OBSERVERS
FOR
LINEAR TIME-INVARIANT SYSTEMS

by

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ABSTRACT

In optimal control theory, the design of a state feedback control law requires the availability of the entire state vector. However, in most of the control systems the measurements cannot provide the entire state vector. Then, the non-available state variables must be estimated. The estimated state variables are combined with the already available state variables to be substituted in the feedback control law.

In this thesis, the observers that are designed to estimate the non-available state variables are considered. In the deterministic case, both for the continuous-time and the discrete-time systems, special emphasis is given to the design of minimal order observers.

The stochastic optimal reduced order observer-estimator and the suboptimal minimal order observer are suggested as alternatives to the Kalman filter. These alternative designs are compared and the inter-relationships are discussed.

Furthermore, a computer package program has been developed for computer aided design of such observers for practical implementation. The user must only supply the necessary data to obtain the values of the parameters of the observer of interest.

ÖZETÇE

Kapalı çevrimli bir sistemin eniyi çalışması için bu sistemin durum değişkenlerinin geri beslenmesi ile yaratılan denetim yasası kullanılır. Denetim yasasının gerçeklenebilmesi için tüm durum değişkenlerinin izlenebilir olması gerekir. Birçok denetim sisteminde bazı durum değişkenlerini izleme olanağı yoktur. Denetim yasasının gerçeklenmesi ancak izlenemeyen bu durum degişkenlerini uygun bir kestirimi yapıldıktan sonra mümkün olur.

Bu tez çalışmasında, izlenemeyen durum degişkenlerinin kestiriminde kullanılan gözlemleyiciler ele alınmıştır. Deterministik sistemlerde özellikle enaz kerteli gözlemleyicilerin tasarımları üzerinde durulmuştur.

Rassal sistemlerde ise, eniyi düşük kerteli gözlemleyici-kestricilerle, eniyiye yakın enaz kerteli gözlemleyiciler ele alınmış ve Kalman süzgeciyle olan ilişkileri incelenmiştir.

Gözlemleyicilerin parametrelerini hesaplamak için bir bilgisayar paket programı hazırlanmıştır. Kullanıcının istediği gözlemleyicinin parametrelerinin hesaplanması için sadece gerekli veriyi hazırlaması yeterlidir.

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INTRODUCTION

Application of optimal control theory to the design of optimal closed loop control systems results in a state feedback control law of the form $u(t) = \phi(x(t),t)$. The state-feedback control law requires that the entire state vector be available from the measurements. However, in practice, it may not be possible to implement the state feedback control law due to the fact that the entire state vector is not available in most cases. Therefore, a suitable approximation to the state vector must be developed that can be substituted into the state-feedback control law.

In the view of the above discussion, the design of the state feedback control law is separated into two phases: The first phase is the design of the state-feedback control law assuming that the entire state vector is available. The second phase is the design of a system that provides an estimate to the state vector.

Kalman and Bucy have treated the state estimation problem for the case where all the measurements are corrupted by white noise |1|. Bryson and Johansen have considered a more general problem assuming that some noise free measurements exist |2|. The assumption that noise free measurements are available is a realistic assumption in many control problems. They have shown that the optimal estimator is a modification of the Kalman-Bucy estimator which contains differentiators and integrators. Since differentiation is not desirable for practical reasons, an alternative to the modified Kalman-Bucy estimator, an observer, namely the Luenberger observer has been suggested.

The observer is a dynamic system driven by the inputs and the outputs of the system whose states are estimated. The order of the observer is less than the order of the modified Kalman-Bucy estimator, hence it is easier to implement it.

Observers are important in the linear system theory since they offer, in addition to their practical utility, a converging setting of the fundamental concepts, such as controllability, observability and stability.

This thesis deals with the observer designs for linear time-invariant systems represented in state space. It considers observers both in the deterministic and in the stochastic cases, and it introduces the computer aided implementation methods in detail.

In the first chapter, the motivating ideas concerning the structure of deterministic observers are introduced. The basic construction methods for one type of observers, i.e., identity observer, are presented. The interactions of classical concepts such as observability and stability with the dynamics of the observer are discussed |3|.

In the second chapter, a class of observers, i.e., the class of canonical minimal order observers for continuous-time systems is considered. The transformation into the canonical form which decouples the state variables at the output is presented and a design procedure which determines the dynamics of the minimal order observer for the system in canonical form, by employing Lyapunov stability theory, is suggested |7|. The last section consists of computer based algotithms which are used in the design of the minimal order observer.

The third chapter deals with the design of minimal order observers for discrete-time systems represented in the canonical form. Lyapunov stability theory for discrete-time systems is used to determine the dynamics of the observer |8|. Some properties of the canonical form are used to reduce the order of the Lyapunov equation considerably compared to the ones developed so far in the literature. In the last section, a computational difficulty and a suggestive solution are discussed.

In the fourth chapter, optimal observer-estimators for stochastic systems which are represented in the Gauss-Markov model are considered. This chapter is mainly concerned with the case that some of the measurements are being corrupted by additive white noise while others being noise free and presents an optimal reduced order observer-estimator to estimate the entire state vector of the system in the minimum mean square error sense |13|. The equations derived for the optimal reduced order observer-estimator are also valid for the extreme cases, those being that all the measurements are corrupted with noise and all the measurements are noise-free. When all the measurements are corrupted with noise the optimal reduced order observer estimator functions like the Kalman filter and in the other extreme case that the measurement noise is not present, it reduces to the minimal order observer-estimator. Besides the optimal reduced order observer-estimator, a suboptimal minimal order observer is discussed as an alternative solution.

The fifth chapter deserves a special importance since it is the practical setting of the theory developed in the previous chapters. The user's manual which has been prepared providing an easy access to the user explains how the data is entered into the related programs in the package and gives the structures of the subprograms.

Finally, the conclusion chapter discusses what has been done in this study and suggests topics for further research in the area of observers.

CHAPTER 1

BASIC THEORY [3]

1.1. INTRODUCTION TO OBSERVERS

The purpose of this chapter is to familiarize the reader with the basic concepts of the observer theory. The continuous time systems are considered in this chapter, but the theory developed here is applicable to the discrete time systems as well.

The linear time invariant dynamical plants are characterized by the set of equations

$$\dot{x}(t) = A\ x(t) + B\ u(t)$$
$$y(t) = C\ x(t) \qquad 1.1$$

where x(t) is the n-dimensional state vector, u(t) is the r-dimensional input vector and y(t) is the m-dimensional output vector. A is the (nxn) state matrix, B is the (nxr) input matrix and C is the (mxn) output matrix.

Let S_1 denote the system in Equation 1.1. If the output and the input of the system S_1 are used to drive a system S_2, the the following theorem announces that the state of S_2 tracks a linear transformation of the state of S_1.

<u>THEOREM 1.1:</u> Let S_1, represented as

$$\dot{x}(t) = A\ x(t) + B\ u(t)$$
$$y(t) = C\ x(t)$$

drive S_2 which is given by

$$\dot{z}(t) = F\,z(t) + G\,y(t) + H\,u(t)$$

where z(t) is a p-dimensional vector, F is a (pxp) matrix, G is a (pxm) matrix and H is a (pxr) matrix, $(p \leq n)$. Suppose there exists a (pxn) transformation matrix T, such that

$$TA - FT = GC$$

and

$$TB = H.$$

If $z(t_o) = Tx(t_o)$, then $z(t) = T\,x(t)$ for all $t \geq t_o$.

<u>PROOF</u>: It is immediately written that

$$\dot{z}(t) - T\dot{x}(t) = Fz(t) + Gy(t) + Hu(t) - TAx(t) - TBu(t)$$

$$\dot{z}(t) - T\dot{x}(t) = Fz(t) + (GC-TA)x(t) + (H-TB)u(t)$$

Since

$$GC - TA = -FT$$

and

$$TB = H$$

the above can be written as

$$\dot{z}(t) - T\dot{x}(t) = Fz(t) - FTx(t)$$

$$\dot{z}(t) - T\dot{x}(t) = F\left[z(t) - Tx(t)\right] \qquad 1.2$$

Letting

$$q(t) = z(t) - Tx(t)$$

$$\dot{q}(t) = \dot{z}(t) - T\dot{x}(t)$$

Equation 1.2 can be written as

$$\dot{q}(t) = F\,q(t)$$

which has the solution

$$q(t) = e^{F(t-t_o)}\, q(t_o) \qquad 1.3$$

But since $z(t_o) = Tx(t_o)$, it follows that

$$q(t_o) = 0$$

Then Equation 1.3 gives

$$q(t) = 0, \; t \geq t_o,$$

or

$$z(t) = T\, x(t) \; .$$

This completes the proof.

One should note that the systems S_1 and S_2 need not have the same dimension. It can be shown that |4| there exists a unique T, for the equation

$$TA - FT = GC$$

provided that A and F do not have common eigenvalues.

DEFINITION 1.1: Any system S_2 which tracks a linear transformation of the states of the system S_1 is an observer for the system S_1 in the sense of Theorem 1.1.

In the above theorem, the initial condition $x(t_o)$ was assumed to be known. If the initial condition $x(t_o)$ is not known then $z(t_o)$, the initial condition of the state of the observer, may be arbitrarily assigned to be

$$z(t_o) = T\, x_g \qquad 1.4$$

for some n-dimensional vector x_g.

In this case, the following differential equation

$$\dot{z}(t) - T\,\dot{x}(t) = F[z(t) - Tx(t)]$$

yields a solution of the form

$$z(t) - Tx(t) = e^{F(t-t_o)}[z(t_o) - Tx(t_o)] \qquad 1.5$$

It follows from Equation 1.4 that

$$z(t_o) \neq T\,x(t_o)$$

then it is evident from Equation 1.5 that z(t) cannot track a linear transformation of x(t) exactly but with some error which is due to the uncertainty in the initial condition. This error, designated by e(t), is known as the observer error and it is defined by

$$e(t) \overset{\Delta}{=} z(t) - T\,x(t) \qquad 1.6$$

Then, is follows from Equation 1.5 that

$$e(t) = {}^{F(t-t_o)}\, e(t_o) \qquad 1.7$$

where

$$e(t_o) = z(t_o) - T\,x(t_o)\ . \qquad 1.8$$

The error caused by the uncertainty in the initial condition will propagate in time as shown in Equation 1.7 and will diminish as time increases if and only if F is a stable matrix. Since the matrix F is the state matrix of the observer, the observer must be an asymptotically stable system so that the observer error decreases with time and becomes zero at steady state.

1.2. IDENTITY OBSERVER

If it is required that the order of the observer S_2 be the same as that of the system S_1, then the most convenient transformation is the identity transformation, i.e., $T = I$. Then,

$$TA - FT = GC$$

becomes

$$F = A - GC \tag{1.9}$$

where F and G are (nxn) and (nxm) matrices respectively.

Since the matrices A and C in Equation 1.9 are fixed by the system S_1, only the (nxm) dimensional matrix G is selected to determine the dynamics of the identity observer, which is given by the following differential equation.

$$\dot{z}(t) = (A-GC)z(t) + G\ y(t) + B\ u(t)$$

We now will state two theorems related to the design of the identity observer.

<u>THEOREM 1.2</u>: For the real matrices A and C, the eigenvalues of (A-GC) can be assigned from a desired set of eigenvalues by a suitable choice of the real matrix G, if and only if (A,C) is a completely observable pair.

For a possible proof one may refer to |5|.

<u>THEOREM 1.3</u>: An identity observer with arbitrary dynamics can be designed for a linear time-invariant system if, and only if the system is completely observable.

Proof is an obvious result of Theorem 1.2.

The eigenvalues of the identity observer are chosen to have more negative real parts than those of the observed system so that the performance of the overall system is not significantly delayed.

An example is presented below to elucidate the design of the identity observer for a completely observable system.

Example 1: A second order system is given in Figure 1.1. It is desired to design an identity observer for the system so that the two states will be available to be used for feedback purposes.

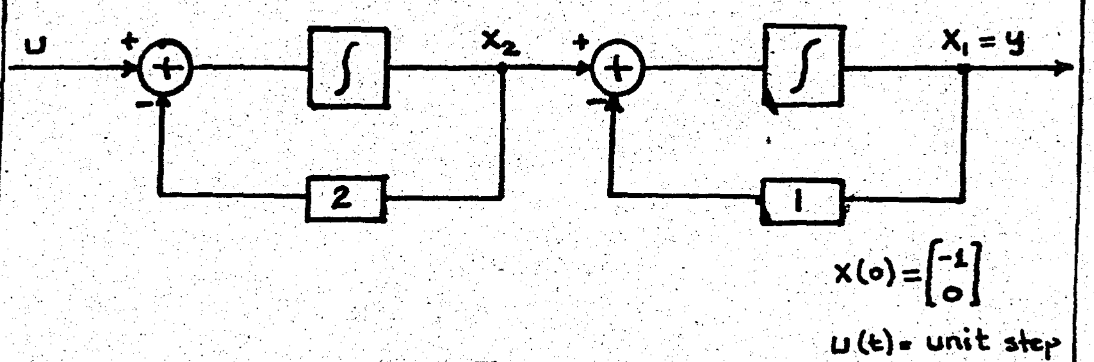


Fig 1.1

The state space representation of the above system is:

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad 1.10$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

The solution of the differential equation in Equation 1.10, for the unit step input

$$x(t) = \begin{bmatrix} \frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2} e^{-2t} \end{bmatrix}, \quad t \geq 0 \qquad 1.11$$

shows the behavior of the states as time elapses. But, only one state, i.e. $x_1(t)$, is available at the output.

It is shown below that an identity observer designed for the system in Equation 1.10 will provide $x_1(t)$ and $x_2(t)$. The dynamics equation of the observer is given by

$$\dot{z}(t) = F\ z(t) + G\ y(t) + B\ u(t)$$

where

$$F = A - GC\ .$$

The matrix G must be selected to determine the matrix F, then letting

$$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

yields

$$F = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -1-g_1 & 1 \\ -g_2 & -2 \end{bmatrix}$$

The characteristic equation of the matrix F is

$$[\lambda I - F] = \lambda^2 + (3+g_1)\lambda + 2 + 2g_1 + g_2 = 0 \qquad 1.12$$

The eigenvalues of F are chosen to be more negative than those of A, so that the system performance is not affected.

The eigenvalues of the original system are -1 and -2, as can be seen in Equation 1.11.
Then, the eigenvalues of F are chosen as

$$\lambda_1 = -3 \quad \text{and} \quad \lambda_2 = -4 .$$

If the characteristic equation corresponding to λ_1 and λ_2,

$$\lambda^2 + 7\lambda + 12 = 0$$

is compared with Equation 1.12, g_1 and g_2 are found to be

$$g_1 = 4 \quad \text{and} \quad g_2 = 2 .$$

Then the differential equation governing the observer given by

$$\dot{z}(t) = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 4 \\ 2 \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

yields the solution for the unit step input as follows

$$z(t) = \begin{bmatrix} \frac{1}{2} - 2e^{-t} + \frac{1}{2} e^{-2t} \\ \frac{1}{2} - \frac{1}{2} e^{-2t} \end{bmatrix} \quad t \geq t_o \qquad 1.13$$

The result found in Equation 1.13 is identical to the one found in Equation 1.11. Figure 1.2 illustrates the overall system.

The eigenvalues of the observer have been selected so far to correspond to a given set of eigenvalues. Another approach considers that the error

$$e(t) = z(t) - T\, x(t)$$

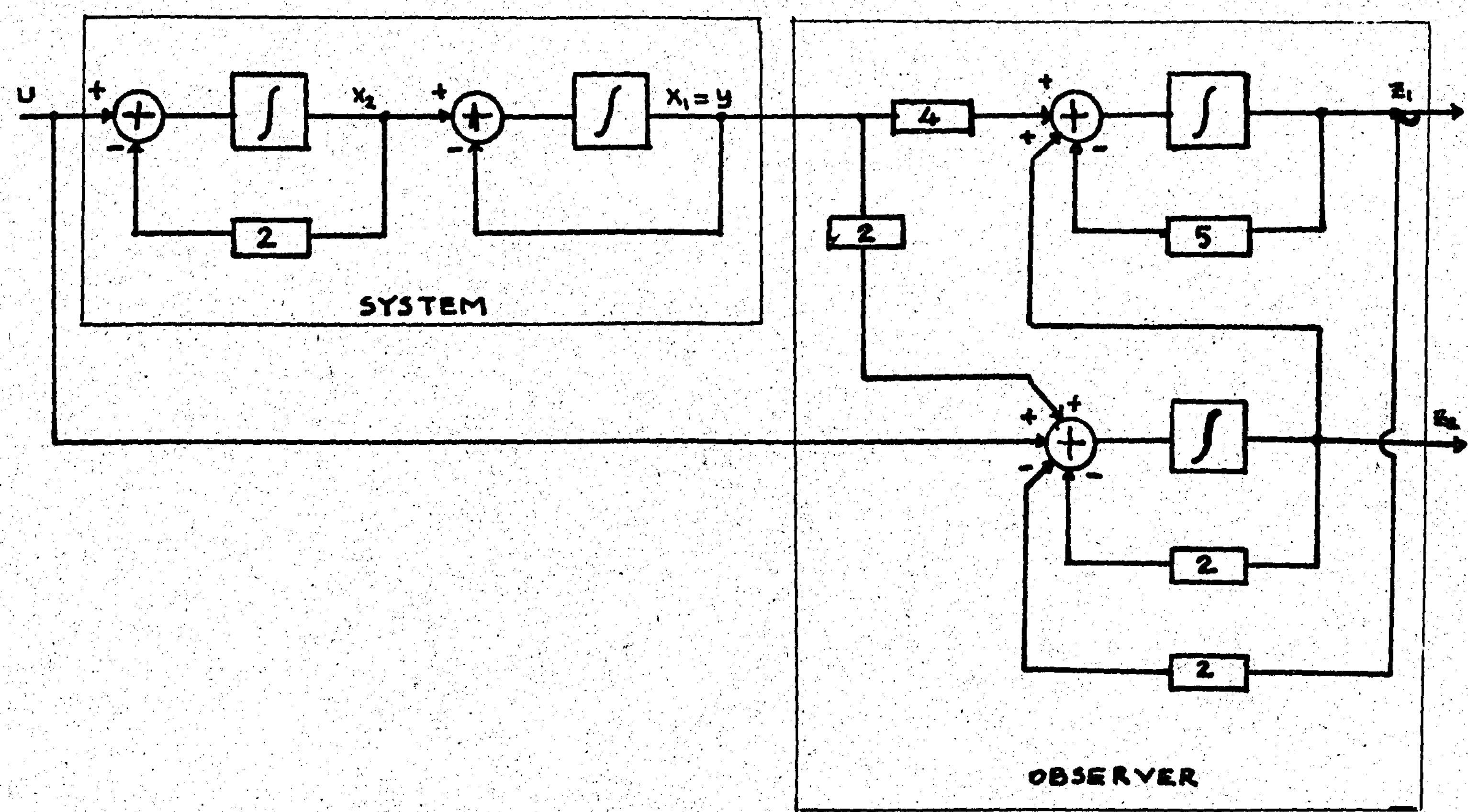


Fig 1.2

due to some uncertainties in the initial value of the state vector $x(t_o)$ dies out exponentially. Since,

$$\dot{e}(t) = F\, e(t) \qquad 1.14$$

the above statement is equivalent to stating that the observer system matrix F must be determined to ensure uniform asymptotic stability |7|.

To this end, a Lyapunov function of the form

$$V(e) = e'(t)\, R\, e(t)$$

where R is an (nxn) real symmetric positive definite matrix, together with Equation 1.14, yields

$$\dot{V}(e) = -e'(t)(F'R + RF)\, e(t) \qquad 1.15$$

or

$$\dot{V} = -e'(t)\, Q\, e(t) \qquad 1.16$$

where Q is a (nxn) real symmetric positive definite matrix.

For asymptotic stability of the matrix F, |6| the following conditions must hold

1. $V(o) = 0$
2. $V(e) > 0$ for $e \neq 0$
3. $\dot{V}(e) < 0$ for $e \neq 0$

Then from Equations 1.15 and 1.16, it follows that

$$F'R + RF + Q = 0 \,. \qquad 1.17$$

Since, it is known that for an identity observer

$$F = A - GC$$

then Equation 1.17 becomes

$$(A-GC)'R + R(A-GC) + Q = 0 \tag{1.18}$$

If G is selected as

$$G = \frac{1}{2} R^{-1} C' K \tag{1.19}$$

where K is an (mxm) arbitrary real symmetric positive semidefinite matrix, where m is the dimension of the output vector. Selection of the arbitrary matrix K is treated in the computational aspects of Chapter 2.

Inserting Equation 1.19 into Equation 1.18 yields

$$(A - \frac{1}{2} R^{-1} C'KC)'R + R(A - \frac{1}{2} R^{-1}C'KC) + Q = 0$$

$$A'R + RA + Q - C'KC = 0 \tag{1.20}$$

Equation 1.20 is solved for the matrix R, then the result is substituted into Equation 1.19 to evaluate the matrix G. Once G is found, F is determined by

$$F = A - GC \ .$$

The matrix F found by this method is a stable matrix, therefore ensures that the error

$$e(t) = z(t) - T\, x(t)$$

will diminish as time increases regardless of the initial uncertainties,

$$e(t_o) = z(t_o) - T\, x(t_o) \ .$$

Example 2: The system is as given in Example 1. It is desired to design an identity observer for the system. The initial value of the state vector is not known by the designer.

The differential equation that governs the identity observer is

$$\dot{z}(t) = F\ z(t) + G\ y(t) + B\ u(t)$$

where

$$F = A - GC\ .$$

G must be determined such that F is a stable matrix. Choosing Q and K as

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and } K = 4$$

and inserting into

$$A'R + RA - C'\ KC + Q = 0$$

yields a symmetric positive definite matrix R as,

$$R = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

Substituting this result into

$$G = \frac{1}{2}\ R^{-1}\ C'\ K$$

it is found that

$$G = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The matrix F is found from

$$F = A - GC$$

as

$$F = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

The eigenvalues of F are -1 and -3, hence F is a stable matrix. The observer is given by

$$\dot{z}(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

As stated earlier, the observer output z(t) approaches to the true value of the state x(t) at steady state regardless of the choice of the initial conditions for the observer. We now will show that an arbitrary initial condition will give satisfactory results.

Choosing,

$$z(o) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

and noting from Example 1 that

$$y(t) = \frac{1}{2} - 2e^{-t} + \frac{1}{2} e^{-2t}$$

the solution of the above differential equation is given by

$$z(t) = e^{Ft} z(o) + \int_o^t \{e^{F(t-\tau)}Gy(\tau) + e^{F(t-\tau)}Bu(\tau)\} d\tau$$

yields, for unit step input

$$z(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} e^{-t} + \frac{1}{2} e^{-2t} + \frac{1}{2} e^{-3t} \\ \frac{1}{2} + \frac{5}{2} e^{-t} - \frac{1}{2} e^{-2t} - \frac{1}{2} e^{-3t} \end{bmatrix}$$

Next the observer's states and the true states of the system are plotted in Figure 1.3 to show the convergence as time increases.

Although the identity observer can easily be implemented, yet it is not very attractive since it possesses redundancy. The redundancy is due to the fact that the identity observer, while estimating the non-available states, estimates the already available states as well. To eliminate this redundancy, the design of an observer of lower dimension is suggested.

1.3. MINIMAL ORDER OBSERVER

The following two chapters consider the design procedure of minimal order observers for continuous time and discrete time linear systems. In this section only the basic structure of such observers is given.

Consider the system

$$\dot{x}(t) = A\, x(t) + B\, u(t)$$
$$y(t) = C\, x(t)$$

An observer of order n-m, where n is the order of the state vector and m is the order of the output vector of the above system, is constructed to estimate the non-available states. The output of the observer and the output of the original system are then used to estimate the entire state vector as follows

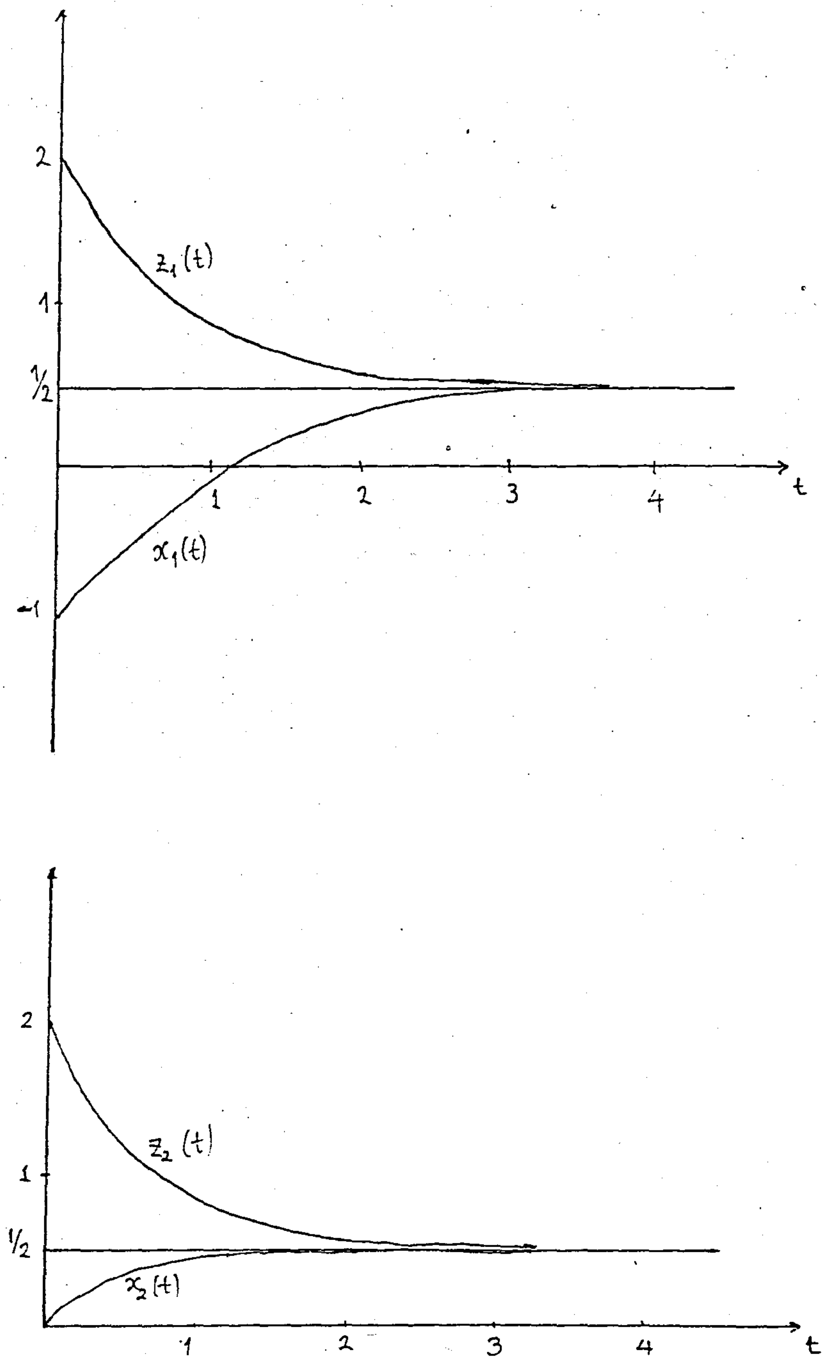


Fig 1.3

$$\hat{x}(t) = \left[\begin{array}{c} T \\ \hline C \end{array}\right]^{-1} \left[\begin{array}{c} z(t) \\ \hline y(t) \end{array}\right]$$

where T is a mxn matrix to be determined. Determination of the matrix T is treated in the following chapter. The block diagram below illustrates the structure of the minimal order observer.

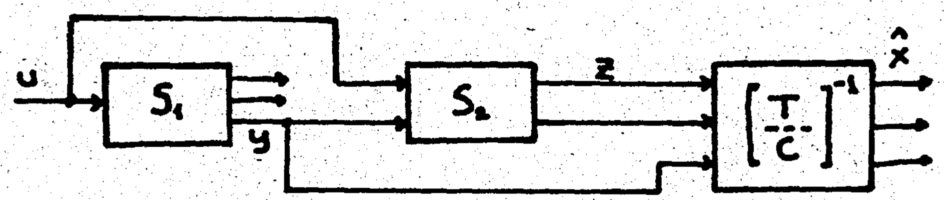


Fig 1.4

CHAPTER 2

MINIMAL ORDER OBSERVERS FOR DETERMINISTIC CONTINUOUS TIME SYSTEMS

Consider the linear-time invariant continuous time system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y &= C\,x(t) \end{aligned} \qquad 2.1$$

where A, B and C are nxn, nxr and mxn matrices respectively. Furthermore, C is of full rank m and (A,C) is a completely observable pair.

From Theorem 1.1, it follows that an observer for the system in Equation 2.1 can be constructed as

$$\begin{aligned} \dot{z}(t) &= F\,z(t) + G\,y(t) + H\,u(t) \\ z(t_o) &= T\,x(t_o) \end{aligned}$$

where F, G and H are pxp, pxm and pxr matrices respectively, $(p \leq n)$, satisfying the constraint equations

$$\begin{aligned} TA - FT &= GC \\ H &= TB \end{aligned} \qquad 2.2$$

for some pxn matrix T. This observer estimates the states x(t) by

$$\hat{x}(t) = \begin{bmatrix} T \\ \hline C \end{bmatrix}^{*} \begin{bmatrix} z(t) \\ \hline y(t) \end{bmatrix}$$

where (*) denotes a special inverse which can be evaluated as follows:

1. If $p+m<n$, $\left[\begin{array}{c} T \\ \hline C \end{array}\right]^{*}$ does not exist.

2. If $p+m=n$, $\left[\begin{array}{c} T \\ \hline C \end{array}\right]^{*} = \left[\begin{array}{c} T \\ \hline C \end{array}\right]^{-1}$

3. If $p+m>n$. Then form a square matrix using any n linearly independent rows from $\left[\begin{array}{c} T \\ \hline C \end{array}\right]$ and apply ordinary matrix inversion procedures.

The upper bound for p is n and when p=n an identity observer may be constructed for the system in Equation 2.1. The designer will naturally try to set p to the smallest value it may attain in order to have the simplest form for the observer. Then the question arises: Does there exist a lower bound for p? The answer is given by the following theorem.

<u>THEOREM 2.1:</u> The order of the observer designed for the system in Equation 2.1 can not be less than n-m.

<u>PROOF:</u> In order for the p-dimensional observer

$$\dot{z}(t) = F\, z(t) + G\, y(t) + H\, u(t)$$

to estimate the state x(t) by an estimate of the form

$$\hat{x}(t) = \left[\begin{array}{c} T \\ \hline C \end{array}\right]^{*} \left[\begin{array}{c} z(t) \\ \hline y(t) \end{array}\right]$$

the indicated inverse must exist. This inverse exists, if $p+m \geq n$ or $p \geq n-m$. The the smallest value for p is n-m, where p is the order of the observer. This completes the proof.

The observer whose order is n-m is called the *Minimal Order Observer*. All the derivations developed in the rest of this chapter concern only the minimal order observers.

It is inevitable that the observer involves the dynamics of the observed system, or equivalently, the observed system constrains the dynamics of the observer. Therefore, the relation between the observer and the observed system is brought out in the form of a set of constraints.

To this end, define

$$\begin{bmatrix} T \\ \hdashline C \end{bmatrix}^{-1} = \left[P \;\vdots\; V \right]$$

where P and V are nx(n-m) and (nxm) matrices respectively, then

$$PT + VC = I_n \qquad 2.3$$

Premultiplying both sides of Equation 2.3 by TA

$$TAPT + TAVC = TA$$

is obtained. This result is then substituted in Equation 2.2

$$TAPT + TAVC - FT - GC = 0$$

$$(TAP - F)T + (TAV - G)C = 0 \qquad 2.4$$

Since rows of T and C are chosen to be linearly independent in Equation 2.4, it follows that

$$F = TAP$$

$$G = TAV \;.$$

The equations for the minimal order observer are:

$$\dot{z}(t) = F\ z(t) + G\ y(t) + H\ u(t)$$

$$\hat{x}(t) = P\ z(t) + V\ y(t)$$

where $z(t)$ is the $(n-m)$ dimensional observer state vector, F is the $(n-m) \times (n-m)$ state matrix, G and H are the $(n-m)\times m$ and $(n-m)\times r$ input matrices respectively.

The constraint equations are:

$$PT + VC = I_n$$

$$F = TAP$$

$$G = TAV$$

$$H = TB$$

From the above constraints it is clearly seen that the design of the minimal order observer hinges on the selection of the matrix T.

If $x(t_o)$ is known, it has been shown that

$$z(t) = T\ x(t)\ , \quad t \geq t_o$$

and that $x(t)$ can be reconstructed exactly by

$$\hat{x}(t) = PT\ x(t) + VC\ x(t)$$

$$\hat{x}(t) = (PT + VC)\ x(t)$$

$$\hat{x}(t) = x(t)\ .$$

If $x(t_o)$ is not known, then there exists an observer error defined by

$$e(t) \triangleq z(t) - T\ x(t) \qquad 2.5$$

which satisfies

$$\dot{e}(t) = F\ e(t), \quad e(t_o) = z(t_o) - T\ x(t_o) \tag{2.6}$$

Our goal is then to find a stable matrix F so that the observer error decreases as time increases.

The next section presents an easy solution for the design of a minimal order observer for the system in Equation 2.1, considering, as a design criterion, the stability of the free system in Equation 2.6.

2.1. A CANONICAL CLASS OF OBSERVERS |7|

Consider the system given by Equation 2.1

$$\dot{x}(t) = A\ x(t) + B\ u(t)$$
$$y(t) = C\ x(t)$$

and recall that C is of full rank m.

Collecting the linearly independent columns, the matrix C may be partitioned as

$$C = \left[C_1 \;\vdots\; C_2 \right]$$

where C_1 is mxm non-singular matrix. One should note that collecting the linearly independent columns of the matrix C to form C_1 as a non-singular matrix may necessitate re-numbering the state variables.

There exists a (nxn) non-singular transformation matrix M given by

$$M = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline 0 & I_{n-m} \end{array}\right]$$

which can be used to define a state transformation

$$q(t) = M\,x(t) \qquad 2.7$$

and

$$\dot{q}(t) = M\,\dot{x}(t) \quad \text{with } q(t_o) = M\,x(t_o) \qquad 2.8$$

The non-singularity of M ensures the existence of the inverse given by

$$M^{-1} = \left[\begin{array}{c|c} C_1^{-1} & -C_1^{-1}C_2 \\ \hline 0 & I_{n-m} \end{array}\right]$$

Solving Equation 2.7 and 2.8 for $x(t)$ and $\dot{x}(t)$ as

$$x(t) = M^{-1}\,q(t)$$

$$\dot{x}(t) = M^{-1}\,\dot{q}(t)$$

and substituting these equalities into Equation 2.1 the following system of equations are obtained.

$$\dot{q}(t) = \tilde{A}\,q(t) + \tilde{B}\,u(t)$$

$$y(t) = \tilde{C}\,q(t) = \left[I_m \,\vdots\, 0\right] q(t) \qquad 2.9$$

where

$$\tilde{A} = MAM^{-1}$$

$$\tilde{B} = MB$$

$$\tilde{C} = CM^{-1}$$

The similarity transformation

$$MAM^{-1}$$

does not alter the stability, observability and controllability of the original system.

Partitioning $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A} = \begin{array}{cc} \begin{array}{cc} m & n-m \end{array} & \\ \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] & \begin{array}{c} m \\ n-m \end{array} \end{array}$$

$$\tilde{B} = \begin{array}{cc} r & \\ \left[\begin{array}{c} \tilde{B}_1 \\ \hline \tilde{B}_2 \end{array}\right] & \begin{array}{c} m \\ n-m \end{array} \end{array}$$

then substituting into Equation 2.9 yields

$$\left[\begin{array}{c} \dot{q}_1(t) \\ \hline \dot{q}_2(t) \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \left[\begin{array}{c} q_1(t) \\ \hline q_2(t) \end{array}\right] + \left[\begin{array}{c} \tilde{B}_1 \\ \hline \tilde{B}_2 \end{array}\right] u(t)$$

$$y(t) = \tilde{C} \left[\begin{array}{c} q_1(t) \\ \hline q_2(t) \end{array}\right] = \left[\begin{array}{c|c} I_m & 0 \end{array}\right] \left[\begin{array}{c} q_1(t) \\ \hline q_2(t) \end{array}\right] \qquad 2.13$$

where $q_1(t)$ and $q_2(t)$ are m and (n-m)-dimensional vectors respectively.

It is seen from Equation 2.13 that the states $q_2(t)$ are not available. The following minimal order observer is designed to estimate the non-available states, $q_2(t)$,

$$\dot{z}(t) = F\ z(t) + G\ y(t) + H\ u(t)$$

This minimal order observer satisfies the following constraints:

$$PT + V\tilde{C} = I_n \ , \quad \text{or} \quad \left[\begin{array}{c} T \\ \hline \tilde{C} \end{array}\right]^{-1} = \left[\begin{array}{c|c} P & V \end{array}\right]$$

$$F = T\tilde{A}P$$

$$G = T\tilde{A}V$$

$$H = T\tilde{B}$$

Since the matrix

$$\left[\begin{array}{c} T \\ \hline \tilde{C} \end{array}\right]$$

must be invertible, the simplest choice of the matrix T is

$$T = \left[\begin{array}{c|c} -L & I_{n-m} \end{array}\right] \qquad 2.14$$

where L is an arbitrary (n-m)xm matrix. This choice of the matrix T ensures the existence of the indicated inverse as shown below

$$\left[\begin{array}{c} T \\ \hline \tilde{C} \end{array}\right]^{-1} = \left[\begin{array}{c|c} -L & I_{n-m} \\ \hline I_m & 0 \end{array}\right]^{-1} \qquad 2.15$$

since the right side of Equation 2.15 is of rank n.

The constraint equation

$$PT + V\tilde{C} = I_n$$

with the substitution of Equation 2.14 and upon being partitioned, may be written as

$$\left[\begin{array}{c} P_1 \\ \hline P_2 \end{array}\right] \left[\begin{array}{c|c} -L & I_{n-m} \end{array}\right] + \left[\begin{array}{c} V_1 \\ \hline V_2 \end{array}\right] \left[\begin{array}{c|c} I_m & 0 \end{array}\right] = \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & I_{n-m} \end{array}\right]$$

Then the following set of matrix equations is obtained.

$$-P_1L + V_1 = I_m \qquad 2.16$$

$$P_1 = 0 \qquad 2.17$$

$$-P_2L + V_2 = 0 \qquad 2.18$$

$$P_2 = I_{n-m} \qquad 2.19$$

From Equations 2.17 and 2.19

$$P = \begin{bmatrix} 0 \\ \hline I_{n-m} \end{bmatrix} \qquad 2.20$$

and from Equations 2.16, 2.17 and 2.18

$$V = \begin{bmatrix} I_m \\ \hline L \end{bmatrix} \qquad 2.21$$

are found.

Straight forward substitution of the matrices T, Equation 2.14, P, Equation 2.20 and V, Equation 2.21 into the following equations

$$F = T\tilde{A}P$$
$$G = T\tilde{A}V$$
$$H = T\tilde{B}$$

the matrices F, G and H are found to be

$$F = \tilde{A}_{22} - L\tilde{A}_{12}$$
$$G = \tilde{A}_{21} - L\tilde{A}_{11} + FL$$
$$H = \tilde{B}_2 - L\tilde{B}_1$$

At this stage the role played by observability must be examined. "Observability" means that one can, in principle, determine the initial state of an observable system from its output measurements. The idea of 'state reconstructability' follows by noting that knowledge of $q(t_o)$ and Equation 2.13 is sufficient to determine $q(t)$ for all $t \geq t_o$. In the context of a 'canonical' class of observers the following definition is appropriate.

DEFINITION 2.1

The system in Equation 2.13 is state reconstructible if there exists an observer

$$\dot{z}(t) = (\tilde{A}_{22} - L\tilde{A}_{12})z(t) + (\tilde{A}_{21} - L\tilde{A}_{11} + FL)y(t) + (\tilde{B}_2 - L\tilde{B}_1)u(t)$$

$$\hat{q}(t) = \begin{bmatrix} 0 \\ \hline I_{n-m} \end{bmatrix} z(t) + \begin{bmatrix} I_m \\ \hline L \end{bmatrix} y(t) \qquad 2.22$$

with $z(t_o) = \left[-L \mid I_{n-m} \right] q_g$ where q_g is arbitrary such that

$$\lim_{t\to\infty} (\hat{q}(t) - q(t)) = 0 .$$

THEOREM 2.3: If the observer is asymptotically stable then the system in Equation 2.13 is state reconstructable.

PROOF: Define

$$\varepsilon(t) \triangleq \hat{q}(t) - q(t) \qquad 2.23$$

and substitute Equation 2.22 into Equation 2.23 to obtain

$$\varepsilon(t) = \begin{bmatrix} 0 \\ \hline I_{n-m} \end{bmatrix} z(t) + \begin{bmatrix} I_m \\ \hline L \end{bmatrix} y(t) - q(t)$$

or

$$\varepsilon(t) = \begin{bmatrix} 0 \\ \hline I_{n-m} \end{bmatrix} z(t) + \begin{bmatrix} I_m \\ \hline L \end{bmatrix} \left[I_m \mid 0 \right] - \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & I_{n-m} \end{array} \right] q(t)$$

$$\varepsilon(t) = \begin{bmatrix} 0 \\ \hline I_{n-m} \end{bmatrix} z(t) - \begin{bmatrix} & 0 & \\ \hline -L & \vdots & I_{n-m} \end{bmatrix} q(t)$$

$$\varepsilon(t) = \begin{bmatrix} 0 \\ \hline z(t) - \begin{bmatrix} -L & \vdots & I_{n-m} \end{bmatrix} q(t) \end{bmatrix}$$

Recalling $\quad e(t) = z(t) - T\, x(t)$

and

$$T = \begin{bmatrix} -L & \vdots & I_{n-m} \end{bmatrix}$$

$$\varepsilon(t) = \begin{bmatrix} 0 \\ \hline e(t) \end{bmatrix} \qquad 2.24$$

is found.

For the state reconstructability

$$\lim_{t\to\infty} (\hat{q}(t) - q(t)) = \lim_{t\to\infty} \varepsilon(t) = 0$$

must hold.

From Equation 2.24 it is seen that

$$\lim_{t\to\infty} e(t) = 0 \qquad 2.25$$

implies that

$$\lim_{t\to\infty} \varepsilon(t) = 0$$

On the other hand the observer error given by

$$\dot{e}(t) = F\, e(t)$$

satisfies the requirement in Equation 2.25 if, and only if F is a stable matrix.

This completes the proof.

Next question is whether there exists a matrix L, such that

$$F = \tilde{A}_{22} - L\tilde{A}_{12}$$

is a stable matrix. A direct application of Theorem 1.2 ensures the existence of a stable matrix F, provided that $(\tilde{A}_{22}, \tilde{A}_{12})$ pair is completely observable.

THEOREM 2.4 |5|: The observability of the pair $(\tilde{A},\tilde{C})$ implies that $(\tilde{A}_{22},\tilde{A}_{12})$ is also an observable pair.

PROOF: Since $(\tilde{A},\tilde{C})$ is a completely observable pair

$$\text{rank}\begin{bmatrix} \tilde{C} \\ \hline \tilde{C}\tilde{A} \\ \hline \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = n .$$

where

$$\tilde{C} = \left[I_m \;\vdots\; 0 \right]$$

then

$$\text{rank}\begin{bmatrix} \tilde{C} \\ \hline \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \text{rank}\left[\begin{array}{c|c} I_m & 0 \\ \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{11} + \tilde{A}_{12}\tilde{A}_{21} & \tilde{A}_{11}\tilde{A}_{12} + \tilde{A}_{12}\tilde{A}_{22} \end{array}\right]$$

Elementary row and column operations do not alter the rank of a matrix. Then

$$\text{rank}\left[\begin{array}{c|c} I_m & 0 \\ \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{11} + \tilde{A}_{12}\tilde{A}_{21} & \tilde{A}_{11}\tilde{A}_{12} + \tilde{A}_{12}\tilde{A}_{22} \\ \vdots & \vdots \end{array}\right] = \text{rank}\left[\begin{array}{c|c} I_m & 0 \\ 0 & \tilde{A}_{12} \\ \vdots & \tilde{A}_{12}\tilde{A}_{22} \\ \vdots & \vdots \\ 0 & \tilde{A}_{12}\tilde{A}_{22}^{\,n-1} \end{array}\right]$$

$$= n \qquad 2.26$$

Irrespective of the left column matrix in Equation 2.26

$$\text{rank} \begin{bmatrix} 0 \\ \tilde{A}_{12} \\ \tilde{A}_{12}\tilde{A}_{22} \\ \vdots \\ \tilde{A}_{12}\tilde{A}_{22}^{\,n-1} \end{bmatrix} = n-m$$

Cayley-Hamilton theorem implies that only the terms upto $\tilde{A}_{12}\tilde{A}_{22}^{\,n-m-1}$ are needed, then

$$\text{rank} \begin{bmatrix} \tilde{A}_{12} \\ \tilde{A}_{12}\tilde{A}_{22} \\ \vdots \\ \tilde{A}_{12}\tilde{A}_{22}^{\,n-m-1} \end{bmatrix} = n-m \; . \qquad 2.27$$

Equation 2.27 shows that $(\tilde{A}_{22},\tilde{A}_{12})$ is a completely observable pair.

Theorem 2.4 guarantees the existence of a matrix L, such that the matrix F given by the following equation is stable.

$$F = \tilde{A}_{22} - L\,\tilde{A}_{12}$$

If one determines the matrix L which makes F a stable matrix, then the dynamics of the observer are known since the matrices G and H are also functions of the matrix L. Then, the design of the minimal order observer reduces to the determination of the matrix L. Next is a suggestion to evaluate the matrix L.

A suitable quadratic Lyapunov function for the free system

$$\dot{e}(t) = F\,e(t)$$

is generated. Choosing

$$\dot{V}_L = -e'(t)\ Q\ e(t)$$

where Q is a (n-m)x(n-m) real symmetric positive definite matrix. It follows that a Lyapunov function of the form

$$V_L = e'(t)\ R\ e(t)$$

exists, where R is a (n-m)x(n-m) real symmetric positive definite matrix satisfying

$$F'R + RF + Q = 0 . \qquad 2.28$$

Replacing F by

$$F = \tilde{A}_{22} - L\ \tilde{A}_{12}$$

Equation 2.28 becomes

$$(\tilde{A}_{22} - L\tilde{A}_{12})'R + R(\tilde{A}_{22} - L\tilde{A}_{12}) + Q = 0 \qquad 2.29$$

Select L as

$$L = \frac{1}{2}\ R^{-1}\ \tilde{A}_{12}'\ K \qquad 2.30$$

where K is an arbitrary (mxm) real symmetric positive semi definite matrix, then substitute in Equation 2.29 to obtain

$$\tilde{A}_{22}'R + R\ \tilde{A}_{22} + Q - \tilde{A}_{12}'\ K\ \tilde{A}_{12} = 0 .$$

This algebraic matrix equation yields a unique solution for the symmetric positive definite matrix R |7|. This solution is then substituted into Equation 2.30 and the matrix L is evaluated. Once the matrix L is found the matrices F, G and H are calculated by

$$F = \tilde{A}_{22} - L\ \tilde{A}_{12}$$

$$G = \tilde{A}_{21} - L\,\tilde{A}_{11} + FL$$
$$H = \tilde{B}_2 - L\,\tilde{B}_1$$

Once the minimal order observer is designed for the canonical system, the estimate of the states of the original system can be obtained by the following transformation.

$$\hat{x}(t) = M^{-1}\,\hat{q}(t)\ .$$

Figure 2.1 illustrates the block diagram of the overall system.

Examples concerning the continuous-time deterministic minimal order observers can be found in Appendix A. The reader is suggested to read "Computational Aspects" section before he refers to the examples.

2.2. COMPUTATIONAL ASPECTS

The design procedure outlined in the previous section considers the case that the steady state estimation error approaches zero. It was also shown that in order to achieve a steady state estimation error equal to zero, the observer must be stable. Since the error in the transient response is not of interest, we design a constant eigenvalued stable observer to satisfy the requirement of the steady state error. It is deduced from this statement that the state and input matrices of the observer are computed once, in other words, off-line.

In view of computational aspects, this off-line design procedure mainly involves:

1. Transformation into the canonical representation
2. Solution of the Lyapunov equation.

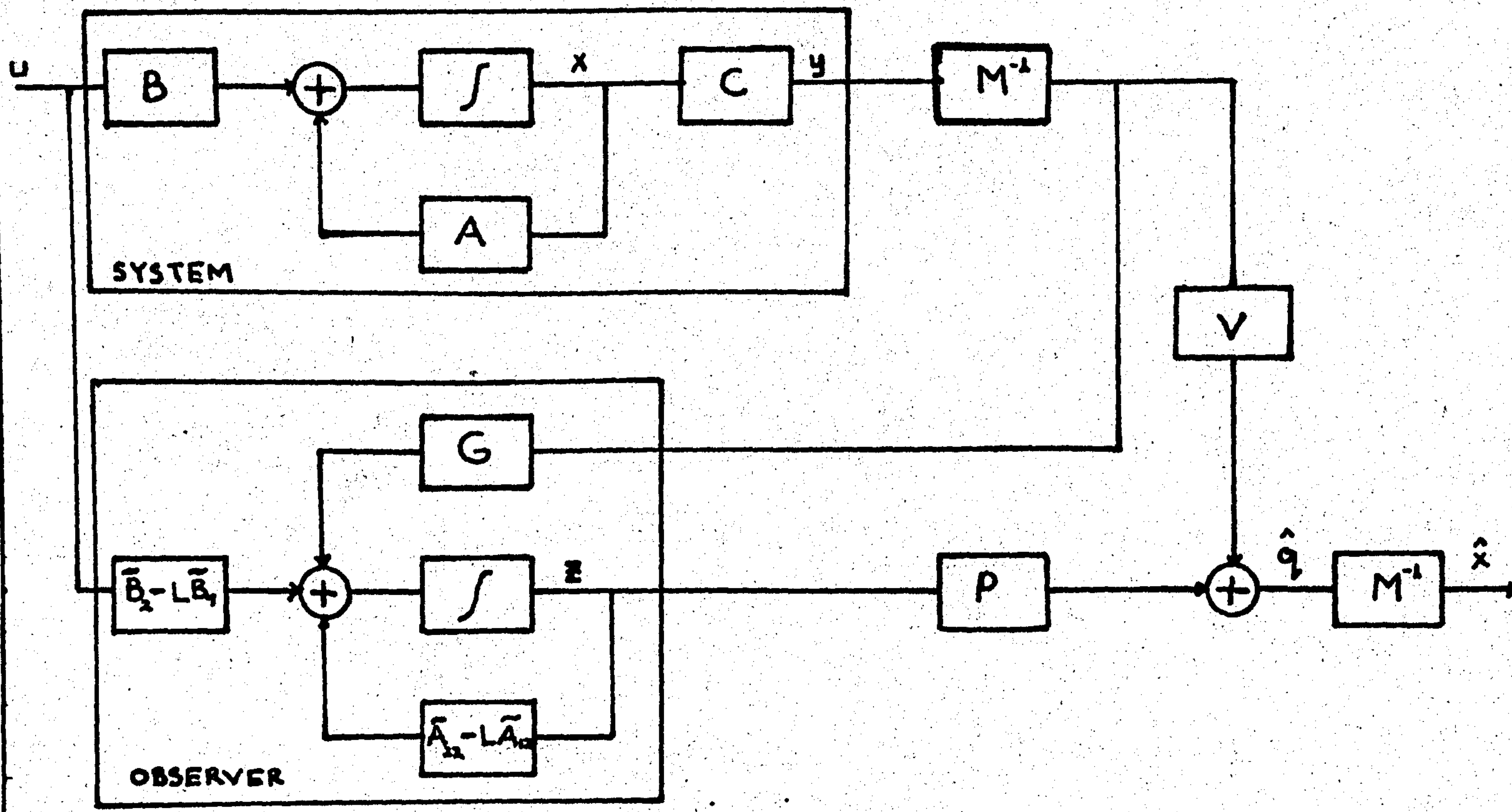

u
B
x
C
y
M^{-1}
A
SYSTEM
V
G
$\bar{B}_2-L\bar{B}_1$
z
$\tilde{A}_{22}-L\tilde{A}_{12}$
OBSERVER
P
$\hat{q}$
M^{-1}
$\hat{x}$

Fig 2.1

2.2.1. Transformation into the Canonical Representation

This transformation hinges upon finding a non-singular matrix M such that the output matrix C of the system equation upon being post multiplied by M^{-1} takes the form shown below

$$\left[I_m \;\vdots\; 0 \right] = C\, M^{-1} \tag{2.31}$$

This problem may be viewed as the reduction of a mxn(m<n) full rank matrix C given as

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \\ c_{m1} & \cdots & \cdots & c_{mn} \end{bmatrix}$$

to the following mxn matrix Φ

$$\Phi = \left[\begin{array}{cccc|c} \Phi_{11} & & & & \\ & \Phi_{22} & & & \\ & & \ddots & & 0 \\ & & & \Phi_{mm} & \end{array} \right]$$

which is a collection of m row vectors each containing only one non-zero element located at the ith column.

The indicated reduction can be obtained by elementary column operations. The ith row vector in the reduced matrix is obtained by subtracting an appropriate multiple of the ith column from the remaining columns. This operation is equivalent to post-multiplying by an (nxn) elementary matrix PM(i) given by

$$
\begin{array}{rc}
 & \begin{array}{c}\text{ith column}\\ \downarrow\end{array} \\
PM(i) = & \left[\begin{array}{cc|c|c}
I & & 0 & 0 \\
\hline
-\dfrac{\gamma_{i1}^{(i-1)}}{\gamma_{ii}^{(i-1)}} & -\dfrac{\gamma_{i2}^{(i-1)}}{\gamma_{ii}^{(i-1)}} & 1 & -\dfrac{\gamma_{in}^{(i-1)}}{\gamma_{ii}^{(i-1)}} \\
\hline
0 & & 0 & I
\end{array}\right] \\
\text{ith row} \rightarrow &
\end{array}
$$

Since the matrix C is post-multiplied successively by the above matrices, i=1,...m, the elements of the matrix C are assigned new values after each multiplication. We have found appropriate to use $\gamma_{ij}^{(i-1)}$ to denote the values of the matrix C which are obtained when the multiplication by PM(i-1) has been performed, that is

$$\Gamma^{(i-1)} = C\ PM(o)\ PM(1)\ \ldots\ PM(i-1)$$

where PM(o) = I.

Upon post-multiplying the matrix C by m elementary matrices evaluated by Equation 2.32, the following equality is obtained

$$\Phi = C \prod_{i=1}^{m} PM(i)$$

If the matrix Φ is further post-multiplied by the matrix PM(m+1) which is given by

$$PM(m+1) = \left[\begin{array}{cccc|c} \frac{1}{\Phi_{11}} & 0 & \cdots & 0 & \\ 0 & \frac{1}{\Phi_{22}} & \cdots & 0 & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & \frac{1}{\Phi_{mm}} & \\ \hline & & 0 & & I_{n-m} \end{array}\right]$$

we obtain

$$\left[I_m \mid 0\right] = \{C \prod_{i=1}^{m} PM(i)\} \quad PM(m+1) \qquad 2.33$$

From Equations 2.31 and 2.33, M^{-1} is found to be

$$M^{-1} = \{ \prod_{i=1}^{m} PM(i)\} \quad PM(m+1)$$

To find the matrix M, evaluating the inverse of the matrix M^{-1} is unnecessary, since the matrix M is given by

$$M = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline 0 & I_{n-m} \end{array}\right]$$

as it was shown in the previous section.

One should note that Equation 2.32 is meaningful if

$$\gamma_{ii} \neq 0$$

The case $\gamma_{ii} = 0$ necessitates the interchange of the columns.

Next is the summary of the reduction scheme explained above.

STEP 1. Initialize $\Gamma(o)$ and $PM(i)$, $i=1,\ldots,m+1$ as follows

$$\Gamma(o) = C$$
$$PM(i) = I$$

STEP 2. $i = 1$

STEP 3. $\ell = i$

STEP 4. If $\gamma_{\ell\ell} \neq 0$ go to step 6; otherwise continue.

STEP 5. Interchange

a) the columns of the matrix $\Gamma(i-1)$

b) the elements of the kth row of the matrices $PM(k)$, $k=1,\ldots,i$

go to step 10.

STEP 6. Form $PM(i)$ as in Equation 2.32

STEP 7. Evaluate

$$\Gamma(i) = \Gamma(i-1) \cdot PM(i)$$

STEP 8. $\ell = \ell+1$

STEP 9. If $\gamma_{\ell\ell} = 0$ go to step 5; otherwise continue

STEP 10. $i = i+1$

STEP 11. If $i \leq m$ go to step 3

STEP 12. $PM = \prod_{i=1}^{m} PM(i)$

STEP 13. $M^{-1} = PM \cdot PM(m+1)$

STEP 14. STOP

2.2.2. Solution of the Lyapunov Equation

A numerical method is presented for solving the Lyapunov equation of the form

$$Y'X + XY = -Z$$

where Y is (nxn) stable matrix, Z is (nxn) positive definite matrix and X is (nxn) positive definite solution matrix |10|.

Consider the linear time-invariant system of differential equations

$$\dot{x} = Y\,x, \quad x(o) = x_o \qquad 2.34$$

and the quadratic form

$$V = x' X\,x \qquad 2.35$$

then

$$\dot{V} = -\,x' Z\,x \qquad 2.36$$

where Z is defined as follows

$$Y'X + XY = -\,Z \qquad 2.37$$

Integrating Equation 2.36 gives

$$V(t) = V(o) - \int_o^t x'Z\,x\,dt \qquad 2.38$$

and as $t \to \infty$, $x(t) \to 0$ so that

$$V(o) = x_o' \,X\,x_o = \int_o^\infty x'Z\,x\,dt \qquad 2.39$$

The numerical integration of Equation 2.39 and 2.34 may be written as

$$\int_o^\infty x'Z\,x\,dt = \sum_{k=0}^{\infty} q x_k'\,Z\,x_k \quad \text{as} \quad q \to 0 \qquad 2.40$$

and

$$x_{k+1} = (I - \frac{q}{2}\,Y)^{-1}\,(I + \frac{q}{2}\,Y)x_k \quad k=0,1,\ldots \qquad 2.41$$

Letting

$$W = (I - \frac{q}{2}\,Y)^{-1}\,(I + \frac{q}{2}\,Y) \qquad 2.42$$

and inserting this last equality into Equation 2.40, one obtains

$$x_o' X x_o = q x_o' [Z + W' Z W + W'^2 Z W^2 + \ldots] x_o$$

or

$$X = q[Z + W' Z W + W'^2 Z W^2 + \ldots] \qquad 2.43$$

Equation 2.43 may be written in the following way

$$X = \lim_{k \to \infty} X_k$$

where X_k satisfies the recursive relation

$$X_{k+1} = W'^{2^k} X_k W^{2^k} + X_k \qquad k=0,1,2,\ldots$$

which is initialized as

$$X_o = q Z \quad \text{with } q \to 0$$

This algorithm is numerically stable, since the stability of Y implies that the eigenvalues of W are inside the unit circle regardless of the value of q and converges for all values of q |17|.

The above algorithm is used to solve the Lyapunov equation

$$(\tilde{A}_{22} - L \tilde{A}_{12})' R + R(\tilde{A}_{22} - L \tilde{A}_{12}) + Q = 0$$

for the matrices L and R. The matrix L must be so chosen that a positive definite matrix R satisfies the above Lyapunov equation. To this end, choosing the matrix L as

$$L = \frac{1}{2} R^{-1} \tilde{A}_{12}' K$$

the above Lyapunov equation becomes

$$\tilde{A}_{22}' R + R \tilde{A}_{22} + Q - \tilde{A}_{12}' K \tilde{A}_{12} = 0 \qquad 2.44$$

for some arbitrary positive semi-definite matrix K. There exists in theory a positive semi-definite matrix K such that a positive definite matrix R satisfies the above equation, but there has not yet been any search method devised to obtain such a matrix K. We consider three cases below for the selection of the matrix K.

<u>CASE 1</u> - If $\tilde{A}_{22}$ is a stable matrix then select K as

$$K = 0$$

Such selection of K reduces the equation in Equation 2.44 to

$$\tilde{A}_{22}'R + R\,\tilde{A}_{22} + Q = 0$$

In this case there exists a positive definite matrix R as far as the algorithm presented above is concerned |21|. The matrix L is then found to be

$$L = 0 .$$

<u>CASE 2</u> - If all the eigenvalues of the matrix $\tilde{A}_{22}$ are positive, then multiplying Equation 2.44 by (-1) yields

$$-\tilde{A}_{22}'R - R\,\tilde{A}_{22} - Q + \tilde{A}_{12}'\,K\,\tilde{A}_{12} = 0$$

The above equation may be rewritten as

$$(-\tilde{A}_{22})'R + R(-\tilde{A}_{22}) + \tilde{A}_{12}'\,K\,\tilde{A}_{12} - Q = 0 \qquad 2.45$$

Then the eigenvalues of $(-\tilde{A}_{22})$ are negative and there exists a positive definite matrix R as a solution of the equation in Equation 2.45 if and only if

$$\tilde{A}_{12}'\,K\,\tilde{A}_{12} - Q > 0$$

Empirical results have shown that there exists an α such that the selection of the matrix K as

$$K = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & & \alpha \end{bmatrix}$$

satifies the requirement

$$\tilde{A}_{12}' \, K \, \tilde{A}_{12} - Q > 0 \, .$$

<u>CASE 3</u> - If the matrix $\tilde{A}_{22}$ is singular or it has both negative and positive eigenvalues then a proper positive semi-definite matrix K of the form

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{1m} \\ k_{12} & k_{22} & \vdots \\ \vdots & \cdot & \vdots \\ k_{1m} & \cdots & k_{mm} \end{bmatrix}$$

which yields a positive definite solution matrix R of the equation 2.44, must be searched . Unfortunately, there has not been any devised search method to select such a matrix. In this study, this case is excluded and is left to further researchers.

CHAPTER 3

MINIMAL ORDER OBSERVERS FOR DETERMINISTIC DISCRETE TIME SYSTEMS

In this chapter, two design criteria for minimal order observers for the deterministic discrete time systems are studied. One of these criteria considers the *stability* of the observer, and the other is interested in the *state estimation error decay*. The former demands solutions to various parameters simultaneously, whereas in the latter, determination of one parameter is sufficient. From this comparative statement it is deduced that the implementation of the second criterion is easier than that of the first one. Consequently, we suggest a design procedure for the second criterion.

3.1. BASIC EQUATIONS

Consider the completely observable linear time invariant discrete time system

$$x(k+1) = A\,x(k) + B\,u(k)$$
$$y(k) = C\,x(k) \qquad 3.1$$

It is assumed that the output matrix C is of full rank m.

<u>THEOREM 3.1</u>: Let a system of order p, which is driven by the inputs and the outputs of the system in Equation 3.1 be given by

$$z(k+1) = F\ z(k) + G\ y(k) + H\ u(k) \qquad 3.2$$

If there exists a matrix T of dimensions pxn which satisfies

$$TA - FT = GC \qquad 3.3$$

$$TB = H \qquad 3.4$$

then z(k) in Equation 3.2 estimates, T x(k), a linear transformation of the states x(k) in Equation 3.1, i.e.,

$$z(k) = T\ x(k) + e(k)$$

where e(k) denotes the observer error.

PROOF: It immediately follows from Equations 3.1 and 3.2 that

$$\begin{aligned} z(k+1) - T\ x(k+1) &= F\ z(k) + G\ y(k) + H\ u(k) \\ &\quad - TA\ x(k) - TB\ u(k) \\ &= F\ z(k) + (GC - TA)x(k) + (H - TB)u(k) \end{aligned}$$

From Equations 3.3 and 3.4 the above equation can be written as

$$z(k+1) - T\ x(k+1) = F\ \left[z(k) - T\ x(k)\right] \qquad 3.5$$

The difference equation in Equation 3.5 yields a solution

$$\begin{aligned} z(k) - T\ x(k) &= F^k \left[z(o) - T\ x(o)\right] \\ z(k) &= T\ x(k) + F^k \left[z(o) - T\ x(o)\right] \end{aligned}$$

The observer error e(k) is

$$e(k) = F^k \left[z(o) - T\ x(o)\right]$$

This completes the proof.

The smallest value p can attain is n-m, see Theorem 2.1. If the value of p is replaced by n-m in the above theorem, the system in Equation 3.2 is the minimal order observer for the system in Equation 3.1 in the sense of the above theorem.

<u>THEOREM 3.2</u>: The minimal order observer constructed in Theorem 3.1, in conjunction with the output of the system in Equation 3.1, reconstructs the states of the system in Equation 3.1 exactly if $z(o) = T\ x(o)$ provided that $\begin{bmatrix} T \\ \hline C \end{bmatrix}^{-1}$ exists.

<u>PROOF</u>: Imposing the condition

$$z(o) = T\ x(o)$$

upon

$$z(k) = T\ x(k) + F^k \left[z(o) - T\ x(o) \right]$$

which has already been obtained, results in

$$z(k) = T\ x(k)\ .$$

Enlarging the observer state vector $z(k)$ with the output vector $y(k)$ of the system in Equation 3.1 which is given by

$$y(k) = C\ x(k)$$

the following is obtained.

$$\begin{bmatrix} z(k) \\ \hline y(k) \end{bmatrix} = \begin{bmatrix} T \\ \hline C \end{bmatrix} x(k)$$

Premultiplying the above equation by

$$\begin{bmatrix} T \\ \hline C \end{bmatrix}^{-1}$$

it is found that

$$x(k) = \begin{bmatrix} T \\ --- \\ C \end{bmatrix}^{-1} \begin{bmatrix} z(k) \\ ------ \\ y(k) \end{bmatrix} \tag{3.6}$$

This completes the proof.

If x(o) is not known then z(o) can not be chosen to be equal to T x(o). In this case, the observer error e(k) exists and, in turn, effects the reconstruction of the state x(k). Then, Equation 3.6 is modified as

$$x(k) = \begin{bmatrix} T \\ --- \\ C \end{bmatrix}^{-1} \begin{bmatrix} z(k) \\ ------ \\ y(k) \end{bmatrix} - \varepsilon(k) \tag{3.7}$$

where $\varepsilon(k)$ is the estimation error defined by

$$\varepsilon(k) = \hat{x}(k) - x(k)$$

where $\hat{x}(k)$ is the estimate of the state x(k). If $\varepsilon(k)$ in Equation 3.7 is replaced with the above equation, then Equation 3.7 becomes

$$\hat{x}(k) = \begin{bmatrix} T \\ --- \\ C \end{bmatrix}^{-1} \begin{bmatrix} z(k) \\ ------ \\ y(k) \end{bmatrix} \tag{3.8}$$

It has been shown so far that the minimal order observer estimates the states of the observed system. How the minimal order observer is constructed is the topic of the next section.

3.2. CONSTRUCTION OF MINIMAL ORDER OBSERVERS

As it was pointed out in Chapter 2, the construction of the observer must involve the dynamics of the observed system since it produces an estimate to the state of the observed system. In other words, the observed system constrains the behavior of the observer. Hence, the starting point with the construction of the observer is to set these constraints.

Letting the indicated inverse in Equation 3.8

$$\begin{bmatrix} T \\ --- \\ C \end{bmatrix}^{-1} = \begin{bmatrix} P & \vdots & V \end{bmatrix} \qquad 3.9$$

where P and V are nxn-m and nxm matrices respectively, $\hat{x}(k)$ is written as

$$\hat{x}(k) = P\ z(k) + V\ y(k)\ .$$

Equation 3.9 implies that

$$PT + VC = I_n\ .$$

Both sides of the last equation are pre-multiplied by TA to obtain

$$TAPT + TAVC = TA$$

which is substituted in the constraint equation, Equation 3.3 and

$$TAPT + TAVC - FT - GC = 0$$

or

$$(TAP - F)T + (TAV - G)C = 0 \qquad 3.10$$

is obtained.

T and C are linearly independent since $\begin{bmatrix} T \\ - \\ C \end{bmatrix}$ has been assumed to be invertible, then Equation 3.10 holds true, if

$$F = TAP$$

and

$$G = TAV\ .$$

The summary of the equations governing the minimal order observer is given below for easy reference:

The minimal order observer given by

$$z(k+1) = F\ z(k) + G\ y(k) + H\ u\ (k)$$

estimates the state $x(k)$ of the observed system as

$$\hat{x}(k) = P\ z(k) + V\ y(k)$$

and satisfies the following constraints:

$$PT + VC = I_n$$

$$F = TAP$$

$$G = TAV$$

and

$$H = TB\ .$$

The next two sub-sections present the already mentioned two design criteria separetely. At the end of the second subsection the interrelationship of these two criteria is mentioned as well.

3.2.1. Stability of the Observer |8|

This design criterion deals with the observer error defined by

$$e(k) = z(k) - T\ x(k) \qquad 3.11$$

and sets the necessary conditions for

$$\lim_{k\to\infty} e(k) = 0\ . \qquad 3.12$$

The derivation below shows how Equations 3.11 and 3.12 are related to the stability of the observer,

From Equation 3.11, it is immediately seen that

$$e(k+1) = z(k+1) - T\ x(k+1) \qquad 3.13$$

Substitution of Equations 3.1 and 3.2 into 3.13 yields

$$e(k+1) = F\ z(k) + G\ y(k) + H\ u(k) - TA\ x(k) - TB\ u(k)$$

Furthermore,

$$e(k+1) = F\ z(k) + (GC - TA)x(k) + (H - TB)u(k)$$

since $y(k) = C\ x(k)$.
Inserting Equations 3.3 and 3.4 into the above equation

$$e(k+1) = F\ [z(k) - T\ x(k)]$$

is obtained. Further substitution of Equation 3.11 yields

$$e(k+1) = F\ e(k) .$$

In order for,

$$\lim_{k\to\infty} e(k) = 0$$

to hold true, F must be a stable matrix.
Then the matrix F must be tuned in such a way so that all its eigenvalues are inside the unit circle.
Since,

$$F = TAP$$

the tuning of the matrix F requires the selection of two matrices T and P which are related through the matrix V.

The difficulty of selecting three matrices T, P and V simultaneously renders this design procedure unattractive.

3.2.2. Estimate Error Decay |8|

As the title implies, this design criterion determines the dynamics of the observer such that the state estimation error defined by

$$\varepsilon(k) = \hat{x}(k) - x(k) \tag{3.14}$$

decays as time increases, or in mathematical terms

$$\lim_{k\to\infty} \varepsilon(k) = 0 .$$

Substituting the output equation of Equation 3.1 into the following equation

$$\hat{x}(k) = P\, z(k) + V\, y(k)$$

yields

$$\hat{x}(k) = P\, z(k) + VC\, x(k) \tag{3.15}$$

The constraint equation

$$PT + VC = I_n$$

may be written as

$$VC = I_n - PT$$

and with the insertion of this result into Equation 3.15, $\hat{x}(k)$ becomes

$$\hat{x}(k) = P\, z(k) + \left[I - PT\right] x(k)$$

or

$$\hat{x}(k) = x(k) + P\left[z(k) - T\, x(k)\right]$$

Recalling that

$$e(k) = z(k) - T\,x(k)$$

the above equation reads

$$\hat{x}(k) = x(k) + P\,e(k)\ .$$

Substitution of this result into Equation 3.14 yields

$$\varepsilon(k) = x(k) + P\,e(k) - x(k)$$

or

$$\varepsilon(k) = P\,e(k) \qquad 3.16$$

It follows from the above equation that

$$\varepsilon(k+1) = P\,e(k+1)$$

or

$$\varepsilon(k+1) = P\,F\,e(k) \qquad 3.17$$

On the other hand,

$$PF = PTAP$$
$$PF = \left[I - VC\right] AP$$
$$PF = \left[A - VCA\right] P \qquad 3.18$$

Equations 3.18 and 3.16 are substituted into Equation 3.17 in the given order as follows, and

$$\varepsilon(k+1) = \left[A - VCA\right] P\,e(k)$$
$$\varepsilon(k+1) = \left[A - VCA\right] \varepsilon(k) \qquad 3.19$$

is obtained.

In order for this free system to satisfy

$$\lim_{k\to\infty} \varepsilon(k) = 0$$

the eigenvalues of the matrix $\left[A - VCA\right]$ must lie in the unit circle. Since the matrices A and C are known, an appropriate choice of V is sufficient to determine a stable matrix A-VCA.

Furthermore, picking V in order to have a stable free system

$$\varepsilon(k+1) = \left[A - VCA \right] \varepsilon(k)$$

ensures the stability of the observer as follows.

$$\lim_{k\to\infty} \varepsilon(k) = P \lim_{k\to\infty} e(k) = 0$$

Since the matrix P is of full column rank the above equation is satisfied only if

$$\lim_{k\to\infty} e(k) = 0 \; .$$

In the next section we suggest a design procedure for the evaluation of the matrix V, and show that the matrices F, G and H of the observer are determined by simple operations once V is evaluated.

3.3. A CANONICAL CLASS OF OBSERVERS

The system given in Equation 3.1

$$x(k+1) = A\, x(k) + B\, u(k)$$
$$y(k) = C\, x(k)$$

can be transformed to

$$q(k+1) = \tilde{A}\, q(k) + \tilde{B}\, u(k)$$
$$y(k) = \tilde{C}\, q(k) = \left[I_m \;\vdots\; 0 \right] q(k) \qquad 3.20$$

through a (nxn) non-singular transformation matrix M,

$$M = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline 0 & I_{n-m} \end{array}\right]$$

such that $q(k) = M\ x(k)$ as it was explained in Section 2.1. The matrices $\tilde{A}$ and $\tilde{B}$ are given by

$$\tilde{A} = M\ A\ M^{-1}$$

$$\tilde{B} = M\ B \quad .$$

As it was pointed out in Chapter 2, the similarity transformation does not alter stability, observability, controllability.

Upon partitioning the matrices $\tilde{A}$ and $\tilde{B}$ appropriately, Equation 3.20 is written as:

$$\left[\begin{array}{c} q_1(k+1) \\ \hline q_2(k+1) \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \left[\begin{array}{c} q_1(k) \\ \hline q_2(k) \end{array}\right] + \left[\begin{array}{c} \tilde{B}_1 \\ \hline \tilde{B}_2 \end{array}\right] u(k)$$

$$y(k) = \left[\begin{array}{c|c} I_m & 0 \end{array}\right] \left[\begin{array}{c} q_1(k) \\ \hline q_2(k) \end{array}\right] \qquad 3.21$$

where $q_1(k)$ and $q_2(k)$ are m- and (n-m)-dimensional parts of the partitioned state vector respectively. $\tilde{A}_{11}$ is (mxm), $\tilde{A}_{12}$ is mx(n-m), $\tilde{A}_{21}$ is (n-m)xm, $\tilde{A}_{22}$ is (n-m)x(n-m), $\tilde{B}_1$ is mxr and $\tilde{B}_2$ is (n-m)xr. The output equation of Equation 3.21 shows that the states $q_2(k)$ are not available. Therefore, the (n-m) order observer to be designed will estimate the states $q_2(k)$.

To this end, Equation 3.19 for the system in Equation 3.20 is written as

$$\varepsilon(k+1) = [\tilde{A} - V\,\tilde{C}\,\tilde{A}]\,\varepsilon(k) \qquad 3.22$$

If the Lyapunov stability theory is employed for discrete time systems to evaluate a stabilizing V for the free system in Equation 3.22, the following Lyapunov equation is obtained |8|, |9|.

$$[\tilde{A} - V\,\tilde{C}\,\tilde{A}]'\,[R]\,[\tilde{A} - V\,\tilde{C}\,\tilde{A}] - R = -Q \qquad 3.23$$

where Q is an (nxn) real symmetric positive definite matrix.
If there exists a (nxn) real symmetric positive definite solution matrix R to the above equation, Equation 3.23, then $[\tilde{A} - V\tilde{C}\tilde{A}]$ is said to be stable. Equation 3.23 is solved for V and R simultaneously with the constraint that R is a symmetric positive definite matrix.

There are several algorithms developed one may use to solve Equation 3.23 for the (nxn) matrix R, but we claim that the order of Equation 3.23 can be reduced considerably. It is unfortunate however, that Equation 3.23 does not admit any further reduction in the order, therefore we modify the problem of seeking out a stable matrix $[\tilde{A} - V\,\tilde{C}\,\tilde{A}]$ for the free system

$$\varepsilon(k+1) = [\tilde{A} - V\,\tilde{C}\,\tilde{A}]\;\varepsilon(k)$$

so that

$$\lim_{k\to\infty} \varepsilon(k) = 0 \qquad 3.24$$

as follows:
Define a new free system given by

$$m(k+1) = [\tilde{A} - V\,\tilde{C}\,\tilde{A}]'\,m(k) \qquad 3.25$$

If the matrix $[\tilde{A} - V\,\tilde{C}\,\tilde{A}]'$ is a stable matrix or equivalently the eigenvalues of $[\tilde{A} - V\,\tilde{C}\,\tilde{A}]'$ are inside the unit circle, then

$$\lim_{k\to\infty} m(k) = 0 \qquad 3.26$$

Equation 3.26 implies Equation 3.24 since transposing a square matrix does not alter eigenvalues.
Employing the Lyapunov stability theory for the free system in Equation 3.25 results in

$$[\tilde{A} - V\tilde{C}\tilde{A}]\, R\, [\tilde{A} - V\tilde{C}\tilde{A}]' - R = -Q \qquad 3.27$$

where both Q and R are (nxn) real symmetric positive definite matrices.

In our case the most appropriate algorithm to solve Equation 3.27 is the "Successive Approximation Algorithm" since it provides V and R simultaneously. The Successive Approximation Algorithm consists of iterative solution as outlined in the following theorem.

<u>THEOREM 3.5:</u> |10| Let N_k, k=0,1,2,..., be the solutions of the equation

$$N_k = S_k N_k S_k' + Q \qquad 3.28$$

where

$$S_k = \tilde{A} - V_k \tilde{C}\tilde{A} \qquad k=0,1,2,...$$

with

$$V_k = \tilde{A} N_{k-1} \tilde{A}'\tilde{C}' (\tilde{C}\tilde{A} N_{k-1} \tilde{A}'\tilde{C}')^{-1} \quad k=1,2,3,.. \qquad 3.29$$

and V_o is chosen such that S_o is a stable matrix. Then the matrix E_k defined as

$$E_k = N_k - N_{k+1}, \qquad k=0,1,...$$

is positive-semidefinite, that is

$$E_k \geq 0$$

$$\lim_{k\to\infty} N_k = R \quad \text{and} \quad \lim_{k\to\infty} V_k = V$$

<u>PROOF</u>: Since S_o is a stable matrix, the unique positive definite solution N_o of Equation 3.28 may be written as

$$N_o = \sum_{N=0}^{\infty} (S_o)^N \, Q \, (S_o')^N$$

It follows from Equation 3.29 that

$$S_o = \tilde{A} - V_o \tilde{C} \tilde{A} \qquad \text{for} \quad k=0$$

and

$$S_1 = \tilde{A} - V_1 \tilde{C} \tilde{A} \qquad \text{for } k=1.$$

The last equation may be rewritten as

$$\tilde{A} = S_1 + V_1 \tilde{C} \tilde{A}$$

and this result is substituted in S_o, then S_o becomes

$$S_o = S_1 + (V_1 - V_o) \tilde{C} \tilde{A} .$$

Then, $$S_o N_o S_o' = (S_1 + (V_1 - V_o)\tilde{C}\tilde{A}) N_o (S_1 + (V_1 - V_o)\tilde{C}\tilde{A})'$$

$$S_o N_o S_o' = S_1 N_o S_1' + (V_1 - V_o)\tilde{C}\tilde{A} N_o \tilde{A}'\tilde{C}'(V_1 - V_o)'$$

$$+ S_1 N_o \tilde{A}'\tilde{C}'(V_1 - V_o)'$$

$$+ (V_1 - V_o) \tilde{C} \tilde{A} N_o S_1' \qquad 3.30$$

is obtained. From Equation 3.29

$$S_1 N_o \tilde{A}'\tilde{C}'(V_1 - V_o)' = (\tilde{A} - V_1 \tilde{C}\tilde{A}) N_o \tilde{A}'\tilde{C}'(V_1 - V_o)'$$

$$S_1 N_o \tilde{A}' \tilde{C}' (V_1 - V_o)' = \tilde{A} N_o \tilde{A}' \tilde{C}' (V_1 - V_o)' - V_1 \tilde{C} \tilde{A} N_o \tilde{A}' \tilde{C}' (V_1 - V_o)'$$

$$S_1 N_o \tilde{A}' \tilde{C}' (V_1 - V_o)' = \tilde{A} N_o \tilde{A}' \tilde{C}' (V_1 - V_o)' - \tilde{A} N_o \tilde{A}' \tilde{C}' (V_1 - V_o)'$$

$$S_1 N_o \tilde{A}' \tilde{C}' (V_1 - V_o)' = 0$$

is found. This result implies

$$(V_1 - V_o)\ \tilde{C}\ \tilde{A}\ N_o S_1' = 0$$

since this term is the transpose of the above term. Then, the identity in Equation 3.30 becomes

$$S_o N_o S_o' = S_1 N_o S_1' + (V_1 - V_o)(\tilde{C} \tilde{A} N_o \tilde{A}' \tilde{C}')(V_1 - V_o)' \qquad 3.31$$

Insertion of Equation 3.31 into Equation 3.28 shows that N_o also satisfies the following equation

$$N_o = S_1 N_o S_1' + K \qquad 3.32$$

where

$$K = (V_1 - V_o)(\tilde{C} \tilde{A} N_o \tilde{A}' \tilde{C}')(V_1 - V_o)' + Q \geq 0$$

Since this implies that S_1 is a stable matrix, the unique positive definite solution N_1 of Equation 3.28 exists and is given by

$$N_1 = \sum_{N=0}^{\infty} (S_1)^N\ Q\ (S_1')^N \qquad 3.33$$

The solution to the equation in Equation 3.32 is

$$N_o = \sum_{N=0}^{\infty} (S_1)^N\ K\ (S_1')^N \qquad 3.34$$

Subtracting Equation 3.33 from Equation 3.34, one obtains

$$N_o - N_1 = \sum_{N=0}^{\infty} (S_1)^N (\tilde{K} - Q)(S_1')^N$$

$$N_o - N_1 = \sum_{N=0}^{\infty} (S_1)^N (V_1 - V_o)(\tilde{C}\tilde{A}N_o\tilde{A}'\tilde{C}')(V_1 - V_o)'(S_1')^N \geq 0$$

If an identity similar to that in Equation 3.31 is employed,

$$N_k - N_{k+1} = \sum_{N=0}^{\infty} (S_{k+1})^N (V_{k+1} - V_k)(\tilde{C}\tilde{A}N_k\tilde{A}'\tilde{C}')(V_{k+1} - V_k)'(S_{k+1}')^N \geq 0$$

is found. Now, let

$$V^* = \tilde{A}R\tilde{A}'\tilde{C}' \; (\tilde{C} \; \tilde{A} \; R \; \tilde{A}'\tilde{C}')^{-1}$$

and again employ an identity similar to Equation 3.31 to obtain

$$N_{k+1} - R = \sum_{N=0}^{\infty} (R)^N (V^* - V_{k+1})(\tilde{C}\tilde{A}N_{k+1}\tilde{A}'\tilde{C}')(V_{k+1} - V^*)'(R')^N \geq 0$$

This completes the proof.

The next theorem proves that the rate of convergence of this algorithm is quadratic.

THEOREM 3.6 |10| If the algorithm in Theorem 3.5 is employed, then the rate of convergence to the steady state R is

$$[R - N_{k+1}] \leq C \; [R - N_k]^2$$

where C is a constant independent of the iteration index k.

<u>PROOF</u>: Let

$$V^* = \tilde{A}\, R\, \tilde{A}'\tilde{C}'(\tilde{C}\,\tilde{A}\, R\, \tilde{A}'\tilde{C}')^{-1}$$

and

$$\tilde{A} = S_* + V^*\tilde{C}\,\tilde{A}\ .$$

Since the steady state R is assumed to exist, S_* is a stable matrix. The matrix difference $R - N_{k+1}$ can be expressed as in the proof of Theorem 3.5, in terms of a convergent series.

$$R-N_{k+1} = \sum_{n=0}^{\infty} S_*^{\,n}(V^*-V_{k+1})(\tilde{C}\tilde{A}R\tilde{A}'\tilde{C}')(V^*-V_{k+1})'(S_*^{\,n})'$$

where V_k is obtained from Equation 3.29.
The expression

$$\begin{aligned} V^*-V_{k+1} = {} & (\tilde{C}\tilde{A}R\tilde{A}'\tilde{C}')^{-1}\tilde{C}\tilde{A}(R-N_k)\tilde{A} \\ & \cdot\ (\tilde{C}\tilde{A}R\tilde{A}'\tilde{C})^{-1}(\tilde{C}\tilde{A}N_k\tilde{A}'\tilde{C}' - \tilde{C}\tilde{A}R\tilde{A}'\tilde{C}')V_{k+1} \end{aligned}$$

can be verified by matrix manipulation. Substituting this expression in the above series, it is seen that

$$[R - N_{k+1}] \le C\ [R - N_k]^2$$

where C is a constant.
This completes the proof.

In order to be able to evaluate V_k in Equation 3.29 the indicated inverse must exist.

<u>THEOREM 3.7</u>: If $\tilde{A}$ and N_{k-1} are nxn non-singular matrices and $\tilde{C}$ is an mxn matrix with full rank m, then

$$(\tilde{C}\ \tilde{A}\ N_{k-1}\ \tilde{A}'\tilde{C}') \text{ is invertible.}$$

PROOF: The matrix $\tilde{C}$ is of full rank m,

$$\text{rank } \tilde{C} = m .$$

When a rectangular matrix is multiplied on the left or on the right by a non-singular matrix, the rank of the original matrix remains unchanged |11|. Then,

$$\text{rank}[\tilde{C}] = \text{rank } [\tilde{C}\ \tilde{A}] = \text{rank } [\tilde{C}\ \tilde{A}\ N_{k-1}] = m$$

If rank $|\tilde{C}\ \tilde{A}| = m$, then

$$\text{rank } |\tilde{A}'\tilde{C}'| = m .$$

Then the (mxm) matrix $[\tilde{C}\ \tilde{A}\ N_{k-1}\ \tilde{A}'\tilde{C}']$ is of rank m, hence invertible.
This completes the proof.

The non-singularity of the matrix $\tilde{A}$ is a sufficient condition for the indicated inverse to exist.
The converse of the above theorem, that is, if

$$(\tilde{C}\ \tilde{A}\ N_{k-1}\ \tilde{A}'\ \tilde{C}')$$

is invertible then $\tilde{A}$ is a non-singular matrix, does not hold true. It can be shown that for some singular matrix $\tilde{A}$ the indicated inverse exists. Considering the case that the indicated inverse may not exist for some singular matrix $\tilde{A}$, we conclude that the minimal order observer can not be constructed by this design procedure. In this case, we suggest constructing an identity observer. For the sake of completeness, we mention how an identity observer is constructed in a separate section later in this chapter.

As we have stated the algorithm derived in Theorem 3.5 to solve Equation 3.27 may be reduced in order if some properties of the canonical system in Equation 3.21 are used. To this end the matrices N_k and V_k are partitioned as follows:

$$N_k = \left[\begin{array}{c|c} N_{k11} & N_{k12} \\ \hline N_{k21} & N_{k22} \end{array}\right]$$

$$V_k = \left[\begin{array}{c} V_{k1} \\ \hline L_k \end{array}\right]$$

where N_{k11} is mxm, N_{k12} is mx(n-m), N_{k21} is (n-m)xm, N_{k22} is (n-m)x(n-m), V_{k1} is mxm and L_k is (n-m)xm. Inserting these partitioned matrices into Equation 3.29, yields

$$\left[\begin{array}{c} V_{k1} \\ \hline L_k \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right]\left[\begin{array}{c|c} N_{k-1,11} & N_{k-1,12} \\ \hline N_{k-1,21} & N_{k-1,22} \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right]' \left[\begin{array}{c} I_m \\ \hline 0 \end{array}\right]$$

$$\cdot\left\{\left[\begin{array}{c|c} I_m & 0 \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right]\left[\begin{array}{c|c} N_{k-1,11} & N_{k-1,12} \\ \hline N_{k-1,21} & N_{k-1,22} \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right]' \left[\begin{array}{c} I_m \\ \hline 0 \end{array}\right]\right\}^{-1}$$

$$\left[\begin{array}{c} V_{k1} \\ \hline L_k \end{array}\right] = \left[\begin{array}{c} \left[\tilde{A}_{11}N_{k-1,11}+\tilde{A}_{12}N_{k-1,21}\right]\tilde{A}'_{11}+\left[\tilde{A}_{11}N_{k-1,12}+\tilde{A}_{12}N_{k-1,22}\right]\tilde{A}'_{12} \\ \hline \left[\tilde{A}_{21}N_{k-1,11}+\tilde{A}_{22}N_{k-1,21}\right]\tilde{A}'_{11}+\left[\tilde{A}_{21}N_{k-1,12}+\tilde{A}_{22}N_{k-1,22}\right]\tilde{A}'_{12} \end{array}\right]$$

$$\cdot\left\{\left[\tilde{A}_{11}N_{k-1,11}+\tilde{A}_{12}N_{k-1,21}\right]\tilde{A}'_{11}+\left[\tilde{A}_{11}N_{k-1,12}+\tilde{A}_{12}N_{k-1,22}\right]\tilde{A}'_{12}\right\}^{-1}$$

$$\left[\begin{array}{c} V_{k1} \\ \hline L_k \end{array}\right] = \left[\begin{array}{c} I \\ \hline \left[\tilde{A}_{21}N_{k-1,11}+\tilde{A}_{22}N_{k-1,21}\right]\tilde{A}'_{11}+\left[\tilde{A}_{21}N_{k-1,12}+\tilde{A}_{22}N_{k-1,22}\right]\tilde{A}'_{12} \end{array}\right]$$

$$\cdot\left[\left[\tilde{A}_{11}N_{k-1,11}+\tilde{A}_{12}N_{k-1,21}\right]\tilde{A}'_{11} + \left[\tilde{A}_{11}N_{k-1,12}+\tilde{A}_{12}N_{k-1,22}\right]\tilde{A}'_{12}\right]^{-1} \qquad 3.35$$

Equation 3.35 shows that

$$V_k = \left[\begin{array}{c} I \\ \hline L_k \end{array}\right]$$

If the above equation is substituted in Equation 3.29, we find S_k as

$$S_k = \left[\begin{array}{c:c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hdashline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] - \left[\begin{array}{c} I \\ \hline L_k \end{array}\right] \left[\begin{array}{c:c} I_m & 0 \end{array}\right] \left[\begin{array}{c:c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hdashline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right]$$

$$S_k = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline \tilde{A}_{21} - L_k\tilde{A}_{11} & \tilde{A}_{22} - L_k\tilde{A}_{12} \end{array}\right]$$

or

$$S_k = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline X_k & Y_k \end{array}\right] \qquad 3.36$$

where X_k and Y_k are $(n-m)xm$ and $(n-m)x(n-m)$ matrices respectively and given by

$$X_k = \tilde{A}_{21} - L_k \tilde{A}_{11}$$

and

$$Y_k = \tilde{A}_{22} - L_k \tilde{A}_{12}$$

Choosing

$$Q = I$$

and substituting Equation 3.36 into Equation 3.28, we obtain

$$\left[\begin{array}{c:c} N_{k11} & N_{k12} \\ \hdashline N_{k21} & N_{k22} \end{array}\right] = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline X_k & Y_k \end{array}\right] \left[\begin{array}{c:c} N_{k11} & N_{k12} \\ \hdashline N_{k21} & N_{k22} \end{array}\right] \left[\begin{array}{c:c} 0 & X_k' \\ \hdashline 0 & Y_k' \end{array}\right] + \left[\begin{array}{c:c} I_m & 0 \\ \hdashline 0 & I_{n-m} \end{array}\right]$$

$$\left[\begin{array}{c|c} N_{k11} & N_{k12} \\ \hline N_{k21} & N_{k22} \end{array}\right] = \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & I_{n-m}+\left[X_k N_{k11}+Y_k N_{k21}\right] X_k' + \left[X_k N_{k12}+Y_k N_{k22}\right] Y_k' \end{array}\right]$$

From this expression we obtain the following identities:

$$\begin{aligned} N_{k11} &= I_m \\ N_{k12} &= 0 \\ N_{k21} &= 0 \end{aligned} \qquad 3.37$$

If these identities are substituted into

$$N_{k22} = I + \left[X_k \; N_{k11}+Y_k N_{k21}\right] X_k' + \left[X_k N_{k12}+Y_k N_{k22}\right] Y_k'$$

N_{k22} becomes,

$$N_{k22} = I_{n-m} + X_k X_k' + Y_k N_{k22} Y_k' . \qquad 3.38$$

We then substitute the above result together with the identities Equation 3.37 into Equation 3.35 and we find that

$$L_k = \left[\tilde{A}_{21}\tilde{A}_{11}'+\tilde{A}_{22}N_{k22}\tilde{A}_{12}'\right]\left[\tilde{A}_{11}\tilde{A}_{11}'+\tilde{A}_{12}N_{k22}\tilde{A}_{12}'\right]^{-1} \qquad 3.39$$

Furthermore,

$$S_o = \left[\begin{array}{c|c} 0 & 0 \\ \hline X_o & Y_o \end{array}\right]$$

or replacing X_o and Y_o by

$$\begin{aligned} X_o &= \tilde{A}_{21} - L_o \tilde{A}_{11} \\ Y_o &= \tilde{A}_{22} - L_o \tilde{A}_{12} \end{aligned}$$

$$S_o = \left[\begin{array}{c|c} 0 & 0 \\ \hline \tilde{A}_{21} - L_o\tilde{A}_{11} & \tilde{A}_{22} - L_o\tilde{A}_{12} \end{array}\right] \qquad 3.40$$

is found.

Then the algorithm given in Theorem 3.5 by Equations 3.28 and 3.29 may be replaced by

$$N_{k22} = Y_k N_{k22} Y_k + X_k X_k + I_{n-m} \qquad k=0,1,2,\ldots$$

where

$$X_k = \tilde{A}_{21} - L_k \tilde{A}_{11} \qquad k=0,1,2,\ldots$$

$$Y_k = \tilde{A}_{22} - L_k \tilde{A}_{12} \qquad k=0,1,2,\ldots$$

$$L_k = \tilde{A}_{21}\tilde{A}_{11} + \tilde{A}_{22} N_{k22} \tilde{A}_{12} \quad \tilde{A}_{11}\tilde{A}_{11} + \tilde{A}_{12} N_{k22} \tilde{A}_{12} {}^{-1} \qquad k=1,2,\ldots$$

In order for S_o in Equation 3.40 to be a stable matrix, L_o must be so chosen that the eigenvalues of $\tilde{A}_{22} - L_o\tilde{A}_{12}$ are inside the unit circle. Then the algorithm converges and N_{k22} is the solution of the following algebraic Matrix Riccati Equation

$$(\tilde{A}_{22} - L\,\tilde{A}_{12})R_{22}(\tilde{A}_{22} - L\,\tilde{A}_{12})' + (\tilde{A}_{21} - L\,\tilde{A}_{11})(\tilde{A}_{21} - L\,\tilde{A}_{11})' + I = R_{22}$$

The table below indicates the dimensions of the matrices to emphasize the considerable reduction.

	Replaced by
N_k nxn	N_{k22} (n-m)x(n-m)
V_k nxm	L_k (n-m)xm
S_k nxn	X_k (n-m)xm and Y_k (n-m)x(n-m)
R nxn	R_{22} (n-m)x(n-m)

TABLE 3.1

Next we determine the matrices F, G and H.

Recalling

$$V = \begin{bmatrix} I_m \\ \hdashline L \end{bmatrix}$$

the constraint equation

$$PT + V\tilde{C} = I_n$$

may be written as

$$\begin{bmatrix} P_1 \\ -- \\ P_2 \end{bmatrix} \begin{bmatrix} T_1 & \vdots & T_2 \end{bmatrix} + \begin{bmatrix} I_m \\ -- \\ L \end{bmatrix} \begin{bmatrix} I_m & \vdots & 0 \end{bmatrix} = \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & I_{n-m} \end{array}\right]$$

where P_1 is $m\text{x}(n-m)$, P_2 is $(n-m)\text{x}(n-m)$, T_1 is $(n-m)\text{x}m$ and T_2 is $(n-m)\text{x}(n-m)$.

Then we obtain the following set of matrix equations:

$$P_1T_1 = 0$$

$$P_1T_2 = 0$$

$$P_2T_1 = -L$$

$$P_2T_2 = I_{n-m} .$$

One set of solutions to this set of matrix equations is

$$P_1 = 0$$

$$P_2 = I_{n-m}$$

$$T_1 = -L$$

$$\text{and} \quad T_2 = I_{n-m} . \qquad 3.41$$

We also have shown that

$$F = T\tilde{A}P$$

$$G = T\tilde{A}V$$

$$\text{and} \quad H = T\tilde{B}$$

If the matrices $\tilde{A}$ and $\tilde{B}$ are partitioned properly and the matrices in Equation 3.41 are inserted in the above matrices F, G and H we obtain

$$F = \tilde{A}_{22} - L\,\tilde{A}_{12}$$

$$G = \tilde{A}_{21} - L\,\tilde{A}_{11} + F\,L$$

and

$$H = \tilde{B}_2 - L\,\tilde{B}_1 \quad .$$

The minimal order observer designed by this procedure estimates the state q(k) as

$$\hat{q}(k) = P\,z(k) + V\,y(k)$$

and one may easily obtain the original system state estimate $\hat{x}(k)$ by

$$\hat{x}(k) = M^{-1}\,\hat{q}(k)$$

where M^{-1} is the inverse of the transformation matrix M.

3.4. IDENTITY OBSERVER

Although constructing the identity observer is unattractive due to the redundancy which was pointed out in Chapter 1, one may have to construct it if the matrix A is singular. Considering this possibility, we suggest a design procedure as outlined below.

Replacing the matrix T with I in Equation 3.3, we obtain

$$F = A - GC \qquad 3.42$$

On the other hand, we have shown that the observer error e(k) satisfies

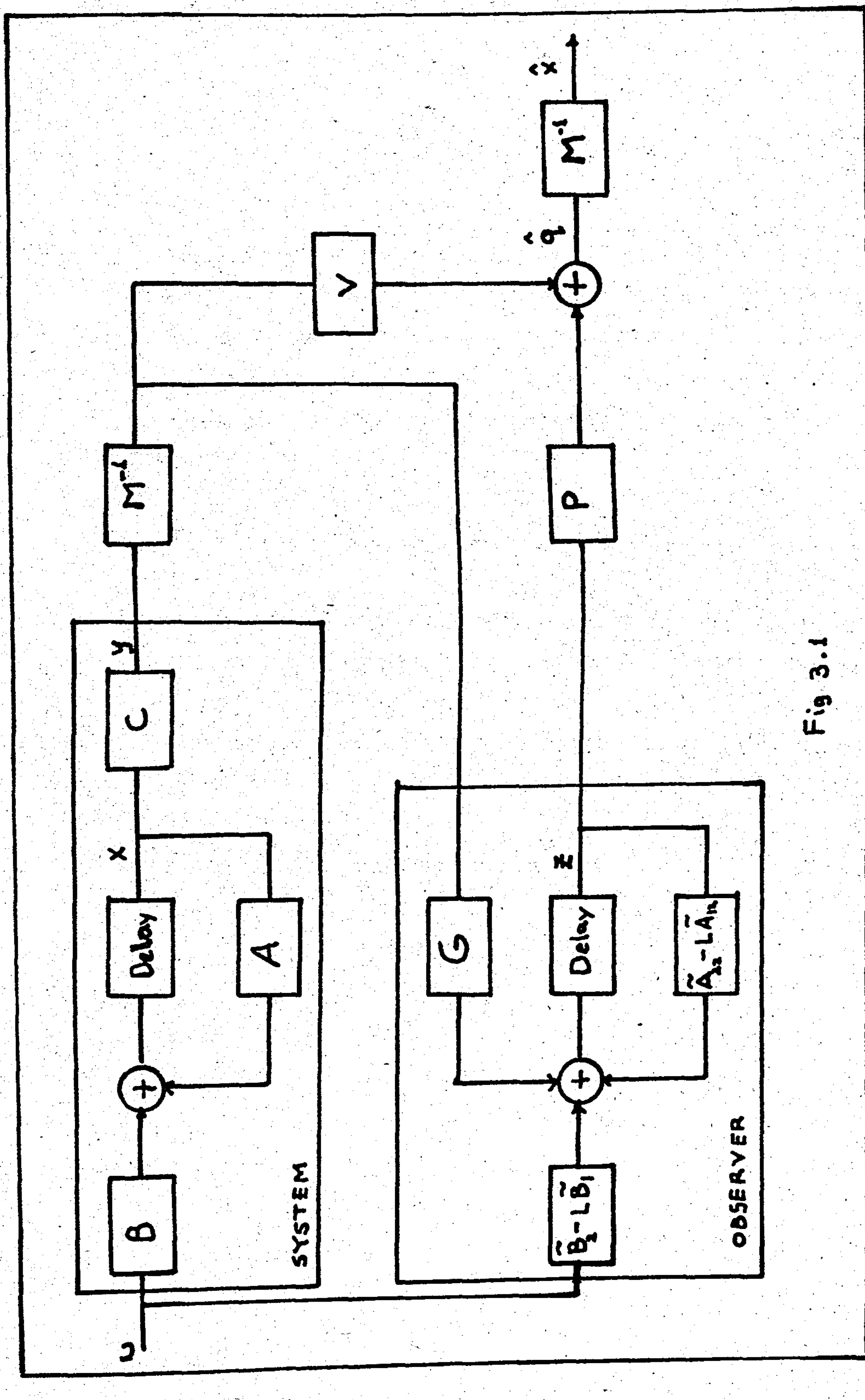


Fig 3.1

$$e(k+1) = F\ e(k)$$

and in order for,

$$\lim_{k\to\infty} e(k) = 0$$

F must be a stable matrix. Since A and C are known matrices in Equation 3.42, the matrix G must be so chosen that F is a stable matrix, i.e., the eigenvalues of the matrix F are inside the unit circle. Then we employ the Lyapunov stability theory and we obtain the following Lyapunov equation |8|, |9|.

$$(A - GC)'R\ (A - GC) - R = - Q \qquad 3.43$$

where Q is (nxn) real symmetric positive definite matrix and R is (nxn) real symmetric positive definite solution matrix.

Equation 3.43 can be solved by the Successive Approximation Method as follows:

Let N_k, k=0,1,2,... be the solutions of the equation

$$N_k = S_k{}'\ N_k\ S_k + Q$$

where

$$S_k = A - G_k C \qquad k=0,1,2,...$$

$$G_k = A\ N_{k-1} C'(C'N_{k-1}C')^{-1} \qquad k=1,2,3,...$$

and G_o is chosen such that S_o is a stable matrix. Then

$$N_k - N_{k+1} \geq 0 \qquad k=0,1,2,...$$

and

$$\lim_{k\to\infty} N_k = R\ .$$

Once G which stabilizes F is found then the observer is constructed according to the following difference equation.

$$z(k+1) = (A - GC)\ z(k) + B\ u(k)$$

to be initialized with

$$z(o) = x_g$$

where x_g is an arbitary matrix.

3.5. COMPUTATIONAL ASPECTS

We have derived the equations for constructing the observer having constant valued matrices. Hence, all the computations are carried out off-line.

The transofrmation into the canonical representation is exactly the same with the continuous-time case presented in Section 2.2.

A numerical solution method for the Lyapunov equation was explained in detail in the previous section. Here we just mention a computational difficulty which was encountered while preparing a computer-software program for the numerical solution of the Lyapunov equation. That is, the matrix S_o given in Equation 3.40 as

$$S_o = \left[\begin{array}{c|c} 0 & 0 \\ \hline \tilde{A}_{21}-L_o\tilde{A}_{11} & \tilde{A}_{22}-L_o\tilde{A}_{12} \end{array}\right]$$

must be a stable matrix so that the algorithm converges. If the eigenvalues of $\left[\tilde{A}_{22}-L_o\tilde{A}_{12}\right]$ are inside the unit circle then so are those of S_o, since the remaining eigenvalues are already zero. Although, in principle, there exists a matrix L_o so that the eigenvalues of $A_{22}-L_oA_{12}$ are chosen to correspond to a given set of eigenvalues, yet there has not been any computer based algorithm developed to realize it. On the other hand, hand-calculation, which may be quite cumbersome for high order systems, is always possible. Therefore, the designer must initialize the algorithm for the numerical solution of the Lyapunov equation with the calculated matrix L_o.

CHAPTER 4

OPTIMAL REDUCED ORDER OBSERVER-ESTIMATORS AND SUBOPTIMAL MINIMAL ORDER OBSERVERS IN STOCHASTIC SYSTEMS

4.1. INTRODUCTION

Most of the control systems are subject to disturbances and in addition, the measurements are corrupted by noise. If all the measurements are corrupted by additive Gaussian white noise then one can use a Kalman filter to estimate the states of the system, such that mean square estimation error, $E\left[\,||x(k) - x(k)||^2\right]$ is minimized. On the other hand, there are many cases in which some measurements are noise-free while others are noisy. In such problems, the measurement noise covariance matrix is singular, therefore one may be confronted with the inversion of a singular matrix in the Kalman filter algorithm. Considering the case that it may not be possible to use the Kalman filter algorithm when the measurement noise covariance matrix is singular, the optimal reduced order observer-estimator has been suggested |12,13|. The optimal reduced order observer estimator casts the state estimation problem to a constrained optimization problem. The optimal reduced order observer-estimator algorithm derived in the sequel is a general one in the sense that both Kalman filter and minimal order observer-estimator algorithms can be obtained for the special cases that all the measurements are noise corrupted or all the measurements are noise free.

Minimal order observer as an alternative to the Kalman filter is also discussed in this chapter. Minimal order observer is less optimal compared to the Kalman filter but considerable reduction in computation time makes the minimal order observer to be attractive in real-time implementations.

4.2. OPTIMAL REDUCED ORDER-ESTIMATOR

The standard Gauss-Markov model is considered:

$$x(k+1) = A\,x(k) + B\,u(k) + D\,w(k)$$

$$y(k) = C\,x(k) + v(k)$$

where w is a p-dimensional disturbance vector and v is a m-dimensional measurement noise vector and D is (nxp) disturbance matrix. The matrices A, B and C and the vectors x(k), u(k) and y(k) are as defined in Chapter 2. x(o), w(k) and v(k) are independent Gaussian random vectors with the following statistics:

$$E[x(o)] = x_o \qquad \text{cov}\,[x(o)] = \Sigma x_o$$

$$E[w(k)] = 0 \qquad E[w(k)w(j)'] = \Sigma w\delta(k-j)$$

$$E[v(k)] = 0 \qquad E[v(k)v(j)'] = \Sigma v\delta(k-j)$$

where Σw and Σv are pxp and mxm matrices respectively and it is assumed that Σv is a non-negative definite matrix.

Suppose that m_1 ($m_1<m$) measurements are noise free or v(k) is of the form

$$v(k) = \begin{bmatrix} 0 \\ \hline v_2(k) \end{bmatrix}$$

where $v_2(k) \epsilon R^{m-m_1}$.
Then, Σv becomes

$$\Sigma v = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Sigma_{v_2} \end{array}\right]$$

where Σv_2 is a $(m-m_1)x(m-m_1)$ dimensional positive definite matrix.
The output equation of the system in Equation 4.1 may be partitioned as

$$\left[\begin{array}{c} y_1(k) \\ \hline y_2(k) \end{array}\right] = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right] \left[\begin{array}{c} x_1(k) \\ \hline x_2(k) \end{array}\right] + \left[\begin{array}{c} 0 \\ \hline v_2(k) \end{array}\right]$$

where $y_1(k)$ is m_1-dimensional, y_2 is $(m-m_1)$ dimensional, $x_1(k)$ is m_1-dimensional and $x_2(k)$ is $(n-m_1)$ dimensional. The blocks C_{11}, C_{12}, C_{21} and C_{22} are of appropriate dimensions. To reduce the complexity of calculations to minimum the system represented in Equation 4.1 is transformed to a canonical form as follows.

There exists a nxn non-singular matrix M given by

$$M = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline 0 & I \end{array}\right], \quad C_{11} > 0$$

such that the transformation

$$q(k) = M\ x(k)$$

yields the following equations

$$q(k+1) = \tilde{A}\ q(k) + \tilde{B}\ u(k) + \tilde{D}\ w(k)$$

and

$$y(k) = \left[\begin{array}{c|c} I_m & 0 \\ \hline \tilde{C}_{21} & \tilde{C}_{22} \end{array}\right] q(k) + \left[\begin{array}{c} 0 \\ \hline v_2(k) \end{array}\right] \qquad 4.2$$

where

$$\tilde{A} = M\ A\ M^{-1}$$

$$\tilde{B} = M\ B$$

$$\tilde{D} = M\ D$$

$$\tilde{C}_{21} = C_{21}C_{11}^{-1}$$

$$\tilde{C}_{22} = -C_{21}C_{11}^{-1}C_{12} + C_{22}$$

With proper partitioning Equation 4.2 becomes

$$\begin{bmatrix} q_1(k+1) \\ \hline q_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \begin{bmatrix} q_1(k) \\ \hline q_2(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \hline \tilde{B}_2 \end{bmatrix} u(k) + \begin{bmatrix} \tilde{D}_1 \\ \hline \tilde{D}_2 \end{bmatrix} w(k)$$

and

$$\begin{bmatrix} y_1(k+1) \\ \hline y_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} I_{m_1} & 0 \\ \hline \tilde{C}_{21} & \tilde{C}_{22} \end{array}\right] \begin{bmatrix} q_1(k) \\ \hline q_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \hline v_2(k) \end{bmatrix} \qquad 4.3$$

where $q_1(k)$ is m_1-dimensional, $q_2(k)$ is $(n-m_1)$-dimensional. The statitics are then changed as follows

$$E[q(o)] = M\ x_o = q_o \qquad \text{cov}\ [q(o)] = M\Sigma x_o M^T = \Sigma q_o$$

$$E[w(k)] = 0 \qquad E[w(k)w(j)'] = \Sigma w\delta(k-j)$$

$$E[v(k)] = 0 \qquad E[v(k)v(j)'] = \Sigma v\delta(k-j)$$

Equation 4.3 may be rewritten as

$$q_1(k+1) = \tilde{A}_{11}q_1(k) + \tilde{A}_{12}q_2(k) + \tilde{B}_1u(k) + \tilde{D}_1w(k) \qquad 4.4$$

$$q_2(k+1) = \tilde{A}_{21}q_1(k) + \tilde{A}_{22}q_2(k) + \tilde{B}_2u(k) + \tilde{D}_2w(k) \qquad 4.5$$

$$y_1(k) = q_1(k) \qquad 4.6$$

$$y_2(k) = \tilde{C}_{21}q_1(k) + \tilde{C}_{22}q_2(k) + v_2(k) \qquad 4.7$$

It is readily seen from Equation 4.6 that the states $q_1(k)$ are available as an output, then the state estimation problem for $q_2(k)$ is that of finding $\hat{q}_2(k+1|k+1)$ subject to Equation 4.5 and the constraint in Equation 4.4. The unmeasurable quantities in Equation 4.5 are $q_2(k)$ and $w(k)$. These quantities are constrained in Equation 4.4. $q_1(k)$ can be handled as a deterministic input.

To solve this problem, defining the $(n-m_1)$ dimensional vector $z(k)$ as

$$z(k) = q_2(k) - L(k)\, q_1(k) \tag{4.8}$$

where $L(k)$ is $(n-m_1)\mathrm{x}m_1$ dimensional, one may easily see that

$$z(k+1) = q_2(k+1) - L(k+1)\, q_1(k+1)$$

Substitution of Equations 4.4 and 4.5 into the above expression yields,

$$z(k+1) = \left[\tilde{A}_{21} - L(k+1)\tilde{A}_{11}\right] q_1(k) + \left[\tilde{A}_{22} - L(k+1)\tilde{A}_{12}\right] q_2(k)$$
$$+ \left[\tilde{B}_2 - L(k+1)\tilde{B}_1\right] u(k) + \left[\tilde{D}_2 - L(k+1)\tilde{D}_1\right] w(k)$$

Adding,

$$\left[\tilde{A}_{22} - L(k+1)\tilde{A}_{12}\right] L(k) q_1(k) - \left[\tilde{A}_{22} - L(k+1)\tilde{A}_{12}\right] L(k) q_1(k) = 0$$

to the above expression and using Equation 4.8

$$z(k+1) = \left[\tilde{A}_{22} - L(k+1)\tilde{A}_{12}\right] z(k) + \left[\tilde{A}_{21} - L(k+1)\tilde{A}_{11} + \tilde{A}_{22}L(k) - L(k+1)\tilde{A}_{12}L(k)\right] q_1(k$$
$$+ \left[\tilde{B}_2 - L(k+1)\tilde{B}_1\right] u(k) + \left[\tilde{D}_2 - L(k+1)\tilde{D}_1\right] w(k)$$

is obtained. Defining:

$$F(k) = \tilde{A}_{22} - L(k+1)\tilde{A}_{12}$$
$$G(k) = \tilde{A}_{21} - L(k+1)\tilde{A}_{11} + F(k)L(k)$$
$$H(k) = \tilde{B}_2 - L(k+1)\tilde{B}_1$$
$$Y(k) = \tilde{D}_2 - L(k+1)\tilde{D}_1$$

the above equation becomes

$$z(k+1) = F(k)z(k) + G(k)q_1(k) + H(k)u(k) + Y(k)w(k) \qquad 4.9$$

On the other hand, adding

$$\tilde{C}_{22}\,L(k)q_1(k) - \tilde{C}_{22}L(k)q_1(k) = 0$$

to Equation 4.7 and using Equation 4.8, one obtains:

$$y_2(k) = \tilde{C}_{22}z(k) + \left[\tilde{C}_{21} + \tilde{C}_{22}L(k)\right]q_1(k) + v_2(k)$$

or

$$y_2(k) = \tilde{C}_{22}z(k) + \Gamma(k)q_1(k) + v_2(k) \qquad 4.10$$

where $\Gamma(k) = \tilde{C}_{21} + \tilde{C}_{22}L(k)$.

Then the problem of estimating the state $q_2(k)$ with the constraint in Equation 4.4 is cast as estimating $z(k)$ given the measurements $y_2(k)$ as formulated in Equations 4.9 and 4.10.

The unbiased estimate of $z(k)$ is given by |14|.

$$\hat{z}(k+1|k+1) = [I-P(k+1)\tilde{C}_{22}]F(k)\hat{z}(k|k) + [I-P(k+1)\tilde{C}_{22}]H(k)u(k) + [I-P(k+1)\tilde{C}_{22}]G(k)q_1(k) + P(k+1)[y_2(k+1) - \Gamma(k+1)q_1(k+1)] \quad 4.11$$

where P(k+1) is $(n-m_1)x(m-m_1)$ gain matrix.

The estimation error vector $\tilde{z}(k+1|k+1)$ is defined as

$$\tilde{z}(k+1|k+1) = \hat{z}(k+1|k+1) - z(k+1)$$

Substitution of Equation 4.11, 4.10 and 4.9 in that order into the above error vector equation yields:

$$\tilde{z}(k+1|k+1) = [I-P(k+1)\tilde{C}_{22}]F(k)\tilde{z}(k|k) - [I-P(k+1)\tilde{C}_{22}]Y(k)w(k) + P(k+1)v_2(k+1) .$$

Then the error covariance matrix satisfies the following: matrix difference equation

$$\Sigma\tilde{z}(k+1|k+1) = [I-P(k+1)\tilde{C}_{22}]F(k)\Sigma\tilde{z}(k|k)F'(k)[I-P(k+1)\tilde{C}_{22}]' + [I-P(k+1)\tilde{C}_{22}]Y(k)\Sigma wY'(k)[I-P(k+1)\tilde{C}_{22}]' + P(k+1)\Sigma v_2P'(k+1) \quad 4.13$$

Defining the one-step prediction error covariance matrix $\Sigma z(k+1|k)$ as

$$\Sigma\tilde{z}(k+1|k) = F(k)\Sigma\tilde{z}(k|k)F'(k) + Y(k)\Sigma wY(k)'$$

and inserting this equation into Equation 4.13 yields

$$\Sigma\tilde{z}(k+1|k+1) = [I-P(k+1)\tilde{C}_{22}]\Sigma\tilde{z}(k+1|k)[I-P(k+1)\tilde{C}_{22}]'$$

$$+ P(k+1)\Sigma v_2 P'(k+1) \qquad 4.14$$

The optimal estimator estimates the non-available states in the minimum mean square error sense, that is

$$E\{||\hat{q}_2-q_2||^2\} = E\{||\hat{z} + Lq_1 - z - Lq_1||^2\} = E\{||\hat{z}-z||^2\}$$

is minimized.

$$E\{||\hat{z}-z||^2\} = E\{(\hat{z}-z)'(\hat{z}-z)\} = E\{\sum_{k=1}^{n-m_1} (\hat{z}_k-z_k)^2\} \qquad 4.15$$

On the other hand,

$$\Sigma\tilde{z} = E\{(\hat{z}-z)(\hat{z}-z)'\}$$

or

$$\Sigma\tilde{z} = E\begin{bmatrix} (\hat{z}_1-z_1)^2 & (\hat{z}_1-z_1)(\hat{z}_2-z_2) & \cdots & (\hat{z}_1-z_1)(\hat{z}_{n-m_1}-z_{n-m_1}) \\ (\hat{z}_2-z_2)(\hat{z}_1-z_1) & (\hat{z}_2-z_2)^2 & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ (\hat{z}_{n-m_1}-z_{n-m_1})(\hat{z}_1-z_1) & & & (\hat{z}_{n-m_1}-z_{n-m_1})^2 \end{bmatrix}$$

$$\text{trace } \Sigma\tilde{z} = E\{\sum_{k=1}^{n-m_1} (\hat{z}_k-z_k)^2\} \qquad 4.16$$

From Equation 4.15 and 4.16 it is deduced that minimizing $E\{||\hat{z}-z||^2\}$ is equivalent to minimizing the trace of the error covariance matrix, $\Sigma\tilde{z}$.

The matrices P and L must then be evaluated in such a way so that trace $\Sigma\tilde{z}$ is minimized. Hence, setting the gradient of trace of $\Sigma\tilde{z}$ with respect to the matrices P and L, the following equations are obtained,

$$P^*(k+1) = \Sigma\tilde{z}(k+1|k)\tilde{C}_{22}'[\tilde{C}_{22}\Sigma\tilde{z}(k+1|k)\tilde{C}_{22}' + \Sigma v_2]^{-1}$$

and the optimal L(k+1) denoted as L*(k+1) is given by

$$L^*(k+1)\ \Lambda_1(k) = \Lambda_2(k)$$

where

$$\Lambda_1(k) = \tilde{A}_{12}\Sigma\tilde{z}(k|k)\tilde{A}_{12}' + \tilde{D}_1\Sigma w\tilde{D}_1'$$

$$\Lambda_2(k) = \tilde{A}_{22}\Sigma\tilde{z}(k|k)\tilde{A}_{12}' + \tilde{D}_2\Sigma w\tilde{D}_2' \quad .$$

According to the values of $\Lambda_1(k)$ the following special cases are of interest.

CASE 1 - $\Lambda_1(k) = 0$. In this case $\Lambda_2(k) = 0$ and thus $L^*(k+1) = R^{(n-m_1)xm_1}$. This case is possible if Equation 4.4 does not contain any information pertaining to the estimation of $q_2(k)$. For example, $\tilde{A}_{12} = 0$, $\tilde{B}_1 = 0$ and $\tilde{D}_1 = 0$, etc.

CASE 2 - $\Lambda_1(k)$ is a singular non-zero matrix. In this case, only some components of $q_1(k)$ contain information on ($q_2(k)$, and w(k)). Then, the transformation matrix M can be defined so as to isolate only those elements in $q_1(k)$ which constitute a constraint on ($q_2(k)$ and w(k)) and, thus L(k) is in $R^{(n-r)xr}$ where r is the number of such constaints ($r<m_1$).

CASE 3 - $\Lambda_1(k)$ is a non-singular matrix. In this case L*(k+1) is uniquely given by

$$L^*(k+1) = \Lambda_2(k)\ \Lambda_1^{-1}(k) \ .$$

The optimal reduced order observer equations are given by

$$\hat{z}(k+1|k+1) = \left[I-P^*(k+1)\tilde{C}_{22}\right]F^*(k)\hat{z}(k|k)$$
$$+ \left[I-P^*(k+1)\tilde{C}_{22}\right]H^*(k)u(k)$$
$$+ \left[I-P^*(k+1)\tilde{C}_{22}\right]G^*(k)q_1(k)$$
$$+ P^*(k+1)\left[y_2(k+1) - \Gamma^*(k+1)q_1(k+1)\right]$$

$$P^*(k+1) = \Sigma^*\tilde{z}(k+1|k)\tilde{C}_{22}'\left[\tilde{C}_{22}\Sigma^*\tilde{z}(k+1|k)\tilde{C}_{22}' + \Sigma v_2\right]^{-1}$$

$$L^*(k+1)\ \Lambda_1(k) = \Lambda_2(k)$$

where $\Lambda_1(k) = \tilde{A}_{12}\Sigma^*\tilde{z}(k|k)\tilde{A}_{12}' + \tilde{D}_1\Sigma w\tilde{D}_1'$

$$\Lambda_2(k) = \tilde{A}_{22}\Sigma^*\tilde{z}(k|k)\tilde{A}_{12}' + \tilde{D}_2\Sigma w\tilde{D}_2'$$

$$\Sigma^*\tilde{z}(k+1|k) = F^*(k)\Sigma^*\tilde{z}(k|k)F^{*\prime}(k) + Y^*(k)\Sigma wY^{*\prime}(k)$$

Equation 4.14, after some matrix manipulations |14|, becomes

$$\Sigma^*\tilde{z}(k+1|k+1) = \left[I-P^*(k+1)\tilde{C}_{22}\right]\Sigma^*\tilde{z}(k+1|k)$$

These equations are initialized by the following values

$$\hat{z}(o|o) = E\left[q_2(o)\right]$$
$$\Sigma\tilde{z}(o|o) = \Sigma q_{2o}$$
$$L(o) = 0 .$$

The above equations are used to calculate $\hat{x}(k+1|k+1)$ and $\Sigma\tilde{x}(k+1|k+1)$ as follows:

$$\hat{q}_2(k+1|k+1) = \hat{z}(k+1|k+1) + L(k+1)q_1(k+1)$$

$$q_1(k+1) = y_1(k+1)$$

$$\Sigma\tilde{q}(k+1|k+1) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Sigma\tilde{z}(k+1|k+1) \end{array}\right]$$

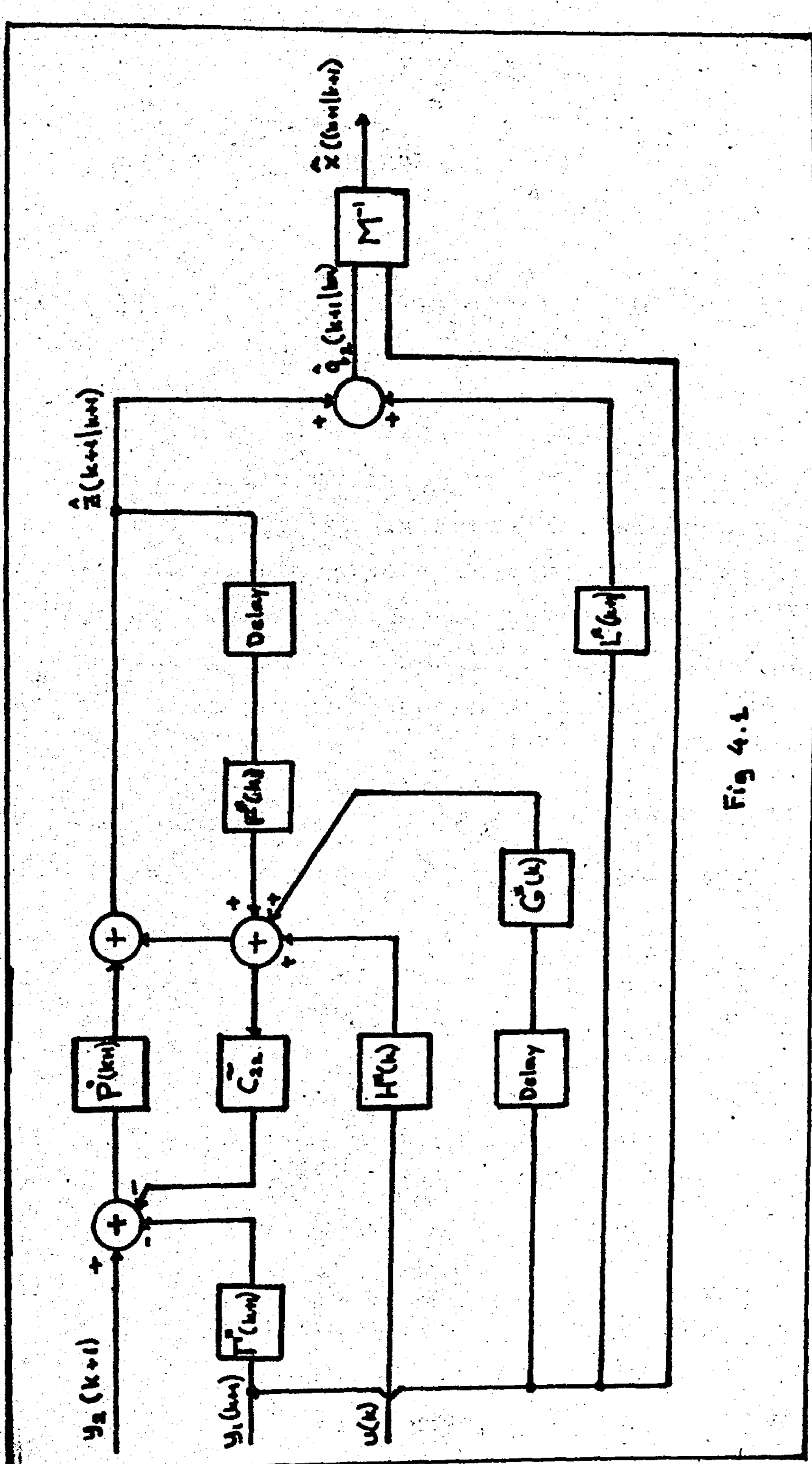


Fig 4.1

$$\hat{x}(k+1|k+1) = M^{-1} \begin{bmatrix} q_1(k+1) \\ \hat{q}_2(k+1|k+1) \end{bmatrix}$$

$$\Sigma\tilde{x}(k+1|k+1) = M^{-1}\ \Sigma\tilde{q}(k+1|k+1)(M^{-1})'$$

<u>SPECIAL CASE 1</u> - All the Measurements are Noise Corrupted ($m_1 = 0$).

In this case the optimal reduced order observer-estimator equations coincide with those of the Kalman filter. The system is again as given in Equation 4.1 except that the measurement noise covariance matrix Σv is positive definite. To show the relationship between the optimal reduced order observer-estimator and the Kalman filter we have formed the following replacement table.

Optimal Reduced Order Observer-Estimator		Kalman Filter	
Variable Name	Dimension	Variable Name	Dimension
M	$n x n$		
$q_1(k)$	$m_1 x1$		
$q_2(k)$	$(n-m_1)x1$		
$z(k)$	$(n-m_1)x1$	$x(k)$	$nx1$
$y_2(k)$	$(m-m_1)x1$	$y(k)$	$mx1$
$\tilde{C}_{22}$	$(m-m_1)x(n-m_1)$	C	mxn
$L(k)$	$(n-m_1)xm_1$		
$F(k)$	$(n-m_1)x(n-m_1)$	A	nxn
$G(k)$	$(n-m_1)xm_1$		
$H(k)$	$(n-m_1)xr$	B	nxr
$Y(k)$	$(n-m_1)xp$	D	nxp
$\Gamma(k)$	$(m-m_1)xm_1$		
$P(k)$	$(n-m_1)x(m-m_1)$	$P(k)$	nxm
$\Sigma\tilde{z}(k\|k)$	$(n-m_1)x(n-m_1)$	$\Sigma\tilde{x}(k\|k)$	nxn
$\Sigma v_2(k\|k)$	$(m-m_1)x(m-m_1)$	Σv	mxm

TABLE 4.1

By the use of Table 4.1 the Kalman filter equations are obtained directly as,

$$\hat{x}(k+1|k+1) = [I-P(k+1)C]A\,\hat{x}(k|k) + [I-P(k+1)C]Bu(k) + P(k+1)\,y(k+1)$$

$$\Sigma\tilde{x}(k+1|k) = A\Sigma\tilde{x}(k|k)A' + D\Sigma wD'$$

$$P(k+1) = \Sigma\tilde{x}(k+1|k)C'\left[C\Sigma\tilde{x}(k+1|k)C' + \Sigma v\right]^{-1}$$

$$\Sigma\tilde{x}(k+1|k+1) = [I-P(k+1)C]\Sigma\tilde{x}(k+1|k)$$

with the initial values

$$\hat{x}(o|o) = x_o$$

$$\Sigma\tilde{x}(o|o) = \Sigma x_o\ .$$

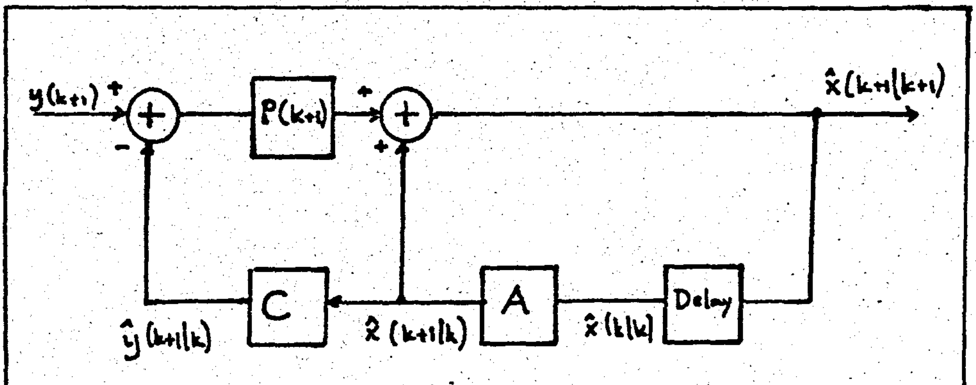


Fig 4.2

SPECIAL CASE 2 - All the Measurements are Noise Free ($m_1 = m$)

In this case, it is assumed that there exists an additive Gaussian white disturbance to the system but the measurements are obtained accurately. We then use minimal order observer-estimator of order n-m to estimate the states x(k) of the system given in Equation 4.1. Again we use a replacement table similar to the one given in the previous section to obtain the equations of minimal order observer-estimator from those of reduced order observer-estimator.

Optimal Reduced Order Observer-Estimator: Variable Name	Dimension	Optimal Minimal Order Observer-Estimator: Variable Name	Dimension
$M = \left[\begin{array}{c:c} C_{11} & C_{12} \\ \hdashline 0 & I \end{array}\right]$	$n \times n$	$M = \left[\begin{array}{c:c} C_1 & C_2 \\ \hdashline 0 & I \end{array}\right]$	$n \times n$
C_{11}	$m_1 \times m_1$	C_1	$m \times m$
C_{12}	$m_1(n-m_1)$	C_2	$m \times (n-m)$
$q_1(k)$	$m_1 \times 1$	$q_1(k)$	$m \times 1$
$q_2(k)$	$(n-m_1) \times 1$	$q_2(k)$	$(n-m) \times 1$
$z(k)$	$(n-m_1) \times 1$	$z(k)$	$(n-m) \times 1$
$y_2(k)$	$(m-m_1) \times 1$		
$\widetilde{C}_{22}$	$(m-m_1) \times (n-m_1)$		
$L(k)$	$(n-m_1) \times m_1$	$L(k)$	$(n-m) \times m$
$F(k)$	$(n-m_1) \times (n-m_1)$	$F(k)$	$(n-m) \times (n-m)$
$G(k)$	$(n-m_1) \times m_1$	$G(k)$	$(n-m) \times m$
$H(k)$	$(n-m_1) \times r$	$H(k)$	$(n-m) \times r$
$Y(k)$	$(n-m_1) \times p$	$Y(k)$	$(n-m) \times p$
$\Gamma(k+1)$	$(m-m_1) \times m_1$		
$P(k+1)$	$(n-m_1) \times (m-m_1)$		
$\Sigma\tilde{z}(k\|k)$	$(n-m_1) \times (n-m_1)$	$\Sigma\tilde{z}(k\|k)$	$(n-m) \times (n-m)$
Σv_2	$(m-m_1) \times (m-m_1)$		

TABLE 4.2

Then, we may write the optimal minimal order observer-estimator equations as

$$\hat{z}(k+1|k+1) = F(k)\hat{z}(k|k) + G(k)q_1(k) + H(k)u(k)$$

$$\Sigma\tilde{z}(k+1|k) = F(k)\ \Sigma\tilde{z}(k|k)F'(k) + Y(k)\ \Sigma wY'(k)$$

$$\Sigma\tilde{z}(k+1|k+1) = \Sigma\tilde{z}(k+1|k)$$

$$L(k+1)\ \Lambda_1(k) = \Lambda_2(k)$$

$$\Lambda_1(k) = \tilde{A}_{12}\Sigma\tilde{z}(k|k)\tilde{A}_{12}' + \tilde{D}_1\Sigma w\tilde{D}_1'$$

$$\Lambda_2(k) = \tilde{A}_{22}\Sigma\tilde{z}(k|k)\tilde{A}_{12}' + \tilde{D}_2'\Sigma w\tilde{D}_2'$$

$$\hat{q}_2(k+1|k+1) = \hat{z}(k+1|k+1) + L(k+1)q_1(k+1)$$

$$q_1(k+1) = y(k+1)$$

with the following initial conditions

$$\hat{z}(o|o) = E[q_2(o)]$$

$$\Sigma\tilde{z}(o|o) = \Sigma q_{20}$$

$$L(o) = 0\ .$$

The estimate of the original state x(k) and the error co-variance matrix are calculated by the following equations

$$\hat{x}(k+1|k+1) = M^{-1}\begin{bmatrix} y(k+1) \\ \\ \hat{q}_2(k+1|k+1) \end{bmatrix}$$

$$\Sigma\tilde{x}(k+1|k+1) = M^{-1}\ \Sigma\tilde{q}(k+1|k+1)(M^{-1})'$$

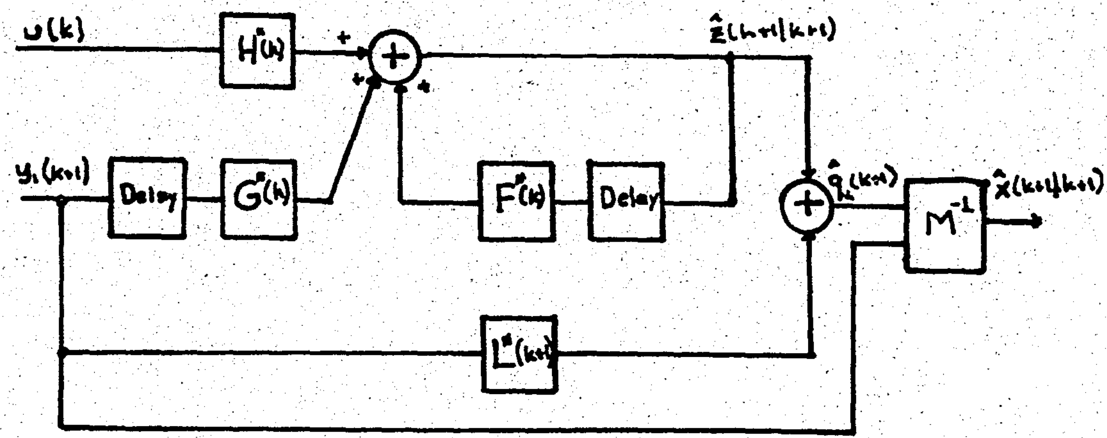


Fig 4.3

4.3. A SUBOPTIMAL MINIMAL ORDER OBSERVER

The minimal order observer which has been developed as an alternative to the Kalman filter, estimates the entire state vector of a stochastic system in the minimum mean square error sense. Its order is restricted to "n-m" where n is the order of the state vector and m is the order of the output vector. The minimal order observer assumes that the estimation error of m components of the state vector which are available at the output, is due to the measurement noise only regardless of the effect of the propagation of the estimation error in time through the dynamics of the system. Under this assumption, the minimal order observer can only serve as a suboptimal solution to the state estimation problem, unlike the Kalman filter.

Construction of the Minimal Order Observer:

Consider the canonical Gauss-Markov model given as

$$x(k+1) = A\ x(k) + B\ u(k) + D\ w(k)$$
$$y(k) = \left[I_m \;\vdots\; 0 \right] x(k) + v(k) \qquad 4.17$$

where $x(k) \varepsilon R^n$, $w(k) \varepsilon R^p$, $v(k) \varepsilon R^m$ are independent Gaussian random vectors with the following statistics

$$E\left[x(o)\right] = x_o \qquad cov\left[x(o)\right] = \Sigma x_o$$
$$E\left[w(k)\right] = 0 \qquad E\left[w(k)w(j)'\right] = \Sigma w \delta(k-j)$$
$$E\left[v(k)\right] = 0 \qquad E\left[v(k)v(j)'\right] = \Sigma v \delta(k-j)$$

The (n-m) dimensional vector z(k) which is the output of the minimal order observer given by

$$z(k+1) = F(k)\ z(k) + G(k)\ y(k) + H(k)\ u(k) \qquad 4.18$$

estimates a linear transformation of the state vector x(k) for some (n-m)xn matrix T(k) as

$$z(k) = T(k)\ x(k) + \varepsilon(k) \qquad 4.19$$

where $\varepsilon(k)$ is the error in the estimate of T(k)x(k), if the following matrix relations are satisfied

$$T(k+1)A = F(k)\ T(k) + G(k) \left[I_m \;\vdots\; 0 \right] \qquad 4.20$$

$$T(k+1)B = H(k) \qquad 4.21$$

and

$$\begin{bmatrix} T(k) \\ \hline I_m \;\vdots\; 0 \end{bmatrix}^{-1}$$

exists.

Then the error $\varepsilon(k)$ satisfies the following:

$$\varepsilon(k+1) = F(k)\varepsilon(k) + G(k)v(k) - T(k+1)Dw(k) \qquad 4.22$$

Rewriting Equation 4.20 in the form

$$T(k+1)A = \left[F(k) \;\vdots\; G(k)\right] \left[\begin{array}{c|c} \multicolumn{2}{c}{T(k)} \\ \hline I_m & 0 \end{array}\right] \qquad 4.23$$

and postulating the existence of the indicated matrix inverse in the form of:

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{T(k)} \\ \hline I_m & 0 \end{array}\right]^{-1} = \left[P(k) \;\vdots\; V(k)\right] \qquad 4.24$$

where P(k) is an nx(n-m) matrix and V(k) is an nxm matrix, one obtains by multiplying Equation 4.23 from the right by the above inverse, the solutions

$$F(k) = T(k+1)A\ P(k)$$

$$G(K) = T(k+1)A\ V(k) \qquad 4.25$$

From Equations 4.18, 4.21, 4.24 and 4.25 it is seen that the design of the minimal order observer reduces to the selection of the matrix T(k+1).

The observer error covariance matrix can be found from the difference equation in Equation 4.22 as follows:

$$\Sigma_\varepsilon(k+1) = E\{\left[F(k)\varepsilon(k) + G(k)\ v(k) - T(k+1)D\ w(k)\right] \cdot \left[\varepsilon'(k)F'(k) + v'(k)G'(k) - w'(k)D'T'(k+1)\right]\}$$

$$\Sigma_\varepsilon(k+1) = F(k)\Sigma_\varepsilon(k)F'(k) + G(k)\ \Sigma_v G'(k) + T(k+1)D\ \Sigma w D'T'(k+1) \qquad 4.26$$

It may be shown that the cross-terms disappear since the observer error is independent of v(k) and w(k) |14|. If F(k) and G(k) are replaced with their equivalents in Equation 4.25, $\Sigma_\varepsilon(k+1)$ becomes

$$\Sigma_\varepsilon(k+1) = T(k+1)\Big[AP(k)\ \Sigma_\varepsilon(k)P'(k)A' + A\ V(k)\ \Sigma_v V'(k)A' + D\ \Sigma w D'\Big]T'(k+1) \qquad 4.27$$

Defining the matrix $\Omega(k)$ as follows

$$\Omega(k) = AP(k)\Sigma_\varepsilon(k)P'(k)A' + AV(k)\Sigma_v V'(k)A' + D\Sigma w D'$$

and partitioning $\Omega(k)$, $\Sigma_\varepsilon(k+1)$ may be written as

$$\Sigma_\varepsilon(k+1) = T(k+1)\left[\begin{array}{c|c} \Omega_{11}(k) & \Omega_{12}(k) \\ \hline \Omega_{21}(k) & \Omega_{22}(k) \end{array}\right] T'(k+1) \qquad 4.28$$

where $\Omega_{11}(k)$ is mxm, $\Omega_{22}(k)$ is (n-m)x(n-m) and $\Omega_{12}(k) = \Omega_{21}'(k)$ is mx(n-m).

Combining the observer output z(k+1) with the output y(k+1) gives the following

$$\left[\begin{array}{c} z(k+1) \\ \hline y(k+1) \end{array}\right] = \left[\begin{array}{c|c} \multicolumn{2}{c}{T(k+1)} \\ \hline I_m & 0 \end{array}\right] x(k+1) + \left[\begin{array}{c} \varepsilon(k+1) \\ \hline v(k+1) \end{array}\right] \qquad 4.29$$

Defining

$$\hat{x}(k+1) = \left[\begin{array}{c|c} \multicolumn{2}{c}{T(k+1)} \\ \hline I_m & 0 \end{array}\right]^{-1} \left[\begin{array}{c} z(k+1) \\ \hline y(k+1) \end{array}\right]$$

or equivalently

$$\hat{x}(k+1) = \left[\begin{array}{c|c} P(k+1) & V(k+1) \end{array}\right] \left[\begin{array}{c} z(k+1) \\ \hline y(k+1) \end{array}\right]$$

and using Equation 4.29 the following relation is obtained

$$\hat{x}(k+1) = x(k+1) + \left[\begin{array}{c|c} P(k+1) & V(k+1) \end{array}\right] \left[\begin{array}{c} \varepsilon(k+1) \\ \hline v(k+1) \end{array}\right] \qquad 4.30$$

The resulting estimation error e(k+1) defined by

$$e(k+1) = \hat{x}(k+1) - x(k+1)$$

is found to be

$$e(k+1) = \left[P(k+1) \mid V(k+1)\right] \begin{bmatrix} \varepsilon(k+1) \\ \hline v(k+1) \end{bmatrix} \qquad 4.31$$

Finally, the estimation error covariance matrix $\Sigma_e(k+1)$ is obtained as follows

$$\Sigma_e(k+1) = E\{\left[P(k+1) \mid V(k+1)\right] \begin{bmatrix} \varepsilon(k+1) \\ \hline v(k+1) \end{bmatrix} \begin{bmatrix} \varepsilon(k+1) \\ \hline v(k+1) \end{bmatrix}' \left[P(k+1) \mid V(k+1)\right]'\}$$

$$\Sigma_e(k+1) = \left[P(k+1) \mid V(k+1)\right] \left[\begin{array}{c|c} \Sigma_\varepsilon(k+1) & E\left[\varepsilon(k+1)v'(k+1)\right] \\ \hline E\left[v(k+1)\varepsilon'(k)\right] & \Sigma_v \end{array}\right] \cdot \left[P(k+1) \mid V(k+1)\right]'$$

But,

$$E\left[\varepsilon(k+1)\ v'(k+1)\right] = 0$$

as it was stated before, the above equation becomes

$$\Sigma_e(k+1) = \left[P(k+1) \mid V(k+1)\right] \left[\begin{array}{c|c} \Sigma_\varepsilon(k+1) & 0 \\ \hline 0 & \Sigma_v \end{array}\right] \left[P(k+1) \mid V(k+1)\right]' \qquad 4.32$$

This last equation may be simplified further with the following choice of the matrix T(k) as

$$T(k) = \left[-L(k) \mid I_{n-m}\right] \qquad 4.33$$

where L(k) is an arbitrary (n-m)xm gain matrix which will be chosen to minimize the norm of the estimation error.

With this choice of the matrix T(k), the matrices P(k) and V(k) are found to be

$$P(k) = \begin{bmatrix} 0 \\ \hline I_{n-m} \end{bmatrix}, \quad V(k) = \begin{bmatrix} I_m \\ \hline L(k) \end{bmatrix} \tag{4.34}$$

Substituting Equation 4.34 into Equation 4.32 yields

$$\Sigma_e(k+1) = \left[\begin{array}{c|c} \Sigma_v & \Sigma_v L'(k+1) \\ \hline L(k+1)\Sigma_v & \Sigma_\varepsilon(k+1) + L(k+1)\,\Sigma_v\,L'(k+1) \end{array}\right]$$

Furthermore, substitution of Equation 4.33 into Equation 4.28 gives

$$\Sigma_\varepsilon(k+1) = L(k+1)\Omega_{11}(k)L'(k+1) - \Omega_{21}(k)L'(k+1)$$
$$- L(k+1)\,\Omega_{12}(k) + \Omega_{22}(k) \tag{4.35}$$

In order to be able to obtain an estimate in the minimum mean square error sense, trace $\Sigma_e(k+1)$ must be minimized with respect to L(k+1) since L(k+1) is the only matrix to be determined. Then, from Equations 4.34 and 4.35 one obtains

$$\text{trace } \Sigma_e(k+1) = \text{trace } \Sigma_v + \text{trace } \{L(k+1)(\Omega_{11}(k)+\Sigma_v)L'(k+1)$$
$$- \Omega_{21}(k)L'(k+1) - L(k+1)\Omega_{12}(k) + \Omega_{22}(k)\}$$

Setting the gradient of the above equation with respect to the gain matrix L(k+1) equal to zero and using the formulas given in |16|, the minimizing L(k+1) is found to be

$$L(k+1) = \Omega_{21}(k)(\Omega_{11}(k) + \Sigma_v)^{-1}$$

if the indicated inverse exists.

The observer matrices F, G and H may be obtained in terms of the matrices A, B and L(k) by straightforward substitution of Equations 4.33 and 4.34 into Equations 4.21 and 4.25 as follows

$$F(k) = A_{22} - L(k+1)\, A_{12}$$

$$G(k) = A_{21} - A_{22}L(k) + L(k+1)(A_{11} - A_{12}L(k))$$

$$H(k) = B_2 - L(k+1)\, B_1$$

Initialization of the observer is done as follows.
Let $z(1) = T(1)x_1$ be the observer initial condition, where X_1 is the expected value of the state vector x(1).
Since $\varepsilon(1) = z(1) - T(1)x_1$, then

$$\Sigma_\varepsilon(1) = T(1)\, E\left[(v(1) - x_1)(x(1) - x_1)'\right]\, T'(1)$$

But $x(1) - x_1 = A(x(o) - x_o) + w(o)$ hence the above equa- becomes

$$\Sigma_\varepsilon(1) = T(1)(A\Sigma x_o A' + D\Sigma w D')\, T'(1)$$

To initialize the observer, define the covariance matrix $\Omega(o)$ to be

$$\Omega(o) = A\ \Sigma x_o\ A' + D\ \Sigma w\ D'$$

and take the gain matrix L(1) to be

$$L(1) = \Omega_{21}(o)(\Omega_{11}(o) + \Sigma_v)^{-1}$$

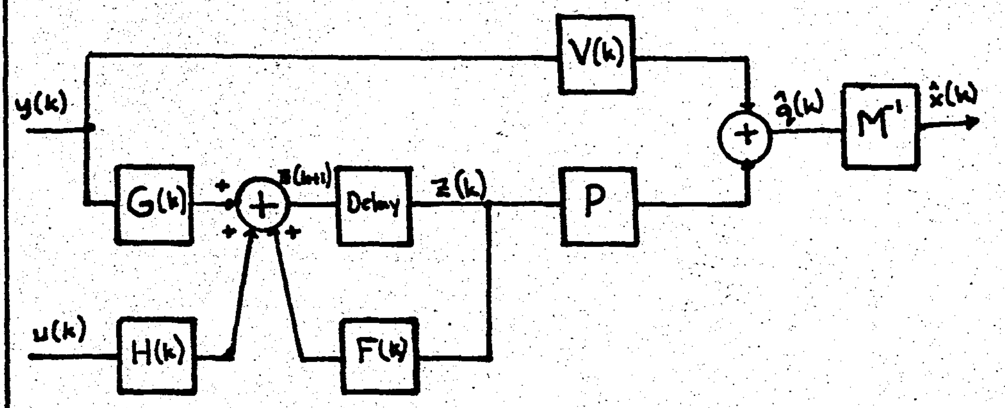


Fig 4.4

4.4. COMPUTATIONAL ASPECTS

In this section we have prepared a table showing the memory space and number of multiplications used in each algorithm so that the user may compare the algorithms according to the given dimensions of the system.

It should be noted that if all the measurements are noise corrupted then the user does not have access to Optimal Reduced-Order Observer-Estimator algorithm with the prepared package program.

	Kalman Filter	Optimal Reduced-Order Observer - Estimator	Suboptimal Minimal Order Observer
Memory Space	$6n^2+(2m+p+r)n+m^2+p^2$	$7n^2+(m-7m_1+2r+p)n$ $+4m_1^2-(m+r)m_1+m^2+p^2$	$7n^2+(2r-p-m)n$ $-m^2-mr$
Number of Multiplication	$3n^4+(p+m)n^3+2m^2n^2$ $+m^3n+m^3$	$2n^4+(r+m-7m_1)n^3+(12m_1^2$ $-6m_1m-rm+rm_1+2m^2+p^2)n^2$ $+(-10m_1^3+9mm_1^2-(4m^2+2p^2)m_1+p^3)n$ $+(2m^2+3m+2p^2-p)m_1^2+m^3$	$3n^4+(p-4m+r)n^3$ $+(2m^2-rm)n^2+3m^4+m^3$

CHAPTER 5

USER'S MANUAL

5.1. INTRODUCTION

The package program consists of one main program and ten subroutines. The function of the main program is the selection of the appropriate subroutine as regards the type of the observer.

The observers are classified as follows:

1. Continuous-time deterministic observers
2. Discrete-time deterministic observers
3. Optimal observer-estimators for stochastic systems.
   a) Kalman filter, if all the measurements are noise corrupted.
   b) Optimal reduced order observer-estimator, if the measurements are partially noise corrupted.
   c) Optimal minimal order observer-estimator, if all the measurements are noise free.

Each type of observer above is examined in a separate subsection as far as supply of data cards, the output variables, error messages and used subroutines are concerned. The user can find all the features that are provided by the prepared package program, when he refers to the section concerning the type of the observer that he chooses.

5.2. DEFINITION OF INPUT VARIABLES

ST	System type, real
N	System dimension, $\leq$ 10, integer
M	Output dimension, < N, integer
NR	Control input dimesion, $\leq$ 10, integer
M1	Noise-free measurement dimension, $\leq$ M, integer
NP	Disturbance input dimension, $\leq$ 10, integer
NSTEP	Number of k (measurement time-points) in stochastic observers, integer
A	State transition matrix, (NxN), real
B	Control input matrix, (NxNR), real
C	Output matrix, (MxN), real
ZL	The matrix L_o in the discrete-time deterministic observers, (N-MxM), real
D	Disturbance input matrix, (NxNP), real
EXPX	$E[x(o)]$ vector, (N), real
COVX	$\mathrm{cov}[x(o)]$ matrix, (NxN), real
COVW	Disturbance covariance matrix, (NPxNP), real
COVV2	Measurement noise covariance matrix, (M-M1xM-M1), real

5.3. SUPPLY OF CONTROL AND DATA CARDS

The following control cards must be supplied in the given order:

```
∂RUN,     ↓           ↓            ↓                  ↓
       Priority    ID Name   Charge Number   Project Name

∂ASG,A  BOGAZICI * BUMP1.
∂ASG,A  EETHESIS * OVSERVER.
∂XQT    OBSERVER.MAIN
 |DATA
∂FIN
```

The READ statements in the package program are in free FORMAT, (see Example 1).

The matrices are read row-wise.

5.3.1. Continuous-Time Deterministic Observers

Data Cards	Remarks
1.	System type indicator (Continuous)
M,N,NR,M,0,0	
A	Original s.t. matrix
B	Original input matrix
C	oringinal output matrix

EAMPLE 1. The following system equation is given.

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0.5 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 3 & 5 \\ 0 & 1 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 4 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} x(t)$$

The user must supply the data cards as follows:

80 th column

1.

3,6,2,3,0,0

A -1.,0.,0.5,1.,2.,1.,0.,0.,2.,1.,3.,5.,0.,1.,-0.5.,0.,0.,0.,0.,0.,1.,2., 4.
0.,0.,0.,0.,1.,1.,1.,1.,2.,1.,0.,1.,0.

B 2.,1.,0.,1.,0.,0.,0.,0.,0.,1.,1.,1.

C 2.,4.,0.,0.,1.,0.,1.,2.,3.,2.,1.,1.,0.,0.,1.,0.,0.,0.,

Important Note

If the input matrix B is not present, then punch

a) 1 in the column of NR

b) N-dimensional zero matrix in the data card concerning the matrix B.

Output

Re-numbered state variables	QX
Re-oriented system matrices	A,B
Transformation matrix	M
Transformation matrix inverse	M^{-1}
Canonical system matrices	A,B,C
Observer's state matrix	F
Observer's input matrices	H,G

Subroutines

MAIN, TRNSFO, CLTIS, CARP, CARP1, PRTIT1, PRTIT2, OBSERV, LIAPUN and MTAMDF, MTVLM, MTMPRT from BOGAZICI*BUMP1 Library.

5.3.2. Discrete-Time Deterministic Observers

The system matrices of observers can be obtained in two steps. The output of the first step is supplied as input data for the second step.

STEP 1

Data Cards	Remarks
2.	System type indicator (Discrete)
M,N,NR,M,0,0	
A	Original s.t. matrix
B	Original input matrix
C	Original output matrix

Output

Re-numbered state variables	QX
Re-oriented system matrices	A,B
Transformation matrix	M
Inverse of transformation matrix	M^{-1}
Canonical system matrices	A,B,C

STEP 2

Canonical state transition matrix $\tilde{A}$ is partitioned as follows:

$$\begin{array}{cc} & \begin{array}{cc} m & n-m \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \end{array}$$

and the matrix L_o is calculated so that the eigenvalues of

$$\tilde{A}_{22} - L_o\tilde{A}_{12}$$

are inside the unit circle. Then the following data cards are supplied.

Data Cards	Remarks
5.	Indicates that the matrix L_o is calculated
M,N,NR,M,0,0	
A	Canonical state transition matrix
B	Canonical input matrix
ZL	L_o

See Example 1 and Important Note in Section 5.3.1.

Output

Observer's state transition matrix F

Observer's input matrices H,G

Error Messages

M exceeds or equal to N

C is not of full-rank

A is singular

Subroutines

MAIN, TRNSFO, DLTIS, CARP, CARP1, PRTIT1, PRTIT2, OBSERV and MTAMDF, MTYLM, MTMPRT from BOGAZICI*BUMP1 Library.

5.3.3. Optimal Observer-Estimators for Stochastic Systems

Case 1. All the Measurements are Noise Corrupted

This case considers that the following system equations are given:

$$x(k+1) = A\ x(k) + B\ u(k) + D\ w(k)$$

$$y(k) = C\ x(k) + v(k)$$

where covariance matrix of the measurement noise is a (mxm) positive-definite matrix.

Data Cards	Remarks
3.	Optimal-estimator
M,N,NR,O,NP,NSTEP	M1=0 indicates that all the measurements are noise corrupted.
A	Original state transition matrix
B	Original input matrix
C	Original output matrix
D	Original disturbance matrix
EXPX	Original $E[x(o)]$
COVX	Original $\mathrm{cov}[x(o)]$
COVW	Covariance matrix of original disturbance matrix
COVV2	Measurement noise covariance matrix (mxm)

See Example 1 and Important Note in Section 5.3.1. Important Note also applies for the matrices D and COVW.

Output

Kalman gain matrix $P(k)$

Error covariance matrix $\Sigma\tilde{x}(k|k)$

Error Messages

M1 eceeds M.

PSI is singular ($[(C\ \Sigma\tilde{x}(k+1|k)C' + \Sigma v]^{-1}$ does not exist)

Subroutines

MAIN, STOKAS, CARP1 and MTAMDF, MTMPRT from BOGAZICI* BUMP1 Library.

Case 2. Measurements are Partially Noise Corrupted

This case considers that the following system equations are given:

$$x(k+1) = A\ x(k) + B\ u(k) + D\ w(k)$$
$$y(k) = C\ x(k) + \begin{bmatrix} 0 \\ \hline v_2(k) \end{bmatrix}$$

where $v_2(k) \varepsilon R^{m-m_1}$. The measurement noise covariance matrix Σv is of the form

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Sigma v_2 \end{array}\right]$$

where $\Sigma v_2 \varepsilon R^{(m-m_1)x(m-m_1)}$ is positive-definite.

Data Cards	Remarks
3.	Optimal-estimator
M,N,NR,M1,NP,NSTEP	M1<M. Partically noise corrupted measurements.
A	Original state transition matrix
B	Original control input matrix
C	Original output matrix
D	Original distrubance matrix
EXPX	Original E[x(o)]
COVX	Original cov[x(o)]

Data Cards	Remarks
COVW	Covariance matrix of original disturbance matrix
COVV2	Measurement noise covariance matrix $(m-m_1)x(m-m_1)$

See Example 1 and Important Note in Section 5.3.1. Important Note also applies for the matrices D and COVW.

Output

Re-numbered state variables	QX
Re-oriented system matrices	A,B,D,EXPX,COVX
Transformation matrix	M
Inverse of transformation matrix	M^{-1}
Canonical system matrices	A,B,C,D,EXPX,COVX
Gain matrix	P(k)
Observer gain matrix	L(k)
Error covariance matrix of the re-oriented system	$\Sigma\tilde{x}(k\|k)$
Observer's state transition matrix	F(k)
Observer's input matrices	H(k), G(k)

Error Messages

M1 exceeds M.

C is not of full-rank

PSI is singular $\left[[C_{22}\Sigma\tilde{z}(k+1|k)C_{22}' + \Sigma v_2]^{-1} \text{ does not exist}\right]$

LAMDA1 is singular $[\Lambda_1(k) \text{ is singular}]$

Subroutines

MAIN, STOKAS, TRNSFO, CARP, CARP1, PRTIT1, PRTIT2, OBSERV and MTAMDF, MTMPRT from BOGAZICI*BUMP1 Library.

Case 3. All the Measurements are Noise Free

This case considers that the following system equations are given:

$$x(k+1) = A\ x(k) + B\ u(k) + D\ w(k)$$

$$y(k) = C\ x(k)$$

Data Cards	Remarks
3.	Optimal-estimator
M,N,NR,M,NP,NSTEP	M1=M indicates that the measurements are noise free
A	Original state transition matrix
B	Original control input matrix
C	Original output matrix
D	Original disturbance matrix
EXPX	Original $E[x(o)]$
COVX	Original $\mathrm{cov}[x(k)]$
COVW	Covariance matrix of original disturbance matrix

See Example 1 and Important Note in Section 5.3.1. Important note also applies for the matrices D and COVW.

Output

Re-numbered state variables	QX
Re-oriented system matrices	A,B,D,EXPX,COVX
Transformation matrix	M
Transformation matrix inverse	M^{-1}
Canonical system matrices	A,B,C,D,EXPX,COVX
Observer gain matrix	L(k)

Error covariance matrix of the re-oriented system	$\Sigma\tilde{x}(k\|k)$
Observer's state transition matrix	F(k)
Observer's input matrices	H(k), G(k)

Error Messages

M1 exceeds M.

C is not of full-rank

LAMDA1 is singular

Subroutines

MAIN, STOKAS, TRNSFO, CARP, CARP1, PRTIT1, PRTIT2, OBSERV and MTAMD, MTMDRT from BOGAZICI*BUMP1 Library.

5.3.4. Suboptimal Minimal Order Observer for Stochastic Systems

Data Cards	Remarks
4.	Sub-optimal estimator
M,N,NR,O,NP,NSTEP	
A	Original state transition matrix
B	Original control input matrix
C	Original output matrix
D	Original disturbance matrix
EXPX	Original E[x(o)]
COVX	Original cov[x(o)]
COVW	Covariance matrix of original disturbance matrix
COVV2	Measurement noise covarinace matrix (mxm)

See Example 1 and Important Note in Section 5.3.1. Important note also applies for the matrices D and COVW.

Output

Re-numbered state variables	QX
Re-oriented system matrices	A,B,D,EXPX,COVX
Transformation matrix	M
Transformation matrix inverse	M^{-1}
Canonical system matrices	A,B,C,D,EXPX,COVX
Observer gain matrix	L(k)
Error covariance matrix of the re-oriented system	$\Sigma\tilde{\tilde{x}}(k\|k)$
Observer's state transition matrix	F(k)
Observer's input matrices	H(k), G(k)

Error Messages

M1 exceeds M.

C is not of full-rank.

The gain matrix L(k) cannot be calculated.

$((\Omega_{11}(k) + \Sigma v)^{-1}$ does not exist)

Subroutines

MAIN, STOKAS, TRNSFO, CARP, CARP1, PRTIT1, PRTIT2 and MTAMDF, MTMPRT from BOGAZICI*BUMP1 Library.

5.4. SUBROUTINES

1. MAIN Program

This main program selects the proper subroutine according to the type of the observer. Input parameters are read from the supplied data cards.

Input parameters

ST,N,M,NR,M1,NP,NSTEP,A,B,C. (See Section 5.2 for the definition of input parameters.)

Output parameters

None.

Error Messages

M exceeds or equal to N.

M1 exceeds M.

A is singular. Discrete-time minimal order observer cannot be realized.

Subroutines Selected

TRNSFO, CLTIS, DLTIS, STOKAS

2. Subroutine TRNSFO (A,B,C,D,EXPX,VOCX,M,MAUX,NR,NP,N,ST)

The subroutine TRNSFO re-numbers the state variables to obtain a non-singular matrix C_1 (See Section 2.1) and evaluates the transformation matrix M that transforms the original system equation into the canonical form.

Input parameters

A,B,C,D,EXPX,COVX,M,MAUX,NR,NP,S,ST

MAUX - Output dimension

M - Noise-free measurement dimension

Rest is as given in Section 5.2.

Output parameters

A,B,C,D,EXPX,COVX

All these matrices obtain canonical values.

Error Message

C is not of full-rank.

Subroutines Used

CARP,CARP1

3. Subroutine CLTIS (A,B,M,N,NR)

The subroutine CLTIS evaluates and prints the matrix L of the continuous-time deterministic observer by solving the Lyapunov equation. The matrix L is then used in SUBROUTINE OBSERV to evaluate the matrices F, G and H of the Observer.
The subroutine CLTIS is called from the main program after the transformation into the canonical form has been performed.

Input parameters
A,B,M,N,NR

Output parameters
ZL - The matrix L.

Error Message
No solution to Lyapunov Equation - see Section 2.2.2.

Subroutines Used
CARP1, LIAPUN, OBSERV, PRTIT1,PRTIT2

4. Subroutine DLTIS (A,B,M,N,NR)

Performs the same operations in SUBROUTINE CLTIS for the discrete-time deterministic observer.

Input parameters
A,B,M,N,NR

Output parameter
ZL - The matrix L.

Error Message

None.

Subroutines Used

CARP1, OBSERV, PRTIT1, PRTIT2

5. Subroutine LIAPUN (ASQ,QSQ,N,R,Y)

The subroutine LIAPUN solves the Lyapunov Equation of the form

$$(ASQ)'R + R(ASQ) + QSQ = 0.$$

Input parameters

ASQ, QSQ, N

ASQ - $-\tilde{A}_{22}$

QSQ - $\tilde{A}_{12}{}'K\tilde{A}_{12} - Q$ (K and Q are determined in the subroutine CLTIS)

Output parameters

R, Y

R - Solution matrix of the above Lyapunov Equation.

Y - R^{-1}

Error Message

None.

(It has a message as, "Inverse does not exist", but solution is found in this case as well.)

Subroutines Used

CARP1

6. Subroutine STOKAS (A,B,C,M1,M,NR,NP,N,NSTEP,ST)

This subroutine evaluates the parameters of optimal-observer-estimators and sub-optimal minimal-order observers for stochastic systems.

Input parameters

A,B,C,M1,M,NR,NP,N,NSTEP,ST

Output parameters

See Sections 5.3.3 and 5.3.4.

Error Messages

See Section 5.3.3 and 5.3.4.

Subroutines Used

TRNSFO,CARP1,OBSERV,PRTIT1,PRTIT2

7. Subroutine OBSERV (ZL,A11,A12,A21,A22,B1,B2,D1,D2,M, NR,NP,N.ST,F,G,H,PN)

The subroutine OBSERV evaluates and prints the observer matrices F,G and H in the deterministic case. This subroutine is also used in optimal reduced-order and in optimal minimal order observer-estimator designs.

Input parameters

ZL,A11,A12,A21,A22,B1,B2,D1,D2,M,NR,NP,N,ST

ZL - L in deterministic case, L(k) in stochastic case

A11,A12,A21,A22 - Blocks of the partitioned matrix A

B1,B2 - Blocks of the partitioned matrix B

D1,D2 - Blocks of the partitioned matrix D

Output parameters

F,G,H,PN

PN - Y(k)

Error Message

None.

Subroutine Used

CARP1

8. Subroutine CARP (CC,PP,MM,NN,KK,D)

The subroutine CARP performs the following multiplication:

$$D = CC * PP(i)$$

This subroutine is used in the reduction to canonical form.

Input parameters

CC,PP,MM,NN,KK

CC - The matrix C

PP - The matrix PM(i) (See Section 2.2.1)

MM - M (output dimension)

NN - N (system dimension)

KK - i

Output parameter

D

Error Message

None

Subroutine Used

None

9. Subroutine CARP1 (CC1,PP1,MM,NN,NN1,D1)

This subroutine is used in matrix multiplication.

Input parameters

CC1 - Pre-multiplying matrix
PP1 - Multiplied matrix
MM - Row dimension of the pre-multiplying matrix
NN - Column dimension of the pre-multiplying matrix or the row dimension of the multiplied matrix
NN1 - Column dimension of the multiplied matrix

Output parameter

D1 - Resultant matrix

Error Message

None

Subroutine Used

None

10. Subroutine PRTIT1 (A,M,N,A11,A12,A21,A22)

This subroutine partitions a square matrix A as follows:

$$A = \begin{array}{cc} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & \begin{array}{c} m \\ n-m \end{array} \end{array}$$

Input parameters

A,M,N

Output parameters

A11,A12,A21,A22

Error Message

None

Subroutine Used

None

11. Subroutine PRTIT2 (AA,M,N,BR,B1,B2)

This subroutine partitions a rectangular matrix AA as follows:

$$AA = \begin{array}{c} r \\ \left[\begin{array}{c} B_1 \\ \\ B_2 \end{array}\right] \end{array} \begin{array}{c} \\ m \\ \\ n-m \end{array}$$

Input parameters

AA,M,N,NR

NR - r

Output parameters

B1,B2

Error Message

None

Subroutine Used

None

CHAPTER 6

CONCLUSIONS

In this study, deterministic and stochastic observers for linear time invariant systems have been discussed. Since, not all the approaches to the design of observers can be utilized by digital computers the theoretical development has been mainly devoted to the design procedures that can be treated by digital computers. The computational problems of other approaches are stated below for the sake of future researchers in this area.

In the deterministic case, the minimal order observers with zero steady state estimate error are considered. The steady state estimate error has been related to the stability of the observer and the dynamics of the observer have been determined by the use of the Lyapunov stability theory |7|. The dynamics of the observer can also be determined by choosing the eigenvalues of the state transition matrix of the observer in such a way that they correspond to a given set of eigenvalues |3|. This method requires that the parametric characteristic equation of the observer be equal to the characteristic equation obtained from the set of eigenvalues. An algorithm has not yet been developed that can be applied on digital computers to evaluate this parametric equation. One may use the method which transforms the system equations into the observable companion form and evaluate the parameters of the observer |18|. But, this method becomes quite cumberson for multi-output systems as far as computational aspects are concerned, therefore the digital computer based solution is not attractive. One other approach is to use the extension of Ackermann's procedure to multivariable systems. This method is not yet utilizable by digital computers and it is under investigation |19|. The methods presented so far result in time-invariant observers, thus the computations are held off-line. One may consider the transient estimate error and employ an on-line procedure which minimizes the norm of the error in the transient period. This suggestion opens a new research area since some systems may not admit large errors in the transient period.

The construction of minimal order observers for continuous time systems has been reduced to the solution of the Lyapunov equation of the form

$$\tilde{A}_{22}'R + R\tilde{A}_{22} + Q-\tilde{A}_{12}'K\tilde{A}_{12} = 0$$

The solution of this equation hinges on the selection of the matrix K. It was shown that the matrix K can easily be selected if the matrix $\tilde{A}_{22}$ is either positive-definite or negative-definite. If $\tilde{A}_{22}$ is an indefinite or a semi-definite matrix then the matrix K can be found by an exhaustive search among positive-semidefinite matrices such that the above equation yields a positive-definite solution matrix R. This thesis work misses any suggestion concerning this exhaustive search method.

The reduction of the order of the Lyapunov equation in the discrete-time case is an important step in the development of the deterministic-time observer theory. This reduction in order reduces the computation time considerably compared to the computation time of the solutions found in the literature. For the discrete-time observers it remains finally to say that the hand-calculation of the matrix L_o in order to have a stable matrix S_o must not be considered as a drawback in the developed study, since men-computer interactions are quite common in recent years.

The stochastic case involves the optimal reduced order observer-estimator and suboptimal minimal order observer. For systems in which optimality is not of utmost importance one may use the suboptimal minimal order observer which provides fast response since the computation time which is a function of the order of the observer reduces considerably. The trade-off between optimality and the computational time may be best decided upon by comparing the results and the computation time of both the optimal reduced-order observer-estimator and the suboptimal minimal order observer for the system of interest.

The user's manual prepared explains how the data must be supplied by the user to evaluate the parameters of the observer of interest. It also provides the structures of the subprograms in the package program. The package program is preserved at Boğaziçi University Computer Center. The programs in the package are listed in the Appendix B. In all programs, single precision is used, if high accuracy is required, by changing matrix and array definitions and supplying necessary cards to declare double precision variables, double precision arithmetic may be used. We finally say that, the disadvantage of this package is that it is only compatible to UNIVAC systems. The problem may be avoided by rewriting some of the subroutines so that they can be used in any computer system.

APPENDIX A

<u>EXAMPLE 1</u>. The following plant is given:

$$\dot{x}(t) = \begin{bmatrix} -3 & 0 & -2 & -1 \\ -2 & -2 & -3 & -3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 & 4 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} x(t)$$

Find an optimal state feedback control u(t) of the form

$$u(t) = -K\, x(t)$$

such that the following performance index is minimized.

$$PI = \int_0^\infty (x'x + u'u)dt$$

<u>Solution</u>

All the states are not available at the output. We will show that an observer can be designed to estimate the non-available states so that the above optimal control problem is solved. The solution, after all the states becomes available, of the above problem can be found in any optimal control book.

The transformation matrix M and its inverse M^{-1}

$$M = \begin{bmatrix} 2 & 1 & 4 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

transfor; the above system into the canonical form as follows:

$$\dot{q}(t) = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} q(t) + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} q(t)$$

where

$$q(t) = M\, x(t)$$

It is seen from the above output equation that the state variables $q_3(t)$ and $q_4(t)$ are not available. To obtain the estimates $q_3(t)$ and $q_4(t)$ the minimal order observer is designed as follows:

The Lyapunov equation

$$\widetilde{A}_{22}{}'R + R\widetilde{A}_{22} + Q - \widetilde{A}_{12}{}'K\widetilde{A}_{12} = 0$$

is solved.

$$\tilde{A}_{22} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

is a negative-definite matrix, then if the matrix K is chosen as

$$K = 0$$

the above Lyapunov equation yields a positive definite solution matrix R, for every $Q > 0$.
Since $K = 0$, the matrix L becomes

$$L = \frac{1}{2} R^{-1} \tilde{A}_{12}' K = 0$$

The the observer matrices are:

$$F = \tilde{A}_{22} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$G = \tilde{A}_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$H = \tilde{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The observer error

$$e(t) = \begin{bmatrix} z_1(t) - q_3(t) \\ z_2(t) - q_4(t) \end{bmatrix}$$

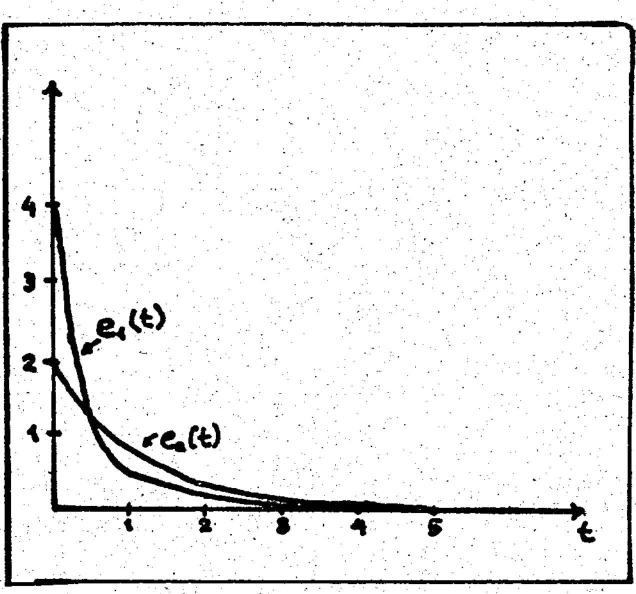
$$\dot{e}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} e(t)$$

The solution of the above differential equation is

$$e(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} e(o)$$

The following graphs show that the error decreases with time regardless of the error in the initial condition.

Choose $e(o) = \begin{bmatrix} +4 \\ +2 \end{bmatrix}$



The matrices P and V are found to be:

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

EXAMPLE 2. A canonical system is described as:

$$q(k+1) = \begin{bmatrix} 0.25 & 0.625 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0.5 \end{bmatrix} q(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} q(k)$$

The poles of this open-loop system are at: k, 0.5 and 0.25. It is desired to find a state feedback control of the form $u(k) = K\, q(k)$ so that the closed-loop poles are at 0, 0.25 and 0.5.

Solution

The state variable $q_3(k)$ is not available at the output hence we will design a first order observer to estimate the state variable $q_3(k)$ so that the state feedback control $u(k)$ can be obtained.

For this system n=3 and m=2 then we partition the matrix A as follows:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad \left[\begin{array}{cc|c} 0.25 & 0.625 & 1 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 0.5 \end{array}\right]$$

We calculate the elements of the matrix L such that the eigenvalues of $A_{22}-LA_{12}$ are inside the unit circle.

$$0.5 - \begin{bmatrix} \ell_1 & \ell_2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0.5 - \ell_1 - 3\ell_2$$

Choosing $\ell_2 = 0$ and $\ell_1 = 0.5$, the eigenvalues of $A_{22}-LA_{12}$ is found to be zero.
With the initial value of the matrix L as

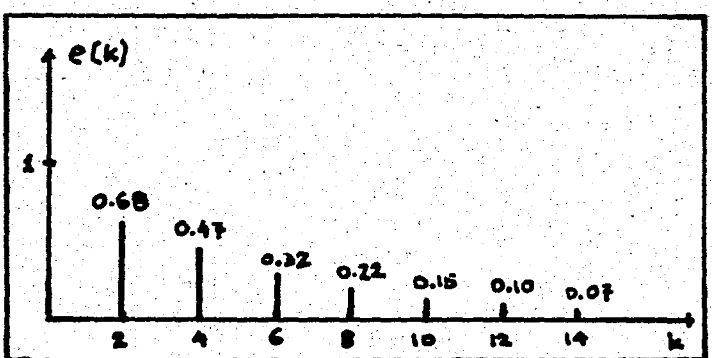
$$L = \begin{bmatrix} 0.5 & 0 \end{bmatrix}$$

the following results have been obtained.

$$F = 0.827$$
$$G = \begin{bmatrix} 18.036 & -17.7 \end{bmatrix}$$
$$H = -31.22$$

The observer error e(k) evolves in time as

$$e(k) = (0.827)^k \, e(o)$$



<u>EXAMPLE 3.</u> Consider the following stochastic system:

$$x(k+1)=\begin{bmatrix} 0.5 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0.5 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 0 & 0 \\ 0 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(k)$$

$$+ \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix} w(k)$$

$$E(x(o))=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathrm{cov}(x(o)) = \mathrm{diag}(10,20,30,40,50,60)$$

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

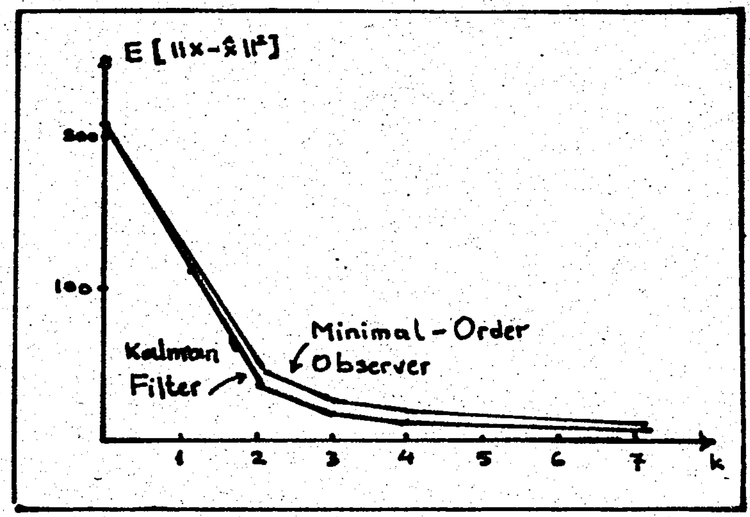
<u>Case 1</u>. All the measurements are noise corrupted.

$$y(k+1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} x(k+1) + v(k+1)$$

and

$$R = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

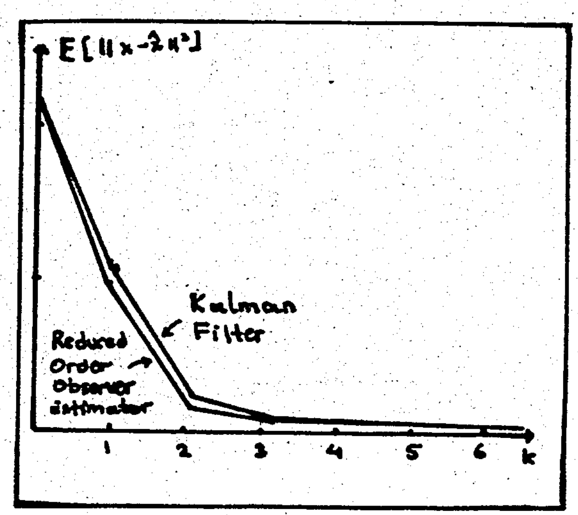
The following graph shows the results obtained from the Kalman filter algorithm and Minimal Order-Observer.



Case 2. Some measurements are noise free.

$$y(k+1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} x(k+1) + \begin{bmatrix} 0 \\ \hline v_2(k+1) \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



APPENDIX B

SUBROUTINES

```
      MAIN/
      MAIN/
02/12-17:22:34-(4,)
      DIMENSION A(10,10),B(10,10),C(10,10),EXPX(10,10),COVX(10,1
      DIMENSION ARB(10,10),JC(10),V(10)
      READ(5,5) ST
      READ(5,1) M,N,NR,M1,,P,NSTEP
      IF ( M .LT. N ) GO TO 20
      WRITE(6,21)
      GO TO 95
   20 IF ( M1 .LE. M ) GO TO 22
      WRITE(6,23)
      GO TO 95
   22 READ(5,2) ((A(I,J),J=1,N),I=1,N)
      READ(5,2) ((B(I,J),J=1,NR),I=1,N)
      IF (ST.EQ.5.) GO TO 3
      READ(5,2) (( C(I,J),J=1,N),I=1,M)
      IF ( ST .NE. 2. ) GO TO 7
      DO 8 I=1,N
      DO 8 J=1,N
    8 ARB(I,J)=A(I,J)
      V(1)=1.
      CALL GJR ( ARB,10,10,N,N,$100,JC,V )
    7 CALL MTAMDE(A,10,10,,S,,N,N,,GEN,)
      CALL MTAMDE(B,10,10,,S,,N,NR,,GEN,)
      CALL MTAMDE(C,10,10,,S,,M,N,,GEN,)
      CALL MTMPRT(A,,(8E15.8),,0,,ORIGINAL STATE TRANS. MATRIX,A
      CALL MTMPRT(B,,(8E15.8),,0,,ORIGINAL INPUT MATRIX, B.,)
      CALL MTMPRT(C,,(8E15.8),,0,,ORIGINAL OUTPUT MATRIX,C.,)
      IF (ST.EQ.3.) GO TO 6
      IF (ST.EQ. 4.) GO TO 6
      CALL TRNSFO(A,B,C,D,EXPX,COVX,M1,M,NR,NR,N,ST)
      IF (ST.EQ.2.) GO TO 95
      IF (ST.EQ.1.) GO TO 4
    3 CALL DLTIS ( A,B,M,N,NR )
      GO TO 95
    4 CALL CLTIS ( A,B,M,N,NR )
      GO TO 95
    6 CALL STOKAS ( A,B,C,M1,M,NR,NR,N,NSTEP,ST )
      GO TO 95
  100 WRITE(6,10)
    1 FORMAT ( )
    2 FORMAT ( )
    5 FORMAT ( )
   10 FORMAT(5X,,A IS SINGULAR - DISCRETE TIME MINIMAL ORDER OBS
     1CANNOT BE REALIZED.,)
   21 FORMAT(10X,,M EXCEEDS ( OR EQUAL ) N,)
   23 FORMAT(10X,, M1 EXCEEDS M,)
   95 STOP
      END
```

```
      SUBROUTINE CLTIS ( A,B,M,N,NR )
      DIMENSION Q(10,10),ZM(10,10),A11(10,10),A12(10,10),A12T(10,
      DIMENSION A21(10,10),A22(10,10),P(10,10),PINV(10,10),ZL(10,
      DIMENSION A(10,10),AR(10,10),EVL(10,10),B(10,10),B1(10,10)
      DIMENSION  B2(10,10),D1(10,10),D2(10,10),F(10,10),G(10,10)
      DIMENSION  H(10,10),PN(10,10),VAL(20,20),ASQ(10,10),VEC(20,
      DIMENSION ARR(100),ARR1(20,20)
      WRITE(6,100)
      NMM=N-M
      NM2=2*NMM                                                  2+
      CALL MTAMDE ( P,10,1 ,,S,,N-M,N-M,,GEN, )
      CALL MTAMDE(EVL,10,10,,S,,N-M,N-M,,DIA,)
      CALL MTAMDE(VAL,10,10,,S,,NMM,NMM,,DIA,)
      CALL MTAMDE(VEC,10,10,,S,,NMM,NMM,,GEN,)
      CALL MTAMDE(ARR1,20,20,,S,,NMM,NMM,,GEN,)
      CALL MTAMDE(ZL,10,10,,S,,N-M,M,,GEN,)
      CALL MTAMDE(F,10,10,,S,,N-M,N-M,,GEN,)
      CALL MTAMDE (G,10,10,,S,,N-M,M,,GEN,)
      CALL MTAMDE (H,10,10,,S,,N-M,NR,,GEN,)
      CALL MTAMDE(ZM,10,10,,S,,N-M,N-M,,GEN,)
      CALL PRTIT1 (A,M,N,A11,A12,A21,A22)
      CALL PRTIT2 (B,M,N,NR,B1,B2)
      K=1
      DO 700 I=1,NMM
            DO 700 J=1,NMM
                  ARR(K)=A22(I,J)
                  K=K+1
  700             CONTINUE
      K=1
      DO 800 I=1,NMM
            DO 800 J=1,NMM
                  ARR1(I,J)=ARR(K)
                  K=K+1
  800             CONTINUE
      CALL MTMPRT(ARR1,,(8E15.8),,0,,A22...)
      DO 26 I=1,NMM
   26 Q(I,I)=2.
      IF ( NMM .NE. 1 ) GO TO 600
      IF ( ARR1(1,1) ) 601,550,603
  600 CALL MTVCU(ARR1,VEC,VAL,*666)
      CALL MTMPRT ( VAL,,(8E15.8),,0,,CHECK...)
      IF ( VAL(1) ) 27,550,28
   27 DO 29 I=3,NM2,2
            IF ( VAL(I) .GE. 0. ) GO TO 550
   29 CONTINUE
  601 WRITE(6,101)
      DO 30 I=1,NMM
            DO 30 J=1,NMM
   30             F(I,J)=A22(I,J)
      DO 31 I=1,NMM
            DO 31 J=1,M
   31             G(I,J)=A21(I,J)
      DO 32 I=1,NMM
            DO 32 J=1,NR
   32             H(I,J)=B2(I,J)
      CALL MTMPRT(F,,(8E15.8),,0,,STATE TRANS. MATRIX ,F...)
      CALL MTMPRT(G,,(8E15.8),,0,,INPUT MATRIX ,G...)
      CALL MTMPRT(H,,(8E15.8),,0,,INPUT MATRIX ,H...)
```

```
0ō          RETURN
06       28 DO 33  I=3,NM2,2
06                IF ( VAL(I)  .LE. 0. ) GO TO 550
06       33       CONTINUE
06      603 WRITE(6,102)
06          CALL MXTRN(A12,A12T,M,N-M,10,10)
06          ALFA=0.1
06          DO 35  I=1,7
06          CALL MXSCA(A12T,NMM,M,10,ALFA)
06          CALL CARP1(A12T,A12,N-M,M,N-M,AR)
06          CALL MXSUB(AR,Q,ZM,NMM,NMM,10)
06          CALL MTVLM(ZM,EVL)
08          DO 900 II=1,NMM
09                IF ( EVL(II) ) 901,901,900
09      900       CONTINUE
09          GO TO 552
09      901 ALFA=10.
06       35 CONTINUE
06          WRITE(6,103)
06          RETURN
06      552 DO 34  I=1,NMM
06                DO 34 J=1,NMM
06       34             ASQ(I,J)=-A22(I,J)
06          IF ( NMM  .EQ. 1 ) GO TO 604
06          CALL LIAPUN (ASQ,ZM,NMM,R,RINV )
06          GO TO 208
07      604 R(1,1)=-ZM(1,1)/(2.*ASQ(1,1))
06          RINV(1,1)=1./R(1,1)
06      208 CALL CARP1 (RINV,A12T,N-M,N-M,M,ZL)
06          CALL MXSCA(ZL,N-M,M,10,0.5)
06          CALL MTMPRT(R,,(8E15.8),,0,,SOLUTION OF MRE R ..,)
06          CALL MTMPRT(ZL,,(8E15.8),,0,, L (N-M)*M ..,)
06          CALL OBSERV(ZL,A11,A12,A21,A22,B1,B2,D1,D2,M,NP,NP,N,ST,F,G,
06          RETURN
06      550 WRITE(6,104)
06          RETURN
06      666 WRITE(6,105)
06      100 FORMAT(5X,,MINIMAL ORDER OBSERVER FOR CONTINOUS TIME SYSTEMS
06      101 FORMAT(20X,,A22 IS NEG. DEF.-NO NEED FOR LIAP. EQ.,)
06      102 FORMAT(20X,,A22 IS POS. DEF.-LIAP. EQ. YIELDS,)
10      103 FORMAT(20X,,THE MATRIX K=DIAG(ALFA) CANNOT BE FOUND- A12 HAS
10         1AST ONE ZERO ROW(COLUMN),)
06      104 FORMAT(20X,,A22 IS INDEF. OR SEMIDEF.-SEE SEC. 2.2.2.,)
06      105 FORMAT(20X,,EIGENVALUE CANNOT BE CALCULATED-SEE MANUAL BUMP1
06          RETURN
06          END

6,,,3.
```

```
      SUBROUTINE DLTIS ( A,B,M,N,NR )
      DIMENSION ZL(10,10),A(10,10),A11(10,10),A12(10,10),A21(10,1
      DIMENSION A22(10,10),A11T(10,10),A11TA(10,10),A12T(10,10)
      DIMENSION A21TA(10,10),Y(10,10),YT(10,10),X(10,10),XT(10,10
      DIMENSION P(10,10),EVL(10,10),JC(10),V(10),ARB(10,10)
      DIMENSION ARB1(10,10),ARB2(10,10),ARB3(10,10)
      DIMENSION B(10,10),B1(10,10),B2(10,10),D1(10,10),D2(10,10)
      DIMENSION F(10,10),G(10,10),H(10,10),PN(10,10)
      WRITE(6,100)
      CALL MTAMDF (P,10,10,,S,,N-M,N-M,,GEN,)
      CALL MTAMDF ( ARB3,10,10,,S,,N-M,N-M,,GEN, )
      CALL MTAMDF(EVL,10,10,,S,,N-M,N-M,,DIA,)
      CALL MTAMDF(ZL,10,10,,S,,N-M,M,,GEN,)
      CALL MTAMDF(F,10,10,,S,,N-M,N-M,,GEN,)
      CALL MTAMDF(G,10,10,,S,,N-M,M,,GEN,)
      CALL MTAMDF(H,10,10,,S,,N-M,NP,,GEN,)
      CALL PRTIT1 (A,M,N,A11,A12,A21,A22)
      NMM=N-M
      READ(5,2) ((ZL(I,J),J=1,M),I=1,NMM)
    2 FORMAT ( )
      CALL MXTRN (A11,A11T,M,M,10,10)
      CALL MXTRN (A12,A12T,M,NMM,10,10)
      CALL CARP1 (A11,A11T,M,M,M,A11TA)
      CALL CARP1 (A21,A11T,NMM,M,M,A21TA)
      DO 520 K=1,20
      CALL CARP1 (ZL,A11,NMM,M,M,X)
      CALL MXSUB (A21,X,X,NMM,M,10)
      CALL MXTRN (X,XT,NMM,M,10,10)
      CALL CARP1 (Y,XT,NMM,M,NMM,ARB)
      CALL CARP1 (ZL,A12,NMM,M,NMM,Y)
      CALL MXSUB (A22,Y,Y,NMM,NMM,10)
      CALL MXTRN (Y,YT,NMM,NMM,10,10)
      CALL CARP1 (Y,YT,NMM,NMM,NMM,ARB1)
      CALL MXADD (ARB,ARB1,ARB,NMM,NMM,10)
      DO 500 I=1,NMM
  500 R(I,I)=1.
      DO 510 I=1,10
      CALL MXADD (R,ARB,R,NMM,NMM,10)
      CALL CARP1 (Y,ARB,NMM,NMM,NMM,ARB1)
      CALL CARP1 (ARB1,YT,NMM,NMM,NMM,ARB)
  510 CONTINUE
      CALL MTMPRT (R,,(8F15.8),,0,,SOLUTION OF RMRE R ...)
      CALL MTVLM (P,EVL)
      CALL MTMPRT(EVL,,(8E15.8),,0,,EIGENVALUES OF P..,)
      CALL CARP1 (P,A12T,NMM,NMM,M,ARB)
      CALL CARP1(A12,ARB,M,NMM,M,ARB1)
      CALL MXADD (A11TA,ARB1,ARB1,M,M,10)
      V(1)=1.
      CALL GJR (ARB1,10,10,M,M,$551,JC,V)
      CALL CARP1 (A22,ARB,NMM,NMM,M,ARB2)
      CALL MXADD (A21TA,ARB2,ARB2,NMM,M,10)
      CALL CARP1 (ARB2,ARB1,NMM,NMM,M,ZL)
      IF (K .EQ. 1) GO TO 530
      CALL MXSUB (ARB3,R,ARB3,NMM,NMM,10)
      CALL MTVLM (ARB3,EVL)
      IF ( EVL(1) .LE. 1.E-4 ) GO TO 550
  530 DO 540 I=1,NMM
      DO 540 J=1,NMM
      ARB3(I,J)=R(I,J)
  540 R(I,J)=0.
  520 CONTINUE
  550 CALL MTMPRT(ZL,,(8F15.8),,0,, L (N-M)*M ..,)
      CALL PRTIT2 ( B,M,N,NR,B1,B2 )
      CALL OBSERV(ZL,A11,A12,A21,A22,B1,B2,D1,D2,M,NR,NP,N,ST,F,G
      GO TO 553
  551 WRITE(6,552)
  552 FORMAT(,INVERSE DOES NOT EXIST,)
  100 FORMAT(5X,,MINIMAL ORDER OBSERVER FOR DISCRETE TIME SYSTEMS
  553 RETURN
      END
```

```
      SUBROUTINE STOKAS ( A,B,C,M1,M,NP,NR,N,NSTEP,ST )
      DIMENSION  A(10,10),B(10,10),C(10,10),D(10,10),EXPX(10),PI(1
      DIMENSION  COVX(10,10),COVW(10,10),COVV2(10,10),F(10,10),V(1
      DIMENSION  CVEROR(10,10),PN(10,10),C22(10,10),C21(10,10),JC(
      DIMENSION B1(10,10),B2(10,10),D1(10,10),D2(10,10),A11(10,10)
      DIMENSION D1T(10,10),ARB(10,10),LAMDA1(10,10),APR1(10,10)
      DIMENSION P(10,10),FNEW(10,10),GNEW(10,10),ZV(10,10)
      DIMENSION  APB2(10,10),LAMDA2(10,10),ZL(10,10),FT(10,10),H(1
      DIMENSION  PNT(10,10),GAMMA(10,10),C22T(10,10),PSI(10,10)
      DIMENSION A12(10,10),A21(10,10),A22(10,10),A12T(10,10),G(10,
      DIMENSION OMEG(10,10),OMEG11(10,10),OMEG21(10,10),SIGMA(10,1
      DIMENSION T(10,10)
      IF ( ST .EQ. 4. ) GO TO 125
      MM1 = M-M1
      NM1 = N-M1
      READ(5,2)   ((D(I,J),J=1,NP),I=1,N)
      READ(5,2)  (EXPX(I),I=1,N)
      READ(5,2)  ((COVX(I,J),J=1,N),I=1,N)
      READ(5,2)   ((COVW(I,J),J=1,NP),I=1,NP)
      IF ( M1 .EQ. M ) GO TO 1
      READ(5,2)  ((COVV2(I,J),J=1,MM1),I=1,MM1)
      CALL MTAMDF(COVV2,10,10,,S,,MM1,MM1,,GEN,)
      CALL MTAMDF(P,10,10,,S,,MM1,MM1,,GEN,)
    1 CALL MTAMDF(CVEROR,10,10,,S,,NM1,NM1,,GEN,)
      IF ( M1 .EQ. 0 ) GO TO 136
       CALL MTAMDF(ZL,10,10,,S,,NM1,M1,,GEN,)
       CALL MTAMDF(G,10,10,,S,,NM1,M1,,GEN,)
  136  CALL MTAMDF(F,10,10,,S,,NM1,NM1,,GEN,)
       CALL MTAMDF(H,10,10,,S,,NM1,NR,,GEN,)
      CALL MTAMDF(D,10,10,,S,,N,NP,,GEN,)
      CALL MTAMDF(EXPX,10,1,,S,,N,1,,GEN,)
      CALL MTAMDF(COVX,10,10,,S,,N,N,,GEN,)
      CALL MTAMDF(COVW,10,10,,S,,NP,NP,,GEN,)
      IF ( M1 .EQ. 0  .OR. M1 .EQ. M ) GO TO 211
       CALL MTAMDF(ZV,10,10,,S,,MM1,M1,,GEN,)
  211 CALL MTMPRT(D,,(8E15.8),,0,,ORIGINAL DISTURBANCE MATRIX, D..
      CALL MTMPRT(EXPX,,(8E15.8),,0,,ORIGINAL E(X(0)),,,)
      CALL MTMPRT(COVX,,(8E15.8),,0,,ORIGINAL COVAR. MATRIX OF X(
      CALL MTMPRT(COVW,,(8E15.8),,0,,DISTURBANCE COVARIANCE MATRI
      IF ( M1 .EQ. M ) GO TO 300
      CALL MTMPRT(COVV2,,(8E15.8),,0,,MEASUREMENT NOISE COV. MATRI
  300 IF ( M1 .NE. 0 ) GO TO 17
      WRITE(6,14)
      WRITE(6,15)
      WRITE(6,100)
      GO TO 18
   17 IF ( M1 .EQ. M ) GO TO 21
      WRITE(6,19)
      WRITE(6,20)
      GO TO 18
   21 WRITE(6,22)
      WRITE(6,23)
   18 DO 32 I=1,NM1
   32 PI(I,I)=1.
      IF ( M1 .NE. 0 ) GO TO 3
      DO 4 I=1,N
           DO 4 J=1,N
                CVEROR(I,J)=COVX(I,J)
```

```
4             F(I,J)=A(I,J)
   DO 5 I=1,N
        DO 5 J=1,NP
5             PN(I,J)=D(-,J)
   DO 6 I=1,M
        DO 6 J=1,N
6             C22(I,J)=C(I,J)
   DO 2? I=1,N
        DO 28 J=1,NR
28            H(I,J)=B(I,J)
   GO TO 7
 3 CALL TRNSFO(A,B,C,D,EXPX,COVY,M1,M,NR,NP,N,ST)
   DO 30 I=1,NM1
   DO 30 J=1,NM1
30 CVEROR(I,J)=COVX(I+M1,J+M1)
   IF ( M1 .EQ. M ) GO TO 8
   DO 9 I=1,MM1
        DO 9 J=1,M1
9             C21(I,J)=C(I+M1,J)
   DO 10 I=1,MM1
        DO 10 J=1,NM1
10            C22(I,J)=C(I+M1,J+M1)
 8 CALL PRTIT1 ( A,M1,N,A11,A12,A21,A22 )
   CALL PRTIT2 ( B,M1,N,NR,B1,B2 )
   CALL PRTIT2 ( D,M1,N,NP,D1,D2 )
   CALL MXTRN ( A12,A12T,M1,NM1,10,10 )
   CALL MXTRN ( D1,D1T,M1,NP,10,10 )
 7 DO 11 K=1,NSTEP
   WRITE(6,101) K
   IF ( M1 .EQ. 0 ) GO TO 12
   DO 1000 I=1,N
          DO 1000 J=1,
                    COVX(I,J)=0
1000 CALL CARP1 ( CVEROR,A12T,NM1,NM1,M1,ARB )
     CALL CARP1 ( A12,ARB,M1,NM1,M1,LAMDA1 )
     CALL CARP1 ( COVW,D1T,NP,NP,M1,ARB1 )
     CALL CARP1 ( D1,ARB1,M1,NP,M1,ARB2 )
     CALL MXADD ( LAMDA1,ARB2,LAMDA1,M1,M1,10 )
   V(1)=1.
   CALL GJR ( LAMDA1,10,10,M1,M1,$545,JC,V )
     CALL CARP1 ( A22,ARB,NM1,NM1,M1,LAMDA2 )
     CALL CARP1 ( D2,ARB1,NM1,NP,M1,ARB2 )
     CALL MXADD ( LAMDA2,ARB2,LAMDA2,NM1,M1,10 )
   CALL CARP1 ( LAMDA2,LAMDA1,NM1,M1,M1,ZL )
      CALL OBSERV ( ZL,A11,A12,A21,A22,B1,B2,D1,D2,M1,NR,NP,N,ST
  1H,PN )
12    CALL MXTRN ( F,FT,NM1,NM1,10,10 )
      CALL MXTRN ( PN,PNT,NM1,NP,10,10 )
    CALL CARP1 ( F,CVEROR,NM1,NM1,NM1,ARB )
    CALL CARP1 ( ARB,FT,NM1,NM1,NM1,GAMMA )
    CALL CARP1 ( PN,COVW,NM1,NP,NP,ARB )
    CALL CARP1 ( ARB,PNT,NM1,NP,NM1,ARB1 )
    CALL MXADD ( GAMMA,ARB1,GAMMA,NM1,NM1,10 )
   IF ( M1 .EQ. M ) GO TO 13
      CALL MXTRN ( C22,C22T,MM1,NM1,10,10 )
      CALL CARP1 ( GAMMA,C22T,NM1,NM1,MM1,ARB )
      CALL CARP1 ( C22,ARB,MM1,NM1,MM1,PSI )
      CALL MXADD ( PSI,COVV2,PSI,MM1,MM1,10 )
        V(1)=1.
        CALL GJR ( PSI,10,10,MM1,MM1,$544,JC,V )
      CALL CARP1 ( ARB,PSI,NM1,MM1,MM1,P )
```

```
      CALL CARP1 ( P,C22,NM1,MM1,NM1,ARR )
      CALL MXSUB ( PI,ARR,ARR,NM1,NM1,10 )
      CALL CARP1 ( ARR,GAMMA,NM1,NM1,NM1,CVEROR )
      IF ( M1 .NE. 0 ) GO TO 16
        CALL MTMPRT ( P,,(8E15.8),,0,,KALMAN GAIN MATRIX ..., )
        CALL MTMPRT ( CVEROR,,(8E15.8),,0,,ERROR COVARIANCE MATRIX
      GO TO 11
   16 IF ( M1 .EQ. M ) GO TO 13
      CALL CARP1 ( C22,ZL,NM1,NM1,M1,ZV )
      CALL MXADD ( ZV,C21,ZV,NM1,M1,10 )
      DO 1001 I=1,NM1
              DO 1001 J=1,NM1
                  COVX(I+M1,J+M1)=CVEROR(I,J)
 1001 CALL TRNSFO(COVX,B,C,D,EXPX,ARR,M1,M,NR,NR,N,6.)
        CALL MTMPRT(ZL,,(8E15.8),,0,,L(K)...,)
        CALL MTMPRT ( P,,(8E15.8),,0,, GAIN MATRIX ...,)
      CALL MTMPRT(COVX,,(8E15.8),,0,,ERROR COVARIANCE MATRIX...,)
        CALL MTMPRT(F,,(8E15.8),,0,,F(K)...,)
        CALL MTMPRT(G,,(8E15.8),,0,,G(K)...,)
        CALL MTMPRT(H,,(8E15.8),,0,,H(K)...,)
        CALL MTMPRT ( ZV,,(8E15.8),,0,,ZV ...,)
      GO TO 11
   13   DO 24 I=1,NM1
              DO 24 J=1,NM1
                  CVEROR(I,J)=GAMMA(I,J)
   24             COVX(I+M1,J+M1)=GAMMA(I,J)
      CALL TRNSFO(COVX,B,C,D,EXPX,ARR,M1,M,NR,NR,N,6.)
        CALL MTMPRT(ZL,,(8E15.8),,0,,L(K)...,)
      CALL MTMPRT(COVX,,(8E15.8),,0,,ERROR COVARIANCE MATRIX...,)
        CALL MTMPRT(F,,(8E15.8),,0,,F(K)...,)
        CALL MTMPRT ( H,,(8E15.8),,0,, H..., )
        CALL MTMPRT ( G,,(8E15.8),,0,, G..., )
   11 CONTINUE
      RETURN
  544 WRITE(6,25)
      RETURN
  545 WRITE(6,26)
      RETURN
  125 NMM=N-M
      MP1=M+1
      READ(5,2) ((D(I,J),J=1,NR),I=1,N)
      READ(5,2) (EXPX(I),I=1,N)
      READ(5,2) ((COVX(I,J),J=1,N),I=1,N)
      READ(5,2) ((COVW(I,J),J=1,NR),I=1,NR)
      READ(5,2) ((COVV2(I,J),J=1,M),I=1,M)
       CALL MTAMDE(ZL,10,10,,S,,NMM,M,,GEN,)
       CALL MTAMDE(F,10,10,,S,,NMM,NMM,,GEN,)
       CALL MTAMDE(G,10,10,,S,,NMM,M,,GEN,)
       CALL MTAMDE(H,10,10,,S,,NMM,NR,,GEN,)
       CALL MTAMDE(CVEROR,10,10,,S,,N,N,,GEN,)
      CALL MTAMDE(D,10,10,,S,,N,NR,,GEN,)
      CALL MTAMDE(EXPX,10,1,,S,,N,1,,GEN,)
      CALL MTAMDE(COVX,10,10,,S,,N,N,,GEN,)
      CALL MTAMDE(COVW,10,10,,S,,NR,NR,,GEN,)
      CALL MTAMDE(COVV2,10,10,,S,,M,M,,GEN,)
      CALL MTMPRT(D,,(8E15.8),,0,,ORIGINAL DISTURBANCE MATRIX,D...,)
      CALL MTMPRT(EXPX,,(8E15.8),,0,,ORIGINAL E(X(0))...,)
      CALL MTMPRT(COVX,,(8E15.8),,0,,ORIGINAL COVAR. MATRIX OF X(0)
      CALL MTMPRT(COVW,,(8E15.8),,0,,DISTURBANCE COVAR. MATRIX...,)
      CALL MTMPRT(COVV2,,(8E15.8),,0,,MEASUREMENT NOISE COV. MATRIX
      WRITE(6,102)
```

```
      WRITE(6,133)
       CALL TRNSFO(A,B,C,D,EXPY,COVX,M,M,NR,NP,N,ST)
      DO 126 I=1,NMM
            T(I,I+M)=1.
  126       P(I+M,I)=1.
      DO 127 I=1,M
  127       ZV(I,I)=1.
      CALL CARP1 ( A,P,N,N,NMM,FNEW )
      CALL MXTRN ( FNEW,FT,N,NMM,10,10 )
       CALL MXTRN  ( A,ARB1,N,N,10,10 )
       CALL CARP1  ( A,COVX,N,N,N,ARB )
       CALL CARP1  ( ARB,ARB1,N,N,N,OMEG )
       CALL MXTRN  ( D,D1T,N,NP,10,10 )
       CALL CARP1  ( D,COVW,N,NP,NP,ARB )
       CALL CARP1  ( ARB,D1T,N,NP,N,D )
       CALL MXADD  ( OMEG,D,OMEG,N,N,10 )
      DO 129 K=1,NSTEP
        WRITE(6,101) K
         CALL PRTITI ( OMEG,M,N,OMEG11,ARB,OMEG21,ARB1 )
         CALL MXADD ( OMEG,1,COVV2,OMEG11,M,M,10 )
         V(1)=1.
         CALL GJR ( OMEG11,10,10,M,M,$546,JC,V )
         CALL CARP1 ( OMEG21,OMEG,1,NMM,M,M,ZL )
       CALL MTMPRT(ZL,,(8E15.8),,0,,OBSERVER GAIN MATRIX, L(K)..,)
        DO 130 I=1,NMM
              DO 130 J=1,M
  130               T(I,J)=-ZL(I,J)
        IF ( K .EQ. 1 ) GO TO 140
         CALL CARP1 ( T,FNEW,NMM,N,NMM,F )
         CALL CARP1 ( T,B,NMM,N,NR,H )
         CALL CARP1 ( T,GNEW,NMM,N,M,G )
       CALL MTMPRT(F,,(8E15.8),,0,,OBSERVER SYSTEM MATRIX, F(K-1).
       CALL MTMPRT(H,,(8E15.8),,0,,OBSERVER INPUT MATRIX, H(K-1).,
       CALL MTMPRT(G,,(8E15.8),,0,,COEFF. MATRIX OF Y(K-1),G(K-1).
  140       DO 131 I=1,NMM
              DO 131 J=1,M
  131               ZV(I+M,J)=ZL(I,J)
          CALL MXTRN  ( T,PNT,NMM,N,10,10 )
          CALL CARP1  ( T,OMEG,NMM,N,N,ARB )
          CALL CARP1  ( ARB,PNT,NMM,N,NMM,SIGMA )
          CALL CARP1  ( ZL,COVV2,NMM,M,M,ARB )
          CALL MXTRN  ( ZL,ARB2,NMM,M,10,10 )
          CALL CARP1  ( ARB,ARB2,NMM,M,NMM,ARB1 )
          CALL MXADD  ( SIGMA,ARB1,ARB1,NMM,NMM,10 )
          CALL MXTRN  ( ARB,ARB2,NMM,M,10,10 )
          DO 132 I=1,M
              DO 132 J=1,M
  132               CVEROR(I,J)=COVV2(I,J)
          DO 133 I=1,NMM
              DO 133 J=1,M
  133               CVEROR(I+M,J)=ARB(I,J)
          DO 134 I=1,M
              DO 134 J=1,NMM
  134               CVEROR(I,J+M)=ARB2(I,J)
          DO 135 I=1,NMM
              DO 135 J=1,NMM
  135               CVEROR(I+M,J+M)=ARB1(I,J)
      CALL TRNSFO(CVEROR,B,C,D,EXPY,COVX,M,M,NR,NP,N,6.)
        CALL MTMPRT(CVEROR,,(8E15.8),,0,,ERROR COVARIANCE MATRIX,
         CALL CARP1 ( FNEW,SIGMA,N,NMM,NMM,ARB )
         CALL CARP1 ( ARB,FT,N,NMM,N,OMEG )
```

```
      CALL CARP1 ( A,Z,,N,N,M,GNEW )
      CALL MXTRN ( GNEW,ARB,N,M,10,10 )
      CALL CARP1 ( GNEW,COV2,N,M,M,ARB1 )
      CALL CARP1 ( ARB1,ARB,N,M,N,ARB2 )
      CALL MXADD ( OMEG,ARB2,OMEG,N,N,10 )
      CALL MXADD ( OMEG,D,OMEG,N,N,10 )
  129 CONTINUE
      RETURN
  546 WRITE(6,27)
    2 FORMAT ( )
   14 FORMAT(1H1,20X,'M1=0 , ALL THE MEASUREMENTS NOISE CORRUPTED
     1AN FILTER IS USED',/ )
   15 FORMAT(2X,'XES(K+1/K+1)=(I-P(K+1)*C)*A*XES(K/K) + I-P(K+1)*C
     1K) + P(K+1)*Y(K+1))',//)
   19 FORMAT(1H1,20X,'M1 < M,SOME MEASUREMENTS NOISE CORRUPTED - R
     1 ORDER OBSERVER-ESTIMATOR OF ORDER N-M1 IS USED',//)
   20 FORMAT(2X,'ZES(K+1/K+1)=(I-P(K+1)*C22)*F(K)*ZES(K/K) + (I-P(
     122)*H(K)*U(K) + (I-P(K+1)*C22)*G(K)*Q1(K) + P(K+1)*(Y2(K+1)-
     2)*Q1(K+1))',//)
   22 FORMAT(1H1,20X,'M1=M , MEASUREMENTS ARE NOISE FREE - MINIMAL
     1 OBSERVER-ESTIMATOR OF ORDER N-M IS USED',//)
   23 FORMAT(2X,'ZES(K+1/K+1)=F(K)*ZES(K/K) + G(K)*Q1(K) + H(K)*U(
   25 FORMAT(5X,'Q2 CANNOT BE ESTIMATED - PSI IS SINGULAR ')
   26 FORMAT(5X,'Q2 CANNOT BE ESTIMATED - LAMDA1 IS SINGULAR ')
   27 FORMAT(5X,'THE GAIN MATRIX L CANNOT BE CALCULATED ')
  100 FORMAT(25X,'K=0')
  101 FORMAT(25X,'K=',I2)
  102 FORMAT(5X,'MINIMAL ORDER OBSERVER OF ORDER N-M IS USED')
  103 FORMAT(5X,'Z(K+1)=F(K)*Z(K)+G(K)*Y(K)+H(K)*U(K)')
      RETURN
      END
```

```
      SUBROUTINE OBSERV ( ZL,A11,A12,A21,A22,B1,B2,D1,D2,M,NR,NP,N
     1G,H,PN )
      DIMENSION ZL(10,10),A11(10,10),A12(10,10),A21(10,10),A22(10,
      DIMENSION B1(10,10),B2(10,10),D1(10,10),D2(10,10),F(10,10)
      DIMENSION H(10,10),AB(10,10),G(10,10),PN(10,10)
      CALL CARP1(ZL,A12,N-M,M,N-M,F)
      CALL MXSUB(A22,F,F,N-M,N-M,10)
      CALL CARP1 (F,ZL,N-M,N-M,M,G)
      CALL MXADD (G,A21,G,N-M,M,10)
      CALL CARP1 (ZL,A11,N-M,M,M,AB)
      CALL MXSUB (G,AB,G,N-M,M,10)
      CALL CARP1 (ZL,B1,N-M,M,NR,H)
      CALL MXSUB (B2,H,H,N-M,NR,10)
      CALL CARP1 ( ZL,D1,N-M,M,NP,PN )
      CALL MXSUB ( D2,PN,PN,N-M,NP,10 )
      IF ( ST .EQ. 3. ) GO TO 2
      CALL MTMPRT(F,'(8E15.8)',0,'F (N-M)*(N-M),OBSERVER SYSTEM MT
      CALL MTMPRT (G,'(8E15.8)',0,'OBSERVER OUTPUT MATRIX..')
      CALL MTMPRT (H,'(8E15.8)',0,'OBSERVER INPUT MATRIX..')
    2 RETURN
      END
```

```
      SUBROUTINE TRUSEQ(A,B,C,D,EXPX,COVX,M,MAUX,NR,NP,N,ST)
      DIMENSION A(10,10),B(10,10),C(10,10),PK(10,10,10),PM(10,10)
      DIMENSION PM1(10,10),AA(10,10),D(10,10),EXPX(10),COVX(10,10)
      DIMENSION PM1T(10,10)
      INTEGER QX(10)
      IF ( ST .EQ. 6. ) GO TO 1000
      CALL MTAMDE(PM,10,10,,S,,N,N,,GEN,)
      CALL MTAMDE(PM1,10,10,,S,,N,N,,GEN,)
      MP1=M+1
      DO 1 I=1,N
    1 QX(I)=I
      DO 3 II=1,MP1
      DO 3 I=1,N
    3 PK(I,I,II)=1.
      DO 14 I=1,M
      DO 14 J=1,N
   14 PM1(I,J)=C(I,J)
      DO 15 I=MP1,N
   15 PM1(I,I)=1.
      DO 4 I=1,M
      L=I
      IF (C(L,L).NE.0.) GO TO 2
   11 LP1=L+1
      DO 5 K=LP1,N
    5 IF (C(L,K).NE.0.) GO TO 55
      GO TO 555
   55 CHPKI=PK(I,L,I)
      PK(I,L,I)=PK(I,K,I)
      PK(I,K,I)=CHPKI
      IF (I.EQ.1) GO TO 6
      CHPK1=PK(1,L,1)
      PK(1,L,1)=PK(1,K,1)
      PK(1,K,1)=CHPK1
    6 DO 7 II=1,N
      IF ( II .GT. MAUX ) GO TO 18
      CHC=C(II,L)
      C(II,L)=C(II,K)
      C(II,K)=CHC
      CHPM1=PM1(II,L)
      PM1(II,L)=PM1(II,K)
      PM1(II,K)=CHPM1
   18 IF ( ST .LT. 3. ) GO TO 12
      CHCOVX=COVX(II,L)
      COVX(II,L)=COVX(II,K)
      COVX(II,K)=CHCOVX
   12 CHA=A(II,L)
      A(II,L)=A(II,K)
    7 A(II,K)=CHA
      DO 13 JJ=1,N
      IF ( ST .LT. 3. ) GO TO 19
      CHCOVX=COVX(L,JJ)
      COVX(L,JJ)=COVX(K,JJ)
      COVX(K,JJ)=CHCOVX
   19 CHA=A(L,JJ)
      A(L,JJ)=A(K,JJ)
   13 A(K,JJ)=CHA
      DO 16 JJ=1,NR
      CHB=B(L,JJ)
```

```
      B(L,JJ)=B(K,JJ)
   16 B(K,JJ)=CHB
      CHO=QX(L)
      QX(L)=QX(K)
      QX(K)=CHQ
      IF ( ST .LT. 3. ) GO TO  4
      CHEXPX=EXPX(L)
      EXPX(L)=EXPX(K)
      EXPX(K)=CHEXPX
      DO 20  JJ=1,NP
      CHD=D(L,JJ)
      D(L,JJ)=D(K,JJ)
   20 D(K,JJ)=CHD
      GO TO 4
    8 DO 10 J=1,N
      IF (J.EQ.I) GO TO 10
      PK(I,J,I)=-C(I,J)/C( ,I)
   10 CONTINUE
      KK=I
      CALL CARP(C,PK,M,N,KK,C)
      IF (I.EQ.M) GO TO 4
      L=L+1
      IF (C(L,L).EQ.0.) GO TO 11
    4 CONTINUE
      DO 200 I=1,N
      DO 200 J=1,N
  200 PM(I,J)=PK(I,J,1)
      DO 202 I=1,M
  202 PK(I,I,M+1)=1./C(I,I)
      DO 201 KK=2,MP1
      CALL CARP(PM,PK,N,N,KK,PM)
  201 CONTINUE
      WRITE(6,95)
      DO 99 I=1,N
      WRITE(6,98) I,QX(I)
   99 CONTINUE
      WRITE(6,96)
      CALL MTMPRT(A,,(8E15.8),,0,,A.,,)
      CALL MTMPRT(B,,(8E15.8),,0,,B.,,)
      IF ( ST .LT. 3. ) GO TO 208
      CALL MTMPRT(D,,(8E15.8),,0,,D.,,)
      CALL MTMPRT(EXPX,,(8E15.8),,0,,E(X(0)).,,)
      CALL MTMPRT(COVX,,(8E15.8),,0,,COV(X(0)).,,)
  208 KK=M+1
      CALL CARP(C,PK,M,N,K ,C)
      CALL CARP1(A,PM,N,N,,,AA)
      CALL CARP1(PM1,AA,N,N,N,A)
      CALL CARP1(PM1,B,N,N,NP,AA)
      DO 206 I=1,N
      DO 206 J=1,NP
  206 B(I,J)=AA(I,J)
      CALL MTMPRT(PM,,(8E15.8),,  ,,TRANSFORMATION MATRIX, M.,,)
      CALL MTMPRT(PM,,(8E15.8),,0.,TRANSFORMATION MATRIX INVERSE M-1
      CALL MTMPRT (A,,(8E15.8),,0.,TRANSFORMED SYSTEM MATRIX.,,)
      CALL MTMPRT(B,,(8E15.8),,0.,TRANSFORMED INPUT MATRIX .,,)
      CALL MTMPRT(C,,(8E15.8),,0.,TRANSFORMED OUTPUT MATRIX C*M-1 ..
      IF ( ST .LT. 3. ) GO TO 556
      CALL CARP1(PM1,D,N,N,NP,AA)
      DO 207 I=1,N
      DO 207 J=1,NP
  207 D(I,J)=AA(I,J)
```

```
      CALL MTMPRT(D,,(8E15.8),,0,,TRANSFORMED DISTURBANCE MATRIX..
      WRITE(6,100)
      CALL MTMPRT(COVX,,(8E15.8),,0,,ERROR COVARIANCE MATRIX..,)
      CALL MXTRN ( PM1,PM1T,N,N,10,10 )
      CALL CARP1 ( PM1,COVX,N,N,N,AA )
      CALL CARP1 ( AA,PM1T,N,N,N,COVX )
      RETURN
  555 WRITE (6,97)
      GO TO 556
 1000 CALL MXTRN ( PM,PM1T,N,N,10,10 )
      CALL CARP1 ( PM,A,N,N,N,AA )
      CALL CARP1 ( AA,PM1T,N,N,N,A )
      CALL CARP1 ( AA,PM1T,N,N,N,A )
   95 FORMAT(1H1,20X,,STATE VARIABLES ARE REORIENTED AS FOLLOWS,,/
   96 FORMAT(20X,,REORIENTED SYSTEM DYNAMICS,,///)
   97 FORMAT(,C IS NOT OF FULL RANK,)
   98 FORMAT(25X,,Q(,,I2,,) = X(,,I2,,),)
  100 FORMAT(25X,,K=0,)
  556 RETURN
      END
```

```
      SUBROUTINE LIAPUN(ASQ,QSQ,N,R,Y)
      DIMENSION ASQ(10,10),QSQ(10,10),R(10,10),Y(10,10),SIDEN(10,1
      DIMENSION U(10,10),UT(10,10),VV(10,10),Z(10,10),ZT(10,10),V(
      DIMENSION ARB(10,10),JC(10),ZEN(10,10)
      V(1)=1.
      DO 300 I=1,N
  300 SIDEN(I,I)=0.1
      CALL MXSUB(SIDEN,ASQ,U,N,N,10)
      CALL GJR(U,10,10,N,N,$50,JC,V)
      GO TO 105
  102 CONTINUE
      DO 305 I=1,N
      DO 305 J=1,N
  305 U(I,J)=0.
      DO 304 I=1,N
  304 U(I,I)=1.
  105 CONTINUE
      CALL MXADD(SIDEN,ASQ,VV,N,N,10)
      CALL CARP1(U,VV,N,N,N,ARB)
      DO 302 I=1,N
      DO 302 J=1,N
  302 Z(I,J)=ARB(I,J)
      CALL CARP1(U,QSQ,N,N,N,ARB)
      CALL MXTRN(U,UT,N,N,10,10)
      CALL CARP1(ARB,UT,N,N,N,Y)
      CALL MXSCA(Y,N,N,10,0.2)
      DO 203 KK=1,10
      IF(KK.EQ.1) GO TO 190
      CALL CARP1(Z,Z,N,N,N,ARB)
      DO 182 I=1,N
      DO 182 J=1,N
  182 Z(I,J)=ARB(I,J)
  190 CALL CARP1(Z,Y,N,N,N,ARB)
      CALL MXTRN(Z,ZT,N,N,10,10)
      CALL CARP1(ARB,ZT,N,N,N,ZEN)
      CALL MXADD(Y,ZEN,Y,N,N,10)
  203 CONTINUE
      DO 204 I=1,N
      DO 204 J=1,N
  204 R(I,J)=Y(I,J)
      V(1)=1.
      CALL GJR(Y,10,10,N,N,$50,JC,V)
      GO TO 51
   50 WRITE(6,140)
      GO TO 102
   51 CONTINUE
  140 FORMAT(20X,,INVERSE DOES NOT EXIST ,,//)
      RETURN
      END
```

```
      SUBROUTINE CARP(CC,PP,MM,NN,KK,D)
      DIMENSION CC(10,10),PP(10,10,10),D(10,10)
      DO 100 I=1,MM
             DO 100 J=1,NN
                    SUM=0.
                    DO 102 K=1,NN
102                        SUM=SUM+CC(I,K)*PP(K,J,KK)
                    D(I,J)=SUM
100                 CONTINUE
      RETURN
      END


      SUBROUTINE CARP1 ( CC1,PP1,MM,NN,NN1,D1 )
      DIMENSION CC1(10,10),PP1(10,10),D1(10,10)
      DO 400 I=1,MM
      DO 400 J=1,NN1
      SUM=0.
      DO 402 K=1,NN
402   SUM=SUM+CC1(I,K)*PP1(K,J)
      D1(I,J)=SUM
400   CONTINUE
      RETURN
      END


      SUBROUTINE PRTIT1 (A,M,N,A11,A12,A21,A22)
      DIMENSION A(10,10),A11(10,10),A12(10,10),A21(10,10),A22(10,
      MP1=M+1
      DO 301 I=1,M
      DO 301 J=1,M
301   A11(I,J)=A(I,J)
      DO 21 I=1,M
      DO 21 J=MP1,N
21    A12(I,J-M)=A(I,J)
      DO 22 I=MP1,N
      DO 22 J=1,M
22    A21(I-M,J)=A(I,J)
      DO 23 I=MP1,N
      DO 23 J=MP1,N
23    A22(I-M,J-M)=A(I,J)
      RETURN
      END


      SUBROUTINE PRTIT2 (AA,M,N,NR,B1,B2)
      DIMENSION AA(10,10),B1(10,10),B2(10,10)
      MP1=M+1
      DO 24 I=1,M
      DO 24 J=1,NR
24    B1(I,J)=AA(I,J)
      DO 25 I=MP1,N
      DO 25 J=1,NR
25    B2(I-M,J)=AA(I,J)
      RETURN
      END
```

REFERENCES

1. R.E. KALMAN and R.S. BUCY, "New results in linear filtering and prediction theory", *Trans. ASME, J. Basic Eng.*, Ser. D, pp. 95-108, March 1961.

2. A.E. BRYSON, Jr. and D.E. JOHANSEN, "Linear filtering for time-varying systems using measurements containing colored noise", *IEEE Trans. Automat. Contr.*, Vol. AC-10, pp. 4-10, January 1965.

3. D.G. LUENBERGER, "An introduction to observers", *IEEE Trans. Automat. Contr.*, Vol. AC-16, pp. 596-602, December 1971.

4. D.G. LUENBERGER, "Observing the state of a linear system", *IEEE Trans. Mil. Electron.*, Vol. MIL-8, pp. 74-80, April 1964.

5. B. GOPINATH, "On the control of linear multiple input-output systems", *Bell Syst. Tech. J.*, March 1971.

6. D.G. SCHULTZ and J.L. MELSA, *State Functions and Linear Control Systems*, McGraw-Hill, 1967.

7. J. O'REILLY and M.M. NEWMANN, "Minimal-order observer-estimators for continuous-time linear systems", *Int. J. Control*, Vol. 22, pp. 573-590.

8. Edison TSE, "Observer-estimators for discrete-time systems", *IEEE Trans. Automat. Contr.*, Vol. AC-18, pp. 10-16, February 1973.

9. K. OGATA, *Modern Control Engineering*, Prentice-Hall Electrical Engineering Series, 1970.

10. G.A. HEWER,"An iterative technique for the computation of the steady-state gains for the discrete optimal regulator",*IEEE Trans. Auto. Con.* Vol.AC-16,pp.382, August 1971.

11. F.R. GANTMACHER, *The Theory of Matrices*, Chelsea Publishing Company, New York, N.Y., 1959.

12. F.W. FAIRMAN, "Reduced order state estimation for discrete-time stochastic systems", *IEEE Trans. Automat. Contr.*, Vol. AC-22, pp. 673-675, August, 1977.

13. E. FOGEL and Y.F. HUANG, "Reduced-order optimal state estimator for linear systems with partially noise corrupted measurement", *IEEE Trans. Automat. Contr.*, Vol. AC-25, pp. 994-996, October, 1980.

14. J.S. MEDITCH, *Stochastic Optimal Linear Estimation and Control*, McGraw-Hill.

15. C.T. LEONDES and L.M. NOVAK, "Optimal minimal-order observers for discrete-time systems - a unified theory", *Automatica*, Vol. 8, pp. 379-387, 1972.

16. M. ATHANS, "The matrix minimum principle", *Inform. Control*, Vol. 11, pp. 592, 1968.

17. L.O. CHUA and P.M. LIN, "Computer aided analysis of electronic circuits: algorithms and computational techniques", Prentice-Hall Series in Electrical and Computer Engineering, 1975.

18. W.A. WOLOVICH, "Linear multivariable systems", *Applied Mathematical Sciences*, Volume 11, 1974.

19. M. ZEYTINOĞLU, "Pole assignment in linear multivariable systems", Master of Science Thesis, Boğaziçi University, Istanbul, 1981.