A HARMONIC BALANCE METHOD AND ITS IMPLEMENTATION FOR THE STEADY-STATE ANALYSIS OF NONLINEAR NETWORKS

by

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Submitted to the Faculty of the
School of Engineering in Partial Fulfillment
of the Requirements for the Degree of
MASTER OF SCIENCE
in
ELECTRICAL ENGINEERING

Boğaziçi University
September - 1981

ÖZETÇE

Bu tezde doğrusal olmayan devrelerin periodik çözümünü dinamik denklemlerin zamana bağımlı çözümünü gerektirmeden bulan bir yöntem incelendi. Yöntemde gözönüne alınan devre bir doğrusal ve birden fazla doğrusal olmayan devreye ayrıştırıldı. Sadece doğrusal devrenin frekans alanındaki çözümü asıl çözümü elde etmek için yetmektedir.

Doğrusal devrenin frekans alanındaki denklemlerini elde etmek için kullanılan Karışık Düğüm denklemlerinin çeşitli şekilleri devrenin gerilim ve akım grafiklerinin elde edilmesini açıklayan yöntemlerin tanıtılması aracılığı ile açıklandı.

Salınımların periodunun bilinmediği durumdaki hata işlevinin eğimi için yeni bir ifade geliştirilerek Nakhla ve Vlach[8] tarafından çıkarılanla karşılaştırıldı.

Yöntem çeşitli örnekler üzerinde denendi. Devrede bulunan doğrusal unsurlardan yararlanarak problemin büyüklüğünde önemli ölçüde indirim sağlandığı gözlendi.

ABSTRACT

A method for finding the periodic solution of non-linear networks which avoids the time domain solution of the dynamic equations has been investigated. In the method, the network under consideration is decomposed into one linear and several nonlinear subnetworks. Only frequency domain solution of the linear subnetwork is required.

Various forms of the Mixed Nodal Tableau equations which are used in formulating the linear subnetwork in the frequency domain are explained through the introduction of a technique for obtaining voltage and current graphs of a network.

A new expression for the gradient of the error function in the case that the period T of the oscillations is unknown has been developed and compared to the one derived by Nakhla and Vlach [8].

The method has been tested on several examples. It is shown that considerable reduction in the size of the computational problem is achieved by taking advantage of the linearities present in the network.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my thesis supervisor, Prof.Dr.İ.Cem Göknar for his helpful discussions without which this thesis would not have been possible. I also appreciate his guidance and encouragement.

Thanks are also due to Dr.Yorgo İstefanopulos and Doç. Dr.Ayşe Küsefoğlu for their interest, encouragement and help during this study.

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I. INTRODUCTION

One of the most important problems in the computer - aided design of nonlinear systems is the steady-state analysis of nonlinear networks with periodic responses.

It is always possible to find the periodic response by integrating the differential equations that describe the system until the transients have died out and the steady state remains. But this approach has been found to be extremely expensive whenever the transients are governed by time constants that are very much larger than the period of the driving forces. In this case, continuous integration may require quite large amounts of computing time, since the transient response may be significant for a hundred cycles or more.

Two types of methods have been described to solve this problem: the harmonic balance method [1], [2] and the "shooting methods" [3] - [7]. In the harmonic balance method, the response is assumed to contain a fundamental component of a known or unknown frequency and several harmonics which are believed to be dominant. In this way each state variable is represented in the form of a Fourier series expansion. This assumed solution is substituted into the differential equations and an optimization algoritm is used to adjust the amplitude and phase of the harmonics, i.e., coefficients of the Fourier series such that a mean-square-error function is minimized.

Although this method avoids the computationally expensive process of numerically integrating the dynamic equations, it has a serious limitation in the large number of optimization variables. If the system of equations contains N state variables and each of them requires 2M+1 Fourier coefficients for the dc component and M harmonics, then there will be N(2M+1) optimization variables. This makes the method impractical for large systems.

Shooting methods [3] - [7] solve a two-point boundary value problem. The basic idea of these methods is to find an initial condition vector $\underset{\sim}{x}(0)$ such that when the system equations $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, t)$ are integrated over one period T, the state vector is $x(T) = x(0)$.

It is obvious that the harmonic balance method as originally implemented by Baily [1] and Lindenlaub [2] has the advantage of avoiding the computationally expensive numerical integration of the system dynamic equations but a main disadvantage in the large number of optimization variables.

Nakhla and Vlach [8] modified the harmonic balance method in such a way as to reduce the number of optimization variables by taking full advantage of the fact that considerable part of the network is usually linear. In the Piecewise Harmonic Balance Method proposed by Nakhla and Vlach [8], the nonlinear network under consideration is decomposed into a minimum possible number of linear subnetworks and a minimum possible number of nonlinear ones. The terminals of the linear subnetworks are excited by periodic sources in the form of Fourier series expansion. Then the linear subnetworks are solved in the frequency domain and the coefficients of the Fourier series are adjusted in such a way as to satisfy the topological and constitutive relations of each subnetwork as well as the topological equations resulting form the interconnection of these subnetworks by using a suitable optimization routine.

The Piecewise Harmonic Balance Method proposed by Nakhla and Vlach[8] has reduced the number of optimization variables of the classical harmonic balance method but lead to a port description which is not suitable for generalized computerized applications.

Göknar and Vlach[9] has extended the Piecewise Harmonic Balance Method for most practically encountered nonlinearities by decomposing the network under consideration into one linear and several nonlinear parts. In this method the port description is avoided so that there is no need to decompose the whole network into special subnetworks.

In this thesis, the theory of the Piecewise Harmonic Balance Method as developed by Göknar and Vlach[9] has been examined and the associated computer program prepared by Göknar and Vlach[9] has been modified in such a way as to solve the circuits containing diodes and transistors as nonlinear elements. Several cases of nonlinear periodic circuits have been solved. The structure of the thesis is as follows.

In chapter II the basis of the Piecewise Harmonic Balance Method is introduced. Constitutive equations of the various nonlinear elements are discussed. The idea of source augmenting procedure to seperate the network into one linear and several nonlinear parts is explained. The systematic procedure for removing nonlinear elements is presented for flux controlled inductors and transistors. Finally the scalar error function to be minimized is defined.

In chapter III the procedure for obtaining the respons of the linear network is introduced. It is shown that the solution of the linear subnetwork is obtained only once for each frequency and for unit augmenting sources which is sufficient for all iterations and for the calculation of the gradient. The formulation of the linear network equations is also introduced in this chapter. The Mixed Nodal Tableau (MNT) formulation of the linear network equations is explained in detail. First the importance of the MNT formulation is explained using single graph technique. Then the power of two graph technique is introduced and a systematic procedure for obtaining the voltage and current graphs of the circuits is investigated separately. Finally the solution of the linear network equations obtained using the MNT formulation is introduced and an LU decomposition algorithm which applies partial pivoting on rows is presented.

In chapter IV the evaluation of the gradient both in the case that the period of the oscillations is known and in the case that the period is unknown is introduced. In the case that the period T is unknown an original expression for the gradient is derived and compared to the one presented by Nakhla and Vlach [8] .

In chapter V the steps of the algorithm using the Piecewise Harmonic Balance Method is given. A main program and several subroutines are presented for the case that the period T of the oscillations is known. Some comments and selection of some important parameters required to start the program are also given.

In chapter VI several examples have been solved by the proposed Piecewise Harmonic Balance Method with known period.

Finally in chapter VII concluding remarks and suggestions for further study are given.

II. BASIS OF THE PIECEWISE HARMONIC BALANCE METHOD

In this chapter the basis of the Piecewise Harmonic Balance Method is introduced.

Types of the two-terminal and multi-terminal nonlinear elements are described in section I.

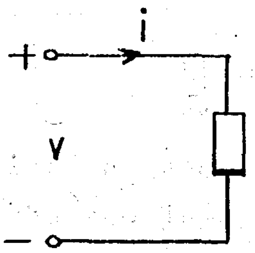
The idea of source augmenting procedure to seperate the network into one linear and several nonlinear parts is presented in section II. First the method of Nakhla and Vlach[8] for seperating the network into linear and nonlinear subnetworks are examined, then their idea is used to explain the method extended by Göknar and Vlach[9].

The systematic procedure for removing nonlinear elements by connecting the augmenting sources is investigated for flux controlled inductors and transistors in section III.

Finally in section IV the error function to be minimized is defined in terms of the complementary variables of the augmenting sources.

II.1. NONLINEAR ELEMENTS

In order to apply Piecewise Harmonic Balance Method for solving nonlinear networks, a restriction is imposed on the type of the nonlinear elements that participate in the given network. It is assumed that all the nonlinear elements present in the network have functional constitutive relations, that is all two-terminal elements have constitutive relations controlled by one variable and all multi-terminal elements have a hybrid description where half of the terminal variables are expressed in terms of the remaining half terminal variables. The various types of the two-terminal nonlinear elements and their constitutive relations are given in the following.



$$i \triangleq \frac{dq}{dt}$$

$$v \triangleq \frac{d\phi}{dt}$$

Figure 1- Two-terminal Nonlinear Element

Current controlled resistors	$V_R = f_R(i_R)$	(1)
Voltage controlled resistors	$i_G = f_G(V_G)$	(2)
Charge controlled capacitors	$V_{\text{ɔ}} = f_{\text{ɔ}}(q_{\text{ɔ}})$	(3)
Voltage controlled capacitors	$q_c = f_c(V_c)$	(4)
Flux controlled inductors	$i_{\text{⅂}} = f_{\text{⅂}}(\phi_{\text{⅂}})$	(5)
Current controlled inductors	$\phi_L = f_L(i_L)$	(6)

Constitutive relation of the voltage controlled capacitors in Eq(4) can be written as

$$i_c = C(v_c)\,\dot{v}_c \tag{7}$$

where

$$C(v_c) \triangleq \frac{\partial f_c}{\partial v_c} \tag{8}$$

and the constitutive relation of the current controlled inductors in Eq(6) can be written as

$$v_L = L(i_L)\,\dot{i}_L \tag{9}$$

where

$$L(i_L) \triangleq \frac{\partial f_L}{\partial i_L} \tag{10}$$

The most general constitutive relation for a multi-terminal element is assumed to be in the form

$$\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x}, \dot{\underset{\sim}{x}}) \tag{11}$$

where $\underset{\sim}{x}$ consists of some of the terminal variables and $\underset{\sim}{y}$ consists of the remaining complementary variables. A two-terminal and a multi-terminal element are illustrated in Fig.1 and Fig.2, respectively.

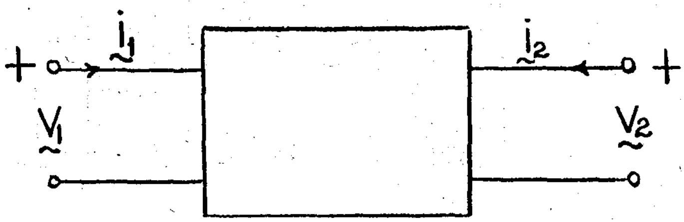


Figure 2- Multiterminal Nonlinear Element

The basic building bloc for modelling transistors is assumed to be of the form given in Fig.3. In Fig.3, the constitutive relations are as follows:

$$i_R = f_R(V_{BC}) \tag{12}$$

$$i_F = f_E(V_{BE}) \tag{13}$$

denotes the diode currents,

$$\alpha_R = \alpha_R(i_R) \tag{14}$$

$$\alpha_F = \alpha_F(i_F) \tag{15}$$

denotes the nonlinear parameters of the controlled sources, and

$$i_{DE} = C_{DE}(V_{BE})\dot{V}_{BE} \tag{16}$$

$$i_{DC} = C_{DC}(V_{BC})\dot{V}_{BC} \tag{17}$$

$$i_{TE} = C_{TE}(V_{BE})\dot{V}_{BE} \tag{18}$$

$$i_{TC} = C_{TC}(V_{BC})\dot{V}_{BC} \tag{19}$$

denotes the constitutive relations for nonlinear capacitors. Then the basic building bloc's terminal equations take the form

$$\begin{bmatrix} i_E \\ i_C \end{bmatrix} = \begin{bmatrix} f_E(V_{BE}, V_{BC}, \dot{V}_{BE}) \\ f_C(V_{BE}, V_{BC}, \dot{V}_{BC}) \end{bmatrix} \tag{20}$$

Defining

$$\underset{\sim}{i}_T = [i_E \quad i_C]^T \tag{21}$$

$$\underset{\sim}{f}_T = [f_E \quad f_C]^T \tag{22}$$

$$\underset{\sim}{V}_T = [V_{BE} \quad V_{BC}]^T \tag{23}$$

Eq(20) can be written as

$$\underset{\sim}{i}_T = \underset{\sim}{f}_T(\underset{\sim}{V}_T, \dot{\underset{\sim}{V}}_T) \tag{24}$$

which is in the form of Eq(11).

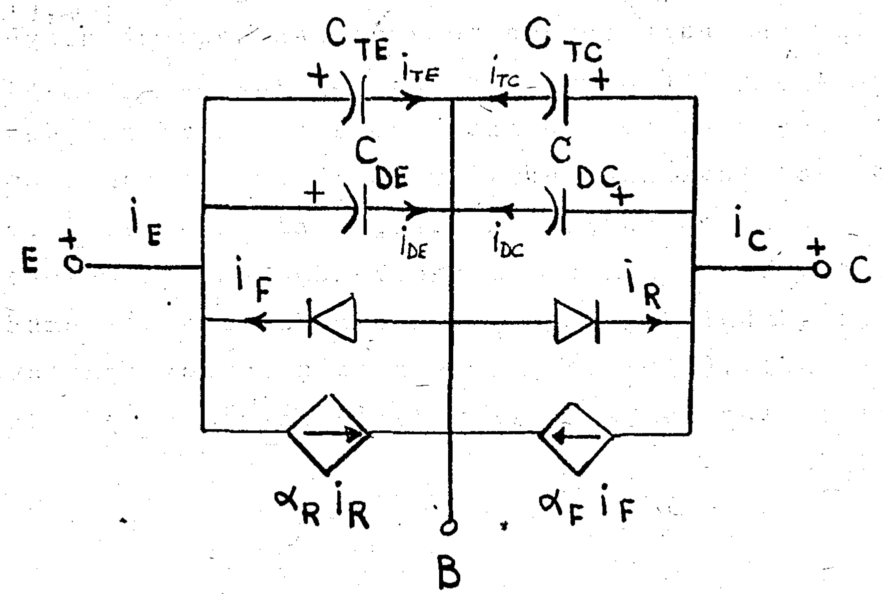


Figure 3- Basic Building Bloc for a Transistor

Any other model for a transistor can be built from this basic bloc by adding elements to the terminals E,B or C. For the more complex model obtained by adding elements to the terminals, either a description of the form(11) is obtained as constitutive relations or the additional elements are considered seperately as individual elements.

II.2. SOURCE AUGMENTING PROCEDURE

In this section, the idea of source augmenting procedure which constitutes the basis of the Piecewise Harmonic Balance Method is introduced by decomposing the network under consideration into nonlinear and linear subnetworks. Then, the procedure is developed in such a way as to seperate the the network into one linear and several nonlinear parts.

Consider first a simple network which consists of two arbitrary subnetworks S_1 and S_2 [Fig.4]. Then, let v(t) and i(t) be the voltage and current waveforms, respectively, at the terminals A-B. Assume that the subnetwork S_1 does not contain any sources dependent on currents or voltages in the subnetwork S_2. Seperate the two subnetworks and augment S_1 by an independent current source with the waveform i(t) [Fig.5]. If the resulting network has a unique solution for all its branch currents and branch voltages, then these currents and voltages will be identical to those of the subnetwork S_1 in Fig.4. The voltage at the terminals A-B in the augmented subnetwork is identical to v(t). On the other hand, if the subnetwork S_2 is augmented by an independent voltage source with the waveform v(t), the resulting current at the terminals A'-B' will be identical to i(t).

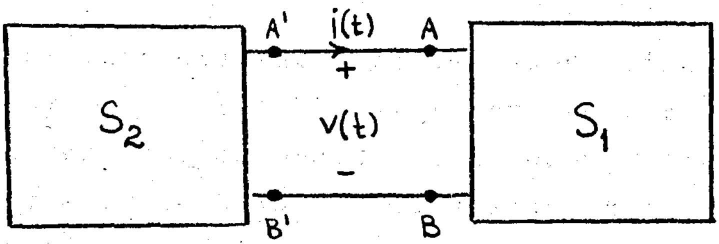


Figure 4- Arbitrary Network

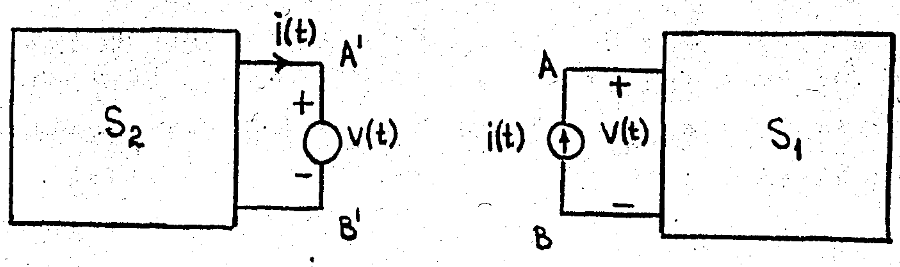


Figure 5- Same network in Fig.4. seperated into two subnetworks each augmented by appropriate source.

Let us now repeat the same procedure with some modifications. We start by augmenting S_1 by an independent current source $i_1(t) \neq i(t)$. Denote the resulting voltage at the terminals A-B by $v_1(t)$. If the subnetwork S_2 is augmented by a voltage source with waveform $v_2(t) = v_1(t)$, some current $i_2(t)$ will flow, in the direction indicated at the terminals A'-B'. Obviously, $i_2(t) \neq i_1(t)$. An error function can be defined as $\varepsilon(t) = i_2(t)-i_1(t)$. If we succeed, by a set of suitable alterations, to make $\varepsilon(t) = 0$ for all t, then the source $i_1(t)$ which reduces the error function to zero for all t is identical to i(t). In the Piecewise Harmonic Balance Method presented here, the alterations are performed by a minimization routine.

The idea presented in the above discussion was generalized by Nakhla and Vlach[8] for the determination of the periodic response of nonlinear networks. Assuming that an arbitrary network S is in the steady state with periodic response of period T, then S can be decomposed into a minimum possible number of linear subnetworks L_i; $i=1,...,\ell$, and nonlinear subnetworks N_i; $i=1,...,n$. All the nonlinear elements that exist in S are assumed to be voltage controlled.

Then, the linear subnetworks can be augmented by periodic current sources of the form

$$\underset{\sim}{I}_L(\underset{\sim}{x}) = \sum_{k=0}^{M} \underset{\sim}{x}_k \cos \omega kt + \sum_{k=1}^{M} \underset{\sim}{x}_k^* \sin \omega kt \qquad (25)$$

where

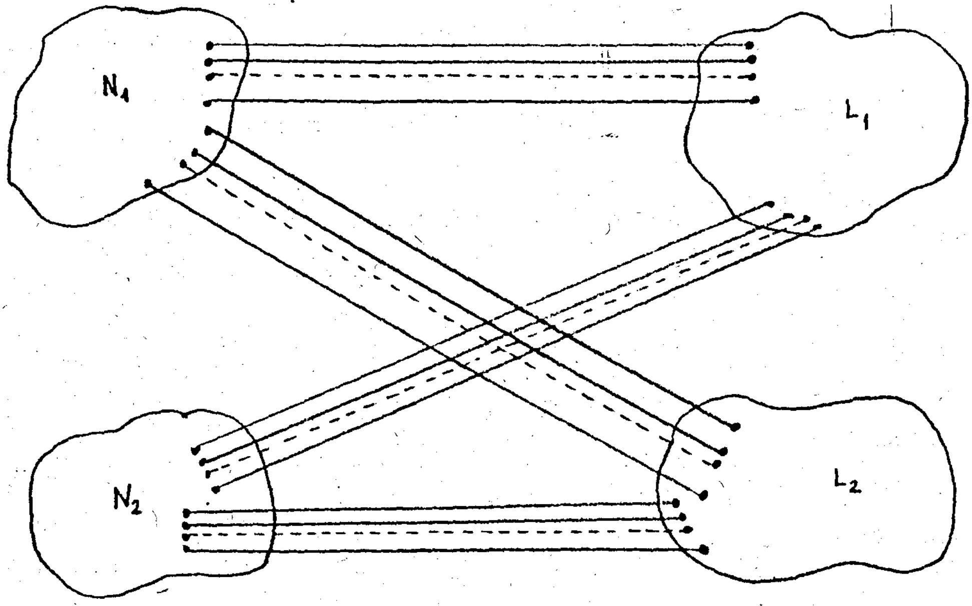
$$\underset{\sim}{x} = [\, \underset{\sim}{x}_k^T \quad \underset{\sim}{x}_k^{*T} \,]^T \quad \text{and} \quad \omega = \frac{2\pi}{T} .$$

Since the linear subnetworks are excited by periodic functions, the solution in the frequency domain can determine the node voltages of these subnetworks. They can be denoted by

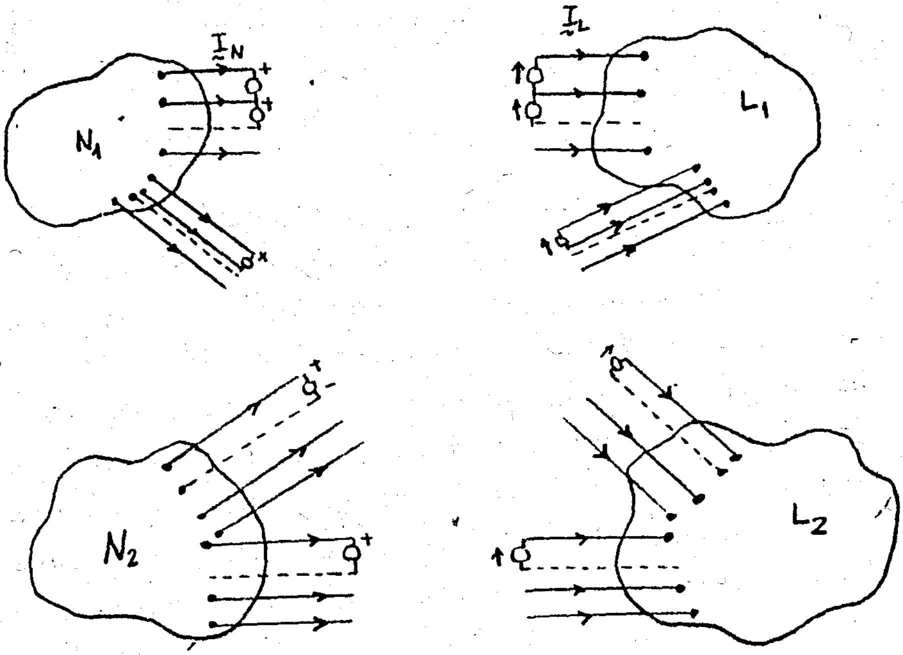
$$\underset{\sim}{e}(t) = \sum_{k=0}^{M} \underset{\sim}{e}_k(t) \qquad (26)$$

where $\underset{\sim}{e}_k(t)$ is the vector of node voltages corresponding to the sources with frequency k. Then, the voltage sources with waveforms identical to those appearing at the respective terminals of the linear subnetworks is applied to the nonlinear subnetworks N_i as shown in Fig.6,b. Since it is assumed that the nonlinear networks contain only resistors and capacitors as nonlinear elements whose controlling variables are the voltages, the resulting terminal currents will be functions of $\underset{\sim}{e}(t)$ which is denoted by

$$\underset{\sim}{I}_N(t) = \underset{\sim}{f}(\underset{\sim}{e}, \dot{\underset{\sim}{e}}) \qquad (27)$$



(a)



(b)

Figure 6- (a) Arbitrary nonlinear network with periodic inputs.
(b) Augmentation of subnetworks with appropriate periodic sources.

For a consistent solution

$$\underset{\sim}{I}_N(t) = \underset{\sim}{I}_L(t) \qquad \text{for} \qquad 0 \leqslant t \leqslant T \tag{28}$$

Thus, the variables in Eq(25) can be chosen such that Eq(28) is satisfied.

Göknar and Vlach [9] extended the theory mentioned above for most practically encountered nonlinear elements presented in section II.1 by decomposing the network under consideration into one linear and several nonlinear parts. Consider the two vectors consisting of the controlling variables of the nonlinear elements

$$\underset{\sim}{V} = [\underset{\sim}{V}_G^T \ \underset{\sim}{V}_C^T \ \underset{\sim}{\phi}_7^T \ \underset{\sim}{V}_{TR}^T \ \cdot \ \cdot \ \cdot]^T \tag{29}$$

$$\underset{\sim}{i} = [\underset{\sim}{i}_R^T \ \underset{\sim}{q}_5^T \ \underset{\sim}{i}_L^T \ \cdot \ \cdot \ \cdot]^T \tag{30}$$

If the controlling variable of the nonlinear element considered is a voltage or a flux, then a voltage or a flux source is connected accordingly across the terminal where the variable is being measured. On the other hand, if the controlling variable of the nonlinear element considered is a current or a charge, then a current or a charge source is connected accordingly in series with the terminal where the variable is being measured. The additional sources connected as explained above are called augmenting sources. Special care should be exercised not to form loops of voltage sources with or without flux sources and cut-sets of current sources with or without charge sources in connecting the augmenting sources.

Let $\underset{\sim}{u}_S$ denote the augmenting source variables and $\underset{\sim}{\varepsilon}$ denote the complementary variables of the augmenting sources.

Then, consider the network with four nonlinearities shown schematically in Fig.7. For this network $\underset{\sim}{u}_s$ and $\underset{\sim}{\varepsilon}$ are given by the following expressions:

$$\underset{\sim}{u}_s = [\,v_{s1} \quad v_{s2} \quad i_{s3} \quad q_{s4}\,]^T \tag{31}$$

$$\underset{\sim}{\varepsilon} = [\,i_{s1} \quad i_{s2} \quad v_{s3} \quad v_{s4}\,]^T \tag{32}$$

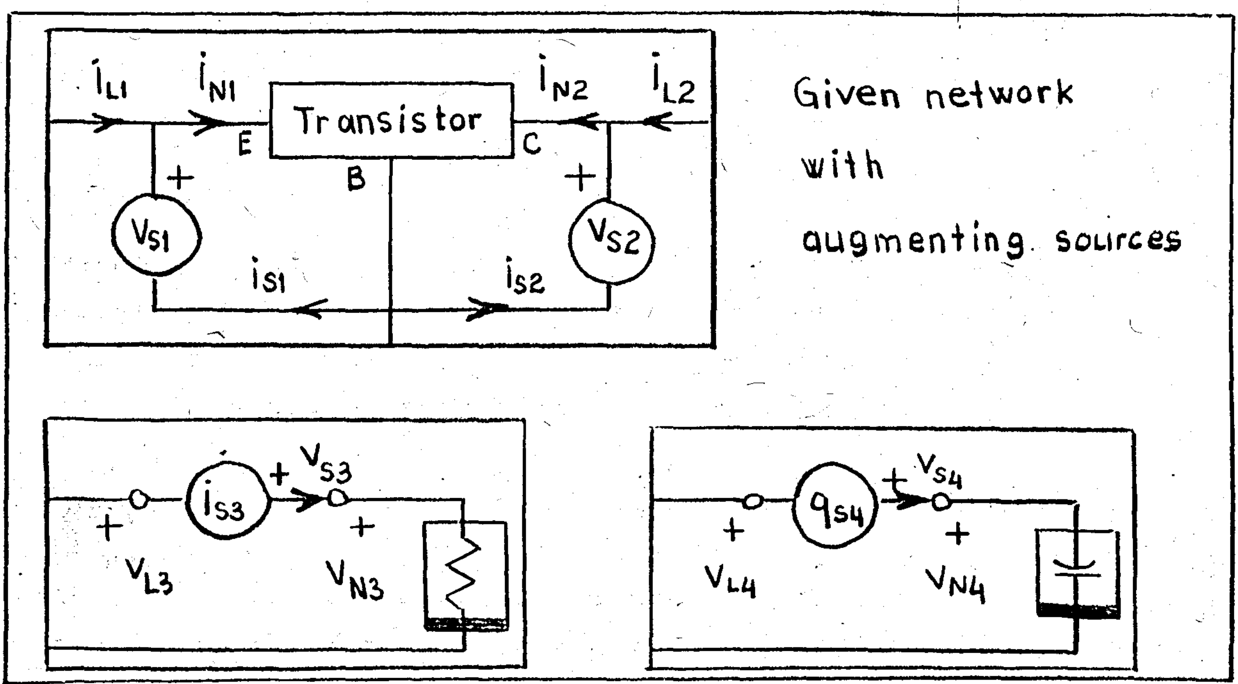


Figure 7- Augmenting sources for a transistor, current controlled resistor and charge controlled capacitor.

Suppose that it is possible to choose the augmenting source variables $\underset{\sim}{u}_s(\cdot) = \underset{\sim}{u}_s^o(\cdot)$ in such a way that $\underset{\sim}{\varepsilon}(t)$ is identically zero for all t. The presence of the augmenting voltage and flux sources connected across the nonlinear elements whose controlling variables are, respectively, a voltage or a flux will not effect the other variables in the circuit when the current through them is identically zero since they all behave as open circuit elements. On the other hand, the presence of the augmenting current and charge sources connected in series with the nonlinear elements whose controlling variables are, respectively, a current or a charge will not effect the other variables in the circuit when the voltage across them is identically zero since they all behave as short circuit elements. Thus, the problem is to find the augmenting source variables $\underset{\sim}{u}_s(.)$ such that the complementary variables of the augmenting sources $\underset{\sim}{\varepsilon}(t)$ become identically zero for all t.

Let $\underset{\sim}{u}_s(\cdot)$ be arbitrarily selected. Then $\underset{\sim}{\varepsilon}(\cdot) \neq 0$ is given by

$$\underset{\sim}{\varepsilon}(t) = \underset{\sim}{y}_N(t) - \underset{\sim}{y}_L(t) \tag{33}$$

where $\underset{\sim}{y}_N$ consists of the controlled variables of the nonlinear elements and $\underset{\sim}{y}_L$ consists of the variables in the network at the terminals where the augmenting sources are connected created by the introduction of these sources.

After augmenting soucres have been introduced and $\underset{\sim}{u}_s(\cdot)$ has been selected, the variables $\underset{\sim}{y}_L(\cdot)$ are not affected by the presence of the nonlinear elements anymore. Therefore all nonlinear elements can be removed from the circuit as shown in Fig.8 and $y_L(t)$ can be calculated from the resulting linear circuit by any technique. Note that the charge and flux sources have been replaced by the current source $\dot{q}(t)$ and voltage source $\dot{\phi}(t)$ in the linear network. The response $\underset{\sim}{y}_N$ of the nonlinear network is obtained by direct substitution of $\underset{\sim}{u}_s(\cdot)$ into the equations describing the nonlinear elements. Thus, for each selection of $\underset{\sim}{u}_s(\cdot)$, $\underset{\sim}{\varepsilon}(\cdot)$ is obtained from Eq(33) and it is forced to zero by a suitable optimization method, to obtain $\underset{\sim}{u}_s^0(\cdot)$.

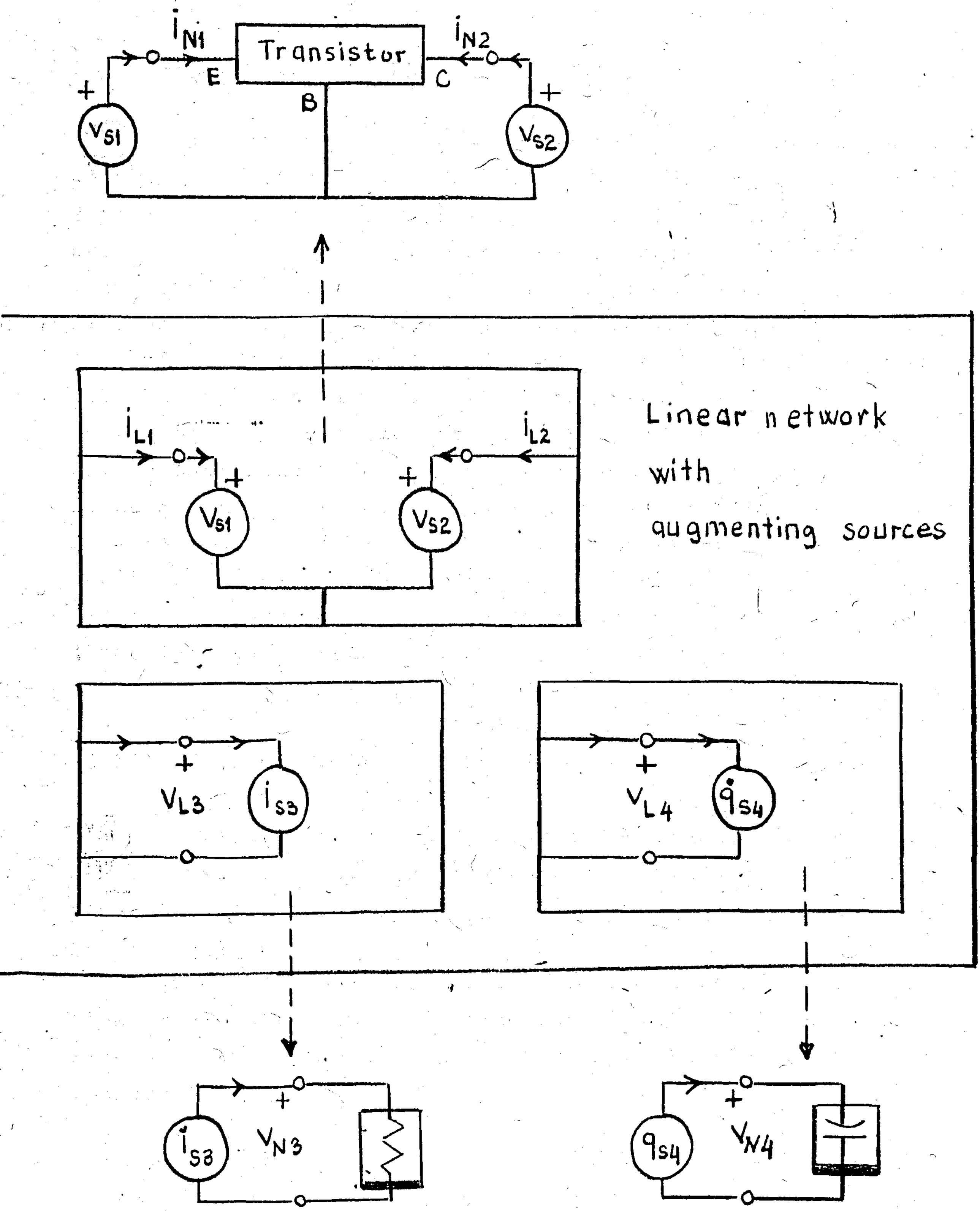


Figure 8- Removal of the nonlinear elements from the given network.

II.3. REMOVAL OF NONLINEARITIES

In order to avoid such degeneracies as loops of voltage sources and cut-sets of current sources and to be able to give a formulation of the nonlinear elements in terms of the primary variables of the augmenting sources, the source augmenting procedure has to be done systematically. In this section the source augmenting procedure for removing non-linearities due to flux controlled inductors and transistors will be explained in detail. The constitutive equations expressing the complementary variables of the augmenting sources due to nonlinear elements will be obtained for flux controlled inductors and transistors, respectively. The constitutive equations expressing the complementary variables of the nonlinear elements other than the flux controlled inductors and transistors that has been obtained in [9] will be presented directly without derivation.

II.3.1. FLUX CONTROLLED INDUCTORS

In this subsection, Systematic procedure for the removal of nonlinearities due to flux controlled inductors will be presented in detail.

EXAMPLE 1

Consider the graph of the given network in Fig.9. The lines represent flux controlled inductors while the dashed lines represent the elements other than the flux controlled inductors.

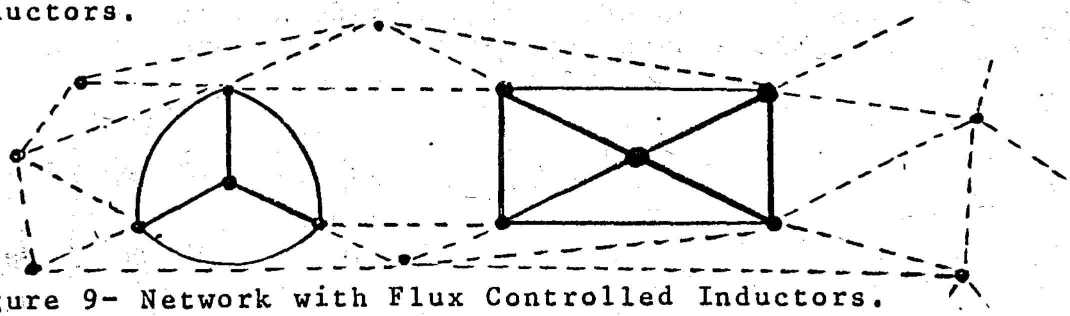


Figure 9- Network with Flux Controlled Inductors.

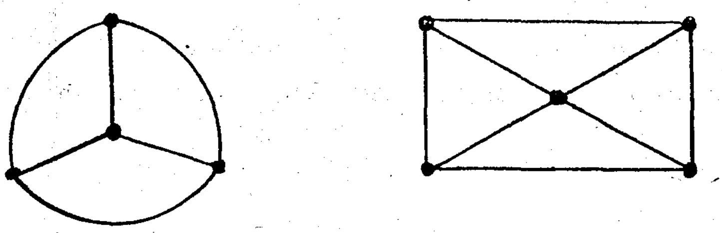


Figure 10- Network obtained from the one in Fig.9 by open circuiting all elements other than Flux Controlled Inductors

Obtain a subcircuit of the given circuit N by open-circuiting all elements other than flux controlled inductors. This is shown in Fig.10. Then choose a forest in this subcircuit. The forest chosen for this circuit consists of the branches numbered from 1 to 7, and augment the subcircuit by connecting a Flux Source across each forest branch as shown in Fig.11.

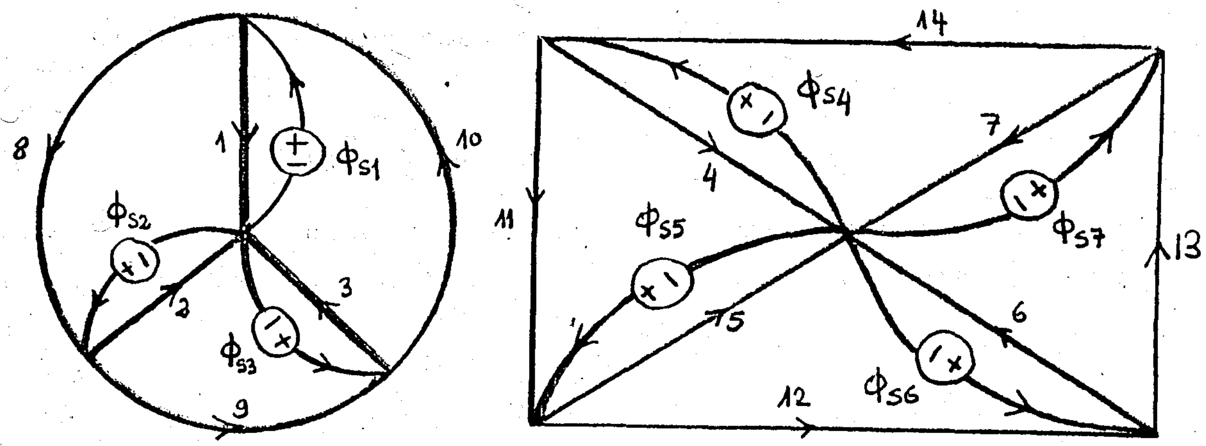


Figure 11- Network obtained from the one in Fig.10. by augmenting flux sources across each tree branch.

In Fig.11

$$\underset{\sim}{\phi}_{ST} = \left[\phi_{S1} \;\; \phi_{S2} \;\; \phi_{S3} \;\; \phi_{S4} \;\; \phi_{S5} \;\; \phi_{S6} \;\; \phi_{S7} \right]^T \tag{34}$$

$$\underset{\sim}{\phi}_{TF} = \left[\phi_{F1} \;\; \phi_{F2} \;\; \phi_{F3} \;\; \phi_{F4} \;\; \phi_{F5} \;\; \phi_{F6} \;\; \phi_{F7} \right]^T \tag{35}$$

$$\underset{\sim}{\phi}_{TC} = \left[\phi_{C8} \;\; \phi_{C9} \;\; \phi_{C10} \;\; \phi_{C11} \;\; \phi_{C12} \;\; \phi_{C13} \;\; \phi_{C14} \right]^T \tag{36}$$

denote the fluxes of augmenting sources, of forest inductors, and of co-forest inductors, respectively.

The fundamental loop equations associated with augmenting source branches of the augmented subcircuit are

$$
\begin{aligned}
\phi_{F1} &= \phi_{S1}\\
\phi_{F2} &= \phi_{S2}\\
\phi_{F3} &= \phi_{S3}\\
\phi_{F4} &= \phi_{S4}\\
\phi_{F5} &= \phi_{S5}\\
\phi_{F6} &= \phi_{S6}\\
\phi_{F7} &= \phi_{S7}
\end{aligned}
\tag{37}
$$

Which can be expressed compactly as

$$\underset{\sim}{\phi}_{S7} = \underset{\sim}{\phi}_{7F} \tag{38}$$

The fundamental loop equations associated with co-forest branches of the augmented subcircuit are

$$
\begin{aligned}
\phi_{C8} + \phi_{S2} - \phi_{S1} &= 0\\
\phi_{C9} + \phi_{S3} - \phi_{S2} &= 0\\
\phi_{C10} + \phi_{S1} - \phi_{S3} &= 0\\
\phi_{C11} + \phi_{S5} - \phi_{S4} &= 0\\
\phi_{C12} + \phi_{S6} - \phi_{S5} &= 0\\
\phi_{C13} + \phi_{S7} - \phi_{S6} &= 0\\
\phi_{C14} + \phi_{S4} - \phi_{S7} &= 0
\end{aligned}
\tag{39}
$$

which can be written in matrix form as

$$\underbrace{\left[\begin{array}{ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right.}_{-\underset{\sim}{B}_7} \vdots \underbrace{\left.\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{\underset{\sim}{I}} \begin{bmatrix} \phi_{S1} \\ \phi_{S2} \\ \phi_{S3} \\ \phi_{S4} \\ \phi_{S5} \\ \phi_{S6} \\ \phi_{S7} \\ \phi_{C8} \\ \phi_{C9} \\ \phi_{C10} \\ \phi_{C11} \\ \phi_{C12} \\ \phi_{C13} \\ \phi_{C14} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

Eq(40) can be expressed in compact form as

$$\left[-\underset{\sim}{B}_7 \quad \underset{\sim}{I}\right] \begin{bmatrix} \underset{\sim}{\phi}_{S7} \\ \underset{\sim}{\phi}_{7C} \end{bmatrix} = \underset{\sim}{0} \tag{41}$$

or

$$-\underset{\sim}{B}_7 \, \underset{\sim}{\phi}_{S7} + \underset{\sim}{\phi}_{7C} = \underset{\sim}{0} \tag{42}$$

Combining the set of equations in (38) and (42), the fundamental loop equations of the augmented subcircuit are obtained as:

$$\begin{bmatrix} -\underset{\sim}{I} & \underset{\sim}{I} & \underset{\sim}{0} \\ -\underset{\sim}{B}_{7} & \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\phi}_{S7} \\ \underset{\sim}{\phi}_{7F} \\ \underset{\sim}{\phi}_{7C} \end{bmatrix} = \underset{\sim}{0} \tag{43}$$

where $\underset{\sim}{I}$ and $\underset{\sim}{0}$ denote the identity and the zero matrices respectively.

The fundamental cut-set equations then become:

$$\begin{aligned}
&\dot{i}_{S1} - \dot{i}_{F1} - \dot{i}_{C8} + \dot{i}_{C10} = 0 \\
&\dot{i}_{S2} - \dot{i}_{F2} + \dot{i}_{C8} - \dot{i}_{C9} = 0 \\
&\dot{i}_{S3} - \dot{i}_{F3} + \dot{i}_{C9} - \dot{i}_{C10} = 0 \\
&\dot{i}_{S4} - \dot{i}_{F4} - \dot{i}_{C11} + \dot{i}_{C14} = 0 \\
&\dot{i}_{S5} - \dot{i}_{F5} + \dot{i}_{C11} - \dot{i}_{C12} = 0 \\
&\dot{i}_{S6} - \dot{i}_{F6} + \dot{i}_{C12} - \dot{i}_{C13} = 0 \\
&\dot{i}_{S7} - \dot{i}_{F7} + \dot{i}_{C13} - \dot{i}_{C14} = 0
\end{aligned} \tag{44}$$

The set of equations in(44) can be put into matrix form as:

$$
\underbrace{\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right.}_{\underset{\sim}{I}}
\underbrace{\begin{array}{|ccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}}_{-\underset{\sim}{I}}
\underbrace{\left|\begin{array}{ccccccc}
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}\right]}_{-\underset{\sim}{B}_{\urcorner}^{T}}
\left[\begin{array}{c}
\dot{i}_{S1} \\ \dot{i}_{S2} \\ \dot{i}_{S3} \\ \dot{i}_{S4} \\ \dot{i}_{S5} \\ \dot{i}_{S6} \\ \dot{i}_{S7} \\ \hline
\dot{i}_{F1} \\ \dot{i}_{F2} \\ \dot{i}_{F3} \\ \dot{i}_{F4} \\ \dot{i}_{F5} \\ \dot{i}_{F6} \\ \dot{i}_{F7} \\ \hline
\dot{i}_{C8} \\ \dot{i}_{C9} \\ \dot{i}_{C10} \\ \dot{i}_{C11} \\ \dot{i}_{C12} \\ \dot{i}_{C13} \\ \dot{i}_{C14}
\end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]
\tag{45}
$$

Eq(45) can be expressed in compact form as:

$$
\left[\begin{array}{ccc} \underset{\sim}{I} & -\underset{\sim}{I} & -\underset{\sim}{B}_{\urcorner}^{T} \end{array}\right]
\left[\begin{array}{c} \dot{\underset{\sim}{i}}_{N\urcorner} \\ \dot{\underset{\sim}{i}}_{\urcorner F} \\ \dot{\underset{\sim}{i}}_{\urcorner C} \end{array}\right] = \underset{\sim}{0}
\tag{46}
$$

where

$$
\dot{\underset{\sim}{i}}_{N\urcorner} = \left[\begin{array}{ccccccc} \dot{i}_{S1} & \dot{i}_{S2} & \dot{i}_{S3} & \dot{i}_{S4} & \dot{i}_{S5} & \dot{i}_{S6} & \dot{i}_{S7} \end{array}\right]^{T}
\tag{47}
$$

$$\underset{\sim}{i}_{\neg F} = \left[i_{F1} \;\; i_{F2} \;\; i_{F3} \;\; i_{F4} \;\; i_{F5} \;\; i_{F6} \;\; i_{F7} \right]^T \tag{48}$$

$$\underset{\sim}{i}_{\neg C} = \left[i_{C8} \;\; i_{C9} \;\; i_{C10} \;\; i_{C11} \;\; i_{C12} \;\; i_{C13} \;\; i_{C14} \right]^T \tag{49}$$

denote the currents of augmenting sources, of forest inductors, and of co-forest inductors, respectively.

Using the constitutive relations

$$\underset{\sim}{i}_{\neg F} = \underset{\sim}{f}_{\neg F}(\underset{\sim}{\phi}_{\neg F}) \tag{50}$$

$$\underset{\sim}{i}_{\neg C} = \underset{\sim}{f}_{\neg C}(\underset{\sim}{\phi}_{\neg C}) \tag{51}$$

in Eq(46)

$$\underset{\sim}{i}_{N\neg} = \underset{\sim}{i}_{\neg F} + \underset{\sim}{B}_{\neg}^{T}\, \underset{\sim}{i}_{\neg C} \tag{52}$$

becomes

$$\underset{\sim}{i}_{N\neg} = \underset{\sim}{f}_{\neg F}(\underset{\sim}{\phi}_{\neg F}) + \underset{\sim}{B}_{\neg}^{T}\, \underset{\sim}{f}_{\neg C}(\underset{\sim}{\phi}_{\neg C}) \tag{53}$$

and using (43) in (53)

$$\underset{\sim}{i}_{N\neg} = \underset{\sim}{f}_{\neg F}(\underset{\sim}{\phi}_{S\neg}) + \underset{\sim}{B}_{\neg}^{T}\, \underset{\sim}{f}(\underset{\sim}{B}_{\neg}\, \underset{\sim}{\phi}_{S\neg}) \tag{54}$$

is obtained. Eq(18) will be written as

$$\underset{\sim}{i}_{N\neg} = \underset{\sim}{\hat{f}}_{\neg}(\underset{\sim}{\phi}_{S\neg}) \tag{55}$$

In the case that there are no loops of flux controlled inductors, Eq(55) reduces to the constitutive relations as

$$\underset{\sim}{i}_{N\neg} = \underset{\sim}{f}_{\neg}(\underset{\sim}{\phi}_{\neg}) \tag{56}$$

The circuit obtained from the given one by replacing all flux controlled inductors by voltage sources as explained above will be denoted by $N_{\neg}$. The voltage function of these sources is taken to be $\underset{\sim}{V}_{s\neg} = \dot{\underset{\sim}{\phi}}_{s\neg}$. This is illustrated in Fig.12 for the special network considered in Fig.9, and in Fig.13 for any network. Note that the derivation in this section is made under the assumption that there exists no loops of flux controlled inductors with actual sources present in the original circuit.

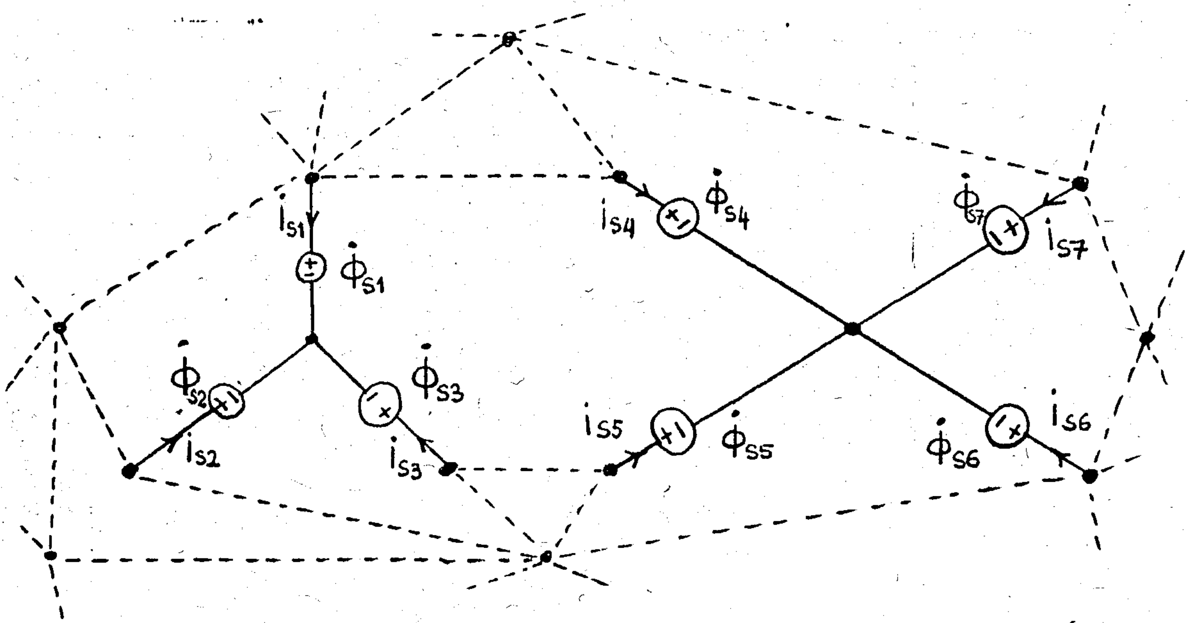


Figure 12- The network $N_{\neg}$ obtained by removing all flux controlled inductors from the network in Fig.9.

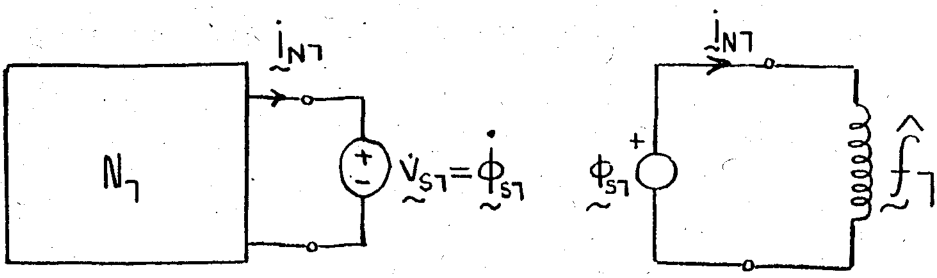


Figure 13- Removal of flux controlled inductors.

II.3.2. TRANSISTORS

In this subsection, systematic procedure for the removal of nonlinearities due to transistors will be explained in an example by augmenting voltage sources into the network, properly.

EXAMPLE 2

Obtain a subcircuit of $N_{\urcorner}$ by open-circuiting all other elements than transistors, voltage sources in N and the augmenting sources introduced in II.3.1. (if neither type of sources form loops with transistor branches, then the sources can also be deleted). Then choose a forest in the subcircuit by including as many sources as possible as forest branches.

Let the subcircuit of $N_{\urcorner}$ obtained by applying the procedure stated above be the circuit shown in Fig.14. (The forest chosen for this example is indicated by heavier lines).

In Fig.14

$$\underset{\sim}{V}_{SA} = \left[V_{SA1} \;\; V_{SA2} \right]^T \tag{57}$$

$$\dot{\underset{\sim}{\Phi}}_{S\urcorner} = \left[\dot{\Phi}_{S\urcorner 1} \;\; \dot{\Phi}_{S\urcorner 2} \right]^T \tag{58}$$

denote the voltages of voltage sources in N that are connected across transistor branches and the voltages of flux sources that are connected across transistor branches respectively.

$$\overline{\underset{\sim}{V}}_{SA} = \left[\overline{V}_{SA1} \;\; \overline{V}_{SA2} \right]^T \tag{59}$$

$$\dot{\overline{\underset{\sim}{\Phi}}}_{S\urcorner} = \left[\dot{\overline{\Phi}}_{S\urcorner 1} \;\; \dot{\overline{\Phi}}_{S\urcorner 2} \right]^T \tag{60}$$

denote voltages of voltage sources in N that form loops with transistor branches and the voltages of flux sources that form loops with transistor branches, respectively.

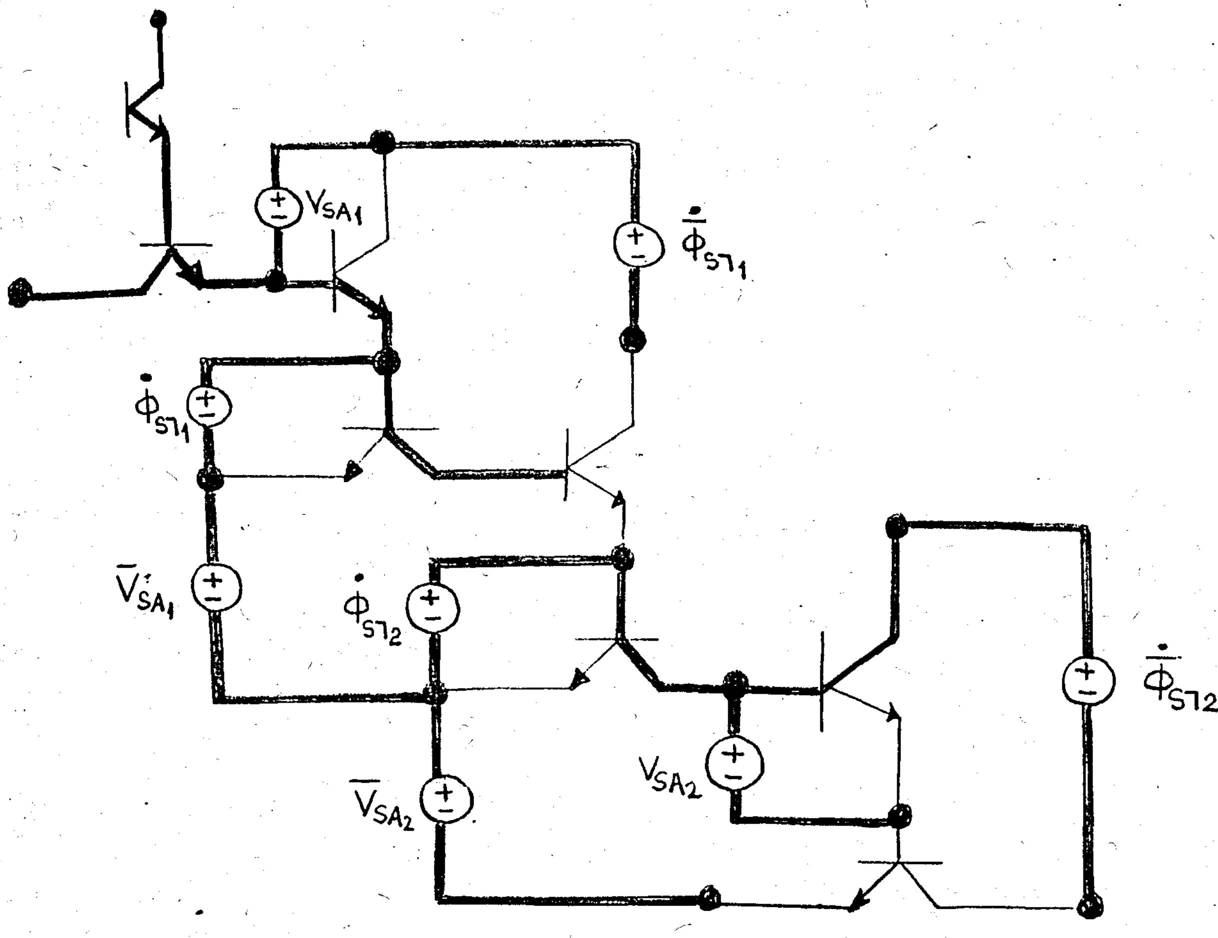


Figure 14- Network obtained from N_T by open circuiting all other elements than transistors, voltage sources in N and the augmenting sources introduced before.

Augment this subcircuit by connecting a voltage source across each transistor branch in the forest. Then the circuit in Fig.14 takes the form shown in Fig.15.

In Fig.15

$$\underset{\sim}{V}_{TA} = \begin{bmatrix} V_{TA1} & V_{TA2} \end{bmatrix}^T \quad (61)$$

$$\underset{\sim}{V}_{T\urcorner} = \begin{bmatrix} V_{T\urcorner 1} & V_{T\urcorner 2} \end{bmatrix}^T \quad (62)$$

denote the voltages of transistor branches which have a voltage source in N across their terminals and the voltages of transistor branches which have a flux source across their terminals, respectively.

$$\underset{\sim}{V}_{TF} = \begin{bmatrix} V_{TF1} & V_{TF2} & V_{TF3} & V_{TF4} & V_{TF5} & V_{TF6} & V_{TF7} & V_{TF8} \end{bmatrix}^T \quad (63)$$

$$\underset{\sim}{V}_{TC} = \begin{bmatrix} V_{TC1} & V_{TC2} & V_{TC3} & V_{TC4} \end{bmatrix}^T \quad (64)$$

denote the voltages of transistor branches that are in the forest and the voltages of transistor branches that are in the coforest, respectively.

$$\underset{\sim}{V}_{ST} = \begin{bmatrix} V_{ST1} & V_{ST2} & V_{ST3} & V_{ST4} & V_{ST5} & V_{ST6} & V_{ST7} & V_{ST8} \end{bmatrix}^T \quad (65)$$

denote the voltages of augmenting sources connected across transistor branches.

The fundamental loop equations associated with transistor branches which have a voltage source in N across their terminals are:

$$\begin{aligned} -V_{SA1} + V_{TA1} &= 0 \\ -V_{SA2} + V_{TA2} &= 0 \end{aligned} \quad (66)$$

The fundamental loop equations associated with transistor branches which have a flux source across their terminals are:

$$-\dot{\Phi}_{ST1} + V_{TT1} = 0$$
$$-\dot{\Phi}_{ST2} + V_{TT2} = 0 \tag{67}$$

The fundamental loop equations associated with augmenting sources which are connected across transistor branches are:

$$-V_{ST1} + V_{TF1} = 0$$
$$-V_{ST2} + V_{TF2} = 0$$
$$-V_{ST3} + V_{TF3} = 0$$
$$-V_{ST4} + V_{TF4} = 0$$
$$-V_{ST5} + V_{TF5} = 0$$
$$-V_{ST6} + V_{TF6} = 0 \tag{68}$$
$$-V_{ST7} + V_{TF7} = 0$$
$$-V_{ST8} + V_{TF8} = 0$$

The fundamental loop equations associated with transistor branches which are in the co-forest are:

$$-V_{SA1} + V_{ST5} + V_{ST6} - \bar{\dot{\Phi}}_{ST1} + V_{TC1} = 0$$
$$-\dot{\Phi}_{ST1} + \dot{\Phi}_{ST2} + V_{ST6} - \bar{V}_{SA1} + V_{TC2} = 0$$
$$V_{SA2} - \Phi_{ST2} + V_{ST7} - \bar{V}_{SA2} + V_{TC3} = 0 \tag{69}$$
$$V_{SA2} - V_{ST8} - \bar{\dot{\Phi}}_{ST2} + V_{TC4} = 0$$

The Eqs(66),(67),(68) and (69) constitute the fundamental loop equations of the augmented circuit in Fig.15.

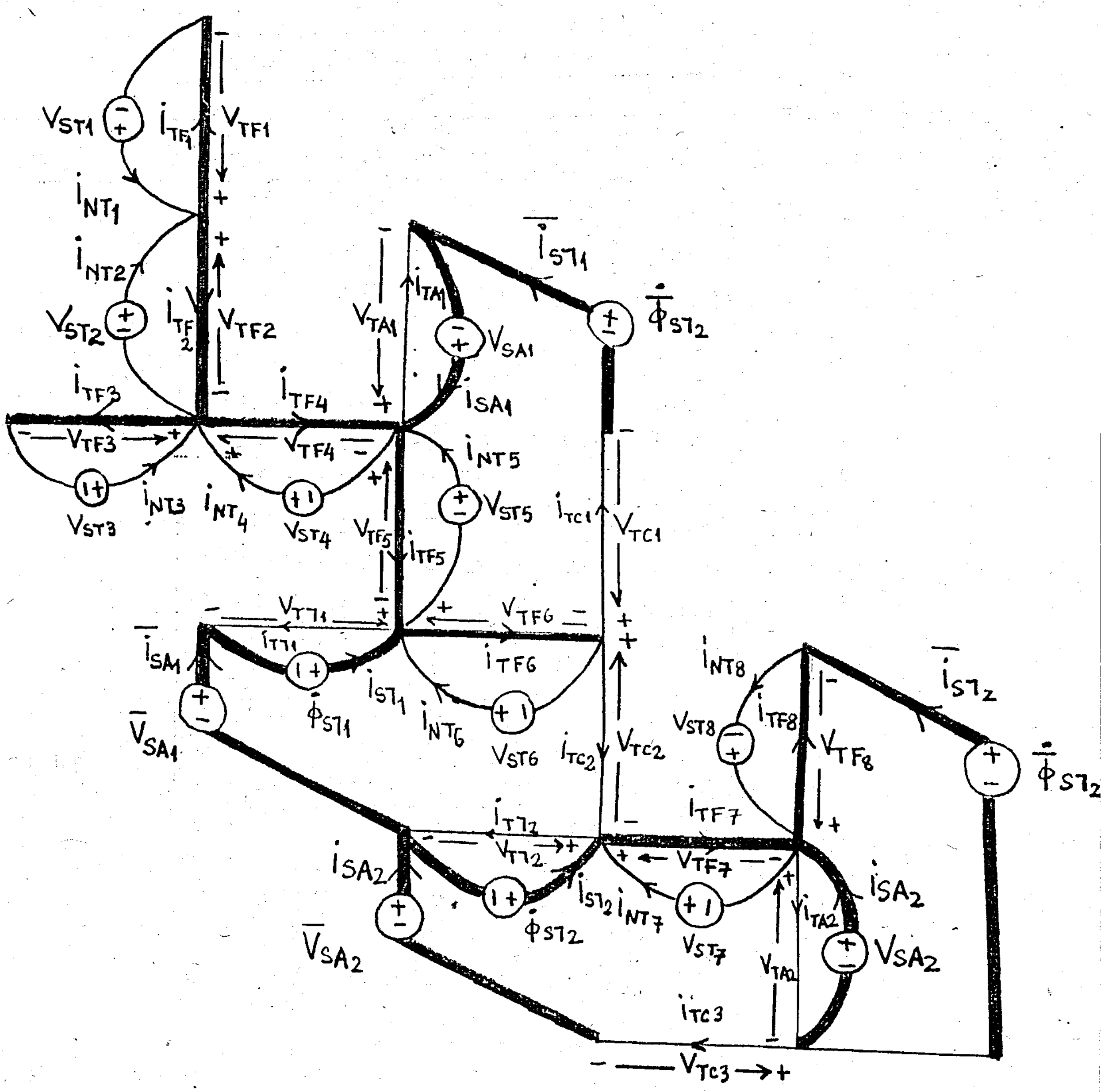


Figure 15- Augmenting a voltage source across each transistor branch which is in the forest in Fig.14.

The fundamental loop equations can be rewritten in the matrix form as:

$$
\left[\begin{array}{cccccccccccccccc|cccccccccccccccc}
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
-1&0&0&0&0&0&0&0&1&1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&-1&1&0&0&0&0&0&1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&1&0&-1&0&0&0&0&0&0&1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0&0&0&-1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right]
\left[\begin{array}{c}
V_{SA1}\\ V_{SA2}\\ \dot{\Phi}_{ST1}\\ \dot{\Phi}_{ST2}\\ V_{ST1}\\ V_{ST2}\\ V_{ST3}\\ V_{ST4}\\ V_{ST5}\\ V_{ST6}\\ V_{ST7}\\ V_{ST8}\\ \bar{V}_{SA1}\\ \bar{V}_{SA2}\\ \dot{\bar{\Phi}}_{ST1}\\ \dot{\bar{\Phi}}_{ST2}\\ \hdashline V_{TA1}\\ V_{TA2}\\ V_{TT1}\\ V_{TT2}\\ V_{TF1}\\ V_{TF2}\\ V_{TF3}\\ V_{TF4}\\ V_{TF5}\\ V_{TF6}\\ V_{TF7}\\ V_{TF8}\\ V_{TC1}\\ V_{TC2}\\ V_{TC3}\\ V_{TC4}
\end{array}\right]
=
\left[\begin{array}{c}
0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0
\end{array}\right]
\qquad (70)
$$

$-\underset{\sim}{B}_{TA}$ $-\underset{\sim}{B}_{TT}$ $-\underset{\sim}{B}_{T}$ $-\underset{\sim}{\bar{B}}_{TA}$ $-\underset{\sim}{\bar{B}}_{TT}$

Eq(70) can be expressed in compact form as:

$$\left[\begin{array}{ccccc:cccc} -\underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I} & \underset{\sim}{0} \\ -\underset{\sim}{B}_{TA} & -\underset{\sim}{B}_{T1} & -\underset{\sim}{B}_{T} & -\underset{\sim}{\bar{B}}_{TA} & -\underset{\sim}{\bar{B}}_{T1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{V}_{SA} \\ \underset{\sim}{\dot{\Phi}}_{S1} \\ \underset{\sim}{V}_{ST} \\ \underset{\sim}{\bar{V}}_{SA} \\ \underset{\sim}{\dot{\bar{\Phi}}}_{S1} \\ \hdashline \underset{\sim}{V}_{TA} \\ \underset{\sim}{V}_{T1} \\ \underset{\sim}{V}_{TF} \\ \underset{\sim}{V}_{TC} \end{array}\right] = \underset{\sim}{0} \tag{71}$$

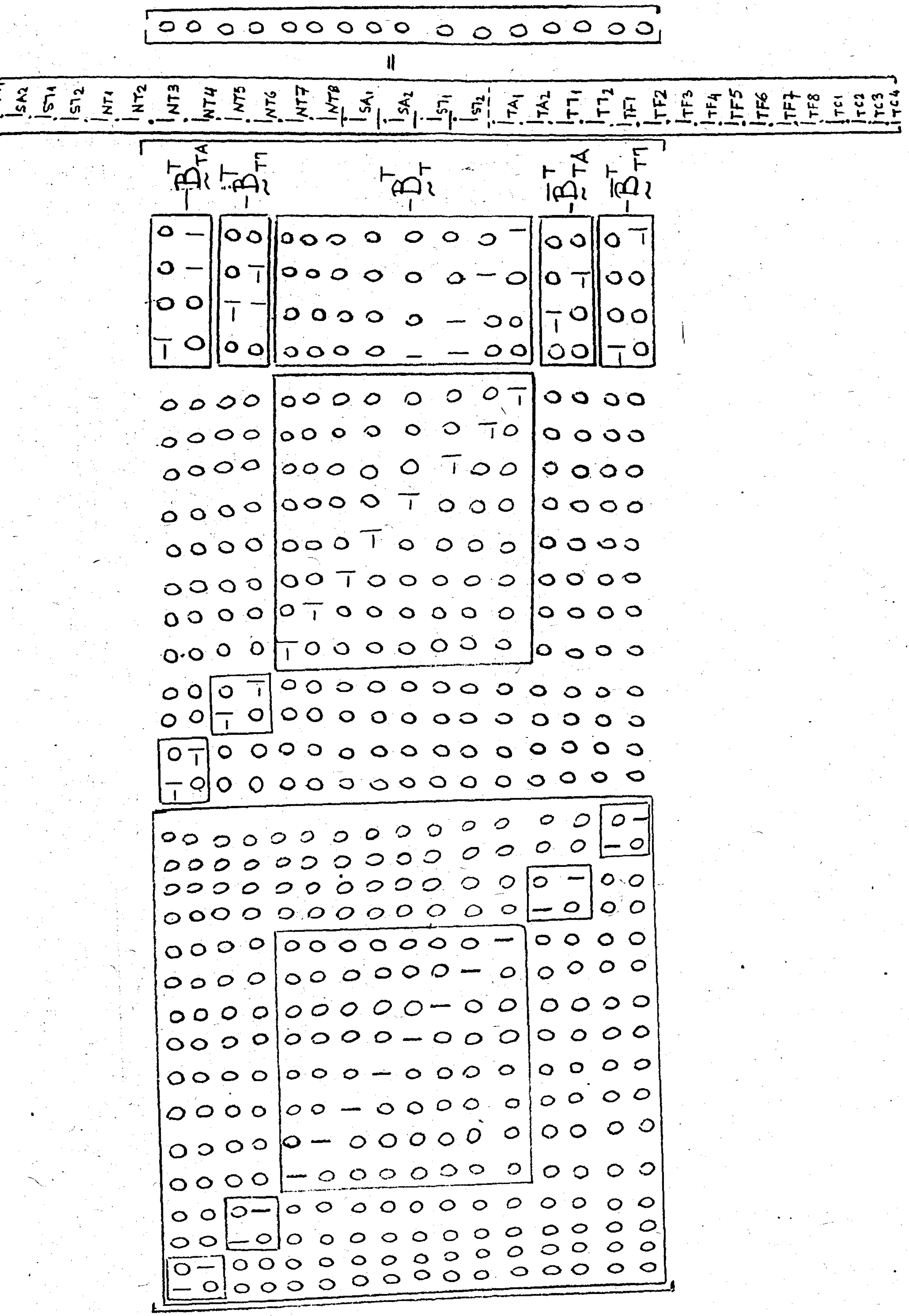
or,

$$\left[\begin{array}{ccccc:c} -\underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \\ \underset{\sim}{0} & -\underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I} \\ \underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{I} & \underset{\sim}{0} & \underset{\sim}{0} & \\ -\underset{\sim}{B}_{TA} & -\underset{\sim}{B}_{T1} & -\underset{\sim}{B}_{T} & -\underset{\sim}{\bar{B}}_{TA} & -\underset{\sim}{\bar{B}}_{T1} & \end{array}\right] \left[\begin{array}{c} \underset{\sim}{V}_{SA} \\ \underset{\sim}{\dot{\Phi}}_{S1} \\ \underset{\sim}{V}_{ST} \\ \underset{\sim}{\bar{V}}_{SA} \\ \underset{\sim}{\dot{\bar{\Phi}}}_{S1} \\ \hdashline \underset{\sim}{V}_{TA} \\ \underset{\sim}{V}_{T1} \\ \underset{\sim}{V}_{TF} \\ \underset{\sim}{V}_{TC} \end{array}\right] = \underset{\sim}{0} \tag{72}$$

The fundamental cut-set equations then become:

$$\begin{aligned}
&i_{SA_1} - i_{TA_1} - i_{TC1} = 0 \\
&i_{SA_2} - i_{TA_2} + i_{TC3} + i_{TC4} = 0 \\
&i_{ST_1} - i_{TT_1} - i_{TC2} = 0 \\
&i_{ST_2} - i_{TT_2} + i_{TC2} - i_{TC3} = 0 \\
&i_{NT_1} - i_{TF_1} = 0 \\
&i_{NT_2} - i_{TF_2} = 0 \\
&i_{NT3} - i_{TF_3} = 0 \\
&i_{NT4} - i_{TF4} = 0 \\
&i_{NT5} - i_{TF5} + i_{TC1} = 0 \\
&i_{NT6} - i_{TF6} + i_{TC1} + i_{TC2} = 0 \\
&i_{NT7} - i_{TF7} + i_{TC3} = 0 \\
&i_{NT8} - i_{TF8} - i_{TC4} = 0 \\
&\bar{i}_{SA_1} - i_{TC_2} = 0 \\
&\bar{i}_{SA_2} - i_{TC3} = 0 \\
&\bar{i}_{ST_1} - i_{TC1} = 0 \\
&\bar{i}_{ST_2} - i_{TC4} = 0
\end{aligned} \qquad (73)$$

The set of equations in(73) can be put into matrix form as:



Each one of the fundamental cut-set equation in (73) associates with a forest branch. Since the augmented circuit in Fig.15 contains 16 forest branches, there exists 16 fundamental cut-set equations in(73).

Eq(74) can be expressed in compact form as:

$$\left[\begin{array}{ccccc:cccc} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & -\mathbf{B}^{T}_{TA} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & -\mathbf{B}^{T}_{T7} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & -\mathbf{B}^{T}_{T} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{B}^{T}_{TA} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{B}^{T}_{T7} \end{array}\right] \left[\begin{array}{c} \mathbf{i}_{SA} \\ \mathbf{i}_{S7} \\ \mathbf{i}_{NT} \\ \hline \mathbf{i}_{SA} \\ \hline \mathbf{i}_{S7} \\ \hdashline \mathbf{i}_{TA} \\ \mathbf{i}_{T7} \\ \mathbf{i}_{TF} \\ \mathbf{i}_{TC} \end{array}\right] = \mathbf{0} \tag{75}$$

or,

$$\left[\begin{array}{c:cccc} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & -\mathbf{B}^{T}_{TA} \\ & \mathbf{0} & -\mathbf{I} & \mathbf{0} & -\mathbf{B}^{T}_{T7} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & -\mathbf{B}^{T}_{T} \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{B}^{T}_{TA} \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{B}^{T}_{T7} \end{array}\right] \left[\begin{array}{c} \mathbf{i}_{SA} \\ \mathbf{i}_{S7} \\ \mathbf{i}_{NT} \\ \hline \mathbf{i}_{SA} \\ \hline \mathbf{i}_{S7} \\ \hdashline \mathbf{i}_{TA} \\ \mathbf{i}_{T7} \\ \mathbf{i}_{TF} \\ \mathbf{i}_{TC} \end{array}\right] = \mathbf{0} \tag{76}$$

where

$$\underset{\sim}{\dot{i}}_{SA} = \left[\dot{i}_{SA_1} \quad \dot{i}_{SA_2} \right]^T \tag{77}$$

$$\underset{\sim}{\dot{i}}_{S7} = \left[\dot{i}_{S7_1} \quad \dot{i}_{S7_2} \right]^T \tag{78}$$

denote the currents through voltage sources in N that are connected across transistor branches and the currents through flux sources that are connected across transistor branches, respectively.

$$\underset{\sim}{\bar{\dot{i}}}_{SA} = \left[\underset{\sim}{\bar{\dot{i}}}_{SA_1} \quad \underset{\sim}{\bar{\dot{i}}}_{SA_2} \right]^T \tag{79}$$

$$\underset{\sim}{\bar{\dot{i}}}_{S7} = \left[\underset{\sim}{\bar{\dot{i}}}_{S7_1} \quad \underset{\sim}{\bar{\dot{i}}}_{S7_2} \right]^T \tag{80}$$

denote the currents of voltage sources in N that form loops with transistor branches and the currents of flux sources that form loops with transistor branches, respectively.

$$\underset{\sim}{\dot{i}}_{TA} = \left[\underset{\sim}{\dot{i}}_{TA_1} \quad \underset{\sim}{\dot{i}}_{TA_2} \right]^T \tag{81}$$

$$\underset{\sim}{\dot{i}}_{T7} = \left[\dot{i}_{T7_1} \quad \dot{i}_{T7_2} \right]^T \tag{82}$$

denote the currents of transistor branches which have a voltage source in N across their terminals and the currents of transistor branches which have a flux source across their terminals, respectively.

$$\underset{\sim}{\dot{i}}_{TF} = \left[\dot{i}_{TF1} \quad \dot{i}_{TF2} \quad \dot{i}_{TF3} \quad \dot{i}_{TF4} \quad \dot{i}_{TF5} \quad \dot{i}_{TF6} \quad \dot{i}_{TF7} \quad \dot{i}_{TF8} \right] \tag{83}$$

$$\underset{\sim}{\dot{i}}_{TC} = \left[\dot{i}_{TC1} \quad \dot{i}_{TC2} \quad \dot{i}_{TC3} \quad \dot{i}_{TC4} \right] \tag{84}$$

denote the currents of transistor branches that are in the forest, and the currents of transistor branches that are in the co-forest.

$$\dot{\underset{\sim}{i}}_{NT} = \left[\dot{i}_{NT1} \; \dot{i}_{NT2} \; \dot{i}_{NT3} \; \dot{i}_{NT4} \; \dot{i}_{NT5} \; \dot{i}_{NT6} \; \dot{i}_{NT7} \; \dot{i}_{NT8} \right]^T \tag{85}$$

denotes the currents of augmenting sources connected across transistor branches.

It should be noticed that the rigth-hand side of the coefficient matrix in(76) is simply the transpose of the left-hand side of the coefficient matrix in(72). This justifies that once the fundamental loop equations have been derived, the fundamental cut-set equations can be written very easily.

The currents of the augmenting sources are obtained from the third bloc row of(76) as:

$$\dot{\underset{\sim}{i}}_{NT} = \dot{\underset{\sim}{i}}_{TF} + \underset{\sim}{B}_T^T \, \underset{\sim}{i}_{TC} \tag{86}$$

Using constitutive relations,

$$\dot{\underset{\sim}{i}}_{TF} = \underset{\sim}{f}_{TF}(V_{TF}, V_{TC}, V_{TA}, V_{T7}, \dot{\underset{\sim}{V}}_{TF}, \dot{\underset{\sim}{V}}_{TC}, \dot{\underset{\sim}{V}}_{TA}, \dot{\underset{\sim}{V}}_{T7}) \tag{87}$$

$$\dot{\underset{\sim}{i}}_{TC} = \underset{\sim}{f}_{TC}(\underset{\sim}{V}_{TF}, \underset{\sim}{V}_{TC}, \underset{\sim}{V}_{TA}, \underset{\sim}{V}_{T7}, \dot{\underset{\sim}{V}}_{TF}, \dot{\underset{\sim}{V}}_{TC}, \dot{\underset{\sim}{V}}_{TA}, \dot{\underset{\sim}{V}}_{T7}) \tag{88}$$

of the transistor in(86)

$$\begin{aligned} \dot{\underset{\sim}{i}}_{NT} &= \underset{\sim}{f}_{TF}(\underset{\sim}{V}_{TF}, \underset{\sim}{V}_{TC}, \underset{\sim}{V}_{TA}, \underset{\sim}{V}_{T7}, \dot{\underset{\sim}{V}}_{TF}, \dot{\underset{\sim}{V}}_{TC}, \dot{\underset{\sim}{V}}_{TA}, \dot{\underset{\sim}{V}}_{T7}) \\ &+ \underset{\sim}{B}_T^T \, \underset{\sim}{f}_{TC}(\underset{\sim}{V}_{TF}, \underset{\sim}{V}_{TC}, \underset{\sim}{V}_{TA}, \underset{\sim}{V}_{T7}, \dot{\underset{\sim}{V}}_{TF}, \dot{\underset{\sim}{V}}_{TC}, \dot{\underset{\sim}{V}}_{TA}, \dot{\underset{\sim}{V}}_{T7}) \end{aligned} \tag{89}$$

is obtained. Rewriting the fundamental loop equations in(71)

$$\underset{\sim}{V}_{TA} = \underset{\sim}{V}_{SA}$$

$$\underset{\sim}{V}_{T7} = \dot{\underset{\sim}{\Phi}}_{S7}$$

$$\underset{\sim}{V}_{TF} = \underset{\sim}{V}_{ST}$$

$$\underset{\sim}{V}_{TC}=\underset{\sim}{B}_{TA}\underset{\sim}{V}_{SA}+\underset{\sim}{B}_{T1}\dot{\underset{\sim}{\Phi}}_{S1}+\underset{\sim}{B}_{T}\underset{\sim}{V}_{ST}+\underset{\sim}{\bar{B}}_{TA}\underset{\sim}{\bar{V}}_{SA}+\underset{\sim}{\bar{B}}_{T1}\dot{\underset{\sim}{\bar{\Phi}}}_{S1} \quad (90)$$

are obtained. Differentiating the equations in(90) with respect to time

$$\dot{\underset{\sim}{V}}_{TA}=\dot{\underset{\sim}{V}}_{SA}$$

$$\dot{\underset{\sim}{V}}_{T1}=\ddot{\underset{\sim}{\Phi}}_{S1}$$

$$\dot{\underset{\sim}{V}}_{TF}=\dot{\underset{\sim}{V}}_{ST}$$

$$\dot{\underset{\sim}{V}}_{TC}=\underset{\sim}{B}_{TA}\dot{\underset{\sim}{V}}_{SA}+\underset{\sim}{B}_{T1}\ddot{\underset{\sim}{\Phi}}_{S1}+\underset{\sim}{B}_{T}\dot{\underset{\sim}{V}}_{ST}+\underset{\sim}{\bar{B}}_{TA}\dot{\underset{\sim}{\bar{V}}}_{SA}+\underset{\sim}{\bar{B}}_{T1}\ddot{\underset{\sim}{\bar{\Phi}}}_{S1} \quad (91)$$

are obtained. Eqs(90) and (91) substituted in (89) gives

$$\begin{aligned}\dot{\underset{\sim}{I}}_{NT}=&\underset{\sim}{f}_{TF}(\underset{\sim}{V}_{ST},\underset{\sim}{B}_{TA}\underset{\sim}{V}_{SA}+\underset{\sim}{B}_{T1}\dot{\underset{\sim}{\Phi}}_{S1}+\underset{\sim}{B}_{T}\underset{\sim}{V}_{ST}+\underset{\sim}{\bar{B}}_{TA}\underset{\sim}{\bar{V}}_{SA}+\underset{\sim}{\bar{B}}_{T1}\dot{\underset{\sim}{\bar{\Phi}}}_{S1},\\&\underset{\sim}{V}_{SA},\dot{\underset{\sim}{\Phi}}_{S1},\dot{\underset{\sim}{V}}_{ST},\underset{\sim}{B}_{TA}\dot{\underset{\sim}{V}}_{SA}+\underset{\sim}{B}_{T1}\ddot{\underset{\sim}{\Phi}}_{S1}+\underset{\sim}{B}_{T}\dot{\underset{\sim}{V}}_{ST}+\underset{\sim}{\bar{B}}_{TA}\dot{\underset{\sim}{\bar{V}}}_{SA}+\underset{\sim}{\bar{B}}_{T1}\ddot{\underset{\sim}{\bar{\Phi}}}_{S1},\dot{\underset{\sim}{V}}_{SA},\ddot{\underset{\sim}{\Phi}}_{S1})\\&+\underset{\sim}{B}_{T}^{T}\underset{\sim}{f}_{TC}(\underset{\sim}{V}_{ST},\underset{\sim}{B}_{TA}\underset{\sim}{V}_{SA}+\underset{\sim}{B}_{T1}\dot{\underset{\sim}{\Phi}}_{S1}+\underset{\sim}{B}_{T}\underset{\sim}{V}_{ST}+\underset{\sim}{\bar{B}}_{TA}\underset{\sim}{\bar{V}}_{SA}+\underset{\sim}{\bar{B}}_{T1}\dot{\underset{\sim}{\bar{\Phi}}}_{S1},\\&\underset{\sim}{V}_{SA},\dot{\underset{\sim}{\Phi}}_{S1},\dot{\underset{\sim}{V}}_{ST},\underset{\sim}{B}_{TA}\dot{\underset{\sim}{V}}_{SA}+\underset{\sim}{B}_{T1}\ddot{\underset{\sim}{\Phi}}_{S1}+\underset{\sim}{B}_{T}\dot{\underset{\sim}{V}}_{ST}+\underset{\sim}{B}_{TA}\dot{\underset{\sim}{\bar{V}}}_{SA}+\underset{\sim}{B}_{T1}\ddot{\underset{\sim}{\bar{\Phi}}}_{S1},\dot{\underset{\sim}{V}}_{SA},\ddot{\underset{\sim}{\Phi}}_{S1})\end{aligned} \quad (92)$$

Eq(92) will be written as

$$\dot{\underset{\sim}{I}}_{NT}=\underset{\sim}{f}_{T}(\underset{\sim}{V}_{ST},\underset{\sim}{V}_{SA},\underset{\sim}{\bar{V}}_{SA},\dot{\underset{\sim}{\Phi}}_{S1},\dot{\underset{\sim}{\bar{\Phi}}}_{S1},\dot{\underset{\sim}{V}}_{ST},\dot{\underset{\sim}{V}}_{SA},\dot{\underset{\sim}{\bar{V}}}_{SA},\ddot{\underset{\sim}{\Phi}}_{S1},\ddot{\underset{\sim}{\bar{\Phi}}}_{S1}) \quad (93)$$

It should be noticed that Eq(93) contain double derivative of augmenting sources when there exists loops of flux sources with transistor branches in the subcircuit of $N_{\urcorner}$.

In the case that there are no loops of transistor branches with or without sources in $N_{\urcorner}$, Eq(93) will reduce to the constitutive relations

$$\dot{\underset{\sim}{i}}_T = \underset{\sim}{f}_T(\underset{\sim}{V}_T, \dot{\underset{\sim}{V}}_T) \tag{94}$$

of the transistors.

II.3.3. OTHER NONLINEAR ELEMENTS

A systematic procedure for the removal of nonlinearities other than flux controlled inductors and transistors can similarly be developed. This has already been done in reference [9] . Then, the complementary variables of the augmenting sources associated with nonlinear elements other than the flux controlled inductors and transistors can easily be obtained. The complementary variables of the augmenting sources associated with these nonlinearities are given as:

$\dot{\underset{\sim}{i}}_{NG}$ Currents of the augmenting voltage sources connected properly across the voltage controlled resistors,

$\dot{\underset{\sim}{i}}_{NC}$ Currents of the augmenting voltage sources connected properly across the voltage controlled capacitors,

$\underset{\sim}{V}_{N\Im}$ Voltages of the augmenting charge sources connected properly in series with charge controlled capacitors,

$\underset{\sim}{V}_{NR}, \underset{\sim}{V}_{NL}$ Voltages of the augmenting current sources connected properly in series with current controlled resistors and inductors, respectively.

II.4. ERROR FUNCTIONS

In this section, the error function defined by Eq(33) in section II.2 will be presented explicitly, and a scalar error function to be minimized will be derived.

The error function to be made zero, can be defined as:

$$\underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \dot{\underset{\sim}{i}}_{S1} \\ \dot{\underset{\sim}{i}}_{ST} \\ \dot{\underset{\sim}{i}}_{SG} \\ \dot{\underset{\sim}{i}}_{SC} \\ \underset{\sim}{V}_{SD} \\ \underset{\sim}{V}_{SR} \\ \underset{\sim}{V}_{SL} \end{bmatrix} = \begin{bmatrix} \dot{\underset{\sim}{i}}_{N1} \\ \dot{\underset{\sim}{i}}_{NT} \\ \dot{\underset{\sim}{i}}_{NG} \\ \dot{\underset{\sim}{i}}_{NC} \\ \underset{\sim}{V}_{ND} \\ \underset{\sim}{V}_{NR} \\ \underset{\sim}{V}_{NL} \end{bmatrix} - \begin{bmatrix} \dot{\underset{\sim}{i}}_{L1} \\ \dot{\underset{\sim}{i}}_{LT} \\ \dot{\underset{\sim}{i}}_{LG} \\ \dot{\underset{\sim}{i}}_{LC} \\ \underset{\sim}{V}_{LD} \\ \underset{\sim}{V}_{LR} \\ \underset{\sim}{V}_{LL} \end{bmatrix} \triangleq \underset{\sim}{y}_N - \underset{\sim}{y}_L \tag{95}$$

where $\underset{\sim}{y}_L$ contains the complementary variables of the augmenting sources calculated from the linear network $\hat{N}$, the variables in $\underset{\sim}{y}_N$ are defined in the previous section. Thus, $\underset{\sim}{\varepsilon}(t)$ corresponds to the complementary variables of the augmenting sources in the original circuit N.

Defining

$$\underset{\sim}{y}_S \triangleq \left[\underset{\sim}{\phi}_{S1}^T \; \underset{\sim}{V}_{ST}^T \; \underset{\sim}{V}_{SG}^T \; \underset{\sim}{V}_{SC}^T \; \underset{\sim}{q}_{SD}^T \; \dot{\underset{\sim}{i}}_{SR}^T \; \dot{\underset{\sim}{i}}_{SL}^T \right]^T \tag{96}$$

and using the expressions derived in section II.3, $\underset{\sim}{y}_N$ can be expressed as

$$
\underset{\sim}{y}_N = \begin{bmatrix}
\hat{\underset{\sim}{f}}_{\reflectbox{L}}(\underset{\sim}{\phi}_{S\reflectbox{L}}) \\
\hat{\underset{\sim}{f}}_T(\underset{\sim}{v}_{ST}, \dot{\underset{\sim}{\phi}}_{S\reflectbox{L}}, \underset{\sim}{v}_{SA}, \dot{\underset{\sim}{v}}_{ST}, \ddot{\underset{\sim}{\phi}}_{S\reflectbox{L}}, \dot{\underset{\sim}{v}}_{SA}) \\
\hat{\underset{\sim}{f}}_G(\underset{\sim}{v}_{ST}, \dot{\underset{\sim}{\phi}}_{S\reflectbox{L}}, \underset{\sim}{v}_{SG}, \underset{\sim}{v}_{SA}) \\
\hat{\underset{\sim}{C}}(\underset{\sim}{v}_{ST}, \dot{\underset{\sim}{\phi}}_{S\reflectbox{L}}, \underset{\sim}{v}_{SG}, \underset{\sim}{v}_{SC}, \underset{\sim}{v}_{SA})\left[\dot{\underset{\sim}{v}}_{ST}^T \; \ddot{\underset{\sim}{\phi}}_{S\reflectbox{L}}^T \; \dot{\underset{\sim}{v}}_{SG}^T \; \dot{\underset{\sim}{v}}_{SC}^T \; \dot{\underset{\sim}{v}}_{SA}^T\right]^T \\
\hat{\underset{\sim}{f}}_{\reflectbox{C}}(\underset{\sim}{q}_{S\reflectbox{C}}) \\
\hat{\underset{\sim}{f}}_R(\dot{\underset{\sim}{q}}_{S\reflectbox{C}}, \underset{\sim}{i}_{SR}, \underset{\sim}{i}_{SA}) \\
\hat{\underset{\sim}{L}}(\dot{\underset{\sim}{q}}_{S\reflectbox{C}}, \underset{\sim}{i}_{SR}, \underset{\sim}{i}_{SL}, \underset{\sim}{i}_{SA})\left[\ddot{\underset{\sim}{q}}_{S\reflectbox{C}}^T \; \dot{\underset{\sim}{i}}_{SR}^T \; \dot{\underset{\sim}{i}}_{SL}^T \; \dot{\underset{\sim}{i}}_{SA}^T\right]^T
\end{bmatrix} \tag{97}
$$

which is given in reference [9] . The additional variables in Eq(97) are defined as follows:

$\underset{\sim}{v}_{SG}$, $\underset{\sim}{v}_{SC}$, $\underset{\sim}{q}_{SC}$ denote voltages of the augmenting voltage sources connected properly across the voltage controlled resistors, voltages of the augmenting voltage sources connected properly across the voltage controlled capacitors, and charges of the augmenting charge sources connected properly in series with the charge controlled capacitors, respectively.

$\dot{\underset{\sim}{i}}_{SR}, \dot{\underset{\sim}{i}}_{SL}, \dot{\underset{\sim}{i}}_{SA}$ denote currents of the augmenting current sources connected properly in series with current controlled resistors, currents of the augmenting current sources connected properly in series with current controlled inductors, and currents of the current sources present in N, respectively. Defining

$$\hat{\underset{\sim}{u}}_A \triangleq \left[v_{SA}^T \quad \dot{i}_{SA}^T \right]^T \tag{98}$$

then

$$\underset{\sim}{u}_A = \left[\hat{\underset{\sim}{u}}_A^T \quad \dot{\hat{\underset{\sim}{u}}}_A^T \right]^T \tag{99}$$

The final expression

$$\underset{\sim}{y}_N = \underset{\sim}{f}(\underset{\sim}{u}_S, \dot{\underset{\sim}{u}}_S, \ddot{\underset{\sim}{u}}_S, \underset{\sim}{u}_A) \tag{100}$$

describing the nonlinear elements is obtained.

Note that $\underset{\sim}{y}_N$ and $\underset{\sim}{y}_L$, therefore $\underset{\sim}{\varepsilon}$ only depend on the waveform selected for $\underset{\sim}{u}_S$. Therefore a scalar error function can be defined as:

$$P(\underset{\sim}{u}_S) \triangleq \int_0^\infty \underset{\sim}{\varepsilon}^T(t)\,\underset{\sim}{\varepsilon}(t)\,dt \tag{101}$$

which also only depends on the waveform $\underset{\sim}{u}_S$. The problem now becomes to find $\underset{\sim}{u}_S$ such that $P(\underset{\sim}{u}_S) = 0$.

Since it is impossible to optimize over all possible waveforms, and as we are interested in the steady-state periodic solution the circuit is assumed to have a periodic solution of period T.

The $\underset{\sim}{u}_S$ can be taken to be of the form:

$$\underset{\sim}{u}_s(t) = \sum_{k=0}^{M} \underset{\sim}{x}_k \operatorname{Cos}\omega kt + \sum_{k=1}^{M} \underset{\sim}{x}_k^* \operatorname{Sin}k\omega t \tag{102}$$

where the coefficients $\underset{\sim}{x}_k, \underset{\sim}{x}_k^*$ and M are to be defermined, and $\omega = \frac{2\pi}{T}$. Defining

$$\underset{\sim}{x} \triangleq \left[\underset{\sim}{x}_0^T \; \underset{\sim}{x}_1^T \ldots \underset{\sim}{x}_M^T \underset{\sim}{x}_1^{*T} \ldots \; . \underset{\sim}{x}_M^{*T} \right]^T \tag{103}$$

The scalar error function now becomes

$$P(\underset{\sim}{x}) = \int_0^T \underset{\sim}{\varepsilon}^T(t) \underset{\sim}{\varepsilon}(t)\, dt \tag{104}$$

and the question is to determine the coefficients $\underset{\sim}{x}$ such that $P(\underset{\sim}{x}) = 0$.

III. CALCULATION OF THE RESPONSE OF THE LINEAR NETWORK

In this chapter, first the procedure for obtaining the response of the linear network is introduced, and then the formulation and solution of the equations describing linear network are presented.

Consider the Eq(102) which can be written in the form

$$\underset{\sim}{u}_s(t) = \mathrm{Re}\left\{\sum_{k=0}^{M} (\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega kt}\right\} \tag{105}$$

where $\underset{\sim}{x}_0^* = \underset{\sim}{0}$ and Re stands for "real part of".

To calculate $\underset{\sim}{y}_L(t)$ first note that $\underset{\sim}{y}_L(t)$ is the response of a linear network. Therefore the linear network can be solved in frequency domain for each frequency ωk seperately to give

$$\underset{\sim}{y}_L(t) = \mathrm{Re}\left\{\sum_{k=0}^{M} \underset{\sim}{y}_{Lk}(j\omega k)\, e^{j\omega kt}\right\} \tag{106}$$

where $\underset{\sim}{y}_{Lk}(j\omega k)$ denotes the phasor response of the linear network at frequency ωk.

For each frequency ωk, the Mixed Nodal Tableau (MNT) equations of the linear network can be written as:

$$\underset{\sim}{T}_k(j\omega k)\begin{bmatrix}\underset{\sim}{y}_k(j\omega k)\\ \underset{\sim}{y}_{Lk}(j\omega k)\end{bmatrix} = \begin{bmatrix}\underset{\sim}{A}_1 & \underset{\sim}{A}_2\end{bmatrix}\begin{bmatrix} j\omega k(\underset{\sim}{x}_{k_1} - jx_{k_1}^*)\\ (\underset{\sim}{x}_{k_2} - jx_{k_2}^*)\end{bmatrix} + \underset{\sim}{B}\,\underset{\sim}{u}_{A_k}(j\omega k) \tag{107}$$

for $k \in \{0,1,2,\dots,M\}$ where

$\underset{\sim}{T}_k(j\omega k)$: the MNT matrix at frequency ωk.

$\underset{\sim}{y}_{L_k}(j\omega k)$: contains the phasors associated with the complementary variables of the augmenting sources.

$\underset{\sim}{y}_k(j\omega k)$: contains the phasors of the other variables (undesired) in the network.

$\underset{\sim}{u}_{A_k}(j\omega k)$: are the phasor values of the actual sources at ωk.

$\underset{\sim}{x}_{k_1}, \underset{\sim}{x}^*_{k_1}$: coefficients of augmenting flux and charge sources at ωk.

$\underset{\sim}{x}_{k_2}, \underset{\sim}{x}^*_{k_2}$: coefficients of the other augmenting sources at ωk.

$\underset{\sim}{A}_1$: incidence matrix of the augmenting flux and charge sources,

$\underset{\sim}{A}_2$: incidence matrix of the augmenting voltage and current sources,

$\underset{\sim}{B}$: incidence matrix of the actual sources,

Eq(107) can be written as:

$$\underset{\sim}{T}_k(j\omega k)\begin{bmatrix}\underset{\sim}{y}_k(j\omega k)\\ \underset{\sim}{y}_{L_k}(j\omega k)\end{bmatrix} = \begin{bmatrix} j\omega k\underset{\sim}{A}_1 & \underset{\sim}{A}_2\end{bmatrix}\begin{bmatrix}\underset{\sim}{x}_{k_1} - j\underset{\sim}{x}^*_{k_1}\\ \underset{\sim}{x}_{k_2} - j\underset{\sim}{x}^*_{k_2}\end{bmatrix} + \underset{\sim}{B}\,\underset{\sim}{u}_{A_k}(j\omega k) \tag{108}$$

for $k \in \{0,1,2,\ldots,M\}$.

Defining a new matrix $\underset{\sim}{A}_k(j\omega k)$ as

$$\underset{\sim}{A}_k(j\omega k) = \left[j\omega k \underset{\sim}{A}_1 \;\; \underset{\sim}{A}_2 \right]$$

which depends explicitly on $j\omega k$, and

$$\underset{\sim}{x}_k - j\underset{\sim}{x}_k^* = \begin{bmatrix} \underset{\sim}{x}_{k_1} - j x_{k_1}^* \\ x_{k_2} - j\underset{\sim}{x}_{k_2}^* \end{bmatrix}$$

as the coefficients of the augmenting sources, Eq(108) can be put into the following form

$$\underset{\sim}{T}_k(j\omega k)\begin{bmatrix} \underset{\sim}{y}_k(j\omega k) \\ \underset{\sim}{y}_{L_k}(j\omega k) \end{bmatrix} = \underset{\sim}{A}_k(j\omega k)\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*) + \underset{\sim}{B}\, u_{A_k}(j\omega k) \tag{109}$$

for $k \in \{0,1,2,\ldots,M\}$.

To solve for $\underset{\sim}{y}_{L_k}(j\omega k)$ in (109) apply superposition and proceed as follows:

(i) Solve the MNT Eq(109) for $\underset{\sim}{u}_{A_k}(j\omega k) = \underset{\sim}{0}$, i.e., solve

$$\underset{\sim}{T}_k(j\omega k)\begin{bmatrix} \underset{\sim}{Y}_k(j\omega k) \\ \underset{\sim}{Y}_{L_k}(j\omega k) \end{bmatrix} = \underset{\sim}{A}_k(j\omega k) \tag{110}$$

for $k \in \{0,1,2,\quad,M\}$.

To obtain the solution matrix $\begin{bmatrix} \underset{\sim}{Y}_k(j\omega k) \\ \underset{\sim}{Y}_{L_k}(j\omega k) \end{bmatrix}$, solve

$$\underset{\sim}{T}_k(j\omega k)\begin{bmatrix}\underset{\sim}{y}_k^i(j\omega k)\\ \underset{\sim}{y}_{L_k}^i(j\omega k)\end{bmatrix} = \underset{\sim}{A}_k(j\omega k)\,\underset{\sim}{e}_i \quad \text{for} \quad i=1,\dots\dots N \tag{111}$$

where N is the number of augmenting sources, and

$$\underset{\sim}{e}_i = \begin{array}{c} [0\ 0\ .\ .\ .\ .\ 0\ 1\ 0\ .\ .\ .\ .\ 0]^T \\ \uparrow \\ i^{th}\ \text{coulumn} \end{array}$$

Then, the solution matrix to(110) can be constructed as

$$\begin{bmatrix}\underset{\sim}{Y}_k(j\omega k)\\ \underset{\sim}{Y}_{L_k}(j\omega k)\end{bmatrix} = \begin{bmatrix}\underset{\sim}{y}_k^1(j\omega k) & \underset{\sim}{y}_k^2(j\omega k) & .\ .\ .\ , & \underset{\sim}{y}_k^N(j\omega k)\\ \underset{\sim}{y}_{L_k}^1(j\omega k) & \underset{\sim}{y}_{L_k}^2(j\omega k) & .\ .\ .\ . & \underset{\sim}{y}_{L_k}^N(j\omega k)\end{bmatrix} \tag{112}$$

Thus, the coulumns of the matrix $\underset{\sim}{Y}_{L_k}(j\omega k)$ contains the response of the linear network to unit augmenting sources.

(ii) The phasor solution due to augmenting sources then is

$$\begin{bmatrix}\underset{\sim}{\bar{y}}_k(j\omega k)\\ \underset{\sim}{\bar{y}}_{L_k}(j\omega k)\end{bmatrix} = \begin{bmatrix}\underset{\sim}{Y}_k(j\omega k)\\ \underset{\sim}{Y}_{L_k}(j\omega k)\end{bmatrix}(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*) \tag{113}$$

Thus, the linear network is solved only once for each frequency and, as $\underset{\sim}{x}_k, \underset{\sim}{x}_k^*$ change the solution $\underset{\sim}{\bar{y}}_{L_k}(j\omega k)$ is obtained by scaling (in the case that the period of the oscillations is known).

(iii) The phasor solution due to actual excitations is obtained from

$$\underset{\sim}{T}_k(j\omega k)\begin{bmatrix}\underset{\sim}{\bar{\bar{y}}}_k(j\omega k)\\ \bar{\bar{y}}_{Lk}(j\omega k)\end{bmatrix}=B\,\underset{\sim}{u}_{A_k}(j\omega k) \tag{114}$$

(iv) The desired solution is calculated as

$$\begin{bmatrix}\underset{\sim}{y}_k(j\omega k)\\ y_{Lk}(j\omega k)\end{bmatrix}=\begin{bmatrix}\underset{\sim}{\bar{y}}_k(j\omega k)\\ \underset{\sim}{\bar{y}}_{Lk}(j\omega k)\end{bmatrix}+\begin{bmatrix}\underset{\sim}{\bar{\bar{y}}}_k(j\omega k)\\ \bar{\bar{y}}_{Lk}(j\omega k)\end{bmatrix} \tag{115}$$

or

$$\begin{bmatrix}\underset{\sim}{y}_k(j\omega k)\\ \underset{\sim}{y}_{Lk}(j\omega k)\end{bmatrix}=\begin{bmatrix}\underset{\sim}{Y}_k(j\omega k)\\ \underset{\sim}{Y}_{Lk}(j\omega k)\end{bmatrix}(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)+\begin{bmatrix}\underset{\sim}{\bar{\bar{y}}}_k(j\omega k)\\ \underset{\sim}{\bar{\bar{y}}}_{Lk}(j\omega k)\end{bmatrix} \tag{116}$$

Hence,

$$\underset{\sim}{y}_{Lk}(j\omega k)=\underset{\sim}{Y}_{Lk}(j\omega k)(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)+\underset{\sim}{\bar{\bar{y}}}_{Lk}(j\omega k) \tag{117}$$

which simply is the phasor response of the linear network at frequency ωk.

Inserting from (117) into (106) to obtain

$$\underset{\sim}{y}_L(t)=\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{Y}_{Lk}(j\omega k)(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)+\underset{\sim}{\bar{\bar{y}}}_{Lk}(j\omega k)\right]e^{j\omega kt}\right\} \tag{118}$$

Eq(118) is the required expression for the response of the linear network in the time domain.

III.1. TABLEAU FORMULATION OF THE LINEAR NETWORK EQUATIONS

In this section, tableau formulation of the linear network equations is introduced, two different types of MNT formulation is presented in detail, their advantages and disadvantages are explained on several examples. In the first approach, the linear network equations are formulated in such a way that all the branch voltages, all the branch currents and all the node voltages are appeared to be unknowns. In the second method, the number of unknowns is decreased in such a way that the equations contain only the node voltages and the currents of the elements other than the resistors, capacitors, and independent current sources.

Let the node-to-branch incidence matrix associated with the graph of a given network be A_a. Then the reduced incidence matrix A is obtained by deleting the row corresponding to the datum node.

Let $\underset{\sim}{V}_n, \underset{\sim}{V}_b$ and $\underset{\sim}{I}_b$ be the node-to-datum voltages, branch voltages, and the branch currents of a given network, respectively.

The branch voltages are obtained from the node valtages by the equation

$$\underset{\sim}{V}_b = \underset{\sim}{A}^T \underset{\sim}{V}_n \qquad (119)$$

and the branch currents are related to each other by the equation

$$\underset{\sim}{A}\, \underset{\sim}{I}_b = \underset{\sim}{0} \qquad (120)$$

The Eqs(119) and (120) are the Kirchhoff's voltage law (KVL) and Kirchhoff's Current Law (KCL) for the given network. They are independent of the nature of the elements of the network.

The constitutive equations representing the behaviour of the elements can be expressed in the form of

$$\underset{\sim}{P}_b \underset{\sim}{V}_b + \underset{\sim}{Q}_b \underset{\sim}{I}_b = \underset{\sim}{W} \tag{121}$$

Suppose that the element being considered is an admittance, then choosing W = 0 and Q = -1 Eq(121) becomes

$$\underset{\sim}{P}_b \underset{\sim}{V}_b - \underset{\sim}{I}_b = \underset{\sim}{0} \tag{122}$$

where P_b represents the admittance. On the other hand, if the element being considered is an impedance, then choosing W = 0 and P = -1 Eq(121) becomes

$$-V_b + Q_b I_b = 0 \tag{123}$$

where Q_b represents the impedance.

Setting $P_b = 1$, $Q_b = 0$, $W = E$ the constitutive relation $V_b = E$ is obtained for an independent voltage source, and $Q_b = 1$, $P_b = 0$, $W = J$ the constitutive relation $I_b = J$ is obtained for an independent current source.

In order to be able to solve the system of Eqs(119), (120) and (121), obtain $\underset{\sim}{I}_b$ from Eq(3) as

$$\underset{\sim}{I}_b = \underset{\sim}{Q}_b^{-1} \underset{\sim}{W} - \underset{\sim}{Q}_b^{-1} \underset{\sim}{P}_b \underset{\sim}{V}_b \tag{124}$$

and replace $\underset{\sim}{V}_b$ in Eq(124) by Eq(119) to obtain

$$\underset{\sim}{I}_b = \underset{\sim}{Q}_b^{-1} \underset{\sim}{W} - \underset{\sim}{Q}_b^{-1} \underset{\sim}{P}_b \underset{\sim}{A}^T \underset{\sim}{V}_n \tag{125}$$

then multiply both sides of (125) by $\underset{\sim}{A}$ to obtain

$$\underset{\sim}{A}\,\underset{\sim}{I}_b = \underset{\sim}{A}\,\underset{\sim}{Q}_b^{-1}\underset{\sim}{w} - \underset{\sim}{A}\,\underset{\sim}{Q}_b^{-1}\underset{\sim}{P}_b\underset{\sim}{A}^T\underset{\sim}{V}_n = \underset{\sim}{0} \tag{126}$$

or,

$$\underset{\sim}{A}\,\underset{\sim}{Q}_b^{-1}\underset{\sim}{P}_b\underset{\sim}{A}^T\underset{\sim}{V}_n = \underset{\sim}{A}\,\underset{\sim}{Q}_b^{-1}\underset{\sim}{w} \tag{127}$$

The node-to-datum voltages can be obtained from(127) if and only if $\underset{\sim}{Q}_b$ is non-singular. In the case that $\underset{\sim}{Q}_b$ is a singular matrix then the solution for $\underset{\sim}{V}_n$ can not be obtained from(127). To overcome this difficulty Eqs(119),(120) and (121) can be put in a system of equations by adding the branch voltages and the currents as additional unknowns. Rewriting (119),(120) and(121) as follows:

$$\begin{aligned} \underset{\sim}{V}_b - \underset{\sim}{A}^T\underset{\sim}{V}_n &= \underset{\sim}{0} \\ \underset{\sim}{P}_b\underset{\sim}{V}_b + \underset{\sim}{Q}_b\underset{\sim}{I}_b &= \underset{\sim}{W} \\ \underset{\sim}{A}\,\underset{\sim}{I}_b &= \underset{\sim}{0} \end{aligned} \tag{128}$$

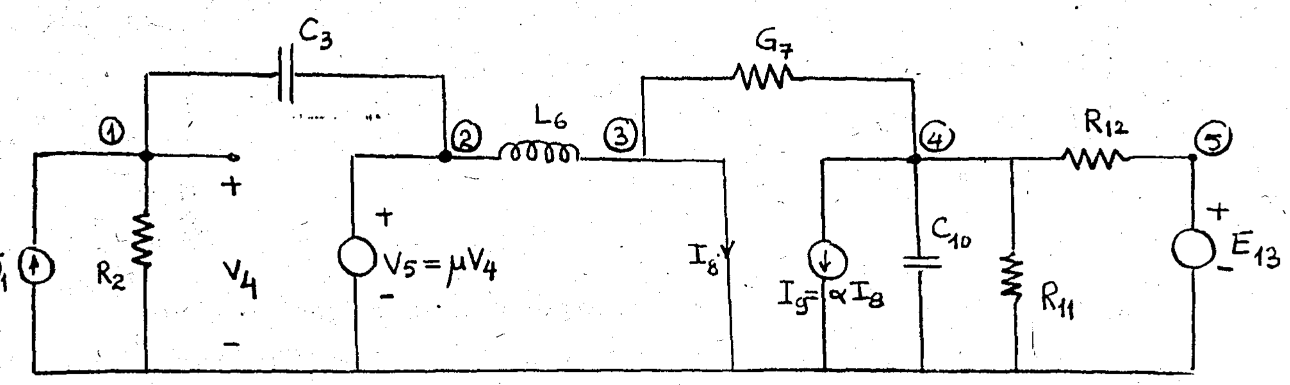
the system of equations to be solved can be expressed as

$$\left[\begin{array}{c:c:c} \underset{\sim}{I} & \underset{\sim}{0} & -\underset{\sim}{A}^T \\ \hdashline \underset{\sim}{P}_b & \underset{\sim}{Q}_b & \underset{\sim}{0} \\ \hdashline \underset{\sim}{0} & \underset{\sim}{A} & \underset{\sim}{0} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{V}_b \\ \hdashline \underset{\sim}{I}_b \\ \hdashline \underset{\sim}{V}_n \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{0} \\ \hdashline \underset{\sim}{w} \\ \hdashline \underset{\sim}{0} \end{array}\right] \tag{129}$$

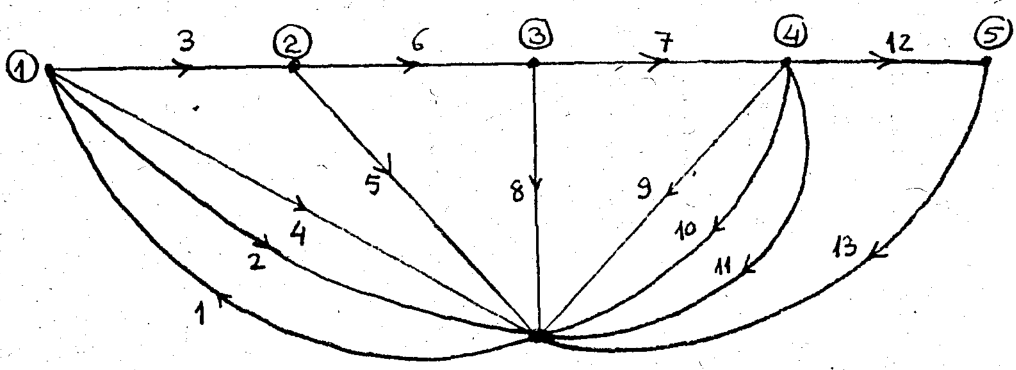
Obtaining the MNT equations in(129) for a given linear circuit will be clarified on an exmaple in the following.

EXAMPLE 3.

Consider the circuit and its graph in Fig.16.



(a)



(b)

Figure 16

The reduced incidence matrix A associated with the graph in Fig.16 can be written as:

$$\underset{\sim}{A} = \begin{bmatrix} -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{130}$$

and the constitutive equations can be written in accordance with(120) as:

$$\begin{aligned}
I_1 &= J_1 \\
-V_2 + R_2 I_2 &= 0 \\
sC_3 V_3 - I_3 &= 0 \\
\begin{bmatrix} 0 & 0 \\ \mu & -1 \end{bmatrix}\begin{bmatrix} V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_4 \\ I_5 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
-V_6 + sL\,I_6 &= 0 \\
-I_7 + G_7 V_7 &= 0 \\
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_8 \\ V_9 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha & -1 \end{bmatrix}\begin{bmatrix} I_8 \\ I_9 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
sC_{10} V_{10} - I_{10} &= 0 \\
R_{11} I_{11} - V_{11} &= 0 \\
R_{12} I_{12} - V_{12} &= 0 \\
V_{13} &= E_{13}
\end{aligned} \tag{131}$$

which can be put into the form of (121) as in(132).

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&sC_3&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&\mu&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&G_7&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&sC_{10}&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} V_1\\ V_2\\ V_3\\ V_4\\ V_5\\ V_6\\ V_7\\ V_8\\ V_9\\ V_{10}\\ V_{11}\\ V_{12}\\ V_{13} \end{bmatrix} + \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&R_2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&sL_6&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\alpha&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&R_{11}&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&R_{12}&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}\begin{bmatrix} I_1\\ I_2\\ I_3\\ I_4\\ I_5\\ I_6\\ I_7\\ I_8\\ I_9\\ I_{10}\\ I_{11}\\ I_{12}\\ I_{13} \end{bmatrix} = \begin{bmatrix} J_1\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ E_{13} \end{bmatrix} \tag{132}$$

Identifying the matrices $\underset{\sim}{P}_b$ and $\underset{\sim}{Q}_b$ in(132), it can be seen that the matrix $\underset{\sim}{Q}_b$ is a singular one since all the elements of its 5^{th} coulumn and 8^{th} row are zero. Therefore the solution of the circuit proposed by the Eq(127) becomes impossible for this example. Instead of Eq(127), the solution of the circuit is obtained by using the Eq(129).

If the constitutive equations(132) of example 3 are examined carefully, it will be seen that the voltage V_1 across the independent current source J_1, and the voltage V_9 across the dependent current source I_9 do not appear in the constitutiv relations since all the elements of 1^{st} and 9^{th} coulumns of the matrix $\underset{\sim}{P}_b$ are zero. Similarly, the current I_5 of the dependent voltage source V_5 and the current I_{13} of the independent voltage source E_{13} do not appear in the constitutive relations since all the elements of 5^{th} and 13^{th} coulumns of the matrix $\underset{\sim}{Q}_b$ are zero. If the solution of the above mentioned variables that do not enter into the constitutive relations are not desired, they can be deleted in the Eqs(129). Since the voltage across the 8^{th} branch and the current through the 4^{th} branch is known to be zero, these variables can also be deleted in the the Eqs(129).

In order to be able to perform this job of deleting the undesired variables from the Eqs(129), a procedure is developed as follows:

1- Obtain a V-Graph from the graph of the network and let the incidence matrix associated with the V-Graph to be $\underset{\sim}{A}v$ then the KVL becomes

$$\underset{\sim}{V}_b = \underset{\sim}{A}v^T \underset{\sim}{V}_n \tag{133}$$

2- Obtain a I-Graph from the graph of the network and denote the incidence matrix associated with the I-Graph as $\underset{\sim}{A}_I$ then the KCL becomes

$$\underset{\sim}{A}_I \, \underset{\sim}{I}_b = \underset{\sim}{0} \tag{134}$$

The V-Graph is obtained as follows:

1- If the voltage across the branch is zero its edge is collapsed in the V-Graph.

2- If the voltage across the branch does not enter the constitutive equations and is of no interest its edge is deleted on the V-Graph.

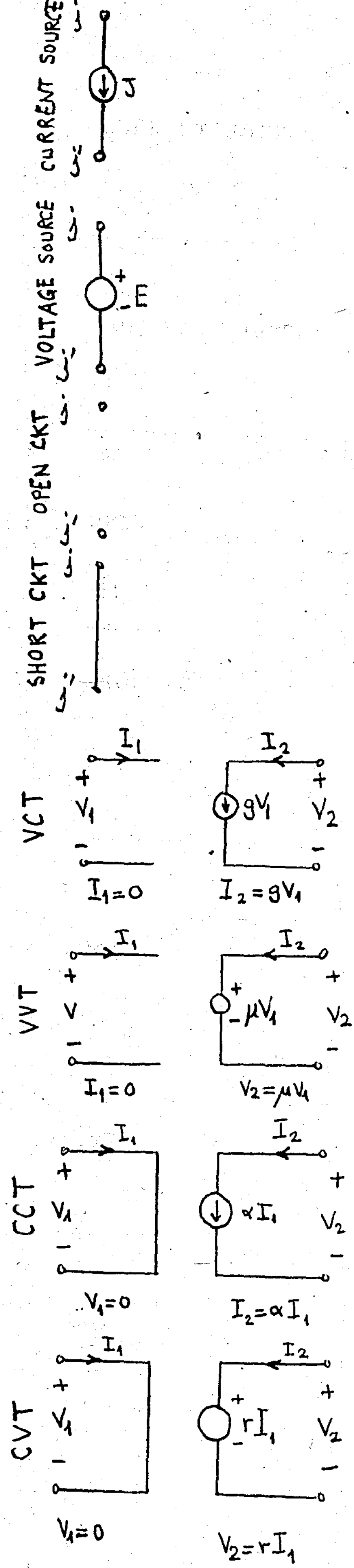
The I-Graph is obtained as follows:

1- If the current in the branch is zero; its edge is deleted from the I-Graph.

2- If the current in the branch does not enter the constitutive equations and is of no interest, its edge is collapsed on the I-Graph.

The word "of no interest" imply that the particular variable will not be needed as the solution. Otherwise the edge must be retained on the graph. For instance, one is generally not interested in the current through a voltage source and a voltage across the current source.

The graphs of some important one-port and two-port elements and their constitutive equations are shown in Fig.17. The current and voltage graphs of these elements can be obtained applying the technique stated above. This is shown in Fig.18.



$I = J$

$V = E$

$I = 0$

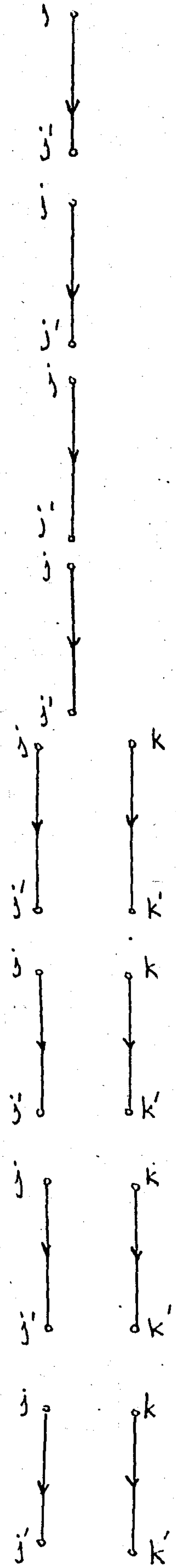
$V = 0$

$$\begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ \mu & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha & -1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

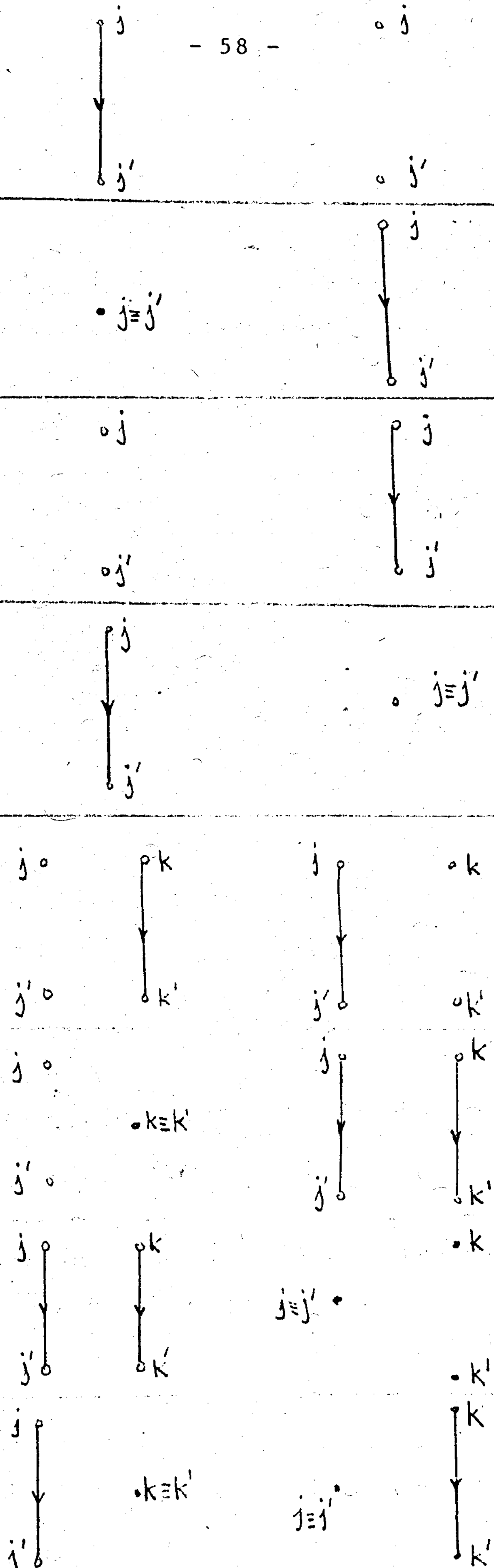


NETWORK ELEMENTS

FIGURE 17

CONSTITUTIVE EQUATIONS

GRAPH

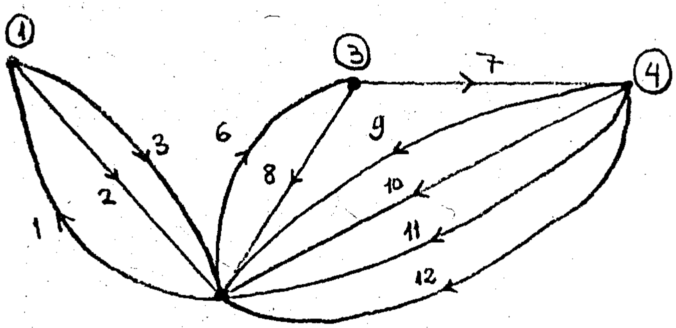


NETWORK ELEMENT	CONSTITUTIVE EQUATIONS
CURRENT SOURCE	$I = J$
VOLTAGE SOURCE	$V = E$
OPEN CKT	$I = 0$
SHORT CKT	$V = 0$
VCT	$gV_1 - I_2 = 0$
VVT	$\begin{bmatrix} \mu & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$
CCT	$\begin{bmatrix} \alpha & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = 0$
CVT	$r_1 I_1 - V_2 = 0$

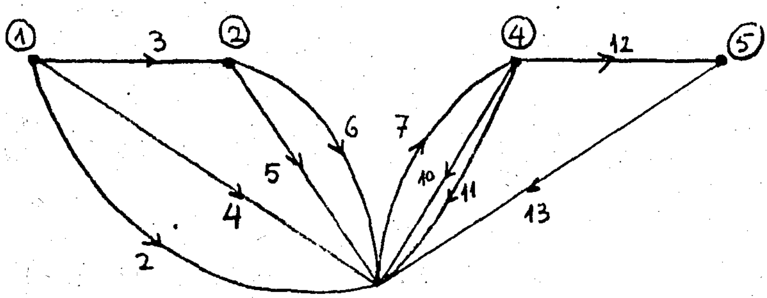
NETWORK ELEMENT CURRENT GRAPH VOLTAGE GRAPH CONSTITUTIVE EQUATIONS

EXAMPLE 4

Using the voltage and the current graphs of the elements shown in Fig.18, the voltage and current graphs of the circuit in Fig. 16 are obtained as shown in Fig.19.



(a) The current graph of the circuit in Figure 16.



(b) The voltage graph of the circuit in Figure 16.

FIGURE 19

Then, the incidence matrix $\underset{\sim}{A}_I$ of the current graph of the circuit in EXAMPLE 3 becomes

$$\underset{\sim}{A}_I = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{135}$$

and the incidence matrix $\underset{\sim}{A}_v$ of the voltage graph of the same circuit becomes

$$\underset{\sim}{A}_v = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{136}$$

then the KVL and KCL for the circuit in Fig.16 become

$$\underset{\sim}{V}_b = \underset{\sim}{A}_v^T \underset{\sim}{V}_n \tag{137}$$

$$\underset{\sim}{A}_I \underset{\sim}{I}_b = \underset{\sim}{0} \tag{138}$$

where

$$\underset{\sim}{V}_b = \begin{bmatrix} V_2 & V_3 & V_4 & V_5 & V_6 & V_7 & V_{10} & V_{11} & V_{12} & V_{13} \end{bmatrix}^T \tag{139}$$

and

$$\underset{\sim}{I}_b = \begin{bmatrix} I_1 & I_2 & I_3 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{11} & I_{12} \end{bmatrix}^T \tag{140}$$

The constitutive equations in this case becomes

$$\underset{\sim}{P} \underset{\sim}{V}_b + \underset{\sim}{Q} \underset{\sim}{I}_b = \underset{\sim}{W} \tag{141}$$

where

$$\underset{\sim}{P} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & sC_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & sC_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{142}$$

and

$$\underset{\sim}{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5L_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{12} \end{bmatrix} \tag{143}$$

and

$$\underset{\sim}{W} = \begin{bmatrix} J_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{13} \end{bmatrix}^T \tag{144}$$

Collecting (137),(138) and (141) together

$$\begin{aligned} \underset{\sim}{V_b} - \underset{\sim}{A_v^T}\, \underset{\sim}{V_n} &= \underset{\sim}{0} \\ \underset{\sim}{P}\, \underset{\sim}{V_b} + \underset{\sim}{Q}\, \underset{\sim}{I_b} &= \underset{\sim}{W} \\ \underset{\sim}{A_I}\, \underset{\sim}{I_b} &= \underset{\sim}{0} \end{aligned} \tag{145}$$

are obtained. Finally the MNT equations become

$$\left[\begin{array}{c|c|c} \underset{\sim}{I} & \underset{\sim}{0} & -\underset{\sim}{A_v^T} \\ \hline \underset{\sim}{P} & \underset{\sim}{Q} & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{A_I} & \underset{\sim}{0} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{V_b} \\ \hline \underset{\sim}{I_b} \\ \hline \underset{\sim}{V_n} \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{0} \\ \hline \underset{\sim}{W} \\ \hline \underset{\sim}{0} \end{array}\right] \tag{146}$$

The MNT equation in (146) obtained by using the two-graph technique contains 25 unknowns whereas the MNT equation in(129) obtained by using single graph technique contains 31 unknowns. This explains the power of using two-graph technique which reduces the number of unknowns.

In the following, a different type of formulation of the MNT equations will be presented in detail. This formulation reduces the number of unknowns in the MNT equation, considerably.

The constitutive equations in(121) can be rewritten in the form of

$$\begin{aligned} P_1 V_1 + Q_1 I_1 &= \underset{\sim}{W}_1 \\ \underset{\sim}{Y}_2 \underset{\sim}{V}_2 &= \underset{\sim}{I}_2 \\ \underset{\sim}{I}_3 &= \underset{\sim}{J}_3 \end{aligned} \tag{147}$$

where $\underset{\sim}{V}_1$ and $\underset{\sim}{I}_1$ denote the branch voltages and currents of the elements other than the admittances and the independent current sources,

$\underset{\sim}{V}_2$ and $\underset{\sim}{I}_2$ denote the branch voltages and currents of the admittances, and

$\underset{\sim}{I}_3$ and $\underset{\sim}{V}_3$ denote the branch currents and voltages of the independent current sources, Then the KCL can be written as

$$\left[\underset{\sim}{A}_{I_1} \;\vdots\; \underset{\sim}{A}_{I_2} \;\vdots\; \underset{\sim}{A}_{I_3} \right] \begin{bmatrix} \underset{\sim}{I}_1 \\ \hdashline \underset{\sim}{I}_2 \\ \hdashline \underset{\sim}{I}_3 \end{bmatrix} = \underset{\sim}{0} \tag{148}$$

where $\left[\underset{\sim}{A}_{I_1} \,\vdots\, \underset{\sim}{A}_{I_2} \,\vdots\, \underset{\sim}{A}_{I_3}\right]$ is the incidence matrix of the I-Graph, and the KVL can be written as

$$\begin{bmatrix} \underset{\sim}{V}_1 \\ \hdashline \underset{\sim}{V}_2 \\ \hdashline \underset{\sim}{V}_3 \end{bmatrix} = \left[\underset{\sim}{A}_{V_1} \,\vdots\, \underset{\sim}{A}_{V_2} \,\vdots\, \underset{\sim}{A}_{V_3}\right]^T \underset{\sim}{V}_n \tag{149}$$

where $\left[\underset{\sim}{A}_{V_1} \,\vdots\, \underset{\sim}{A}_{V_2} \,\vdots\, \underset{\sim}{A}_{V_3}\right]$ is the incidence matrix of the V-Graph, and $\underset{\sim}{V}_n$ denotes the node voltages of the given network. Rewriting the Eq(147), explicitly

$$\underset{\sim}{A}_{I_1} \underset{\sim}{I}_1 + \underset{\sim}{A}_{I_2} \underset{\sim}{I}_2 + \underset{\sim}{A}_{I_3} \underset{\sim}{I}_3 = \underset{\sim}{0} \tag{150}$$

using $\underset{\sim}{Y}_2 \underset{\sim}{V}_2 = \underset{\sim}{I}_2$ and $\underset{\sim}{I}_3 = \underset{\sim}{J}_3$ in (150)

$$\underset{\sim}{A}_{I_1} \underset{\sim}{I}_1 + \underset{\sim}{A}_{I_2} \underset{\sim}{Y}_2 \underset{\sim}{V}_2 + \underset{\sim}{A}_{I_3} \underset{\sim}{J}_s = \underset{\sim}{0} \tag{151}$$

or

$$\underset{\sim}{A}_{I_1} \underset{\sim}{I}_1 + \underset{\sim}{A}_{I_2} \underset{\sim}{Y}_2 \underset{\sim}{V}_2 = - \underset{\sim}{A}_{I_3} \underset{\sim}{J}_s \tag{152}$$

is obtained. Substituting $\underset{\sim}{V}_2 = \underset{\sim}{A}_{V_2}^T \underset{\sim}{V}_n$ into (152)

$$\underset{\sim}{A}_{I_1} \underset{\sim}{I}_1 + \underset{\sim}{A}_{I_2} \underset{\sim}{Y}_2 \underset{\sim}{A}_{V_2}^T \underset{\sim}{V}_n = - \underset{\sim}{A}_{I_3} \underset{\sim}{J}_s \tag{153}$$

is obtained. Inserting(149) into(147), the first equation in (147) becomes

$$\underset{\sim}{P}_1 \underset{\sim}{A}_{V_1}^{T} \underset{\sim}{V}_n + \underset{\sim}{Q}_1 \underset{\sim}{I}_1 = \underset{\sim}{W}_1 \tag{154}$$

Combining (153) and (154) into single equation, the MNT equations are obtained as

$$\left[\begin{array}{c|c} \underset{\sim}{A}_{I_2} \underset{\sim}{Y}_2 \underset{\sim}{A}_{V_2}^{T} & \underset{\sim}{A}_{I_1} \\ \hline \underset{\sim}{P}_1 \underset{\sim}{A}_{V_1}^{T} & \underset{\sim}{Q}_1 \end{array}\right] \left[\begin{array}{c} \underset{\sim}{V}_n \\ \hline \underset{\sim}{I}_1 \end{array}\right] \left[\begin{array}{c} -\underset{\sim}{A}_{I_3} \underset{\sim}{J}_s \\ \hline \underset{\sim}{W}_1 \end{array}\right] \tag{155}$$

The MNT equation in (155) contains only the node voltages and the branch currents of the elements other than the admittances and the independent current sources as unknown variables. Therefore, it is the best one among the MNT equations mentioned here.

In the following two examples, the MNT equations governing the linear network under consideration will be formulated in the form of Eq(155).

EXAMPLE 5

Consider the circuit in Fig.20. The current and voltage graphs of the circuit in Fig.20 can be obtained as shown in Fig.22 and 23.

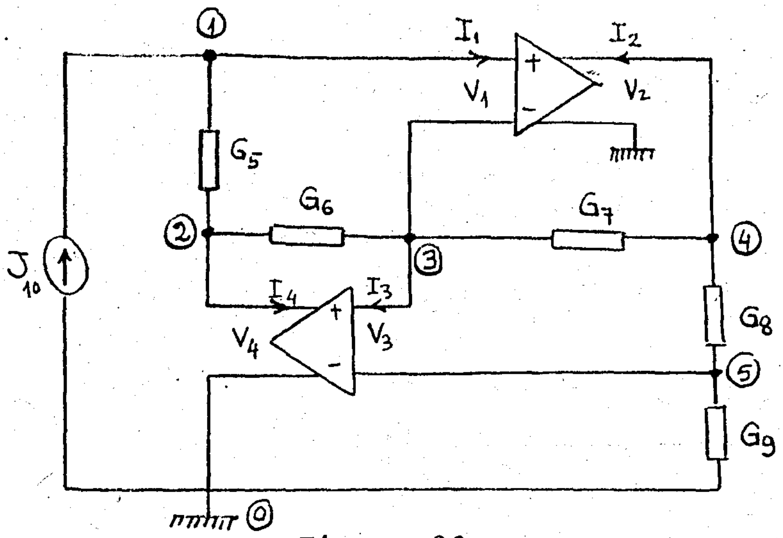


Figure 20

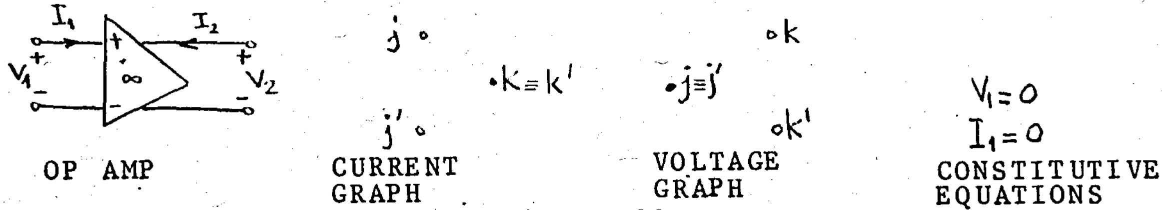


Figure 21

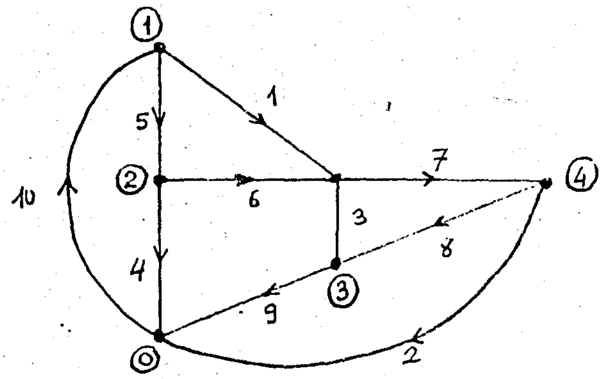
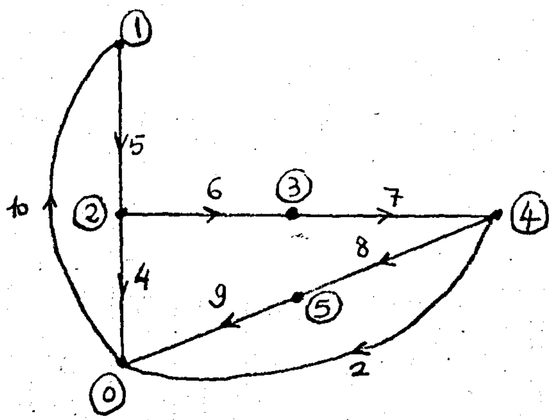
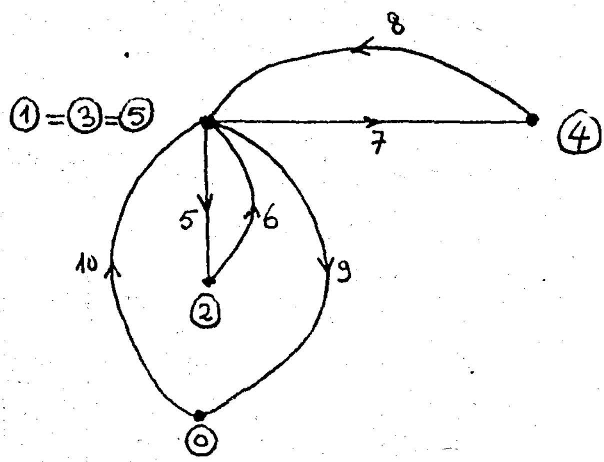


Figure 22- The graph of the circuit in Fig. 20.



(a) I-GRAPH of the circuit in Figure 20.



(b) V-GRAPH of the circuit in Figure 20.

FIGURE 23

The constitutive equations of the circuit become

$$
\begin{aligned}
&\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&I_5 = G_5 V_5 \\
&I_6 = G_6 V_6 \\
&I_7 = G_7 V_7 \\
&I_8 = G_8 V_8 \\
&I_9 = G_9 V_9 \\
&I_{10} = J_{10}
\end{aligned}
\qquad (156)
$$

Writing the constitutive equations in the form of (147)

$$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} V_1 \\ V_2 \\ \hdashline V_3 \\ V_4 \end{array}\right] + \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]\left[\begin{array}{c} I_1 \\ I_2 \\ \hdashline I_3 \\ I_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hdashline 0 \\ 0 \end{array}\right] \qquad (157)$$

which is in the form of $\underset{\sim}{P}_1 \underset{\sim}{V}_1 + \underset{\sim}{Q}_1 \underset{\sim}{I}_1 = \underset{\sim}{W}_1$ and

$$\begin{bmatrix} I_5 \\ I_6 \\ I_7 \\ I_8 \\ I_9 \end{bmatrix} = \begin{bmatrix} G_5 & & & & \\ & G_6 & & 0 & \\ & & G_7 & & \\ & 0 & & G_8 & \\ & & & & G_9 \end{bmatrix}\begin{bmatrix} V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \end{bmatrix} \qquad (158)$$

which is in the form of $\underset{\sim}{I}_2 = \underset{\sim}{Y}_2 \underset{\sim}{V}_2$, and

$$I_{10} = J_{10} \qquad (159)$$

which is in the form of $\underset{\sim}{I}_3 = \underset{\sim}{J}_5$.

Writing the incidence matrix A associated with the graph of the circuit in Fig.22

$$A = \left[\begin{array}{cccc|ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right] \tag{160}$$

and the KCL becomes

$$\Bigg[\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{matrix}}_{\underset{\sim}{A}_1} \Bigg| \underbrace{\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{matrix}}_{\underset{\sim}{A}_2} \Bigg| \underbrace{\begin{matrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}}_{\underset{\sim}{A}_3} \Bigg] \left[\begin{array}{c} I_1 \\ I_2 \\ I_3 \\ I_4 \\ \hline I_5 \\ I_6 \\ I_7 \\ I_8 \\ I_9 \\ \hline I_{10} \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{161}$$

and the $\underset{\sim}{A}_1$, $\underset{\sim}{A}_2$, $\underset{\sim}{A}_3$ matrices are defined as shown in (161). Since the normal graph is used for this example, the equations (155) can simply be reduced to

$$\left[\begin{array}{c|c} \underset{\sim}{A}_2 \underset{\sim}{Y}_2 \underset{\sim}{A}_2^T & \underset{\sim}{A}_1 \\ \hline \underset{\sim}{P}_1 \underset{\sim}{A}_1^T & \underset{\sim}{Q}_1 \end{array}\right] \left[\begin{array}{c} \underset{\sim}{V}_n \\ \hline \underset{\sim}{I}_1 \end{array}\right] = \left[\begin{array}{c} -\underset{\sim}{A}_3 \underset{\sim}{I}_3 \\ \hline \underset{\sim}{W}_1 \end{array}\right] \tag{162}$$

The submatrices in (162) are obtained by direct substitution as

$$\underset{\sim}{A}_2 \underset{\sim}{Y}_2 \underset{\sim}{A}_2^T = \begin{bmatrix} G_5 & G_5 & 0 & 0 & 0 \\ -G_5 & G_5+G_6 & -G_6 & 0 & 0 \\ 0 & -G_6 & G_6+G_7 & -G_7 & 0 \\ 0 & 0 & -G_7 & G_7+G_8 & -G_8 \\ 0 & 0 & 0 & -G_8 & G_8+G_9 \end{bmatrix} \tag{163}$$

and

$$\underset{\sim}{P}_1 \underset{\sim}{A}_1^T = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{164}$$

and

$$-\underset{\sim}{A}_3 \underset{\sim}{I}_3 = - \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} J_{10} = \begin{bmatrix} J_{10} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{165}$$

Then, the MNT equations become

$$
\left[\begin{array}{ccccc|cccc}
G_5 & -G_5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-G_5 & G_5+G_6 & -G_6 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -G_6 & G_6+G_7 & -G_7 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & -G_7 & G_7+G_8 & -G_8 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -G_8 & G_8+G_9 & 0 & 0 & -1 & 0 \\
\hline
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\begin{bmatrix} V_{n1} \\ V_{n2} \\ V_{n3} \\ V_{n4} \\ V_{n5} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}
=
\begin{bmatrix} J_{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\qquad (166)
$$

where only the node voltages and the branch currents other than the admittances and the independent current sources are unknowns. The other variables in the circuit can be obtained by direct substitution of these variables into appropriate equations if desired.

EXAMPLE 6

As a second example to the MNT formulation presented in (155), consider the network in Fig.24.

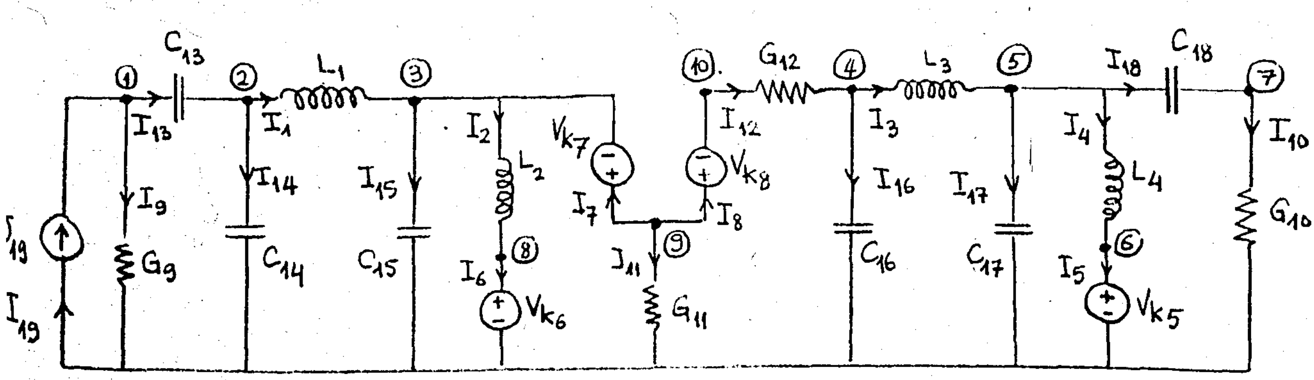


Figure 24

The constitutive equations of the circuit becomes

$$\left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ \hline V_5 \\ V_6 \\ V_7 \\ V_8 \end{array}\right] + \left[\begin{array}{cccc|cccc} sL_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sL_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & sL_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} I_1 \\ I_2 \\ I_3 \\ I_4 \\ \hline I_5 \\ I_6 \\ I_7 \\ I_8 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline V_{k5} \\ V_{k6} \\ V_{k7} \\ V_{k8} \end{array}\right] \tag{167}$$

which are in the form of $\underset{\sim}{P}_1 \underset{\sim}{V}_1 + \underset{\sim}{Q}_1 \underset{\sim}{I}_1 = \underset{\sim}{W}_1$, and

$$\left[\begin{array}{c} I_9 \\ I_{10} \\ I_{11} \\ I_{12} \\ \hline I_{13} \\ I_{14} \\ I_{15} \\ I_{16} \\ I_{17} \\ I_{18} \end{array}\right] = \left[\begin{array}{cccc|cccccc} G_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & sC_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & sC_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & sC_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & sC_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & sC_{17} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & sC_{18} \end{array}\right] \left[\begin{array}{c} V_9 \\ V_{10} \\ V_{11} \\ V_{12} \\ \hline V_{13} \\ V_{14} \\ V_{15} \\ V_{16} \\ V_{17} \\ V_{18} \end{array}\right] \tag{168}$$

which are in the form of $\underset{\sim}{I}_2 = \underset{\sim}{Y}_2 \underset{\sim}{V}_2$, and

$$I_{19} = J_{19} \tag{169}$$

which is in the form of $\underset{\sim}{I}_3 = \underset{\sim}{J}_s$. Writing the incidence matrix $\underset{\sim}{A}$ associated with the graph of the circuit in Fig. 24

$$\underset{\sim}{A} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{170}$$

Then the KCL becomes

$$\left[\begin{array}{c:c:c}
\underbrace{\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{matrix}}_{\underset{\sim}{A}_1} &
\underbrace{\begin{matrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}}_{\underset{\sim}{A}_2} &
\underbrace{\begin{matrix}
-1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{matrix}}_{\underset{\sim}{A}_3}
\end{array}\right]
\left[\begin{array}{c}
I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \\ \hdashline I_9 \\ I_{10} \\ I_{11} \\ I_{12} \\ I_{13} \\ I_{14} \\ I_{15} \\ I_{16} \\ I_{17} \\ I_{18} \\ \hdashline I_{19}
\end{array}\right] = \underset{\sim}{0} \tag{171}$$

and the $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \underset{\sim}{A}_3$ matrices are defined as shown in(171). The submatrices in (162) are obtained by direct substitution as:

$$\underset{\sim}{A}_2 \underset{\sim}{Y}_2 \underset{\sim}{A}_2^T = \begin{bmatrix} G_9+sC_{13} & -sC_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -sC_{13} & sC_{13}+sC_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sC_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12}+sC_{16} & 0 & 0 & 0 & 0 & -G_{12} \\ 0 & 0 & 0 & 0 & sC_{17}+sC_{18} & 0 & -sC_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sC_{18} & 0 & G_{10}+sC_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12} & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix} \quad (172)$$

$$\underset{\sim}{A}_1^T = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad (173)$$

$$-\underset{\sim}{A}_3 \underset{\sim}{I}_3 = -\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} J_{19} = \begin{bmatrix} J_{19} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (174)$$

The MNT equations become

$$
\left[\begin{array}{cccccccccc|cccccccc}
G_9+sC_{13} & -sC_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-sC_{13} & sC_{13}+sC_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & sC_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & G_{12}+sC_{16} & 0 & 0 & 0 & 0 & 0 & -G_{12} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & sC_{17}+sC_{16} & 0 & -sC_{18} & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -sC_{18} & 0 & G_{10}+sC_{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -G_{12} & 0 & 0 & 0 & 0 & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & sL_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & sL_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & sL_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & sL_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
V_{n1} \\ V_{n2} \\ V_{n3} \\ V_{n4} \\ V_{n5} \\ V_{n6} \\ V_{n7} \\ V_{n8} \\ V_{n9} \\ V_{n10} \\ \hline I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8
\end{array}\right]
=
\left[\begin{array}{c}
J_{19} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ V_{k5} \\ V_{k6} \\ V_{k7} \\ V_{k8}
\end{array}\right]
\qquad (175)
$$

III.2. SOLUTION OF THE LINEAR NETWORK EQUATIONS: LU DECOMPOSITION

The MNT equations that have been used in formulating the behaviour of the linear network in the previous section are solved by using the LU decomposition technique.

In this section, first the LU decomposition technique for solving linear system of equations will be introduced and then the algorithm for obtaining the lower triangular matrix $\underset{\sim}{L}$ and the upper triangular matrix $\underset{\sim}{U}$ will be explained.

Consider the following MNT equation

$$\underset{\sim}{T}\,\underset{\sim}{x} = \underset{\sim}{b} \qquad (176)$$

where $\underset{\sim}{T}$ is an (nxn) matrix, and $\underset{\sim}{x}$ and $\underset{\sim}{b}$ are (nxl) unknown and known vectors, respectively. The matrix $\underset{\sim}{T}$ can be decomposed into a product of two matrices $\underset{\sim}{L}$ and $\underset{\sim}{U}$, i.e, lower and upper triangular matrices which are defined by

$$\begin{bmatrix} \ell_{11} & & & & 0 \\ \ell_{21} & \ell_{22} & & & \\ \ell_{31} & \ell_{32} & \ell_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & & \ell_{nn} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & \cdots & U_{1n} \\ & 1 & U_{23} & \cdots & U_{2n} \\ & & 1 & \cdots & U_{3n} \\ & & & \ddots & \vdots \\ 0 & & & & U_{nn} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1n} \\ t_{21} & t_{22} & t_{23} & \cdots & t_{2n} \\ t_{31} & t_{32} & t_{33} & \cdots & t_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & t_{n3} & \cdots & t_{nn} \end{bmatrix} \qquad (177)$$

and if the product of $\underset{\sim}{L}$ and $\underset{\sim}{U}$ in(177) is evaluated,

$$\underset{\sim}{T}=\begin{bmatrix} l_{11} & l_{11}u_{12} & \cdots\cdots & l_{11}u_{1n} \\ l_{21} & l_{21}u_{12}+l_{22} & \cdots\cdots & l_{21}u_{1n}+l_{22}u_{2n} \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n1}u_{12}+l_{n2} & \cdots\cdots & l_{n1}u_{1n}+l_{n2}u_{2n}+\cdots+l_{nn} \end{bmatrix} \tag{178}$$

is obtained. From the set of equations in(178) the elements of the $\underset{\sim}{L}$ and $\underset{\sim}{U}$ matrices can easily be calculated applying the order indicated below for obtaining the elements of the $\underset{\sim}{L}$ and $\underset{\sim}{U}$ matrices.

$$\begin{bmatrix} 1 & n+1 & n+2 & \longrightarrow & 2n-1 \\ 2 & 2n & 3n-1 & \longrightarrow & 4n-4 \\ 3 & 2n+1 & 4n-3 & \longrightarrow & \\ \downarrow & \downarrow & \downarrow & & \\ n & 3n-2 & 5n-6 & & \end{bmatrix}$$

Then, the elements of the upper and lower triangular matrices become

$$u_{ij}=\frac{t_{ij}-\sum_{k=1}^{i-1} l_{ik}u_{kj}}{l_{ii}} \qquad \begin{array}{l} j=i+1,\ i+2,\ \ldots\ldots,\ n \\ i=1,\ 2,\ \ldots\ldots,\ n-1 \end{array} \tag{179}$$

and

$$l_{ij} = t_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad \begin{array}{l} i = j, j+1, j+2, \ldots, n \\ j = 1, 2, \ldots, n; \end{array} \tag{180}$$

After the matrices $\underset{\sim}{L}$ and $\underset{\sim}{U}$ are obtained, Eq(176) becomes

$$\underset{\sim}{L}\,\underset{\sim}{U}\,\underset{\sim}{x} = \underset{\sim}{b} \tag{181}$$

Defining

$$\underset{\sim}{U}\,\underset{\sim}{x} = \underset{\sim}{y} \tag{182}$$

Eq(181) becomes

$$\underset{\sim}{L}\,\underset{\sim}{y} = \underset{\sim}{b} \tag{183}$$

or

$$\begin{bmatrix} l_{11} & & & & \\ l_{21} & l_{22} & & \bigcirc & \\ l_{31} & l_{32} & l_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \tag{184}$$

Eq(184) can easily be solved by forward elimination and the elements of $\underset{\sim}{y}$ can be found as:

$$y_k = \frac{b_k - \sum_{j=1}^{k-1} l_{kj} y_j}{l_{kk}} \qquad k = 1, \ldots, n. \tag{185}$$

Inserting the elements of $\underset{\sim}{y}$ calculated in(185) into(182)

$$\begin{bmatrix} 1 & U_{12} & U_{13} & \cdots & U_{1n} \\ & 1 & U_{23} & \cdots & U_{2n} \\ & & \ddots & & \vdots \\ & & & 1 & U_{n-1,n} \\ O & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \tag{186}$$

is obtained. Eq(186) can easily be solved for the vector $\underset{\sim}{x}$ by backward substitution as:

$$x_k = y_k - \sum_{j=k+1}^{n} U_{kj} x_j \quad , \; k = n-1, n-2, \ldots, 1. \tag{187}$$

which are the desired unknowns in Eq(176).

In the following, an LU decomposition technique which applies partial pivoting on rows will be explained on an example for a 4x4 matrix. This technique which is equivalent to the one presented above reduces the MNT matrix into the lower and upper triangular matrices, $\underset{\sim}{L}$ and $\underset{\sim}{U}$, respectively.

Consider the 4x4 MNT matrix T in the following.

$$\underset{\sim}{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \tag{188}$$

Find the largest element among the elements of the first coulumn. Let the largest element of the first coulumn be t_{k1} where k is a fixed integer such that $1 \leqslant k \leqslant n = 4$. Then, replace the first row with the k^{th} row if $k \neq 1$ and assume the matrix take the form in(188). Then divide all the elements on the right of t_{11} of the first row by t_{11} to obtain

$$\begin{bmatrix} t_{11} & t_{12}/t_{11} & t_{13}/t_{11} & t_{14}/t_{11} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \quad (189)$$

Then obtain a new matrix from this one by performing the following operations or(189) as

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22} - t_{21}t_{12}^{(1)} & t_{23} - t_{21}t_{13}^{(1)} & t_{24} - t_{21}t_{14}^{(1)} \\ t_{31} & t_{32} - t_{31}t_{12}^{(1)} & t_{33} - t_{31}t_{13}^{(1)} & t_{14} - t_{31}t_{14}^{(1)} \\ t_{41} & t_{42} - t_{41}t_{12}^{(1)} & t_{43} - t_{41}t_{13}^{(1)} & t_{44} - t_{41}t_{14}^{(1)} \end{bmatrix} \quad (190)$$

which can be rewritten in the form

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22}^{(1)} & t_{23}^{(1)} & t_{24}^{(1)} \\ t_{31} & t_{32}^{(1)} & t_{33}^{(1)} & t_{34}^{(1)} \\ t_{41} & t_{42}^{(1)} & t_{43}^{(1)} & t_{44}^{(1)} \end{bmatrix} \quad (191)$$

In the second step, find the largest element among those elements of the second coulumn that remains after deleting the elements above its diagonal element. Let the largest element found be $t_{k2}^{(1)}$ where k is a fixed integer such that $2 \leqslant k \leqslant n=4$. Then, replace the second row with the k^{th} row and let the matrix takes the form presented in (191). Then, divide the elements of the second row to the left of the diagonal element by the diagonal element of it, namely, $t_{22}^{(1)}$ to get

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22}^{(1)} & t_{23}^{(1)}/t_{22}^{(1)} & t_{24}^{(1)}/t_{22}^{(1)} \\ t_{31} & t_{32}^{(1)} & t_{33}^{(1)} & t_{34}^{(1)} \\ t_{41} & t_{42}^{(1)} & t_{43}^{(1)} & t_{44}^{(1)} \end{bmatrix} \tag{192}$$

Then obtain a new matrix from this one by performing the operations indicated below.

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22}^{(1)} & t_{23}^{(1)} & t_{24}^{(1)} \\ t_{31} & t_{32}^{(1)} & t_{33}^{(1)}-t_{32}^{(1)}t_{23}^{(2)} & t_{34}^{(1)}-t_{32}^{(1)}t_{24}^{(2)} \\ t_{41} & t_{42}^{(1)} & t_{43}^{(1)}-t_{42}^{(1)}t_{24}^{(2)} & t_{44}^{(1)}-t_{42}^{(1)}t_{24}^{(2)} \end{bmatrix} \tag{193}$$

which can be rewritten as

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22}^{(1)} & t_{23}^{(2)} & t_{24}^{(2)} \\ t_{31} & t_{32}^{(1)} & t_{33}^{(2)} & t_{34}^{(2)} \\ t_{41} & t_{42}^{(1)} & t_{43}^{(2)} & t_{44}^{(2)} \end{bmatrix} \qquad (194)$$

In the third step, find the largest element among those elements of the third coulumn that remains after deleting the elements above its diagonal element. Let the largest element found be $t_{k3}^{(2)}$ where k is a fixed integer such that $3 \leqslant k \leqslant n = 4$. Then, replace the 3^{rd} row with the k^{th} row if $k \neq 3$ and let the matrix takes the form in(194) after performing the operations stated above.

Then divide the elements of the third row to the left of the diagonal element by the diagonal element of it, namely, $t_{33}^{(2)}$ to get

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22}^{(1)} & t_{23}^{(2)} & t_{24}^{(2)} \\ t_{31} & t_{32}^{(1)} & t_{33}^{(2)} & t_{34}^{(2)} / t_{33}^{(2)} \\ t_{41} & t_{42}^{(1)} & t_{43}^{(2)} & t_{44}^{(2)} \end{bmatrix} \qquad (195)$$

Then obtain a new matrix from this one by performing the operations indicated below

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22}^{(1)} & t_{23}^{(2)} & t_{24}^{(2)} \\ t_{31} & t_{32}^{(1)} & t_{33}^{(2)} & t_{34}^{(3)} \\ t_{41} & t_{42}^{(1)} & t_{43}^{(2)} & t_{44}^{(2)} - t_{43}^{(2)} t_{34}^{(3)} \end{bmatrix} \qquad (196)$$

and defining the elements of(196)

$$\begin{bmatrix} t_{11} & t_{12}^{(1)} & t_{13}^{(1)} & t_{14}^{(1)} \\ t_{21} & t_{22}^{(1)} & t_{23}^{(2)} & t_{24}^{(2)} \\ t_{31} & t_{32}^{(1)} & t_{33}^{(2)} & t_{34}^{(3)} \\ t_{41} & t_{42}^{(1)} & t_{43}^{(2)} & t_{44}^{(3)} \end{bmatrix} \qquad (197)$$

is obtained. The matrix in(197) contains the lower and upper triangular matrices, $\underset{\sim}{L}$ and $\underset{\sim}{U}$ respectively.

The steps of the LU factorization algorithm is summarized as follows:

1- Copy coulumn 1.

2- In row 1, divide all nondiagonal elements by the diagonal element.

3- For each element (i,j), i> 1 and j> 1, subtract from it the product of (i,1) and (1,j) elements.

4- If the order of the submatrix subject to subtractions in step 3 is 2 or higher, refer to this submatrix and go back to step 1. Otherwise, the algorithm is completed, and the updated matrix is the desired Q matrix.

IV. GRADIENT EVALUATION

To minimize the error function P, an optimization algorithm is used. All efficient minimization routines require the knowledge of the gradient vector. In this chapter a method is described for the determination of the gradient $\partial P/\partial \underset{\sim}{x}$.

Both the case in which the period T of the oscillations is known and the period T of the oscillations is unknown are investigated seperately.

For the case that the period T of the oscillations are unknown, the derivation performed by Nakhla and Vlach in [8] and a new method presented here will be compared with each other.

IV.1. GRADIENT EVALUATION: PERIOD T OF THE OSCILLATIONS IS KNOWN

To find $\dfrac{\partial P}{\partial \underset{\sim}{x}_k}$ differentiate(104) with respect to $\underset{\sim}{x}_k$

(198)

$$\frac{\partial P}{\partial \underset{\sim}{x}_k} = 2\int_0^T \left(\frac{\partial \underset{\sim}{\varepsilon}}{\partial \underset{\sim}{x}_k}\right)^T \underset{\sim}{\varepsilon}(\tau)\, d\tau$$

is obtained. Differentiating(33)

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial \underset{\sim}{x}_k} = \frac{\partial \underset{\sim}{y}_N}{\partial \underset{\sim}{x}_k} - \frac{\partial \underset{\sim}{y}_L}{\partial \underset{\sim}{x}_k} \qquad k \in \{0, 1, \ldots\ldots, M\} \tag{199}$$

Considering $\dfrac{\partial \underset{\sim}{y}_N}{\partial \underset{\sim}{x}_k}$ first, from(100)

$$\frac{\partial \underset{\sim}{y}_N}{\partial \underset{\sim}{x}_k} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s} \frac{\partial \underset{\sim}{u}_s}{\partial \underset{\sim}{x}_k} + \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s} \frac{\partial \dot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k} + \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s} \frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k} \tag{200}$$

is obtained. The form assumed for $\underset{\sim}{u}_s(t)$ in(105) gives

$$\dot{\underset{\sim}{u}}_s(t) = \mathrm{Re}\left\{\sum_{k=0}^{M} j\omega k\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t}\right\} \tag{201}$$

$$\ddot{\underset{\sim}{u}}_s(t) = \mathrm{Re}\left\{\sum_{k=0}^{M} -\omega^2 k^2 (\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t}\right\} \tag{202}$$

(105),(201) and (202) respectively give

$$\frac{\partial \underset{\sim}{u}_s}{\partial \underset{\sim}{x}_k} = \mathrm{Re}\left\{\underset{\sim}{I}\, e^{j\omega k t}\right\} \tag{203}$$

$$\frac{\partial \dot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k} = \mathrm{Re}\left\{j\omega k\, \underset{\sim}{I}\, e^{j\omega k t}\right\} \tag{204}$$

$$\frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k} = \mathrm{Re}\left\{-\omega^2 k^2\, \underset{\sim}{I}\, e^{j\omega k t}\right\} \tag{205}$$

where $\underset{\sim}{I}$ denotes the identity matrix.

(203),(204) and (205) used in (200) yield

$$\frac{\partial \underset{\sim}{y}_N}{\partial \underset{\sim}{x}_k} = \mathrm{Re}\left\{\left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s} + j\omega k \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\dot{u}}_s} - \omega^2 k^2 \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\ddot{u}}_s}\right] e^{j\omega k t}\right\} \tag{206}$$

To find $\frac{\partial \underset{\sim}{y}_L}{\partial \underset{\sim}{x}_k}$ differentiate(118) with respect to $\underset{\sim}{x}_k$

$$\frac{\partial \underset{\sim}{y}_L}{\partial \underset{\sim}{x}_k} = \mathrm{Re}\left\{ \underset{\sim}{Y}_{Lk}(j\omega k)\, e^{j\omega k t} \right\} \tag{207}$$

$$k \in \{0,1,\ldots\ldots,M\}$$

Combining (199), (206) and (207)

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial \underset{\sim}{x}_k} = \mathrm{Re}\left\{\left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s} + j\omega k \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\dot{u}}_s} - \omega^2 k^2 \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\ddot{u}}_s} - \underset{\sim}{Y}_{Lk}(j\omega k)\right] e^{j\omega k t}\right\} \tag{208}$$

$$k \in \{0,1,\ldots\ldots,M\}$$

Inserting from (208) into (198) obtain

$${}_k = 2\,\mathrm{Re}\left\{\int_0^T \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s} + j\omega k \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\dot{u}}_s} - \omega^2 k^2 \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\ddot{u}}_s} - \underset{\sim}{Y}_{Lk}(j\omega k)\right]^T \underset{\sim}{\varepsilon}(\tau)\, e^{j\omega k \tau}\, d\tau\right\}$$

$$k \in \{0,1,\ldots,M\} \tag{209}$$

Equation (209) is the required expression for the gradient component $\partial P/\partial \underset{\sim}{x}_k$.

To find $\partial P/\partial \underset{\sim}{x}_k^*$ differentiate (104) with respect to $\underset{\sim}{x}_k^*$

$$\frac{\partial P}{\partial \underset{\sim}{x}_k^*} = 2\int_0^T \left(\frac{\partial \underset{\sim}{\varepsilon}}{\partial \underset{\sim}{x}_k^*}\right)^T \underset{\sim}{\varepsilon}(\tau)\, d\tau \tag{210}$$

Going through the same steps as for (199) and (200)

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial \underset{\sim}{x}_k^*} = \frac{\partial \underset{\sim}{y}_N}{\partial \underset{\sim}{x}_k^*} - \frac{\partial \underset{\sim}{y}_L}{\partial \underset{\sim}{x}_k^*} \qquad k \in \{1,2,\ldots\ldots,M\} \tag{211}$$

$$\frac{\partial \underset{\sim}{y}_N}{\partial \underset{\sim}{x}_k^*} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\,\frac{\partial \underset{\sim}{u}_s}{\partial \underset{\sim}{x}_k} + \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\,\frac{\partial \dot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k^*} + \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\,\frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k^*} \tag{212}$$

are obtained, where

$$\frac{\partial \underset{\sim}{u}_s}{\partial \underset{\sim}{x}_k^*} = \mathrm{Im}\left\{ \underset{\sim}{I} e^{j\omega kt} \right\} \tag{213}$$

$$\frac{\partial \dot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k^*} = \mathrm{Im}\left\{ j\omega k \underset{\sim}{I} e^{j\omega kt} \right\} \tag{214}$$

$$\frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial \underset{\sim}{x}_k^*} = \mathrm{Im}\left\{ -\omega^2 k^2 \underset{\sim}{I} e^{j\omega kt} \right\} \tag{215}$$

$$k \in \{1,2,\ldots,M\}$$

$\underset{\sim}{y}_L(t)$ is still as given by (106), and differentiating (106) with respect to $\underset{\sim}{x}_k^*$

$$\frac{\partial \underset{\sim}{y}_L}{\partial \underset{\sim}{x}_k^*} = \mathrm{Im}\left\{ \underset{\sim}{Y}_{Lk}(j\omega k)\, e^{j\omega k t} \right\} \qquad k \in \{1,2,\ldots,M\} \tag{216}$$

is obtained. Here Im stands for "imaginary part of". Finally, substituting (213),(214),(215) and (216) into(211)

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial \underset{\sim}{x}_k^*} = \mathrm{Im}\left\{ \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s} + j\omega k \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s} - \omega^2 k^2 \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s} - \underset{\sim}{Y}_{Lk}(j\omega k) \right] e^{j\omega k t} \right\} \tag{217}$$

$$k \in \{1,2,\ldots\ldots,M\}$$

Inserting from (217) into (210) obtain

$$\frac{\partial P}{\partial \underset{\sim}{x}_k^*} = 2\,\mathrm{Im}\left\{ \int_0^T \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s} + j\omega k \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s} - \omega^2 k^2 \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s} - \underset{\sim}{Y}_{Lk}(j\omega k) \right]^T \underset{\sim}{\varepsilon}(\tau)\, e^{j\omega k \tau}\, d\tau \right\}$$

$$k \in \{1,2,\ldots\ldots,M\} \tag{218}$$

Denoting the expression in the bracets of (218) by

$$\underset{\sim}{z}_k = 2\int_0^T \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s} + j\omega k \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s} - \omega^2 k^2 \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s} - \underset{\sim}{Y}_{Lk}(j\omega k) \right]^T \underset{\sim}{\varepsilon}(\tau)\, e^{j\omega k \tau}\, d\tau \tag{219}$$

$$k \in \{0,1,\ldots\ldots\ldots,M\}$$

it can be seen that

$$\underset{\sim}{Z}_k = \frac{\partial P}{\partial \underset{\sim}{x}_k} + j \frac{\partial P}{\partial \underset{\sim}{x}_k^*} \tag{220}$$

as follows from (209) and (218).

The advantage of the formula (220) is that the gradient components can simply be obtained as Fourier Coefficients of the integrand through Discrete or Fast Fourier Transform techniques. Thus the gradient is

$$\frac{\partial P}{\partial \underset{\sim}{x}} = \left[\frac{\partial P^T}{\partial \underset{\sim}{x}_0} \; \frac{\partial P^T}{\partial \underset{\sim}{x}_1} \cdots\cdots \frac{\partial P^T}{\partial x_M} \; \frac{\partial P^T}{\partial x_1^*} \cdots\cdots \frac{\partial P^T}{\partial x_M^*} \right]^T \tag{221}$$

IV.2. GRADIENT EVALUATION: PERIOD T OF THE OSCILLATIONS IS UNKNOWN

In this part, the gradient of the error function p will be evaluated for the case that the period T of the oscillations unknown. Two different type of evaluation will be presented in detail. The first one which is proposed by Nakhla and Vlach in [8] makes a numerical approximation to the error function P before finding the gradient of P, whereas the second one presented originally here yields the exact expression for the gradient.

With the period T as an additional variable, the gradient vector involves both the term $\partial P / \partial \underset{\sim}{x}$ and the term $\partial P / \partial T$. Using the estimate of the period, the first term is computed in the same manner as in the case that the period T of the oscillations is known. The second term can be computed in two different ways.

Nakhla and Vlach have approximated the integral

$$P(\underset{\sim}{x},T) = \int_0^T \underset{\sim}{\varepsilon}^T(\tau,T)\underset{\sim}{\varepsilon}(\tau,T)\,d\tau \tag{222}$$

using a suitable integration method in the form

$$P(\underset{\sim}{x},T) \simeq \frac{T}{N}\sum_{i=1}^{N} \alpha_i\, \underset{\sim}{\varepsilon}^T(t_i,T)\underset{\sim}{\varepsilon}(t_i,T) \tag{223}$$

where

N is the number of sampling points used to define the integrand,

α_i depends on the numerical integration method used;

and $$t_i = \frac{iT}{N} .$$

Differentiating (223) with respect to T

$$\begin{aligned}\frac{\partial P}{\partial T} &= \frac{1}{N}\sum_{i=1}^{N} \alpha_i\, \underset{\sim}{\varepsilon}^T(t_i,T)\underset{\sim}{\varepsilon}(t_i,T) \\ &+ \frac{2T}{N}\sum_{i=1}^{N} \alpha_i \left(\frac{\partial \underset{\sim}{\varepsilon}}{\partial T}\right)^T \underset{\sim}{\varepsilon}(t_i,T) \\ &+ \frac{2T}{N}\sum_{i=1}^{N} \alpha_i \frac{i}{N}\left(\frac{\partial \underset{\sim}{\varepsilon}}{\partial t}\right)^T \underset{\sim}{\varepsilon}(t_i,T)\end{aligned} \tag{224}$$

The last term in (224) appears because t_i is a function of T.

Rearranging (224)

$$\frac{\partial P}{\partial T} = \frac{1}{N}\sum_{i=1}^{N} \alpha_i \, \underset{\sim}{\varepsilon}^T(t_i,T)\, \underset{\sim}{\varepsilon}(t_i,T)$$
$$+\frac{2T}{N}\sum_{i=1}^{N} \alpha_i \left(\frac{\partial \underset{\sim}{\varepsilon}}{\partial T} + \frac{i}{N}\frac{\partial \underset{\sim}{\varepsilon}}{\partial t}\right)^T \underset{\sim}{\varepsilon}(t_i,T) \tag{225}$$

To find $\partial \underset{\sim}{\varepsilon}/\partial T$ differentiate (33) with respect to T

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial T} = \frac{\partial \underset{\sim}{y}_N}{\partial T} - \frac{\partial \underset{\sim}{y}_L}{\partial T} \tag{226}$$

Considering $\partial \underset{\sim}{y}_N/\partial T$ first, from (100)

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial T} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\frac{\partial \underset{\sim}{u}_s}{\partial T} + \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\frac{\partial \dot{\underset{\sim}{u}}_s}{\partial T} + \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial T} - \frac{\partial \underset{\sim}{y}_L}{\partial T} \tag{227}$$

is obtained. Differentiating (105),(201) and (202) with respect to T and noting that $\omega = \frac{2\pi}{T}$

$$\frac{\partial \underset{\sim}{u}_s}{\partial T} = -\mathrm{Re}\left\{\frac{\omega t}{T} j \sum_{k=0}^{M} k(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)e^{j\omega kt}\right\} \tag{228}$$

$$\frac{\partial \dot{\underset{\sim}{u}}_s}{\partial T} = \mathrm{Re}\left\{\frac{\omega^2 t}{T}\sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)e^{j\omega kt}\right\}$$
$$-\mathrm{Re}\left\{\frac{\omega}{T} j \sum_{k=0}^{M} k(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)e^{j\omega kt}\right\} \tag{229}$$

$$\frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial T} = \mathrm{Re}\left\{\frac{\omega^3 t}{T} j \sum_{k=0}^{M} k^3(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)e^{j\omega kt}\right\}$$
$$+\mathrm{Re}\left\{\frac{2\omega^2}{T}\sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)e^{j\omega kt}\right\} \tag{230}$$

are obtained. Using(106)

$$\frac{\partial \underset{\sim}{y}_L}{\partial T} = \mathrm{Re}\left\{ \sum_{k=0}^{M} \frac{\partial \underset{\sim}{y}_{Lk}(j\omega k)}{\partial T} e^{j\omega kt} \right\} - \mathrm{Re}\left\{ \frac{\omega t}{\partial T} j \sum_{k=0}^{M} k \underset{\sim}{y}_{Lk}(j\omega k)\, e^{j\omega kt} \right\} \tag{231}$$

To find $\dfrac{\partial \underset{\sim}{y}_{Lk}(j\omega k)}{\partial T}$, first note that in

the case that the period of the oscillations is unknown the equations(109) reduces to

$$\underset{\sim}{T}_k(j\omega k)\underset{\sim}{y}(j\omega k) = \underset{\sim}{A}_k(j\omega k)(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*) \tag{232}$$

since $\underset{\sim}{u}_{A_k}(j\omega k) = \underset{\sim}{0}$ for $k > 0$. Rewriting(232) explicitly

$$\underset{\sim}{T}_k(j\omega k)\underset{\sim}{y}(j\omega k) = j\omega k \underset{\sim}{A}_1(\underset{\sim}{x}_{k_1} - j\underset{\sim}{x}_{k_1}^*) + \underset{\sim}{A}_2(\underset{\sim}{x}_{k_2} - j\underset{\sim}{x}_{k_2}^*) \tag{233}$$

where $\underset{\sim}{y}(j\omega k) = \begin{bmatrix} \underset{\sim}{y}_k(j\omega k) \\ \underset{\sim}{y}_{Lk}(j\omega k) \end{bmatrix}$

Differentiating both side of the equation (233) with respect to T

$$\frac{\partial \underset{\sim}{T}_k}{\partial T}\underset{\sim}{y} + \underset{\sim}{T}_k \frac{\partial \underset{\sim}{y}}{\partial T} = -j\frac{\omega k}{T}\underset{\sim}{A}_1(\underset{\sim}{x}_{k_1} - j\underset{\sim}{x}_{k_1}^*) \tag{234}$$

$$\frac{\partial \underset{\sim}{y}}{\partial T} = -\underset{\sim}{T}_k^{-1}\frac{\partial \underset{\sim}{T}_k}{\partial T}\underset{\sim}{y} - j\frac{\omega k}{T}\underset{\sim}{A}_1(\underset{\sim}{x}_{k_1} - j\underset{\sim}{x}_{k_1}^*) \tag{235}$$

expressing $\underset{\sim}{y}$ in(235) as $\underset{\sim}{y} = \left[\underset{\sim}{y}_k^T \;\; \underset{\sim}{y}_{Lk}^T\right]^T$

$$\begin{bmatrix} \dfrac{\partial \underset{\sim}{y}_k}{\partial T} \\ \dfrac{\partial \underset{\sim}{y}_{Lk}}{\partial T} \end{bmatrix} = -\underset{\sim}{T}_k^{-1} \frac{\partial \underset{\sim}{T}_k}{\partial T} \begin{bmatrix} \underset{\sim}{y}_k \\ \underset{\sim}{y}_{Lk} \end{bmatrix} - j\frac{\omega k}{T} \underset{\sim}{A}_1 (\underset{\sim}{x}_{k_1} - j\underset{\sim}{x}_{k_1}^*) \tag{236}$$

Redefining $\underset{\sim}{T}_k^{-1} \dfrac{\partial \underset{\sim}{T}_k}{\partial T}$ as

$$\underset{\sim}{T}_k^{-1} \frac{\partial \underset{\sim}{T}_k}{\partial T} \triangleq \underset{\sim}{M}_k = \left[\begin{array}{c|c} M_k^{(1)} & M_k^{(2)} \\ \hline M_k^{(3)} & M_k^{(4)} \end{array}\right] \tag{237}$$

and

$$j\frac{\omega k}{T} \underset{\sim}{A}_1 (\underset{\sim}{x}_{k_1} - j\underset{\sim}{x}_{k_1}^*) \triangleq \begin{bmatrix} \underset{\sim}{m}_k^{(1)} \\ \underset{\sim}{m}_k^{(2)} \end{bmatrix} \tag{238}$$

Substituting (237) and (238) into (236)

$$\begin{bmatrix} \dfrac{\partial \underset{\sim}{y}_{Lk}}{\partial T} \\ \dfrac{\partial \underset{\sim}{y}_{Lk}}{\partial T} \end{bmatrix} = -\begin{bmatrix} \underset{\sim}{M}_k^{(1)} & \underset{\sim}{M}_k^{(2)} \\ \underset{\sim}{M}_k^{(3)} & \underset{\sim}{M}_k^{(4)} \end{bmatrix} \begin{bmatrix} \underset{\sim}{y}_k \\ \underset{\sim}{y}_{Lk} \end{bmatrix} - \begin{bmatrix} m_k^{(1)} \\ m_k^{(2)} \end{bmatrix} \tag{239}$$

from (239) $\dfrac{\partial \underset{\sim}{y}_{Lk}(j\omega k)}{\partial T}$ is derived as

$$\frac{\partial \underset{\sim}{y}_{Lk}(j\omega k)}{\partial T} = -\left[\underset{\sim}{M}_k^{(3)} \underset{\sim}{y}_k(j\omega k) + \underset{\sim}{M}_k^{(4)} \underset{\sim}{y}_{Lk}(j\omega k) + \underset{\sim}{m}_k^{(2)}\right]$$

Inserting from (228), (229), (230) and (236) into (227)

$$\begin{aligned}
\frac{\partial \underset{\sim}{\varepsilon}}{\partial T} = & -\mathrm{Re}\left\{\frac{\omega t}{T}\left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right) j \sum_{k=0}^{M} k(x_k - j x_k^*) e^{j\omega kt}\right\} \\
& + \mathrm{Re}\left\{\frac{\omega^2 t}{T}\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right) \sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j \underset{\sim}{x}_k^*) e^{j\omega kt}\right\} \\
& - \mathrm{Re}\left\{\frac{\omega}{T}\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right) j \sum_{k=0}^{M} k(\underset{\sim}{x}_k - j \underset{\sim}{x}_k^*) e^{j\omega kt}\right\} \\
& + \mathrm{Re}\left\{\frac{\omega^3 t}{T}\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right) j \sum_{k=0}^{M} k^3(\underset{\sim}{x}_k - j \underset{\sim}{x}_k^*) e^{j\omega kt}\right\} \\
& + \mathrm{Re}\left\{\frac{2\omega^2}{T}\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right) \sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j \underset{\sim}{x}_k^*) e^{j\omega kt}\right\} \\
& + \mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)} \underset{\sim}{y}_k(j\omega k) + \underset{\sim}{M}_k^{(4)} \underset{\sim}{y}_{Lk}(j\omega k) + \underset{\sim}{m}_k^{(2)}\right] e^{j\omega kt}\right\} \\
& + \mathrm{Re}\left\{\frac{\omega t}{T} j \sum_{k=0}^{M} k\, \underset{\sim}{y}_{Lk}(j\omega k)\, e^{j\omega kt}\right\}
\end{aligned} \qquad (241)$$

Equation(241) is used to determine the second term of (224). For the third term we need $\partial \underset{\sim}{\varepsilon}/\partial t$, Using(33).

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial t} = \frac{\partial \underset{\sim}{y}_N}{\partial t} - \frac{\partial \underset{\sim}{y}_L}{\partial t} \tag{242}$$

Using(100)

$$\frac{\partial \underset{\sim}{\varepsilon}}{\partial t} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\frac{\partial \underset{\sim}{u}_s}{\partial t} + \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\frac{\partial \dot{\underset{\sim}{u}}_s}{\partial t} + \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial t} - \frac{\partial \underset{\sim}{y}_L}{\partial t} \tag{243}$$

Setting

$$\begin{aligned}
\frac{\partial \underset{\sim}{u}_s}{\partial t} &= \dot{\underset{\sim}{u}}_s \\
\frac{\partial \dot{\underset{\sim}{u}}_s}{\partial t} &= \ddot{\underset{\sim}{u}}_s \\
\frac{\partial \ddot{\underset{\sim}{u}}_s}{\partial t} &= \dddot{\underset{\sim}{u}}_s
\end{aligned} \tag{244}$$

Differentiating(202) with respect to t

$$\dddot{\underset{\sim}{u}}_s(t) = \mathrm{Re}\left\{\sum_{k=0}^{M} -j\omega^3 k^3 (\underset{\sim}{\chi}_k - j\underset{\sim}{\chi}_k^*)\, e^{j\omega k t}\right\} \tag{245}$$

Differentiating (106) with respect to t

$$\frac{\partial \underset{\sim}{y}_L}{\partial t} = \mathrm{Re}\left\{\sum_{k=0}^{M} j\omega k\, \underset{\sim}{y}_{Lk}(j\omega k)\, e^{j\omega k t}\right\} \tag{246}$$

Inserting from (201),(202),(245) and (246) into (243)

$$\frac{\partial \underset{\sim}{\mathcal{E}}}{\partial t} = \mathrm{Re}\left\{\omega\left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right) j \sum_{k=0}^{M} k(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t}\right\}$$

$$- \mathrm{Re}\left\{\omega^2\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right) \sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t}\right\} \qquad (247)$$

$$- \mathrm{Re}\left\{\omega^3\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right) j \sum_{k=0}^{M} k^3(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t}\right\}$$

$$- \mathrm{Re}\left\{\omega j \sum_{k=0}^{M} k\, \underset{\sim}{y}_{L_k}(j\omega k)\, e^{j\omega k t}\right\}$$

Combining (241) and (247) the second term of the equation (225) can be determined as follows:

$$\frac{\partial \underset{\sim}{\mathcal{E}}}{\partial T} + \frac{i}{N}\frac{\partial \underset{\sim}{\mathcal{E}}}{\partial t_i} =$$

$$- \mathrm{Re}\left\{\omega \frac{i}{N}\left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right) j \sum_{k=0}^{M} k(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\} \qquad (*)$$

$$+ \mathrm{Re}\left\{\omega^2 \frac{i}{N}\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right) \sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\} \qquad (**)$$

$$- \mathrm{Re}\left\{\frac{\omega}{T}\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right) j \sum_{k=0}^{M} k(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\}$$

$$+ \mathrm{Re}\left\{\omega^3 \frac{i}{N}\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right) j \sum_{k=0}^{M} k^3(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\} \qquad (***)$$

$$+\mathrm{Re}\left\{\frac{2\omega^2}{T}\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)\sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\}$$

$$+\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k) + \underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{Lk}(j\omega k) + \underset{\sim}{m}_k^{(2)}\right] e^{j\omega k t_i}\right\}$$

$$+\mathrm{Re}\left\{\omega\frac{i}{N} j\sum_{k=0}^{M} k\,\underset{\sim}{y}_{Lk}(j\omega k)\, e^{j\omega k t_i}\right\} \qquad \left(\begin{smallmatrix}**\\ **\end{smallmatrix}\right)$$

$$+\mathrm{Re}\left\{\omega\frac{i}{N}\left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right) j\sum_{k=0}^{M} k(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\} \qquad (*)$$

$$-\mathrm{Re}\left\{\omega^2\frac{i}{N}\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)\sum_{k=0}^{M} k^2(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\} \qquad (**)$$

$$-\mathrm{Re}\left\{\omega^3\frac{i}{N}\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right) j\sum_{k=0}^{M} k^3(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\} \qquad (***)$$

$$-\mathrm{Re}\left\{\omega\frac{i}{N} j\sum_{k=0}^{M} k\,\underset{\sim}{y}_{Lk}(j\omega k)\, e^{j\omega k t_i}\right\} \qquad \left(\begin{smallmatrix}**\\ **\end{smallmatrix}\right)$$

Cancelling the identical terms,

the expression for $\frac{\partial \underset{\sim}{\varepsilon}}{\partial T} + \frac{i}{N}\frac{\partial \underset{\sim}{\varepsilon}}{\partial t_i}$ becomes

$$-\mathrm{Re}\left\{\frac{1}{T}\sum_{k=0}^{M} j\omega k\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\}$$

$$+\mathrm{Re}\left\{\frac{2}{T}\sum_{k=0}^{M}\omega^2 k^2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\}$$

$$+\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k) + \underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{Lk}(j\omega k) + \underset{\sim}{m}_k^{(2)}\right] e^{j\omega k t_i}\right\} \qquad (248)$$

Rearranging the terms in (248) $\frac{\partial \underset{\sim}{\varepsilon}}{\partial T} + \frac{i}{N}\frac{\partial \underset{\sim}{\varepsilon}}{\partial t_i} =$

$$-\frac{1}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}\left[j\omega k\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right) - 2\omega^2 k^2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)\right](\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)\, e^{j\omega k t_i}\right\}$$

$$+\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k) + \underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{Lk}(j\omega k) + \underset{\sim}{m}_k^{(2)}\right] e^{j\omega k t_i}\right\} \tag{249}$$

Inserting from (249) into (224) obtain

$$\frac{\partial P(\underset{\sim}{x},T)}{\partial T} = \frac{1}{N}\sum_{i=1}^{N}\alpha_i\, \underset{\sim}{\varepsilon}^T(t_i,T)\underset{\sim}{\varepsilon}(t_i,T)$$

$$-\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)^T\cdot\frac{T}{N}\sum_{i=1}^{N}\alpha_i\left[j\omega k\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right) - 2\omega^2 k^2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)\right]^T\underset{\sim}{\varepsilon}(t_i,T)e^{j\omega k t_i}\right\}$$

$$+2\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k) + \underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{Lk}(j\omega k) + \underset{\sim}{m}_k^{(2)}\right]^T\cdot\frac{T}{N}\sum_{i=1}^{N}\alpha_i\,\underset{\sim}{\varepsilon}(t_i,T)e^{j\omega k t_i}\right\} \tag{250}$$

Equation (250) is the required expression for the gradient component $\partial P/\partial T$. The summation

$$\frac{1}{N}\sum_{i=1}^{N}\alpha_i\, \underset{\sim}{\varepsilon}^T(t_i,T)\,\underset{\sim}{\varepsilon}(t_i,T) , \tag{251}$$

the internal summation in the second term of (250)

$$\frac{T}{N}\sum_{i=1}^{N}\alpha_i\left[j\omega k\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)-2\omega^2k^2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)\right]^T\underset{\sim}{\varepsilon}(t_i,T)\,e^{j\omega k t_i} \quad , \tag{252}$$

and the internal summation in the third term of (250)

$$\frac{T}{N}\sum_{i=1}^{N}\alpha_i\,\underset{\sim}{\varepsilon}(t_i,T)\,e^{j\omega k t_i} \tag{253}$$

are the numerical approximations of the integrals:

$$\frac{1}{T}\int_0^T \underset{\sim}{\varepsilon}^T(\tau,T)\,\underset{\sim}{\varepsilon}(\tau,T)\,d\tau \tag{254}$$

$$\int_0^T\left[j\omega k\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)-2\omega^2k^2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)\right]^T\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau \tag{255}$$

and

$$\int_0^T \underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau \tag{256}$$

respectively. Finally, substituting from (253). (254) and (255) into (250) we obtain:

$$\frac{\partial P(\underset{\sim}{x},T)}{\partial T}=\frac{1}{T}\int_0^T \underset{\sim}{\varepsilon}^T(\tau,T)\,\underset{\sim}{\varepsilon}(\tau,T)\,d\tau$$

$$\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)^T\cdot\int_0^T\left[j\omega k\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)-2\omega^2k^2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)\right]^T\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\}$$

$$2\,\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k)+\underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{L_k}(j\omega k)+\underset{\sim}{m}_k^{(2)}\right]^T\int_0^T\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\}$$

(257)

It can easily be observed from (219), (255) and (256) that the integrals to be evaluated for the calculation of the gradient component $\partial P/\partial \underset{\sim}{x}$ in two cases are

$$\int_0^T\left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right)^T\underset{\sim}{\varepsilon}(\tau)\,e^{j\omega k\tau}\,d\tau \qquad (258)$$

$$\int_0^T\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T\underset{\sim}{\varepsilon}(\tau)\,e^{j\omega k\tau}\,d\tau \qquad (259)$$

$$\int_0^T\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T\underset{\sim}{\varepsilon}(\tau)\,e^{j\omega k\tau}\,d\tau \qquad (260)$$

and

$$\int_0^T\underset{\sim}{\varepsilon}(\tau)\,e^{j\omega k\tau}\,d\tau \qquad (261)$$

It is shown in the equation (257) that the additional integrals to be evaluated for the calculation of the gradient component $\partial P/\partial T$ in the cases in which the period of the oscillations is unknown are

$$\int_0^T \underset{\sim}{\varepsilon}^T(\tau,T)\,\underset{\sim}{\varepsilon}(\tau,T)\,d\tau \tag{262}$$

$$\int_0^T \left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau \tag{263}$$

$$\int_0^T \left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau \tag{264}$$

$$\int_0^T \underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau \tag{265}$$

These integrals have been used by Nakhla and Vlach[8] for the calculation of the objective function P and the gradient component $\partial P/\partial \underset{\sim}{\chi}$.

The method of approximating the objective function P by a numerical integration algorithm for the calculation of the gradient component $\partial P/\partial T$ as proposed by Nakhla and Vlach [8] does not yield a reliable expression for $\partial P/\partial T$. The error introduced in approximating the expression for $\partial P/\partial T$ may be amplified since the approximate expression for P is differentiated. This stiuation can intuatively be explained as follows. Let P^* be a numerical approximation to P , the function to be minimized.

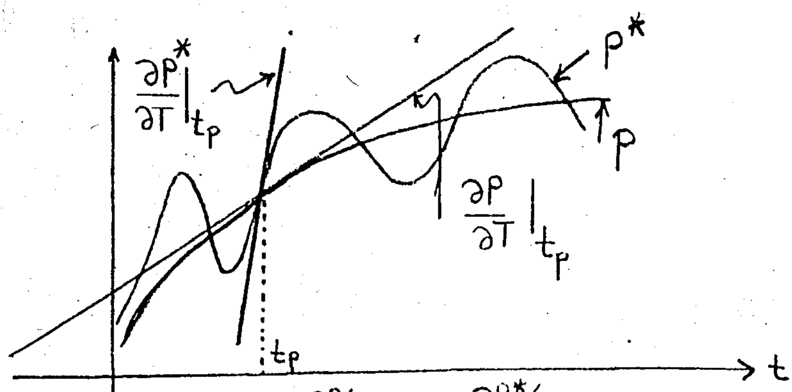


Figure 25- Comparison of $\partial P/\partial T$ and $\partial P^*/\partial T$

Suppose that the error function P approximated in the form of P^* as illustrated in Fig.25. Then, the slope of the approximated function at a specific period t_p will considerably differ from the slope of the exact value of the objective function P at t_p. This fact indicates that the method proposed by Nakhla and Vlach [8] for calculating $\partial P/\partial T$ is not a reliable one.

In order to overcome this difficulty and to obtain a reliable expression for the value of $\partial P/\partial T$, we propose in this thesis to differentiate the exact expression of the error function P with respect to T without doing any approximation for P.

Differentiating (222) with respect to T

$$\frac{\partial P}{\partial T} = \underset{\sim}{\varepsilon}^T(T,T)\,\underset{\sim}{\varepsilon}(T,T) + 2\int_0^T \left(\frac{\partial \underset{\sim}{\varepsilon}}{\partial T}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\,d\tau \tag{266}$$

Taking the transpose of (241)

$$\begin{aligned}\left(\frac{\partial \underset{\sim}{\varepsilon}}{\partial T}\right)^T = &-\frac{1}{T}\mathrm{Re}\left\{\sum_{k=0}^{M} j\omega k\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)^T \cdot \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right)^T e^{j\omega k\tau}\right\}\\ &+\frac{1}{T}\mathrm{Re}\left\{\sum_{k=0}^{M} \omega^2 k^2\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)^T \cdot \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T e^{j\omega k\tau}\right\}\\ &-\frac{1}{T}\mathrm{Re}\left\{\sum_{k=0}^{M} j\omega k\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)^T \cdot \left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T e^{j\omega k\tau}\right\}\\ &+\frac{1}{T}\mathrm{Re}\left\{\sum_{k=0}^{M} j\omega^3 k^3\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*)^T \cdot \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T e^{j\omega k\tau}\right\}\end{aligned}$$

$$+\frac{1}{T}\mathrm{Re}\left\{2\sum_{k=0}^{M}\omega^2k^2(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)^T\cdot\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T e^{j\omega k\tau}\right\}$$

$$+\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k)+\underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{L_k}(j\omega k)+\underset{\sim}{m}_k^{(2)}\right]e^{j\omega k\tau}\right.$$

$$+\frac{1}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}j\omega k\,\underset{\sim}{y}_{L_k}^T(j\omega k)\cdot\tau\, e^{j\omega k\tau}\right\} \tag{267}$$

Substituting from (267) into (266)

$$\frac{\partial P}{\partial T}=\underset{\sim}{\varepsilon}^T(T,T)\,\underset{\sim}{\varepsilon}(T,T)$$

$$-\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}j\omega k(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)^T\cdot\int_0^T\left[\tau\left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right)^T+\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T\right]\underset{\sim}{\varepsilon}(\tau,T)e^{j\omega k\tau}d\tau\right\}$$

$$+\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}\omega^2k^2(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)^T\cdot\int_0^T\left[\tau\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T+2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T\right]\underset{\sim}{\varepsilon}(\tau,T)e^{j\omega k\tau}d\tau\right\}$$

$$+\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}j\omega^3k^3(\underset{\sim}{x}_k-j\underset{\sim}{x}_k^*)^T\cdot\int_0^T\tau\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}d\tau\right\}$$

$$+2\,\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k)+\underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{L_k}(j\omega k)+\underset{\sim}{m}_k^{(2)}\right]^T\cdot\int_0^T\underset{\sim}{\varepsilon}(\tau,T)e^{j\omega k\tau}d\tau\right\}$$

$$+\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}j\omega k\,\underset{\sim}{y}_{L_k}^T(j\omega k)\cdot\int_0^T\tau\,\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\right\} \tag{268}$$

Rearranging the terms in (268)

$$
\begin{aligned}
\frac{\partial P}{\partial T} = {} & \underset{\sim}{\varepsilon}^{T}(T,T)\,\underset{\sim}{\varepsilon}(T,T) \\
& -\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^{*})^{T}\cdot\int_0^T\left[j\omega k\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)-2\omega^2k^2\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)\right]^{T}\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\} \\
& +2\,\mathrm{Re}\left\{\sum_{k=0}^{M}\left[\underset{\sim}{M}_k^{(3)}\underset{\sim}{y}_k(j\omega k)+\underset{\sim}{M}_k^{(4)}\underset{\sim}{y}_{Lk}(j\omega k)+\underset{\sim}{m}_k^{(2)}\right]^{T}\cdot\int_0^T\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\} \\
& -\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}j\omega k\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^{*})^{T}\cdot\int_0^T\tau\left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^{T}\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\} \\
& +\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}\omega^2k^2\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^{*})^{T}\cdot\int_0^T\tau\left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^{T}\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\} \\
& +\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}j\omega^3k^3\,(\underset{\sim}{x}_k - j\underset{\sim}{x}_k^{*})^{T}\cdot\int_0^T\tau\left(\frac{\partial \underset{\sim}{f}}{\partial \dddot{\underset{\sim}{u}}_s}\right)^{T}\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\} \\
& +\frac{2}{T}\mathrm{Re}\left\{\sum_{k=0}^{M}j\omega k\;\underset{\sim}{y}_{Lk}^{T}(j\omega k)\cdot\int_0^T\tau\,\underset{\sim}{\varepsilon}(\tau,T)\,e^{j\omega k\tau}\,d\tau\right\}
\end{aligned}
\tag{269}
$$

Eq(269) represents the exact expression for $\partial P/\partial T$. It can be observed that the second and the third terms of the exact expression for $\partial P/\partial T$ are identical to those derived by Nakhla and Vlach [8] in (257). It is also shown that the exact expression for $\partial P/\partial T$ includes four additional terms which does not exist in the expression derived by Nakhla and Vlach [8] in (257).

This fact indicates that the method proposed by Nakhla and Vlach [8] leads us to a very crude approximation for $\partial P/\partial T$.

The exact expression in (269) for $\partial P/\partial T$ contains four additional integrals in addition to those existing in the approximate expression in (257).

These are

$$\int_0^T \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau \qquad (270)$$

$$\int_0^T \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau \qquad (271)$$

$$\int_0^T \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau \qquad (272)$$

$$\int_0^T \tau\, \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau \qquad (273)$$

So the cost of the computation is considerably increased using (269) instead of (257).

V. PROGRAMMING CONSIDERATION

In this chapter first the steps of the method presented in chapters II, III, and IV are collected in a general algorithm. Then a computer program is presented for the case that the period T of the oscillations is known.

The functions of the main program and several other subroutines will be explained in detail. The steps of the main algorithm and a subroutine that evaluates the error function P and its gradient $\frac{\partial P}{\partial \underset{\sim}{x}}$ is presented. Their flow-graphs, and comments about the functions of some important parameters are explained in detail.

The steps of the method can be collected in a general algorithm which presents the way of solving the problem both in the case that the period of the oscillations is known and in the case that the period of the oscillations is unknown.

1) Decompose the network into a linear network and nonlinear subnetworks.

2) Estimate the variables in (103) and T(for the case in which the period of the oscillations is unknown)

3) Solve the linear network described by (109),(110) and (114) in the frequency domain to determine $\underset{\sim}{Y}_{Lk}(j\omega k)$ and $\bar{\bar{\underset{\sim}{y}}}_{Lk}(j\omega k)$ .Let $u_{Ak}(j\omega k)=0$ for the case that the period of the oscillations is unknown such that $\bar{\bar{\underset{\sim}{y}}}_{Lk}(j\omega k)$ becomes zero. Here M is the number of harmonics to be considered.

4) Evaluate(105) and (106) using Discrete Fourier Transform (DFT) techniques.

5) Evaluate (104) using trapezoidal rule.

6) If the error function P is less than a prespecified small value stop. Otherwise go to step 7.

7) Use the DFT algorithm again to evaluate the integrals.

$$\int_0^T \left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

$$\int_0^T \left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

$$\int_0^T \left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

and

$$\int_0^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

if the period of the oscillations is already known.

In addition to these integrals, calculate

$$\int_0^T \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

$$\int_0^T \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

$$\int_0^T \tau \left(\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}\right)^T \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

$$\int_0^T \tau\, \underset{\sim}{\varepsilon}(\tau,T)\, e^{j\omega k\tau}\, d\tau$$

if the period of the oscillations is unknown.

8) Transfer the error function P and the gradient vector as arguments to the optimization routine.

9) Use the correction vector $\Delta\underset{\sim}{\chi}$ returned by the optimization routine to compute the next estimate of $\underset{\sim}{\chi}$. For the case in which the period of the oscillations is unknown, use the incremental change T to readjust the value of the period for the next iteration.

10) Go to step (4) if the period of the oscillations is already known, otherwise go to step(3).

The computer program associated with the Piecewise Harmonic Balance Method for the case that the period T of the oscillations is known is prepared so that it can solve the networks containing linear and nonlinear elements as long as the required subroutines which define the characteristic behaviour of each nonlinear element are supplied. A main program which includes the preparation of the matrix $\underset{\sim}{Y}_{LK}(j\omega k)$ in (110) and an additional subroutine which arranges the interconnection of the variables among the subroutines characterizing nonlinear elements should also be provided.

The program which is complied to test some networks with nonlinearities is not a general purpose program. It permits the network to contain voltage-controlled junction diodes with a nonlinear junction capacitor, and transistors with nonlinear junction and diffusion capacitors and dependent sources as nonlinear elements simultaneously. In addition the network must contain a unique actual sinusoidal voltage or current source. It is also allowed to contain some actual dc voltage sources. The preperation of the matrix $\underset{\sim}{Y}_{Lk}(j\omega k)$ which is stored into the three dimensional matrix CRESP is also performed in the main program auomatically. The only job to be done by the user to start the runstream after specifying some parameters is to read the circuit description.

V.1. THE ALGORITHM
MAIN PROGRAM

(i) Read-in the circuit description

1) Specify the parameter values which arranges the dimensions of the variables in the main program as follows:

MNNS : number of augmenting sources

MNNS1 : MNNS + 1

MNEL : number of elements in the linear circuit with augmenting sources.

MQ : dimension of the MNT matrix to be used for the solution of the linear network with augmenting sources.

MOUTN : number of outputs which are desired to be computed in addition to the augmenting source variables and to the complementary variables of the augmenting sources.

MOUTN2 : MOUTN/2

MOUT : MOUTN + 4

MNHAR1 : NHAMAX + 1

where NHAMAX is the maximum number of harmonics to be allowed for continuing the iteration

MNP1 : (NHAMAX + NP1AD) * 2 + 1

where NP1AD is the number of additional points considered in calculating time values.

MNOPT : (NHAMAX * 2 + 1) * MNNS

MWORK : (MNOPT + 7) * MNOPT/2

2) Select the initial number of harmonics (NHAR), number of additional harmonics (NAD), maximum number of harmonics (NHAMAX) to be allowed for continuing the iteration the initial estimate (XOPTIN) of the vector $\underset{\sim}{x}$ given by (103). In addition, read

NTR : number of transistors

NDI : number of diodes

NDC : number of actual dc sources

NODES: number of nodes

NEL : number of elements
ISR : 0 if there is an actual sinusoidal voltage source
1 if there is an actual sinusoidal current source in the circuit
AMP : amplitude of the actual sinusoidal source
FREQ : frequency of the oscillations in Hz, and some other parameters whose meanings will be explained later.

3) Read network elements and interconnectiors in the following order:

A. Insert the linear elements such as resistors, capacitors and inductors etc,
B. Insert the actual sinusoidal source,
C. Insert the actual dc source
D. Read the augmenting voltage sources replaced in nonlinear elements in the following order:

(i) Insert the augmenting source corresponding to the emitter - base branch of a transistor
(ii) Insert the augmenting source corresponding to the collector-base branch of the same transistor,continue in this manner for all the transistors
(iii) Insert the diode branches.

4) Read the parameters defining the characteristics of the transistors and diodes respectively.

5) Read NOUTN, the number of additional outputs which are desired to be computed in addition to the augmenting source variables. NOUTN should be an even number, i.e., the number of additional outputs desired to be computed is required to be even since the time values of the outputs in the network are calculated by superimposing two function into one (This point will be further elaborated in the appendix). In addition,

specify the outputs which are desired to be calculated and plotted by introducing the number corresponding to the position of the unknown in the MNT equations.

6) Also, read

IXOPT: 0 if there is no individual initialization for $\underset{\sim}{\chi}$, and 1 if $\underset{\sim}{\chi}$ is initially specified.

INP1: 0 if the integrals are to be evaluated at 2*NP1+1 points, and 1 if the integrals are to be evaluated at more than 2*NP1 + 1 points.

NP1AD: Number of additional points provided for the evaluation of the integrals and the time functions in the case that INP1 = 1

NXOPT: dimension of the vector for the initial estimates,

IPRINT: controls printing of the optimization subroutine,

MAXFN: an upper limit on the number of iterations to be performed,

IWORK: should be equal to NOPT*(NOPT + 7)/2.

(ii) From the network description, obtain the linear subnetwork by connecting the augmenting sources properly.

(iii) Solve the linear network as described in chapter III for each frequency $k\omega$ and for unit sources to obtain given by (112).

(iv) Using the subroutine FUNCT, evaluate the error function P given by (104) and its gradient $\partial P/\partial \underset{\sim}{\chi}$ given by (219) for provided XOPT(I).

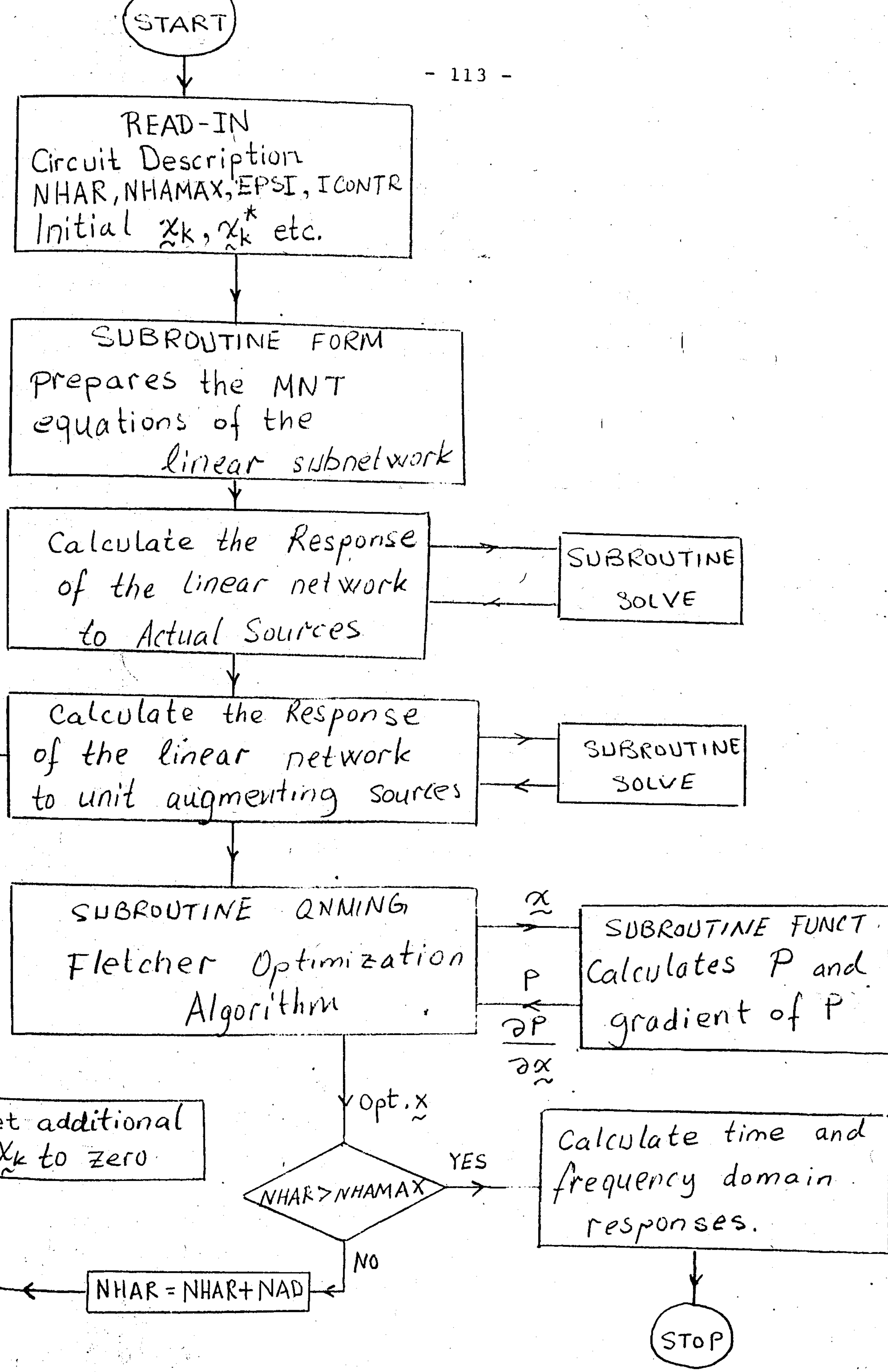


Figure 26- Flow-graph of the main algorithm.

(v) Transfer P and $\partial P/\partial \underset{\sim}{\chi}$ to the optimization subroutine to calculate the new estimate XOPTN(I) of $\underset{\sim}{\chi}$. If $|XOPT(I) - XOPTN(I)| < EPS(I)$ for $I \in \{1,2,\ldots,2NHAR+1$ go to step (vi), otherwise go back to (iv) with XOPT (I) XOPTN(I). The optimization subroutine used in this step is based on Fletcher's algorithm described in[10].

(vi) If NHAR $\geqslant$ NHAMAX go to (ix). Otherwise go to (vii).

(vii) If P is less than the prespecified value EPS go to (ix). Otherwise go to (viii).

(viii) Set NHAR = NHAR + NAD and the additional coefficients $\underset{\sim}{\chi}_k$ to zero and go to step (iii) to solve the linear network only for the additional frequencies.

(ix) Calculate time values and frequency domain response of the desired variables in the network other than the variables associated with the augmenting sources.

(x) Plot all the time values that are computed in the given network.

These steps of the algorithm MAIN are illustrated in the flow graph in Fig. 26.

SUBROUTINE FORM

This subroutine prepares the MNT equations for the linear augmented subcircuit as explained in section III.1 when the circuit elements are read as described in the MAIN PROGRAM. The parameters MQ and MNEL shoud be specified as in the MAIN PROGRAM at the top of the subroutine.

SUBROUTINE SOLVE

This subroutine solves the MNT equations prepared by SUBROUTINE FORM. It provides forward and backward substitution on the matrices $\underset{\sim}{L}$ and $\underset{\sim}{U}$ as described in section III.2. The value of the parameter MQ should be specified as before.

SUBROUTINE LU

This subroutine creates the matrices $\underset{\sim}{L}$ and $\underset{\sim}{U}$ as explained in section III.2 from the MNT equations prepared by SUBROUTINE FORM. The only parameter that has to be specified is MQ.

SUBROUTINE FUNCT

This subroutine takes the values of XOPT (I) provided by the optimization subroutine and returns to it the function value P and its gradient $\partial P/\partial \underset{\sim}{x}$. It consists of the following steps.

(i) Rearrange the values XOPT (I) of the vector given by (103) to correspond to the form $\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*$

(ii) Scale the matrix $\underset{\sim}{Y}_{Lk}$ provided by (iii) of MAIN with the coefficients $\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*$ to obtain $\overline{\overline{\underset{\sim}{y}}}_{Lk}$ as in (113). Then obtain $\underset{\sim}{y}_{Lk}$ as in (115), for $k \in \{0, 1, \ldots, \mathrm{NHAR}\}$.

(iii) From the phasor values $\underset{\sim}{y}_{Lk}$ and $\underset{\sim}{x}_k - j\underset{\sim}{x}_k^*$ obtain the time values $\underset{\sim}{y}_L(t_i)$ as given by (106) and $\underset{\sim}{u}_s(t_i)$, $\dot{\underset{\sim}{u}}_s(t_i)$, $\ddot{\underset{\sim}{u}}_s(t_i)$ as given by (203),(204),(205) at the sampling instants t_i . The DFT described in the appendix is used in this step.

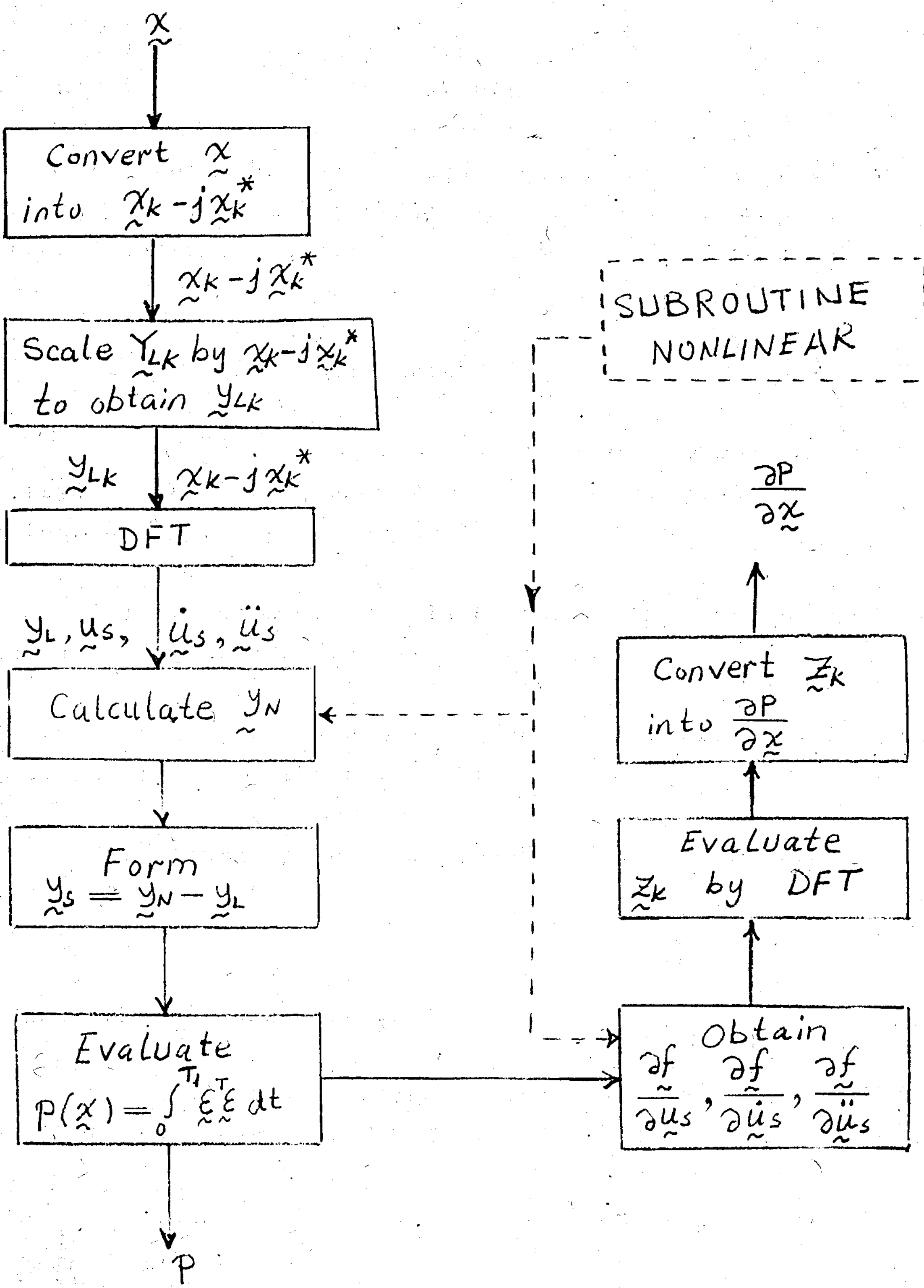


Figure 27- Flow-graph of subroutine FUNCT.

(iv) Evaluate $\underset{\sim}{y}_N(t_i)$ using $\underset{\sim}{u}_s(t_i)$, $\dot{\underset{\sim}{u}}_s(t_i)$, $\ddot{\underset{\sim}{u}}_s(t_i)$ obtained in step (iii) and relation (100).

(v) Form the vector error function $\underset{\sim}{\varepsilon}(t_i)$ defined by (33) and evaluate P($\underset{\sim}{\chi}$) using (104). The trapezoidal rule is selected for the numerical integration of the error function P.

(vi) Calculate the partial derivatives $\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}(t_i)$ $\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}(t_i)$, $\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}(t_i)$ of the functions describing the nonlinearities. The subroutine NONLNR is responsible for the evaluation of these derivatives.

(vii) Evaluate the complex gradient $\underset{\sim}{Z}_k$ for $k \in \{0,1,\ldots,M\}$ according to (219) using DFT techniques.

(viii) Reorder the complex gradient vector, into the real gradient vector as shown in (221).

The flow-graph of the SUBROUTINE FUNCT is illustrated in Fig.27.

SUBROUTINE NONLINEAR (NONLNR)

The subroutine Nonlinear required by Subroutine FUNCT provides the function f in (100) and its partial derivatives $\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}$, $\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}$, $\frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}$ used in evaluating the gradient. These are filled-in one 2 dimensional and two 3 dimensional matrices using $\underset{\sim}{u}_s(t)$, $\dot{\underset{\sim}{u}}_s(t)$, and $\ddot{\underset{\sim}{u}}_s(t)$ which are already prepared in FUNCT and stored in the matrices XS(I,K), XSD(I,K), XSDD(I,K) for $I \in \{1,2,\ldots,NNS\}$ and $K \in \{1,2,\ldots,2NHAR + 1\}$. For each

I and K, XS(I,K), XSD(I,K), XSDD(I,K) provide one with the value at the time instant t_k of the I-th augmenting source its first time derivative and second time derivative respectively.

The function f in(100) and its partial derivatives $\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}_s}, \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{u}}_s}, \frac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{u}}_s}$ are filled-in the matrices by the subroutine NONLNR as described in the following:

YS(I,K): containing the complementary variable of the I-th augmenting source (in the nonlinear subnetwork) at the K-th time instant,

PAR(I,J,K): containing the partial derivative of the complementary variable of the I-th augmenting source at the K-th time instant (i.e. $\frac{\partial \underset{\sim}{f}_i}{\partial \underset{\sim}{u}_{sj}}(t_k)$)

PARD(I,J,K) and PARDD(I,J,K) are filled-in as is PAR(I,J,K) except that the partial derivatives are taken with respect to $\dot{\underset{\sim}{u}}_s$ and $\ddot{\underset{\sim}{u}}_s$ respectively, instead of $\underset{\sim}{u}_s$.

If the nonlinearities in the network are not dynamical (i.e. $\underset{\sim}{y}_N = \underset{\sim}{f}(\underset{\sim}{u}_s, \underset{\sim}{y}_A)$) then there is no need to evaluate XSD, XSDD as well as PARD and PARDD; moreover the expression for the gradient becomes simpler. Setting ICONTR = 0 in the programme these unrequired steps are omitted. If ICONTR $\neq$ 0, but $\ddot{\underset{\sim}{u}}_s$ is not present the matrix PARDD = $\underset{\sim}{0}$ must be provided.

In this program, the matrices YS(I,K), PAR(I,J,K), PARD(I,J,K) and PARDD(I,J,K) are provided by subroutine DIODE for a junction diode with a nonlinear junction capacitor and subroutine TRNSTR for a transistor with nonlinear junction and diffusion capacitors and dependent sources. For other types of nonlinear elements a subroutine providing the information described in the following must be supplied.

SUBROUTINE TRANSISTOR

This subroutine provides the complementary variables of the augmenting sources connected across transistor branches, and the partial derivatives of the complementary variables of the augmenting sources connected across transistor branches with respect to the augmenting source variables and the first time derivative of the augmenting sources, respectively.

Two different kinds of subroutine describing the behaviour of the transistors are used in this program. Both of them are based on the same model of the transistor, namely, the Ebers-Moll model. In the following, the models that were used in the program will be presented, seperately. The Ebers-Moll model of the transistor is illustrated in Fig.28.

MODEL 1

In this model, the equations governing the behaviour of the transistor are given as follows:

(i) Source Current

$$I_{\bar{X}C} = I_0\left(e^{v_{BX}/V_T \cdot m} - 1\right) \tag{274}$$

where $$I = I_S\left(\frac{T}{300}\right)^3 e^{\left[\frac{-1.11}{k}\left[\frac{1}{T} - \frac{1}{300}\right]\right]}$$

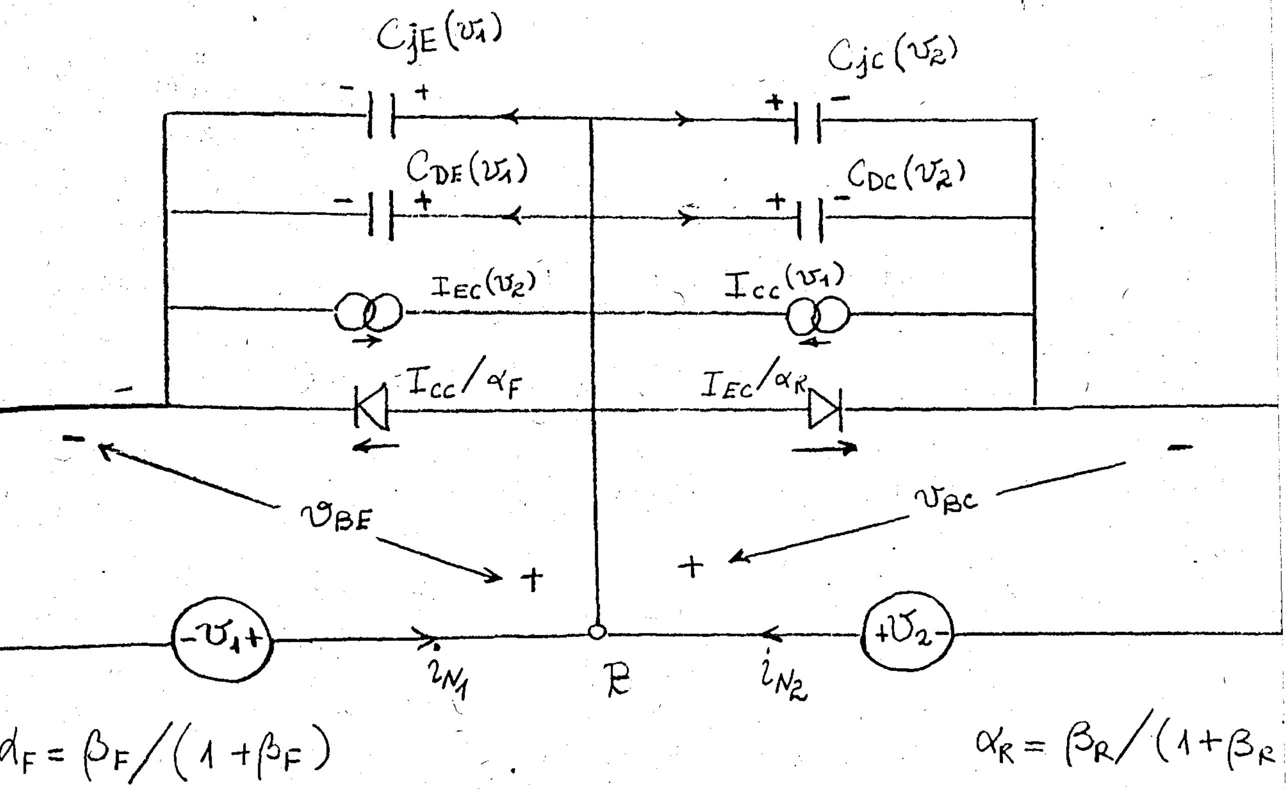


Figure 28- Ebers-Moll Model of the transistor

$k =$ Boltzman const.

$V_T = k.T/q \quad (k/q = 8.6164 \cdot 10^{-5})$

(ii) Junction Capacitances

$$C_{jx} = \frac{C_{jox}}{(1 - v_{Bx}/\phi_x)^{\gamma_x}} \tag{27}$$

(iii) Diffusion Capacitances

$$\begin{aligned} C_{DE} &= \tau_F \cdot g_E & v_{BE} > 0 \\ &= 0 & v_{BE} \leqslant 0 \end{aligned} \tag{27}$$

$$C_{DC} = \tau_R \cdot g_c \qquad v_{BC} > 0$$
$$= 0 \qquad v_{BC} \leq 0 \tag{277}$$

where V_T, m, ϕ_x , γ_x and β_R are some constants,

$$\tau_F = 1./(2\pi f_t) \qquad I_{cf} = 0$$

$$\tau_F = \frac{1.}{2\pi f_t} - \left[C_{jE}(V_{Ef}) + C_{jc}(V_{cf})\right] \cdot \frac{V_T}{I_{cf}} \qquad I_{cf} \neq 0 \tag{278}$$

if $\tau_F < 0$ from above expression
it is set to zero

$C_{jE}(V_{Ef})$ and $C_{jc}(V_{cf})$ are calculated according to (278)

$$\tau_R \doteq 0 \qquad \beta_R = 0$$

$$\tau_R = \frac{\tau_{SAT} \cdot \dfrac{1 + \beta_R + \beta_{FMAX}}{\beta_{FMAX}} - \tau_F(1+\beta_R)}{} \qquad \beta_R \neq 0$$

if $\tau_R < 0$ from above expression
it is set to zero

(279)

$$g_X = \frac{dI_{\bar{X}C}}{dV_{BX}} = \frac{I_{\bar{X}C} + I_0}{V_T.m} = \frac{I_0}{V_T.m} e^{V_{BX}/V_T.m} \tag{280}$$

The nonlinear transistor model whose describing equations presented above is linearized for some values of v_{BX} in the computer program since the source currents will overflow for very large or small values of v_{BX} in (274) and the junction capacitances will be undefined for $v_{BX} > \phi_X$ in(275)

In this case, the describing equations of the source currents become

$$I_{\bar{x}c} = \begin{cases} \frac{1}{V_T.m} e^{(V_1/V_T.m + X)} (v_{BX} - V_1) + e^{(V_1/V_T.m + X)} - I_o & V_1 \leq v_{BX} \\ e^{(v_{BX}/V_T.m + X)} - I_o & V_2 \leq v_{BX} \leq \\ \frac{1}{V_T.m} e^{(V_2/V_T.m + X)} (v_{BX} - V_2) + e^{(V_2/V_T.m + X)} - I_o & v_{BX} \leq \end{cases} \tag{281}$$

where $X = \ln I_o$

Taking the derivative of (281) with respect to v_{BX}

$$\frac{I_{\bar{x}c}}{v_{BX}} = \begin{cases} \frac{1}{V_T.m} e^{(V_1/V_T.m + X)} & V_1 < v_{BX} \\ \frac{1}{V_T.m} e^{(v_{BX}/V_T.m + X)} & V_2 \leq v_{BX} \leq V_1 \\ \frac{1}{V_T.m} e^{(V_2/V_T.m + X)} & v_{BX} < V_2 \end{cases} \tag{282}$$

are obtained. In (281) the equations describing the source currents are linearized for $v_{BX} > V_1$ and $v_{BX} \leq V_2$ in such a way that the overflow of the values for soruce currents is prevented. In the same manner, the diffusion capacitances are linearized for $v_{BX} > \phi_X - .05 = U_X$ as follows:

$$C_{jX} = \begin{cases} \frac{C_{joX}}{(1 - v_{BX}/\phi_X)^{\gamma_X}} & v_{BX} \leq U \\ \frac{\gamma_X C_{joX}}{\phi_X (1 - U_X/\phi_X)^{\gamma_X + 1}} (v_{BX} - U_X) + \frac{C_{joX}}{(1 - U_X/\phi_X)^{\gamma_X}} & v_{BX} > U_X \end{cases} \tag{283}$$

Taking the derivative of (283) with respect to v_{BX}

$$\frac{dC_{jx}}{dv_{BX}} = \begin{cases} \dfrac{\gamma_x C_{jox}}{\phi_x (1 - v_{BX}/\phi_x)^{\gamma_x + 1}} & v_{BX} \leq U_x \\[2ex] \dfrac{\gamma_x C_{jox}}{\phi_x (1 - U_x/\phi_x)^{\gamma_x + 1}} & v_{BX} > U_x \end{cases} \qquad (284)$$

is obtained. Similarly, g_x and $\dfrac{dg_x}{dv_{BX}}$ are linearized as

$$g_x = \begin{cases} \left(\dfrac{1}{V_{T}.m}\right)^2 e^{(V_1/V_T.m + X)} (v_{BX} - V_1) + \dfrac{1}{V_T.m} e^{(V_1/V_T.m + X)} & V_1 \leq v_{BX} \\[2ex] \dfrac{1}{V_T.m} e^{(v_{BX}/V_T.m + X)} & V_2 \leq v_{BX} \leq V_1 \\[2ex] \left(\dfrac{1}{V_T.m}\right)^2 e^{(V_2/V_T.m + X)} (v_{BX} - V_2) + \dfrac{1}{V_T.m} e^{(V_2/V_T.m + X)} & v_{BX} \leq V_2 \end{cases} \qquad (285)$$

Differentiating (285) with respect to v_{BX}

$$\frac{dg_x}{dv_{BX}} = \begin{cases} \left(\dfrac{1}{V_T.m}\right)^2 e^{(V_1/V_T.m + X)} & V_1 \leq v_{BX} \\[2ex] \left(\dfrac{1}{V_T.m}\right)^2 e^{(v_{BX}/V_T.m + X)} & V_2 \leq v_{BX} \leq V_1 \\[2ex] \left(\dfrac{1}{V_T.m}\right)^2 e^{(V_2/V_T.m + X)} & v_{BX} < V_2 \end{cases} \qquad (286)$$

are obtained.

MODEL 2

In this model, the equations of the source currents and capacitances are given as follows:

(i) Soruce currents

$$I_{\bar{X}C} = I_o\left(e^{\lambda v_{BX}} - 1\right) \qquad (287)$$

where $\lambda = \dfrac{1}{V_{T}.m}$ in the model 1.

(ii) Junction capacitances

$$C_{BX} = \frac{C_{BX_o}}{\left(1 - v_{BX}/\phi_X\right)^{\gamma_X}} \qquad (288)$$

(iii) Diffusion capacitances

$$C_{DX} = \lambda k\, e^{\lambda v_{BX}} \qquad (289)$$

where K is the Boltzman's constant.

The equations in (287), (288) and (289) are linearized in the region $v_{BX} > V_1$ and $v_{BX} < V_2$ for the same reasons explained in model 1. In this case, the expressions for (287), (288) and (289) become

$$I_{\bar{X}C} = \begin{cases} \lambda e^{(\lambda V_1 + X)}\left(v_{BX} - V_1\right) + e^{(\lambda V_1 + X)} - I_o & V_1 \leq v_{BX} \\ e^{(\lambda v_{BX} + X)} - I_o & V_2 \leq v_{BX} \leq V_1 \\ \lambda e^{(\lambda V_2 + X)}\left(v_{BX} - V_2\right) + e^{(\lambda V_2 + X)} - I_o & v_{BX} \leq V_2 \end{cases} \qquad (290)$$

$$
\begin{cases}
\dfrac{\gamma_x C_{BX_0}}{\phi_x\left(1-\dfrac{V_1}{\phi_x}\right)^{\gamma_x+1}}\left(v_{BX}-V_1\right)+\dfrac{C_{BX_0}}{\left(1-\dfrac{V_1}{\phi_x}\right)^{\gamma_x}} & V_1 < v_{BX} \\[2ex]
\dfrac{C_{BX_0}}{\left(1-v_{BX}/\phi_x\right)^{\gamma_x}} & V_2 \leq v_{BX} \leq V_1 \\[2ex]
\dfrac{\gamma_x C_{BX_0}}{\phi_x\left(1-\dfrac{V_2}{\phi_x}\right)^{\gamma_x+1}}\left(v_{BX}-V_2\right)+\dfrac{C_{BX_0}}{\left(1-\dfrac{V_2}{\phi_x}\right)^{\gamma_x}} & v_{BX} < V_2
\end{cases}
\qquad (291)
$$

$$
C_{DX} = \begin{cases}
C_{BX_0}\left[\lambda^2 k e^{\lambda V_1}\left(v_{BX}-V_1\right)+\lambda k e^{\lambda V_1}\right] \\
C_{BX_0}\,\lambda k e^{\lambda v_{BX}} \\
C_{BX_0}\left[\lambda^2 k e^{\lambda V_2}\left(v_{BX}-V_2\right)+\lambda k e^{\lambda V_2}\right]
\end{cases}
\qquad (292)
$$

are obtained.

Taking the derivative of (290), (291) and (292) with respect to v_{BX}

$$
\frac{\dot{x}_c}{v_{BX}} = \begin{cases}
\lambda e^{(\lambda V_1 + x)} & V_1 \leq v_{BX} \\
\lambda e^{(\lambda v_{BX} + x)} & V_2 \leq v_{BX} \leq V_1 \\
\lambda e^{(\lambda V_2 + x)} & v_{BX} \leq V_2
\end{cases}
\qquad (293)
$$

$$\frac{\partial C_{BX}}{\partial v_{BX}} = \begin{cases} \dfrac{\gamma_X C_{BX_0}}{\phi_X\left(1-\dfrac{V_1}{\phi_X}\right)^{\gamma_X+1}} & V_1 \leq v_{BX} \\[2ex] \dfrac{\gamma_X C_{BX_0}}{\phi_X\left(1-\dfrac{v_{BX}}{\phi_X}\right)^{\gamma_X+1}} & V_2 \leq v_{BX} \leq V_1 \\[2ex] \dfrac{\gamma_X C_{BX_0}}{\phi_X\left(1-\dfrac{V_2}{\phi_X}\right)^{\gamma_X+1}} & v_{BX} \leq V_2 \end{cases} \tag{294}$$

$$\frac{\partial C_{DX}}{\partial v_{BX}} = \begin{cases} \lambda^2 k\, C_{BX_0}\, e^{\lambda V_1} & V_1 \leq v_{BX} \\ \lambda^2 k\, C_{BX_0}\, e^{\lambda v_{BX}} & V_2 \leq v_{BX} \leq V_1 \\ \lambda^2 k\, C_{BX_0}\, e^{\lambda V_2} & V_2 \leq v_{BX} \end{cases} \tag{295}$$

are obtained. In the above equations γ_X, ϕ_X, C_{BX_0} are some constants, and $\lambda = \dfrac{1}{V_T m}$

The expressions for $I_{\bar{X}C}$, C_{BX}, C_{DX}, $\dfrac{\partial I_{\bar{X}C}}{\partial v_{BX}}$, $\dfrac{\partial C_{BX}}{\partial v_{BX}}$ and $\dfrac{\partial C_{DX}}{\partial v_{BX}}$ will be used in calculating the complementary variables of the augmenting source variables due to nonlinear network, namely, transistors. To find the augmenting source currents due to transistors consider the circuit shown in Fig.28 . Writing the KCL for the emitter and collector nodes of the transistor we obtain

$$i_{N_1}(v_1, v_2, \dot{v}_1) = i_1(v_1, \dot{v}_1) + \frac{1}{\alpha_F} I_{CC}(v_1) - I_{EC}(v_2) \triangleq f_1(v_1, v_2, \dot{v}_2)$$

$$i_{N_2}(v_1, v_2, \dot{v}_2) = i_2(v_2, \dot{v}_2) + \frac{1}{\alpha_R} I_{EC}(v_2) - I_{CC}(v_1) \triangleq f_2(v_1, v_2, \dot{v}_2)$$

(296)

or

$$_1(v_1, v_2, \dot{v}_1) = \left[C_{BE}(v_1) + C_{DE}(v_1)\right]\dot{v}_1 + \frac{1}{\alpha_F} I_{CC}(v_1) - I_{EC}(v_2)$$

$$_2(v_1, v_2, \dot{v}_2) = \left[C_{BC}(v_2) + C_{DC}(v_2)\right]\dot{v}_2 + \frac{1}{\alpha_R} I_{EC}(v_2) - I_{CC}(v_1)$$

(297)

Differentiating f in (297) with respect to $\underset{\sim}{v}, \underset{\sim}{\dot{v}}$ and $\underset{\sim}{\ddot{v}}$

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial v_2} \\[2ex] \dfrac{\partial f_2}{\partial v_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial C_{BE}(v_1)}{\partial v_1}\dot{v}_1 + \dfrac{1}{\alpha_F}\dfrac{\partial I_{CC}(v_1)}{\partial v_1} & -\dfrac{\partial I_{EC}(v_2)}{\partial v_2} \\[2ex] -\dfrac{\partial I_{CC}(v_1)}{\partial v_1} & \dfrac{\partial C_{BC}(v_2)}{\partial v_2}\dot{v}_2 + \dfrac{1}{\alpha_R}\dfrac{\partial I_{EC}(v_2)}{\partial v_2} \end{bmatrix}$$

(298)

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial \dot{v}_1} & \dfrac{\partial f_1}{\partial \dot{v}_2} \\[2ex] \dfrac{\partial f_2}{\partial \dot{v}_1} & \dfrac{\partial f_2}{\partial \dot{v}_2} \end{bmatrix} = \begin{bmatrix} C_{BE}(v_1) & 0 \\[2ex] 0 & C_{BC}(v_2) \end{bmatrix}$$

(299)

$$\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\ddot{v}}} = \underset{\sim}{0}$$

(300)

are obtained. The values of $\underset{\sim}{f}(\underset{\sim}{v}, \dot{\underset{\sim}{v}}, \ddot{\underset{\sim}{v}})$, $\dfrac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{v}}$, $\dfrac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{v}}}$, $\dfrac{\partial \underset{\sim}{f}}{\partial \ddot{\underset{\sim}{v}}}$ are stored into YS(I,K), PAR(I,J,K), PARD(I,J,K) and PARDD(I,J,K), respectively.

SUBROUTINE DIODE

This subroutine provides the complementary variables of the augmenting sources connected across diode braches. The diode model used in this program behaves like voltage controlled nonlinear resistor with a junction capacitor connected across it. The diode currents and the value of the junction capacitors are given as defined for transistors. The model of the diode with a voltage source connected across it shown in Fig. 29.

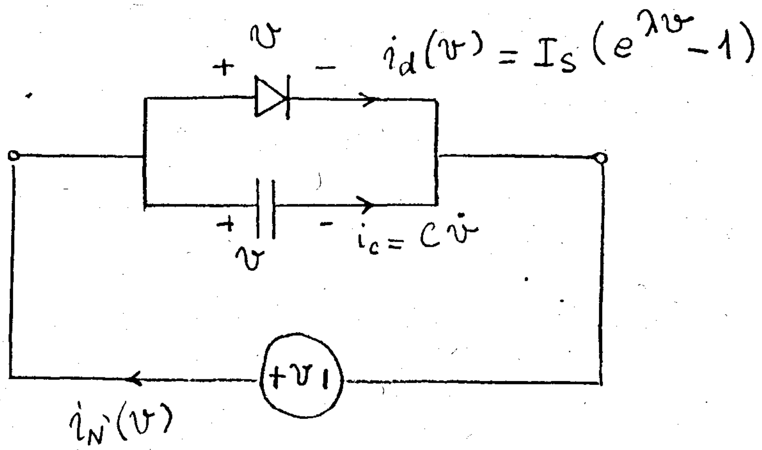


Figure 29- The Model of the diode with a voltage source

The current through the diode will be given by

$$i_d(v) = I_o(e^{\lambda v} - 1) \tag{301}$$

and the nonlinear capacitor is given by

$$c(v) = \frac{C_0}{\left(1 - \frac{v}{\phi}\right)^{\gamma}} \tag{302}$$

The equations for i_d and C are linearized outside the region $V_1 \leq v \leq V_2$. In this case the equations (301) and (302) become

$$i_d(v) = \begin{cases} \lambda e^{(\lambda V_1 + x)} (v - V_1) + e^{(\lambda V_1 + x)} - I_0 & V_1 \leq v \\ e^{(\lambda v + x)} - I_0 & V_2 \leq v \leq V_1 \\ \lambda e^{(\lambda V_2 + x)} (v - V_2) + e^{(\lambda V_2 + x)} - I_0 & v \leq V_2 \end{cases} \tag{303}$$

where $x = \ln I_0$,

and

$$(v) = \begin{cases} \dfrac{C_0 \gamma}{\phi\left(1 - \frac{V_1}{\phi}\right)^{\gamma+1}} (v - V_1) + \dfrac{C_0}{\left(1 - \frac{V_1}{\phi}\right)^{\gamma}} & V_1 \leq v \\ \dfrac{1}{\left(1 - \frac{v}{\phi}\right)^{\gamma}} & V_2 \leq v \leq V_1 \\ \dfrac{C_0 \gamma}{\phi\left(1 - \frac{V_2}{\phi}\right)^{\gamma+1}} (v - V_2) + \dfrac{C_0}{\left(1 - \frac{V_2}{\phi}\right)^{\gamma}} & v \leq V_2 \end{cases} \tag{304}$$

Differentiating (303) and (304) with respect to v;

$$\frac{\partial i_d(v)}{\partial v} = \begin{cases} \lambda e^{(\lambda V_1 + x)} & V_1 \leq v \\ \lambda e^{(\lambda v + x)} & V_2 \leq v \leq V_1 \\ \lambda e^{(\lambda V_2 + x)} & v \leq V_2 \end{cases}$$

(305)

and

$$\frac{\partial C(v)}{\partial v} = \begin{cases} \dfrac{C_0 \gamma}{\phi\left(1 - \dfrac{V_1}{\phi}\right)^{\gamma+1}} & V_1 \leq v \\ \dfrac{C_0 \gamma}{\phi\left(1 - \dfrac{v}{\phi}\right)^{\gamma+1}} & V_2 \leq v \leq V_1 \\ \dfrac{C_0 \gamma}{\phi\left(1 - \dfrac{V_2}{\phi}\right)^{\gamma+1}} & v \leq V_2 \end{cases}$$

(306)

are obtained. Then the complementary variable of the augmenting voltage source corresponding to the diode model in Fig. 29 can be calculated as:

$$i_N(v) = i_d(v) + C(v)\dot{v} \triangleq f(v, \dot{v}) \qquad (307)$$

This is stored into YS as

$i_N[v(t_k)] = YS(K)$ where $K = 1, 2, \ldots\ldots, N2HAR1$

In addition,

$$\frac{\partial f}{\partial v} = \frac{\partial i_d(v)}{\partial v} + \frac{\partial c(v)}{\partial v}\,\dot{v} \tag{308}$$

$$\frac{\partial f}{\partial \dot{v}} = c(v) \tag{309}$$

$$\frac{\partial f}{\partial \ddot{v}} = 0 \tag{310}$$

are obtained. These are stored in

$$\frac{\partial f}{\partial v}(t_k) = \mathrm{PAR}(K) \tag{311}$$

$$\frac{\partial f}{\partial \dot{v}}(t_k) = \mathrm{PARD}(K) \tag{312}$$

$$\frac{\partial f}{\partial \ddot{v}}(t_k) = \mathrm{PARDD}(K) \tag{313}$$

V.2. COMMENTS

1- Selection of XOPT(I), EPS(I), MAXFN.

EPS(I) is required by Fletcher's Algorithm in order to stop the optimization subroutine whenever |XOPT(I)-XOPTN(I)|<EPS(I) for $I \in \{1,2,\ldots,2NHAR+1\}$ for two consecutive iterations. It was observed that to choose the initial estimates XOPTIN for XOPT(I) very small (of the order of 10^{-3} or 10^{-4}) will result in a good convergence when EPS(I) has been chosen very small-(of the order of 10^{-4}) and the maximum number of iterations (MAXFN) has been limited to a maximum allowable value (of the order of 200) provided the value of CTPER, whose meaning and determination will be explained later, is selected appropriately. The limitation of MAXFN is also important in

most of the networks including transistors with nonlinear capacitors. If CTPER is adjusted appropriately the value of the scalar error function P decreases as the number of iterations increase. But if the value of the scalar error function decreases very slowly, than it may take so many iteration for the algorithm to reach a value that the difference between two consecutive iterations of the XOPT(I) becomes less than EPS(I). In this case, if the algorithm is allowed to get the desired small EPS(I) value for the estimation error between two consecutive iterations; even the value of the scalar error function P becomes very small, the algorithm may diverge or the value of P has a very large value (of the order of 10^{20} in some cases) when the additional number of harmonics are tried to be estimated in the next step. This is not a desirable phenomena since it becomes impossible to obtain the estimates of the additional harmonics (NAD). Therefore MAXFN is limited to a fixed relatively small number, say 200, to prevenet the errors due to a large number of iterations.

For the initial NHAR, the initial XOPT (I) are selected arbitrarily, but as NHAR is small the number of optimization variables is also small and to impose a tighter bound on XOPT(I) is less costly. After the initial set of iterations when NHAR is increased by NAD, the additional XOPT(I) are set equal to zero and the XOPT(I), of the previous iteration are taken as initial guess for the new itreation. Finally, it is understood that to choose EPS(I) very small for the initial number of harmonics and to obtain as good an estimate as possible for XOPT(I) at the beginning then to increase (or keep constant) EPS(I) as NHAR is increased is a good stratgey. It is also known that the Fourier coefficients decrease with increasing harmonic number. Therefore a best estimate of the first harmonic coefficients followed by zero estimate for the later harmonic coefficients will provide a good initial guess when NHAR is increased. Thus the number of iterations at a costlier stage of optimization is considerably reduced when the number of variables are relatively large.

2- Determination of CTPER

In some networks the computation can not yield an estimate of the first harmonic coefficients since the original value of the scalar error function which is desired to be minimized is considerably large. To overcome this difficulty, the error function P and its gradient $\partial P/\partial \underset{\sim}{x}$ are multiplied by a constant CTPER in the computer program. Thus the minimization of the function CTPER*P corresponds to the minimization of the error function, P.

Then the original value of the scaled scalar error function CTPER*P can be adjusted by choosing an apropriate value for CTPER such that the original value of CTPER P*is about 1. It is observed that there is no iteration when the value of CTPER is considerably large and the computation time is not finite. As CTPER is decreased the number of iterations increases and the solutions are obtained. Finally as CTPER approaches to zero, the number of iterations becomes to decrease rapidly and the solutions become unreliable for these values of CTPER.

3- Evaluation of the error function P.

The trapezoidal rule for evaluating P is applied at 2NHAR+1 points since all time functions are evaluated at 2NHAR+1 time points. It was observed that P could be driven as small as desired with 2NHAR+1 time points. If P is evaluated by applying the trapezoidal rule at a larger number of points, then the value $\bar{P}$ assumed by P was found to be quite large (about 10 times P). But $\bar{P}$ approaches to P as NHAR increases. Therefore the stopping criteria on the main program is based on $\bar{P}$.

VI. EXAMPLES

In this chapter several networks were solved by the proposed Piecewise Harmonic Balance Method using the algorithm presented in chapter V for the case that the period T of the oscillations was known.

EXAMPLE 7

In the first example, the proposed method was used to determine the periodic response of a square wave generator shown in Fig. 30. The network was decomposed into a linear and a nonlinear subnetwork. Two augmenting sources were applied to the linear subnetwork between the points E-B and C-B. The algorithm was started with 12 harmonics and 35 sampling points. Convergence occured with 15 harmonics and 41 sampling points. The final value of the error function was 0.53181×10^{-3}. The value of CTPER and EPSI chosen for this example was 10^{-3} and 10^{-5} respectively. The time values of the augmenting source variables are shown in Fig. 31 and Fig. 32. The excecution time for this example was about 10 minutes. In the equivalent circuit of the transistor used here, there is no nonlinear capacitors. Therefore ICONTR equals to zero.

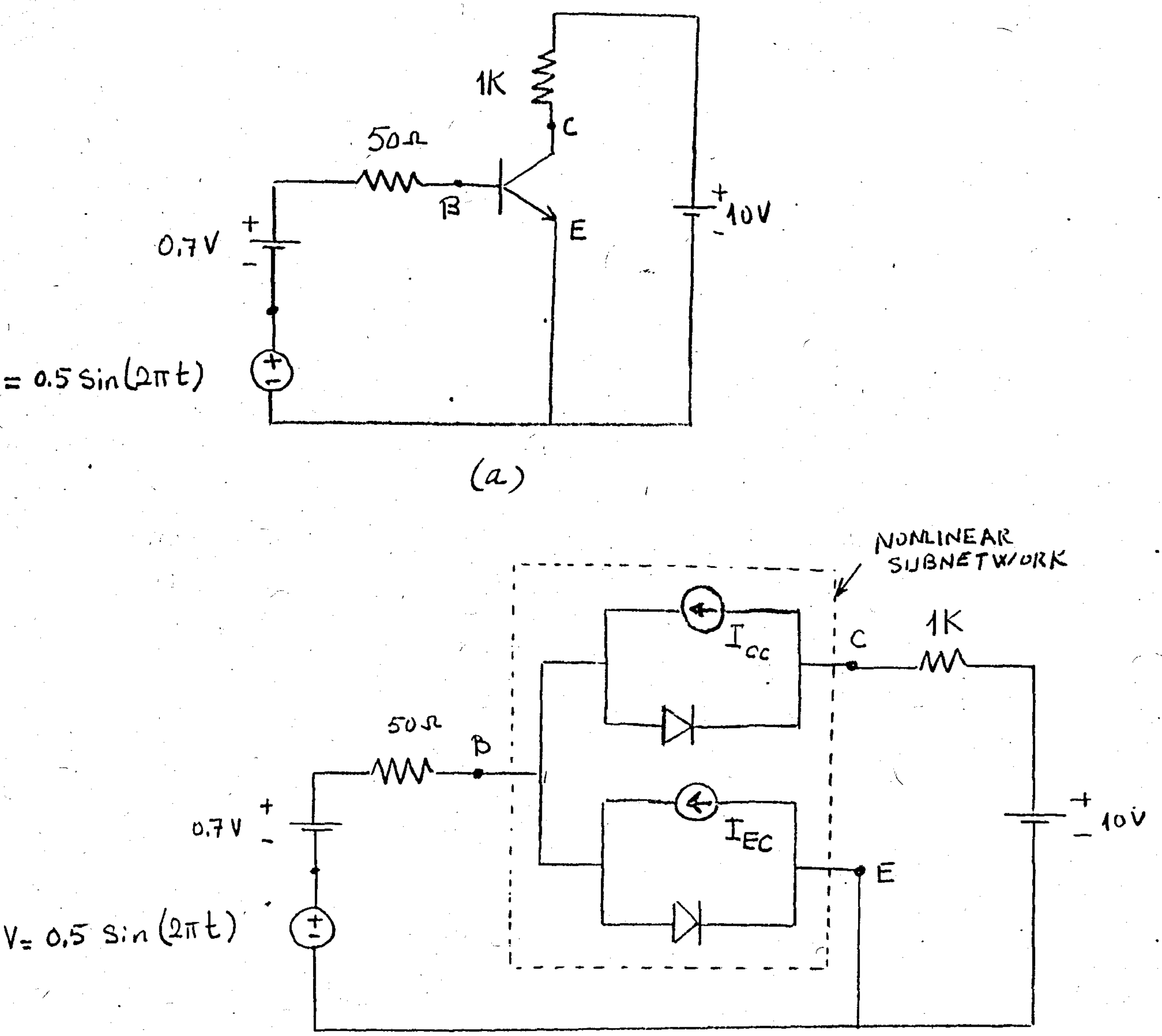
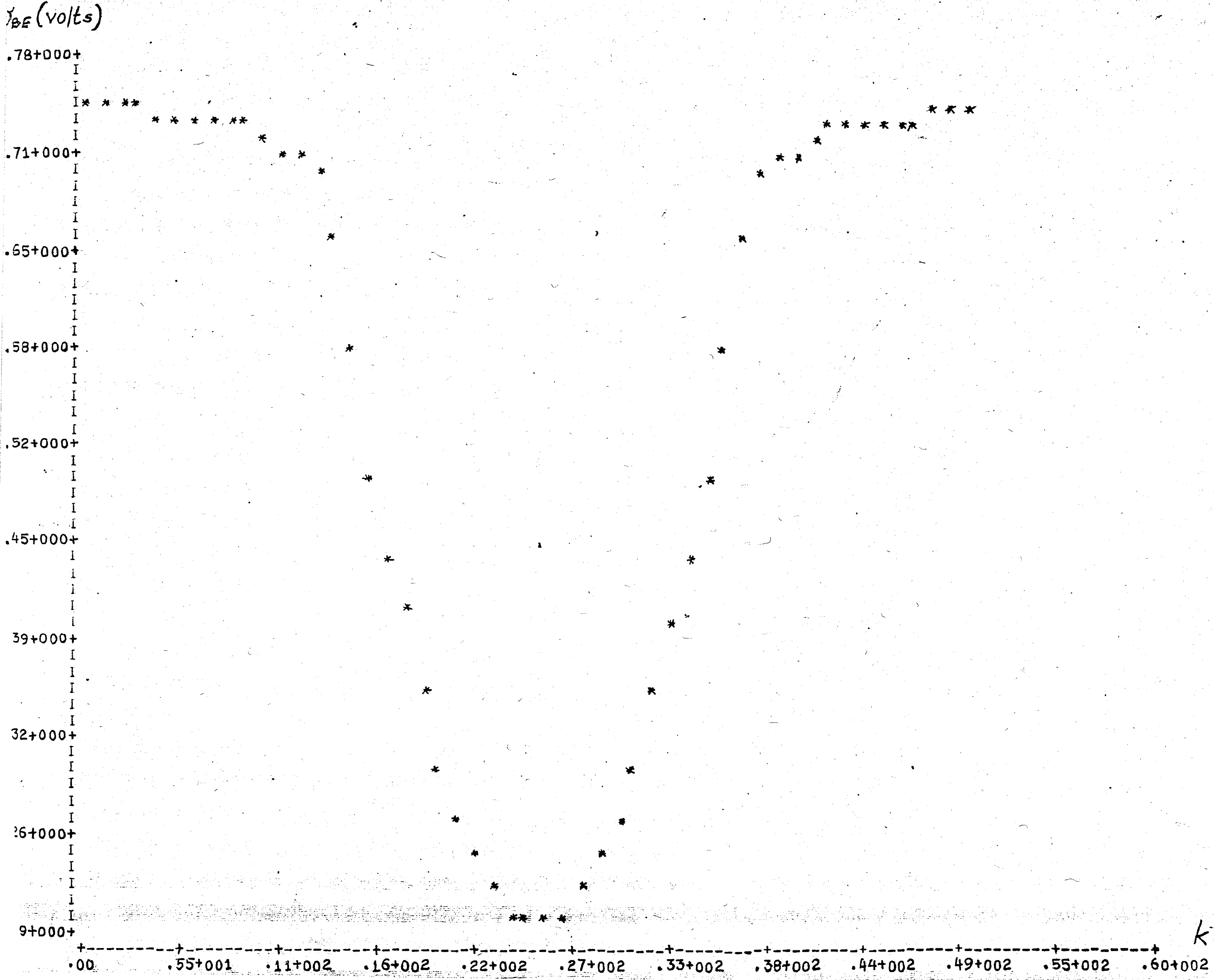


Figure 30 (a) Square wave generator with input v(t) = 0.5 Sin(2πt);
(b) Transistor replaced by its equivalent circuit.

EXAMPLE 8

In the second example, the Piecewise Harmonic Balance Method was used to determine the periodic response of a class C amplifier circuit shown in Fig. 33. The network under consideration was decomposed into a linear and a nonlinear subnetwork. Two augmenting sources were applied to the linear subnetwork between the points E-B and C-B. The algorithm was started with 7 harmonics and 21 sampling points. Convergence occured with 15 harmonics and 37 sampling points. The final



V_{BE} (volts)
.78+000
.71+000
.65+000
.58+000
.52+000
.45+000
39+000
32+000
26+000
9+000
k
.00
.55+001
.11+002
.16+002
.22+002
.27+002
.33+002
.38+002
.44+002
.49+002
.55+002
.60+002

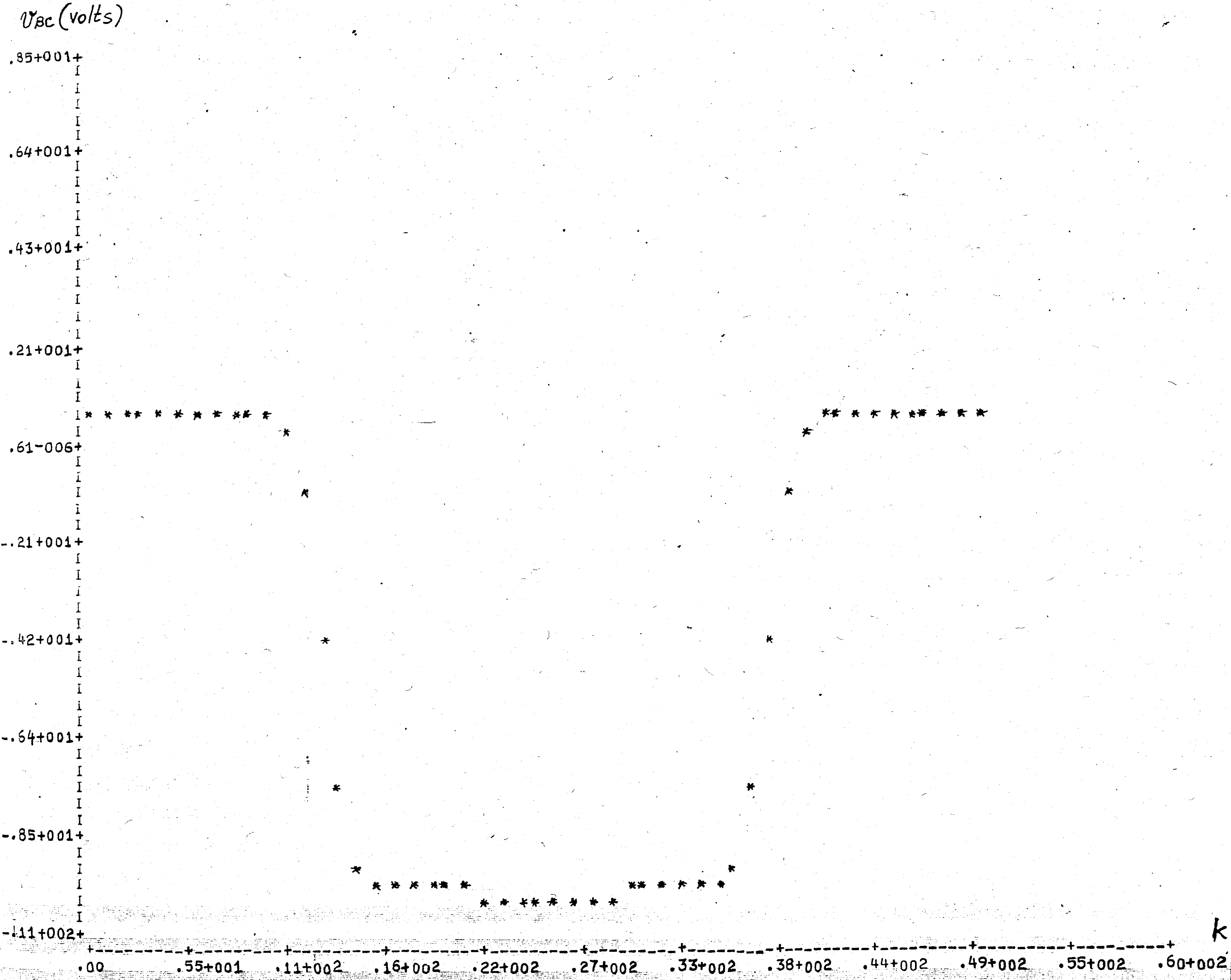


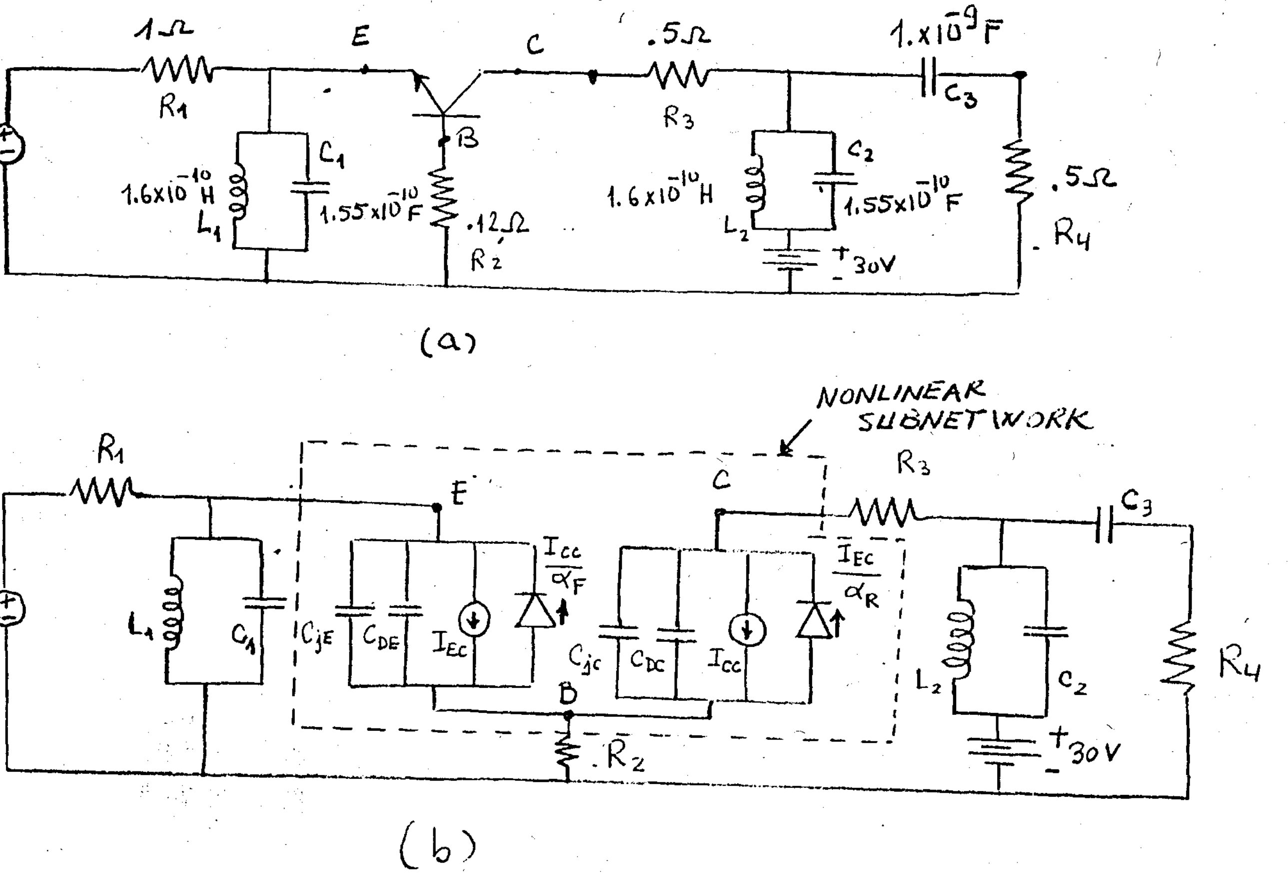
Figure 32. Periodic response of network in Fig. 30.

BE=1
AC=2
UES

R.: 1	2
,4074-001	7.1136-001
,4049-001	7.1101-001
,4006-001	7.1032-001
,3954-001	7.0942-001
,3847-001	7.0786-001
,3709-001	7.0536-001
,3576-001	7.0303-001
,3368-001	7.0009-001
3104-001	6.9213-001
2899-001	6.9406-001
2498-001	6.7541-001
1646-001	2.7704-001
0833-001	-1.2287+000
9855-001	-4.1031+000
6260-001	-7.2905+000
8526-001	-9.2322+000
9808-001	-9.6295+000
4122-001	-9.5082+000
0431-001	-9.5869+000
5614-001	-9.6879+000
0505-001	-9.6638+000
7358-001	-9.7058+000
5079-001	-9.7927+000
2259-001	-9.7722+000
0437-001	-9.7549+000
0374-001	-9.8230+000
0374-001	-9.8224+000
0437-001	-9.7545+000
2259-001	-9.7730+000
5079-001	-9.7928+000
7358-001	-9.7050+000
0505-001	-9.6641+000
5614-001	-9.6886+000
0429-001	-9.5863+000
4123-001	-9.5076+000
9814-001	-9.6298+000
8537-001	-9.2298+000
6270-001	-7.2824+000
9860-001	-4.0932+000
0835-001	-1.2226+000
1646-001	2.7879-001
2498-001	6.7544-001
2899-001	6.9403-001
3104-001	6.9215-001
3368-001	7.0008-001
3576-001	7.0303-001
3709-001	7.0536-001
3847-001	7.0786-001
3954-001	7.0942-001
4006-001	7.1032-001
4049-001	7.1101-001

```
ITN=  179      NFNS=  187        F=        .53181-003
X(I)   .5589+000   -.4358+001    .2722+000    .6559+001
X(I)  -.1985-002    .9945+000   -.1077-001   -.1784+000
X(I)  -.2712-002   -.1135+000    .5080-002   -.1319+000
X(I)  -.6453-003    .1185-001   -.1301-004   -.1109-002
X(I)  -.5279-005   -.8559-003    .8914-005    .2261-003
X(I)   .3214-005    .1755-003   -.3781-005    .3471-003
X(I)  -.4671-006   -.8144-004
G(I)   .2046-004   -.1361-004    .2169-004   -.1409-004
G(I)  -.7260-005    .8178-005    .1685-005    .1599-006
G(I)   .4771-004   -.2176-004    .4475-004   -.1850-004
G(I)   .7065-004   -.3839-004   -.6356-004    .2104-004
G(I)   .5848-004   -.2424-004    .2347-004   -.1132-004
G(I)  -.4028-004    .2082-004   -.7630-004    .3435-004
G(I)   .7622-004   -.2438-004

EXIT FROM QNMING,   IER = 0
FOPT=      1.82271-003
      -.1044+000   -.2986+000   -.1399-002   -.1943+001
       .3790-002   -.5497+000    .5926-002    .1483+000
      -.1841-003    .7570-001   -.4041-002    .4419-001
       .4995-005    .1103-003    .1009-004    .1017-002
       .4541-006    .6640-003   -.6928-005   -.2133-003
       .6164-006   -.1591-003    .1006-005   -.2551-003

       .2066-004   -.9632-005    .1047-004   -.1755-005
       .2502-004   -.1405-004    .4802-004   -.2463-004
       .5218-004   -.1992-004    .6027-004   -.2699-004
      -.6301-004    .2133-004   -.3827-005   -.2101-006
      -.4383-005   -.3341-006   -.7162-005    .3641-005
      -.7398-004    .3300-004   -.2305-004    .1128-004

                                  .2279-001    .2084+000
                                 -.4834-002    .2903+000
                                  .3100-002   -.4784-001
                                 -.8355-005   -.1918-003
                                  .2972-005   -.4835-003
                                 -.1757-006    .2712-003

                                 -.2037-005    .6202-005
                                  .5468-004   -.2611-004
                                  .6353-004   -.3363-004
                                  .5267-004   -.2203-004
                                 -.1114-004    .8568-005
FREQUENCY DOMAIN RESPONSE         .4330-004   -.1385-004
NONLIN NR.:         1                          2
           REAL         IMAG           REAL         IMAG
 DC:   5.5889-001    .0000       -4.3578+000    .0000
  1:   2.7216-001   1.3007-005    6.5592+000   1.1086-003
  2:  -1.0443-001  -4.9952-006   -2.9864-001  -1.1033-004
  3:  -1.3988-003  -1.0090-005   -1.9431+000  -1.0169-003
  4:   2.2795-002   5.3549-006    2.0836-001   1.9177-004
  5:  -1.9827-003   5.2788-006    9.9446-001   8.5594-004
  6:  -1.0766-002  -8.9144-006   -1.7845-001  -2.2606-004
  7:   3.7899-003  -4.5414-007   -5.4968-001  -6.6397-004
  8:   5.9259-003   6.9278-006    1.4825-001   2.1329-004
  9:  -4.8343-003  -2.9718-006    2.9030-001   4.8354-004
 10:  -2.7126-003  -3.2139-006   -1.1354-001  -1.7551-004
 11:   5.0796-003   3.7812-006   -1.3195-001  -3.4705-004
 12:  -1.8416-004  -6.1640-007    7.5705-002   1.5914-004
 13:  -4.0415-003  -1.0062-006    4.4188-002   2.5506-004
 14:   3.1005-003   1.7570-007   -4.7844-002  -2.7116-004
 15:  -6.453[illegible]-004   4.6769-007    1.1846-002   8.1436-005
```

value of the error function was $0.74\text{x}10^{-2}$. The value of CTPER and EPSI chosen for this example was 1 and 10^{-4}, respectively. The excecution time for this example was 8 minutes. The augmenting source variables are shown in Fig. 34 and Fig. 35.



igure 33 (a) Class C amplifier circuit with input $v(t) = 2.6\ \mathrm{Sin}(2\pi 10^{9} t)$;
(b) Transistor replaced by its equivalent circuit.

EXAMPLE 9

In the third example, the algorithm was used to determine the periodic response of the class C amplifier circuit shown in Fig. 36. The network was decomposed into a linear and a nonlinear subnetwork. Then two augmenting sources were applied to the linear subnetwork between the points E-B and C-B. The algorithm was started with 4 harmonics and 21 sampling

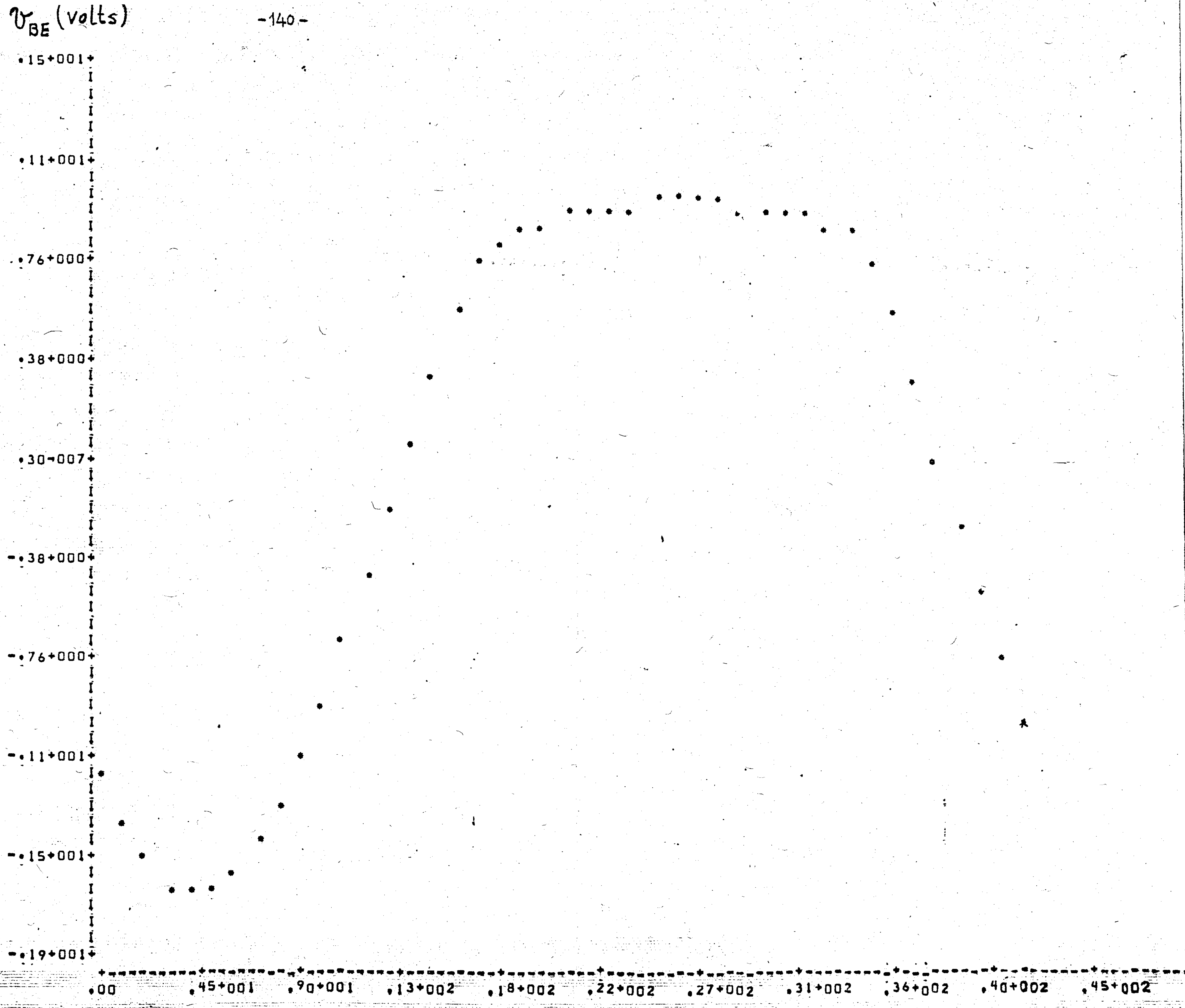


Figure 34. Periodic response of network in Fig. 33.

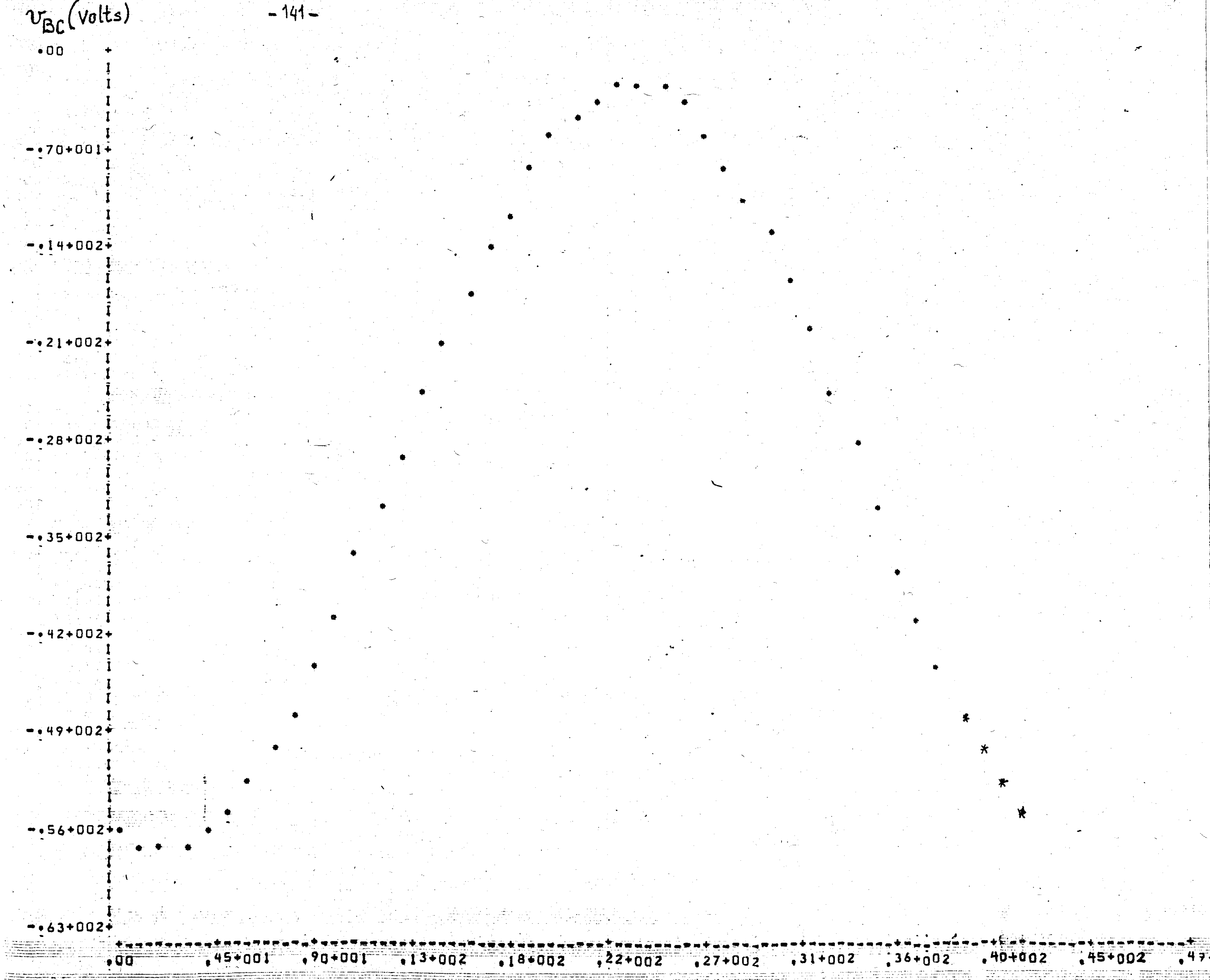


Figure 35. Periodic response of network in Fig. 33.

ES

$v_{BE} = 1$

$v_{BC} = 2$

```
:         1              2
2347+000  -5.6400+001
4032+000  -5.7229+001
5353+000  -5.7485+001
6253+000  -5.7153+001
6595+000  -5.6212+001
6520+000  -5.4733+001
5980+000  -5.2723+001
4913+000  -5.0209+001
3509+000  -4.7301+001
1719+000  -4.4015+001
5492-001  -4.0428+001
2267-001  -3.6665+001
6860-001  -3.2745+001
9816-001  -2.8785+001
0887-002  -2.4901+001
2185-001  -2.1101+001
7577-001  -1.7535+001
6478-001  -1.4272+001
4180-001  -1.1201+001
8461-001  -8.4619+000
1431-001  -6.2384+000
3709-001  -4.4869+000
5521-001  -3.2550+000
6906-001  -2.5968+000
7972-001  -2.4631+000
8716-001  -2.9126+000
9113-001  -3.9528+000
9188-001  -5.5217+000
8897-001  -7.6454+000
8236-001  -1.0250+001
7158-001  -1.3250+001
5654-001  -1.6658+001
3504-001  -2.0343+001
0714-001  -2.4223+001
6768-001  -2.8367+001
6625-001  -3.2635+001
5563-001  -3.6777+001
8708-001  -4.0686+001
4763-002  -4.4307+001
4674-001  -4.7616+001
2671-001  -5.0552+001
8260-001  -5.3008+001
0205+000  -5.4970+001
```

```
ITN=  102     NFNS=  113         F=        .74110-002
X(I)   -.5460-002   -.2982+002   -.1086+001   -.2633+002
X(I)    .2700-001   -.5178-002    .1311-002   -.8683-002
X(I)    .4647-002   -.2099-002    .1663-002   -.6119-003
X(I)   -.1412-002   -.1123-001   -.8231+000   -.8113+001
X(I)    .2190-002   -.2450-001    .1973-003   -.1952-001
X(I)    .7511-003   -.7744-002    .7661-003   -.6237-002
X(I)    .4582-004   -.1423-002
G(I)    .1047-004    .1728-006   -.9330-005   -.7565-007
G(I)    .3717-005   -.3570-006   -.1081-004    .2450-006
G(I)    .1021-004   -.5587-006   -.1088-005    .1571-006
G(I)    .2063-004   -.8915-006   -.3744-005    .1063-006
G(I)   -.4160-005   -.1396-006    .4566-005   -.3414-007
G(I)    .1632-004   -.8041-007   -.1707-004    .8318-007
G(I)    .1661-004   -.2960-006

EXIT FROM QNMING,  IER = 0
FOPT=       6.79044-002
 -.1122+000   -.3159+000   -.2377-001    .4597-001
 -.3245-002   -.1465-001   -.2299-002    .1677-002
 -.7257-003   -.6005-002   -.2162-003    .5027-003
 -.3421+000    .1807+000    .4871-001    .1908-001
 -.1235-001    .1931-001    .3922-002    .3197-002
 -.3532-002    .6475-002   -.4476-003   -.9961-003

  .4667-005    .3202-006    .1337-005   -.1273-006
  .9865-005   -.4117-007    .3709-005   -.5404-008
 -.1016-004    .2508-006    .2024-004   -.3151-006
  .7303-005   -.3352-007   -.4169-005    .9211-007
  .4250-005   -.9334-007   -.3212-005   -.1677-006
  .9900-005   -.4313-006    .7818-005   -.9428-007
                           -.2[illegible]2-001    .4660-001
                           -.4[illegible]77-002   .1382-001
                           -.2390-002    .5433-002
                            .1[illegible]0-001    .1080-001
                            .[illegible]-002      .4962-002
                            .[illegible]16-002    .2734-002

                           -.1[illegible]7-005   -.1390-006
                           -.1[illegible]-004    -.8031-007
                           -.1[illegible]2-004    .5854-006
                           -.94[illegible]-006    .9873-007
                           -.8301-005    .1476-006
                           -.2[illegible]-004     .7592-006
FREQUENCY DOMAIN RESPONSE
NONLIN NR.:              1                          2
            REAL          IMAG           REAL          IMAG
 DC:  -5.4597-003    .0000        -2.9820+001     .0000
  1:  -1.0864+000   8.2306-001    -2.6329+001    8.1128+000
  2:  -1.1219-001   3.4207-001    -3.1594-001   -1.8070-001
  3:  -2.3766-002  -4.8714-002     4.5965-002   -1.9075-002
  4:  -2.6918-002  -1.7296-002     4.6601-002   -1.0799-002
  5:   2.6999-002  -2.1902-003    -5.1775-003    2.4503-002
  6:   1.3114-003  -1.9733-004    -8.6829-003    1.9516-002
  7:  -3.2453-003   1.2351-002    -1.4648-002   -1.9307-002
  8:  -2.2986-003  -3.9222-003     1.6771-003   -3.1967-003
  9:  -4.3769-003  -3.2238-003     1.3818-002   -4.9619-003
 10:   4.6466-003  -7.5106-004    -2.0989-003    7.7440-003
 11:   1.6633-003  -7.6611-004    -6.1195-004    6.2366-003
 12:  -7.2571-004   3.5325-003    -6.0049-003   -6.4753-003
 13:  -2.1616-004   4.4761-004     5.0275-004    [illegible]
```

points. Convergence occured with 24 harmonics and 61 sampling points. The final value of the error function was $.182\times10^{-4}$. The value of CTPER and EPSI was 1 and 10^{-4} respectively. The execution time was 20 minutes. The transistor model does not contain the nonlinear capacitors in the Ebers-Moll model therefore the value of ICONTR is also zero for this example. The time values of the augmenting sources and the output are shown in Figs. 37, 38, 39. The time values of the output voltage at discrete points was compared to the one obtained by Nakhla and Vlach [8]. It is found that the solution obtained here is a similar waveform to the one obtained by Nakhla and Valch [8].

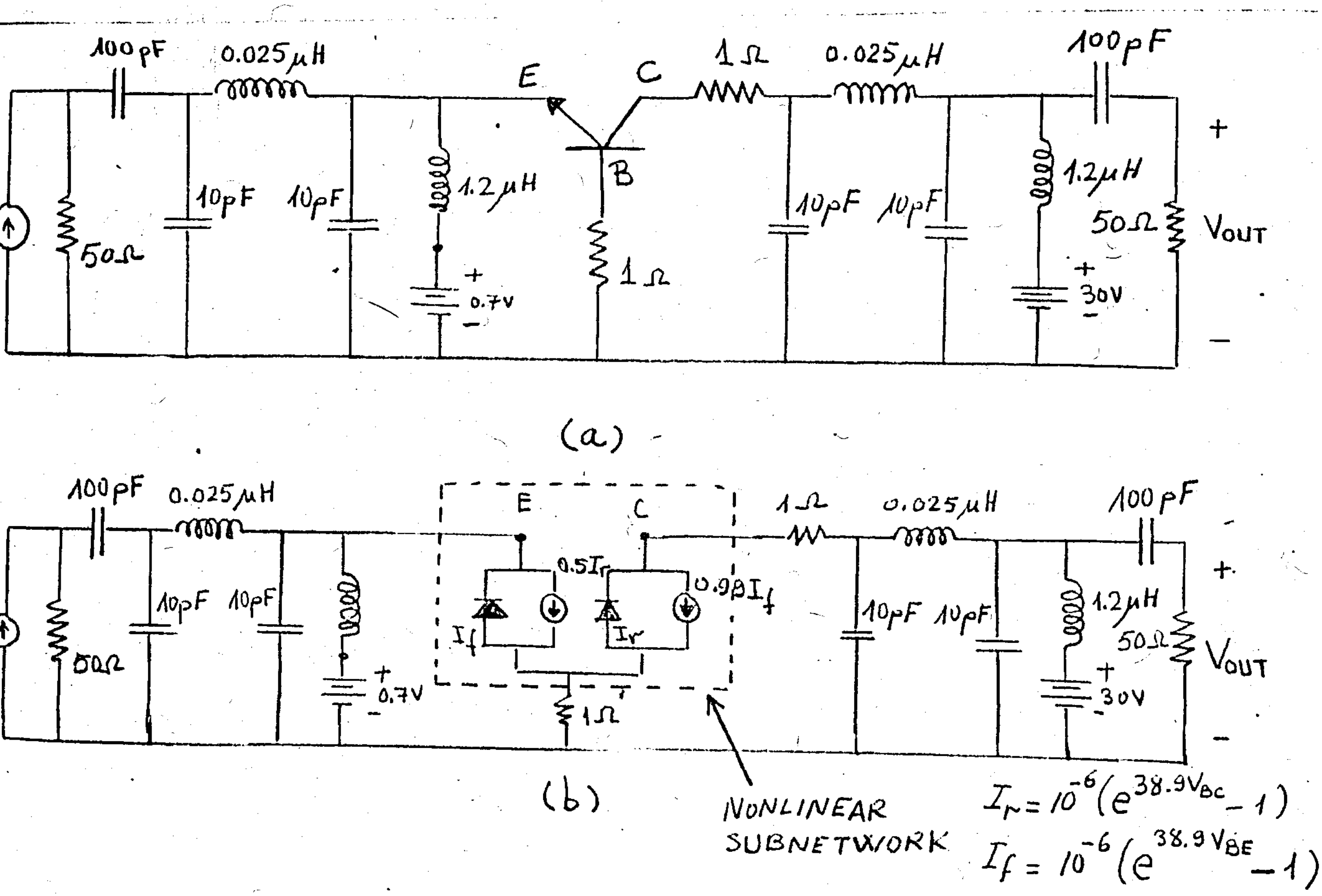


Figure 36 (a) Class C amplifier with input $J(t) = 0.1\ \mathrm{Sin}(2\pi10^8 t)$;
(b) Transistor replaced by its equivalent circuit.

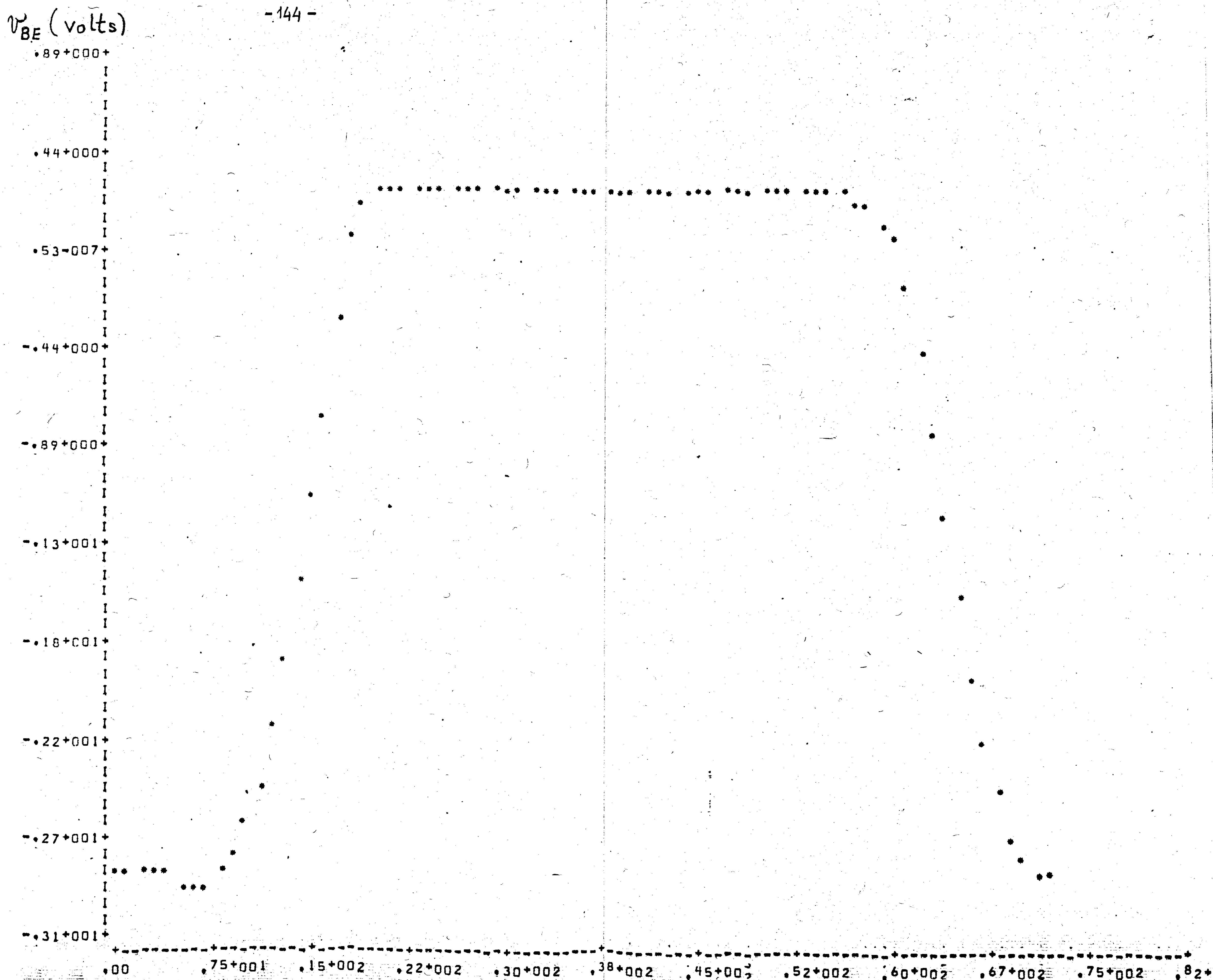


Figure 37. Periodic response of network in Fig. 36.

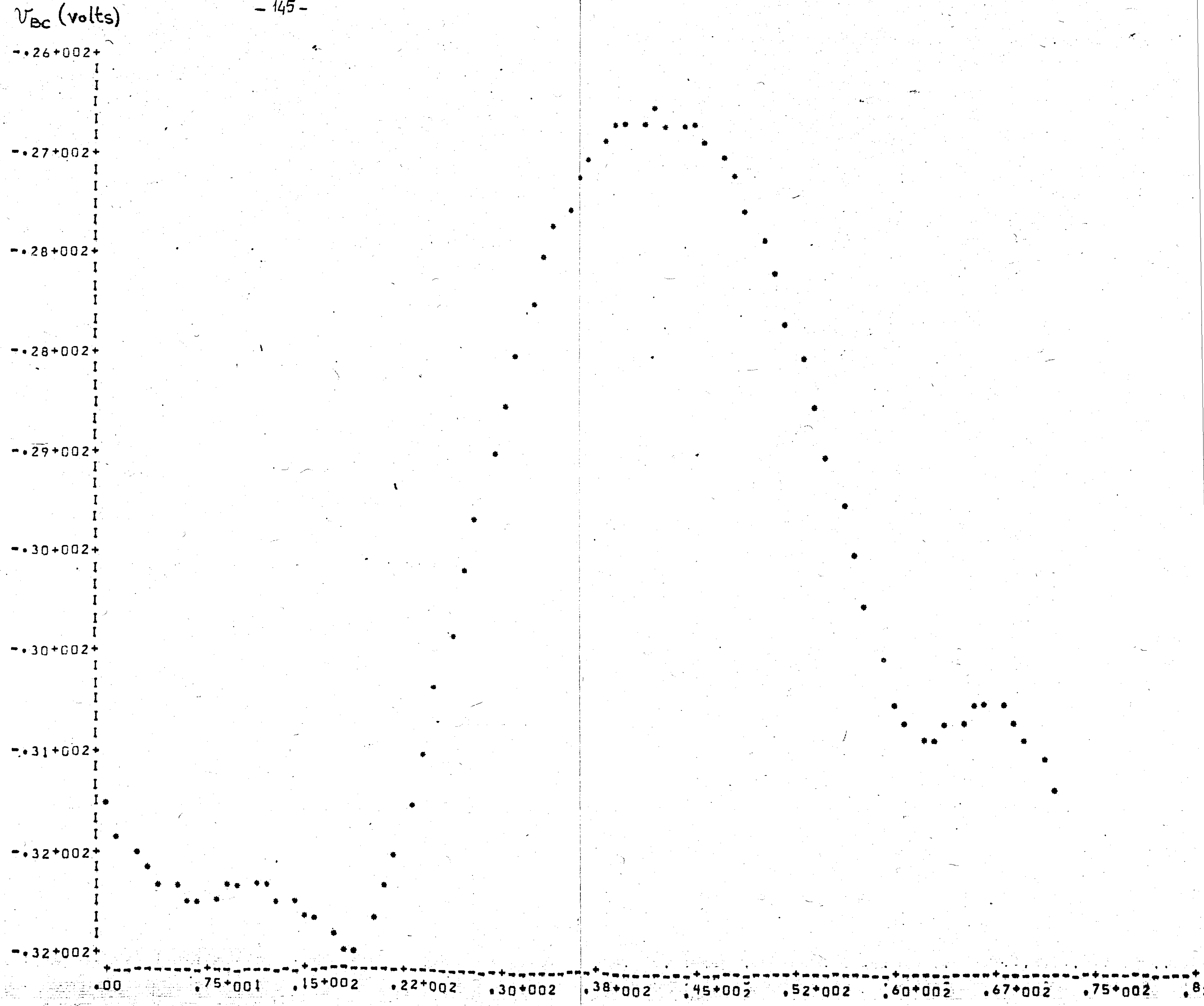


Figure 38. Periodic response of network in Fig. 36.

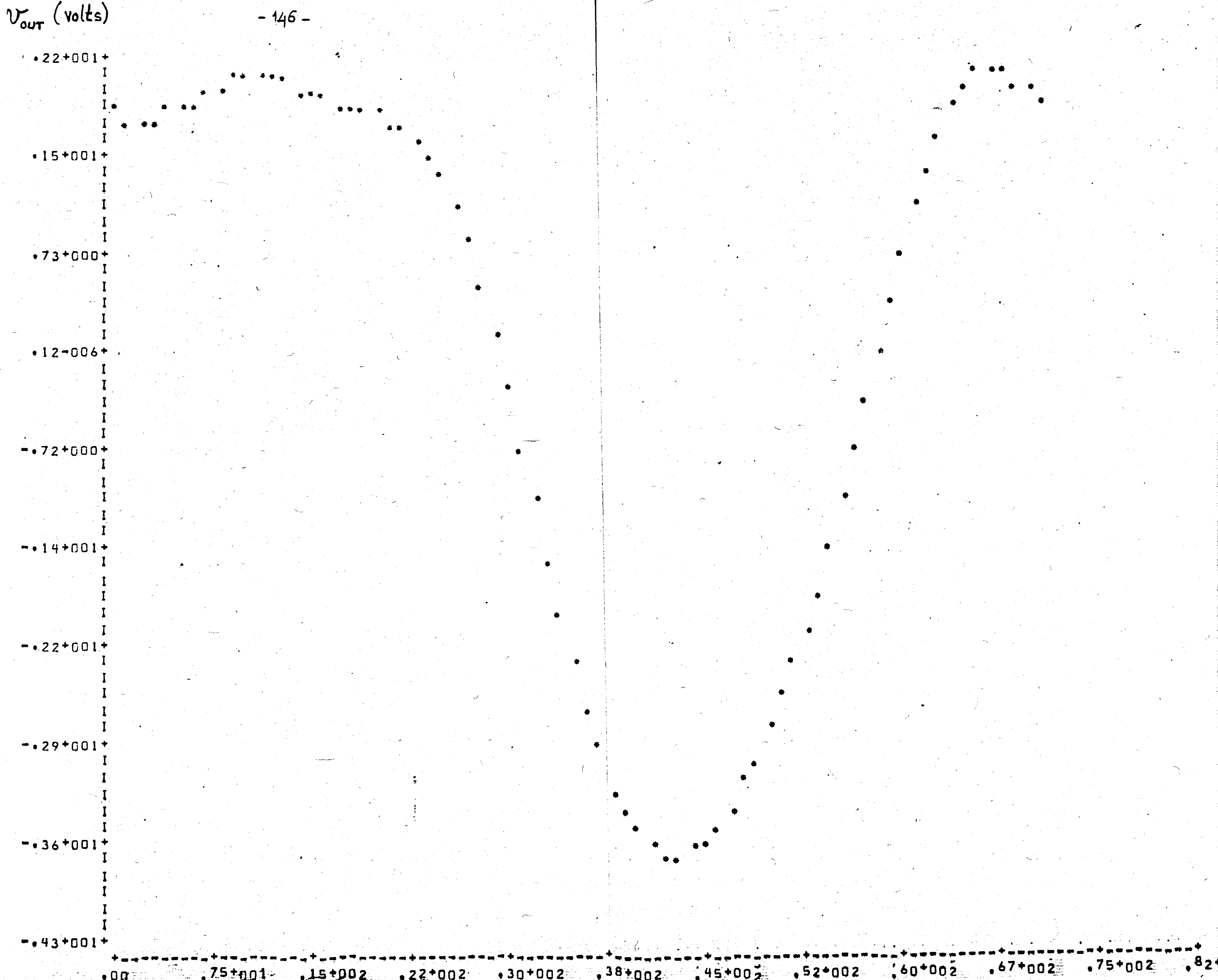


Figure 39. Periodic response of network in Fig. 36.

```
IRMONICS= 24
)INTS    = 61
PSI      = .1-003

TN=    38      NFNS=    45      F=      .18242-004
(I)  -.7000+000  -.3000+002  -.1577+001  -.2153+001  -.7003+000
(I)   .5521-001  -.1781-003  -.2539-002  -.9393-002   .1096-001
(I)  -.1346-001   .1041-001   .4940-002  -.6949-002   .1025-001
(I)   .1413-002  -.2805-002   .4747-002  -.3452-002  -.1013-002
(I)   .2911-002  -.1159-002  -.4501-003   .2294-002  -.2577-002
(I)  -.3878+000  -.1491+001  -.3788+000   .4145+000  -.6399-001
(I)   .2332-001  -.5059-003  -.3966-001   .3083-001  -.9288-002
(I)  -.7997-002   .5838-002   .8662-003  -.2883-002   .5070-002
(I)   .2285-002  -.3676-002   .3553-002  -.2482-002  -.1880-002
(I)   .2901-002  -.1301-002   .1279-003   .2335-002  -.1681-002
(I)   .3830-005   .2452-003  -.1237-003   .2289-004  -.2360-004
(I)   .8722-004   .9129-005  -.1792-004  -.8438-005  -.4291-004
(I)  -.9329-005  -.6988-005  -.8289-004  -.9116-005   .8461-004
(I)   .1300-003  -.1071-004  -.1330-003   .1013-004   .8508-004
(I)  -.8712-004   .3349-004   .9664-004   .2720-004  -.1094-003
(I)   .7418-005  -.9675-005   .5657-004   .1387-004  -.1434-003
(I)   .1611-003   .1298-004  -.1392-003  -.5598-005   .1098-003
(I)   .7274-004  -.2840-005   .8612-005  -.1074-004  -.1067-003
(I)  -.2484-004  -.1048-004  -.3695-004   .4630-005   .7373-004
(I)   .5335-005   .2982-004  -.5661-004   .2008-004   .1878-003

XIT FROM QNMING,  IER = 0
OPT=     2.81831-005

       .6990+000  -.5088-001   .3768-001   .1319+000  -.4484-001
      -.2024-001   .1435-001  -.9738-002  -.9042-002   .1235-001
      -.8273-002  -.2363-002   .4115-002  -.6976-002   .5634-002
       .2096-002  -.3164-002   .2271-002   .1216-002  -.2093-002
       .1182-002  -.1347-002  -.2413-002  -.1025-003  -.8022-003
       .4720-001   .1819+000  -.2729+000   .1646+000  -.1020+000
       .2524-002   .1550-001  -.1209-001   .2198-002   .8060-003
      -.3839-002  -.2243-002   .3839-002  -.4126-002   .2965-002
       .3007-002  -.3154-002   .2111-002   .1206-002  -.2702-002
       .1224-002  -.8628-003  -.2556-002
       .3241-004   .3965-004  -.1356-004  -.2040-004  -.7946-005
      -.1687-004   .1145-004   .7894-005   .3768-004   .1263-004
       .3705-005   .2296-004   .8707-005  -.9172-004  -.4745-005
       .2973-005  -.3251-004  -.6166-005   .1077-003  -.4473-005
       .9436-005   .6319-004  -.2806-004  -.1803-003  -.5919-004
       .2037-004   .7764-004  -.1355-004  -.7127-004   .2405-005
      -.1307-004  -.7389-004   .4654-005  -.3136-004   .1165-004
       .1750-006   .7320-004   .1435-004   .1087-004  -.3700-005
       .3209-005  -.9764-004  -.8992-005   .1060-003  -.1004-004
       .6437-005  -.1811-003  -.2900-004
```

TIME VALUES

17 $v_{BC} = 18$ $v_{OUT} = 8$

7	18			8	
				1.7838+000	
136+000	-3.1484+001			1.7410+000	
204+000	-3.1647+001			1.7223+000	
125+000	-3.1793+001			1.7289+000	
212+000	-3.1917+001			1.7586+000	
126+000	-3.2005+001			1.8058+000	
508+000	-3.2063+001			1.8635+000	
768+000	-3.2096+001			1.9237+000	
747+000	-3.2098+001			1.9787+000	
337+000	-3.2086+001			2.0220+000	
521+000	-3.2074+001			2.0494+000	
253+000	-3.2054+001			2.0583+000	
114+000	-3.2044+001			2.0488+000	
572+000	-3.2059+001			2.0234+000	
509+000	-3.2079+001			1.9858+000	
780+000	-3.2119+001			1.9406+000	
994+000	-3.2190+001			1.8936+000	
843-001	-3.2259+001			1.8494+000	
875-001	-3.2345+001			1.8127+000	
785-002	-3.2458+001			1.7870+000	
813-001	-3.2461+001			1.7673+000	
526-001	-3.2273+001			1.7386+000	
924-001	-3.2012+001			1.6856+000	
679-001	-3.1754+001			1.5981+000	
236-001	-3.1460+001			1.4683+000	
597-001	-3.1101+001			1.2908+000	
133-001	-3.0738+001			1.0636+000	
518-001	-3.0346+001			7.8734-001	
849-001	-2.9938+001			4.6529-001	
184-001	-2.9541+001			1.0390-001	
430-001	-2.9151+001			-2.8831-001	
658-001	-2.8779+001			-7.0112-001	
852-001	-2.8443+001			-1.1227+000	
995-001	-2.8136+001			-1.5410+000	
127-001	-2.7866+001			-1.9442+000	
219-001	-2.7643+001			-2.3210+000	
288-001	-2.7452+001			-2.6617+000	
345-001	-2.7291+001			-2.9593+000	
360-001	-2.7168+001			-3.2081+000	
370-001	-2.7069+001			-3.4053+000	
364-001	-2.6989+001	2.6705-001	-2.9155+001	-3.5505+000	-1.4589+000
318-001	-2.6936+001	2.5897-001	-2.9497+001	-3.6449+000	-1.1283+000
276-001	-2.6900+001	2.4649-001	-2.9841+001	-3.6913+000	-7.8123-001
212-001	-2.6881+001	2.2879-001	-3.0187+001	-3.6936+000	-4.2098-001
112-001	-2.6887+001	1.8400-001	-3.0506+001	-3.6562+000	-5.1510-002
019-001	-2.6914+001	6.7754-002	-3.0758+001	-3.5835+000	3.2112-001
892-001	-2.6964+001	-1.4435-001	-3.0916+001	-3.4796+000	6.8723-001
733-001	-2.7048+001	-4.4282-001	-3.0980+001	-3.3478+000	1.0335+000
581-001	-2.7160+001	-8.0352-001	-3.0979+001	-3.1906+000	1.3451+000
377-001	-2.7304+001	-1.1914+000	-3.0942+001	-3.0097+000	1.6082+000
144-001	-2.7485+001	-1.5727+000	-3.0887+001	-2.8061+000	1.8133+000
906-001	-2.7696+001	-1.9282+000	-3.0840+001	-2.5802+000	1.9554+000
583-001	-2.7938+001	-2.2352+000	-3.0825+001	-2.3322+000	2.0354+000
236-001	-2.8212+001	-2.4735+000	-3.0845+001	-2.0621+000	2.0596+000
847-001	-2.8507+001	-2.6472+000	-3.0907+001	-1.7707+000	2.0387+000
303-001	-2.8820+001	-2.7612+000	-3.1015+001		1.9842+000
		-2.8159+000	-3.1153+001		1.9171+000
		[illegible]	[illegible]		[illegible]

FREQUENCY DOMAIN RESPONSE

17		18		8	
REAL	IMAG	REAL	IMAG	REAL	IMAG
-7.0004-001	.0000	-2.9996+001	.0000	.0000	.0000
-1.5772+000	3.8776-001	-2.1533+000	1.4911+000	2.2906+000	-1.6671+000
-7.0034-001	3.7880-001	6.9905-001	-4.1448-001	-3.2390-001	9.3432-001
-5.0885-002	6.3993-002	3.7683-002	-4.7203-002	4.4084-002	4.6822-002
1.3192-001	-1.8189-001	-4.4842-002	2.7293-001	-1.8937-001	4.6880-002
5.5206-002	-1.6461-001	-1.7812-004	1.0204-001	-3.4938-002	3.0310-002
-2.5387-003	-2.3318-002	-9.3933-003	5.0591-004	-2.3878-003	-1.6820-003
1.0963-002	3.9658-002	-2.0243-002	-3.0834-002	-2.2681-004	-8.1965-003
1.4347-002	9.2880-003	-9.7385-003	-2.5242-003	-1.2593-003	-1.1204-003
-9.0420-003	-1.5503-002	1.2346-002	1.2091-002	8.5132-004	2.0981-003
-1.3457-002	-2.1982-003	1.0411-002	-8.0600-004	1.0534-003	3.1182-004
4.9398-003	7.9969-003	-6.9495-003	-5.8377-003	-4.0700-004	-6.6990-004
1.0247-002	-8.6619-004	-8.2733-003	2.8832-003	-6.3217-004	2.4425-005
-2.3626-003	-5.0703-003	4.1152-003	3.8389-003	1.8024-004	2.9370-004
-6.9758-003	2.2432-003	5.6338-003	-3.8391-003	3.3724-004	-1.2280-004
1.4132-003	4.1264-003	-2.8053-003	-2.9649-003	-9.3905-005	-1.6104-004
4.7468-003	-2.2853-003	-3.4523-003	3.6764-003	-1.6641-004	1.1464-004
-1.0126-003	-3.5527-003	2.0960-003	2.4822-003	5.5247-005	1.0082-004
-3.1639-003	1.8799-003	2.2713-003	-3.0068-003	8.7924-005	-7.9963-005
1.2158-003	3.1540-003	-2.0933-003	-2.1113-003	-4.7754-005	-6.9043-005
2.9108-003	-1.2055-003	-1.1588-003	2.7023-003	-4.0078-005	6.2854-005
-4.5009-004	-2.9007-003	2.2943-003	1.3011-003	4.7467-005	3.7454-005
-2.5772-003	-1.2788-004	1.1815-003	-2.3352-003	3.1164-005	-4.4785-005
-1.3472-003	1.6808-003	-2.4128-003	-1.2245-003	-4.2813-005	-2.8846-005
-1.0248-004	8.6278-004	-8.0223-004	2.5561-003	-1.9197-005	4.3331-005

17		18		8	
MAG.	ARG.	MAG.	ARG.	MAG.	ARG.
7.0004-001	.0000	2.9996+001	.0000	.0000	.0000
1.6242+000	-2.4107-001	2.6191+000	-6.0567-001	2.8330+000	-6.2915-001
7.9622-001	-4.9582-001	8.1269-001	-5.3520-001	9.8887-001	-1.2371+000
8.1758-002	-8.9901-001	6.0399-002	-8.9708-001	6.4309-002	8.1551-001
2.2469-001	-9.4332-001	2.7659-001	-1.4080+000	1.9508-001	-2.4268-001
1.7362-001	-1.2472+000	1.0204-001	-1.5691+000	4.6253-002	-7.1458-001
2.3456-002	1.4624+000	9.4069-003	-5.3806-002	2.9207-003	6.1368-001
4.1146-002	1.3011+000	3.6885-002	9.8987-001	8.1997-003	1.5431+000
1.7091-002	5.7454-001	1.0060-002	2.5361-001	1.6856-003	7.2713-001
1.7947-002	1.0428+000	1.7280-002	7.7499-001	2.2643-003	1.1853+000
1.3636-002	1.6192-001	1.0442-002	-7.7266-002	1.0986-003	2.8779-001
9.3996-003	1.0174+000	9.0760-003	6.9867-001	7.8385-004	1.0248+000
1.0284-002	-8.4327-002	8.7613-003	-3.3534-001	6.3264-004	-3.8618-002
5.5937-003	1.1347+000	5.6279-003	7.5068-001	3.4460-004	1.0204+000
7.3276-003	-3.1113-001	6.8175-003	-5.9816-001	3.5890-004	-3.4923-001
4.3617-003	1.2408+000	4.0817-003	8.1305-001	1.8642-004	1.0429+000
5.2683-003	-4.4870-001	5.0433-003	-8.1682-001	2.0208-004	-6.0322-001
3.6942-003	1.2931+000	3.2488-003	8.6955-001	1.1497-004	1.0695+000
3.6803-003	-5.3611-001	3.7682-003	-9.2384-001	1.1885-004	-7.3802-001
3.3802-003	1.2029+000	2.9732-003	7.8967-001	8.3949-005	9.6569-001
3.1506-003	-3.9265-001	2.9403-003	-1.1657+000	7.4544-005	-1.0032+000
2.9354-003	1.4169+000	2.6375-003	5.1589-001	6.0464-005	6.6803-001
2.5804-003	4.9577-002	2.6171-003	-1.1024+000	5.4561-005	-9.6286-001
2.1541-003	-8.9512-001	2.7057-003	4.6963-001	5.1624-005	5.9290-001
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

VII. CONCLUSION

A Piecewise Harmonic Balance Method for finding the periodic response of networks both in the case that the period T of the oscillations is known and in the case that the period T of the oscillations is unknown has been investigated.

The method has been tested computationally on several problems for the case that the period T of the oscillations is known to find the steady-state solution. The advantages of the method can be summarized as follows:

(i) No special formulation of network equations is required. Most of the computer algorithms such as those using the conventional harmonic balance technique or shooting methods requires the formulation of the state equations or some differential equation describing the behaviour of the network. This is not an easy way of formulation for general purpose computer algorithms.

(ii) In the Piecewise Harmonic Balance Method, there is no need to solve a nonlinear, algebraic or dynamic equation since all the nonlinearities are treated appropriately.

(iii) The linear subnetwork is solved only once for each frequency in frequency domain and the response of the linear network is obtained by scaling. Therefore full advantage of the linearities which constitute larger portion of the network have been utilized.

(iv) Since all the nonlinear branches are replaced by augmenting sources the topology of the network is preserved. This means that a connected linear subnetwork can always be obtained from a given connected network in most of the cases except if there exist nonlinear n-ports which can not be modeled as n+1 terminal elements.

(v) As the number of harmonics are increased after balancing initial estimate of the harmonics, the problem becomes numerically better conditioned, and the number of iterations for the next estimate of the harmonics are considerably decreased.

(vi) The value of the error function to be minimized can be multiplied by a fixed integer in such a way as to allow the algorithm be operating. The adjustment of the value of the scalar error function P to be minimized can be changed to obtain better estimates of the initial harmonics.

(vii) The number of optimization variables in the Piecewise Harmonic Balance Method is reduced considerably as compared to the optimization variables in the conventional harmonic balance method since most of the practical circuits contain more capacitors and inductors than nonlinear elements.

(viii) The speed of the computation of the gradient is increased by taking the DFT of two functions at a time.

A new expression for the gradient evaluation has been developed and compared to the one derived by Nakhla and

Vlach [8] for the case that the period of the oscillations is unknown. It has been observed that the expression proposed by Nakhla and Vlach [8] is a very crude approximation to the gradient whereas the one presented in this study is the exact one.

APPENDIX

AI. GRAPHICAL DEVELOPMENT OF THE DISCRETE FOURIER TRANSFORM

Consider the example function h(t) and its Fourier Transform H(f) illustrated in Fig (Ala). It is desired to modify this Fourier transform pair in such a way that the pair is applicable to digital computer computation. This modified pair, called the discrete Fourier transform is to approximate as closely as possible the continious Fourier transform.

To determine the Fourier transform of h(t) by means of digital analysis techniques, it is necessary to sample h(t). Sampling is accomplished by multiplying h(t) by the sampling function shown in Fig (Alb). The sampling interval is T. Sampled function $\hat{h}(t)$ and its Fourier transform are illustrated in Fig(Alc). This Fourier transform pair represents the first modification to the original pair which is necessary in defining a discrete transform pair. The modified transform pair in Fig (Alc) differs from the original transform pair only by the aliasing effect which results from sampling. If the waveform h(t) is sampled at a frequency of at least twice the largest frequency component of h(t), there is no loss of information as a result of sampling. If the function h(t) is not band limited, i.e., $H(f) \neq 0$ for some $|f| > fc$, then sampling will introduce aliasing as illustrated in Fig(Alc). To reduce this error, T should be chosen smaller. The Fourier transform pair shown in Fig(Alc) is not suitable for machine computation because an infinity of samples of h(t) is considered; it is necessary to truncate the sampled function h(t) so that only a finite number of points, say N,

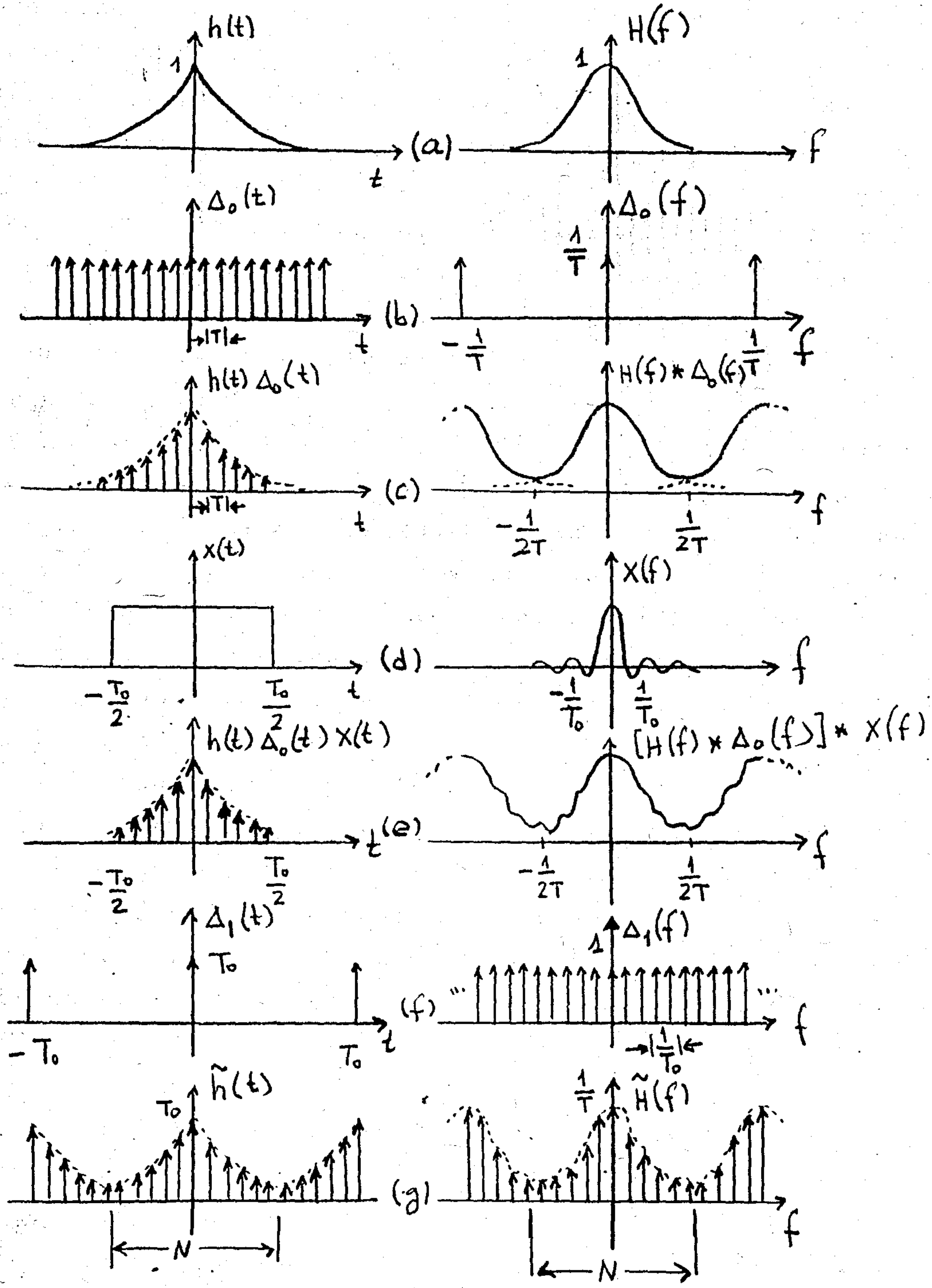


Figure A1. Graphical development of the discrete Fourier transform.

are considered. The rectangular truncation function and its Fourier transform are illustrated in Fig (Ald). The product of the infinite sequence of impulse functions representing h(t) and the truncation function yields the finite length time function illustrated in Fig (Ale). Truncation introduces the second modification of the original Fourier transform pair, this effect is to convolve the aliased frequency transform of Fig (Ald) with the Fourier transform of the truncation function in Fig (Ald),

As shown in Fig (Ale), the frequency transform has a ripple to it. If the truncation function is increased in length, then the Sinf/f function in Fig (Ale) will approach an impulse; the more closely the sinf/f function approximates an impulse, the less ripple or error will be introduced by the convolution which results from truncation. Therefore it is desireable to choose the length of the truncation function as long as possible. The modified transform pair of Fig (Ale) is still not an acceptable discrete Fourier transform pair because the frequency transform is a continious function. For machine computation only sample values of the frequency function can be computed; it is necessary to modify the frequency transform by the frequency sampling function illustrated in Fig (Alf). The frequency sampling interval is $1/T_o$.

The discrete Fourier transform pair of Fig (Alg) is acceptable for the purposes of digital machine computation since both the time and frequency domains are represented by discrete values. As shown in Fig (Alg) the original time function h(t) and the original Fourier transform H(f) are approximated by N samples. These N samples define the discrete Fourier transform pair. The sampling in the time domain resulted in a periodic function of frequnecy and the sampling in the frequency domain resulted in a periodic function of time. Hence the discrete Fourier transform requires that both the original time and frequency functions be modified such that they become periodic functions.

AII. THEORETICAL DEVELOPMENT OF THE DISCRETE FOURIER TRANSFORM

Consider the Fourier transform pair illustrated in Fig (A2a). To discretize this transform pair it is first necessary to sample the waveform h(t); the sampled waveform can be written as $h(t)\Delta_0(t)$ where

$$\Delta_0(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) \tag{A1}$$

is the time domain sampling function illustrated in Fig (A2b). The sample interval is T. The sampled function can be written as

$$h(t)\,\Delta_0(t) = \sum_{k=-\infty}^{\infty} h(kT)\,\delta(t-kT) \tag{A2}$$

The result of this multiplication is illustrated in Fig (A2c). The aliasing effect has occured from the choice of T as shown. The sampled function is truncated by multiplication with the rectangular function x(t) illustrated in Fig (A2d):

$$\begin{aligned} x(t) &= 1 \quad \text{for} \quad -\frac{T}{2} < t < T_0 - \frac{T}{2} \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{A3}$$

where it has been assumed that there are N equidistant impulse functions lying within the truncation interval, that is, $N = \frac{T_0}{T}$.

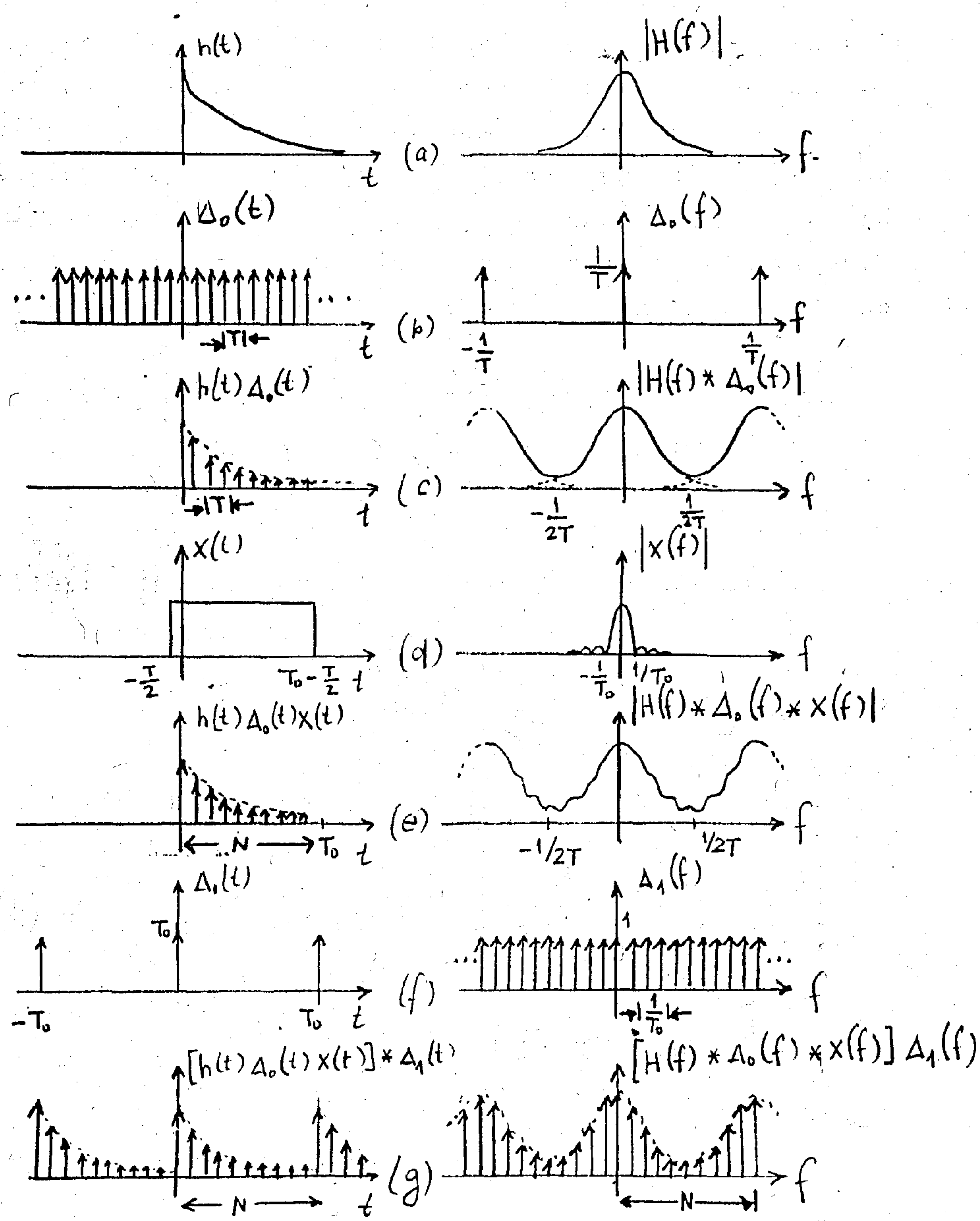


Figure A2. Graphical derivation of the discrete Fourier transform pair.

The sampled truncated waveform and its Fourier transform are illustrated in Fig (A2e). As in the explanation of graphical development, truncation in the time domain results in rippling in the frequency domain. The final step in modifying the original Fourier transform pair to a discrete Fourier transform pair is to sample the Fourier transform of Eq (A4). In the time domain this product is equivalent to convolving the sampled truncated waveform of Eq (A3) and the time function

$$\Delta_1(t) = T_0 \sum_{r=-\infty}^{\infty} \delta(t - rT_0) \tag{A5}$$

as illustrated in Fig (A2e). The desired relationship for $\tilde{h}(t)$ becomes

$$\tilde{h}(t) = \hat{h}(t) * \Delta_1(t) \tag{A6}$$

Substituting Eq (A4) and (A5) into A6)

$$\begin{aligned} \tilde{h}(t) &= \left[\sum_{k=0}^{N-1} h(kT)\,\delta(t-kT)\right] * \left[T_0 \sum_{r=-\infty}^{\infty} \delta(t-rT_0)\right] \\ &= T_0 \sum_{r=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} h(kT)\,\delta(t-kT-rT_0)\right] \end{aligned} \tag{A7}$$

is obtained. Note that Eq (A7) is periodic with period T_0. The continious function $\tilde{h}(t)$ obtained in Eq (A7) is an approximation to the original time function h(t). The choice of the rectangular function $\tilde{x}(t)$ as described by Eq (A3) has a special importance. The convolution result of Eq (A7) is a periodic function with period T_0 which consists of N samples. If the rectangular function had been chosen such that a sample value coincided with each end point of the rectangular function, the convolution of the rectangular function with impulses spaced at intervals of T_o would result in time domain aliasing. That is, the N^{th} point of one period would coincide with the first point of the next period T_o insure that time domain aliasing not occur, it is necessary to choose the truncation interval as illustrated in Fig (A2d).

The Fourier transform $\tilde{H}(f)$ of Eq (A7) can be developed in two different ways. The first method is to use the time convolution theorem. Since the waveform $\tilde{h}(t)$ in Fig.(A2g) is the convolution of $\hat{h}(t)$ with $\Delta_1(t)$, its Fourier transform becomes

$$\tilde{H}(f) = \hat{H}(f)\,\Delta_1(f) \tag{A8}$$

where

$$\begin{aligned}\hat{H}(f) &= \int_{-\infty}^{\infty} \hat{h}(t)\, e^{-j2\pi f t}\, dt \\ &= \int_{-\infty}^{\infty} \Big[\sum_{k=0}^{N-1} h(kT)\,\delta(t-kT)\Big] e^{-j2\pi f t}\, dt \\ &= \sum_{k=0}^{N-1} h(kT) \int_{-\infty}^{\infty} \delta(t-kT)\, e^{-j2\pi f t}\, dt \\ &= \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi f kT}\end{aligned} \tag{A9}$$

and

$$\Delta_1(f) = \int_{-\infty}^{\infty} \Delta_1(t)\, e^{-j2\pi f t}\, dt = \sum_{r=-\infty}^{\infty} \delta\Big(f - \frac{r}{T_o}\Big) \tag{A10}$$

Inserting Eqs (A9) and (A10) into (A11)

$$\tilde{H}(f) = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi fkT} \sum_{r=-\infty}^{\infty} \delta\left(f - \frac{r}{T_0}\right) \tag{A11}$$

or, equivalently

$$\tilde{H}(f) = \sum_{r=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi fkT} \right] \delta\left(f - \frac{r}{T_0}\right) \tag{A12}$$

is obtained. Using the property of impulse Eq (A12) becomes

$$\tilde{H}(f) = \sum_{r=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi krT/T_0} \right] \delta\left(f - \frac{r}{T_0}\right) \tag{A13}$$

Using $\frac{T}{T_0} = \frac{1}{N}$ in Eq(A13).

$$\tilde{H}(f) = \sum_{r=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kr/N} \right] \delta\left(f - \frac{r}{T_0}\right) \tag{A14}$$

is obtained. Eq (A14) represents the continious Fourier transform of the periodic time function $\tilde{h}(t)$, which is an infinite sequence of equidistant impulses with amplitudes given by

$$\alpha_r = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kr/N} \tag{A15}$$

The discrete Fourier transform can then be obtained from the continious Fourier transform $\tilde{H}(f)$ by evaluating $\tilde{H}(f)$

at $f=\frac{n}{T_0}$. Substituting $f=\frac{n}{T_0}$ into Eq (A14)

$$\tilde{H}\left(\frac{n}{T_0}\right)=\sum_{r=-\infty}^{\infty}\left[\sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kr/N}\right]\delta\left(\frac{n-r}{T_0}\right) \tag{A16}$$

is obtained. Considering the meaning of

$$\delta(r-n)=\begin{cases}1 & r=n\\ 0 & r\neq n\end{cases} \tag{A17}$$

which is the Kronecker delta sequence. Then Eq (Alb) becomes

$$\tilde{H}\left(\frac{n}{NT}\right)=\sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kn/N} \tag{A18}$$

which is the desired discrete Fourier transform.

The second method to obtain the discrete Fourier transform can be described as follows:

It is known that

$$\tilde{h}(t)=T_0\sum_{r=-\infty}^{\infty}\left[\sum_{k=0}^{N-1} h(kT)\,\delta(t-kT-rT_0)\right] \tag{A19}$$

is periodic with period T_0. Since the Fourier transform of a periodic function is a sequence of equidistant impulses

$$\tilde{H}(f)=\sum_{r=-\infty}^{\infty}\alpha_r\,\delta\left(f-\frac{r}{T_0}\right) \tag{A20}$$

is obtained. Evaluating the continuous Fourier transform at $f=\frac{n}{T_0}$

$$\tilde{H}\left(\frac{n}{T_0}\right) = \sum_{r=-\infty}^{\infty} \alpha_r \, \delta\left(\frac{n-r}{T_0}\right) \qquad \text{(A21)}$$

where n is an integer constant. Once more considering the Kronecker delta sequence in Eq (A17) Eq (A22) becomes

$$\tilde{H}\left(\frac{n}{T_0}\right) = \alpha_n \qquad \text{(A23)}$$

where α_n's are the Fourier coefficients of the continious function $\tilde{h}(t)$, and can be given by

$$\alpha_n = \frac{1}{T_0} \int_{-T/2}^{T_0 - T/2} \tilde{h}(t)\, e^{-j2\pi nt/T_0}\, dt$$

$$= \frac{1}{T_0} \int_{-T/2}^{T_0 - T/2} T_0 \sum_{r=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} h(kT)\, \delta(t - kT - rT_0) \right] e^{-j2\pi nt/T_0}\, dt$$

(A24)

let r=0, in Eq (A24) since the integration is only over one period, then

$$\alpha_n = \frac{1}{T_0} \int_{-T/2}^{T_0 - T/2} T_0 \sum_{k=0}^{N-1} h(kT)\, \delta(t - kT)\, e^{-j2\pi nt/T_0}\, dt$$

$$= \sum_{k=0}^{N-1} h(kT) \int_{-T/2}^{T_0 - T/2} e^{-j2\pi nt/T_0}\, \delta(t - kT)\, dt$$

$$= \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kn/N} \qquad \text{(A26)}$$

is obtained. Then

$$\tilde{H}\left(\frac{n}{NT}\right) = \alpha_n = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kn/N} \qquad n = 0, \pm 1, \pm 2, \ldots \qquad \text{(A27)}$$

which is identical with the result obtained in Eq (A18).

From Eq (A27) it is seen that the Fourier transform $\tilde{H}\left(\frac{n}{NT}\right)$ is periodic as shown in Fig (A2g). There are only N distinct complex values computable from Eq (A27).

Letting n=r where r is an arbitrary integer in Eq (A27)

$$\tilde{H}\left(\frac{r}{NT}\right) = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kr/N} \qquad \text{(A28)}$$

is obtained. On the other hand, letting $n=r+N$ in Eq (A27)

$$\tilde{H}\left(\frac{r+N}{NT}\right) = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi nk(r+N)/N}$$

$$= \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kr/N} \cdot e^{-j2\pi k}$$

$$= \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kr/N}$$

$$= \tilde{H}\left(\frac{r}{NT}\right) \tag{A29}$$

is obtained. Therefore there are only N distinct values for which Eq (A27) can be evaluated; $\tilde{H}$ (n/NT) is periodic with a period of N samples. The Fourier transform Eq (A27) can be expressed as

$$\tilde{H}\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} h(kT) e^{-j2\pi nk/N} \qquad n=0,1,\ldots,N-1 \tag{A30}$$

The discrete Fourier transform Eq (A30) relates N samples of time and N samples of frequency by means of the continious Fourier transform. If it is assumed that the N samples of the original function h(t) are one period of a periodic waveform, the Fourier transform of this periodic function is given by the N samples as computed by Eq (A30). The notation $\tilde{H}\left(\frac{n}{NT}\right)$ is used to indicate that the discrete Fourier transform obtained in Eq (A30) is an approximation to the continious Fourier transform. Eq (A30) is written as

$$G\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} g(kT)\, e^{-j2\pi nk/N} \qquad n=0,1,\ldots,N-1 \tag{A31}$$

Since the Fourier transform of the sampled periodic fnuction g(kT) is identically G(n/NT).

AIII. DISCRETE INVERSE FOURIER TRANSFORM

The inverse discrete Fourier transform is given by

$$g(kT) = \frac{1}{N} \sum_{n=0}^{N-1} G\left(\frac{n}{NT}\right) e^{j2\pi nk/N}$$

$$k = 0, 1, \ldots, N-1 \qquad \text{(A32)}$$

Substituting Eq (A32) into Eq (A31)

$$G\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} \left[\frac{1}{N} \sum_{r=0}^{N-1} G\left(\frac{r}{NT}\right) e^{j2\pi rk/N}\right] e^{-j2\pi nk/N}$$

$$= \frac{1}{N} \sum_{r=0}^{N-1} G\left(\frac{r}{NT}\right) \left[\sum_{k=0}^{N-1} e^{j2\pi rk/N} e^{-j2\pi nk/N}\right] \qquad \text{(A33)}$$

is obtained. Using the orthogonality relationship

$$\sum_{k=0}^{N-1} e^{j2\pi rk/N} e^{-j2\pi nk/N} = \begin{cases} N & \text{if } r = n \\ 0 & \text{otherwise} \end{cases} \qquad \text{(A34)}$$

in the Eq (A33), the Eq (A33) becomes an identity. The periodicity of the discrete inverse Fourier transform follows from the periodic nature of $e^{j2\pi nk/N}$. g(kT) is actually defined on the complete set of integers $k = 0, \pm 1, \pm 2, \ldots$ and is constraint by the identity

$$g(kT) = g[(rN + k)T] \qquad r = 0, \pm 1, \pm 2, \ldots \qquad \text{(A35)}$$

Then the discrete Fourier transform pair is given by

$$g(kT)=\frac{1}{N}\sum_{n=0}^{N-1} G\left(\frac{n}{NT}\right) e^{j2\pi nk/N} \leftrightarrow G\left(\frac{n}{NT}\right)=\sum_{k=0}^{N-1} g(kT)\, e^{-j2\pi nk/N} \quad \text{(A36)}$$

The pair in Eq (A36) requires both the time and frequency domain functions be periodic, i.e.

$$G\left(\frac{n}{NT}\right)=G\left[\frac{(rN+n)}{NT}\right] \qquad r=0,\pm1,\pm2,\ldots \quad \text{(A37)}$$

$$g(kT)=g[(rN+k)T] \qquad r=0,\pm1,\pm2,\ldots \quad \text{(A38)}$$

The discrete Fourier transform approximates the continious Fourier transform. Validity of this approximation is a function of the wave form being analyzed.

In the following, the error introduced in approximating the Fourier transform of a fnuction by means of the discrete Fourier transform will be examined for two special types of waveform.

EXAMPLE 10: BAND-LIMITED PERIODIC WAVEFORMS (Truncation interval equal to period)

Consider the function h(t) and its Fourier transform illustrated in Fig (A3a). It is desired to sample h(t), truncate the sampled function to N samples, and apply the discrete Fourier transform Eq (A31). Rather than applying this equation directly, its application will be developed graphically. Waveform h(t) is sampled by multiplication with the sampling function illustrated in Fig (A3b). Sampled waveform h(kT) and its Fourier transform are shown in Fig (A3c). It

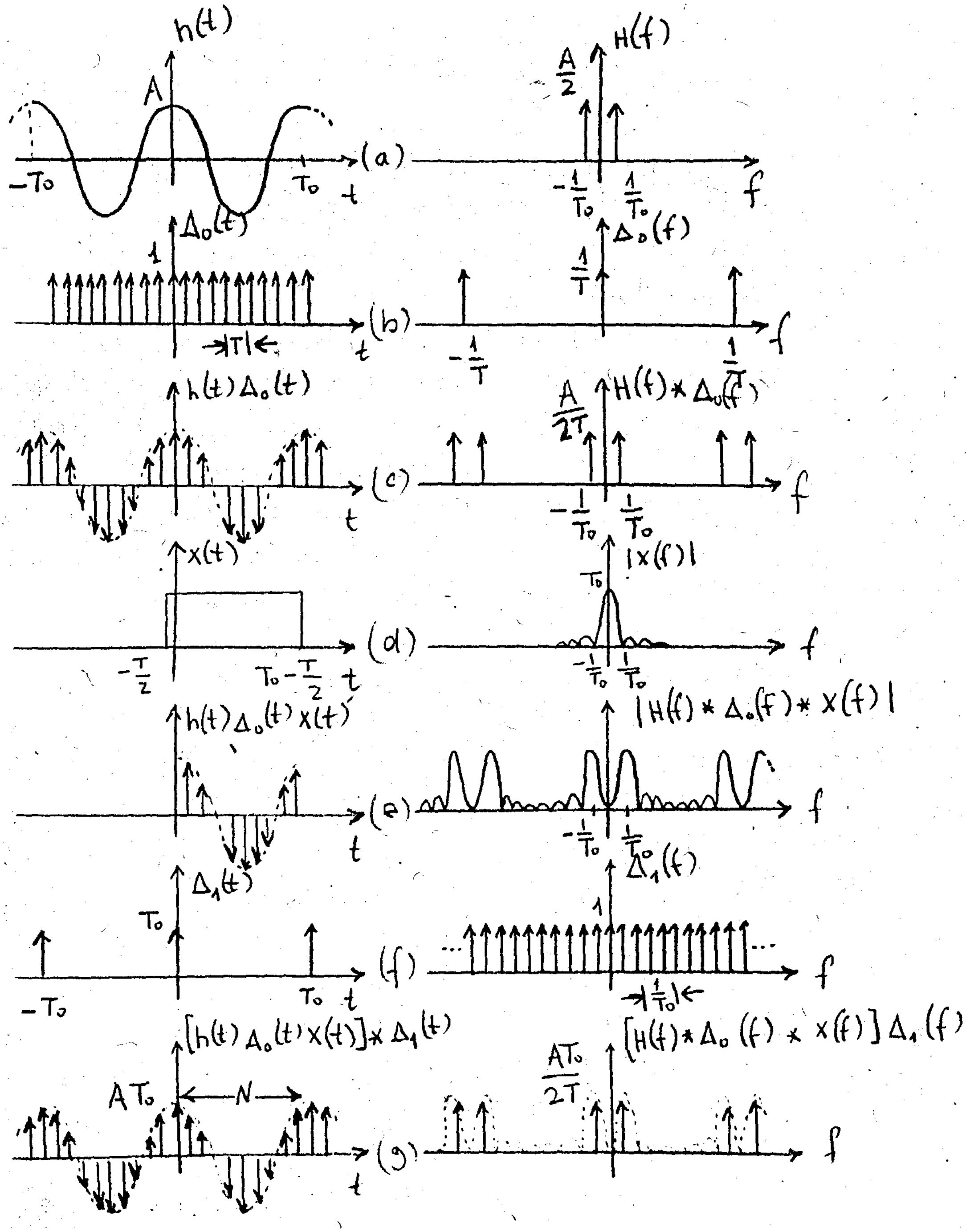


Figure A3. Discrete Fourier transform of a band-limited periodic waveform: truncation interval equal to period.

should be noted that there exists no aliasing in Fig (A3c) for this example and as a result of time domain sampling, the frequency domain has been scaled by the factor 1/T, the Fourier transform impulse has an area of A/2T rather than the original area of A/2. The sampled waveform is truncated by multiplication with the rectangular function illustrated in Fig (A3d). Fig (A3e) illustrates the sampled and truncated waveform. As shown, the rectangular truncation function is chosen so that the N sample value has remained after truncation interval is equated to one period of the original waveform h(t). The Fourier transform of the finite length sampled waveform Fig (A3e) is obtained by convolving the frequency domain impulse functions of Fig (A3c) and the sinf/f function of Fig(A3d). Fig (A3e) illustrated the convolution results. A sinf/f function is centered on each impulse of Fig (A3e) and the resultant waveforms are additively combined to form the convolution result. With respect to the original transform H(f), the convolved frequency function Fig (A3e) is significiantly distorted. However, when this function is sampled by the frequency sampling function illustrated in Fig. (A3f) the distortion is eliminated. This follows because the equidistant impulses of the frequency sampling function are seperated by $1/T_0$, at those frequencies the convolved frequency function in Fig (A3e) is zero except at the frequency $\pm 1/T_0$. Frequency $\pm 1/T_0$ corresponds to the frequency domain impulses of the original frequency function H(f). Because of time domain truncation, these impulses now have an area of $AT_0/2T$ rather than the original area of A/2. Multiplication of the frequency function of Fig (A3e) and the frequency sampling function $\Delta_1(t)$ implies the convolution of the time functions shown in Fig (A3e) and (A3f). Because the sampled truncated waveform Fig (A3e) is exactly one period of the original waveform h(t) and since the time domain impulse functions of Fig (A3f) are seperated by T_0, then their convolution yields a periodic Function as illustrated in Fig (A3g). The time function of Fig (A3g) has a maximum amplitude of AT_0, compared to the original maximum value of A as a result of frequency domain sampling. Examination of Fig (A3g) indicates that the original time function is sampled and multiplied by T_o. The Fourier transform of this function is related to the original frequency function by the factor $AT_o/2T$. Factor T_o is common and can be eliminated. If it is desired to compute the Fourier transform by means of the discrete Fourier transform, it is necessary to multiply the discrete time function by the factor T which yields the desired A/2 area for the frequency function; Eq (A31) then becomes

$$H\left(\frac{n}{NT}\right) = T\sum_{k=0}^{N-1} h(kT)e^{-j2\pi nk/N} \tag{A39}$$

This result is expected since the relationship (A39) is simply the rectangular rule for integration of the continious Fourier transform. This example represents the only class of waveforms for which the discrete and continious Fourier transform are exactly the same within a scaling constant. Equivalence of the two transforms requires that the time function h(t) must be periodic and band-limited, the sampling rate must be at least two times the largest frequency component of h(t), and the truncation function x(t) must be non-zero over exactly one period or integer multiple period of h(t).

EXAMPLE 11: BAND-LIMITED PERIODIC WAVEFORMS (Truncation interval not equal to period)

If a periodic, band-limited function is sampled and truncated to consist of rather than an integer multiple of the period, the resulting discrete and continious transform as shown in Fig (A4) will differ considerably. This example differs from the previous one only in the frequency of the

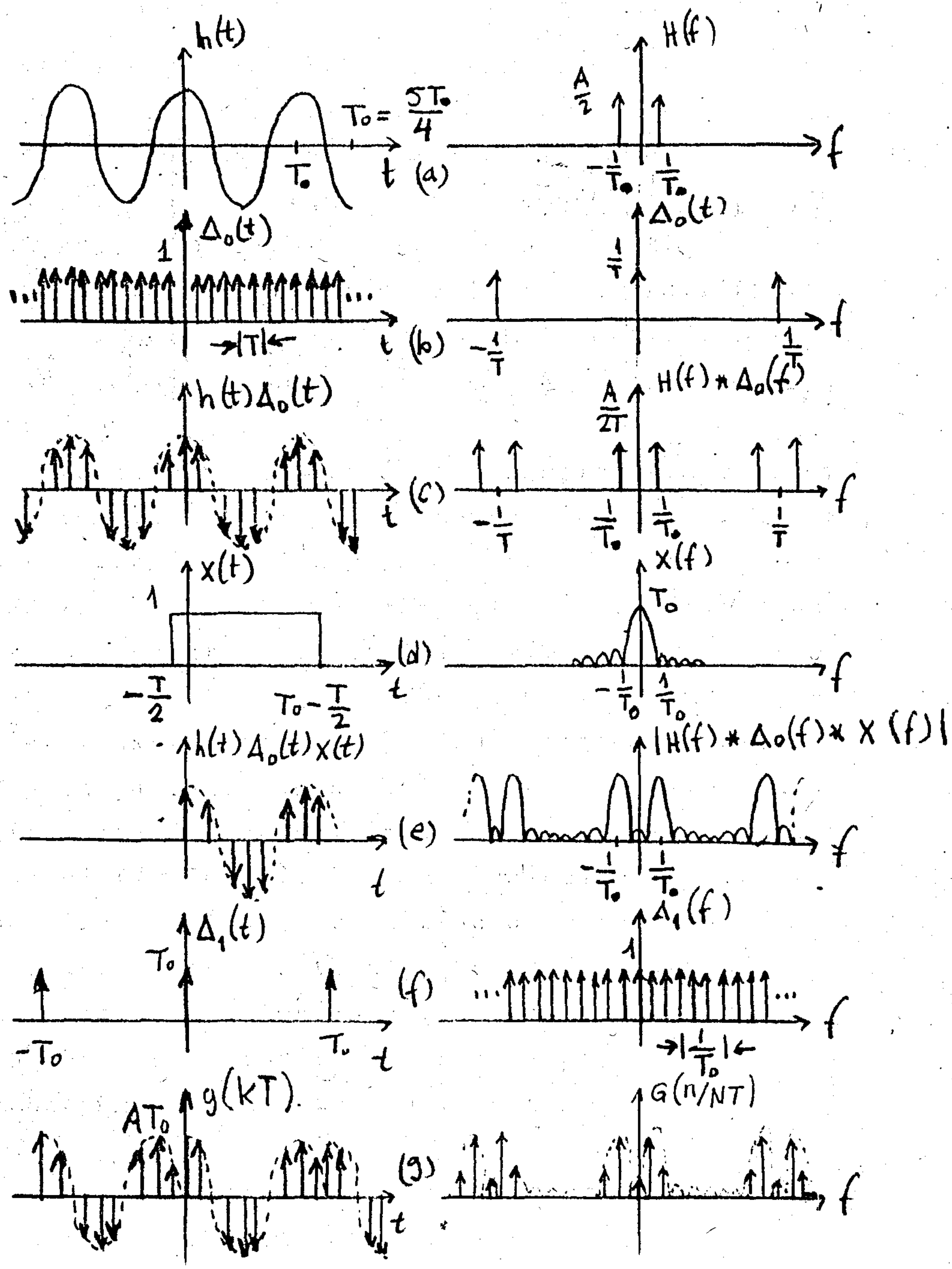


Figure A4. Discrete Fourier transform of a band-limited periodic waveform: truncation interval not equal to period.

sinusoidal waveform h(t). The function h(t) is sampled [Fig(4Ac)] and truncated [Fig(A4e)]. The sampled, truncated function is not an integer multiple of the period of h(t). Therefore, when the time functions of Figs(A4e) and (A4f) are convolved, the periodic waveform of Fig(A4g) results. Although this function is periodic, it is not a replica of the original periodic function h(t). Hence, the Fourier transform of the time waveforms of Figs(A4a) and (A4g) are not equivalent.

Fourier transform of the sampled truncated waveform of Fig (4Ae) is obtained by convolving the frequency domain impulse functions of Fig (A4c) and the sinf/f function illustrated in Fig (A4d). Sampling of the resulting convolution at frequency intervals of $1/T_o$, yields the impulses as illustrated in Fig (A4g). These sample values represent the Fourier transform of the periodic time waveform of Fig (A4g). In Fig (A4g) there is an impulse at zero frequency. The remaining frequency domain impulses occur because the zeros of the sinf/f function are not coincident with each sample value. The effect of truncation at other than a multiple of the period is to create a periodic function with sharp discontinuities as shown in Fig (A4g). The introduction of these sharp changes in the time domain is a result of the additional frequency components in the frequency domain.

Time domain truncation is equivalent to the convolution of a sinf/f function with the single impulse representing the original frequency function H(f) in the frequency domain. Consequently, the frequency function is no longer a single impulse but rather a continious function of frequency with a local maximum centered at the original impulse and a series of other peaks which are said sidelobes. These sidelobes are responsible for the additional frequency components which occur after frequency domain sampling.

AIV. EVEN AND ODD FUNCTIONS

If $h_e(k)$ is an even function then $h_e(k) = h_e(-k)$ and the DFT of $h_e(k)$ becomes

$$H_e(n) = \sum_{k=0}^{N-1} h_e(k)\, e^{-j2\pi nk/N}$$

$$= \sum_{k=0}^{N-1} h_e(k) \operatorname{Cos} \frac{2\pi nk}{N} + j \sum_{k=0}^{N-1} h_e(k) \operatorname{Sin} \frac{2\pi nk}{N}$$

$$= \sum_{k=0}^{N-1} h_e(k) \operatorname{Cos} \frac{2\pi nk}{N} = R_e(n) \qquad \text{(A 40)}$$

which is an real even function.

If $h_o(k)$ is an odd function then $h_o(k)=-h_o(-k)$ and the DFT of $h_o(k)$ becomes

$$H_o(n) = \sum_{k=0}^{N-1} h_o(k)\, e^{-j2\pi nk/N}$$

$$= \sum_{k=0}^{N-1} h_o(k) \operatorname{Cos} \frac{2\pi nk}{N} - j \sum_{n=0}^{N-1} h_o(k) \operatorname{Sin} \frac{2\pi nk}{N}$$

$$= -j \sum_{k=0}^{N-1} h_o(k) \operatorname{Sin} \frac{2\pi nk}{N} = j\, I_o(n) \qquad \text{(A 41)}$$

which is an imaginary function.

AV. WAVEFORM DECOMPOSITION

An arbitrary function $h(k)$ can be decomposed into an even and an odd function by adding and subtracting the common function $h(-k)/2$.

$$h(k) = \left[\frac{h(k)}{2} + \frac{h(-k)}{2}\right] + \left[\frac{h(k)}{2} - \frac{h(-k)}{2}\right]$$

$$= h_e(k) + h_o(k) \qquad \text{(A 42)}$$

Terms in brackets satisfy the definition of an even and an odd function, respectively. Since $h(k)$ is periodic with period N then

$$h(-k) = h(N-k) \qquad \text{(A 43)}$$

and

$$h_e(k) = \frac{h(k)}{2} + \frac{h(N-k)}{2}$$

$$h_o(k) = \frac{h(k)}{2} - \frac{h(N-k)}{2} \qquad \text{(A 44)}$$

(A 44) is the desired relationship for decomposition of the discrete periodic functions. From (A 40) and (A 41) the DFT of (A 42) is

$$H(n) = R(n) + jI(n) = H_e(n) + H_o(n) \qquad \text{(A 45)}$$

where

$$H_e(n) = R(n) \qquad \text{and} \qquad H_o(n) = jI(n) \qquad \text{(A 46)}$$

AVI. COMPLEX TIME FUNCTIONS

The DFT of a complex time function $h(k) = h_r(k) + jh_i(k)$, where $h_r(k)$ and $h_i(k)$ are respectively the real and imaginary part of the complex time function $h(k)$, becomes

$$H(n) = \sum_{k=0}^{N-1} \left[h_r(k) + jh_i(k)\right] e^{-j2\pi nk/N}$$

$$= \sum_{k=0}^{N-1} \left[h_r(k) \cos \frac{2\pi nk}{N} + h_i(k) \sin \frac{2\pi nk}{N}\right]$$

$$-j \sum \left[h_r(k) \sin \frac{2\pi nk}{N} - h_i(k) \cos \frac{2\pi nk}{N}\right]$$

(A 47)

The first expression of (A 47) is R(n), the real part of the DFT, and the second expression is I(n), the imaginary part of the DFT. If h(k) is real, then $h(k) = h_r(k)$, and from (A 47)

$$R_e(n) = \sum_{k=0}^{N-1} h_r(k) \cos \frac{2\pi nk}{N} \tag{A 48}$$

$$I_o(n) = -j \sum_{k=0}^{N-1} h_r(k) \sin \frac{2\pi nk}{N} \tag{A 49}$$

where $R_e(n)$ is an even function, and $I_o(n)$ is an odd function. If h(k) is purely imaginary, then $h(k) = jh_i(k)$, and from (A 47)

$$R_o(n) = \sum_{k=0}^{N-1} h_i(k) \sin \frac{2\pi nk}{N} \tag{A 50}$$

$$I_e(n) = \sum_{k=0}^{N-1} h_i(k) \cos \frac{2\pi nk}{N} \tag{A 51}$$

where $R_o(n)$ is an odd function and $I_e(n)$ is an even function.

AVII. COMPUTATION OF THE DFT OF TWO REAL FUNCTIONS SIMULTANEOUSLY

To compute the DFT of the real time functions h(k) and g(k) form the complex function

$$y(k) = h(k) + jg(k) \tag{A 52}$$

From the linearity property the DFT of y(k) is given by

$$\begin{aligned} Y(n) &= H(n) + jG(n) \\ &= [H_r(n) + jH_i(n)] + j[G_r(n) + jG_i(n)] \\ &= [H_r(n) - G_i(n)] + j[H_i(n) + G_r(n)] \\ &= R(n) + jI(n) \end{aligned} \tag{A 53}$$

Using the frequency domain equivalent of (A 42) we decompose both R(n), the real part of Y(n), and I(n), the imaginary part of Y(n), into even and odd components

$$\begin{aligned} Y(n) = &\left[\frac{R(n)}{2} + \frac{R(N-n)}{2}\right] + \left[\frac{R(n)}{2} - \frac{R(N-n)}{2}\right] \\ &+ j\left[\frac{I(n)}{2} + \frac{I(N-n)}{2}\right] + j\left[\frac{I(n)}{2} - \frac{I(N-n)}{2}\right] \end{aligned} \tag{A 54}$$

From Eqs (A 48) and (A 49)

$$\begin{aligned} H(n) &= R_e(n) + jI_o(n) \\ &= \left[\frac{R(n)}{2} + \frac{R(N-n)}{2}\right] + j\left[\frac{I(n)}{2} - \frac{I(N-n)}{2}\right] \end{aligned} \tag{A 55}$$

Similarly from Eqs (A 50) and (A 51)

$$jG(n) = R_o(n) + jI_e(n)$$

or

$$\begin{aligned} G(n) &= I_e(n) - jR_o(n) \\ &= \left[\frac{I(n)}{2} + \frac{I(N-n)}{2}\right] - j\left[\frac{R(n)}{2} - \frac{R(N-n)}{2}\right] \end{aligned} \tag{A 56}$$

Thus, if real and imaginary parts of the DFT of a complex time function are decomposed according to Eqs (A 55) and (A 56), then the simultaneous discrete transform of two real time functions can be accomplished.

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```
      ***********************************************************************
      ***********************************************************************
      ***  A   H A R M O N I C   B A L A N C E   T E C H N I Q U E         **
      ***                               F O R                              **
      ***       T H E   S T E A D Y   S T A T E   A N A L Y S I S          **
      ***                                O F                               **
      ***            N O N L I N E A R   N E T W O R K S                   **
      ***                              W I T H                             **
      ***              P E R I O D I C   E X C I T A T I O N S             **
      ***********************************************************************
      ***********************************************************************
C**********************************************************************************
             M A I N   P R O G R A M   F O R   B A L A N C E
C**********************************************************************************
      READ INDEPENDENT SOURCE AS LAST ELEMENT
      NODES=NUMBER OF NODES
      READ NETWORK ELEMENTS AND INTERCONNECTIONS
      1.RESISTORS,CAPACITORS,INDUCTORS,ETC...
      2.ACTUAL SIN (VOLTAGE OR CURRENT) SOURCE
      3.ACTUAL  DC  VOLTAGE SOURCES
      4.AUGMENTING  VOLTAGE SOURCES
           4-A.TRANSISTORS  Q*  EMITTER-BASE BRANCH
                                Q*COLLECTOR-BASE BRANCH
           4-B.DIODES           W*BRANCHES
      USE TYPE = 1 FOR A CONDUCTANCE
      USE TYPE = 2 FOR A CAPACITANCE
      USE TYPE = 3 FOR A VCT
      USE TYPE = 5 FOR A VVT
      USE TYPE = 6 FOR A CURRENT SOURCE
      USE TYPE = 7 FOR A VOLTAGE SOURCE
      USE TYPE = 8 FOR AN OPERATIONAL AMPLIFIER
      USE TYPE = 9 FOR AN INDUCTOR
      USE TYPE = 10 FOR A RESISTOR
C**********************************************************************************
      PARAMETER MNNS=2,MNNS1=3,MNEL=19,MQ=18,MOUTN=18,MOUTN2=9,MOUT=22,
     *MNHAR1=26,MNP1=75,MNOPT=102,MWORK=5559
C**********************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX*16 Y(MQ,MQ),DETMAT,X(MQ),PAUX1(MNP1),PAUX2(MNP1),
     *OUT(MOUTN,MNNS1,MNHAR1),COUT(MOUTN,MNHAR1),COUTX(MOUTN2,MNP1),
     *CRESP,XCORD,CX,CYL
      REAL*8 XOPT(MNOPT),GRAD(MNOPT),EPS(MNOPT),H(MWORK),OUTX(MOUT,MNP1
     *,XS,YL
      REAL*4 ITM(102),IVAN(102)
      COMMON/MAINTR/ATR(25)
      COMMON/MAINDI/ADI(14)
      COMMON/TRANS/CRESP(MNNS,MNNS1,MNHAR1),XCORD(MNP1),
     *CX(MNNS,MNHAR1),CYL(MNNS,MNHAR1),XS(MNNS,MNP1),YL(MNNS,MNP1),
     *DOMEG,NP1,NNS,NHAR,NFRQ,ICONTR,CTPER
      COMMON/MATR/G(MQ,MQ),C(MQ,MQ),B(MQ),N,NODES
      INTEGER TYPE,NOUT(MOUTN)
      COMMON/INPUT/VALUE(MNEL),TYPE(MNEL),NO1(MNEL),NO2(MNEL),NO3(MNEL)
     *NO4(MNEL),NEL
      COMMON/MAINON/NTR,NDI
      LOGICAL UNITH
      EXTERNAL FUNCT
      DATA SREAL/'REAL'/,SIMAG/'IMAG'/,SMAG/'MAG.'/,SARG/'ARG.'/
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C...IEPS=0 NO CHANGE IN EPSI,DEPS=DECREASE IN EPSI
      READ(5,86) ICONTR,ISR,IEPS,IXOPT,INP1,NTR,NDI,NDC,NP1AD,NAD,NHAR,
     *HAMAX,NODES,NEL,IPRINT,NXOPT,MAXFN,IWORK,AMP,FREQ,XOPTIN,EPSI,DE
86    FORMAT(8I1,2I2,6I3,2I5,5D8.0)
      DOMEG=FREQ*6.2831853071795860
```

```
      UNITH=.TRUE.
      DFN=0.D0
      NFRQ=1
      NP1=N2HAR1
C...INCREASE IN THE NUMBER OF POINTS NP1
C...INP1=0 MEANS NP1=N2HAR1
      IF(INP1.EQ.1) NP1=NP1+NP1AD
C...INITIALIZATION OF XOPT
      DO 625 I=1,NOPT
      EPS(I)=EPSI
25    XOPT(I)=XOPTIN
C...IXOPT=0 MEANS NO INDIVIDUAL INITIALIZATION
      IF(IXOPT.EQ.0) GO TO 405
      READ(5,91) (XOPT(I),I=1,NXOPT)
1     FORMAT(8D10.4)
      WRITE(6,92) (XOPT(I),I=1,NXOPT)
2     FORMAT(1X,D15.8)
05    CONTINUE
      WRITE(6,14)
4     FORMAT(//,5X,'****CIRCUIT    DESCRIPTION****',///,5X,'K TYPE NO1
     *NO2 NO3 NO4   VALUE',/)
      DO 40 II=1,NEL
      N=NODES
      READ(5,30) K,TYPE(K),NO1(K),NO2(K),NO3(K),NO4(K),VALUE(K)
      WRITE(6,70)K,TYPE(K),NO1(K),NO2(K),NO3(K),NO4(K),VALUE(K)
0     FORMAT(6I5,D15.8)
0     FORMAT(3X,6I4,D15.8)
0     IF(TYPE(K).EQ.10) VALUE(K)=1./VALUE(K)
      CALL FORM
      DO 80 I=1,N
0     WRITE(6,81) I,B(I)
1     FORMAT(5X,'B(',I2,')=',D15.8)
      READ(5,98) ITRANS,IPLOT,CTPER
8     FORMAT(2I2,D10.0)
      WRITE(6,99) ITRANS,IPLOT,CTPER
9     FORMAT(/'   ITRANS = ',I1,'  IPLOT = ',I2,' CTPER = ',D10.4)
      READ(5,100) NOUTN,(NOUT(I),I=1,NOUTN)
00    FORMAT(21I3)
      WRITE(6,101) NOUTN,(NOUT(I),I=1,NOUTN)
01    FORMAT('    NOUTN  = ',I2,'  NOUTS = ',21I3)
      IF(NTR.EQ.0) GO TO 12
      IF(ITRANS-1) 95,95,96
5     NTR12=12*NTR
      READ(5,84) (ATR(I),I=1,NTR12)
4     FORMAT(8D10.0)
      DO 87 I=1,NTR
7     WRITE(6,88) I,ATR(I),ATR(NTR+I),ATR(2*NTR+2*I-1),ATR(2*NTR+2*I),A
     *R(4*NTR+2*I-1),ATR(4*NTR+2*I),ATR(6*NTR+2*I-1),ATR(6*NTR+2*I),ATR
     *8*NTR+2*I-1),ATR(8*NTR+2*I),ATR(10*NTR+2*I-1),ATR(10*NTR+2*I)
8     FORMAT(///,' ***TRANSISTOR',6X,'Q',I1,6X,'PARAMETERS***',//,' LAM
     *A =',D11.4,6X,'IS  =',D11.4,/,' RE     =',D11.4,6X,'RC  =',D11.4,/
     *' FE     =',D11.4,6X,'FC  =',D11.4,/,' V1     =',D11.4,6X,'V2  =',D
     *1.4,/,' CBE   =',D11.4,6X,'CBC =',D11.4,/,' BF     =',D11.4,6X,'BR
     * =',D11.4)
      GO TO 12
6     NTR25=25*NTR
      READ(5,84) (ATR(I),I=1,NTR25)
      WRITE(6,97) (ATR(I),I=1,NTR25)
7     FORMAT(1X,D15.8)
2     IF(NDI.EQ.0) GO TO 13
      NDI7=7*NDI
      READ(5,84) (ADI(I),I=1,NDI7)
      DO 89 I=1,NDI
```

```
      *',D11.4,/,6X,'IS          =',D11.4,/,6X,'R           =',D11.4,/,6X,'F          =',D
      *11.4,/,6X,'V1         =',D11.4,/,6X,'V2          =',D11.4,/,6X,'CJ         ='D11
      *4)
3      WRITE(6,93) IEPS,IXOPT,INP1,NP1AD,NXOPT,XOPTIN,DEPS
3      FORMAT(////,' IEPS=',I1,' IXOPT=',I1,' INP1=',I1,' NP1AD=',I2,' N
      *OPT=',I2,' XOPTIN=',D10.3,' DEPS=',D10.3)
       WRITE(6,11) NHAR,NAD,NHAMAX,NODES,NEL,NTR,NDI,NDC,IPRINT,MAXFN,IWO
      *RK,ICONTR,ISR,AMP,FREQ,DOMEG,NHAR,NP1,EPSI
1      FORMAT(///,' NHAR=',I2,' NAD=',I1,' NHAMAX=',I2,' NODES=',I2,' ELE
      *MENTS=',I2,' NTR=',I1,' NDI=',I1,' NDC=',I1,' IPRINT=',I2,' MAXFN=
      *',I4,' IWORK=',I4,' ICONTR=',I1,' ISR=',I1,' AMP=',F5.2,//,' FREQU
      *ENCY=',D21.16,/,' OMEGA        =',D21.16,///,' # HARMONICS=',I3,/,' #
      *POINTS    =',I3,/,'    EPSI        =',D7.1)
**************************************************************************
       THE RESPONSE CRESP(I,1,K) OF THE LINEAR NETWORK TO ACTUAL SOURCES,
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
...ISR=0 DENOTES ACTUAL VOLTAGE SOURCE
       IF(ISR.EQ.0) GO TO 1
       B(1)=0.D0
       GO TO 2
       B(N-NNS-NDC)=0.D0
       DO 3 I=1,NNS
       B(N-NNS+I)=0.D0
       NFRQ1=NFRQ+1
       DO 306 K=1,NFRQ1
       DO 305 I=1,N
       DO 305 J=1,N
05     Y(I,J)=G(I,J)+DCMPLX(0.D0,DOMEG*(K-1))*C(I,J)
       CALL LU(Y,DETMAT,N)
       CALL SOLVE(Y,B,X,N)
       DO 307 IOUT=1,NOUTN
07     OUT(IOUT,1,K)=X(NOUT(IOUT))
       DO 4 I=1,NNS
       CRESP(I,1,K)=X(N-NNS+I)
       IF(ISR.EQ.0) GO TO 5
       B(1)=AMP
       GO TO 6
       B(N-NNS-NDC)=AMP
       DO 7 I=1,NDC
       B(N-NNS-NDC+I)=0.D0
06     CONTINUE
**************************************************************************
       NNS1=NNS+1
       NHAR1=NHAR+1
       IF(ISR.EQ.0) GO TO 8
       B(1)=0.D0
       GO TO 9
       B(N-NNS-NDC)=0.D0
       IK=1
7      CONTINUE
**************************************************************************
       THE RESPONSE CRESP(I,J,K) OF THE LN TO UNIT AUGMENTING SOURCES.
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
       B(N-NNS+1)=1.D0
       DO 324 L=2,NNS1
       DO 325 K=IK,NHAR1
       DO 330 I=1,N
       DO 330 J=1,N
30     Y(I,J)=G(I,J)+DCMPLX(0.D0,DOMEG*(K-1))*C(I,J)
       CALL LU(Y,DETMAT,N)
       CALL SOLVE(Y,B,X,N)
       DO 326 IOUT=1,NOUTN
26     OUT(IOUT,L,K)=X(NOUT(IOUT))
```

```
      IF(L.EQ.NNS1) GO TO 10
24    B(N-NNS+L)=1.D0
*************************************************************************
...OPTIMIZATION STARTS HERE
0     CALL COORD(XCORD,NP1)
      CALL QNMING(XOPT,GRAD,EPS,NOPT,P,DFN,UNITH,IPRINT,MAXFN,IER,H,
     *IWORK,FUNCT)
...FUNCTION EVALUATION WITH MORE POINTS(FOPT SKIPPED FOR NHAR=0)
      IF(NHAR.EQ.0) GO TO 120
      NP1=NP1+NP1AD
      CALL COORD(XCORD,NP1)
      CALL FUNCT(NOPT,XOPT,F,GRAD)
      WRITE(6,350) F
50    FORMAT(' FOPT=',1PE15.5)
      IF(NHAMAX.LE.NHAR) GO TO 71
      IF(EPSI.LT.1.D-06) GO TO 71
...INCREASE IN HARMONICS
20    CONTINUE
      NOPT=N2HAR1*NNS
      IK=NHAR1+1
      NI1=NHAR*NNS
      NI2=NNS*(NHAR1+NAD)+NI1+1
      NI3=NNS*NHAR1
      NOPT1=NOPT+1
      DO 123 I=1,NI1
23    XOPT(NI2-I)=XOPT(NOPT1-I)
      NI2=NNS*(N2HAR1+NAD)
      NI1=NNS*NAD
      DO 124 I=1,NI1
      XOPT(NI3+I)=XOPTIN
24    XOPT(NI2+I)=XOPTIN
      NHAR=NHAR+NAD
      NHAR1=NHAR+1
      N2HAR1=NHAR*2+1
      NP1=N2HAR1
...INCREASE IN THE NUMBER OF POINTS NP1
      IF(INP1.EQ.1) NP1=NP1+NP1AD
...NO DECREASE IN EPSI IF IEPS=0
      IF(IEPS.EQ.1) EPSI=EPSI/DEPS
      WRITE(6,83) NHAR,NP1,EPSI
3     FORMAT(///,' # HARMONICS=',I3,/,' # POINTS    =',I3,/,3X,'EPSI
     *=',D7.1)
      NOPT=N2HAR1*NNS
      DO 121 I=1,NOPT
21    EPS(I)=EPSI
      GO TO 77
...EXIT FROM BALANCE
1     CONTINUE
...OUTPUTS:TIME-VALUES,MAGNITUDE AND PHASE******************************
      NFRQ1=NFRQ+1
      DO 700 K=1,NFRQ1
      DO 700 I=1,NOUTN
      COUT(I,K)=OUT(I,1,K)
      DO 700 J=1,NNS
00    COUT(I,K)=COUT(I,K)+OUT(I,J+1,K)*CX(J,K)
      NFRQ2=NFRQ+2
      NHAR1=NHAR+1
      DO 701 K=NFRQ2,NHAR1
      DO 701 I=1,NOUTN
      COUT(I,K)=DCMPLX(0.D0,0.D0)
      DO 701 J=1,NNS
01    COUT(I,K)=COUT(I,K)+OUT(I,J+1,K)*CX(J,K)
      NOUTN2=NOUTN/2
```

```
      DO 703 I=1,NOUTN2
03    COUTX(I,1)=COUT(I,1)+DCMPLX(0.D0,1.D0)*COUT(I+NOUTN2,1)
      DO 704 I=1,NOUTN2
      DO 704 J=2,NHAR1
04    COUTX(I,J)=COUT(I,J)+DCMPLX(0.D0,1.D0)*COUT(I+NOUTN2,J)
      DO 705 I=1,NOUTN2
      DO 705 J=1,NHAR
05    COUTX(I,NP1-NHAR+J)=DCONJG(COUT(I,NHAR-J+2))
     *+DCMPLX(0.D0,1.D0)*DCONJG(COUT(I+NOUTN2,NHAR-J+2))
      DO 706 I=1,NOUTN2
      PAUX1(1)=COUTX(I,1)
      DO 707 J=2,NP1
07    PAUX1(J)=COUTX(I,J)/2.D0
      CALL DFTF(XCORD,PAUX1,PAUX2,NP1)
      DO 708 J=1,NP1
      OUTX(I+NOUTN2,J)=DIMAG(PAUX2(J))
08    OUTX(I,J)=DREAL(PAUX2(J))
06    CONTINUE
      DO 709 I=1,NNS
      DO 709 J=1,NP1
      OUTX(I+NOUTN,J)=YL(I,J)
09    OUTX(I+NOUTN+NNS,J)=XS(I,J)
      DO 710 I=1,NNS
      DO 710 J=1,NHAR1
      COUT(I+NOUTN,J)=CYL(I,J)
10    COUT(I+NOUTN+NNS,J)=CX(I,J)
      NSS=NOUTN+2*NNS
      DO 465 I1=1,NSS,10
      WRITE(6,455)
55    FORMAT('1TIME VALUES')
      I2=I1+9
      IF(I2.GT.NSS) I2=NSS
      WRITE(6,475) (I,I=I1,I2)
75    FORMAT(///' OUT=',I2,9I12)
      DO 461 I=1,NP1
61    WRITE(6,470) (OUTX(J,I),J=I1,I2)
70    FORMAT(1P10E12.4)
65    CONTINUE
      DO 500 NN=1,NP1
00    ITM(NN)=FLOAT(NN-1)
      DO 501 JJ=1,NSS
      DO 502 NN=1,NP1
02    IVAN(NN)=OUTX(JJ,NN)
01    CALL GRAPH4(11.,9.,NP1,ITM,IVAN)
      WRITE(6,403)
03    FORMAT('1FREQUENCY DOMAIN RESPONSE')
      DO 430 I1=1,NSS,5
      I2=I1+4
      IF(I2.GT.NSS) I2=NSS
      WRITE(6,404) (I,I=I1,I2)
04    FORMAT(//' OUT=',I12,4I24)
      WRITE(6,406) (SREAL,SIMAG,I=I1,I2)
06    FORMAT(10(8X,A4))
      WRITE(6,401) (COUT(J,1),J=I1,I2)
01    FORMAT(' DC:',1P10E12.4)
      DO 430 I=2,NHAR1
      IM1=I-1
30    WRITE(6,402)IM1,(COUT(J,I),J=I1,I2)
02    FORMAT(I3,':',1P10E12.4)
      DO 450 I=1,NSS
      DO 450 J=1,NHAR1
50    COUT(I,J)=DCMPLX(CDABS(COUT(I,J)),DATAN(DIMAG(COUT(I,J))
     */DREAL(COUT(I,J))))
```

```
      IF(I2.GT.NSS) I2=NSS
      WRITE(6,404) (I,I=I1,I2)
      WRITE(6,406) (SMAG,SARG,I=I1,I2)
      WRITE(6,401) (COUT(J,1),J=I1,I2)
      DO 460 I=2,NHAR1
      IM1=I-1
460   WRITE(6,402)IM1,(COUT(J,I),J=I1,I2)
      STOP
      END
************************************************************************
```

```
************************************************************************
      SUBROUTINE FORM
      PARAMETER MQ=18,MNEL=19
************************************************************************
REATES THE MATRICES GAND C AND THE RIGHTHAND SIDE VECTORFROM INTERCON,O
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 G,C,B,VALUE ,VALVEC
      COMMON/MATR/G(MQ,MQ),C(MQ,MQ),B(MQ),N,NODES
      INTEGER TYPE,NO1,NO2,NO3,NO4,NEL,N,IVEC,JVEC,NPOS
      COMMON/INPUT/VALUE(MNEL),TYPE(MNEL),NO1(MNEL),NO2(MNEL),NO3(MNEL),
     *NO4(MNEL),NEL
      N=NODES
INITIALIZE THE MATRICES AND THE SOURCE VECTOR
      DO 10 I=1,MQ
      B(I)= 0.D0
      DO 10 J=1,MQ
      G(I,J)=0.D0
0     C(I,J)=0.D0
      DO 330 K=1,NEL
      KJ=TYPE(K)
      I=NO1(K)
      IP=NO2(K)
      J= NO3(K)
      JP=NO4(K)
      V=VALUE(K)
      GO TO (30,60,90,140,150,200,220,250,300,20),KJ
      ENTER RESISTANCE
0     CONTINUE
***   ENTER CONDUCTANCE ***********************************************
0     IF(I.EQ.0)  GO TO 40
      G(I,I)=G(I,I)+V
      IF (IP.EQ.0) GO TO 50
      G(I,IP)=G(I,IP)-V
      G(IP,I)=G(IP,I)-V
0     G(IP,IP)=G(IP,IP)+V
0     GO TO 330
***   ENTER CAPACITANCE ***********************************************
0     IF(I.EQ.0) GO TO 70
      C(I,I)= C(I,I)+V
      IF (IP.EQ.0) GO TO 80
      C(I,IP)=C(I,IP)-V
      C(IP,I)=C(IP,I)-V
0     C(IP,IP)=C(IP,IP)+V
0     GO TO 330
***   ENTER VCT *******************************************************
0     IF(J.EQ.0) GO TO 110
      IF (I.EQ.0) GO TO 100
      G(J,I)=G(J,I)+V
00    IF (IP.EQ.0) GO TO 110
      G(J,IP)= G(J,IP)-V
10    IF (JP.EQ.0) GO TO 130
      IF (I.EQ.0) GO TO 120
      G(JP,I)= G(JP,I)-V
20    IF (IP.EQ.0) GO TO 130
      G(JP,IP)= G(JP,IP)+V
30    CONTINUE
40    GO TO 330
***   ENTER VVT *******************************************************
50    N=N+1
      IF (I.EQ. 0) GO TO 160
      G(N,I)= G(N,I)+V
60    IF (IP.EQ.0) GO TO 170
      G(N,IP)=-V+G(N,IP)
```

```
      IF (JP.EQ.0) GO TO 190
      G(N,JP)= 1.DO+G(N,JP)
      G(JP,N)=-1.DO
190   GO TO 330
C**** ENTER CURRENT SOURCE *******************************************
200   IF (I.EQ.0) GO TO 210
      B(I)= B(I)-V
210   IF (IP.EQ.0) GO TO 330
      B(IP)=B(IP)+V
      GO TO 330
C**** ENTER VOLTAGE SOURCE *******************************************
220   N=N+1
      IF (I.EQ.0) GO TO 230
      G(N,I)= 1.DO
      G(I,N)=1.DO
230   IF (IP.EQ.0) GO TO 240
      G(N,IP) =-1.DO
      G(IP,N)= -1.DO
240   B(N)=V
      GO TO 330
C**** ENTER AN OPERATIONAL AMPLIFIER *********************************
250   N=N+1
      VALUE(K)= 0.DO
      V=0.DO
      IF (I.EQ.0) GO TO 260
      G(N,I)= 1.DO
260   IF (IP.EQ.0) GO TO 270
      G(N,IP) =-1.DO
270   IF (J.EQ.0) GO TO 280
      G(J,N)= 1.DO
      G(N,J)= 0.DO
280   IF (JP.EQ.0) GO TO 290
      G(JP,N) =-1.DO
      G(N,JP)= ODO
290   GO TO 330
C**** ENTER INDUCTANCE ***********************************************
300   N=N+1
      IF(I.EQ.0) GO TO 310
      G(I,N)= 1.DO
      G(N,I)= -1.DO
310   IF (IP.EQ.0) GO TO 320
      G(IP,N) = -1.DO
      G(N,IP)= 1.DO
320   C(N,N)=V
      GO TO 330
330   CONTINUE
      RETURN
      END
```

```
C****************************************************************************
      SUBROUTINE  SOLVE (Y,B,X,N)
      PARAMETER MQ=18
C****************************************************************************
C* PROVIDES THE FORWARD AND BACKWARD SUBSTITITION ON THE MATRIX Y
C* DECOMPOSED BY 'LU' .
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER IPOINT
      COMMON/FORLU/IPOINT(MQ)
      COMPLEX*16 Y(MQ,MQ),Z(MQ),X(MQ),SUM,DETMAT
      REAL*8 B(MQ),BR(MQ)
      DO 10 I=1,N
      K=IPOINT(I)
 10   BR(I)=B(K)
      DO 30 I=1,N
      IM1=I-1
      SUM=BR(I)
      IF (IM1.EQ.0) GO TO 30
      DO 20 J=1,IM1
 20   SUM=SUM-Y(I,J)*Z(J)
 30   Z(I)= SUM/Y(I,I)
      DO 50 IP1=1,N
      I=IP1-1
      SUM=Z(N-I)
      IF(I.EQ.0) GO TO 50
      DO 40 JP1=1,I
      J=JP1-1
 40   SUM=SUM-Y(N-I,N-J)*X(N-J)
 50   X(N-I)= SUM
      RETURN
       END
```

```
C**********************************************************************
      SUBROUTINE LU (Y,DETMAT,N)
      PARAMETER MQ=18
C**********************************************************************
C*ROUTINE FOR TRIANGULAR DECOMPOSITION THIS ROUTINE PERFORMS PARTIAL
C*PIVOTING ON ROWS VECTOR IPOINT KEEPS TRACK OF ROW-CHANGES DURING
C*PIVOTING... DETMAT IS THE VALUE OF THE DETERMINANT
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER IPOINT
      COMMON/FORLU/IPOINT(MQ)
      COMPLEX*16 Y(MQ,MQ),YDN,Z(MQ),DETMAT
      PROD =1.D0
      DO 10 I=1,N
10    IPOINT(I)= I
      DO 80 M=2,N
      MM1=M-1
      IND=MM1
      POINT = CDABS(Y(MM1,MM1))
      DO 20 L=M,N
      IF(CDABS(Y(L,MM1)).LE.POINT) GO TO 20
      IND=L
      POINT=CDABS(Y(L,MM1))
20    CONTINUE
      IF(MM1.EQ.IND) GO TO 40
      IF(MM1.NE.IND) PROD=-PROD
      KP=IPOINT(MM1)
      IPOINT(MM1)= IPOINT(IND)
      IPOINT(IND)= KP
      DO 30 LL=1,N
      YDN= Y(MM1,LL)
      Y(MM1,LL)=Y(IND,LL)
30    Y(IND,LL)=YDN
40    DO 50 J=M,N
50    Y(MM1,J)=Y(MM1,J)/Y(MM1,MM1)
      IF(POINT.GT.1.D-6) GO TO 60
      WRITE(6,100)
60    DO 70 I=M,N
      DO 70 J=M,N
70    Y(I,J)=Y(I,J)-Y(I,MM1)*Y(MM1,J)
80    CONTINUE
      DETMAT=(1.D0,0D0)
      DO 90 I=1,N
90    DETMAT=DETMAT*Y(I,I)
      DETMAT=DETMAT*PROD
      RETURN
100   FORMAT(1X,'PIVOT TOO SMALL,RESULTS UNRELIABLE ')
      END
```

```
      SUBROUTINE DFTF (X,A,F,NP1)
      COMPLEX*16 SUM,X(NP1),F(NP1),A(NP1)
      DO 20 LL=1,NP1
      SUM=(0.D0,0.D0)
      J=LL-1
      DO 10 I=1,NP1
      K=1+MOD(J*(I-1),NP1)
10    SUM=SUM+A(I)*X(K)
20    F(LL)=SUM
      RETURN
      END
```

```
      SUBROUTINE DFTC (X,A,F,NP1)
      COMPLEX*16  SUM,X(NP1),F(NP1),A(NP1)
      NP2=NP1+1
      DO 20 LL=1,NP1
      SUM=(0.D0,0.D0)
      J=NP2-LL
      DO 10 I=1,NP1
      K=1+MOD(J*(I-1),NP1)
 10   SUM=SUM+F(I)*X(K)
 20   A(LL)=SUM/NP1
      RETURN
      END
```

```
      SUBROUTINE COORD(X,NP1)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMPLEX*16 CDEXP,X(NP1),DCMPLX
      Z=6.283185307179586D0/DFLOAT(NP1)
      DO 10 I=1,NP1
 10   X(I)=CDEXP(DCMPLX(0.D0,Z*(I-1)))
      RETURN
      END
```

```
      SUBROUTINE DIODE(M,XS,XSD,XSDD,NP1,YS,PAR,PARD)
      IMPLICIT  REAL*8(A-H,O-Z)
      REAL*8 XS(NP1),XSD(NP1),XSDD(NP1),YS(NP1),PAR(NP1),PARD(NP1),
     *YZ(2),YZD(2),YEXP1(2),YEXP2(2),S1(2),S2(2)
      COMMON/MAINDI/VL(2),YSO(2),R(2),F(2),V(2,2),CO(2)
      Y=DLOG(1.38D-23)
      X=DLOG(YSO(M))
      X=-16.DO*DLOG(10.DO)+DLOG(2.156DO)
      DO 1 I=1,2
      YZ(I)=DEXP(VL(M)*V(I,M)+X)
      YZD(I)=VL(M)*YZ(I)
      YZ(I)=YZ(I)-YSO(M)
      YEXP1(I)=DEXP(VL(M)*V(I,M)+Y)
      YEXP2(I)=VL(M)*YEXP1(I)
      ZEXP=1.DO-V(I,M)/F(M)
      S1(I)=R(M)/(F(M)*ZEXP**(R(M)+1.DO))+YEXP2(I)
1     S2(I)=1.DO/ZEXP**R(M)+YEXP1(I)
      DO 5 K=1,NP1
      IF(XS(K).GE.V(1,M)) GO TO 2
      IF(XS(K).LE.V(2,M)) GO TO 3
      XEXP=DEXP(VL(M)*XS(K)+X)
      YX=XEXP-YSO(M)
      YXD=VL(M)*XEXP
      YEXP=VL(M)*DEXP(VL(M)*XS(K)+Y)
      ZEXP=1.DO-XS(K)/F(M)
      C=CO(M)*(1.DO/ZEXP**R(M)+YEXP)
      CD=CO(M)*(R(M)/(F(M)*ZEXP**(R(M)+1.DO))+VL(M)*YEXP)
      GO TO 4
2     YX=YZD(1)*(XS(K)-V(1,M))+YZ(1)
      YXD=YZD(1)
      C=CO(M)*(S1(1)*(XS(K)-V(1,M))+S2(1))
      CD=CO(M)*S1(1)
      GO TO 4
3     YX=YZD(2)*(XS(K)-V(2,M))+YZ(2)
      YXD=YZD(2)
      C=CO(M)*(S1(2)*(XS(K)-V(2,M))+S2(2))
      CD=CO(M)*S1(2)
4     CONTINUE
      YS(K)=YX+C*XSD(K)
      PAR(K)=YXD+CD*XSD(K)
5     PARD(K)=C
      RETURN
      END
```

```
      SUBROUTINE TRNSTR(M,XS,XSD,XSDD,NP1,YS,PAR,PARD)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 XS(2,NP1),XSD(2,NP1),XSDD(2,NP1),YS(2,NP1),PAR(2,2,NP1),
     *PARD(2,2,NP1)
      COMMON/MAINTR/CUR(1),COEF(1),EXP(1),VT(1),EM(1),CJO(1,2),PHI(1,2
     *ENU(1,2),BEF(1),BER(1),FT(1),CIF(1),VEF(1),VCF(1),BEMAX(1),
     *TAUSAT(1),U1(1,2),U2(1,2),U3(1,2)
CALCULATED...TRANSISTOR PARAMETERS(EMITTER)...
      CUR(1)=1.D-6
      COEF(1)=1.D0
      EXP(1)=-6.D0
      VT(1)=.258492D-1
      EM(1)=1.D0
      CJO(1,1)=0.D0
      CJO(1,2)=0.D0
      PHI(1,1)=.75D0
      PHI(1,2)=.79D0
      ENU(1,1)=.463D0
      ENU(1,2)=.489D0
      BEF(1)=1.D2
      BER(1)=1.D0
      FT(1)=3.D9
      CIF(1)=.7D0
      VEF(1)=.78D0
      VCF(1)=-3.D1
      BEMAX(1)=1.D2
      TAUSAT(1)=3.4D-7
      U1(1,1)=.7D0
      U1(1,2)=.7D0
      U2(1,1)=-.7D0
      U2(1,2)=-.7D0
      U3(1,1)=.7D0
      U3(1,2)=.74D0
      TAU=1.D0/(6.283185307179586*FT(M))
      TAU=TAU-((ENU(M,1)*CJO(M,1)*(VEF(M)-U3(M,1)))/(PHI(M,1)*(1.D0-U3
     *,1)/PHI(M,1))**(ENU(M,1)+1.D0))+(CJO(M,1)/(1.D0-U3(M,1)/PHI(M,1)
     ***ENU(M,1))+(CJO(M,2)/(1.D0-VCF(M)/PHI(M,2))**ENU(M,2)))*VT(M)/
     *CIF(M)
      ALFF=1.01D0
      ALFR=2.D0
      VTM=1.D0/38.9D0
      X=-6.D0*DLOG(10.D0)
      I=1
40    CONTINUE
      DO 15 K=1,NP1
      V=XS(I,K)
      IF(V.GE.U1(M,I)) V=U1(M,I)
      IF(V.LE.U2(M,I)) V=U2(M,I)
      CURCC=DEXP(V/VTM+X)-CUR(M)
      DICCDV=(CURCC+CUR(M))/VTM
      TAU=0.D0
      CDE=TAU*DICCDV
      DCDEDV=CDE/VTM
      VCAPJ=XS(I,K)
      IF(VCAPJ.GE.U3(M,I)) VCAPJ=U3(M,I)
      CJE=CJO(M,I)/((1.D0-VCAPJ/PHI(M,I))**ENU(M,I))
      DCJEDV=(ENU(M,I)*CJE)/(PHI(M,I)*(1.D0-VCAPJ/PHI(M,I)))
      IF(V.EQ.XS(I,K)) GO TO 10
      CURCC=CURCC+DICCDV*(XS(I,K)-V)
      CDE=CDE+DCDEDV*(XS(I,K)-V)
10    IF(VCAPJ.EQ.XS(I,K)) GO TO 20
      CJE=CJE+DCJEDV*(XS(I,K)-VCAPJ)
20    IF(I.EQ.2) GO TO 30
```

```
      PAR(1,1,K)=ALFF*DICCDV+(DCJEDV+DCDEDV)*XSD(I,K)
      PAR(2,1,K)=-DICCDV
      PARD(1,1,K)=CJE+CDE
      IF(K.LT.NP1) GO TO 15
CALCULATED...TRANSISTOR PARAMETERS(COLLECTOR)...
      TAU=(TAUSAT(M)*(1.DO+BER(M)+BEMAX(M))/BEMAX(M)-
     *TAU*(1.DO+BER(M)))/BER(M)
      IF(TAU.LT.O.DO) TAU=O.DO
C...ALLOCATION INTO YS,PAR   ETC...
      I=I+1
      GO TO 15
30    YS(1,K)=YS(1,K)-CURCC
      YS(2,K)=YS(2,K)+ALFR*CURCC+(CJE+CDE)*XSD(I,K)
      PAR(1,2,K)=-DICCDV
      PAR(2,2,K)=ALFR*DICCDV+(DCJEDV+DCDEDV)*XSD(I,K)
      PARD(2,2,K)=CJE+CDE
      PARD(1,2,K)=0.DO
      PARD(2,1,K)=0.DO
      IF(K.EQ.NP1) I=I+1
15    CONTINUE
      IF(I.EQ.2) GO TO 40
      RETURN
      END
```

```
C*****************************************************************
      SUBROUTINE NONLNR(NNS,YS,XS,XSD,XSDD,NP1,PAR,PARD,PARDD)
      PARAMETER MNNS=2,MNP1=75
C*****************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 YS(NNS,NP1),XS(NNS,NP1),XSD(NNS,NP1),XSDD(NNS,NP1),
     *PAR(NNS,NNS,NP1),PARD(NNS,NNS,NP1),PARDD(NNS,NNS,NP1),
     *QXS(MNNS,MNP1),QXSD(MNNS,MNP1),QXSDD(MNNS,MNP1),
     *QYS(MNNS,MNP1),QPAR(MNNS,MNNS,MNP1),QPARD(MNNS,MNNS,MNP1),
     *WXS(MNP1),WXSD(MNP1),WXSDD(MNP1),WYS(MNP1),WPAR(MNP1),WPARD(MNP1)
      COMMON/MAINON/NTR,NDI
      DO 1 K=1,NP1
      DO 1 I=1,NNS
      DO 1 J=1,NNS
      PAR(I,J,K)=0.D0
      PARD(I,J,K)=0.D0
1     PARDD(I,J,K)=0.D0
      IF(NTR.EQ.0) GO TO 5
      DO 4 M=1,NTR
      DO 2 I=1,2
      J=I+2*(M-1)
      DO 2 K=1,NP1
      QXS(I,K)=XS(J,K)
      QXSD(I,K)=XSD(J,K)
2     QXSDD(I,K)=XSDD(J,K)
      CALL TRNSTR(M,QXS,QXSD,QXSDD,NP1,QYS,QPAR,QPARD)
      DO 4 K=1,NP1
      DO 3 I=1,2
      J=I+2*(M-1)
      YS(J,K)=QYS(I,K)
      PAR(J,J,K)=QPAR(I,I,K)
3     PARD(J,J,K)=QPARD(I,I,K)
      J=2*(M-1)
      PAR(1+J,2+J,K)=QPAR(1,2,K)
      PAR(2+J,1+J,K)=QPAR(2,1,K)
      PARD(1+J,2+J,K)=QPARD(1,2,K)
4     PARD(2+J,1+J,K)=QPARD(2,1,K)
5     IF(NDI.EQ.0) GO TO 8
      DO 7 M=1,NDI
      N=2*NTR+M
      DO 6 K=1,NP1
      WXS(K)=XS(N,K)
      WXSD(K)=XSD(N,K)
6     WXSDD(K)=XSDD(N,K)
      CALL DIODE(M,WXS,WXSD,WXSDD,NP1,WYS,WPAR,WPARD)
      DO 7 K=1,NP1
      YS(N,K)=WYS(K)
      PAR(N,N,K)=WPAR(K)
7     PARD(N,N,K)=WPARD(K)
8     RETURN
      END
```

```
C*********************************************************************
      SUBROUTINE FUNCT(NOPT,XOPT,F,GRAD)
      PARAMETER MNNS=2,MNNS1=3,MNHAR1=26,MNP1=75
C*********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX*16 CX,CYL,CRESP,XCORD,DCMPLX,PAU1(MNP1),PAU2(MNP1),
     *CYLX(MNNS,MNP1),CDXDDX(MNNS,MNP1),CINT(MNNS,MNP1),
     *CAD(MNHAR1),CALDE(MNHAR1),CBC(MNHAR1),CBEGA(MNHAR1),
     *CLIGR(MNNS1,MNHAR1),CGRAD(MNNS,MNHAR1)
      REAL*8 XOPT(NOPT),GRAD(NOPT),EPSNRM(MNP1),XS,YL,DREAL,DIMAG,DOMEG
     *PAR(MNNS,MNNS,MNP1),PARD(MNNS,MNNS,MNP1),PARDD(MNNS,MNNS,MNP1),
     *A(MNNS,MNHAR1),B(MNNS,MNHAR1),ALFA(MNNS,MNHAR1),BETA(MNNS,MNHAR1)
     *C(MNNS,MNHAR1),D(MNNS,MNHAR1),GAMA(MNNS,MNHAR1),DETA(MNNS,MNHAR1)
     *YS(MNNS,MNP1),XSD(MNNS,MNP1),XSDD(MNNS,MNP1),VEPS(MNNS,MNP1),
     *AINT(MNNS,MNP1)
      COMMON/TRANS/CRESP(MNNS,MNNS1,MNHAR1),XCORD(MNP1),
     *CX(MNNS,MNHAR1),CYL(MNNS,MNHAR1),XS(MNNS,MNP1),YL(MNNS,MNP1),
     *DOMEG,NP1,NNS,NHAR,NFRQ,ICONTR,CTPER
C**************TPER=2PI/DOMEG************
C     TPER=6.2831853071795 86/DOMEG
      TPER=6.283185307179586*CTPER
C**************************************
      NNS1=NNS+1
C...JUMP FOR DC OPERATING POINT (NHAR=0)
      IF(NHAR.EQ.0) GO TO 215
C*********************************************************************
C                          C O M M A T  1
C     CONVERTS THE VARIABLES XOPT(I) OF OPTIMIZATION FROM A REAL VALUED
C     STRING INTO A COMPLEX VALUED ARRAY CX(J,I).
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 5 J=1,NNS
5     CX(J,1)=XOPT(J)
      NJUMP=NHAR*NNS
      NHAR1=NHAR+1
      DO 10 K=2,NHAR1
      JKB=(K-1)*NNS
      JKE=K*NNS
      JKBP1=JKB+1
      DO 10 J=JKBP1,JKE
10    CX(J-JKB,K)=DCMPLX(XOPT(J),-XOPT(NJUMP+J))
C*********************************************************************
C*********************************************************************
C                          C O M M A T  2
C     SCALES THE RESPONSE CRESP(I,J,K) OF THE LINEAR NETWORK TO UNIT
C     SOURCES WITH THE COMPLEX VARIABLES CX(J,K).
C     CYL(I,K) CONTAINS THE RESPONSE OF THE LINEAR NETWORK IN W-DOMAIN
C     AT THE AUGMENTING SOURCES.
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      NFRQ1=NFRQ+1
      DO 15 K=1,NFRQ1
      DO 15 I=1,NNS
      CYL(I,K)=CRESP(I,1,K)
      DO 15 J=1,NNS
15    CYL(I,K)=CYL(I,K)+CRESP(I,J+1,K)*CX(J,K)
      CONTINUE
      NFRQ2=NFRQ+2
      NHAR1=NHAR+1
      DO 20 K=NFRQ2,NHAR1
      DO 20 I=1,NNS
      CYL(I,K)=DCMPLX(0.D0,0.D0)
      DO 20 J=1,NNS
20    CYL(I,K)= CYL(I,K)+CRESP(I,J+1,K)*CX(J,K)
C*********************************************************************
```

```
C     FROM PHASOR VALUES CX(I,J) AND CYL(I,J),INSTANTANEOUS VALUES
C     XS(I,J) OF THE AUGMENTING SOURCES AND YL(I,J) OF THE OUTPUTS
C     ARE OBTAINED RESPECTIVELY USING INVERSE DFT.
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 35 I=1,NNS
      DO 35 J=1,NP1
35    CYLX(I,J)=DCMPLX(0.D0,0.D0)
      DO 40 I=1,NNS
40    CYLX(I,1)=CX(I,1)+DCMPLX(0.D0,1.D0)*CYL(I,1)
      DO 45 I=1,NNS
      DO 45 J=2,NHAR1
45    CYLX(I,J)=CX(I,J)+DCMPLX(0.D0,1.D0)*CYL(I,J)
      DO 50 I=1,NNS
      DO 50 J=1,NHAR
50    CYLX(I,NP1-NHAR+J)=DCONJG(CX(I,NHAR-J+2))
     *+DCMPLX(0.D0,1.0D0)*DCONJG(CYL(I,NHAR-J+2))
      DO 55 I=1,NNS
      PAU1(1)=CYLX(I,1)
      DO 60 J=2,NP1
60    PAU1(J)=CYLX(I,J)/2.D0
      CALL DFTF (XCORD,PAU1,PAU2,NP1)
      DO 65 J=1,NP1
      YL(I,J)=DIMAG(PAU2(J))
65    XS(I,J)=DREAL(PAU2(J))
55    CONTINUE
C************************************************************************
      IF(ICONTR.EQ.0) GO TO 86
C************************************************************************
C                                 C O M M A T   4
C     FIRST AND SECOND DERIVATIVES XSD(I,K) AND XSDD(I,K) OF THE
C     AUGMENTING SOURCES ARE OBTAINED USING INVERSE DFT.
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 70 I=1,NNS
      DO 70 K=1,NP1
70    CDXDDX(I,K)=DCMPLX(0.D0,0.D0)
      DO 75 I=1,NNS
      DO 75 K=1,NHAR1
75    CDXDDX(I,K)=DCMPLX(0.D0,DOMEG)*(K-1)*CX(I,K)
     *-DCMPLX(0.D0,1.D0)*(((K-1)*DOMEG)**2)*CX(I,K)
      DO 80 I=1,NNS
      DO 80 K=1,NHAR
80    CDXDDX(I,NP1-NHAR+K)=DCONJG(DCMPLX(0.D0,DOMEG)*
     *(NHAR-K+1)*CX(I,NHAR-K+2))+DCMPLX(0.D0,1.D0)*DCONJG
     *(-((DOMEG*(NHAR-K+1))**2)*CX(I,NHAR-K+2))
      DO 85 I=1,NNS
      PAU1(1)=CDXDDX(I,1)
      DO 90 K=2,NP1
90    PAU1(K)=CDXDDX(I,K)/2.D0
      CALL DFTF(XCORD,PAU1,PAU2,NP1)
      DO 95 K=1,NP1
      XSDD(I,K)=DIMAG(PAU2(K))
95    XSD(I,K)=DREAL(PAU2(K))
85    CONTINUE
C************************************************************************
86    CONTINUE
C*************************CALL TO NONLINEARITIES*************************
      CALL NONLNR(NNS,YS,XS,XSD,XSDD,NP1,PAR,PARD,PARDD)
C************************************************************************
C                                 C O M M A T   5
C     CALCULATES THE VECTOR ERROR VEPS(I,J)=YS(I,J)-YL(I,J)WHERE YS(I,
C     HAS TO BE PROVIDED FROM THE NONLINEARITIES BY DIRECT SUBSTITUTIO
C     IT ALSO CALCULATES THE SCALAR ERROR F (OR P) BY INTEGRATING NORM
C     SQUARED OF VEPS(I,J) USING THE TRAPEZOIDAL RULE.
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

```
      DO 105 K=1,NP1
      EPSNRM(K)=0.D0
      DO 105 I=1,NNS
105   EPSNRM(K)=EPSNRM(K)+VEPS(I,K)**2
      F=0.D0
      DO 110 K=1,NP1
110   F=F+EPSNRM(K)
      F=TPER*F/DFLOAT(NP1)
C*****************************************************************************
C*****************************************************************************
C                              C O M M A T   6
C     CALCULATES THE REAL AND IMAGINARY PARTS OF INT&VEPS(I,N)EXP(JWKT)
C     AND OF INT&PAR(I,M,N)VEPS(I,N)EXP(JWKT)$ USING DFT.
C     THESE ARE TO BE USED IN THE EVALUATION OF THE GRADIENT.
C     PAR(I,M,N) IS TO BE PROVIDED BY THE NONLINEARITIES.
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 115 K=1,NNS
      DO 115 J=1,NP1
115   AINT(K,J)=0.D0
      DO 120 K=1,NNS
      DO 120 J=1,NP1
      DO 120 I=1,NNS
120   AINT(K,J)=AINT(K,J)+PAR(I,K,J)*VEPS(I,J)
      DO 125 K=1,NNS
      DO 125 J=1,NP1
125   CINT(K,J)=DCMPLX(AINT(K,J),VEPS(K,J))
      DO 130 I=1,NNS
      DO 135 K=1,NP1
135   PAU2(K)=CINT(I,K)
      CALL DFTC(XCORD,PAU1,PAU2,NP1)
      A(I,1)=TPER*DREAL(PAU1(1))
      D(I,1)=TPER*DIMAG(PAU1(1))
      DO 140 J=2,NHAR1
      CAD(J)=.5D0*TPER*(PAU1(J)+PAU1(NP1-J+2))
      CALDE(J)=.5D0*TPER*(PAU1(J)-PAU1(NP1-J+2))
      A(I,J)=DREAL(CAD(J))
      D(I,J)=DIMAG(CAD(J))
      ALFA(I,J)=-DIMAG(CALDE(J))
140   DETA(I,J)=-DREAL(CALDE(J))
130   CONTINUE
C*****************************************************************************
      IF(ICONTR.EQ.0)  GO TO 161
C*****************************************************************************
C                              C O M M A T   7
C     PREPARES THE REAL AND IMAGINARY PARTS OF INT&PARD(I,M,N)VEPS(I,N)
C     EXP(JWKT)$ AND OF INT&PARDD(I,M,N)VEPS(I,N)EXP(JWKT)$ USING DFT.
C     THESE ARE TO BE USED IN THE EVALUATION OF THE GRADIENT.PARD(I,K,
C     AND PARDD(I,M,N) ARE TO BE PROVIDED BY THE NONLINEARITIES.
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 145 K=1,NNS
      DO 145 J=1,NP1
145   CINT(K,J)=(0.D0,0.D0)
      DO 150 K=1,NNS
      DO 150 J=1,NP1
      DO 150 I=1,NNS
150   CINT(K,J)=CINT(K,J)+DCMPLX(PARD(I,K,J)*
     *VEPS(I,J),PARDD(I,K,J)*VEPS(I,J))
      DO 160 I=1,NNS
      DO 165 K=1,NP1
165   PAU2(K)=CINT(I,K)
      CALL DFTC(XCORD,PAU1,PAU2,NP1)
      B(I,1)=DREAL(PAU1(1))*TPER
      C(I,1)=DIMAG(PAU1(1))*TPER
```

```
      CBEGA(J)=(PAUT(J)-PAUT(NPT-J+2))*0.5D0*TPER
      B(I,J)=DREAL(CBC(J))
      C(I,J)=DIMAG(CBC(J))
      BETA(I,J)=-DIMAG(CBEGA(J))
 70   GAMA(I,J)=-DREAL(CBEGA(J))
 60   CONTINUE
C***********************************************************************
 61   CONTINUE
C***********************************************************************
C                         C O M M A T  8
C     PROVIDES THE COMPLEX GRADIENT CGRAD(I,K) FROM THE DATA PREPARED
C     IN COMMAT6 AND COMMAT7.
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 175 J=2,NNS1
      CLIGR(J,1)=CRESP(1,J,1)*D(1,1)
      IF(NNS.EQ.1) GO TO 175
      DO 177 I=2,NNS
177   CLIGR(J,1)=CLIGR(J,1)+CRESP(I,J,1)*D(I,1)
175   CONTINUE
      DO 180 K=2,NHAR1
      DO 180 J=2,NNS1
      CLIGR(J,K)=CRESP(1,J,K)*(D(1,K)-DCMPLX(0.D0,DETA(1,K)))
      IF(NNS.EQ.1) GO TO 180
      DO 181 I=2,NNS
181   CLIGR(J,K)=CLIGR(J,K)+CRESP(I,J,K)*(D(I,K)-DCMPLX(0.D0,DETA(I,K)))
180   CONTINUE
      DO 185 I=1,NNS
185   CGRAD(I,1)=2.D0*(A(I,1)-CLIGR(I+1,1))
      DO 190 I=1,NNS
      DO 190 K=2,NHAR1
190   CGRAD(I,K)=2.D0*(A(I,K)+DCMPLX(0.D0,ALFA(I,K))-CLIGR(I+1,K))
      IF(ICONTR.EQ.0) GO TO 200
      DO 195 I=1,NNS
      DO 195 K=2,NHAR1
      CGRAD(I,K)=CGRAD(I,K)+2.D0*(DCMPLX(0.D0,DOMEG)*(K-1)*B(I,K)-
     *DOMEG*(K-1)*BETA(I,K)-(DOMEG*(K-1))**2*C(I,K)+
     *DCMPLX(0.D0,(DOMEG*(K-1))**2)*GAMA(I,K))
195   CONTINUE
200   CONTINUE
C***********************************************************************
C***********************************************************************
C                         C O M M A T  9
C     CONVERTS THE COMPLEX GRADIENT CGRAD(I,K) IN ARRAY FORM INTO A REA
C     GRADIENT GRAD(I) IN STRING FORM.
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 205 K=1,NHAR1
      IS=(K-1)*NNS+1
      IE=K*NNS
      DO 205 I=IS,IE
205   GRAD(I)=DREAL(CGRAD(I-IS+1,K))
      DO 210 K=2,NHAR1
      IS=(NHAR+(K-1))*NNS+1
      IE=(NHAR+K)*NNS
      DO 210 I=IS,IE
210   GRAD(I)=DIMAG(CGRAD(I-IS+1,K))
C***********************************************************************
      RETURN
C...OPERATING POINT EVALUATION (NHAR=0)
215   DO 220 I=1,NNS
      YL(I,1)=CRESP(I,1,1)
      DO 220 J=1,NNS
220   YL(I,1)=YL(I,1)+CRESP(I,J+1,1)*XOPT(J)
      DO 225 I=1,NNS
      XS(I,1)=XOPT(I)
```

```
      CALL NONLNR(NNS,YS,XS,XSD,XSDD,NP1,PAR,PARD,PARDD)
      F=0.D0
      DO 230 I=1,NNS
      VEPS(I,1)=YS(I,1)-YL(I,1)
230   F=F+VEPS(I,1)**2
      DO 235 I=1,NNS
      GRAD(I)=0.D0
      DO 235 J=1,NNS
235   GRAD(I)=GRAD(I)+2.D0*(PAR(J,I,1)-CRESP(J,I+1,1))*VEPS(J,1)
      RETURN
      END
```

MO(1).SUB12

```
      SUBROUTINE QNMING(X,G,EPS,N,F,FEST,UNITH,IPRINT,MAXIT,IEXIT,H,NM,
     *FUNCT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(N),G(N),H(NM),EPS(N)
      LOGICAL CONV,UNITH,LPRINT
      LPRINT=IPRINT.EQ.0
      CALL FUNCT(N,X,F,G)
      IF(F.LT.FEST) GO TO 98
      NFNS=1
      ITN=0
      STEP=1.D0
      IDX=N
      IDG=N+N
      IH=IDG+N
      IF(.NOT.UNITH) GO TO 1
      IJ=IH+1
      DO 2 I=1,N
      DO 2J=I,N
      H(IJ)=0.D0
      IF(I.EQ.J) H(IJ)=1.D0
      IJ=IJ+1
      CONV=.TRUE.
      IEXIT=2
      IF(ITN.EQ.MAXIT) GO TO 99
      GDX =0.D0
      DO 3 I=1,N
      Z=0.D0
      IJ=IH+I
      IF(I.EQ.1) GO TO 21
      II=I-1
      DO 4 J=1,II
      Z=Z-H(IJ)*G(J)
      IJ=IJ+N-J
1     DO 5 J=I,N
      Z=Z-H(IJ)*G(J)
      IJ=IJ+1
      IF(DABS(Z).GT.EPS(I)) CONV=.FALSE.
      H(IDX+I)=Z
      GDX=GDX+G(I)*Z
      IF(LPRINT) GO TO 6
      IF(MOD(ITN,IPRINT).NE.0) GO TO 6
      PRINT 130,ITN, NFNS,F
30    FORMAT (/1X,'ITN=',I5,5X,'NFNS=',I5,5X,'F=',G15.5)
      PRINT 131,(X(I),I=1,N)
31    FORMAT(1X,'X(I)',10G12.4)
      PRINT 132,(G(I),I=1,N)
32    FORMAT(1X,'G(I)',10G12.4)
      IEXIT=0
      IF(CONV) GO TO 99
      IEXIT=1
      IF(GDX.GE.0.D0) GO TO 99
      Z=1.D0
      IF(ITN.LT.N.AND.UNITH) Z=STEP
      W=2.D0*(FEST-F)/GDX
      IF(W.LT.Z) Z=W
      STEP=Z
3     GDX=GDX*Z
      DO 7 I=1,N
      H(IDX+I)=H(IDX+I)*Z
      X(I)=X(I)+H(IDX+I)
      CALL FUNCT(N,X,FP,H)
      IF(FP.LT.FEST) GO TO 98
```

```
      GPDX=GPDX+H(I)*H(IDX+I)
      DGDX=GPDX-GDX
      IF(F.GT.FP-0.0001D0*GDX) GO TO 9
      IEXIT=3
      IF(GPDX.LT.0.D0.AND.ITN.GT.N) GO TO 99
      Z=3.D0*(F-FP)+GPDX+GDX
      W=DSQRT(1.D0-GDX/Z*GPDX/Z)*DABS(Z)
      Z=1.D0-(GPDX+W-Z)/(DGDX+2.D0*W)
      IF(Z.LT..1D0) Z=.1D0
      DO 10 I=1,N
1     X(I)=X(I)-H(IDX+I)
      GO TO 11
      F=FP
      DO 12 I=1,N
2     G(I)=H(I)
      IF(DGDX.GT. 0.D0) GO TO 14
      GDX=GPDX
      Z=4.D0
1     STEP=Z*STEP
      GO TO 13
4     IF(GPDX.LT..5D0*GDX) STEP=2.D0*STEP
      DGHDG= 0.D0
      DO 15 I=1,N
      Z=0.D0
      IJ=IH+I
      IF(I.EQ.1) GO TO 22
      II=I-1
      DO 16 J=1,II
      Z=Z+H(IJ)*H(IDG+J)
6     IJ=IJ+N-J
2     DO 17 J=I,N
      Z=Z+H(IJ)*H(IDG+J)
7     IJ=IJ+1
      DGHDG=DGHDG+Z*H(IDG+I)
5     H(I)=Z
      IF(DGHDG.LT.0.D0)DGHDG=DGDX*0.01D0
      IF(DGDX.LT.DGHDG) GO TO 18
      W=1.D0+DGHDG/DGDX
      DO 19 I=1,N
9     H(IDX+I)=W*H(IDX+I)-H(I)
      DGDX=DGDX+DGHDG
      DGHDG=DGDX
8     IJ=IH
      DO 20 I=1,N
      W=H(IDX+I)/DGDX
      Z=H(I)/DGHDG
      DO 20 J=I,N
      IJ=IJ+1
0     H(IJ)=H(IJ)+W*H(IDX+J)-Z*H(J)
      ITN=ITN+1
      GO TO 1
8     IEXIT=4
9     IF(LPRINT) GO TO 103
      PRINT 130,ITN,NFNS,F
      PRINT 131,(X(I),I=1,N)
      PRINT 132,(G(I),I=1,N)
      PRINT 101,IEXIT
01    FORMAT(/'0EXIT FROM QNMING,  IER = ',I1)
03    RETURN
      END
INTS
```