## - MODERN CONTROL PROBLEMS IN LINEAR TIME-INVARIANT MÚLTIVARIABLE SYSTEMS


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Progresses in any field of scientific research always bring simple and powerful approaches to the solutions of sophisticated problems. In recent years, the developping technology has greatly increased the interest in System Theory.Especially, the problems encountered in the design of complicated control systems have emphasized the necessity of simple and unified approaches to the structural analysis of linear time-invariant multivariable systems.In the sequel to this work, the results obtained in two new frameworks, namely the geometric approach and the frequency domain approach to the treatment of modern control problems, are considered and used.

This work consists of two parts. The first part is a detailed litterature survey on the newly introduced geometric approach and its frequency domain translation. $I_{1}$ Chapter I , " the mathematical preliminairies ", some basic concepts of Linear Algebra are reviewed and new geometric concepts (such as (A,B)-invariant subspaces, stabilizability subspaces ) are given. A frequency domain characterization of each geometric property is also presented. Chapter II takes into consideration some of modern control problems ( such as Disturbance Decoupling,Output stabilization with respect to Disturbance ). . .ach problem is defined first, then the geometric and the related frequency domain formulations of its solvability are given. Special comments, remarks and alternate proofs are also made, whenever it is possible.

In the second part a special structure of linear multivariable systems is considered.This system is a coupling of two basic multiports, the first composed of " lossless " components and the second characterizing the " algebraic " components.I Chapter III
some properties of this structure are introduced, based on [10]. Then using strict system equivalence [13] , the existence of different " lossless " multjports for which the transfer matrix of the system is invariant when the " algebraic " multiport is kept constant,is investigated. The results are given as theorems and illustrated with an example. In Chapter IV using the results of Chapter III, the solvability of D D P for two different disturbance structures is formulated.In one of these cases a larger degree of freedom, to nullify the effect of disturbances,is obtained. This case is also illustrated with an example.

One general remarl is that most of the decoupling problems considered are generically unsolvable;that is the solution space of these problems consist of isolated points in the neighborhood of which the problem is unsolvable.

It should be emphasized here that, the problem stated and formulated in Chapter III is totally new and answer obtained is that this problem is generically solvable. The new properties that are introduced may find many areas of application in System Theory, especially in electrical circuit design. A letter from M $I$ 产 Hautus also states that the problems are very interesting and relevant.

Finally, I'm personally gratefull to my thesis supervisor C. Göknar, for his orientation and helps in the preparation of this thesis.

M. Salim Arslanalp

## I MATHEMATICAL PRELIMINARIES

In order to give the frequency domain characterisations of some geometric concepts widely used in the geometric formulations of modern control problems we will first consider the $(\mathcal{F}, \omega)$-representtation newly introduced in $\left[7^{-}, 9\right]$.
(1) - $(z, w)$-Representation :

We consider the linear time-invariant system $\sum$ given in its state space form by

$$
\begin{align*}
& \underline{x}=A \underline{x}+B \underline{u}  \tag{1}\\
& \underline{y}=C \underline{x} \tag{2}
\end{align*}
$$

where $x(t) \in X \triangleq \mathbb{R}^{n}, \underline{\mu}(t) \in U \triangleq \mathbb{R}^{m}, \underline{y}(t) \in \mathcal{Y} \triangleq \mathbb{R}^{r}$ and $A: X \rightarrow X, B: U \rightarrow X, C: X \rightarrow Y \quad$ are linear maps. Now let $\underline{x}(t)$ be a time-domain solution of (1) subject to an input function $\underline{\mu}(t)$ and to an initial state $\underset{\sim}{x}\left(t_{0}\right) \triangleq x_{0}$. Then

$$
\underline{x}(t)=A \underline{x}(t)+B \underline{u}(t)
$$

and the same equations (1), (2) in Laplace domain can be written as

$$
\begin{equation*}
{ }_{s \underline{G}}(s)-\underline{x}_{0}=A_{\underline{\xi}}(s)+B \underline{w}(s) \tag{3}
\end{equation*}
$$

where $\underline{\xi}(s), \underline{w}(s)$ are the Laplace transforms of $x(t), \underline{u}(t)$ respectively. The initial state $x\left(t_{0}\right)=x_{0}$ is obviously an element of the state space $\mathcal{X}$, hence $\underline{x}_{0} \in X$ and $\underline{\mathcal{G}}(s)$ and $\underline{\omega}(s)$ are strictly proper rational functions since $\underline{x}(t)$ and $\underline{u}(t)$ are real functions.

## DETINITION 1.1 [7]

Let $x_{0} \in X$; the formula

$$
\begin{equation*}
\underline{x}_{0}=(s \bar{I}-A) \underline{\xi}(s)-B \underline{w}(s) \tag{4}
\end{equation*}
$$

is called a ( $\mathcal{E}, \underline{w}$ )-representation of $\underline{x}_{0}$, if $\underline{\xi}(s)$ and $\underline{w}(s)$ are strictly proper rational functions.
(2) - A-invariance:

A-invariance is an important property of subspaces, widely used in linear Algebra. It $w$ ill be discussed briefly, since it helps to understand the related but more complicated concept of ( $A, B$ )-invafiance.

Consider the linear space $\mathcal{Z}$, the linear map $A: \mathcal{Z} \rightarrow \mathcal{Z}$ and a subspace $\mathcal{W} \mathcal{Z}$.If $\mathscr{G} \triangleq A W$, then for $w_{i} \in W, i \in\{1, \ldots, \rho\}$ a basis for $W$, the set of vectors $t_{i} \in \mathscr{C}_{\infty}^{\infty}$ such that $\underline{t}_{i}=A \underline{\omega}_{i}$, , $i \dot{\in}\{1,2, \ldots, \rho\} \quad$ spans the subspace $\mathscr{Q}^{\boldsymbol{Q}}$.

## DEFINITION 1.2

$\omega$ is said to be $\Lambda$-invariant if and only if $A W \subseteq W_{-}$. With this definition we will consider the system $\sum$ and always talk about the A-invariance of a subspace $\mathcal{V}$ of the state space $\mathcal{X}$.It's also possible to obtain a matrix characterisation of the above property as follows. Let $V$ be a matrix whose columns are basis vectors for the subspace $V \subseteq X$, that is $V \triangleq\left[\underline{v}_{i}, \underline{v}_{2}, \ldots \underline{v}_{k}\right] \quad$ where $\underline{v}_{i} \in V$ for $i \in\{1, \ldots, k\} \quad$ are linearly independent and $\operatorname{span}\left\{\underline{v}_{i}\right\}=V$. $V$ will be called a basis matrix for $V$.

THEOREM 1.3
Given $\sum$ and $V \subseteq \chi$ with a basis matrix $V, V$ is A-invariant if and only if a solution $P$ of the matrix equation

$$
A V=V P \quad \text { exists }
$$

Proof:
$\| "$
Let $A V \leq V$ and $V=\left[\begin{array}{llll}\underline{v}_{1} & \underline{v}_{2} & \ldots & v_{k}\end{array}\right]$ be a basis matrix
for $V$. Then $A V=\left[\begin{array}{llll}A \underline{v}_{1} & A \underline{v}_{2} & A \underline{v}_{k}\end{array}\right]=\left[\begin{array}{lll}\underline{y}_{1} & \underline{y}_{2} & \ldots \underline{y}_{k}\end{array}\right] \in V$

Hence $\underline{y}_{i} \in V$, for $i \in\{1, \ldots, k\} \Rightarrow \underline{y}_{i}=V_{p_{i}}$ for $i \in\{1, \ldots, k\}$ for some $p_{i} \in \mathbb{R}^{k} \quad$.Then $A V=V\left[\underline{p}_{1} \underline{p}_{2} \ldots \underline{p}_{k}\right]$ and $P \triangleq\left[\underline{p}_{1} \cdots p_{k}\right]$ is a solution of the matrix equation.

Let $A V=V P$ with $V$ a basis matrix for $V$ and let $\underline{x} \in \mathcal{V}$, then $x$ can be written as a linear combination of basis vectors of $V$ such as

$$
\underline{x}=V_{\underline{r}} \text {, for some } r \in \mathbb{R}^{k} \text {. Then as }
$$

$A \underline{x}=A V \underline{r}=V P_{\underline{r}} \quad$ we see that $A \underline{x}$ is a linear combination of columns of $V$.Hence $A \underline{x} \in V$, which implies $A V \subseteq V$ since $\underline{x} \in \mathcal{V}$ was arbitrary.

In modern control problems, an extension of the idea of invariance has found an area of application. The ( $A, B$ )-invariant subspaces of a state space $X$ as introduced by Wonham and Norse will be the essential mathematical tool in handling the problems to be stated in section II.

## (3) $-(A, B)$ - invariant subspaces

We consider again the system $\sum$ with its state space $X$ its input space $U$ and its output space $Y$. Let $\mathcal{V}$ be a subspace of $\mathcal{X}$ If $W \subseteq X$ and $\mathscr{B} \omega+V$; then $\forall \underline{E}, \underline{t}=\omega+v$ for some $w \in W$ and $v \in V$.

## DEFINITION 1.4 [5]

$V_{\text {is }}$ an ( $A, B$-invariant subspace of $\mathcal{X}$ if it's A-invarian $\bmod (\mathrm{BUL})$ ie

$$
\begin{equation*}
A V \subseteq V+B U \tag{5}
\end{equation*}
$$

The importance of (A, B) -invariant subspaces is that these subspaces can be made $(A+B F)$-invariant for a suitable choice of the feedback matrix F. This property is very $y$ useful since it helps to change a feedback problem to an existence of a subspace which is (A,B)-invaw riant.

## LEMMA 1.5 [1]

Let $V \subseteq X$. There exists $F: X \rightarrow \mathcal{U}$ such that $(A+B F) V \leq V$ if and only if $V$ is $(A, B)$-invariant.

## PROOF: [5]

$" \Longrightarrow 1$
Let $(A+B F) V \subseteq V \quad$ and $x \in V$.
Then $(A+B F) \underline{x}=\underline{v}$ and $\underline{v} \in V$; then $A \underline{x}=\underline{v}-B F_{\underline{x}} \in V+B U$ since $F_{\underline{x}} \in \mathcal{U}$
" $\&$ Let $V$ be $(A, B)$-invariant and $\left\{V_{1}, \ldots, v_{\mu}\right\}$ be a basis for $V$. As $A V \subseteq V+B U \quad$, there exist $\underline{w}_{i} \in V$ and $\underline{u}_{i} \in \mathcal{U}$ for $i \in\{1, \ldots, \mu\}$ such that $A \underline{v} i=\underline{\omega}_{i}-B F \underline{\mu}_{i}, i \in\{1, \ldots, \underline{\mu}\}$
Defining $F_{0}: V \rightarrow U$ by $F_{0} \underline{v}_{i}=\underline{u}_{i}, i \in\{1, \ldots, \mu\}$ and letting $F$ be any extension of the map $F_{0}$ to $\mathcal{X}$ we obtain

$$
\begin{equation*}
A \underline{v}_{i}=\underline{w}_{i}-B \mathcal{F}_{i}, \quad i \in\{1, \ldots, \mu\} . \tag{四}
\end{equation*}
$$

By the above lemma it's seen that there always exists a feedback $F$ by which an ( $A, B$ )-invariant subspace $V$ in the openloop characterisation can be made $\bar{A}$-invariant in closed loop characterisation; where $\bar{A} \triangleq A+B F$
. This fact will be later used in finding a solution of disturbance decoupling problem by state feedback.

For frequency domain applications, the frequency domain characterisation of ( $A, B$ )-invariant subspaces is needed. This characterisation is given in terms of ( $\mathcal{Z}, \omega)$-representation as in [7].
(However we have to mention that a polynomial characterisation of ( $A, B$ )-invariant subspaces making use of the Rosenbrock system matrix is discussed in detail in [6]).

THEOREM $1.6 \quad[7,9]$
A subspace $V \subseteq X$ is an ( $A, B$ )-invariant subspace if and only if every $\underline{x}_{0} \in \mathcal{V}$ has a ( $\underline{U}, \underline{\mathcal{V}}$ )-representation satisfying $\underline{\xi}(s) \in V$ for all.

## PR OOF:

[9]

Let $\underline{x}_{0} \in V \subseteq X \quad$ and $\underline{x}_{0}=(s I-A) \underline{\underline{z}}(s)-B \underline{w}(s)$ with strictly proper $\underline{w}(s) \in \mathcal{U}$ and strictly proper $\underline{\xi}(s) \in \mathcal{V}$ for all $s$. Since $s_{\underline{G}}(s) \in V$

$$
A_{\xi}(s)=s \underline{\xi}(s)-x_{0}-B_{\underline{w}}(s) \in V+B U \text { for all. }
$$

The functions $\bar{\xi}(s)$ and $w(s)$ being strictly proper, it is possible to go back to time domain by inverse Laplace transform and for $\mathcal{L}^{-1}\{\underline{\xi}(s)\}=\underline{x}(t), \mathcal{L}^{-1}\{\underline{\omega}(s)\}=\underline{u}(t)$ for $t \geqslant 0$ we have $\underline{x}(t) \in V$ for $t \geqslant 0$. Then $\underline{x}^{( }\left(0^{+}\right)=\lim _{t \rightarrow 0} t^{-1}\left(\underline{x}(t)-x_{0}\right) \in V$. Hence for $t=0^{+} \quad A x_{0}=\dot{x}\left(0^{+}\right)-B \underline{u}\left(0^{+}\right) \in V^{+}+B u$.
$" \Rightarrow$ " Let $V \subseteq X$ and $A V \subseteq V+B U$, then by I H M 1.5 there exist an $F$ such that $(A+B F) V \subseteq V$. For $x_{0} \in \mathcal{V}$ choosing $\underline{\xi}(s) \triangleq(s I-A-B F)^{-1} \underline{x}_{0}$ and $\underline{w}(s) \triangleq F_{\underline{\mathcal{E}}}(s)$ we have $x_{0}=(s I-A-B F) \underline{\mathcal{E}}(s)$ $=(s I-A) \underline{\xi}(s)-B \underline{w}(s)$ with $\xi(s)$ and $\underline{\omega}(s)$ strictly proper because of the property of $(S I-A-B F)^{\prime}$.

When we are interested in ( $A, B$ )-invariant subspaces contained in a subspace of $X$, the property of maximality becomes very important in expressing the solvability criteria of various problems.
(4) Maximal ( $A, B$ )-invariant subspaces in her $\mathcal{C}$

## DERINITTOM 1.7

Let $V \subseteq X ; \mathcal{V}$ is an (A,B )-invariant subspace contained in $\operatorname{Ker} C_{\text {if }}$ and only if the following two conditions hold:
(i) $A V \subseteq V+B U$
(ii) $V \subseteq \operatorname{ker} C$

In general one can talk about an (A,B )-invariant subspace contained in any subspace of $X$.However for application purposes ( D DP, etc) we are interested with the inclusion in Ger C. It can be shown that it's not necessary that Ger $C_{\text {contains }}$ a unique (A,B)-invarient subspace. This leads us to talk about the maximal ( largest ) (A, B) -invariant subspace contained in Kor $C$, in order to judge correctly if a problem has a solution or not. In [2] a geometric construction of the maximal ( $\mathrm{A}, \mathrm{B}$ )-invariant subspace in $\operatorname{Ker} C$ is given as part of theorem. We will consider only the iterafive construction formula and will not discuss the proof which is in $[1,2]$.

## THEOREM 1.8 [2]

Let $V^{(0)} \triangleq \operatorname{Ker} C$ and define

$$
\begin{equation*}
V^{(i)}=V^{(i-1)} \cap A^{-1}\left(B 2 L+V^{(i-1)}\right) \quad i=1,2, \ldots, \mu \tag{6}
\end{equation*}
$$

where $\mu=\operatorname{dim} \operatorname{Ker} C \quad$. Then $V_{*}=V^{(\mu)}$ is the maximal $(A, B)$-invariant subspace contained in $\operatorname{Ker} C$. A matrix whenever $A$ is singular. The geometric construction of the maximal ( $A, B$ )-invariant subspace contained in $K e r C_{i s}$ a practical method when working on specific problems. To illustrate $\mathbb{T} H M 1.8$ the following flowchart will be considered together with an example.


FLOWCHART 1.8: GEOMETRIC CONSTRUCTION OF $V_{*}$

## Example 1.8

Given

$$
A=\left[\begin{array}{lllll}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{array}\right] \quad B=\left[\begin{array}{ll}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 0
\end{array}\right] \quad C=\left[\begin{array}{llll}
1 & 0 & 0 & 0
\end{array} 0\right.
$$

we will construct the maximal ( $A, B$ )-invariant subspace contained in $\operatorname{Ker} C$. For this we begin by computing $\operatorname{Ker} \mathcal{C}$ :
$\underline{x}=\left[\begin{array}{lllll}1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0\end{array}\right]\left[\begin{array}{l}x_{1} \\ x_{2} \\ x_{3} \\ x_{4} \\ x_{5}\end{array}\right]=\left[\begin{array}{l}x_{1} \\ x_{4}\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right] \Rightarrow \operatorname{Ker} C=\operatorname{sp}\left\{\left[\begin{array}{l}0 \\ 1 \\ 0 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 0 \\ 1\end{array}\right]\right\}$

Following the steps of the flowchart 1.8 ,
$i=0 \quad V^{(0)}=\operatorname{Ker} C$
$i=1$

$$
\begin{aligned}
& A^{-1}\left(B U+V^{(0)}\right)=s p \cdot\left(\left[\begin{array}{c}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array},\left[\begin{array}{c}
0 \\
0
\end{array}\right],\left[\begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right)\right.\right. \\
& V^{(1)}=V^{(0)} \cap A^{-1}\left(B U+V^{(0)}\right)=s p \cdot\left\{\left[\begin{array}{c}
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right)\right.
\end{aligned}
$$

heck $V^{(1)} \subset V^{(0)}$ but $V^{(1)} \neq V^{(0)}$
$i=2$

$$
\begin{aligned}
& B U+V^{(1)}=s p .\left(\left[\begin{array}{l}
0 \\
0 \\
1 \\
0 \\
0
\end{array}\right]\left[\begin{array}{c}
0 \\
0 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right)\right\} \\
& \left.A^{-1}\left(B U l+V^{(1)}\right)=s p .\left\{\begin{array}{l|l|l}
1 \\
0 & & 0 \\
0 & 0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 & 0 \\
0 & 0 \\
0
\end{array}\right]\right\}
\end{aligned}
$$

cheek

$$
\begin{aligned}
& V^{(2)}=V^{(1)} \cap A^{-1}\left(B U+V^{(1)}\right)=s p .\left\{\left(\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right)\right. \\
& V^{(2)} \subset V^{(1)} \text { but } V^{(2)} \neq V^{(1)}
\end{aligned}
$$

$i=3$

$$
B U+V^{(2)}=s p \cdot\left(\left[\begin{array}{l}
0 \\
0 \\
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
1 \\
1
\end{array}\right)\right.
$$

check

$$
\left.\begin{array}{l}
A^{-1}\left(B U+V^{(2)}\right)=s p \cdot\left\{\left[\begin{array}{l}
1 \\
0 \\
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right]\right) \\
V^{(3)}=V^{(2)} \cap A^{-1}\left(B U+V^{(2)}\right)=\text { sp }\left\{\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right]\right. \\
V^{(3)}=V^{(2)}=\text { sp }\left(\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right]\right.
\end{array}\right)=V_{*} \quad .
$$

As seen from the above example, to obtain the maximal (A, B)-invarient subspace contained in $\operatorname{Ker} C_{\text {will }}$ be a more difficult problem when systems of larger dimension are considered.

To give a frequency domain characterisation of the maximal ( $A, B$ )-invariant subspace contained in $\operatorname{Ker} C$, the ( $(, \omega)$-reprosentation will play the basic role. In [9] , this formulation is given by a definition and a theorem which are combined here.

THEOREM 1.9 [9]
$\bigcup_{*}$ is the largest ( $A, B$ )-invariant subspace contained in $\operatorname{Ker} C_{i f}$ and only if $V_{*}$ denotes the space of all points for which there exists a $(\underline{Z}, \underline{w})$-representation satisfying $\underline{Z}(s) \in \operatorname{Ker} C$ for all s. (ie. $C_{\mathcal{Z}}(s)=0$ )

Proof:
$\Longrightarrow "$
For $\underline{x}_{0} \in V_{*} \quad$, we have by TH MI 1.6

$$
\underline{x}_{0}=(s I-A) \underline{\underline{G}}(s)-B \underline{w}(s) \text { with } \underline{\underline{E}}(s), w(s)
$$

strictly proper and $\underline{\underline{G}}(\mathrm{~s}) \in V_{*} \quad \forall s$. since $V_{*} \subseteq \operatorname{KerC}$ by definition $\xi(s) \bar{\epsilon} \operatorname{Ker} C$.

$$
"
$$

$I_{e t} \underline{x}_{0} \in V_{*}$ and show first that $\underline{x}_{0} \in \operatorname{Ker} C$.

- For this expand $\underline{\xi}(s)$ and $\underline{\omega}(s)$ as power series in $s^{-1}$,

$$
\begin{aligned}
& \underline{\mathcal{Z}}(s)=\sum_{k=1}^{\infty} \underline{\xi}_{k} s^{-k} \\
& \underline{w}(s)=\sum_{k=1}^{\infty} \underline{w}_{k} s^{-k}
\end{aligned}
$$

and note that $\underline{Z}(s) \in \operatorname{KerC} \forall s \Rightarrow \underline{G}_{k} \in \operatorname{Ker} C \quad \forall K \in\{1,2, \ldots\}$
Then $\underline{x}_{0}$ having a $(\underline{\xi}, \underline{\omega})$-representation with $\underline{\mathcal{G}}(s) \in \operatorname{Ker} \mathcal{C}$,

$$
\underline{x}_{0}=(s I-A) \underline{\xi}(s)-B \underline{w}(s)
$$

or $\quad x_{0}=(s I-A) \sum_{k=1}^{-\infty} \underline{\xi}_{k} s^{-k}-B \sum_{k=1}^{\infty} \underline{w}_{k} s^{-k}$
and equating the constant terms

$$
\underline{x}_{0}=\underline{z}_{1} \in \operatorname{Ker} C
$$

are obtained.

To complete the proof there remains only to show that $V_{*}$ is (A,B )-invariant (or equivalently, to show that $\underline{\underline{\xi}}(s) \in V_{*}$ for all $S$ because of T II M 1.6 ). Regarding $\mathcal{E}^{(s)}$ as power series in $S^{-1}$, it's sufficient again to proove that $\underline{\underline{Z}}_{k} \in V_{*} \forall k \in\{1,2, \ldots\}$ For this we take the proper parts of both sides of the equation

$$
\begin{equation*}
s^{k} \underline{x}_{0}=s^{k}(s I-A) \underline{\underline{\underline{G}}}(s)-s^{k} B \underline{w}(s) \tag{7}
\end{equation*}
$$

that is

$$
\begin{aligned}
& {\left[s^{k} \underline{x}_{0}\right]_{\text {proper }}=\left[(s I-A)\left(s^{k-1} \underline{\underline{\xi}}_{1}+s^{k-2} \underline{\underline{\underline{Z}}}_{2}+\cdots+\underline{\underline{Z}}_{k}+s^{-1} \underline{\underline{\xi}}_{k+1}+\cdots\right)\right]_{\text {proper }} } \\
&-\left[B\left(s^{k-1} \underline{\omega}_{1}+\cdots+\underline{\omega}_{k}+s^{-1} \omega_{k+1}+\cdots\right)\right]_{\text {proper }}
\end{aligned}
$$

which is equivalent to

$$
\begin{equation*}
0=-A \underline{\underline{\xi}}_{k}+(s I-A)\left(s^{-1} \underline{\underline{\xi}}_{k+1}+\cdots\right)-B w_{k}-B\left(s_{w_{k+1}}^{-1}+\cdots\right) \tag{9}
\end{equation*}
$$

Hen if $(V(s))$ denotes the strictly proper part of the rational vector $V(s)$, by equation ( 9 )

$$
\begin{aligned}
(s I-A) & \left(s^{k-1} \underline{\underline{G}}(s)\right)-B\left(s^{k-1} \underline{w}(s)\right) \\
& =\underline{\xi}_{k}+s^{-1}\left[-A \underline{\xi}_{k}+(s I-A)\left(s^{-1} \xi_{k+1}+\cdots\right)-B\left(w_{k}+s^{-1} \underline{w}_{k+1}+\cdots\right)\right]=\xi_{k}
\end{aligned}
$$

$$
\text { equation ( } 10 \text { ) implies }
$$

$$
\begin{equation*}
\underline{\underline{Z}}_{k}=(s I-A) \eta(s)-B \varphi(s) \tag{11}
\end{equation*}
$$

Now $\eta(s), \varphi(s)$ are strictly proper functions ; furthermore $\eta(s)^{-} \in \operatorname{Ker} C$ since for $\underline{\underline{\zeta}}(s) \in \operatorname{Ker} C, \underline{\xi}_{k}$ is also an element of $\operatorname{Ker} C($ for $k \in\{1,2, \ldots\})$. Then $\xi_{k}$ has a $(\xi, \underline{\omega})$-representation with $\eta(s) \in \operatorname{Ker} C$, and by definition $\underline{\underline{Z}}_{k} \in V_{*}^{-}$. The maximality of $\mathcal{V}_{*}$ is immediate since it denotes all such points.

Using $T$ HM 1.9 the following result which is more convenient to use in most cases can easily be obtained.

## COROLLARY 1.10 [9]

$\underline{x}_{0} \in V_{*}$ if and only if there exists strictly proper
rational functions $\mathcal{\xi}(s)$ and $\underline{\omega}(s)$ such that

$$
\left[\begin{array}{cc}
s I-A & -B  \tag{12}\\
C & 0
\end{array}\right]\left[\begin{array}{l}
\underline{Z}(s) \\
\underline{w}(s)
\end{array}\right]=\left[\begin{array}{c}
x_{0} \\
0
\end{array}\right]
$$

Also the following corollary can be obtained by eliminating from equations ( 12 ).

COROLLARY 1.11

## [9]

$\underline{x}_{0} \in U_{*} \quad$ if and only if there exists $\underline{\omega}(S)$ strictly proper such that

$$
C(s I-A)^{-1} \underline{x}_{0}=-R(s) \underline{w}(s)
$$

where $R(S) \triangleq C(s I-A)^{-1} B \quad$ is the transfer function matrix of the system $\sum$ described by (1), (2).

In fact COROLLARY 1. II is very explanatory about $V_{*}$ and tells that for this $\underline{\omega}(s)$ the output corresponding to the initial state $x_{0}$ is zero. In other words, whatever be the initial state chosen in $V_{*}$, the trajectories followed by the state of the system remain at any instant in $V_{*}$, and thus the corresponding output at any instant is zero.
(5) - Stabilizability subspaces:

In most of the design problems stability considerations play an important role.A new type of subspace,stabilizability subspaces introduced in $[7,8,9]$ are very useful to treat such problems. There is a close relation between the stabilizability and the controllability subspaces. To emphasize this point later on, it is useful to explain with two brief definitions what is a contollability subspace.

Consider again the system $\sum$ described by (1), (2) and its state space $\mathcal{X}$; we have the following definitions

Given the system $\sum$ the controllable subspace $R_{0} \subset X$ of the pair $(A, B)$ is

$$
\begin{equation*}
R_{0}=B+A B+\cdots+A^{n-1} B \triangleq\langle A / B\rangle \tag{13}
\end{equation*}
$$

where

$$
B \triangleq \operatorname{Im} B \triangleq B U
$$

In otherwords $R_{0}$ is the set of states which are reachable from $\underline{x}_{0}=0$ and is a linear subspace of $\mathcal{X}[1]$.

Now given the pair ( $A, B$ ), consider all pairs ( $A+B F, B G$ ) which can be obtained by using a state feedback $F$ and connecting again matrix $G$ at the system input. (fig. 1.13). The controllable subspace of the new system pair ( $A+B F, B G$ ) is called a controllability subspace of the original pair ( $A, B$ ). The following definition w ill make the concept clearer.


FIGURE 1.13: System ( $A+B F, B G$ ) obtained from $Z$
DEFINITION 1.13 [1]
Let $A: X \rightarrow X$ and $B: X\left(X\right.$ be as described in system $\sum$.
A subspace $R \subset X$ is a controllability subspace of the pair $(A, B)$ if there exist maps $F: X \rightarrow \mathcal{U}$ and $G: \mathcal{Z} \rightarrow \mathcal{U}$ such that

$$
\begin{equation*}
R=\langle A+B F / \operatorname{Im} B G\rangle \tag{14}
\end{equation*}
$$

Note that

$$
R=I_{m} B G+(A+B F) \operatorname{Im} B G+\cdots+(A+B F)^{n^{-1}} I_{m} B G
$$

and

$$
(A+B F) R=(A+B F) \operatorname{Im} B G+\cdots+(A+B F)^{n} \operatorname{Im} B G ;
$$ hence $(A+B F) R \subset \mathcal{R}$ by the Cayley-Hamilton theorem. By Lemma 1.5

the family of controllability subspaces of a fixed pair ( $A, B$ ) is a subfamily of the family of ( $A, B$ )-invariant subspaces.

We now our attention to stability and stabilizability and we consider stability from a general point of view. We denote by $\mathbb{C}$ any subset of $\mathbb{C}$ satisfying the condition $\mathbb{C} \cap \mathbb{R} \neq \phi$. This condition is brought in, recalling the property that no stable system having only complex conjugate poles in $\mathbb{C}^{-}$exists. As understood $\mathbb{C}^{-}$denotes our " stability region" in the general sense and we say that $A$ is a stability man ( matrix) if $\sigma(A) \subseteq \mathbb{C}^{-}$, where $\sigma(\cdot)$ means " spectrum of $A$ ". Again we will say that a rational function is stable if it has no poles outside of $\mathbb{C} \bullet \mathbb{C}^{+} \subset \mathbb{C}$ ja the set with tho mounties $\mathbb{C}^{-} \cap \mathbb{C}^{+}=\phi$ and $\mathbb{C}^{-} \cup \mathbb{C}^{+}=\mathbb{C}^{\text {. }}$

Dentition - $-1.14 . \quad[1]$
Given the system $\sum$, we say that the pair ( $A, B$ ) is stabilezable if there exists a map $F: X \rightarrow 2 C$ such that

$$
\sigma(A+B F) \subset \mathbb{C}^{-}
$$

Hence " stabilizing " the pair ( $A, B$ ) is equivalent to change the unstable map $A$ to a stable map $\bar{A}$ by simply using a state feedback $F$. In [I], the close relationship between the existence of $F$ and controllability is given. Let the minimal polynomial of $A$ be $\alpha(s)$ and factor it as $\alpha(s)=\alpha^{-}(s) \alpha^{+}(s)$ where zeroes of $\alpha \overline{(s)} \in \mathbb{C}^{-}$and zeroes of $\alpha^{+}(s) \in \mathbb{C}^{+}$. The subspace $\operatorname{Ker} \alpha^{+}(A)$ of $X$ is called the subspace of " unstable modes " of $A$.As shown in [1] , the pair ( $A, B$ ) is stabilizable if and only if the " unstable modes " of A are controllable. The proof of this conclusion will not be discussed but the follwing theorem will be stated for further reference.

Given the system $\sum$, the pair $(A, B)$ is stabilizable if and only if $X_{b} \triangleq \operatorname{Ker} \alpha^{+}(A) \subset\langle A \mid B\rangle$
Consequently there exist a feedback $F$ such that $\sigma(A+B F) \subseteq \mathbb{C}-$ if and only if the subspace of " unstable modes " of $A$ is included in the contollable subspace of the pair ( $A, B$ ), $[1]$. Recalling a property of controllability for linear systems, we can state the follwing theorem.

тнеовм $1.16 \quad[7,9]$
( $A, B$ ) is stabilizable if and only if for every complex $S \in \mathbb{C}^{-}$
we have

$$
\operatorname{rank}[s I-A: B]=n \quad, \text { where } n \text { is the system }
$$

dimension.
We notice that rank $[s I-A \vdots B]$ is always equal to "n" except at the eigenvalue of $A$. Then when we restrict the controllability to the unstable modes the eigenvalues are also restricted to those which lie in $\mathbb{C}^{+}$. Another useful conclusion is that a completely controllable system pair is always stabilizable.

We are now ready to give the definition of stabilizability subspace.

1

## Definition 1.17 [7]

$V \subseteq X$ is called a stabilizability subspace if there exists $F: X \rightarrow U$ such that $(A+B F) V \subseteq V$ and $\sigma((A+B F) \mid V) \subseteq \mathbb{C}$. In ${ }^{-\pi t h e}$ above definition $\sigma((A+B F) \mid V)$ means the $"$ spectrum of the map $(A+B F)$ restricted to the subspace $V$. To be clearer, consider the map of: $X \rightarrow X$. With a basis $\left\{x_{1}, x_{2}, \ldots x_{n}\right\}$ for $X$, let $A_{n \times n}$
be an $n \times n$ matrix representing of with respect to this basis. Then let $\mathcal{V} \subseteq X$ be an $A$-invariant subspace of $\mathcal{X}$ such that $A \mathcal{V} \subseteq \mathcal{V}$ If $\left\{v_{1}, \ldots v_{r}\right\}$ is a basis for $V$, completing this basis to a basis for $X$ we obtain $\left\{v_{1}, \ldots, v_{r}, x_{r+1}, \ldots x_{n}\right\}$. In this new basis the map $f$ is characterised by the matrix

$$
\bar{A}=\left[\begin{array}{cc}
\hat{A}_{1} \hat{A}^{(x+} & \bar{A}_{3} \\
0 & \bar{A}_{2}^{(n-1) \times(n-1]}
\end{array}\right]
$$

and $\sigma(t \mid v) \triangleq \sigma\left(\bar{A}_{1}\right) \quad,[i]$.
To follow the idea introduced by BNF l. 17, consider a stabiliability subspace $V \subseteq X$; $V$ xi $\in \mathcal{V}$, $\mathcal{V}$ ga suitable feedback $F$ the response

$$
C e^{t(A+B F)} \underline{x}_{0} \rightarrow 0 \quad \text { for } t \rightarrow \infty
$$

In other words we don't require that

$$
C e^{t(A+B F)} \rightarrow 0 \quad \text { for } t \rightarrow \infty
$$

but obtain a decay to zero only for some of the state vectors that we are interested in. This is equivalent to say that we are interested in stabilizing a part of the state space $\mathcal{X}$, if it's not possible to stabilize the whole state space. This property will be later used to stabilize the output corresponding to a set of disturbance inputs by using state feedback.

By definition a stabilizability subspace is ( $\mathrm{A}, \mathrm{B}$ ) -invariant according to LEMTA 1.5. If $\mathcal{V}$ is any ( $A, B$ ) -invariant subspace with $F: X \rightarrow \mathcal{U}$ satisfying $(A+B F) U \subseteq \mathcal{U}_{\text {we }}$ can construct a system $\sum_{F, G}$ such that

$$
\begin{equation*}
\underline{x}=(A+B F) \underline{x}+B G \underline{w} \tag{14}
\end{equation*}
$$

where $G: W \rightarrow 2 L$ with $G=R^{l}$ for some $l$.This is so since there exists $G: W \rightarrow \mathcal{L}$ such that the time domain response to the input $u \in \mathcal{U}, \mathcal{G}_{u}\left(t, x_{0}\right) \in \mathcal{V}_{\text {for }}$ all $t \geqslant 0$ if and only if $x_{0} \in V^{\prime}$ and $u$ is of the form $\underline{u}=F \underline{x}+G \underline{w} \quad$ for some $w:[0, \infty) \rightarrow \omega, \quad[7]$. $S_{0}$ taking any $G$ such that $B G C V=(B \bar{L}) \cap V$ we can obtain a
restricted system $\sum_{F, G}$ with an input value space $C$ and a state space $V$.Obviously $\mathcal{Z}_{F, G}$ is not necessarily unique; then let $\Phi_{Z}(V)$ denote the set of such pairs ( $F, G_{r}$ ) given rise to systems $\sum_{F_{j}} G$ with a state space $V$.

LEMMA $1.18 \quad[7]$
Let $V \subseteq X$ be $(A, B)$-invariant and let $(F, G) \in \Phi_{\Sigma}(V)$ then $V$ is a stabilizability subspace if and only if $(A+B F, B G) / V$ is stabilizabl

The proof is in [7]
国
The following lemma shows the existence conditions of such
$(F, G)$ pairs.

LEMMA $1.18 \mathrm{~A} \quad[7]$
There exists $(F, G) \in \Phi_{\Sigma}(V)$ such that $\sigma(A+B F) \subseteq \mathbb{C}$ if and only if the system $\sum_{1}$ is stabilizable and $\mathcal{V}$ is a stabilizability subspace.

Hence the system $\sum_{\mathcal{L}}, G$ must be stabilizable, in other words the unstable modes of $\sum_{F_{1} G}$ must be included in the controllable subspace of $\sum_{F, C}$
$\sum_{F, G}$. In conclusion, we use a linear state feedback $F$ and a gain matrix $G$ to obtain a new system which restricted to a subspace $V^{C}$ has controllable unstable modes. Since $\langle A+B F| v|\operatorname{Im}(B G)\rangle=\langle A+B F \mid \operatorname{Im}(B G)\rangle=R$ for $V^{\text {to }}$ to a stabilizability subspace there must exist a controllability subspace $R$ of ( $A, B$ ) such that it contains the unstable modes of $(A+B F$, where $F$ matrix is not necessarily unique and induced by the condition. $(A+B F) V \subseteq V$.

To give the frequency domain characterisation of a stabilizebility subspace we again use the $(\underline{z}, \underline{w})$-representation.
$V \subseteq X$ is a stabilizability subspace if and only if every $\underline{x} \in \mathcal{V}$ has a $(\underline{\mathcal{Z}}, \underline{\omega})$-representation such that $\underline{\xi}(s) \in \mathcal{V}$ and $\underline{\underline{\xi}}(s), \underline{\omega}(s)$ are stable. As is the case of ( $\mathrm{A}, \mathrm{B}$ )-invariant subspaces, the maximal stabilizability subspace contained in Ger is important to characterize the solutions of various control problems.

## DeFINITION $1.20 \mathrm{~A}[7,9]$

Given the system $\Sigma$ and the subset $\mathbb{C}^{-}$of $\mathbb{C}, V_{*}^{-}$denotes the set of points for which there exists a stable ( $\underline{\mathcal{Z}}, \underline{\omega}$ )-representation satinfying $\underset{G}{ }(s) \in \operatorname{Ker} C$.

THEOREM 1.20 [9]
$V_{*}^{-}$is the maximal stabilizability subspace contained in Ken $C$.

## REMARK $\quad 1.20 \mathrm{~B}$

The DEF 1.20 A and THM 1.20 can be stated for the maximal stabilizability subspace contained in a space $\mathcal{K} \subseteq \mathcal{X}$, in general, by simply changing $\operatorname{Ker} C$ to $K$.

COROLLARY 1.21
The system $\sum$ is stabilizable if and only if $V_{*}^{-}(x)=\mathcal{X}$, where $V_{*}^{-}(X)$ denotes the maximal stabilizability subspace contained in $X$.

We believe that this simple introduction to stabilizability subspaces will be sufficient to follow the analysis of the decoupling problems which takes stability criteria in consideration, and this completes the first chapter on mathematical preliminaries. In the next chapter we will discuss in detail control problems in which the concepts stated here will play a basic role.

## II DISTURBANCE DECOUPLING \& STABILIZATION

The aim of this chapter is to give a detailed presentation of various formulations concerning the disturbance decoupling and stabilization problems. We have to point out that the problems we are going to analyze have recently been introduced with the help of a new approach, the geometric approach to the structural synthesis of linear time-invariant multivarjable systems. For each problem being analyzed the goal is to formulate the solvability criteria. In each case, the formulation will be basicly geometric, then its frequency domain translation will be given using the ( $\mathcal{Z}, \underline{w}$ )-representation and the work $d^{\text {one }}$ in $[6,7,8,9]$. Most of the time we $w$ ill also try to give a matrix polynomial formulation for some comparaison burposes.

## (1) Disturbance Decoupling Problem (DDP) :

In system simulation and mathematical modelling, the unwanted effects imposed on the system are known as disturbance paranatens. In very simple terms, disturbance decoupling is to decouple the effect of disturbances acting at the system's input parts" from the system output, using state feedback. Examination of this problem involves essentially the fundamental geometric concept of ( $A, B$ )-invariant subspaces.

$$
\begin{align*}
& C_{\text {onsider }} \text { the system } \bar{\sum} \\
& \underline{\dot{x}}(t)=A \underline{x}(t)+B \underline{u}(t)+E \underline{q}(t)  \tag{15}\\
& \underline{y}(t)=C \underline{x}(t) \tag{16}
\end{align*}
$$

for $t \geqslant 0$; where again $\underline{\chi}(t)$ is the state vector, $\underline{u}(t)$ is the input vector, $\underline{q}(t)$ represents the disturbance vector and $y(t)$ the output vector.

Find a state feedback $F: X \rightarrow \mathcal{U}$ such that $q(0)$ has no influence off the output $\underline{y}(\cdot)$. Note that we denote again the state space by $X$, the input space by $\mathcal{L}$, the output space by $\mathcal{Y}$ and the space where $g(\cdot)$ takes its values by $Q$.

A definition that takes place in [1] states that " the system $\sum$ is said to be disturbance decoupled if and only if the forced response

$$
\begin{equation*}
\underline{y}(t)=C \int_{0}^{t} e^{(t-\tau) A} E q(\tau) d \tau \tag{17}
\end{equation*}
$$

due to the disturbance is zero for all $q(\cdot) \in \underset{\sim}{Q}$ and for all $t \geqslant 0$, where $\underset{\sim}{Q}$ is a function class. Geometricly, the system $\sum$ is distrbance decoupled if and only if

$$
\langle A \mid \varepsilon\rangle<\operatorname{Ker} C \text {, where } \mathcal{G} \triangleq \operatorname{Im} E,[1,2]
$$

Now, consider the case where the system is not originelly disturbance decoupled and where the linear state feedback law $F: X \rightarrow U$ is being used to change the map $A: X \rightarrow X$ to $\bar{A} \leqq A+B F:$ $\chi$ ne $X$. The so obtained system is disturbance decoupled if and only if

$$
\langle A+B F \mid \xi\rangle \subset \operatorname{Ker} C .
$$

## тнвовві е.1 [1, 2]

Given system $\sum$ described by (15), (16), DDP is solvable if and only if $\mathcal{G} \subseteq U_{*}$, where $V_{*}$ is the maximal ( $A, B$ )-invariant subspace contained in $\operatorname{Ker} C$.

PROOF: "
$\stackrel{ }{\prime}$
Given the system $\sum$, DDP is solvable implies that $V \triangleq\langle A+B F \mid \varepsilon\rangle \subset K e r C$ Since $\xi \subset V, V$ is $(A, B)$-invariant and $V \subset K e r C$; $V \subseteq V_{*}$ hence $\xi \subseteq V_{*}$.
$" \Longleftarrow "$
Given $\mathcal{E} \subseteq V_{*}$, since $V_{*}$ is $(A, B)$-invariant, by Lima 1.5 there exists a state feedback $F$ such that $V_{*}$ is $(A+B F)$-invariant, 1.0. $(A+B F) V_{*} \subseteq V_{*}$. Then since $\varepsilon \subseteq V_{*}$,

$$
\langle A+B F / E\rangle \subset\left\langle A+B F / V_{*}\right\rangle \subseteq V_{*} \subseteq \operatorname{Ker} C,
$$

hence $\langle A+B F \mid E\rangle \subseteq \operatorname{Ker} C$.

So far the analysis of DDP has been abstract. However using the algorithm of THill 1.8 we can always compute $V_{*}$ and check if DDP is solvable or not, remembering that $E$ is a given matrix. The following block diagram gives a clear picture of the disturbance decoupling.


FIGURE 2.1: DDP
Once it is known how to characterize $V_{*}$ in frequency domain it's easy to obtain the frequency domain formulation of DDP.As assumed previously, the class of function $\mathbb{Q}$, where $q(0) \in \mathbb{Q}$, is large enough not to give them a special configuration, and the only restriction is assumed to be on the structure of $E$ matrix. Thus it's wanted in general the disturbance to output transfer function matrix be nullified by using state feedback. The geometric condition being $\mathcal{E} \triangleq E Q \triangleq \operatorname{Im} E \subseteq V_{*}$ we use LTMMA 1.10 to obtain to frequency domain formulation of DDP.

THEOREM $2.2 \quad[9]$
DDP is solvable if and only if there exists strictly proper matrices $X(S)$ and $U(S)$ such that

$$
\left[\begin{array}{cc}
s I-A & -B  \tag{18}\\
C & 0
\end{array}\right]\left[\begin{array}{l}
X(s) \\
U(s)
\end{array}\right]=\left[\begin{array}{l}
E \\
0
\end{array}\right]
$$

The proof is straight forward. By applying Lema 1.10 to each column of $E$, we obtain that (18) implies each column of Eqofor $\forall \underline{q}_{0} \in Q$ be an element of $V_{*}$, and we use Thil 2.1 to complete the proof.

At this point it is useful to consider another problem known as the exact model matching problem [3]. Our purpose is to show that the existence of a solution for $\operatorname{DDP}$ is equivalent to the existence of asolution for the corresponding EIMP.

Exact model matching problem is motivated in general by the notion of " model following control ", To be more explicite we may think of a model system as a system having all desirable qualitios. The large scale realisation of it may introduce unwanted side-effect The compensation scheme is then used to modiify the realised system such that it behaves just like its model.

## PROBLEM ( EMMP):

Given a system with the ( $p \times m$ ) strictly proper, rational transfer matrix. $G_{1}(s)$ and a model swstem with the ( $p \times q$ ) strictly proper, rational transfer matrix $G_{2}(s)$, does there exist a cornpensation scheme which employs linear state variable feedback in combination with input dynamics such that the transfer matrix of the given system is equal to $G_{2}(s)$ and when ?

THEORMA $2.3 \quad[3]$
EMIP is solvable if and only if there exists a strictity proper rational ( $m \times q$ ) matrix $Q(s)$ such that

$$
G_{1}(s) Q(s)=G_{2}(s) .
$$



EIGURE 2.3: MODEL FOLLOWING CONTROL
Now to show first the one to one correspondence between ERMP and DDP we make the following definitions. Given the observable system $(A,[B, E], C):$

$$
\begin{align*}
& G_{1}(S) \triangleq C(S I-A)^{-1} B  \tag{19}\\
& G_{2}(S) \triangleq C(S I-A)^{-1} E \tag{20}
\end{align*}
$$

In fact (19) characterize the transfer function of the "given system" and (20) the transfer function of the model system, and we want to exactly match the input/output transfer matrix $G_{1}(s)$ to the disturbance/ output transfer matrix $G_{2}(S)$. Conversely when data for EMAP is given as $G_{1}(s)$ and $G_{2}(s)$, the corresponding data for $D D P$ is constructed as observable realisations ( $A, B, C$ ) and ( $A, E, C)$ combined as $(A,[B, E), C)$. By these definitions we have to

$$
\left\{\begin{array}{c}
\text { SOLVE DDP } \\
\text { FOR }\left(A,\left[B_{1} E\right], C\right)
\end{array}\right\} \stackrel{\text { SOLVE EMMA }}{ } \Leftrightarrow\left\{\begin{array}{l}
\left.G_{1} C S\right) \triangleq C(S I-A)^{-1} B \\
G_{2}(S) \triangleq C(S I-A)^{-1} E
\end{array}\right\}
$$

THEOREM $2.4 \quad[6]$

EMP is solvable if and only if the corresponding DDP is solvable

## PROOF

For system: $\sum$ defined by (15), (16) let $G_{1}(s) \triangleq C(S I-A)^{-1} B$ and $G_{2}(s) \triangleq C(s I-A)^{-1} E$. Then DDP issolvable if and only if THA.2.2 holds. By eliminating $X(s)$ in the equations of (18) we have

$$
C(S I-A)^{-1} E=-C(S I-A)^{-1} B U(S)
$$

which implies $G_{2}(s)=-G_{1}(s)(\mid(s)$, where $(l(s)$ is srticly proper by THM:2.2.

The relation between $D D P$ and EMAP is attractive because it gives a second framework to treat the $\operatorname{EMMP}$. In order to solve EMMP we have always the possibility to construct the data $\left(A,\left[B_{1} E\right], C\right)$ for the corresponding $D D P$. Then by solving $D D P$ we will have the realisation of the compensation transfer matrix $Q(s)$.

## (2) Modified Disturbance Decoupling Problem (MDDP)

We consider again the system $\Sigma$, but this time we assume that the disturbance $q(t)$ is also directly available for measurement such that a feedback from the disturbance input is possible.

## PROBLEM (ADP):

Given the system $\sum$ determine constant matrices $F$ and $D$ such that when the linear state feedback law $\underline{u}(t)=F \underline{x}(t)+D \underline{g}(t)$ is used the output doesnot depend on $q(t)$, ie. $C(s I-A-B F)^{-1}(B D+E) \equiv 0$.

The solvability of MDDP is formulated in a similar way to the one of DDP.

## THEOREM 2.5

MDDP is solvable if and only if

$$
\begin{equation*}
\varepsilon \subseteq V_{*}+\nrightarrow \tag{21}
\end{equation*}
$$

where $V_{*}$ is the maximal ( $A, B$ )-invariant subspace contained in Ken.

## PROOF: $"$

Given $\mathcal{E} \leq V_{\neq}+\mathcal{B}$ since $\mathcal{E} \leq \operatorname{Im} E$ and $\mathcal{B} \triangleq \operatorname{Im} B$, for any basis matrix $V$ of $V_{*}$ there exists mans $D_{1}$ and $G$ such that $E=V G+B D_{1}$
which implies that

$$
E-B D_{1}=V G \text {, hence } \operatorname{Im}\left\{E-B D_{1}\right\} \subseteq V_{*} \subseteq \operatorname{Ker} C \text {. }
$$

Then by LEMMA 1.5 there exists $F: X \rightarrow U$ such that
$(A+B F) I_{m}\left\{E-B D_{1}\right\} \subseteq V_{*}$. Hence sufficiency holds by $C(S I-A-B F)^{-1}\left(E-B D_{1}\right) \equiv O$ and bytaking $D=-D_{1}$.

Let $\operatorname{RDDP}$ be solvable, then there exist $F$ and $D$ such that by $\underline{u}=F \underline{x}+D \underline{q}$ the system is disturbance decoupled. Then by the definition of disturbance decoupling

$$
\begin{equation*}
\langle A+B F \mid \operatorname{Im}(E+B D)\rangle \subseteq \operatorname{Ker} C \tag{22}
\end{equation*}
$$

and $\operatorname{Im}(E+B D) \subseteq V_{*}$.
Now since $\quad \operatorname{Im}(E+B D) \triangleq(E+B D) Q=E Q+B D Q$ and since $\quad-B D Q \subseteq \operatorname{Im} B$
(22) implies that $E Q+B D Q-B D Q \subseteq V_{*}+\operatorname{Im} B$ completing the proof.

Similar to the DDP case, we can talk about the relation between MDDP and the modified exact model matching problem (WIMMP). The ITMMP is defined as follows:


FIGURE 2.5: MDDP

Given the strictly proper transfer matrices $G_{1}(s)$ and $G_{2}(s)$ find a compensation scheme that employs a transfer matrix $Q(s)$ proper rational.

## RTMARK:

Infact in [3] Wolovich defines the EMP, in the most general case by putting the condition of the existence of a proper rational $Q(s)$. However in [6] E. Mrere and miv. Hautus have splitted this general formulation into two parts: i) a strictly proper $Q(s)$ (EMIP) and ii) a proper $Q(s)$ (MEMP).

THEOREM 2.6 [6]

MDDP is solvable if and only if the corresponding Immil has a solution.

The one to one corresponding and the proof is completely analogous to the one of 1 mith 2.4 and avaible in [6].

The problems of disturbance decoupling and modified disturbance decoupling intend only to reduce the effect of disturbance at the output,but they don't consider how the system dynamics are changed by the used feedback. Stability is the most important property which should be considered when investigating the effect of feedback on the system dynamics.Next topics will take this into considerations.

## (3) Stable Disturobance Decoupling Problem

As inplied by the title of the section the problem that we are going to analyze is the staple version of DDP. For this, we again consider the system $\Sigma$ described by the equations (15),(16).

Find (if possible) a state feodback law $\underline{u}=$ Fx such that the effect of the disturbance at the output is anihilated and the closed loop system with $(A+B F)$ is stable, i.e find $F: X \rightarrow U$ such that $\langle A+B F \mid \Sigma\rangle \subset \operatorname{Ker} C$ and $\sigma(A+B F) \subset \mathbb{C}^{-}$.

In [I] we sen the definition of a family of (A,B)-invariant subspaces as follows:

$$
\underline{v} \triangleq\{v: \exists F: x \rightarrow u(A+B F) V \subseteq v, v \subseteq \operatorname{Ker} C \text { and }
$$

One can immediatly notice that $\mathcal{V} \in \underline{V}$ is a " stabilizability subspace: contained in $\operatorname{Ker} C$; so the maximal element of $\underline{V}$ is $V_{*}$, which was defined in DEF 1.17 combined with DER 1.20 A , as the maximal stabilizability subspace contained in $\operatorname{Ker} C$.

The formulation of the solvability of SDDP is given by the following theorem.

THEORAM 2.7 [1]
Given $\sum$, suppose $(A, B)$ is controllable. Then SDDP is solvable if and only if $\mathcal{E} \subseteq V_{*}^{-}$.
$\mathbb{N}_{\text {Ow, }}$, notice that the formulation of the THiti2.7 brings the hypothesis of $(A, B)$ to be controllable, which implies that $(A, B)$ is alwaysstabilizable according to $\mathbb{T H}$ 1.16. However, as pointed out in the sequel of the samework in [1] , this hypothesis is too strong, since by LEMA 1.18A it is only necessary that ( $A, B$ ) is stabilizable. Then the following formulation bring less restriction.
$\begin{array}{lll}\text { THEORTM } & 2.8 \quad[7]\end{array}$
Given $\Sigma$,SDDP is solvable if and only if $(A, B)$ is stabilizable and $\mathcal{E} \subseteq V_{*}^{-}$.

The difference between the formulations of THI 2.7 and THI 2.8 is that in the first one it is wanted that $\sum$ be completely control= lable; but in the second, only the unstable modes of $\Sigma$ are required to be controllaple. (Since $(A, B)$ is stabilizable iff $X_{b}(A) \subset\langle A \mid \beta\rangle$ ).

We wwill turn our attention now, to the frequency domain characterisation of SDDP.Again this will be a direct translation from the geometric formulation. First consider the following LEMA characterizing the stabilizability of the pair $(A, B)$ in frequency domain.

IEMA $\quad 2.9 \quad[7]$
Given $\sum,(A, B)$ is stabilizable if and only if there exists strictly proper stable matrices $\tilde{X}(s), \tilde{U}(s) \quad$ such that

$$
\begin{equation*}
(S I-A) \tilde{X}(s)-B \tilde{U}(s)=I \tag{24}
\end{equation*}
$$

## Proof :

By COROLLARY 1.21 ( $A, B$ ) is stabilizable if and only if $V_{*}^{-}=X$. Then applying trM 1.19 to $X$, with a basis $\left\{\underline{e}_{1}, \ldots, \underline{e}_{n}\right\}$ for $X$ where $\underline{e}_{i}$ $i \in\{1, \ldots, n\}$ are columns of $I_{n}$ matrix we obtain (24) with $\tilde{X}(s)$ and $\tilde{U}(s)$ strictly proper stable matrices.

THEOREM $\quad 2.10 \quad[7,9]$
Given $\Sigma$, SDDP is solvable if and only if (i) there exists (s)and $\tilde{U}(S)_{\text {strictly }}$ proper stable such that

$$
(S I-A) \tilde{X}(S)-B \tilde{U}(S)=I
$$

and (ii) there exists $X(s)$ and $U(s)$ strictly proper stable such that

$$
\left[\begin{array}{cc}
s I-A & -B \\
C & 0
\end{array}\right]\left[\begin{array}{l}
x(s) \\
u(s)
\end{array}\right]=\left[\begin{array}{l}
E \\
0
\end{array}\right] .
$$

Until now we have analyzed three problems DDP, MDDP and SDDP. We have to mention that this three problems are generically unsolvable since the localisation of the disturbances in to the wanted subspace $V_{*}$
of $X$ is very difficult. It is also rather difficult to have a sufficiently large ( $A, B$ )-invariant subspace in KerC. We also have to notice that the constraints brought in the stable version of $\operatorname{DDP}$ are stronger. The modified problem brings the difficulty of measuring the disturbance which is normally not possible in most of the physical systems.However we think again that all of these three problems are very useful when a clever modelling is of consideration, and they lead to more efficient design techniques such as partial decoupling of disturbances.

In the next section we will consider more realistic problems which are generically solv able.
(4) - Output Stabilisation with Respect to Disturbance (OSDP)

Our purpose now is to obtain a stable response depending on the disturbance. More clearly the problem is:

PROBLEM (OSDP):
Given $\Sigma$ determine the linear state feedback law $\underline{\mu}=F \underline{Z}$ such that the disturbance/output transfer matrix

$$
C(s I-A-B F)^{-1} E \quad \text { is stable. }
$$

We face two problems together, the problem of finding a feedbac (in other words a suitable ( $A, B$ )-invariant subspace) and the problem of making the output to disturbances stable using this feedback. Recalling that in the geometric approach we think of $X$ as the set of all possible initial states; the response corresponding to initial state $\underline{x}_{0}$ is stable if $\underline{x}_{0}$ has a ( $\underline{\underline{Z}}, \underline{w}$ )-representation with $C_{\underline{Z}}(s)$ stable. Hence $x_{0}$ gives a stable response when $\underline{\xi}^{(s)}$ is stable or when the unstable poles of $\underline{\xi}(s)$ are in $\operatorname{KerC}$ and decoupled at the output.

## DEPITTRION 2.11 [9]

Given $\sum, \mathcal{Q}$ denotes the subspace of points $\underline{x}_{0} \in X$ for which there exists a ( $\underline{\underline{Z}}, \underline{w}$ )-representation with $\mathcal{C}_{\underline{\underline{Z}}}$ (s)stable;

The above explications make it clear that in extreme cases such points in $\mathcal{X}$ may be chosen from a stabilizability subspace of $X$ or from $V_{y}$. In general $x_{0} \in \mathscr{S}^{-}$can always be written as a sum

$$
\underline{x}_{0}=\underline{x}_{01}+\underline{x}_{02}
$$

where $\underline{x}_{0}, V_{*}$ and $\underline{x}_{02} \in V_{*}^{-}(x) ; V_{*}^{-}(x)$ being the maximal stabilizability subspace contained in $X$ and $V_{*}$ being the maximal ( $A, B$ )-invariant subspace contained in $\operatorname{Ker} C$ as stated previously. The following theorem characterizes $\delta^{-}$in terms of $V_{*}^{-}(x)$ and $V_{*}$.

THEOREM 2.12 [7]

$$
\mathscr{S}^{-}=v_{*}+v_{*}^{-}(x)
$$

The complete proof is in 7 . In the sequel we also need:

## THEOREM: 2.13 [7]

There exists a feedback $F: X \rightarrow \mathcal{U}$ such that $(A+B F) V_{*} \subseteq V_{*}$ and $C(S I-A-B F)^{-1} \underline{x}_{0} \quad$ is stable for all $\underline{x}_{0} \in \mathcal{S}$.

Combining the results of THM 2.12 , THE 2.13 and the DEP 2.11 we mo state the solvability of OSDP as follows:

THEOREM $2.14 \quad[8,9]$
Given $\sum, O S D P$ is solvable if and only if $\mathscr{S}^{-} \supseteq \mathcal{E}$.
The above theorem gives the geometric formulation. When the DEF 2.11 is applied to each colum of $E$ we obtain the frequency domain formulaion.

MeEOREM 2.1.5 [9]
Given $\sum$ OSDP is solvaple if and only if there exists strictly proper matrices $X(s)$ and $U(s)$ such that

$$
(S I-A) X(S)-B U(S)=E
$$

and $C X(s)$ is stable.
Compared with the decoupling problems previously analyzed, OSDP differs by beeing gonerically solvable. This is since controllability is a generic property. Then according to COROLLARY 1.21, $S=X$ is generically satisfied, which immediately satisfies the condition $\xi \subseteq \mathscr{S}^{-}$trivially.

## (5) Output Stabilization Problem (OSP):

Given the system $\sum$, we pose the problem as follows: PROBLIM (OSP):

Stabilize the output $\underline{\underline{y}}$ by means of a state feedback $\boldsymbol{F}$; more precisely find the conditions for the existence of a state feedback matrix $F$ which can be calculated in terms of the systen parameters $(A, B, C)$ and such that the response $C e^{t(A+B F)}$
tends to zero, as $t \rightarrow \infty$

An altemate interpretation of the problern statement is to find such a feedback matrix $F$, that the characteristic exponents appearing in the response are in the stable subset $\mathbb{C}^{-}$of $\mathbb{C}$. The response function, which is written for arbitrary initial state, make us understand that OSP is a generalisation of OSDP, in which initial states are bounded by the subspace. $\mathcal{E} \subseteq X$. In [1] the geometric formulation is given as follows:

## throned 2.16

Given $\sum$, Asp is solvable if and only if
$x_{b}(A) \subseteq\langle A \mid B\rangle+v_{*}$, where $x_{b}(A) \triangleq \operatorname{Ker} \alpha+(A)$.
$\mathrm{H}_{\text {enc }}$ it's wanted that the unstable modes of $A$ be either controltable or unobservable. The proof given in [I] is long and tedious as the formulation is based again on the controllability subspaces. However, using the stabilizability subspace the formulation becomes easier. In $[9]$ the following procedure is used:

The largest stabilizability subspace $X$ can be written as $V_{*}^{-}(x)=\langle A \mid \beta\rangle+X_{g}(A)$, where $X_{g}(A) \triangleq \operatorname{Ker} \alpha^{-}(A)$. So $V_{*}^{-}(x)$ is the space of all initial conditions for which the response can be made stable by means of a suitable feedback matrix $F$ or is already stable. The if the condition of THII 2.16 is satisfied, we have by

$$
\begin{aligned}
& V_{*}^{-}(x)=\langle A \mid B\rangle+x_{g}(A) \text { and } x_{b}(A) \subseteq\langle A \mid B\rangle+V_{*} \\
& \mathscr{S}^{-}=V_{*}+V_{*}^{-}(x)=\langle A \mid B\rangle+V_{*}+x_{g}(A)=x_{b}(A)+x_{g}(A)
\end{aligned}
$$

and since $X_{b}(A)+X_{g}(A)=X$, we have
$\delta=X$ appearing as the condition. We obtained the
formulation by direct translation, however using the information brought by OSDP we notice that if the initial states are not bounded by $\mathcal{K} \subseteq X$, but assumed totally arbitrary, we obtain the condition for the solvability of OSP as $\mathscr{S}=X$; since OSDP is transformed to OSP.

## THEOREM 2.17 <br> [9]

OSP is solvable if and only if $\quad \mathscr{S}^{-}=x$.
Now, since the column of $I_{n}$ matrix span the state :space $\chi$, the following gives the frequency domain formulation.

## THEOREM <br> 2.18 <br> [9]

OSP is solvable if and only if there exists stricly proper matrices $P(s)$ and $Q(s)$ such that
$(s I-A) P(s)-B Q(s)=I$
and $C P(s)$ is stable.

The stabilization of output as posed by OSP guarantees only the output $\underline{y}(\cdot) \quad$ is well-behaved, but it brings no restriction on the behaviour of system map on the unobservable subspace of $\mathcal{X}$; and it's a possibility this map be unstable.

## (6) Output Null Control

Ever it sounds as a completely new problem, the output null controllability has been derived from applications widely used in control theory, especially on discrete-time systems.

## PROBLEM ( ONE ):

Given $x_{0} \in X$ find a condition for the existence of a control function $\underset{\sim}{\mu}(\cdot)$ such that for solution of $\sum$ with $\underline{x}_{0}$ as the initial state there exists $T>0$ such that $\underset{\sim}{y}(t)=0 \quad(\forall t>T)$.

- We first notice that the conditions is weaker than the one required by output controllability, in which any arbitrary state is reachable from $x_{0}=0$.However in ONC we have more freedom on choosing $\underset{\sim}{u}(\cdot)$ such that it only brings the response of a given state $\underline{x}_{0}$ to zero at the end of some $T>0$.
$\underline{x}_{0} \in X$ is output null controllable if and only if $\underline{x}_{0} \in \Psi \triangleq\langle A \mid \xi\rangle+V_{*}$.

We think that the following explanation will be sufficient. Since we understand by ONC to bring the initial state response to zero after some $t>T$ where $T>0$, we have two choices:either we choose the initial state in the controllable subspace of the system $\Sigma$ or in the largest ( $A, B$ )-invariant subspace contained in $\operatorname{Ker} C$, to decouple of completely at the output even if $\underline{x}_{0}$ is not controllable.

LELIA 2.20
$\psi$ is an 1 -invariant subspace of $X$.

Since by definition $\psi \triangleq V_{*}+\langle A \mid \beta\rangle$, we see that $\langle A \mid \beta\rangle$ is A-invariant and $\langle A \mid B\rangle \supset \beta$ implies

$$
A \psi=A U_{*}+A\langle A \mid B\rangle \subseteq V_{*}+\beta+\langle A \mid \beta\rangle=\psi
$$

as $A V_{*} \subseteq V_{*}+B$.

The following lemma is used in the characterisation of the frequency domain formulation.

LEMMA 2.21 [9]

$$
\underline{x}_{0} \in\langle A \mid \beta\rangle \text { if and only if there exists polynomial }
$$

vectors $\underline{\underline{G}}(s)$ and $\underline{w}(S)$ such that

$$
\underline{x}_{0}=(s I-A) \underline{\xi}(s)-B \underline{w}(s) .
$$

PROOF:
Let $\mathcal{G}(s), \underline{\omega}(s)$ be polynomial vectors such that

$$
\underline{x}_{0}=(s I-A)_{\underline{\xi}}(s)-B \underline{\omega}(s),
$$

which can be written explicitely as

$$
\underline{x}_{0}=(s I-A)\left[\underline{\xi}_{n} s^{n}+\cdots+\underline{\xi}_{0}\right]-B\left[\underline{\omega}_{m} s^{m}+\cdots+\underline{\omega}_{0}\right] .
$$

How $X_{0}$ being a constant vector, obviously $m=n+1$ and we have by equating the coefficients

$$
\begin{aligned}
& \xi_{n}=B \underline{w}_{n+1} \\
& \xi_{n-1}-A \xi_{n}=B w_{n} \\
& \vdots \\
& \underline{\xi}_{0}-A A_{1} \\
&=B \underline{w}_{1} \\
&-A \bar{\xi}_{0}=B w_{0}+x_{0}
\end{aligned}
$$

Solving for $\underline{\underline{G}}_{0} \underline{x}_{0} b y$ successive substitutions

$$
\underline{x}_{0}=-B \underline{w}_{0}-A \underline{\underline{\xi}}_{0}=-B \underline{w}_{0}-A^{2} \underline{\xi}_{1}-A B \underline{w}_{1}=\cdots
$$

finally we obtain

$$
\underline{x}_{0}=-B \underline{w}_{0}-A B \underline{w}_{1}-A^{2} B \underline{w}_{2}-\cdots-A^{p-1} B\left(w_{p-1}+\cdots+w_{n+1}\right)
$$

where $p=\operatorname{dim}(A) \quad$.Since

$$
\langle A \mid B\rangle \triangleq \beta+A \beta+\cdots+A P^{-1} \beta
$$

${ }^{\prime \prime} \Rightarrow x_{0} \in\langle A \mid B\rangle$ as $B \underline{\omega}_{0} \in \mathcal{B}, A B \omega_{1} \in A B, \cdots \quad A^{p-1} B\left(\omega_{p-1}+\cdots \omega_{n+1}\right) \in$
Let $x_{0} \in\langle A / B\rangle$, then we can write

$$
\begin{equation*}
x_{0}=B C_{1}+A B C_{2}+\cdots+A^{P-1} B C_{p} \tag{25}
\end{equation*}
$$

Define $-\underline{w}_{0} \triangleq \underline{C}_{1}$ and in general $-\underline{w}_{i-1} \triangleq \underline{C}_{i}$ for $i \in\{1, \ldots, p-1\}$ then also defining

$$
\begin{aligned}
& \xi_{p-2} \triangleq B \underline{w}_{p-1} \\
& \underline{\xi}_{p-3} \triangleq A \underline{\varepsilon}_{p-2}+B \underline{w}_{p-2} \\
& \\
& \underline{\xi}_{0} \triangleq A \underline{\xi}_{1}+B \underline{w}_{1}
\end{aligned}
$$

(25) can be written as

$$
\begin{gathered}
\underline{x}_{0}=(S I-A)\left[\underline{\xi}_{0}+\cdots \underline{\xi}_{p-2} s^{p-2}\right]-B\left[\omega_{0}+\cdots+\underline{w}_{p-1} s^{p-1}\right], \\
\text { completing the proof. }
\end{gathered}
$$

Combining this result with phi 2.19 we obtain:

THEOREM
$x_{0} \in \psi$ if and only if there exists rational vectors $\underline{\mathcal{Z}}(s)$ and $\underline{w}(s)$ such that $\underline{x}_{0}=(s I-A) \underline{\xi}(s)-B w\left(s\right.$ nd $C_{\underline{z}}(s)$ is a polynomial.

PROOF:

$$
\text { Let } \underline{x}_{0} \in \psi \triangleq\langle A \mid \beta\rangle+V_{*} \quad, \text { then } \underline{x}_{0} \text { can be }
$$

writeh as $\quad \underline{x}_{0}=\underline{x}_{01}+\underline{x}_{02}$
, where $\underline{x}_{01} \in V_{*}^{\prime}, \underline{x}_{02} \in\langle A \mid \beta\rangle$
which implies that
and $C_{\underline{\xi}_{1}}(s)=0$

$$
\underline{x}_{01}=(s I-A)_{\underline{\underline{Z}}_{1}}(s)-B \underline{w}_{1}(s) \quad \text { with } \underline{\underline{\xi}}_{1}, \underline{w}_{1} \text { strictly proper }
$$

$$
\begin{aligned}
& \underline{x}_{02}=(s I-A) \underline{\xi}_{2}(s)-B \underline{w}_{2}(s) \quad \text { with } \underline{\underline{z}}_{2}, \underline{w}_{2} \\
& =(s I-A)\left[\underline{\underline{\xi}}_{1}(s)+\underline{\xi}_{2}(s)\right]-B\left[\underline{w}_{1}(s)+\underline{w}_{2}(s)\right]
\end{aligned}
$$

Hence $\quad \underline{\underline{z}}_{0}=(s I-A)\left[\underline{\underline{z}}_{1}(s)+\underline{\xi}_{2}(s)\right]-B\left[\underline{\omega}_{1}(s)+\underline{\omega}_{2}(s)\right]$
then $\quad \underline{\xi}(s) \underline{\underline{G}}_{1}(s)+\underline{\xi}_{2}^{-}(s) \quad$ is rational and

$$
C_{\underline{Z}}(s)=C_{\underline{b}_{-}}(s)^{0}+C_{\underline{\underline{G}}_{2}}(s)=C_{\underline{\underline{G}}_{2}}(s) \quad \text { is polynomial }
$$

and

$$
\begin{array}{lr}
\underline{w}(s)=w_{1}(s)+w_{2}(s) & \text { is rational, and } \\
\underline{x}_{0}=(s I-A) \underline{\xi}(s)-B \underline{w}(s) & \text { with } \underline{\xi}(s), w(s)
\end{array}
$$

rational and $C_{\underline{\xi}}(S)$ polynomial.


$$
I_{\text {et }} \underline{x}_{0}=(s I-A) \underline{\xi}(s)-B \underline{\omega}(s) \text { with } \underline{\xi}(s), \omega(s) \text { rational }
$$

and $C_{\underline{E}}(s)$ polynomial. Defining

$$
\underline{\underline{\xi}}_{1}(s) \triangleq(\underline{\underline{z}}(s)) \quad, \quad \text { we have } \underline{\xi}^{( }(s)=\underline{\underline{\xi}}_{1}(s)+\underline{\underline{\xi}}_{2}(s)
$$

where $\underline{\xi}_{2}(s)$ polynomial. And since $C \underline{\xi}(s)$ is polynomial and $C$ is a constant matrix, obviously $C_{z_{1}}(s)=0$. Also defining $\underline{\omega}_{1}(s) \triangleq(\underline{\omega}(s))$ we:have $\underline{w}(s)=\underline{w}_{1}(s)+\underline{w}_{2}(s)$ with $\underline{\omega}_{2}(s)$ polynomial. Then $\underline{x}_{0}=(s I-A) \underline{\xi}_{1}(s)-B \underline{w}_{1}(s)+(s I-\bar{A}) \underline{\xi}_{2}(s)-B \underline{w}_{2}(s)=\underline{x}_{01}+\underline{x}_{02}$ where $x_{01} \triangleq(s I-A)_{\underline{\underline{Z}}_{1}}(s)-B \underline{w}_{1}(s) \in V_{*}$.

$$
\underline{x}_{02} \triangleq(s I-A) \underline{\underline{q}}_{2}(s)-B \underline{w}_{2}(s) \in\langle A \mid \beta\rangle
$$

(7 )-Concluding Remarks
We want to end this chapter with the following table summarising the conditions corresponding to the solvability of the problems that have been considered.

$$
\text { Given } \begin{aligned}
\sum: \underline{x} & =A \underline{x}+B u \underline{u}+E \underline{q} \\
\underline{y} & =C \underline{x}
\end{aligned}
$$



1
We have mentioned before that disturbance decoupling problems are generically unsolvable. In many system descriptions a large enough $V_{*}$ does not exists. However if our purpose is to obtain a partial decoupling we can achievethigvery easily. Le us define ${ }^{2}$ by $\xi_{p}$ the subspace

$$
\xi_{p} \triangleq \varepsilon \cap v_{*} .
$$

How choosing a suitable feedback $F_{p}$ it's possible to decouple the noise components $q$ such that $E_{q} \in \xi_{p}$. Also we notice that if $\mathcal{G} \subseteq \ell^{-}$is satisfied (generally it is) we have always the possibility to choose the feedback $F_{p}$ such that $\xi_{p}$ is decoupled and the remaining noise components is stabilized at the output.
$\qquad$
O] SEPARATED "JOSSIBSS" AND "ALGEBRAIC" PARIS

In many applications, a linear m-port system determined by a state space description consists of a "lossless component $N_{D}$ " and a "algebraic component $N_{A}$ ", as in Pig. 3.I. As an example consider an electrical network, the lossless components are inductors, capacitors and the algebraic components are resistors, dependent sources, etc. In $[10]$ a detailed work on obtaining the state space description of such an m-port and on the observability and controllability conditions has been done. Here our aim is to investigate further properties of this m-port system; the interconnection of $N_{D}$ and $N_{A}$ brings a larger degree of freedom in the salution of the problems considered in chapter II. We will concentrate our investigation on the improvements obtained for the solution of $D D P$ in such an m-port. We begin by giving the system description.
(1) F First Level System Decomposition

We consider an m-port obtained by interconnecting an algebmaia $(m+n)$-port $N_{A}$ and a lossless $n$-port $N_{D}$, as shown in rig. 3.1.


Figure 3.1: System obtained by interconnecting Na \& Nd

The defining equations of $N_{D}$ and $N_{A}$ are

$$
\begin{align*}
& \underset{\sim}{w}=-F_{1} \bar{z}-F_{2} \underset{\sim}{u}  \tag{26}\\
& \underset{\sim}{u}=C_{\bar{z}}+G_{\underset{\sim}{u}}^{\underset{\sim}{u}}=H_{\sim}^{x}  \tag{27}\\
& \dot{\sim}=\underset{\sim}{w} \tag{28}
\end{align*}
$$

The input vector ( $\underset{\sim}{\sum}, \underset{\sim}{u}$ ) and the output vector ( $\underset{\sim}{w}, \underset{\sim}{y}$ ) of $N_{A}$ are usually hyorid pairs (i.e the $i^{t h}$ element of ( $\underset{\sim}{\underset{\sim}{*}}, \underset{\sim}{u}$ ) is a current, then the $i^{\text {th }}$ element of $(\underset{\sim}{w}, \underset{\sim}{y})$ is a voltage variable). Same observation can be made for the input vector $\underset{\sim}{w}$ and output vector $z$ of $\mathrm{I}_{\mathrm{D}}$; when equations (26), (27), (28), (29) are combined the state space description of this system is given by

$$
\begin{align*}
& \underset{\sim}{x}=F_{1} H \underset{\sim}{x}+F_{2} \underset{\sim}{u}  \tag{30}\\
& \underset{\sim}{y}=C H \underset{\sim}{x}+G_{\sim}^{u} \tag{31}
\end{align*}
$$

where $\underset{\sim}{x} \in \tilde{X}$ is the state vector, $u \in \tilde{\mathcal{L}}$ is the input vector and $y \in \tilde{y}$ is the output vector. These equations give the state equations as a function of charges and flux linkages.A more convenient description can be obtained in terms of capacitor voltages and, inductor currents, a.s

$$
\begin{align*}
& \underset{\sim}{z}=H F_{1} \underset{\sim}{z}+H F_{2} \underset{\sim}{u}  \tag{32}\\
& \underset{\sim}{y}=C \underset{\sim}{z}+G \underset{\sim}{u} \tag{33}
\end{align*}
$$

An inportant remark is that $H$ must be invertible. If this condition is not satisfied the above equations are no longer the state equations Qfothe system of Fig.3.1. For simplicity we will neglect the direct coupling of $\underset{\sim}{c}$ to the output $\underset{\sim}{y}$ and also denote by $D$ the inverse of the matrix $H$, hence $D \triangleq H^{-1}$. We will call the system $\sum$ the following system described by the state equations

$$
\begin{align*}
D \underline{x} & =F_{1} \underline{x}+F_{2} \underline{u}  \tag{34}\\
\underline{y} & =C \underline{x} \tag{35}
\end{align*}
$$

where $\underline{x}$ is the state vector, $\underline{u}$ is the input vector, $\underline{y}$ is the output vector. When analyzing the disturbance decoupling problem we will introduce the noise component $E_{\underline{q}}$ to the equation (34) to give a description
of the disturbance. We also immediately notice that the matrix $D$ is totally a function of the internal properties of $N_{D}$. So we can say that in an electrical network the entries of $D$ are determined by the values of inductors and capacitors.

In the sequel we :will also need the conditions for which. the system $\sum$ is controllable, observable. The following tables, [10], gives us the necessary and sufficient conditions.

| rank $F_{2}$ | rank $\left[F_{1}: F_{2}\right]$ | $\Sigma$ is |  |
| :---: | :---: | :---: | :---: |
| $\neq 0$ | $=n$ | $\Longleftrightarrow$ | $\exists D$ such that $\Sigma$ is cont. |
| $=n$ | $=n$ | $\Leftrightarrow$ | $\forall D, \Sigma$ is cont. |
| $=0$ | arbitrary | $\nexists D$ such that $\Sigma$ is |  |
| cont. |  |  |  |

TABLE 3.2a: CONTROLLABILITY COND. OF $\Sigma$

| $\operatorname{rank} C$ | $\operatorname{rank}\left[\begin{array}{c}F_{1} \\ c\end{array}\right]$ |  | $\sum$ is |
| :---: | :---: | :---: | :---: |
| $\neq 0$ | $=n$ | $\Leftrightarrow$ | $\exists D$ such that $\Sigma$ is observe. |
| $=n$ | $=n$ | $\Leftrightarrow$ | $\forall D, \Sigma$ is observe. |
| $=0$ | arbitrary | $\nRightarrow D$ such that $\Sigma$ is |  |
| observ. |  |  |  |
| $\neq 0$ | $<n$ |  |  |

TABLE 3.2.b: OBSERVABILITY COND. OF $\sum$

As can be seen from tables 3.2.a,3.2.b the controllability and observability conditions are derived independently. However in general we expect a system to be controllable and observable at the same time. The necessary and sufficient conditions for this case ja given by the following theorem.

## THEOREM 3.1 [10]

Given $P_{1}, P_{2}$ and $C$ there exist matrix $D$ (not necassarily unique) such the systera $\sum$ is observablo and controllable if and only if both of the following conditions are satisfied:

$$
\begin{aligned}
& \text { (i) } F_{1} \neq O, \operatorname{rank}\left[F_{1}: F_{2}\right]=n \\
& \text { (ii) } C \neq \underset{\sim}{O}, \operatorname{rark}\left[\begin{array}{l}
F_{1} \\
-c
\end{array}\right]=n .
\end{aligned}
$$

In the sane work, the proof of this theorem also gives an algorithm that shows haw such $D$ matrices can be selected when $F_{1}, F_{2}, C$ are given. In the sequel we will assume that. we know how to choose $D$ matrices and concentrate an other properties of $\Sigma$.
(2) - Polynomial System Hatrices [13]

Consider the state space equations of a linear syster, as

$$
\begin{align*}
& \dot{x}=A \underline{x}+B \underline{u}  \tag{36}\\
& \underline{y}=C \underline{x}+G \underline{u} \tag{37}
\end{align*}
$$

it's obvious that these equations are linear time -invariant differential equations.In control theory, the advantage of using the frequency domain approach leads us to consider the Laplace transform of the state variables for many application purposes. Then assuming zero initial state the equations (36),(37) turns out to be

$$
\begin{align*}
s \bar{x} & =A \bar{x}+B \bar{u}  \tag{38}\\
\bar{y} & =C \bar{x}+G \bar{u} \tag{39}
\end{align*}
$$

where $\bar{x}=\mathcal{L}(\underline{x}(t)) \quad, \bar{u}=\mathcal{L}(\underline{u}(t)) \quad$ and $\bar{y}=\mathcal{L}(\underline{y}(t))$.
Rearranging (33) and (39) we obtain

In $[13]$, the coefficient matrix $P(s)$ of the vector $[\bar{x} \bar{u}]^{\top}$ is called a polynomial system matrix. Although the results seen to be trivial, the represetation of a linear time invariant system as a single polynomial matrix has many advantages.First, in the case of general linear constant
differential systems it is most of the time difficult to obtain a suitable state space description. Secondly, using the system matrix $P(s)$ all transformations of the system equations can be expressed as operatior on $P(s)$. ( Third they appear naturally, by simply taking the Laplace trans form of the describing differential equationg.) Therefore the properties of the operations on $P(s)$ can be more systematically studied. In general a linear constant differential equations system gives the following system matrix

$$
P(s)=\left[\begin{array}{cc}
T(s) & u(s) \\
-V(s) & W(s)
\end{array}\right]
$$

We restrict ourselves to the case of system $\sum$ and the corresponding system matrix $P_{z}(s)$ is

$$
P_{\Sigma}(s)=\left[\begin{array}{cc}
D s-F_{1} & F_{2} \\
-C & 0
\end{array}\right]
$$

## (3)- Problem Statement

In a first level system decomposition one can notice that given the triple ( $\mathbb{F}_{1}, F_{2}, C$ ) a $D$ matrix givingrise to a controllable and observable system $\Sigma$ is not necessarily unique. Hence, by changing $D$ matrix in such a way that $\sum$ reméing controllable and observable we obtain different state descriptions leading probably to different transt fer function matrices. Then the following questions can be asked for realisation purposes:
a) Given ( $\left.F_{1}, F_{2}, C\right)$ fixed, are the matrices $D$ giving rise to controllable and observable systems having same transfer function matrices unique ?
b) Assuming that such matrices $D$ are not unique,is it possible to decauple a noise component Eq using a special $D$ matrix from the above equivalence class w hich leaves the transfer function matrix invariant.

The answer to the first question obviously prepares the
investigation of the second one. In fact,the idea is to investigate if a first level system decomposition brings any extra degree of freedom to treat several control problems, and more specifically the disturbance decoupling problem. This degree of freodor can also be useful for realisations of a given transfer function, if many $D$ matricos are available to describe the "lossless" n-port $N_{D}$ one can choose a proper one for a more suitable physical realisation.

## (4) Equivalence Transformations of System Hatrices

We shall be particularly interested in a tranformation which leaves unchanged the transfer function matrix and the system order:

Strict System Equivalence (SSE)
Let an $(r+m) \times(r+l)$ polynomial system matrix $P(s)$ be given. Let $M(s), N(s)$ be (rxr ) unimodular matrices; that is their determinants ane nonzero and independent of $s$. Also let $X(s), Y(s)$ be polynomial matrices respectively $(m \times r)$ and ( $r \times l$ ). If two system matrices

$$
P(s)=\left[\begin{array}{cc}
T(s) & U(s) \\
-V(s) & W(s)
\end{array}\right] \text { and } P_{1}(s)=\left[\begin{array}{cc}
T(s) & U(s) \\
-V(s) & W(s)
\end{array}\right] \text {. }
$$

are related by the transformation

$$
\left[\begin{array}{cc}
M(s) & 0  \tag{43}\\
X(s) & I_{m}
\end{array}\right] \cdot\left[\begin{array}{cc}
T(s) & U(s) \\
-V(s) & W(s)
\end{array}\right]\left[\begin{array}{cc}
N(s) & Y(s) \\
0 & I_{l}
\end{array}\right]=\left[\begin{array}{cc}
T_{1}(s) & U_{1}(s) \\
-V_{1}(s) & W(s)
\end{array}\right]
$$

then $P(s)$ and $P_{1}(s)$ are said to be strictly system equivalent.
A very important property of SSE transformation is given
as follows.

THEOREM 3.2

Two system matrices which are strictly system equivalent give rise to the same transfer function matrix and have same order.

- Hence, because of the property stated by THM 3.2 , it's very probable that for the system $\Sigma$ described by the equations (34), (35), $P_{\boldsymbol{Z}}(s)$ matrices giving rise to the same transfer function matrix (i.e. D matrices, since they are the only changing parameter of $P_{\boldsymbol{Z}}(s)$ ) are related by SSE transformations.More precisely the question is:


## QUESTION 3.3:

Given the system $\Sigma_{\alpha}$ described by the state equations

$$
\begin{equation*}
D_{\alpha} \underline{\underline{x}}=F_{1} \underline{x}+F_{2} \underline{u} \tag{44}
\end{equation*}
$$

$$
\begin{equation*}
\underline{y}=C \underline{x} \tag{45}
\end{equation*}
$$

and the corresponding $P_{\Sigma_{\alpha}}$ (s), are all system matrices giving rise to the same transfer function matrix related by SSE transformations ? To give the answer of this question we need the following results which are instrumental.
(5) - Decoupling Zeroes and Relatively Prime Polynomials

Let's consider first the Smith Form of a polynomial matrix $A(s)$.

THEOREM $3.4 \quad[13,14]$
By a combination of elementary row and columin operations, an ( $m \times n$ ) polynomial matrix $\mathrm{A}(\mathrm{s})$ can be reduced to its Smith form

$$
S(s)=M(s) A(s) N(s)
$$

where $M, N$ are unimodular and represent the elementary row and column operations, and

Here $Q(s)$ is a diagonal matrix having as entries on its principal dia-
gonal the invariant polymomials $\varepsilon_{i}(s)$. Fach non-zero invariant polymomial has the coefficient of its higest power of $s$ equal to $I$. In Smith form if the rank of $\Lambda(s)$ is $r$, there are $r$ non-zero invariant polynomials occupying the leading positions and the remaining invariant polynomials are zero.

THT 3.4 fives a detailed definition of the Smith Form. Now let's consider a systen matrix $P(s)$ in which the matrix $[T(s) U(s)]$ has a Smith form $S(s)=[Q(s) \quad 0]$. Then the determinant $|Q(s)| \triangleq D_{r}(s)$ is called the greatestmonic common divisor of minors of order $r$ in $[T(s) U(s)]$. The roots $\left\{\beta_{i}\right\}, C \in\left\{1_{1} \ldots, b\right\}_{o f}$ the equation $D_{r}(s)=0$ will be called zeroes of $S(s)$. The complete set of these roots is called inputdecounling zerioes [13]. The removal of these roots from $[T(s) U(s)]$ by dividing one by one by factors of ( $s-\beta_{i}$ ), the Smith form turns out to be $S(s)=\left[\begin{array}{cc}I_{p} & 0 \\ 0 & 0\end{array}\right]$, $p<r$. In $[13,14]$ we equivalently find the definition for output-decoupling zeroes where the $\left[T^{\top}(s)-V^{\top}(s)\right]$ submatrix of $P(s)$ is taken into consideration. Similarly by removing the output-decoupling zeroes from $\left[T^{\top}(s)-V^{\top}(s)\right]$ the Smith form becomes $S_{1}^{\prime}(s)=\left[\begin{array}{cc}I_{q} & 0 \\ 0 & 0\end{array}\right], q<r$. Hence by these definitions the following lema is inmediate.

## IRMTA: 3.5

A system described by the polynomial system matrix $P(s)$ has no decoupling zeroes if and only if the following conditions hold:
(i) the sinith form of $[T(s) \quad U(s)]$ is $S(s)=\left[\begin{array}{ll}I_{r} & 0\end{array}\right]$
(ii) the Smith form of $\left[T^{\top}(s)-V^{\top}(s)\right]$ is $S_{1}(s)=\left[I_{r}^{\top} O^{\top}\right]$, where $r \triangleq\{$ dimension of the spuare matrix $T(s)\}$.

The following theorem is basic to find an answer to the quest.: tion 3.3.

Iet $P(s)$ and $P_{1}(s)$ be two $(r+m) \times(r+l)$ polynomial syotom matrices having no decoupline zeroes. Then $P(s)$ and $P_{1}(s)$ are strictly system equivalent if and only in they give rise to the same transfer function matrix.

We also nesd the following definition of relatively prine polynomial.

## DEFINITION 3.6 [13]

Polynomial matrices $T(s)$ and $V(s)$ are called relatively left (right) prime if and only if thejr greatest common left divisor $G_{L}(s)$ (g.c.r.d. $G_{R}(s)$ is unimodular.

- A property of the relatively prime polynomials is given by the following theoren.


## THEOR 3.7 [14]

The polynomial matrices $T(s), V(s)$ respectively ( $r \times r$ ) and ( $r \times P$ ) are relatively left prime if and only if the Smith form of $[T(s) \quad V(s)]$ is $\left[I_{r} O\right]$. A similar result can be stated for right primeness. Thus if $\mathrm{H}(\mathrm{s}), \mathrm{V}(\mathrm{s})$ are the submatrices of a given polynomial system matrix $P(s)$; they are relatively left prime if and only if this system has no input decoupling zeroes.

## $\begin{array}{lll}\text { MHMORTI } 3.8 & {[3]}\end{array}$

Given a systen described by the polynomial matrix
$P(s)=\left[\begin{array}{ll}T(s) & U(s) \\ -V(s) & W(s)\end{array}\right]$, it is
(a) completely controllable if and only ir any E.c.l.d $G_{L}(s) \circ f \quad\{T(s), V(s)\}$ is unimodular
(b) completely observable if and only if any g.c.r.d $G_{R}(s) \circ f \quad\{-V(s), T(s)\}$ is unimodular
(c) completely controllable and completely observable if and only if both (a) and (b) holds.

The properties of least order systems that we have mentioned one by one lead to a very important result that gives an answer to question 3.3. The next part of the investigation is mainly based on the follwing result.

THEOREM 3.9

Consider two completely controllable and completely observable systems described by the polynomial matrices $P_{1}(s)$ and $P_{2}$ (s). Then $P_{1}(s)$ and $P_{2}(s)$ give rise to the same transfer function matrix if and only if. they are strictly system equivalent.

## PROOF:

- After TH M 3.6 , all we need to show is that a system is completely controllable and completely observable if and only if it has no decoupling zeroes. For this: $P(s)$ is completely controllable \& completely observable $\stackrel{T H M 3.8}{\rightleftarrows}$ g.c.l.d of $\{T(s), V(s)\}$ is unimodular and g.c.r.d of $\{-V(s), T(s)\}$ is unimodular $\stackrel{\text { DEF } 3.6}{\Longleftrightarrow} T(s), V(s)$ are relatively left prime and $-V(s), T(s)$ are relatively right prime $\stackrel{\text { THE } 3.7}{\longleftrightarrow}$ Smith form of $[T(s) V(s)]=\left[\begin{array}{ll}I_{r} & 0\end{array}\right]$ and Smith form of $\left[\begin{array}{ll}T^{\top}(s) & -V^{\top} c s\end{array}\right]=\left[\begin{array}{ll}I_{r} & 0^{\top}\end{array}\right]$
 $P(s)$ has no decoupling zeroes.
Now consider the systems $\Sigma_{\alpha}$ described by the equations (44), (45).

COROLLARY 3.10
Let system $\Sigma_{\alpha}$ be described by

$$
\begin{aligned}
& D_{\alpha} \dot{x}=F_{1} \underline{x}+F_{2} \underline{u} \\
& \underline{y}=C \underline{x}
\end{aligned}
$$

where $D$ and $F_{1}$ are $(n \times n), F_{2}$ is $(n \times q)$ and $C$ is ( $q \times n$ ). Let the corresponding system matrices be

$$
P_{\alpha}(s)=\left[\begin{array}{cc}
D_{\alpha} s-F_{1} & F_{2} \\
-c & 0
\end{array}\right]
$$

Let also rank $\left[F_{1} \vdots F_{2}\right]=n \quad$ and rank $\left[F_{1}^{\top} C^{\top}\right]=n$.
Consider all systems $\sum_{\alpha}$ which are controllable and observable; then these systems have the same transfer function matrix if and only if they are strictly system equivalent.
(5) - Equivalence Class of Da Matrices Which Leaves The Transfer Function İ variant

The answer to the question 3.3 is given by COR. $3 \cdot 10$. Assuming that we have a controllable and observable first level decomposition with an initial $D_{\alpha_{0}}$, we can obtain all other $D_{\alpha}$ giving the same transfer function by using SSE transformations. For this we will use the following procedure.

$$
\text { Let } P_{\alpha_{0}}(s)=\left[\begin{array}{cc}
D_{\alpha_{0}} s-F_{1} & F_{2} \\
-c & 0
\end{array}\right]
$$

be the initial controllable
and observable decomposition. $A P_{\alpha_{i}}(s)$ which is strictly system equivalent to $P_{\alpha_{0}}(s)$ will also be completely controllable and observable and related to $P_{\alpha_{0}}(s)$ by equation (4.3), such that

$$
\left[\begin{array}{ll}
M(s) & 0  \tag{46}\\
X(s) & I_{q}
\end{array}\right]\left[\begin{array}{cc}
D_{\alpha_{0}}-F_{1} & F_{2} \\
-C & 0
\end{array}\right]\left[\begin{array}{cc}
N(s) & Y(s) \\
0 & I_{q}
\end{array}\right]=\left[\begin{array}{cc}
D_{\alpha_{i}} s-F_{1} & F_{2} \\
-C & 0
\end{array}\right]
$$

where $M(s), N(s)$ are unimodular and $X(s), X(s)$ are polynomial.

Two system matrices $P_{1}(s)$ and $P_{2}(s)$ which are in state space form are strictly systen equivalent if and only if they are system similar.

Then using Tirt 3.10.a and the nonsingularity of $D_{\alpha_{0}}$ and $D_{\phi_{i}}$ we can restrict $\mathbb{M}(s), \mathbb{N}(s)$ to be constant nonsingular matrices and $X(s)=Y(s)=0$.

## THEORTM $3.10 . \mathrm{b}$

Given $P_{\alpha_{0}}(s)$ and $P_{\alpha_{i}}(s)$, they are strictly system equivalent if and only if there exists nonsingular constant matrices $M$ and $\mathbb{N}$ such that

$$
\left[\begin{array}{cc}
M & 0  \tag{47}\\
0 & I_{q}
\end{array}\right]\left[\begin{array}{cc}
D_{\alpha_{0}} s-F_{1} & F_{2} \\
-C & 0
\end{array}\right]\left[\begin{array}{cc}
N & 0 \\
0 & I_{q}
\end{array}\right]=\left[\begin{array}{cc}
D_{\alpha_{i}} s-F_{1} & F_{2} \\
-C & O
\end{array}\right]
$$

Proof:
The equation (46) is equivalent to

$$
\left[\begin{array}{cc}
D_{\alpha}^{-1} & 0 \\
0 & I_{q}
\end{array}\right]\left[\begin{array}{cc}
M(s) & 0 \\
X(s) & I_{q}
\end{array}\right]\left[\begin{array}{cc}
D_{\alpha_{0}} & 0 \\
0 & I_{q}
\end{array}\right]\left[\begin{array}{cc}
s I-D_{\alpha_{0}}^{-1} F_{1} & D_{\alpha_{0}}^{-1} F_{2} \\
-C & 0
\end{array}\right]\left[\begin{array}{cc}
N(s) & Y(s) \\
0 & I_{q}
\end{array}\right]=\left[\begin{array}{cc}
s I-D_{\alpha_{1}}^{-1} F_{1} & D_{\alpha_{1}}^{-1} F_{1} \\
-c & 0
\end{array}\right]
$$

since $D_{\alpha_{0}} D_{\alpha_{i}}$ are nonsingular matrices. Now letting $\hat{M}(s)=D_{\alpha_{i}}^{-1} M(s) D_{\alpha_{0}}$

$$
\left[\begin{array}{cc}
\hat{M}(s) & 0  \tag{48}\\
X(s) & I_{q}
\end{array}\right]\left[\begin{array}{cc}
s I-D_{\alpha_{0}}^{-1} F_{1} & D_{\alpha_{0}}^{-1} F_{2} \\
-C & 0
\end{array}\right]\left[\begin{array}{cc}
N(s) & Y(s) \\
0 & I_{q}
\end{array}\right]=\left[\begin{array}{cc}
s I-D_{\alpha_{i}}^{-1} F_{1} D_{\alpha i 2}^{-1} \\
-C & 0
\end{array}\right]
$$

In the matrix equation (48), the SSE transformation is preserved; further more the polynonial system matrices are in state spacc form. Then by THM 3.10.a they are SSE if and only if they are system similar. Therefore there exist a constant nonsingular matrix $H$ such that

$$
\left[\begin{array}{cc}
H^{-1} & 0  \tag{49}\\
0 & I_{q}
\end{array}\right]\left[\begin{array}{cc}
s I-D_{\alpha_{0}}^{-1} F_{1} & D_{d_{0}}^{-1} F_{2} \\
-C & 0
\end{array}\right]\left[\begin{array}{cc}
H & 0 \\
0 & I_{q}
\end{array}\right]=\left[\begin{array}{cc}
s I-D_{\alpha_{i}}^{-1} F_{1} & D_{\alpha_{i}}^{-1} F_{2} \\
-C & 0
\end{array}\right]
$$

or equivalently

$$
\left[\begin{array}{cc}
D_{\alpha_{i}} H^{-1} D_{\alpha_{0}}^{-1} & 0  \tag{50}\\
0 & I_{q}
\end{array}\right]\left[\begin{array}{cc}
D_{\alpha_{0}} S-F_{1} & F_{2} \\
-C & 0
\end{array}\right]\left[\begin{array}{cc}
H & 0 \\
0 & I_{q}
\end{array}\right]=\left[\begin{array}{cc}
D_{\alpha_{i}} S-F_{1} & F_{2} \\
-C & 0
\end{array}\right]
$$

Thus without loss of generality we can restrict the unimodular matrices $I(s)$ and $I(s)$ to being constant and nonsingular, also choose $X(s)=0$ and $Y(s)=0$.

COROLTARY $3.10 . \mathrm{C}$
Given $P_{\alpha_{0}}(s)$ and $P_{\alpha_{1}}(s)$, they are strictly systern equivalent and give rise to the same transfer function matrix if and only ir there exist $M$ and $N$ constant and nonsingular such that the following equations are satisfied:
(i) $\quad \mathbb{M} \quad D_{\alpha_{0}} N=D_{d_{1}}$
(ii) $M \quad F_{1} N=F_{1}$
(iii) $M F_{2}=F_{z}$
(iv) $C \cdot \mathbb{N}=C$

Obviously, since $\left(F_{1}, F_{2}, C, D_{\alpha_{0}}\right)$ are given we can compute in and $N$ matrices from equation (52),(53),(54) then using equation (51) we can compute $D_{\alpha_{1}}$. The degree of freedom obtained on (in, $\mathbb{N}$ ) counle will also determine the degree of freedom in choosing the matrices $D_{\alpha i}$. in important remark is that (52), (53), (54) always have a trivial sulution, that is the ( $M, N$ ) couple where $M=I$ and $N=I$. In this case all four equations hold but giving $D_{\alpha_{0}}=D_{\alpha_{1}}$. The following theorem gives the necessary and sufficient conditions for which a nontrivial solution of equations (52), (53) (54) exists.

MHEORM 3.11

Given the matrices $\mathbb{P}_{1}, F_{2}, C$ respectively $(n \times n)$, ( $n \times q$ ) , $(q \times n)$ and such that rant $\left[F_{1}: F_{2}\right]=n$ and rant $\left[F_{1}^{\top} \vdots C^{\top}\right]=n$, there exist nontrivial solutions of the matrix equations

$$
\left[\begin{array}{ll}
M & O  \tag{55}\\
O & I_{q}
\end{array}\right]\left[\begin{array}{ll}
F_{1} & F_{2} \\
C & O
\end{array}\right]\left[\begin{array}{cc}
N & O \\
O & I_{q}
\end{array}\right]=\left[\begin{array}{ll}
F_{1} & F_{2} \\
C & O
\end{array}\right]
$$

with nonsjngular $M, N$ matrices if and only if rank $F_{2}<n$ and rank $C<n$.

Due to THM 3.11 the existence of nontrivial solutions of (52), (53), (54) is a generic property since the theorem excludes only the special cases where rank $F_{2}=n$ and/ or rank $C=m$.

The following lemma is needed in the proof of TH M 3.11.

LETA 3.12
Given the matrix $X(q \times n), q \geqslant n$; there exists a nonsingular and nontrivial (ie. $\ddagger I$ ) solution of the matrix equation

$$
\begin{equation*}
X=X Y \tag{56}
\end{equation*}
$$

where $Y$ is $(n \times n)$, if and only if $\operatorname{Ker} X \neq\{0\}$.

PROOF:" "
Given $\operatorname{Ker} X \neq\{0\}$; rank $[X]<n$ since $X$ has less than $n$ linearly independent columns. By elementary row and column operations represented by $T_{1}(q \times q)$ and $T_{2}(n \times n)$, we bring $X$ to its Smith form

$$
T_{1} \times T_{2}=\left[\begin{array}{ll}
I_{r} & 0 \\
0 & 0
\end{array}\right] \text {, where } \quad r=\operatorname{rank}[X]<n
$$

Hence (56) becomes

$$
T_{1}^{-1}\left[\begin{array}{cc}
I_{r} & 0  \tag{57}\\
0 & 0
\end{array}\right] T_{2}^{-1} Y=T_{1}^{-1}\left[\begin{array}{cc}
I_{r} & 0 \\
0 & 0
\end{array}\right] T_{2}^{-1}
$$

where for $\hat{y}=T_{2}^{-1} y \cdot T_{2}, \quad\left[\begin{array}{cc}I_{r} & 0 \\ 0 & 0\end{array}\right] \hat{y}=\left[\begin{array}{cc}I_{r} & 0 \\ 0 & 0\end{array}\right]$
Then partitionning we obtain

$$
\left[\begin{array}{cc}
I_{r} & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{ll}
\hat{Y}_{11} & \hat{Y}_{12} \\
\hat{Y}_{21} & \hat{Y}_{22}
\end{array}\right]=\left[\begin{array}{cc}
I_{r} & 0 \\
0 & 0
\end{array}\right]
$$

which gives $\hat{Y}_{11}=I_{r}, \hat{Y}_{12}=0$ with $\hat{Y}_{21} \hat{Y}_{22}$ totally arbitrary; therfore taking $\hat{Y}_{11}$ to be identity matrix $\hat{y}$ can be made nonsingular by carefully choosing $\hat{Y}_{21}$ and $\hat{Y}_{22}$.

Finally is obtained as $Y=T_{2} Y T_{2}^{-1}$.
$" \Longrightarrow "$
Let $Y$ be a nontrivial solution of (56), then $X(Y-I)=0$.
since $Y \neq I$ we have $Y-I \neq 0$ then $\operatorname{Ker} X \neq\{0\}$.

## PROOF OR THEOREM 3.11

We will consider all possible cases one by one.
(i) Let rank $F_{2}=n$ and rank $C=n$, then $\operatorname{Ker} F_{2}{ }^{\top}=\{0\}$ and $(M-I) F_{2}=0 \Rightarrow M=I \quad$ Also $\operatorname{Ker} C=\{0\}$ and
and $C(N-I)=0 \Rightarrow N=I$.
(ii) Let rank $F_{2}<n$ and rank $C=n$, then as in (i)
$N=I$ hence the matrix equations (52), (53) give $\left.\begin{array}{l}M F_{1}=F_{1} \\ M F_{2}=F_{2}\end{array}\right\} \Rightarrow M\left[F_{1}: F_{2}\right]=\left[F_{1} ; F_{2}\right]$
Since rank $\left[F_{1} ; F_{2}\right]=n$ for $(M-I)\left[F_{1} ; F_{2}\right]=0 \Rightarrow M=I$.
(iii) Let rank $F_{2}=n$ and rank $C<n$. Similarly, as in (i) $M=I$ and (52),(54) give

$$
\left.\begin{array}{l}
F_{1} N=F_{1} \\
C N=C
\end{array}\right\} \Rightarrow\left[\begin{array}{l}
F_{1} \\
-C
\end{array}\right](N-I)=0
$$

Then since rank $\left[\begin{array}{c}F_{1} \\ \dot{C}^{-}\end{array}\right]=n$, we have $(N-I)=0$ or $N=I$. "

Let rank $F_{2}<n$ and rank $C<n$. Proceed as follows:

* Put $\mathrm{F}_{2}, \mathrm{C}$ to Smith form keeping the structure of the matrix equations (55).

$$
\left[\begin{array}{c:c}
F_{1} & F_{2}  \tag{58}\\
\hdashline C & 0
\end{array}\right]=\left[\begin{array}{c:c}
T_{1} & 0 \\
\hdashline O & \vdots \\
\hline O
\end{array}\right]\left[\begin{array}{c:c}
T_{A} & I_{a} \\
F_{1} & \vdots \\
\hdashline I_{b} & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{c:c}
P_{1} & 0 \\
\hdashline 0 & 0 \\
0 & P_{A}
\end{array}\right]
$$

Save matrices $T_{1}, P_{1}$

$$
a\left[\begin{array}{c:c:c:c}
\tilde{F}_{11} & \vdots & \tilde{F}_{12} & I_{a}
\end{array}\right] 0
$$

* Put $\tilde{F}_{14}$ to $S_{\text {mith }}$ form.

Save $T_{2}, P_{2}$ matrices.


Reduce $\hat{F}_{131}$ and $\hat{F}_{121}$ by elementary row or column operations.

Save $T_{3}, P_{3}$ matrices.

$$
\begin{aligned}
& \text { b m n-b-m a } \quad \mathrm{a}-\mathrm{a}
\end{aligned}
$$

where

$$
\begin{aligned}
& \hat{M}=T_{3}^{-1} T_{2}^{-1} T_{1}^{-1} M T_{1} T_{2} T_{3} \\
& \hat{N}=P_{3} P_{2} P_{1} N P_{1}^{-1} P_{2}^{-1} P_{3}^{-1} .
\end{aligned}
$$

* Block multiplications in (63) give first:

$$
a\left[\begin{array}{c:c}
\hat{M}_{11} & \hat{M}_{12}  \tag{64}\\
\hdashline \hat{M}_{21} & \hat{\hat{M}}_{22}
\end{array}\right] a\left[\begin{array}{c:c}
n-a & I_{a} \\
\hdashline 0 & 0 \\
\hdashline & a \\
n-a
\end{array}\right]=\left[\begin{array}{c:c}
I_{a} & 0 \\
\hdashline 0 & 0
\end{array}\right]
$$

Thus $\hat{M}_{14}=I_{a}$ and $\hat{M}_{21}=0$, and by partitioning

$$
\hat{M}=\left[\begin{array}{c:c}
I_{a} & \hat{M}_{12}  \tag{65}\\
\hdashline 0 & \hat{M}_{22}
\end{array}\right]_{n-m-a}^{a}\left[\begin{array}{c:c:c}
I_{a} & \hat{M}_{121} & \hat{M}_{122} \\
\hdashline 0 & \hat{M}_{221} & \hat{M}_{222} \\
\hdashline 0 & \hat{M}_{223} & \hat{M}_{224}
\end{array}\right]^{a}
$$

The second equation in $\hat{N}$ is obtained as :
$q-b\left[\begin{array}{c:c}I_{b} & 0 \\ \hdashline 0 & 0\end{array}\right] \quad b\left[\begin{array}{c:c}\hat{N}_{1-b} & \hat{N}_{12} \\ \hdashline \hat{N}_{21} & \hat{N}_{22}\end{array}\right]=\left[\begin{array}{c:c}I_{b} & 0 \\ \hdashline 0 & 0\end{array}\right]$
Thus $\hat{N}_{11}=I_{b}, \hat{N}_{12}=0$ and by partitioning

Replacing $\hat{I I}$ and $\hat{i v}$ obtained in (65) and (67) in the third equation $\hat{M} \bar{F}_{1} \hat{N}=\bar{F}_{1}$, we obtain

$$
\left[\begin{array}{c:c:c}
\mathcal{I}_{0} & \hat{M}_{121} & \hat{M}_{122}  \tag{68}\\
\hdashline 0 & \hat{M}_{221} & \hat{M}_{222} \\
\hdashline 0 & \hat{M}_{223} & \hat{M}_{224}
\end{array}\right]\left[\begin{array}{c:c:c}
\vec{F}_{11} & 0 & \bar{F}_{122} \\
\hdashline 0 & I_{m} & 0 \\
\hdashline \bar{F}_{132} & 0 & 0
\end{array}\right]\left[\begin{array}{c:c:c}
I_{b} & 0 & 0 \\
\hdashline \hat{N}_{21} & \hat{N}_{221} & \hat{N}_{222} \\
\hdashline \hat{N}_{212} & \hat{N}_{223} & \hat{N}_{224}
\end{array}\right]=\left[\begin{array}{c:c:c}
\bar{F}_{11} & 0 & \bar{F}_{122} \\
\hdashline 0 & I_{m} & 0 \\
\hdashline \mathcal{F}_{132} & 0 & 0
\end{array}\right]
$$

* Block multiplications in equation (68) give:

$$
\begin{align*}
& \bar{F}_{11}+\hat{M}_{122} \bar{F}_{132}+\hat{M}_{121} \hat{N}_{211}+\bar{F}_{122} \hat{N}_{212}=\bar{F}_{11}  \tag{69}\\
& \hat{M}_{121} \hat{N}_{221}+\bar{F}_{122} \hat{N}_{223}=0  \tag{70}\\
& \hat{M}_{121} \hat{N}_{222}+\bar{F}_{122} \hat{N}_{224}=\bar{F}_{122}  \tag{71}\\
& \hat{M}_{222} \bar{F}_{132}+\hat{M}_{221} \hat{N}_{211}=0  \tag{72}\\
& \hat{M}_{221} \hat{N}_{221}=I_{m}  \tag{73}\\
& \hat{M}_{221} \hat{N}_{222}=0  \tag{74}\\
& \hat{M}_{224} \bar{F}_{132}+\hat{M}_{223} \hat{N}_{211}=\bar{F}_{132}  \tag{75}\\
& \hat{M}_{223} \hat{N}_{221}=0  \tag{76}\\
& \hat{M}_{223} \hat{N}_{222}=0 \tag{77}
\end{align*}
$$

Then the following 3 cases covers all possible structures that $\overline{F_{1}}$ matrix can take.

## CASE $A: m=n-a=n-b$

We have immediately $\bar{F}_{122}=0 ; \bar{F}_{132}=0$ in $\bar{F}_{1}$. Then $\hat{M}, \hat{N}$ to be nonsjingular a choice

$$
\hat{M}_{22}=\hat{N}_{22}^{-1} \neq I \quad \text { with } \hat{M}_{12}=0 \quad \text { and } \quad \hat{M}_{21}=0
$$

always exists. Then computing $M$ and. $N$ matrices by inverse transformations ( $\mathbb{T}, \mathrm{P}$, etc.) we end up with a nontrivial ( $M, N$ ) couples, since ( $\hat{M}, \hat{\Pi}$ ) couple is nontrivial.

## CASE $B: a \neq b, m \neq 0$

In this we have the following implications:

$$
\begin{aligned}
& E_{q .}(73) \Rightarrow \hat{N}_{221}=\hat{M}_{221}^{-1} \\
& E_{q .}(74) \Rightarrow \hat{M}_{223}=0 \\
& E_{q .}(76) \Rightarrow \hat{N}_{222}=0
\end{aligned}
$$

Then $\hat{M}, \hat{N}$ are nonsingular if and only if $\hat{M}_{224}$ and $\hat{N}_{224}$ are nonsingular.Now
$-(73),(74),(76)$ determine $\hat{N}_{221}, \hat{M}_{223}, \hat{M}_{221}, \hat{N}_{222}$, and (77) holds immediately since (74), (76).

- Choosing $\hat{N}_{224}=I, \hat{M}_{224}=I$ : we see that $(7 I),(75)$ are triviality satisfied.
- Choosing $\hat{M}_{122}=0, \hat{M}_{121}=0 \quad, \hat{N}_{212}=O$ (69) is trivially satisfied.
- Choosing $\hat{M}_{222}=0, \hat{N}_{211}=0 \quad$ (72) is satisfied trivially
- Choosing $\hat{N}_{223}=0$ (70) is satisfied trivially.

However even after these choices we have

$$
\hat{N}=\left[\begin{array}{ccc}
I_{b} & 0 & 0 \\
0 & \hat{N}_{22} & 0 \\
0 & 0 & I
\end{array}\right] \quad \text { and } \quad \hat{M}=\left[\begin{array}{ccc}
I_{a} & 0 & 0 \\
0 & \hat{N}_{22}^{-1} & 0 \\
0 & 0 & I
\end{array}\right]
$$

which shows that ( $\hat{\mathbb{M}}, \hat{\mathrm{N}}$ ) is nontrivial. Hence for $a \neq b$ and $m \neq \theta$ there always exists a nontrivial solutions for ( $\hat{\mathrm{H}}, \hat{\mathbb{N}}$ ) couple.

CASE $C: m=0$
If $m=0$, then $\bar{F}_{14}=0$. Then the resulting equation is

$$
\left[\begin{array}{c:c}
I_{a} & \hat{M}_{12}  \tag{78}\\
\hdashline 0 & \hat{M}_{22}
\end{array}\right] \text { a } \quad\left[\begin{array}{c:c}
\tilde{F}_{11} & \tilde{F}_{12} \\
\hdashline \tilde{F}_{13} & 0
\end{array}\right]\left[\begin{array}{c:c}
I_{b} & 0 \\
\hdashline \hat{N}_{21} & \hat{N}_{22}
\end{array}\right]=\left[\begin{array}{c:c}
\tilde{F}_{11} & \tilde{F}_{12} \\
\hdashline \tilde{F}_{13} & 0
\end{array}\right]
$$

and block multiplication gives

$$
\begin{align*}
& \widetilde{F}_{11}+\hat{M}_{12} \tilde{F}_{13}+\tilde{F}_{12} \hat{N}_{21}=\widetilde{F}_{11}  \tag{79}\\
& \widetilde{F}_{12} \hat{N}_{22}=\tilde{F}_{12}  \tag{80}\\
& \hat{M}_{22} \tilde{F}_{13}=\widetilde{F}_{13} \tag{81}
\end{align*}
$$

The block triangular form of $\hat{M}$ and $\hat{N}$ imposes $\hat{M}_{22}$ and $\hat{N}_{22}$ to be nonsingular. Now

- Let $\operatorname{Ker} \tilde{F}_{12} \neq\{0\}$ and $\operatorname{Ker} \tilde{F}_{13}{ }^{\top} \neq\{0\}$, then by Lima 3.12 the solutron of the equation (80) and (81) are nontrivial and nonsingular. Choosing $\hat{M}_{12}=0$ and $\hat{N}_{21}=0$ (79) is trivially satisfied. Then since $\hat{M}_{22}$ and $\hat{N}_{22}$ are nontrivial there exists always a nontrivial solution of ( $\hat{\mathrm{H}}, \hat{\mathrm{N}}$ ) couple.
- Let $\operatorname{Ker} \tilde{F}_{12}=\{0\}$ and $\operatorname{Ker} \tilde{F}_{13}^{\top}=\{0\}$, then we have rank: $\tilde{F}_{12}=n-b$ and rank $\tilde{F}_{13}=n-a \quad$ By LiARIAA $3.12 \quad \hat{N}_{22}=I$ and $\hat{M}_{22}=I$.Hence the solution of ( $\hat{\mathrm{f}}, \hat{\mathrm{i}}$ ) couple is nontrivial if and only if (79) is nontrivially satisfied, that is

$$
\hat{M}_{12} \tilde{F}_{13}+\tilde{F}_{12} \hat{N}_{21}=0
$$

for $\hat{M}_{12} \neq 0$ and $\hat{N}_{21} \neq 0$. For this by nonsingular transformations we bring $\tilde{F}_{13}$ and $\tilde{F}_{12}$ to their Smith forms

$$
\hat{M}_{12} T_{4}\left[I_{n-a} \vdots 0\right] T_{5}=-P_{4}\left[\begin{array}{c}
I_{n-b}  \tag{82}\\
\hdashline 0
\end{array}\right] P_{5} \hat{N}_{21}
$$

or

$$
\begin{equation*}
P_{4}^{-1} \hat{M}_{12} T_{4}\left[I_{n-a}: 0\right]=-\left[\frac{I_{n-b}}{0}\right] P_{5} \hat{N}_{21} T_{5}^{-1} \tag{83}
\end{equation*}
$$

Therefore
where

$$
\mathcal{M}_{12}\left[I_{n-a}: 0\right]=-\left[\begin{array}{c}
I_{n-b}  \tag{84}\\
\cdots \cdots \\
0
\end{array}\right] \tilde{N}_{21}
$$

Equation (84) implies

$$
\begin{align*}
& (84) \text { implies }  \tag{85}\\
& {\left[\tilde{M}_{12}: 0\right]=-\left[\begin{array}{c}
\tilde{N}_{21} \\
\hdashline 0 \\
0
\end{array}\right]}
\end{align*}
$$

By partitioning (35) gives

$$
a+b-n\left[\begin{array}{c:c}
\tilde{M}_{121} & 0  \tag{86}\\
\hdashline \tilde{M}_{122} & 0
\end{array}\right]=-\left[\begin{array}{c:c}
\tilde{N}_{211} & \tilde{N}_{212} \\
\hdashline 0 & \vdots \\
0 & 0
\end{array}\right]_{n-b} a+b-n
$$

The equation (86) implies that $\tilde{M}_{121}=-\tilde{N}_{211}$ and $\tilde{M}_{122}=0, \tilde{N}_{212}=0 \quad$. Since $a<n, b<n ; \tilde{M}_{121}$ and $\hat{N}_{21} \quad$ are never zero. So by inverse transformations nonzero $\hat{M}_{12}, \hat{N}_{21}$ can be calculated.
Therfore when $\operatorname{Ker} \tilde{F}_{12}=\{0\}$ and $\operatorname{Ker} \tilde{F}_{13}^{\top}=\{0\}$ nontrivial solutions always exist. To complete, when $m=0$ a nontrivial solution for ( $N, N$ ) couple can always be found.

Tinally we have considered all possible cases in which a nontrivial solution of the matrix equation (55) alvays exists if and only if rank $F_{2}<n$ and rank $C<n$. This completes the proof.

We will close this chapter by the following result which is immediate after the previous theorems.

## THEOREM 3.13

$D_{\alpha}$ giving rise to controllable and observable systens $\sum_{d}$ having same transferfunctiomatrix are not unique if and only if rank $F_{2}<n$ and rank $C<n$.

## EXAMPLE $\quad 3.13$

Consider the system $\Sigma_{\alpha_{0}}$

$$
\begin{aligned}
{\left[\begin{array}{ccc}
1 & -1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \underline{\dot{x}} } & =\left[\begin{array}{lll}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{array}\right] \underline{x}+\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
1 & 1
\end{array}\right] \underline{\underline{u}} \\
\underline{y} & =\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right] \underline{x}
\end{aligned}
$$

and note that rank $F_{2}=2<3$ and rank $C=2<3$. The system is completely controllable and completely observable. The transfer function matrix cen be calculated easily and is equal to

$$
T(s)=C\left(D_{\alpha 0} s-F_{1}\right)^{-1} F_{2}=\left[\begin{array}{lc}
1 / s & 1 / s \\
\frac{(s-1)^{2}}{s\left(s^{2}-3 s+1\right)} & \frac{1-s}{s\left(s^{2}-3 s+1\right)}
\end{array}\right]
$$

Now we want to find all matrices $D_{\alpha_{i}}$ such that the new systems are controllable, observable and have the same transfer function matrix $T(s)$. First we have to determine the ( $\mathrm{M}, \mathrm{N}$ ) -couples satisfying equations of COROLLARY 3.10.c, that is

$$
\begin{equation*}
M F_{1} N=F_{1} \tag{87}
\end{equation*}
$$

$$
\begin{align*}
& M F_{2}=F_{2}  \tag{88}\\
& \left.C N=C \quad \text { (or } C N^{-1}=C\right) \tag{89}
\end{align*}
$$

Then the matrices $D_{d_{i}}$ will be obtained from

$$
\begin{equation*}
D_{d_{i}}=M D_{\alpha_{0}} N \tag{90}
\end{equation*}
$$

The equation (89) gives:

$$
\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{lll}
\hat{n}_{1} & \hat{n}_{2} & \hat{n}_{3} \\
\hat{n}_{4} & \hat{n}_{5} & \hat{n}_{6} \\
\hat{n}_{7} & \hat{n}_{8} & \hat{n}_{5}
\end{array}\right]=\left[\begin{array}{lll}
\hat{n}_{7} & \hat{n}_{8} & \hat{n}_{9} \\
\hat{n}_{4} & \hat{n}_{5} & \hat{n}_{6}
\end{array}\right]=\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right]
$$

Therföre

$$
\grave{N}^{-1}=\left[\begin{array}{ccc}
\hat{n}_{1} & \hat{n}_{2} & \hat{n}_{3} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] .
$$

Similarly equation (SB) gives

$$
\left[\begin{array}{lll}
m_{1} & m_{2} & m_{3} \\
m_{4} & m_{5} & m_{6} \\
m_{7} & m_{8} & m_{3}
\end{array}\right]\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
1 & 1
\end{array}\right]=\left[\begin{array}{ll}
m_{2}+m_{3} & m_{3} \\
m_{5}+m_{6} & m_{6} \\
m_{8}+m_{9} & m_{9}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
1 & 1
\end{array}\right]
$$

Therefore

$$
M=\left[\begin{array}{lll}
m_{1} & 0 & 0 \\
m_{4} & 1 & 0 \\
m_{7} & 0 & 1
\end{array}\right]
$$

Substituting $M$ and $I T$ in (57)

$$
\left[\begin{array}{ccc}
m_{1} & 0 & 0 \\
m_{4} & 1 & 0 \\
m_{7} & 0 & 1
\end{array}\right]\left[\begin{array}{lll}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]=\left[\begin{array}{lll}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]\left[\begin{array}{ccc}
\hat{n}_{1} & \hat{n}_{2} \hat{n}_{3} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

equivalently.

$$
\left[\begin{array}{ccc}
m_{1} & 0 & m_{1} \\
m_{4}+1 & 1 & m_{4} \\
m_{7} & 0 & m_{7}
\end{array}\right]=\left[\begin{array}{ccc}
\hat{n}_{1} & \hat{n}_{2} & \hat{n}_{3}+1 \\
\hat{n}_{1} & \hat{n}_{2}+1 & \hat{n}_{3} \\
0 & 0 & 0
\end{array}\right]
$$

then we have

$$
\begin{aligned}
& m_{1}=\hat{n}_{1} \\
& m_{4}=\hat{n}_{1-1} \\
& \hat{n}_{3}=\hat{n}_{1-1} \\
& \hat{n}_{2}=0 \\
& m_{7}=0
\end{aligned}
$$

Which gives for $\hat{n}_{1}=1 / 8-(\delta \neq 0)$

$$
M=\left[\begin{array}{ccc}
1 / \delta & 0 & 0 \\
\frac{1-\delta}{\delta} & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \quad N^{-1}=\left[\begin{array}{ccc}
1 / \delta & 0 & \frac{1-\delta}{\delta} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \quad N=\left[\begin{array}{ccc}
\delta & 0 & \delta-1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

Therefore

$$
D_{\alpha} \cdot(\delta)=\left[\begin{array}{ccc}
1 / \delta & 0 & 0 \\
\frac{1-\delta}{\delta} & 1 & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
1 & -1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]\left[\begin{array}{ccc}
\delta & 0 & \delta-1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

$$
D_{\alpha}(\delta)\left[\begin{array}{ccc}
1 & -1 / \delta & 1 \\
1-\delta & \frac{2 \delta-1}{\delta} & 1-\delta \\
0 & 0 & 1
\end{array}\right] \quad \text { with } \delta \neq 0
$$

We notice that $D_{\alpha}(1)=D_{\alpha_{0}}$ and the upper triangular fora or the matrix $D_{\alpha_{0}}$ is not preserved for $\delta \neq 1$. Let us check if for $D_{\alpha}(\delta)$ in general, the transfer function matrix equals the given $\mathbb{T}(s)$. For this we first form

$$
\left(D_{\alpha}(\delta) s-F_{i}\right)=\left[\begin{array}{ccc}
s-1 & -1 / \delta^{s} & s-1 \\
(1-\delta)_{s-1} & \frac{2 \delta-1}{\delta} s-1 & (1-\delta) s \\
0 & 0 & s
\end{array}\right]
$$

and

$$
\left(D_{\alpha}(\delta) s-F_{1}\right)^{-1}=\frac{1}{s\left(s^{2}-3 s+1\right)}\left[\begin{array}{ccc}
\frac{2 \delta-1}{\delta} s^{2}-s & s^{2} / \delta & -s^{2}+\frac{3 \delta-1}{\delta}-1 \\
-(1-\delta) s^{2}+s & s(s-1) & 1-s \\
0 & 0 & s^{2}-3 s+1
\end{array}\right]
$$

finally

$$
T^{\prime}(s)=C\left(D_{\alpha}(\delta) s-F_{1}\right)^{-1} F_{2}=\left[\begin{array}{cc}
1 / s & 1 / s \\
\frac{(s-1)^{2}}{s\left(s^{2}-3 s+1\right)} & \frac{(1-s)}{s\left(s^{2}-3 s+1\right)}
\end{array}\right]=T(s) \quad \forall \delta \neq 0 .
$$

Before considering the DDP for two different disturbance structuxes we need to characterize the largest $\left(D_{\alpha}^{-1} F_{1}, D_{\alpha}^{-1} F_{2}\right.$ ) invariant subspace contained in Kor.

IMIT 4.1
Given the system $\Sigma_{\alpha}$

$$
\begin{aligned}
D \alpha \underline{x} & =F_{1} \underline{x}+F_{2} \underline{u} \\
\underline{y} & =C_{\underline{x}}
\end{aligned}
$$

$\underline{x}_{0} \in V_{*}\left(D_{\alpha}\right)$ if and only if there exist $\underline{\underline{G}}(s)$ and $\underline{\omega}(s)$ strictly proper rational such that

$$
\left[\begin{array}{cc}
D_{\alpha} s-F_{1} & F_{2}  \tag{1}\\
-C & 0
\end{array}\right]\left[\begin{array}{c}
\underline{\underline{z}}(s) \\
\underline{\omega}(s)
\end{array}\right]=\left[\begin{array}{c}
D_{\alpha} \underline{x}_{0} \\
0
\end{array}\right]
$$

where $U_{n}\left(D_{\alpha}\right)$ denotes the largest $\left(D_{\alpha}^{-1} F_{1}, D_{\alpha}^{-1} F_{2}\right)$-invariant subspace contained in Ger $C$.

The proof can be achieved by inverting $D_{\alpha}$ and applying LEMA I. IO.
Now let $\Sigma_{\alpha_{0}}$ be a controllable no observable system as described before and let rank $F_{2}<n$ and rank $C<n$. Then by COROILARY 3.10 and TH i 3.13 it is known that $D_{\alpha}$ matrices leaving the transfer matrix invariant under SSE transformations are not unique. Now let a decomposition ( $D_{\alpha_{0}}, F_{1}, F_{2}, C$ ) be given and satisfy the above conditions. Then applying SSE

$$
\left[\begin{array}{cc}
M_{\alpha} & 0 \\
0 & I
\end{array}\right]\left[\begin{array}{cc}
D_{\alpha_{0}}-F_{1} & F_{2} \\
C & 0
\end{array}\right]\left[\begin{array}{cc}
N_{\alpha} & 0 \\
0 & I
\end{array}\right]=\left[\begin{array}{cc}
D_{\alpha} S-F_{1} & F_{2} \\
C & 0
\end{array}\right]
$$

( $\mathrm{H}_{\alpha}, M_{\alpha}$ ) -couples and by $M_{\alpha_{i}} D_{\alpha_{0}} N_{\alpha_{i}}=D_{\alpha_{i}}$ all $D_{\alpha_{i}}$ matrices can be conputed giving the following leman.

InTern 4.2
Let $V_{*}\left(D_{\alpha_{0}}\right)$ be the largest $\left(D_{\alpha_{0}}^{-1} F_{1}, D_{\alpha_{0}}^{-1} F_{2}\right)$-invariant subspace contained in $\operatorname{Ker} C$ of $\Sigma_{\alpha_{0}}$, then

$$
\begin{equation*}
N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right)=V_{*}\left(D_{\alpha_{i}}\right) \tag{92}
\end{equation*}
$$

where $N_{\alpha_{i}}$ is such that $D_{\alpha_{i}}=M_{\alpha_{i}} D_{\alpha_{0}} N_{\alpha_{i}}$.

Pioof: $L_{\text {et }} x_{0} \in V_{*}\left(D_{\alpha_{i}}\right)$, by ITRTiAA 4.1 we have

$$
\left[\begin{array}{cc}
D_{\alpha i} s-F_{1} & F_{2} \\
C . & 0
\end{array}\right]\left[\begin{array}{l}
\underline{\underline{\zeta}}(s) \\
-\underline{w}(s)
\end{array}\right]=\left[\begin{array}{c}
D_{\alpha i} \underline{x}_{0} \\
0
\end{array}\right]
$$

where $\underline{\xi}(s), \underline{w}(s)$ are strictly proper. Using SSE transformation we obtain

$$
\left[\begin{array}{cc}
M_{\alpha i} & 0 \\
0 & I
\end{array}\right]\left[\begin{array}{cc}
D_{\alpha_{0}} s-F_{1} & F_{2} \\
C & 0
\end{array}\right]\left[\begin{array}{cc}
N_{\alpha i} & 0 \\
0 & \pm
\end{array}\right]\left[\begin{array}{l}
\underline{\xi}(s) \\
\underline{\omega}(s)
\end{array}\right]=\left[\begin{array}{c}
M_{\alpha i} D_{\alpha_{0}} N_{\alpha_{i} x_{0}} \\
0
\end{array}\right]
$$

or

$$
\left[\begin{array}{cc}
D_{\alpha_{0}} s-F_{1} & F_{2}  \tag{93}\\
C & 0
\end{array}\right]\left[\begin{array}{c}
N_{\alpha_{i}} \underline{\xi}(s) \\
-\omega(s)
\end{array}\right]=\left[\begin{array}{c}
D_{\alpha_{0}} N_{\alpha_{i}} \underline{x}_{0} \\
0
\end{array}\right]
$$

Since $N_{d i} \underline{Z}_{\underline{( }}(s)$ is strictly proper we have by LEMMA 4.1

$$
N_{\alpha_{i}} x_{0} \in V_{*}\left(D_{\alpha_{0}}\right) \Rightarrow x_{0} \in N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right)
$$

equivalently

$$
\begin{equation*}
V_{*}\left(D \alpha_{i}\right) \subset N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right) \tag{94}
\end{equation*}
$$

Similarly let $\underline{x}_{0} \in N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right)$, we have by nonsingularjty of $N_{\alpha_{i}}$, $N_{\alpha_{i} \underline{x}_{0}} \in V_{*}\left(D_{\alpha_{0}}\right)$. Using the above procedure we obtain

$$
\begin{equation*}
\underline{x}_{0} \in V_{*}\left(D_{\alpha_{i}}\right) \Rightarrow N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right) \subseteq V_{*}\left(D_{\alpha_{i}}\right) \tag{95}
\end{equation*}
$$

Hence (94) and (95) imply:

$$
V_{*}\left(D_{\alpha_{i}}\right)=N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right)
$$

How we will consider two different disturbance component structures for which we will formulate the solvability of DDP.

## CASE I:

Consider the system

$$
\begin{align*}
D_{\alpha_{0}} \underline{x} & =F_{1} \underline{x}+F_{2} \underline{u}+E_{d_{0}} \underline{q}  \tag{96}\\
\underline{y} & =C \underline{x} \tag{97}
\end{align*}
$$

where $E_{\alpha_{0}} \underline{9}$ is the disturbance component. We assume that the application of SSE transformations also changes the matrix $E_{\alpha_{0}}$. Hence while the input/output transfer function remains the same, disturbance/output transfer function may change.

More precisely

where $E_{\alpha_{i}}=M_{\alpha_{i}} E_{\alpha_{0}}$. The system: $\Sigma_{\alpha}$ is shown in Figure 4.1.


Figure 4.1: Structure of case.
Now in general, by THA 2.1 we have the following lemma for the solvabilite of DDE in CASE I.

LETA 4.3
DDP is solvable if and only jiff

$$
\mathrm{D}_{\alpha_{i}}^{-1} \varepsilon_{\alpha_{i}} \subseteq V_{*}\left(\mathrm{D}_{\alpha_{i}}\right)
$$

This trivial rosul.t can be completed by

IRMA 4.4

$$
D_{\alpha_{0}}^{-1} \varepsilon_{\alpha_{0}} \subseteq V_{*}\left(D_{\alpha_{0}}\right) \text { if and only if } D_{\alpha_{i}}^{-1} \varepsilon_{\alpha_{i}} \subseteq V_{*}\left(D_{\alpha_{i}}\right)
$$

## Proof:

$$
\text { Let } D_{\alpha_{i}}^{-1} \varepsilon_{\alpha_{i}} \subseteq v_{*}\left(D_{\alpha_{i}}\right)=N_{\alpha_{i}}^{-1} v_{*}\left(D_{\alpha_{0}}\right) \quad \text { by LImit } 4.2
$$

Since $\quad E_{\alpha_{i}}=M_{\alpha_{i}} E_{\alpha_{0}}$

$$
D_{\alpha_{i}}^{-1} \varepsilon_{\alpha_{i} .} \subseteq N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right) \Leftrightarrow D_{\alpha_{i}}^{-1} M_{\alpha_{i}} \varepsilon_{\alpha_{0}} \subseteq N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right)
$$

Since $\quad D_{\alpha_{i}}, M_{\alpha_{i}}$ are nonsingular

$$
\begin{aligned}
& D_{\alpha_{i}}^{-1} M_{\alpha_{i}} \varepsilon_{\alpha_{0}} \subseteq N_{\alpha_{i}}^{-1} v_{*}\left(D_{\alpha_{0}}\right) \Leftrightarrow N_{\alpha_{i}}^{-1} D_{\alpha_{0}}^{-1} M_{\alpha_{i}}^{-1} M_{\alpha_{i}} \varepsilon_{\alpha_{0}} \subseteq N_{\alpha_{i}}^{-1} v_{*}\left(D_{\alpha_{0}}\right) \\
& \Leftrightarrow D_{\alpha_{0}}^{-1} \varepsilon_{\alpha_{0}} \subseteq V_{*}\left(D_{\alpha_{0}}\right)
\end{aligned}
$$

Hence by LBINA 4.4 we show, assuming the structure of ChaSE I, that we bring no improvement in the solvability range of DDI. In this case DDP is either solvable in all systems $\Sigma_{\alpha}$ obtained by SSI or not solvable at all.

## CASE II:

$$
\begin{align*}
& \text { Now let the system be described by } \\
& D_{\alpha_{i}} \underline{\dot{x}}=F_{1} \underline{x}+F_{2} \underline{u}+E_{\underline{q}}  \tag{98}\\
& \underline{y}=C \underline{x} \tag{99}
\end{align*}
$$

Ye assume that we want to decouple a noise component $E_{q}$ without changing input/output transfer matrix and without violating the observability and the controllability, by property selecting $D_{\alpha_{i}}$. In other words the general description for the SSH systems is shown in Fig.4.2.


FIGURE 4.2: STRUCTURE OF CASEI

The following lemmas clearify the range of the solutions for $\operatorname{DDP}$ $n$ the CASE II.

MA 4.5
Given the systems $\Sigma_{\alpha}$ of CASP II, DDP is solvable for both $\Sigma_{\alpha_{0}}$ and $\Sigma_{\alpha_{i}}$ if and only if $\mathcal{K} \subseteq V_{M_{k i}}$, where $V_{M_{i}}$ is the $M_{\alpha_{i}}$ invarint subspace of $D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right)$.
roof ${ }^{\prime \prime} \Rightarrow$ "
DDP is solvable for $D_{\alpha_{i} i f}$ and only if $D_{\alpha_{i}}^{-1} \xi \subseteq V_{*}\left(D_{\alpha_{i}}\right)$
$r \mathcal{G} \subseteq D_{\alpha_{i}} V_{*}\left(D_{\alpha_{i}}\right) \quad$ By Lem 4.2

$$
\begin{aligned}
& V_{*}\left(D_{\alpha_{i}}\right)=N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right) \\
& \varepsilon \subseteq M_{\alpha_{i}} D_{\alpha_{0}} N_{\alpha_{i}} N_{\alpha_{i}}^{-1} V_{*}\left(D_{\alpha_{0}}\right)=M_{\alpha_{i}} D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right)
\end{aligned}
$$

inge $D D P$ is solvable for $D \alpha_{0}$

$$
\varepsilon \subseteq D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right)
$$

ene $D D P$ is solvable simultaneously for $D_{\alpha_{0}}$ and $D_{\alpha_{1}}$ implies

$$
\xi \subseteq M_{\alpha_{i}} D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right) \cap D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right) \triangleq V_{M_{\alpha_{i}}}
$$



$$
\varepsilon \subseteq V_{M_{\alpha_{i}}} \Rightarrow\left\{\begin{array}{l}
\varepsilon \subseteq M_{\alpha_{i}} D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right) \Rightarrow \varepsilon \subseteq D_{\alpha_{i}} V_{*}\left(D_{\alpha_{i}}\right) \\
\varepsilon \subseteq D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right) \Rightarrow \varepsilon \subseteq D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right)
\end{array}\right.
$$

Thus by IMITA 4.5 we understand that, given ( $D_{\alpha_{0}}, F_{1}, F_{2}, E, C$ ) for which DDP is solvable, DDF will also be solvable for all ( $D_{\alpha i}, F_{1}, F_{2}$, $E, C$ ) obtained using ( $M_{\alpha i}, N_{\alpha}$ ) couples for which $\mathcal{E}$ is $M_{\alpha i}-i n v a-$ riant.

IEIMA 4.6
Given $\Sigma_{\alpha_{0}}$, I.et DDP be unsolvable for $\Sigma_{\alpha_{0}}$. Then DDP jis not solvable for $\Sigma_{\alpha_{i}}$ obtajned using ( $M_{\alpha_{i}} N_{\alpha}$ )-couples for which the subspace $D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right) \quad$ is $M_{d i-i n v a r i a n t}$.

The proof of LEPA 4.6 is straight forward since $\xi \notin D_{\alpha_{0}} v_{*}\left(D_{\alpha_{0}}\right)=M \alpha_{i} D_{\alpha_{0}} v_{*}\left(D_{\alpha_{0}}\right)=D_{\alpha_{i}} V_{*}\left(D_{\alpha_{i}}\right)$.

Hence by limma 4.6 we see that we may have an inprovement for the sdlvability of $D D P$.If for the initial decomposition ( $D_{\alpha}, F_{1}$, $F_{2}, E_{1} C$ ) DDP is not solvable, we will find out all ( $M_{\alpha_{i}}, N_{\alpha_{i}}$ ) couples for which $D_{\alpha_{0}} V_{X}\left(D_{\alpha_{0}}\right)$ is $M_{\alpha_{i}}$-invariant and exclude thern. The condition for the solvability of DDP can be sa.tisfied in systems $\Sigma_{\alpha}$ obtained using the remaining ( $\left.M \alpha_{i}, N_{\alpha_{i}}\right)$-couples.

In fact it will be wise to determine first ( $M_{\alpha_{i}}, N_{\alpha_{i}}$ )-couples parametrically and check if the condition $\mathcal{E} \subseteq M_{d_{i}} D_{\alpha_{0}} \mathcal{V}_{*}\left(\Gamma_{\alpha_{k}}\right)$ is satisfied for any choice of free paraneters.We will close this chapter by the following example illustrating the above results.

## EXAMPIE 4.1.:

Consider the controllable and observable systern $\Sigma_{\alpha_{0}}$ described by the equations:

$$
\left[\begin{array}{cccc}
1 & -1 & 1 & -2 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{array}\right] \underline{\underline{x}}=\left[\begin{array}{llll}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right] \underline{x}+\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 0
\end{array}\right] \underline{u}+\left[\begin{array}{l}
e_{1} \\
e_{2} \\
e_{3} \\
e_{4}
\end{array}\right] \underline{q}
$$

$$
\underline{y}=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \underline{x}
$$

where $\underline{x}$ is the state vector, $\underline{\mu}$ is the input vector, $\underline{q}$ is the distrbonce vector and $\underline{y}$ is the output vector. We will first try to find out-the constraints in the structure of $\mathbb{E}$ matrix for DDP to be solvable in $\Sigma_{\alpha_{0}}$.According to LJMAA 4.7 DDP is solvable if and only if there exists strictly proper $\underline{X}(s)$ and $\underline{U}(s)$ such that

$$
\left[\begin{array}{cccc:cc}
s-1 & -s & s & -2 s-1: & 0 & 0 \\
0 & s-1 & -s & s & 1 & 0 \\
0 & 0 & s & -s & 0 & 1 \\
0 & 0 & -1 & s & 0 & 0 \\
\hdashline 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
x_{1}(s) \\
x_{2}(s) \\
x_{3}(s) \\
x_{4}(s) \\
\hdashline u_{1}(s) \\
u_{2}(s)
\end{array}\right]=\left[\begin{array}{c}
e_{1} \\
e_{2} \\
e_{3} \\
e_{4} \\
\cdots \\
0 \\
0
\end{array}\right]
$$

explicitly we obtain the Following equations

$$
\begin{array}{r}
(s-1) x_{1}(s)-s x_{2}(s)+s x_{3}(s)-(2 s+1) x_{4}(s)=e_{1} \\
(s-1) x_{2}(s)-s x_{3}(s)+s x_{4}(s)+y_{1}(s)=e_{2} \\
s x_{3}(s)-s x_{4}(s)+u_{2}(s)=e_{3} \\
-x_{3}(s)+s x_{4}(s)=e_{4} \\
x_{1}(s)=0 \\
x_{4}(s)=0 \tag{105}
\end{array}
$$

$$
(100)
$$

$$
(101)
$$

(104)

Then (104), (105) implies that
$E_{q}(100) \Leftrightarrow-S x_{2}(s)+s x_{3}(s)=e_{1}$
$E_{q} \cdot(101) \Leftrightarrow(s-1) x_{2}(s)-s x_{3}(s)+u_{1}(s)=e_{2}$
$E_{q} .(102) \Leftrightarrow s x_{3}(s)+u_{2}(s)=e_{3}$
$E_{q}(103) \Leftrightarrow \quad-x_{3}(s)=e_{4}$

How for DDT to be solvable

$$
E_{q}(109) \Rightarrow-x_{3}(s)=e_{4}=0
$$

since $x_{3}(s)$ is required to be strictly proper,

$$
E_{q} \cdot(108) \Rightarrow u_{2}(s)=e_{3}=0
$$

since $U_{2}(s)$ must be strict iv proper and then

$$
\text { Eq. } 106) \Rightarrow x_{2}(s)=-\frac{e_{1}}{s} \quad \text { which is striction proper, }
$$

## finally

$E_{q .}(107) \Rightarrow u_{1}(s)=\frac{s e_{2}+s e_{1}-e_{1}}{s}$ which is strictly proper only fore $\quad\left[e_{2}=-e_{1}\right.$.
Hence DDP is solvable if and only if


In fact one can immediately check that

$$
I_{m} E=\operatorname{span}\left[\begin{array}{ccc}
1-1 & 0 & 0
\end{array}\right]^{\top}=D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right) \text {. To be more explicit, we can }
$$ construct the subspace $V_{*}\left(D_{\alpha_{0}}\right)$ by using the Flowchart 1.3. We define first

$$
A \triangleq D_{\alpha_{0}}^{-1} F_{1}=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{array}\right] \quad B \triangleq D_{\alpha_{0}}^{-1} F_{2}=\left[\begin{array}{ll}
1 & 0 \\
1 & 1 \\
0 & 1 \\
0 & 0
\end{array}\right]
$$

Then we obtain

Setting

$$
\operatorname{Ker} C=\operatorname{span}\left\{\left[\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right]\left[\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right]\right\}
$$

$$
V^{\circ} \triangleq K_{e r} C
$$

from

$$
V^{i}=V^{i-1} \cap A^{-1}\left(V^{i-1}+B\right)
$$

iteration

$$
V^{\prime}=V^{0} \cap A^{-1}\left(V^{0}+\beta\right) \quad \text { where }
$$

$$
=\operatorname{span}\left\{\left[\begin{array}{l}
1 \\
1 \\
0 \\
0
\end{array}\right]\left[\begin{array}{l}
0 \\
1 \\
1 \\
0
\end{array}\right]\right\} \text { and } V^{0}+B=\operatorname{span}\left\{\left[\begin{array}{lll}
1 \\
0 \\
0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right\}
$$

ne $A^{-1}$ denotes the "functional inverse of map $A$ ", by forming

$$
A \underline{x}=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right]=\left[\begin{array}{c}
x_{1}+x_{2}+x_{3}+x_{4} \\
x_{2} \\
x_{3} \\
x_{3}
\end{array}\right]
$$

$A \underline{x} \in V^{0}+B$ if and only if $x_{3}=0$

$$
\left.j^{0}+B\right)=\operatorname{span}\left\{\left[\begin{array}{l}
1 \\
1 \\
0 \\
0 \\
0 \\
0
\end{array}\right]\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\left[\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right]\right) \text { and } V^{0} \cap A^{-1}\left(V^{0}+B\right)=\operatorname{span}\left\{\left[\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right]\right\}=V^{1}
$$

iteration

$$
\begin{aligned}
& V^{2}=V^{1} \cap A^{-1}\left(V^{\prime}+\beta\right)=\operatorname{span}\left\{\left[\begin{array}{llll}
0 & 1 & 0 & 0
\end{array}\right]^{\top}\right\}=V^{1}=V_{*}\left(D_{\alpha_{0}}\right) . \\
& \\
& V_{*}\left(D_{\alpha_{0}}\right)=\operatorname{span}\left\{\left[\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right]\right\} \quad \text { and according to the }
\end{aligned}
$$

II 2.1 , DDP is solvable for $\Sigma_{\alpha_{0}}$ if and only if $\mathcal{G} \subseteq D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right)$, ie.

$$
\mathcal{G} \subseteq \operatorname{span}\left\{\left[\begin{array}{llll}
-1 & 1 & 0 & 0
\end{array}\right]^{\top}\right\}
$$

Now we will try to out if we have any important improvement to enlarge the subspace $D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right)$, using the procedure defined in CASE II.

> Let

$$
M_{d_{i}} \triangleq\left[\begin{array}{llll}
m_{1} & m_{2} & m_{3} & m_{4} \\
m_{5} & m_{6} & m_{7} & m_{8} \\
m_{9} & m_{10} & m_{11} & m_{12} \\
m_{13} & m_{14} & m_{15} & m_{16}
\end{array}\right] \quad N_{\alpha i} \triangleq\left[\begin{array}{llll}
n_{1} & n_{2} & n_{3} & n_{4} \\
n_{5} & n_{6} & n_{7} & n_{8} \\
n_{9} & n_{10} & n_{11} & n_{12} \\
n_{13} & n_{44} & n_{15} & n_{16}
\end{array}\right]
$$

Then according to COROLLARY 3.10.c, we determine the free parameters of $M_{d_{i}}$. and $N_{\alpha_{i}}$ using the equations

$$
\begin{align*}
M_{\alpha_{i}} F_{2} & =F_{2}  \tag{110}\\
C N_{d_{i}}^{-1} & =C  \tag{111}\\
M_{\alpha_{i}} F_{1} & =F_{1} N_{\alpha_{i}}^{-1} \tag{112}
\end{align*}
$$

where
$E_{q}(110) \Rightarrow M_{\alpha_{i}}=\left[\begin{array}{cccc}m_{1} & 0 & 0 & m_{4} \\ m_{5} & 1 & 0 & m_{8} \\ m_{9} & 0 & 1 & m_{12} \\ m_{13} & 0 & 0 & m_{16}\end{array}\right]$ and $E_{q}(111) \Rightarrow N_{\alpha_{i}}^{-1}=\left[\begin{array}{cccc}1 & 0 & 0 & 0 \\ \hat{n}_{5} & \hat{n}_{6} & \hat{n}_{7} & \hat{n}_{8} \\ \hat{n}_{7} & \hat{n}_{10} & \hat{n}_{11} & \hat{n}_{12} \\ 0 & 0 & 0 & 1\end{array}\right]$
$\Lambda_{\alpha_{i}}=\left[\begin{array}{cccc}1 & 0 & 0 & 0 \\ \frac{n_{7} n_{12}-n_{5} n_{11}}{n_{11}} & 1 & 0 & \frac{-n_{7}}{n_{11}} \\ 0 & 0 & 1 & 0 \\ -\frac{n_{12}}{n_{11}} & 0 & 0 & \frac{1}{n_{11}}\end{array}\right] \quad N_{\alpha_{i}}=\left[\begin{array}{cccc}1 & 0 & 0 & 0 \\ n_{5} & 1 & n_{7} & n_{5} \\ n_{12} & 0 & n_{11} & n_{12} \\ 0 & 0 & 0 & 1\end{array}\right]$
where. $n_{5}, n_{7}, n_{12} \in \mathbb{R} \quad$ and $n_{11} \in \mathbb{R}-\{0\}$
And in general with
$D_{\alpha_{i}}=M_{\alpha_{i}} D_{\alpha_{0}} N_{\alpha_{i}}$, keeping $F_{1}, F_{2}, C$ fixed we obtain the same transfer function matrix. To judge whether there is an improvement to solve $\operatorname{DDP}$ we check

$$
D_{\alpha_{i}} V_{*}\left(D_{\alpha_{i}}\right)=M_{\alpha_{i}} D_{\alpha_{0}} V_{*}\left(D_{\alpha_{0}}\right)
$$

$$
\left.=\text { span }\left\{\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\frac{n_{7} \cdot n_{12}-n_{5} n_{11}}{n_{11}} & 0 & -\frac{n_{7}}{n_{11}} \\
0 & 0 & 1 & 0 \\
-\frac{n_{12}}{n_{11}} & 0 & 0 & \frac{1}{n_{11}}
\end{array}\right] 00\right]\left[\begin{array}{c}
1 \\
0
\end{array}\right]\right\}=\text { span }\left\{\left[\begin{array}{c}
-1 \\
\frac{n_{7} n_{21}-n n_{5} n_{1}}{n_{11}} \\
0 \\
\frac{n_{12}}{n_{11}}
\end{array}\right]\right)
$$

How

$$
D_{d_{i}} V_{*}\left(D_{\alpha_{i}}\right) \subseteq D_{d_{0}} V_{\neq}\left(D_{\alpha_{0}}\right)
$$

$n_{12}=0$ and $n_{5}=0$. Other wise we obtain

$$
D_{\alpha_{i}} v_{*}\left(D_{\alpha_{i}}\right)=s p a n
$$

$$
\left\{\left[\begin{array}{c}
-1 \\
\alpha \\
0 \\
\beta
\end{array}\right]\right\}
$$

with $(\alpha, \beta) \in \mathbb{R}_{-}\{0\}$

- equivalently

$$
\xi \subseteq \operatorname{span}\left\{\left[\begin{array}{c}
-1 \\
\alpha \\
0 \\
\beta
\end{array}\right]\right\}
$$

and the structure of F matrix for DDP to be solvable is

$$
E=\left[\begin{array}{c}
-q \\
\alpha q \\
0 \\
\beta q
\end{array}\right]=\left[\begin{array}{c}
e_{1} \\
e_{2} \\
0 \\
e_{4}
\end{array}\right] \quad \text { with } e_{1}, e_{2}, e_{4} \in \mathbb{R} \text {, arbitrary. }
$$

Hence choosing suitable ( $M_{\alpha_{i}}, N_{\alpha_{i}}$ )-couples we can solve DDP for $E=\left[\begin{array}{llll}e_{1} & e_{2} & 0 & e_{4}\end{array}\right]^{\top}$ by simply changing $D_{\alpha_{i}}$ matrices and keeping the
the transfer function matrix unchanged.
Thus the subspace that $\mathcal{E}$ must be included is determined by the product of $D_{\alpha_{0}} V_{*}\left(O_{\alpha_{0}}\right)$ and the degree of freedom obtained in $M_{\alpha i}$ in general.

## V. CONCIITSION

In this work,it is presented a report which consists of two seperated parts:a survey and an investigation. While completing,it will be usefull to give some extra notes to emphasize the importance of various concepts encountered.

The basic reference for the "Geometric Approach" is [1]. However it is full of sophisticated mathematics and concepts which make the study for a beginner, difficult. So, the aim of the survey is to give the essential idea introduced by this new approach. For thin, basic concepts (such as A-invariance, ( $A, B$ ) -invariance, Stabilizability) and basic problems (such as DDP, SDDP, OSDP, ONC) are studied as simply as possible.Thinking of the fact that most of the readers are familiar with the frequency domain approach,frequency domain treatment of these basic concepts and problems is also studied as introduced in $[6,7,8,9]$. So, a survey of the modern control problems chosen is obtained, complete with their solutjons the geometric framework and in the frequency domain.

The "geometric approach" will tend to be the "exponent" of the "Modern Control Theory", in the future. In this new framework, we believe that it will be possible to easily investigate the properties of the solvability criteria of new control problems and the other relations existing in the diverging methods of the Control Theory.

The second part which is an investigation is believed to be new. The problem is essentially based on the work introduced in [10]. The considered linear m-part has a special configuration
which is obtained using a first level decomposition, and the obtained state equations are not in stete-space form. Hence the essential idea is to investigate if the "extra paraneter:matrix D " brings any degree of freedon. It has to be mentioned here that in this study it is only considered completely contollable and observable systems of the above structure, in which statespace form of the state equations can be obtained by simply inverting D matrix. In Chapter III, the mathematical tools used are presented in full detail, and an important property of such syrstems is presented by THM 3.11. This result is totally new. Based on this property of "the existence of D matrices leaving transfer function invariant while $F_{1}, F_{2}, C$ kept unchanged" another question is asked and answered in Chapter IV : To use D matrices for decoupling the disturbance at the output. The result obtained is that in the disturbance structure of $\mathrm{C} \Lambda \mathrm{ST}$ II the dimension of $V_{*}$, the maximal. ( $A, B$ )-invariant subspace in Ker C, does not increased, however a larger dagree of freedom may be obtained for some $D$. As seen in the example 4.1 a $V_{*}$ which is initially equal to $\operatorname{span}\left\{\left[\begin{array}{lll}-1 & 1 & 0\end{array}\right]^{T}\right\}$ can be mapped to $\operatorname{span}\left\{\left[\begin{array}{lll}1 \alpha & \circ \beta\end{array}\right]^{T}\right\}$ with $\alpha, \beta \in \mathbb{R}$-\{o\}arbitrary for a suitable choice of $D$;furthermore we also know that this choice $D$ does not change the transfer function matrix of the system.

As we stated in the context,it can be expected that it finds an efficient area of application in the electrical an electronic circuitry designes. The analysis related to the solvability of DDP may be extended, to more generic decoupling problems. For further investigations the determination of the boundaries of the equivalence class of ( $M, N$ ) matrix couples using the information brought by the stucture of $X_{1}, F_{2}, C$ matrices, can be suggested.

In general, the study includes an introductive survey, concerning both the geometric and frequency domain treatment of some new concepts; and the second part may be considered as a partial application of the methods that make part of the survey.

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