)$ is detected or an optimal solution is computed. It can be shown that if the algorithm does not stop in finitely many steps then the sequence $\{w^*_{h,m}\}_{h=1}^{\infty}$, generated by solving the dual problem $D^{(h)}(m)$ at each iteration h, converges to an optimal solution of $P_1(m)$, i.e., to t^*_m (Altınel, 1994). Therefore it is possible to stop the algorithm in finite iterations by replacing the stopping condition of step 4 with the following ε-perturbed one, which we refer as "ε-stopping condition": If $(z^*_{I,h}(m) \geq \lambda_{\alpha,m} - \varepsilon$ *and* $z^*_{II,h}(m) \leq \lambda_{1-\beta,m} + \varepsilon)$ then STOP. Here ε is an arbitrarily small positive number.

Following theorem initially stated with its proof in Altınel (1990) for systems with one mission, indicates the correctness of the General Column Generation Algorithm. It can be easily generalized to the situation where there are more than one missions. Hence, the proof is not given.

Theorem 5.1 *(Altınel, 1990) General Column Generation Algorithm either detects the infeasibility of $P_1(m)$ for the given data or computes a sequence of component test times which eventually converges to a set of component test times solving $P_1(m)$ to optimality.*

Finally, last step of the optimization procedure is the search for the optimum value of the m. It is known that optimal objective value of $max\{z = c^T x : Ax \leq b,\, x \geq 0\}$ is a piecewise and a convex function of the cost vector c and a concave function of the requirement vector b due to Charnes and Cooper (1962). Then, the optimal objective value of dual problem $D(m)$ is a convex function of $\lambda_{\alpha,m}$ and $\lambda_{1-\beta,m}$. Unfortunately, $\lambda_{\alpha,m}$ and $\lambda_{1-\beta,m}$ can be any discrete function of m. However, they can be efficiently approximated by two linear functions in m as $\lambda_{\alpha,m} \approx p_1 + q_1 m$ and $\lambda_{1-\beta,m} \approx p_2 + q_2 m$, with $q_1 > 0$ and $q_2 > 0$ (Altınel, 1990). After this approximation, $z^*(m)$ becomes a convex function of m. Consequently, it becomes possible to search for the first value of m for which $z^*(m) < z^*(m+1)$ starting from $m = 0$. Note that General Column Generation Algorithm can be employed to find $z^*(m)$ values. We can assume $z^*(m) = \infty$ for any value of m, $D(m)$ is unbounded, or equivalently $P_2(m)$ is infeasible, by convention. Although this strategy does not guarantee the optimal solution, it does guarantee an approximate solution.

5.2. Sub-Problems

If the structure of General Column Generation Algorithm is analyzed, it is easy to see that type I and type II problems are to be solved many times in Step 2 and Step 3 of the algorithm. However, these problems may have difficult structures depending on the system at hand, and the mission reliability function of the system . Hence, it is a good idea to concentrate on the stuctures of these problems. Consequently, general structures of type I problem depending on the system reliability function $R(\lambda)$ can be given as

$$z_I^*(m) = \quad min \sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \tag{5.11}$$

$$s.t.$$

$$R(\lambda) \ \leq R_0 \tag{5.12}$$

$$0 \leq \lambda_j(i) \leq u_j(i) \qquad j \in C(i),\ i \in M \tag{5.13}$$

while type II problem's structure is

$$z_{II}^*(m) = \quad \max \sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \tag{5.14}$$

$$s.t.$$

$$R(\lambda) \ \geq R_1 \tag{5.15}$$

$$0 \leq \lambda_j(i) \leq u_j(i) \qquad j \in C(i),\ i \in M. \tag{5.16}$$

Observe that the decision variables of type I and type II problems are component failure rates in each mission whereas the coefficients of failure rates are the test times of components in each mission. Moreover, type I and type II problem structures are related with the system structure because of the reliability constraints, (5.12) and (5.15). Depending on the system reliability function $R(\lambda)$ of the system at the moment, type I and type II problems may lie within the subjects of the linear programming, convex programming, reverse convex programming or global optimization. For instance if the system reliability function $R(\lambda)$ is a convex function of λ, (5.12) defines a convex region whereas (5.15) indicates a concave, or reverse convex area. Thus, even for the same model, the structure of type I and type II problems may differ, and the solution methods applied for type I problem may become invalid for type II problem. The system reliability also depends on whether the missions follow a deterministic or Markovian sequence. In the following sections, solution methods of type I and type II problems for several systems are presented. For deterministic sequence of missions, the series systems and serial connection of redundant subsystems are considered, while for the Markovian case the series system is handled.

5.2.1. Deterministic Sequence of Missions

Series System. In chapter 4, the reliability function (4.2) for a series system with deterministic sequence of missions is found to be convex due to lemma 4.1. If the more compact form of reliability function given by (4.3) is considered, type I problem becomes as follows;

$$z_I^*(m) = \quad min \sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \tag{5.17}$$

$$s.t.$$

$$\prod_{i \in M} \left(\frac{\mu(i)}{\mu(i) + \sum_{j \in C(i)} \lambda_j(i)} \right) \leq R_0 \tag{5.18}$$

$$0 \leq \lambda_j(i) \leq u_j(i) \qquad j \in C(i), \; i \in M \tag{5.19}$$

Since system reliability function (i.e., left-hand side of (5.18)) is convex, the feasible set (5.18)- (5.19) is a convex set (Rockafellar, 1970). Feasible region of type I problem is

$$\rho(R_0) \equiv \{\lambda_j(i) \in \mathbb{R}_+^{k_n} : \prod_{i \in M} \left(\frac{\mu(i)}{\mu(i) + \sum_{j \in C(i)} \lambda_j(i)} \right) \leq R_0, \;\; 0 \leq \lambda_j(i) \leq u_j(i), \;\; j \in C(i), i \in M\} \tag{5.20}$$

where prior information set Γ is given by constraint (5.19).

Hence, a linear objective function is to be minimized over a convex set. There has been accumulated a vast literature about the convex optimization problems. For instance, (Bazaraa *et al.*, 1993; Bertsekas, 1999) includes several methods. For the general case, we refer to Sequential Quadratic Programming (SQP) (MATLAB

Optimization User Guide, 2000). SQP methods presents the state-of-the-art for convex programming solution techniques. For instance, Schittkowski (1986) provides a performance and comparison analysis in terms of accuracy, efficiency and percentage of successful solutions over a large number of test problems. He proves that SQP outperforms over all other methods with respect to mentioned criteria. We use SQP implementation of Matlab (MATLAB 6.5 User's Guide, 2002) in our experiments.

On the other hand, type II problem given below poses a structure requiring a very different technique for its solution.

$$z^*_{II}(m) = \quad max \sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \tag{5.21}$$

$$s.t.$$

$$\prod_{i \in M} \left(\frac{\mu(i)}{\mu(i) + \sum_{j \in C(i)} \lambda_j(i)} \right) \geq R_1 \tag{5.22}$$

$$0 \leq \lambda_j(i) \leq u_j(i) \qquad j \in C(i),\ i \in M \tag{5.23}$$

Constraint (5.22) is a reverse convex inequality. This fact complicates the solution procedure and makes the methods employed for the solution of type I problem invalid for type II problem case. Problems with linear objective functions and feasible regions defined by a reverse convex and some convex constraints, are of the special optimization problem forms evaluated under global optimization general title, and named as Linear Reverse Convex Programming Problems (LRCP). Before preeceding to the algorithmic solution procedure that we offer for the solution of type II problem, we believe it is beneficial to briefly expose the topics and tools of LRCP problems.

An LRCP problem is also a difference of convex (d.c.) optimization problem which is in canonical form. Canonical d.c. problem which is abbreviated as CDC problem, in general is an optimization problem of the form (Horst *et al.*, 2000)

$CDC:$

$$
\begin{aligned}
z^*_{CDC} = \quad & min\ c^T x && (5.24)\\
s.t. \quad & \\
& g(x) \geq 0 && (5.25)\\
& x \in D && (5.26)
\end{aligned}
$$

where $c \in \mathbb{R}^n$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a convex function and D is a closed convex subset of $\mathbb{R}^n$.

Let the feasible region of CDC be denoted by $F = \{x \in D : g(x) \geq 0\}$. In the sequel, it will frequently be required that problem CDC fulfills some of the following natural additional assumptions (Horst *et al.*, 2000):

Assumption 5.1 *D is compact and $intD \neq \varnothing$*

Assumption 5.2 *There exists a point $x^0 \in D$ satisfying $g(x) \leq 0$ and $c^T x^0 < min\left\{c^T x : x \in D, g(x) \geq 0\right\}$.*

This assumption ensures that the reverse convex constraint $g(x) \leq 0$ is essential, because if Assumption 5.2 is not satisfied, CDC problem will be equivalent to following convex optimization problem

$$
\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & && (5.27)\\
& x \in D
\end{aligned}
$$

which can be easily solved to optimality by applying the same method that solves the type I problem.

One final assumption presented below concerns the robustness of the feasible set

$F = \{x \in D : g(x) \geq 0\}$.

Assumption 5.3 $F = cl(intF)$. *In other words, the feasible set* F *equals closure of its interior set, namely it is robust.*

This assumptions provides that CDC has a certain regularity ensuring F has full dimension. In other words, the case where intersection of the sets $\{x : g(x) \geq 0\}$ and D is part of boundary of D, is excluded from the consideration.

In addition to Assumptions 5.1, 5.2 and 5.3, two more definitions are required to proceed to Theorem 5.2 due to CDC theory (Horst *et al.*, 2000), which contructs the idea that we base the algorithmic solution procedure of type II problem on. Let $G = \{x : g(x) \leq 0\}$ and let ∂A denote the boundary of a set $A \subset \mathbb{R}^n$.

Theorem 5.2 *In the CDC program, assume that* D *is bounded,* F *is nonempty and the reverse convex contraint* $g(x) \geq 0$ *is essential. Then there exists an optimal solution on the intersection* $\partial D \cap \partial G$ *of the boundaries of* D *and* G*. Moreover, when* D *is a polytope, there exists an optimal solution on the intersection of an edge of* D *with the boundary of* G*.*

In addition, following two propositions state the necessary and sufficient conditions for a vector $\bar{x} \in \mathbb{R}^n$ to be an optimal solution of CDC problem (Horst *et al.*, 2000).

Proposition 5.1 *(Necessary Optimality Condition). In problem CDC, let the Assumptions 5.1 and 5.2 be satisfied. Then every optimal solution* $\bar{x}$ *of CDC satisfies*

$$\max\{g(x) : x \in D, c^T x \leq c^T \bar{x}\} = 0. \tag{5.28}$$

Proposition 5.2 *(Sufficient Optimality Condition). Assume that in problem CDC, the feasible set F is robust (Assumption 5.3 is satisfied) and that a point x^0 satisfying $g(x^0) < 0$ exists. Let $\bar{x} \in F$ and $S \supseteq F$ such that*

$$\max\{g(x) : x \in S, c^T x \leq c^T \bar{x}\} = 0. \tag{5.29}$$

Then $\bar{x}$ is an optimal solution of CDC.

It is easy to observe that, type II problem can be turned into a CDC problem if the objective function (5.21) is replaced with

$$z^*_{II}(m) = \quad -min \sum_{i \in M} \sum_{j \in C(i)} -\lambda_j(i) t_{j,m}(i). \tag{5.30}$$

Therefore, the closed convex subset D of CDC corresponds to the prior information Γ of type II problem, and reverse convex constraint $g(x) \geq 0$ corresponds to (5.22). Moreover, if the prior information is in the form of upperbounds as given by (5.23), then the closed convex set also becomes a polytope. Hence, due to Theorem 5.2 optimal solution of the problem lies on an edge of the polytope defined by $\Gamma = \{\lambda_j(i) \in \mathbb{R}^{k_n}_+ : \lambda_j(i) \leq u_j(i) \quad j \in C(i),\ i \in M\}$ and on the boundary of the set $G = \{\prod_{i \in M} (\frac{\mu(i)}{\mu(i) + \sum_{j \in C(i)} \lambda_j(i)}) \ - R_1 \leq 0\}$.

This observation makes it possible to come up with an algorithmic procedure that simply enamurates all the points that lie both on at least one edge of Γ and on the boundary of G. Then, an edge following algorithm given below due to an earlier algorithm of Altınel (1990), guarantees to find the optimal solution of type II problem under Assumptions 5.1, 5.2 and 5.3. Algorithm identifies the edge of Γ and checks whether the edge intersects with the boundary of G or not, and finds the point of the edge that intersects the boundary of G by a simple line seach technique if there is an intersection. Before giving the algorithm, in order to be sure that the algorithm is an appropriate tool for type II problem, an analysis of type II problem in terms of satisfying the assumptions is made below.

Assumption 5.1 is surely satisfied unless all of the upperbounds $u_j(i)$'s are set to zero which is clearly an exceptional situation. One can also easily observe that Assumption 5.2 should also be satisfied for every practical situation. Briefly, for real life situations people expect that the acceptable reliability level R_1 should be high enough such that the reliability level attained to upperbounds on the failure rates are less than this value. In mathematical words, $R(u) \leq R_1$ should hold where u is the vector of upperbounds on failure rates which is a reasonable assumption. However, if this holds then Assumption 5.2 also holds since the optimum point within Γ that minimizes the (5.30) is simply the one which is the largest in magnitude, which is u, and u does not satisfy constraint (5.22). Finally, Assumption 5.3 holds for Γ because it is closed and convex by definition implying it is robust. Therefore, the edge following algorithm submitted in Figure 5.2 below can be employed for the solution of the type II problem.

Serial Connection of Redundant Subsystems. The situation is completely different for the serial connection of redundant subsystems. System reliability function $R(\lambda)$, is found to be a d.c. function because it is previously stated that the system reliability function (4.4) is consist of linear combination of multiplication of linear laplace transforms. By Lemma 4.1, it has been proven that the multiplication of Laplace transform of linear functions is a convex function. Therefore, the reliability function (4.4) actually includes linear combination of convex functions. It is a well known fact from convexity theory (Rockafellar, 1970) that positive linear combination of convex functions result in a convex function. Therefore, because some of the coefficients of the Laplace transforms are negative in equation (4.7), some of the coefficients of the Laplace transform multiplications in overall system reliability function (4.4) will be negative as well. Then, one can separate the terms having negative coefficients and the ones having positive coefficients, end up a d.c. function in which one of the convex function is the total of convex terms having positive coefficients. The other convex function is simply the total of the convex terms having negative coefficients, and since this begin

1. Set $f_j(i) = 0$ for $\forall j \in C(i), i \in M$;

2. Initialize the vertex list L by inserting f and set the current value of the objective function $z = +\infty$;

3. while $L \neq \varnothing$ do {

Delete the front entry of the list L and call it current search vertex v;

for $j \in C(i)$ do; {

for $i \in M$ do; {

If $v_j(i) = 0$ {

Increase $v_j(i)$ untill $R(v) - R_1 = 0$ holds, and name the value of $v_j(i)$ as step;

Set $v_j(i) = max\{step, u_j(i)\}$

If $R(v) - R_1 > 0$ { then v is a vertex which is strictly above the surface $R(v) - R_1 = 0$ and feasible. Therefore, it can be an endpoint of an edge which intersects this surface and it must be searched. Insert v at the end of list L. }

Else {

Evaluate the objective function $Q(\lambda)$ at point v, namely compute $Q(v)$

If $Q(v) < z$ { then

Save v as the current best solution;

Set $z = Q(v)$; } } } } } }

4. The current best vertex is the global optimum solution f^{II} and z is the optimum objective value $Q(f^{II})$

end;

Figure 5.2. Edge Following Algorithm

convex function is extracted from the other one, resulting system reliability function is a d.c. function. To have a clearer identification, it would be beneficial to formulate these ideas mathematically. For instance let the convex terms having negative coefficients are $g'_1(\lambda), g'_2(\lambda), ..., g'_{m_1}(\lambda)$, and the convex terms having pozitive coefficients are $h'_1(\lambda), h'_2(\lambda), ..., h'_{m_2}(\lambda)$. Then the system reliability function can be equivalently written as,

$$R(\lambda) = h'(\lambda) - g'(\lambda), \tag{5.31}$$

where $h'(\lambda) = \sum_{e=1}^{m_2} h'_e(\lambda)$ and $g'(\lambda) = \sum_{f=1}^{m_1} g'_f(\lambda)$. Clearly, $h', g' : \mathbb{R}^{k_n} \longrightarrow \mathbb{R}$ are convex functions. This representation of the reliability function which is based on the convex functions g' and h', is called as d.c. decomposition of $R(\lambda)$ (Horst *et al.*, 2000).

Because we have a d.c. reliability function $R(\lambda)$, we will be dealing with d.c. type I and d.c. type II problems. Hence, before preeceding further, it would be beneficial to state the general form of the d.c. optimization problems. Difference of convex optimization problems are in general defined as the optimization problems in which a d.c. function is to be minimized under d.c. constraints. The generic form of a D.C. Problem (DCP) is

$DCP:$

$$z^*_{DCP} = \quad min\ f_0(x) \tag{5.32}$$

$$s.t.$$

$$x \in D \tag{5.33}$$

$$f_i(x) \leq 0 \quad i = 1, ..., m \tag{5.34}$$

where z_{DCP} is the optimum solution of DCP problem, D is a closed convex subset of $\mathbb{R}^n$, and all functions $f_i,\ i = 0, 1, .., m$ are d.c. functions. Observe that type I and type II problems arising for the serial connection of redundant subsystems are d.c. optimization problems because $R(\lambda)$ is a d.c. function, and the linear objective function is a special type d.c. function. Moreover, the convex area defined by Γ corresponds to D within the body of CDP. Then, based on the d.c. decomposition of $R(\lambda)$, one can obtain following forms of type I and type II problems both of which fit in DCP form.

Type I :

$$z_I^*(m) = \quad min \sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \tag{5.35}$$

$$s.t.$$

$$h(\lambda) - g(\lambda) \leq 0 \tag{5.36}$$

$$0 \leq \lambda_j(i) \leq u_j(i) \qquad j \in C(i),\ i \in M \tag{5.37}$$

where $g(\lambda) = g'(\lambda) + R_0$ and $h(\lambda) = h'(\lambda)$.

Type II :

$$z_{II}^*(m) = \quad - min \sum_{i \in M} \sum_{j \in C(i)} - \lambda_j(i) t_{j,m}(i) \tag{5.38}$$

$$s.t.$$

$$h(\lambda) - g(\lambda) \leq 0 \tag{5.39}$$

$$0 \leq \lambda_j(i) \leq u_j(i) \qquad j \in C(i),\ i \in M \tag{5.40}$$

where $h(\lambda) = g'(\lambda) + R_1$ this time and $g(\lambda) = h'(\lambda)$.

Note that the type I and type II problems have the same form, and the same solution procedure can be applied to solve both of them. Therefore, from now on we concentrate on the type I problem.

Many methods have been offered for the solution of d.c. problems especially for the ones that are of the canonical form. CDC problems as mentioned previously, are convex programs with an additional reverse convex constraint. However, as stated in Horst *et al.* (2000), every d.c. optimization problems can be reduced to CDC form.

Below, the method of reducing the type I problem into CDC form, and the solution method proposed for its solution is presented.

In order to reduce the problem into CDC form, a new variable say $d \in \mathbb{R}$ is added to the formulation, and by making use of the new variable, the constraint $h(\lambda) - g(\lambda) \leq 0$ is separated into two parts one of which is $h(\lambda) - d \leq 0$ and the other is $g(\lambda) - d \geq 0$. Note that the area defined by the constraint $h(\lambda) - g(\lambda) \leq 0$ and the region specified by the constraints $h(\lambda) - d \leq 0$ and $g(\lambda) - d \geq 0$ are equivalent. After this transformation, type I problem is reduced to the following CDC.

CDC Type I :

$$z_I^*(m) = \quad min \sum_{i \in M} \sum_{j \in C(i)} \lambda_j(i) t_{j,m}(i) \tag{5.41}$$

$$s.t.$$

$$h(\lambda) - d \leq 0 \tag{5.42}$$

$$g(\lambda) - d \geq 0 \tag{5.43}$$

$$d \in \mathbb{R} \tag{5.44}$$

$$0 \leq \lambda_j(i) \leq u_j(i) \qquad j \in C(i),\ i \in M \tag{5.45}$$

Note that constraint (5.43) is reverse convex, and the closed convex subset of $\mathbb{R}^n$, which is denoted as D within CDC formulation corresponds to intersections of closed convex areas defined by (5.42) and (5.45). It should be underlined that D is not a polytope this time, as it is in the case of type II problem of series system. This observation actually makes edge following algorithm an inappropriate tool for solution of the type I problem we have in the CDC form. The reason for this is the fact that, the optimal solution of the type I CDC problem does not have to occur on an edge of D as it is used to be for the type II problem of series system case since D is not a polytope anymore. Therefore, one cannot construct a strategy on following the edges of D and enumerating the points on which boundary of G intersects, where

$G = \{\lambda \in \mathbb{R}^{k_n} : g(\lambda) - d \leq 0\}$ for this case.

Among many algorithmic procedures proposed for the solution of CDC problem, we refer to one which is an outer approximation method proposed by Tuy (1995). Before giving the algorithm, we believe that discussing the general conceptual framework of outer approximation methods employed for CDC problems would be helpful.

Suppose we are to find an element of (unknown) closed set Ω which is a subset of a compact set F, where $F \subset \mathbb{R}^n$. If this is a difficult problem to solve by conventional methods, one may attempt to build a sequence of easier relaxed problems whose solutions x^1,x^2,..., form a sequence that eventually converges to $\bar{x} \in \Omega$. Suppose that there exists a family Ξ of polytopes containing F, and suppose these polytopes satisfy the following assumptions (Tuy, 1995);

Assumption 5.4 *For every polytope $P \in \Xi$ we can select a point $w(P) \in P$, called* distinguished *point associated with P, such that $w(P)$ does not exist only if $\Omega = \varnothing$, and whenever a sequence of distinguished points $x^1 = w(P^{(1)})$,$x^2 = w(P^{(2)})$,..., converges to a point $\bar{x} \in F$ then $\bar{x} \in \Omega$.*

Assumption 5.5 *For every distinguished point $z = w(P)$ $(P \in \Xi)$ we can recognize whether z belongs to F or not and if $z \notin F$, we can construct an affine function $l(x)$ such that $P' = P \cap \{x : l(x) \leq 0\} \in \Xi$ and $l(x)$ strictly separates z from F, i.e. satisfies $l(z) > 0$ and $l(x) \leq 0$ for $\forall x \in F$.*

Under Assumptions 5.4 and 5.5, the general outer approximation scheme can be displayed in Figure 5.3 (Tuy, 1995).

General Outer Approximation Scheme given in Figure 5.3 provides a sequence of polytopes such that $P^{(1)} \supset P^{(2)} \supset ... \supset P^{(i)} \supset ... \supset F$, approximating F more and more closely from outside. In most of the applications, the distinguished point $x^i = w(P^{(i)})$ is selected among the vertices of the polytope $P^{(i)}$. Therefore, a scheme begin

1. Start with an initial polytope $P^{(1)} \in \Xi$. Set $i = 1$;
2. Find the distinguished point $x^i = w(P^{(i)})$ (by Assumption 5.4)

    If $w(P^{(i)})$ does not exist then terminate $\Omega = \varnothing$;

    If $x^i \in F$ then terminate: x^i solves the problem (by Assumption 5.4).
3. Otherwise, referring Assumption 5.5, construct an affine function $l_i(x)$ such that $P^{(i+1)} = P^{(i)} \cap \{x : l_i(x) \leq 0\} \in \Xi$ and $l_i(x^i) > 0$, $l_i(x) \leq 0 \ \forall x \in F$.
4. Set $i \longleftarrow i + 1$ and go back to Step 1.

end;

Figure 5.3. General Outer Approximation Scheme

that enables getting the knowledge of vertices of $P^{(i+1)}$ from the vertex set of $P^{(i)}$ is required. Indeed, some subroutines like (Chen *et al.*, 1991; Horst *et al.*, 1988) are developed for this procedure where the next polytope is obtained by adding an additional linear constraint to the current polytope, which actually fits into an outer approximation scheme.

Now, the following outer approximation algorithm which is given in Figure 5.4 is developed for the solution of CDC problems and it is due to Tuy (1995) and it approximately solves CDC problems to optimality. In order to ensure the convergency of the algorithm, concept of ϵ-approximate solution is put forward, and the algorithm finds a solution $\bar{x}$ such that $c^T\bar{x} \leq min\{c^Tx : x \in F,\ g(x) \geq 0\} + \epsilon$ and inequalities $h(\bar{x}) \leq \epsilon$, $g(\bar{x}) \geq -\epsilon$ hold instead of $h(\bar{x}) \leq 0$, $g(\bar{x}) \geq 0$. Note that ϵ can be taken as a very little positive real number such that the deviation from the optimality is negligible.

It should be noted that inner multiplication of two vectors, say a and b, is denoted as $\langle a, b\rangle$ within the body of the algorithm. The converge of the algorithm is shown by Tuy (1995).

One may observe that, in order for the Outer Approximation Algorithm for

Input: An instance of CDC problem: ($g(x)$: Reverse convex constraint, $h(x)$: Convex Constraint, Positive real number ϵ)

Output: An ϵ-optimal solution of CDC problem

begin

Initialization. Let $\gamma_1 = c^T\hat{x}^1$ where $\hat{x}^1$ is the best feasible solution available (If no feasible solution is obtained, set $\hat{x}^1 = \varnothing$ and $\gamma_1 = +\infty$). Take a polytope $P^{(1)}$ such that $\{x \in F : c^T x \leq \gamma_1 - \epsilon\} \subset P^{(1)} \subset \{x : c^T x \leq \gamma_1 - \epsilon\}$ which has a known vertex set $V^{(1)}$. Set $i = 1$;

1. Compute $x^i \in argmax\{g(x) : x \in V^{(i)}\}$. If $g(x^i) < 0$ then terminate;

   a) If $\gamma_i < +\infty$, then $\hat{x}^i$ is an ϵ-approximate solution of CDC;

   b) If $\gamma_i = +\infty$, then CDC is infeasible;

2. Select $w^i \in V^{(i)}$ such that $c^T w^i \leq min\{c^T x : x \in V^{(i)}\} + \epsilon$. If $h(w^i) \leq \epsilon$, $g(w^i) \geq -\epsilon$, then terminate: w^i is an ϵ-approximate optimal solution.

3. If $h(w^i) > \frac{\epsilon}{2}$, then define $\hat{x}^{i+1} = \hat{x}^i$, $\gamma_{i+1} = \gamma_i$. Let $p^i \in \partial h(w^i)$, $l_i(x) = \langle p^i, x - w^i\rangle + h(w^i)$ and go to Step 6.

4. Determine $y^i \in [w^i; x^i]$ such that $g(y^i) = -\epsilon$ (y^i exists because $g(x^i) \geq 0$, $g(w^i) < -\epsilon$).

   a) If $h(y^i) > \epsilon$, then define $\hat{x}^{i+1} = \hat{x}^i$, $\gamma_{i+1} = \gamma_i$. Determine $u^i \in [w^i; y^i]$ such that $h(u^i) = \epsilon$ (u^i exists because $h(w^i) \leq \frac{\epsilon}{2}$ and, $h(y^i) > \epsilon$). Let $p^i \in \partial h(u^i)$, $l_i(x) = \langle p^i, x - u^i\rangle$ and go to Step 6.

5. If $h(y^i) \leq \epsilon$, then define $\hat{x}^{i+1} = y^i$, $\gamma_{i+1} = c^T y^i$.

   a) If $c^T w^i \geq c^T y^i$ then terminate: $\hat{x}^{i+1}$ is an ϵ-approximate global optimal solution.

   b) Otherwise, let $l_i(x) = \langle c, x - y^i\rangle + \epsilon$ and go to Step 6.

6. Compute the vertex set $V^{(i+1)}$ of the polytope $P^{(i+1)} = P^{(i)} \cap \{x : l_i(x) \leq 0\}$. Set $i \longleftarrow i + 1$ and go back to Step 1.

end;

Figure 5.4. Outer Approximation Algorithm for Canonical D.C. Problems

Canonical D.C. Problems to work in an efficient manner, there is a need for a subroutine that enables finding the vertex set $V^{(i+1)}$ of the polytope $P^{(i+1)}$ from the knowledge of previous vertex set $V^{(i)}$. We refer to *Adjacency list algorithm* of Chen *et al.* (1991) for this purpose. Before introducing the algorithm, some more terminology needs to be defined. Suppose the polytope we have at the step i of the Outer Approximation Algorithm for Canonical D.C. Problems is defined as $P^{(i)} = \{x \in \mathbb{R}^n : l_j(x) = a_j^T x - b_j \leq 0,\ j = 1, 2, ..., i-1\}$ where $a_j \in \mathbb{R}^n \backslash \{0\}$, $b_j \in \mathbb{R}$ $(j = 1, 2, ..., i-1)$ and new linear cut introduced is defined as $l_i(x)$ as defined in the body of the Outer Approximation Algorithm for Canonical D.C. Problems. Define $V^+ = \{v \in V^i : l_i(v) > 0\}$ and $V^- = \{v \in V^i : l_i(v) < 0\}$ as the sets of vertices that lie above and below the hyperlane defined by the new cut $l_i(x)$ respectively. Following theorem that is due to Horst *et al.* (2000) defines a framework that the adjacency list algorithm is built on.

Theorem 5.3 *(Horst* et al.*, 2000)A point $w \in \mathbb{R}^n$ is a member of vertex set $V^{(i+1)}$ of $P^{(i+1)}$, if and only if w is either a vertex of $P^{(i)}$ lying on $l_i(x)$ or a point where an edge $[u, v]$ of $P^{(i)}$, $u \in V^-$, $v \in V^+$ intersects $l_i(x)$, or w is already a member of V^-.*

Furthermore, let $N(u)$ is the set of vertices that are adjacent to vertex u. By two adjacent vertices, it is meant that there lies one of the linear cuts defining the polytope binding on both of the vertices. In addition, let $J(u)$ define the index set including the indices of the cuts that are active on the vertex u. In mathematical words, $J(u) = \{j \in \{1, ..., i\} : l_j(u) = 0\}$. One final remark before the algorithm is the way of finding the point w, where new cut intersects the edge $[u, v]$ of the polytope. If $u \in V^-$ and $v \in V^+$ holds, then $w = \alpha u + (1-\alpha)v$ where $\alpha = l_i(v)/(l_i(v) - l_i(u))$ (Horst *et al.*, 2000).

5.2.2. Markovian Sequence of Missions and Series System

Reliability function (4.8) dependent on the beginning mission i, is previously stated to be a convex function for the series systems with markovian sequence of missions where the converted transition matrix $\tilde{P}$ is defined as in equation (4.9). Therefore,

1. (Initialization) Determine V^+,V^-,v^+,v^- ;
2. (New vertices) $V^{(i+1)} \longleftarrow \varnothing$;

    for all $u \in V^-$ do

        for all $v \in N(u) \cap V^+$ do

            determine $w = [u, v] \cap l_i(x)$;

            $V^{(i+1)} \longleftarrow V^{(i+1)} \cup \{w\}$;

            $N(u) = (N(u) \backslash \{v\}) \cup \{w\}$;

            $N(w) \longleftarrow \{u\}$;

            $J(w) = (J(u) \cap J(v)) \cup \{i\}$;

        end for

    end for

3. (Neighbors of new vertices)

    for all $u \in V^{(i+1)}$, $v \in V^{(i+1)}$ do

        if $|J(u) \cap J(v)| = n - 1$ then

            $N(u) \longleftarrow N(u) \cup \{v\}$;

            $N(v) \longleftarrow N(v) \cup \{u\}$;

        end if

    end for

4. (Construction of the new vertex set)

    $V^{(i+1)} \longleftarrow V^{(i+1)} \cup V^-$

Figure 5.5. Adjacency List Algorithm

similar to the series system case with deterministic sequence of missions, the type I problem is a simple convex programming problem whereas type II problem is a canonical d.c. programming problem in which the compact and closed set that is meant to exist in the body of CDC type I problem, is a polytope. These observations clearly indicate that the solution methods employed for the solutions of type I and type II problems for the case of series systems with deterministic sequence of missions still remain valid for this case also. Hence, the method of SQP submitted within MATLAB Optimization User Guide (2000) is applied for type I problem. Furthermore, edge following algorithm of Altınel (1990) is used for the solution of the type II problem.

6. NUMERICAL EXAMPLES

In this chapter some numerical examples are provided for series systems with deterministic and markovian sequence of missions and serial connection of redundant subsystems with deterministic sequence of missions. Solution methods explained in the previous chapter is coded in Matlab (MATLAB 6.5 User's Guide, 2002) and run on a computer having an AMD Sempron 2800 Processor and 512 Megabyte random access memory.

6.1. Deterministic Sequence of Missions

6.1.1. Series System

First of all, we consider a series system of five components which is worked under three missions. We choose $\alpha = 0.05$, $\beta = 0.05$, $R_0 = 0.8$ and $R_1 = 0.98$. Moreover, component test costs are chosen randomly as follows; ($c_1(1)$, $c_2(1)$, $c_3(1)$, $c_4(1)$, $c_5(1)$, $c_1(2)$, $c_2(2)$, $c_3(2)$, $c_4(2)$, $c_5(2)$, $c_1(3)$, $c_2(3)$, $c_3(3)$, $c_4(3)$, $c_5(3)$) = (1925, 846, 1410, 1229, 1836, 1526, 760, 823, 1957, 1624, 303, 542, 1560, 1938, 1040). We also assume that prior knowledge on the component failure rates in each mission is given as upperbounds limiting the failure rates. We take upperbounds as ($u_1(1)$, $u_2(1)$, $u_3(1)$, $u_4(1)$, $u_5(1)$, $u_1(2)$, $u_2(2)$, $u_3(2)$, $u_4(2)$, $u_5(2)$, $u_1(3)$, $u_2(3)$, $u_3(3)$, $u_4(3)$, $u_5(3)$) = (0.092, 0.102, 0.082, 0.102, 0.1, 0.2041, 0.2341, 01841, 0.1741, 0.2101, 0.2961, 0.2803, 0.2901, 0.3161, 0.2806). We also assume that duration of the mission i, which is $D(i)$, follows from an exponential distribution with rate $\mu(i)$, which is given as follows for $i = 1, 2, 3$; ($\mu(1)$, ($\mu(2)$, ($\mu(3)$) = (5, 10, 15). Finally, the set of the components required in mission i, which is $C(i)$, is given as follows for $i = 1, 2, 3$; ($C(1)$, $C(2)$, $C(3)$) = ($\{1, 2, 3, 4, 5\}$, $\{2, 4, 5\}$, $\{1, 2, 3, 5\}$). Under these problem parameters optimum component test times are found as ($t_1(1)$, $t_2(1)$, $t_3(1)$, $t_4(1)$, $t_5(1)$, $t_1(2)$, $t_2(2)$, $t_3(2)$, $t_4(2)$, $t_5(2)$, $t_1(3)$, $t_2(3)$, $t_3(3)$, $t_4(3)$, $t_5(3)$) = (0, 0, 0, 0, 0, 0, 9.6537, 0, 0, 0, 6.4358, 6.4358, 5.4099, 0, 6.4358). Optimum m, is found as $m^* = 4$. On the other hand, initial feasible m is 2, with objective function value 33589.5721. These results are obtained after producing 99 columns in 3.109 seconds. Total cost of testing the components is also found to be 27907.732.

For the second and third cases, in order to observe how the results are sensitive to changes in the acceptable and unacceptable reliability levels, the same model is run exactly with same parameters but $R_1 = 0.9$ for the former one, and $R_0 = 0.9$ for the latter case respectively. For the prior one the optimum m equals to the first initial m, and is found as $m^* = 19$, and total cost of testing equals to 248846.8252. Optimum component test times are ($t_1(1)$, $t_2(1)$, $t_3(1)$, $t_4(1)$, $t_5(1)$, $t_1(2)$, $t_2(2)$, $t_3(2)$, $t_4(2)$, $t_5(2)$, $t_1(3)$, $t_2(3)$, $t_3(3)$, $t_4(3)$, $t_5(3)$) = (20.04, 24.7390, 24.1868, 24.1868, 24.1868, 0, 12.2685, 0, 12.2685, 12.2685, 8.2452, 8.0677, 8.0677, 0, 8.0677). These solutions are found after producing 522 columns in 56.937 seconds. On the other hand, for the latter case optimum m which is also the initial feasible m, is found as $m^* = 4$ as it is in the original example. Total test cost is equal to 177297 and component test times are; ($t_1(1)$, $t_2(1)$, $t_3(1)$, $t_4(1)$, $t_5(1)$, $t_1(2)$, $t_2(2)$, $t_3(2)$, $t_4(2)$, $t_5(2)$, $t_1(3)$, $t_2(3)$, $t_3(3)$, $t_4(3)$, $t_5(3)$) = (8.0102, 18.3982, 18.3971, 18.3996, 18.3929, 0, 9.6348, 0, 9.6331, 9.6346, 6.4354, 6.4342, 6.4354, 0.0004, 6.4346). From the dramatic changes in the total cost of testing and in the optimum component testing times, one can easily extract the result that the model is very sensitive to aceptable reliability level R_1 and the unacceptable reliability level R_0.

6.1.2. Serial Connection of Redundant Subsystems

Since the reliability function (4.4) has a very complicated nature and the computation time becomes extensive, we decided to provide a small example. Consider a system with two subsystems each of which has two identical components in parallel which have same failure rate and test cost. Suppose also that the system is going to work for two missions. Five instances of the problem is provided below by changing the problem parameters.

First of all, let $R_0 = 0.3$, $R_1 = 0.9999$, $\alpha = 0.05$, $\beta = 0.05$. Suppose cost of testing components in each mission is given as follows; ($c_1(1)$, $c_2(1)$, $c_1(2)$, $c_2(1)$) = (55, 65, 75, 60). Moreover, suppose both of the subsystems are required to work in both of the missions, e.g., $(C(1),\ C(2)) = (\{1, 2\},\ \{1, 2\})$. Duration of the mission i is assumed to follow from an exponential distribution with rate $\mu(i)$ as usual, and these rates are given as $(\mu(1),\ \mu(2)) = (5,\ 10)$. Finally, let the prior knowledge about the component failure rates are given in the form of upperbounds that limits the failure rates from above, as $(u_1(1),\ u_2(1),\ u_1(2),\ u_2(2)) = (5.6709,\ 7.2911,\ 8.1011,\ 4.8607)$. Under these problem parameters, optimum m is found equal to 17, e.g., $m^* = 17$. Initial feasible m value is also 17. Optimum component test times are found as $(t_1(1),\ t_2(1),\ t_1(2),\ t_2(2)) = (2.4765,\ 2.7010,\ 1.7235,\ 1.3818)$. On the other hand, testing components at optimum test times result in a total cost of 523.9396. These results are obtained after producing 36 columns in 10369.984 seconds.

In the second example, unacceptable reliability level R_0 is increased to 0.5, and upperbounds on the failure rates are tightened. Mathematically, $(u_1(1),\ u_2(1),\ u_1(2),\ u_2(2)) = (3.2394,\ 4.1649,\ 4.6276,\ 2.7766)$. Other problem parameters are kept as the same. Initial feasible m equals optimum m which is found to be 376, e.g., $m^* = 376$. Optimum component test times are found as $(t_1(1),\ t_2(1),\ t_1(2),\ t_2(2)) = (77.1368,\ 95.9951,\ 45.9780,\ 25.8903)$. Under the optimum component test times found and the cost vector c, total cost of testing components at optimum component test times equals 15333.9723. Finally, these results are found by producing 53 columns in 6865.547 seconds.

In the third example, acceptable reliability level R_1 is reduced to 0.99, and all other problem parameters remain the same as with the second example. This time $m^* = 640$ and the initial feasible m again equals to optimum m. Optimum component test times are $(t_1(1),\ t_2(1),\ t_1(2),\ t_2(2)) = (120.3843,\ 156.1827,\ 77.7272,\ 47.1150)$. Total optimum testing cost turns out to be 25429.4561 which is found after 26 column productions in 1612.671 seconds.

Forth and fifth examples illustrate the effect of values of α and β on the optimum results. In order to observe these effects, α and β values are increased to 0.2 from 0.05 for second and third examples while keeping other parameters same. Such high allowable number 376 of component failures found in second example reduces to optimum $m^* = 105$ for the forth example. However, initial feasible m value remains equal to the optimum one. Significant decrease effect can also be observed on the optimum component test times, which are $(t_1(1), t_2(1), t_1(2), t_2(2)) = (22.7430, 26.0119, 11.7504, 8.1927)$, and naturally on the total optimum testing cost which is equal to 4314.4817. These results are obtained after producing 31 columns in total in 2161.734 seconds. Same effects are obtained for the fifth example case as well. For instance optimum m is reduced to 198 from 640 that is found in the third example. However, initial feasible m still equals to the optimum one. Moreover, optimum test time as are also significantly reduced to $(t_1(1), t_2(1), t_1(2), t_2(2)) = (42.4291, 47.4942, 21.1728, 15.0231)$ which lead to total optimum component testing cost of 7910.2664, after producing 49 columns in 2834.297 seconds.

One may realize that the initial feasible m value equals to the optimum m for all the given examples for serial connection of redundant subsystems. Complicated nature of the problem narrows the feasible region and makes it difficult to reach feasible m values. This fact explains why the initial feasible m values are relatively higher than the previous cases. As previously stated and observed for the series system case, numerical examples presented for the serial connection of redundant subsystems reveals a high sensitivity to the changes in the values of acceptable and unacceptable reliability levels R_0 and R_1. Additionally, model is also sensitive to the values of α and β. These results are indeed predictable because the main instruments that define the feaible regions for type I and type II problems are the levels of acceptable and unacceptable reliability levels whereas the levels for α and β determines how system reliability constraints (3.3) and (3.4) are tight which affects the general feasible region of the overall model.

6.2. Markovian Sequence of Missions and Series System

For the series system with markovian sequence of missions, four numerical examples are obtained with different problem parameters. As a first example, let the unacceptable system reliability level R_0 equal to 0.5 while acceptable system reliability level R_1 is 0.98. A sistem of five components is designed to perform two missions and the transition matrix of the markovian process lying behind the model is accepted as

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}$$

On the other hand, α and β which are the bounds for consumer and producer risks are both taken as 0.05. Cost of testing each component in each mission is given as $(c_1(1), c_2(1), c_3(1), c_4(1), c_5(1), c_1(2), c_2(2), c_3(2), c_4(2), c_5(2))$ = (984, 1925, 1326, 846, 756, 1410, 1235, 1229, 1623, 1836), and upperbounds limiting the component failure rates are accepted as $(u_1(1), u_2(1), u_3(1), u_4(1), u_5(1), u_1(2), u_2(2), u_3(2), u_4(2), u_5(2))$ = (1.4019, 1.015, 0.2636, 0.6696, 0.9817, 1.4094, 0.2408, 0.6658, 0.4632, 0.4321). Durations of missions are exponentially distributed with rates $(\mu(1), \mu(2)) =$ (5, 10), and sets of components required to perform in each mission is given as $(C(1), C(2)) = (\{1, 2, 3, 4, 5\}, \{1, 2, 3, 5\})$. Finally, the initial mission which is the first state of the markovian process behind the system is taken as 2. With these problem parameters, both m^* and the initial feasible m are found to be 1, and optimum component test times are found as $(t_1(1), t_2(1), t_3(1), t_4(1), t_5(1), t_1(2), t_2(2), t_3(2), t_4(2), t_5(2))$ = (2.0610, 0, 0, 0, 2.0157, 0, 0, 0, 0, 0). Total optimal cost is found to be equal to 3551.8565. In order to reach these results, 0.5 second time is required and a total of 24 columns are produced.

For the second example, R_0 is rised to 0.8, and upperbounds are tightened a little bit to $(u_1(1), u_2(1), u_3(1), u_4(1), u_5(1), u_1(2), u_2(2), u_3(2), u_4(2), u_5(2))$ = (0.38404, 0.27805, 0.07228, 0.18345, 0.26894, 0.38609, 0.065768, 018233, 0.12684, 0.11838). Optimum number of allowable failures is found as $m^* = 2$ and optimum component test times are found as $(t_1(1), t_2(1), t_3(1), t_4(1), t_5(1), t_1(2), t_2(2), t_3(2), t_4(2), t_5(2)) =$ (4.7423, 0, 0, 4.7423, 4.7423, 5.6172, 0, 1.9645, 0, 0). Initial feasible m value also equals to the m^*. Total cost of testing turns out to be 22598.34 which is found in 1032 seconds after producing 52 columns in total. One may observe the increase in the total cost as well as in the computational required time.

As a third example, again a system of five components is constructed. The values of R_0, R_1, α and β is kept same with the previous example, while the number of missions is increased to 3. Initial mission is accepted again as 2 and the system has the following transition matrix

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}$$

Cost of testing each component in each mission is also given as ($c_1(1)$, $c_2(1)$, $c_3(1)$, $c_4(1)$, $c_5(1)$, $c_1(2)$, $c_2(2)$, $c_3(2)$, $c_4(2)$, $c_5(2)$, $c_1(3)$, $c_2(3)$, $c_3(3)$, $c_4(3)$, $c_5(3)$) = (984, 1925, 1326, 846, 756, 1410, 1235, 1229, 1623, 1836, 1625, 945, 598, 1050, 1123), and the upper-bounds are taken as ($u_1(1)$, $u_2(1)$, $u_3(1)$, $u_4(1)$, $u_5(1)$, $u_1(2)$, $u_2(2)$, $u_3(2)$, $u_4(2)$, $u_5(2)$, $u_1(3)$, $u_2(3)$, $u_3(3)$, $u_4(3)$, $u_5(3)$) = (0.384, 0.27802, 0.0722, 0.18343, 0.26892, 0.38606, 0.065762, 0.18231, 0.12683, 0.11837, 0.62494, 0.45795, 0.23958, 0.19798, 0.23498). Additionally, duration of mission i, which is $D(i)$, follows from exponential rate with rates ($\mu(1)$, ($\mu(2)$, ($\mu(3)$) = (5, 10, 15). Finally, the set of the components required in mission i, which is $C(i)$, is given as follows for i = 1, 2, 3; ($C(1)$, $C(2)$, $C(3)$) = ($\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 5\}$, $\{2, 3, 4\}$). Optimum number of allowable failures turns out to be equal to initial feasible m value which is 1 and optimum test times are found as ($t_1(1)$, $t_2(1)$, $t_3(1)$, $t_4(1)$, $t_5(1)$, $t_1(2)$, $t_2(2)$, $t_3(2)$, $t_4(2)$, $t_5(2)$, $t_1(3)$, $t_2(3)$, $t_3(3)$, $t_4(3)$, $t_5(3)$) = (3.3974, 0, 0, 3.3974, 3.3974, 2.6510, 0, 2.1767, 0, 0, 0, 0.5675, 0.5672, 0.5675, 0). Total optimum cost is also found as 16670.0602 in computational time of 1375 seconds, after producing 49 columns.

As a final example, we think a sequence of five missions, and a system of five components is operated in order to complete the missions. The values for R_0, R_1, α and β remain same with the previous example. Transition matrix this time a 5×5 one is composed as

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0 & 0.3 & 0.1 \\ 0.6 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 & 0 \\ 0.1 & 0.15 & 0.25 & 0.3 & 0.2 \end{bmatrix}$$

and cost of testing each component in each mission is taken as ($c_1(1)$, $c_2(1)$, $c_3(1)$, $c_4(1)$, $c_5(1)$, $c_1(2)$, $c_2(2)$, $c_3(2)$, $c_4(2)$, $c_5(2)$, $c_1(3)$, $c_2(3)$, $c_3(3)$, $c_4(3)$, $c_5(3)$, $c_1(4)$, $c_2(4)$, $c_3(4)$, $c_4(4)$, $c_5(4)$, $c_1(5)$, $c_2(5)$, $c_3(5)$, $c_4(5)$, $c_5(5)$) = (984, 653, 1925, 785, 1326, 1125, 1623, 1836, 968, 1625, 1146, 945, 598, 964, 1050, 1123, 846, 789, 756, 1410, 1265, 1235, 1040, 1587, 1229), while upperbounds are given as ($u_1(1)$, $u_2(1)$, $u_3(1)$, $u_4(1)$, $u_5(1)$, $u_1(2)$, $u_2(2)$, $u_3(2)$, $u_4(2)$, $u_5(2)$, $u_1(3)$, $u_2(3)$, $u_3(3)$, $u_4(3)$, $u_5(3)$, $u_1(4)$, $u_2(4)$, $u_3(4)$, $u_4(4)$, $u_5(4)$, $u_1(5)$, $u_2(5)$, $u_3(5)$, $u_4(5)$, $u_5(5)$) = (0.384, 0.564, 0.278, 0.6249, 0.658, 0.458, 0.215, 0.236, 0.0722, 0.445, 0.1834, 0.259, 0.2689, 0.3861, 0.198, 0.235, 0.215, 0.0658, 0.118, 0.1823, 0.1268, 0.1184, 0.219, 0.568, 0.925). Rates of the exponential mission durations are accepted as ($\mu(1)$, ($\mu(2)$, ($\mu(3)$, ($\mu(4)$, ($\mu(5)$) = (7, 9, 8, 10, 6), while the set of components required for each mission is given as ($C(1)$, $C(2)$, $C(3)$, $C(4)$, $C(5)$) = ($\{1,2,3,4,5\}$, $\{1,2,4\}$, $\{1,2,3,5\}$, $\{2,3,4,5\}$, $\{2,4,5\}$). Finally, fourth mission is taken as the initial mission. Optimum number for maximum allowable component failures is found as 1, i.e., $m^* = 1$. Initial feasible m value is also the optimum one. Optimum component test results are found under these problem parameters as ($t_1(1)$, $t_2(1)$, $t_3(1)$, $t_4(1)$, $t_5(1)$, $t_1(2)$, $t_2(2)$, $t_3(2)$, $t_4(2)$, $t_5(2)$, $t_1(3)$, $t_2(3)$, $t_3(3)$, $t_4(3)$, $t_5(3)$, $t_1(4)$, $t_2(4)$, $t_3(4)$, $t_4(4)$, $t_5(4)$, $t_1(5)$, $t_2(5)$, $t_3(5)$, $t_4(5)$, $c_5(5)$) = (0, 3.0403, 0, 3.0403, 0.3565, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.8667), and total cost of optimum testing is 5909.9394. These results are obtained after producing 26 columns in total, in 33.203 seconds.

7. CONCLUSIONS

This thesis provides a new way of approach to the component testing problem by deviating from the conventional fixed time based reliability definition to a more general one which is suitable for the systems designed to perform a sequence of missions rather than a single mission. Systems are supposed to be maintained to replace failed components during intermissions. Moreover, the sequence of the missions can also be Markovian as well as deterministic. In such conditions, reliability of the system is defined as the probability for the system to be functional until the end of the last mission.

This new reliability definition enabled us to construct a means for determining component test plans that ensures type I and type II error bounds for system reliability at minimum possible total cost for systems designed to perform a sequence of missions that are possibly in random order and have possibly random durations. We named this method as *mission based component testing* and we managed to adapt it for series systems with deterministic and markovian sequence of missions and for serial connection of redundant subsystems with deterministic sequence of missions. We dealt with convex and reverse convex optimization subproblems for series systems with deterministic and markovian sequence of missions and d.c. optimization subproblems for serial connection of redundant subsystems with deterministic sequence of missions.

We witnessed that as the complexity of the system increases, the initial feasible m value increases and the system reveals more sensitivity to problem parameters because of the narrower feasible regions. In addition, we observed that for almost every problem instance we tested but one, the initial feasible m also turned out to be the optimum one. Because of extensive computational time, a detailed sensitivity analysis of the systems could not be achieved within the scope of this thesis. However, we believe that such an analysis would enlight the characteristics of mission based component testing method even more.

We believe that mission based component testing idea will lead interesting research discussions in the future. For instance as a natural extension of this thesis, serial connection of redundant subsystems with markovian sequence of missions come. Moreover, the idea of mission based component testing can be applied for the systems having different topologies and different working styles such as serial connection of standby redundant subsystems, or serial connection of k-out-of-m systems. Additionally, another interesting idea can be put on practice by changing the reliability definition from probability of completing all the missions to probability of completing a critical mission. This thesis naturaly will trigger such discussions and contribute to the component testing literature.

APPENDIX A: Proof of Lemma 4.1

Laplace transform of a linear equation can be extended as follows

$$\begin{aligned}
L_i^k(c_i^T\alpha_i) &= E\left[e^{-(c_i^T\alpha_i)D(i)}D(i)^k\right] \\
&= \int_0^\infty e^{-(c_i^T\alpha_i)D(i)}D(i)^k\mu(i)e^{-\mu(i)D(i)}dD(i) \\
&= \mu(i)\int_0^\infty e^{-(c_i^T\alpha_i+\mu(i))D(i)}D(i)^k dD(i)
\end{aligned}$$

Then applying the substitution , $\left(c_i^T\alpha_i+\mu(i)\right)D(i)=x$, we have

$$\begin{aligned}
L_i^k(c_i^T\alpha_i) &= \mu(i)\int_0^\infty e^{-x}\left(\frac{x}{c_i^T\alpha_i+\mu(i)}\right)^k\frac{dx}{c_i^T\alpha_i+\mu(i)} \\
&= \frac{\mu(i)}{(c_i^T\alpha_i+\mu(i))^{k+1}}\int_0^\infty e^{-x}x^k dx \\
&= \frac{\mu(i)}{(c_i^T\alpha_i+\mu(i))^{k+1}}\Gamma(k+1) \\
&= \frac{\mu(i)k!}{(c_i^T\alpha_i+\mu(i))^{k+1}}
\end{aligned}$$

Then following equality holds;

$$R=\prod_{i=1}^{n}L_i^k(c_i^T\alpha_i)=\prod_{i=1}^{n}\frac{\mu(i)k!}{(c_i^T\alpha_i+\mu(i))^{k+1}} \tag{A.1}$$

and if we take partial derivatives of equation (A.1), we obtain

$$\begin{aligned}
\frac{\partial R}{\partial\alpha_{xv}} &= \frac{-c_{xv}(k+1)}{(c_x^T\alpha_x+\mu(x))}\frac{\mu(x)k!}{(c_x^T\alpha_x+\mu(x))^{k+1}}\prod_{i\in\{1,\ldots,n\}/\{x\}}\frac{\mu(i)k!}{(c_i^T\alpha_i+\mu(i))^{k+1}} \\
&= \frac{-c_{xv}(k+1)}{(c_x^T\alpha_x+\mu(x))}\prod_{i=1}^{n}\frac{\mu(i)k!}{(c_i^T\alpha_i+\mu(i))^{k+1}}
\end{aligned}$$

where $x\in M$ and $v\in C(x)$.

Therefore, second derivatives happen to be as

$$
\begin{aligned}
\frac{\partial R}{\partial \alpha_{xv} \partial \alpha_{xt}} &= \frac{c_{xv} c_{xt} (k+2)(k+1)}{(c_x^T \alpha_x + \mu(x))^2} \prod_{i=1}^{n} \frac{\mu(i) k!}{(c_i^T \alpha_i + \mu(i))^{k+1}} \\
\frac{\partial R}{\partial \alpha_{xv} \partial \alpha_{sr}} &= \frac{c_{xv} c_{sr} (k+1)^2}{(c_x^T \alpha_x + \mu(x))(c_s^T \alpha_s + \mu(s))} \prod_{i=1}^{n} \frac{\mu(i) k!}{(c_i^T \alpha_i + \mu(i))^{k+1}}
\end{aligned}
$$

Here we have x, $s \in M$; v, $t \in C(x)$, $r \in C(s)$.

Note that $R = \prod_{i=1}^{n} \frac{\mu(i) k!}{(c_i^T \alpha_i + \mu(i))^{k+1}} \geq 0$, then we can eliminate the term from the second order derivatives and constitute the Hessian matrix H'. If H'can be shown to be positive semi-definite, then the real hessian H will also be positive semi-definite as well. Matrix H' can be obtained as

$$
H' = \begin{bmatrix}
F_{1111} & \cdot & F_{111l_1} & F_{1121} & \cdot & F_{112l_2} & \cdot & \cdot & F_{11n1} & \cdot & F_{11nl_n} \\
F_{1211} & \cdot & F_{121l_1} & F_{1221} & \cdot & F_{122l_2} & \cdot & \cdot & F_{12n1} & \cdot & F_{12nl_n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
F_{1l_111} & \cdot & F_{1l_11l_1} & F_{1l_121} & \cdot & F_{1l_12l_2} & \cdot & \cdot & F_{1l_1n1} & \cdot & F_{1l_1nl_n} \\
F_{2111} & \cdot & F_{211l_1} & F_{2121} & \cdot & F_{212l_2} & \cdot & \cdot & F_{21n1} & \cdot & F_{21nl_n} \\
F_{2211} & \cdot & F_{221l_1} & F_{2221} & \cdot & F_{222l_2} & \cdot & \cdot & F_{22n1} & \cdot & F_{22nl_n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
F_{2l_211} & \cdot & F_{2l_21l_1} & F_{2l_221} & \cdot & F_{2l_22l_2} & \cdot & \cdot & F_{2l_2n1} & \cdot & F_{2l_2nl_n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
F_{n111} & \cdot & F_{n11l_1} & F_{n121} & \cdot & F_{n12l_2} & \cdot & \cdot & F_{n1n1} & \cdot & F_{n1nl_n} \\
F_{n211} & \cdot & F_{n21l_1} & F_{n221} & \cdot & F_{n22l_2} & \cdot & \cdot & F_{n2n1} & \cdot & F_{n2nl_n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
F_{nl_n11} & \cdot & F_{nl_n1l_1} & F_{nl_n21} & \cdot & F_{nl_n2l_2} & \cdot & \cdot & F_{nl_nn1} & \cdot & F_{nl_nnl_n}
\end{bmatrix}
$$

where $F_{xvsd} = \frac{c_{xv}c_{sd}(k+1)^2}{A_xA_s}$ for the cases where x is not equal to s, and $F_{xvsd} = \frac{c_{xv}c_{sd}(k+2)(k+1)}{A_xA_s}$ while $x = s$, where $x, s \in M$ and $v \in C(x)$, $d \in C(s)$ in which the term $A_i = c_i^T\alpha_i + \mu(i)$ is employed. Moreover, $l_i = |C(i)|$ for all $i \in M$, and k_n is the sum of l_i's for all $i \in M$. Observe also that since $\alpha_i, c_i, \mu(i) \geq 0$, $A_i \geq 0$ holds for all $i \in M$.

Let $x \in R^{k_n}$ such that $x^T = \{x_{11}, x_{12}, .., x_{1l_1}, x_{21}, x_{22}, .., x_{2l_2}, ..., x_{n1}, x_{n2}, .., x_{nl_n}\}$. Then, after opening F_{xvsd} terms again, and multiplying transpose of x vector with H' we obtain

$$(x^TH')^T = \begin{bmatrix}
\frac{c_{11}(k+2)(k+1)}{A_1^2}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{11}(k+1)^2}{A_2A_1}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{11}(k+1)^2}{A_nA_1}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
\frac{c_{12}(k+2)(k+1)}{A_1^2}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{12}(k+1)^2}{A_2A_1}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{12}(k+1)^2}{A_nA_1}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
. \\
. \\
\frac{c_{1l_1}(k+2)(k+1)}{A_1^2}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{1l_1}(k+1)^2}{A_2A_1}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{1l_1}(k+1)^2}{A_nA_1}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
\frac{c_{21}(k+1)^2}{A_1A_2}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{21}(k+2)(k+1)}{A_2^2}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{21}(k+1)^2}{A_nA_2}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
\frac{c_{22}(k+1)^2}{A_1A_2}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{22}(k+2)(k+1)}{A_2^2}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{22}(k+1)^2}{A_nA_2}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
. \\
. \\
\frac{c_{2l_2}(k+1)^2}{A_1A_2}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{2l_2}(k+2)(k+1)}{A_2^2}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{2l_2}(k+1)^2}{A_nA_2}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
. \\
. \\
. \\
\frac{c_{n1}(k+1)^2}{A_1A_n}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{n1}(k+1)^2}{A_2A_n}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{n1}(k+2)(k+1)}{A_n^2}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
\frac{c_{n2}(k+1)^2}{A_1A_n}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{n2}(k+1)^2}{A_2A_n}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{n2}(k+2)(k+1)}{A_n^2}\sum_{i=1}^{l_n}c_{ni}x_{ni} \\
. \\
. \\
\frac{c_{nl_n}(k+1)^2}{A_1A_n}\sum_{i=1}^{l_1}c_{1i}x_{1i} + \frac{c_{nl_n}(k+1)^2}{A_2A_n}\sum_{i=1}^{l_2}c_{2i}x_{2i} + ... + \frac{c_{nl_n}(k+2)(k+1)}{A_n^2}\sum_{i=1}^{l_n}c_{ni}x_{ni}
\end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned}
x^T H' x &= \sum_{i=1}^{l_1} c_{1i}x_{1i}\left[\frac{(k+2)(k+1)}{A_1^2}\sum_{i=1}^{l_1} c_{1i}x_{1i} + \frac{(k+1)^2}{A_2A_1}\sum_{i=1}^{l_2} c_{2i}x_{2i} + ... + \frac{(k+1)^2}{A_nA_1}\sum_{i=1}^{l_n} c_{ni}x_{ni}\right] \\
&+ \sum_{i=1}^{l_2} c_{2i}x_{2i}\left[\frac{(k+1)^2}{A_1A_2}\sum_{i=1}^{l_1} c_{1i}x_{1i} + \frac{(k+2)(k+1)}{A_2^2}\sum_{i=1}^{l_2} c_{2i}x_{2i} + ... + \frac{(k+1)^2}{A_nA_2}\sum_{i=1}^{l_n} c_{ni}x_{ni}\right] \\
&+ .... + \\
&+ \sum_{i=1}^{l_n} c_{ni}x_{ni}\left[\frac{(k+1)^2}{A_1A_n}\sum_{i=1}^{l_1} c_{1i}x_{1i} + \frac{(k+1)^2}{A_2A_n}\sum_{i=1}^{l_2} c_{2i}x_{2i} + ... + \frac{(k+2)(k+1)}{A_n^2}\sum_{i=1}^{l_n} c_{ni}x_{ni}\right]
\end{aligned}$$

then, because $k \geq 0$ we have

$$\begin{aligned}
x^T H' x &\geq \sum_{i=1}^{l_1} c_{1i}x_{1i}\left[\frac{(k+1)^2}{A_1^2}\sum_{i=1}^{l_1} c_{1i}x_{1i} + \frac{(k+1)^2}{A_2A_1}\sum_{i=1}^{l_2} c_{2i}x_{2i} + ... + \frac{(k+1)^2}{A_nA_1}\sum_{i=1}^{l_n} c_{ni}x_{ni}\right] \\
&+ \sum_{i=1}^{l_2} c_{2i}x_{2i}\left[\frac{(k+1)^2}{A_1A_2}\sum_{i=1}^{l_1} c_{1i}x_{1i} + \frac{(k+1)^2}{A_2^2}\sum_{i=1}^{l_2} c_{2i}x_{2i} + ... + \frac{(k+1)^2}{A_nA_2}\sum_{i=1}^{l_n} c_{ni}x_{ni}\right] \\
&+ .... + \\
&+ \sum_{i=1}^{l_n} c_{ni}x_{ni}\left[\frac{(k+1)^2}{A_1A_n}\sum_{i=1}^{l_1} c_{1i}x_{1i} + \frac{(k+1)^2}{A_2A_n}\sum_{i=1}^{l_2} c_{2i}x_{2i} + ... + \frac{(k+1)^2}{A_n^2}\sum_{i=1}^{l_n} c_{ni}x_{ni}\right] \\
&= \left(\sum_{j=1}^{n}\frac{(k+1)K_j}{A_j}\right)^2 \geq 0
\end{aligned}$$

where $K_j = \sum_{i=1}^{l_j} c_{ji}x_{ji}$ for all $j \in M$.

Therefore, $x^T H' x \geq 0$, implying that H' is positive semi-definite. Then, we also have H positive semi-definite. Then R is convex. Q. E. D.

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